Week 6: Collaborative Based RSs – Part II

CSX4207/ITX4207: Decision Support and Recommender Systems

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Objectives

- To introduce model-based collaborative based filtering algorithms

Outlines

- Model-based collaborative based filtering algorithms

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Recall the Main Idea of Collaborative Approach

- To exploit information about the past behavior/opinions of an existing user community.
- To predict which items the current user will most probably like.

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How to Generate Recommendation Using Collaborative Based Filtering Approach

Target
User's
Preferences

Recommendation
Algorithm(s)

Prediction
(numerical value)

Rank

Recommendations
(A list of top-k suggested items)

Community's
Preferences

Collaborative: "Tell me what's popular among my peers"

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Memory-based VS Model-based Approaches

Memory-based approach

- Modeless (no model created)
- Directly applying rating matrix to find neighbors and to recommend items.
- Time consuming and not scalable
- Example, user-based NN

Model-based approach

- Offline process the raw data.
- At run time, require only the “learned” model to make prediction.
- Update/retrained the model periodically.
- Example, matrix factorization methods, association rule mining, etc.

Model-based Algorithms

- Association rule mining
- Probabilistic recommendation approaches
- Slope one predictor
- Matrix factorization methods

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Association Rule Mining

- **Basic idea:** identify **rule-like relationship patterns** (X -> Y) in large-scale sales transactions.

Example,

- ***If*** a customer **buys** baby food ***then*** he/she **also buys** diapers in 70 percent of the cases.
  {baby food} -> {diapers}

- In RSs, "**If** user *X* **liked** both *item1* and *item4*, **then** *X* will **most probably also like** *item3*."

  {Item1, Item4} -> {Item3}

Transaction ID	**Baby Food**	Bread	Beer	**Diaper**	Coke
1	**Y**		Y	**Y**	
2		Y			Y
3	**Y**		Y	**Y**	
4	**Y**	Y	Y	**Y**	
5		Y			
6					Y
7	**Y**		Y	**Y**	Y
8	**Y**	Y	Y	**Y**	
9	**Y**	Y	Y	**Y**	
10	**Y**	Y	Y	**Y**	

Generating Association Rules

- Step 1: Frequent Itemset Generation
  - Generate all **itemsets whose support ≥ min_support_threshold**

- Step 2: Rule Generation
  - Generate **high confidence rules** from each frequent itemset, where each rule is a binary partitioning of a frequent itemset

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Step 1: Frequent Itemset Generation

User-Item rating matrix

	Item1	Item2	Item3	Item4
User1	5	3	3	
User2	4	5	4	5
User3	5	5		4
User4	4	3	3	5
User5	5	4		4
User6				4

Terminologies:

$Itemset_{length\_k}$ = a set of k-items

e.g., $\text{Itemset}_{\text{length\_2}}$ = { {Item1, Item2}, {Item1, Item3},
{Item1, Item4 }, {Item2, Item3},
{Item2, Item4}, {Item3, Item4} }

min_support_threshold: a user-defined value in the range [0 – 1].

$FrequentItemset_{length\_k}$ = a set of k-items whose **support** is greater than or equal to *min_support_threshold*

$$support(x) = \frac{count(x)}{n},$$

where
count(x): no. of rows that *x* present in a dataset
n: no. of rows in a dataset
x: an itemset in a dataset

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Step 1: Frequent Itemset Generation – *Cont.*

User-Item rating matrix

	Item1	Item2	Item3	Item4
User1	5	3	3	
User2	4	5	4	5
User3	5	5		4
User4	4	3	3	5
User5	5	4		4
User6				4

Suppose that *min_support_threshold* = 4/6 = 0.67
$Itemset_{length\_2}$ = { {Item1, Item2}, {Item1, Item3}, {Item1, Item4}, {Item2, Item3}, {Item2, Item4}, {Item3, Item4} }
$FrequentItemset_{length\_2}$ = { {Item1, Item2}, {Item1, Item4}, {Item2, Item4} }
support({Item1, Item2}) = 5/6 = 0.83
support *({Item1, Item3})* = 3/6 = 0.5
support({Item1, Item4}) = 4/6 = 0.67
...
support({Item2, Item4}) = 4/6 = 0.67

Terminologies:

$Itemset_{length\_k}$ = a set of *k*-items

min_support_threshold: a user-defined value in the range [0 – 1].

$FrequentItemset_{length\_k}$ = a set of *k*-items whose support is greater than or equal to *min_support_threshold*

$$support(x) = \frac{count(x)}{n},$$

where *count(x)*: no. of rows that *x* presents in a dataset

n: no. of rows in a dataset

x: an itemset in a dataset

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Step 2: Rule Generation

- Generate frequent rules *(high confidence rules)* from each frequent itemset, where each rule is a binary partitioning of a frequent itemset

User-Item rating matrix

	Item1	Item2	Item3	Item4
User1	5	3	3	
User2	4	5	4	5
User3	5	5		4
User4	4	3	3	5
User5	5	4		4
User6				4

Terminologies:
a rule, $x \rightarrow y$ = a **co-occurrence** of two itemsets, x and y

e.g., the *candidate rules* created from the frequent itemset;

Given
$\text{Frequent\_Itemset}_{\text{length\_2}}$ = {{Item1, Item2}, {Item1, Item4}, {Item2, Item4}}

∴ Candidate Rules = { Item1 -> Item2,
Item2 -> Item1,
Item1 -> Item4,
Item4 -> Item1,
Item2 -> Item4,
Item4 -> Item2 }

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Step 2: Rule Generation – *Cont.*

- Generate frequent rules *(high confidence rules)* from each frequent itemset, where each rule is a binary partitioning of a frequent itemset

User-Item rating matrix

	Item1	Item2	Item3	Item4
User1	5	3	3	
User2	4	5	4	5
User3	5	5		4
User4	4	3	3	5
User5	5	4		4
User6				4

Terminologies:

a rule, $x \rightarrow y$ = a co-occurrence of two itemsets, x and y

$$confident\ of\ the\ rule, conf(x \rightarrow y) = \frac{support(x \cap y)}{support(x)} = \frac{count(x \cap y)}{count(x)}$$

min_confident_threshold: a user-defined value in the range [0 – 1].

Given

Frequent_Itemset$_{length2}$ = {{Item1, Item2}, {Item1, Item4}, {Item2, Item4}} ,

and suppose that Min_confident_threshold = 0.8

∴ Frequent Rules = {
Item1 -> Item2 (conf = 5/5 =1),
Item2 -> Item1 (conf = 5/5 = 1),
Item1 -> Item4 (conf = 4/5 = 0.8),
Item4 -> Item1 (conf = 4/5 = 0.8),
Item2 -> Item4 (conf = 4/5 = 0.8),
Item4 -> Item2 (conf = 4/5 = 0.8)
}

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Step 2: Rule Generation – *Cont.*

- Another example,

Alter the dataset

	Item1	Item2	Item3	Item4
User1	5	3	3	4
User2	4	5	4	5
User3	5	5		4
User4	4	3	3	5
User5	5	4		4
User6				4

Given

Frequent_Itemset$_{length_3}$ = { {Item1, Item2, Item4} },

and suppose that min_confident_threshold = 0.9

Candidate Rules = {
Item1 -> Item2, Item4 (conf = 5/5 =1.0),
Item2 -> Item1, Item4 (conf = 5/5 = 1.0),
Item4 -> Item1, Item2 (conf = 5/6 = **0.8**), << *not frequent!!*
Item1, Item2 -> Item4 (conf = 5/5 = 1.0),
Item1, Item4 -> Item2 (conf = 5/5 = 1.0),
Item2, Item4 -> Item1 (conf = 5/5 = 1.0)
}

Frequent Rules = {
Item1 -> Item2, Item4 (conf = 5/5 =1.0),
Item2 -> Item1, Item4 (conf = 5/5 = 1.0),
Item1, Item2 -> Item4 (conf = 5/5 = 1.0),
Item1, Item4 -> Item2 (conf = 5/5 = 1.0),
Item2, Item4 -> Item1 (conf = 5/5 = 1.0)
}

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Applying Association Rules in RSs – 1/4

- Calculation of the set of interesting association rules with high confidence and support is performed offline.

1. Determine the set of $X \rightarrow Y$ association rules that are relevant for the target user.
2. Compute the union of items appearing in the consequent Y of these association rules not previously experienced by the target user.
3. Sort the products according to the confidence of the rule.
4. Return the first N elements of this ordered list as a recommendation.

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Applying Association Rules in RSs – 2/4

- Calculation of the set of interesting association rules with high confidence and support is performed offline.

1. Determine **the set of $X \rightarrow Y$ association rules** that are relevant for the target user (Item1).

   Frequent Rules = {r1: **Item1** -> Item2, r2: Item2 -> **Item1**, r3: **Item1** -> Item4, r4: Item4 -> **Item1**, r5: Item2 -> Item4, r6: Item4 -> Item2}

	Item1	Item2	Item3	Item4
User7	4			

2. Compute the union of items appearing in the consequent Y of these association rules not previously experienced by the target user.
3. Sort the products according to the confidence of the rule.
4. Return the first N elements of this ordered list as a recommendation.

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Applying Association Rules in RSs – 3/4

- Calculation of the set of interesting association rules with high confidence and support is performed offline.

1. Determine the set of $X \rightarrow Y$ association rules that are relevant for the target user.

   Frequent Rules = {r1: Item1 -> **Item2**, r2: Item2 -> **Item1**, r3: Item1 -> **Item4**, r4: Item4 -> **Item1**, r5: Item2 -> Item4, r6: Item4 -> Item2}

2. Compute the **union (∪)** of items appearing in the consequent Y of these association rules NOT previously experienced by the target user.

	Item1	Item2	Item3	Item4
User7	4			

   {Item1 -> **Item2**} ∪ {Item1 -> **Item4**}={Item2, Item4}

	Item1	Item2	Item3	Item4
User1	5	3	3	
User2	4	5	4	5
User3	5	5		4
User4	4	3	3	5
User5	5	4		4
User6				4

3. Sort the products according to the confidence of the rule.
4. Return the first N elements of this ordered list as a recommendation.

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Applying Association Rules in RSs – 4/4

- Calculation of the set of interesting association rules with high confidence and support is performed offline.

1. Determine the set of $X \rightarrow Y$ association rules that are relevant for the target user.
2. Compute the union of items appearing in the consequent Y of these association rules not previously experienced by the target user.
3. **Sort** the products according to the confidence of the rule.

   Rank 1: Item2 (because Conf(r1: Item1 -> **Item2**)= **1**)

   Rank 2: Item4 (because Conf(r3: Item1 -> **Item4**)= **0.8**)
4. Return the first N elements of this ordered list as a recommendation.
   1. Recommended items = {**Item2**, **Item4**}

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Probabilistic Recommendation Approaches

- **Basic idea:** Assigning an object to one of several predefined categories (classification problem)
- Example, Naïve Bayes classifiers

Naïve Bayes Classifiers

1. With conditionally independent attributes, calculate conditional probabilities using Bayes Theorem for each possible rating value (Y) given the target user's other ratings.

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \frac{\prod_{i=1}^{d} P(X_i|Y)P(Y)}{P(X)}$$

2. Select the rating value with the highest probability as a prediction.

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Example, given Alice's rating = {1, 3, 3, 2}

Predict Alice's rating for item 5

	Item 1	Item 2	Item 3	Item 4	Item 5
Alice	1	3	3	2	?

	Item 1	Item 2	Item 3	Item 4	Item 5
User 1	2	4	2	2	4
User 2	**1**	**3**	**3**	**5**	**1**
User 3	4	5	2	3	3
User 4	**1**	**1**	**5**	**2**	**1**

- P(Y)
  - P(Item5=1) = 2/4
  - P(Item5=2) = 0
  - P(Item5=3) = ¼
  - P(Item5=4) = ¼
  - P(Item5=5) = 0
- P(X|Item5=1) =

  P(Item1=**1**|Item5=**1**) *

  P(Item2=**3**|Item5=**1**) *

  P(Item3=**3**|Item5=**1**) *

  P(Item4=**2**|Item5=**1**)

  = 2/2 * 1/2 * 1/2 * 1/2

  = 0.125
- P(X|Item5=2) =

  P(Item1=**1**|Item5=2) *

  P(Item2=**3**|Item5=2) *

  P(Item3=**3**|Item5=2) *

  P(Item4=**2**|Item5=2)

  = 0/0 * … * … * …

  = 0

…

…

- P(X|Item5=5) =

  …

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Example - Cont.

- $$P(Y|X) = \frac{\prod_{i=1}^{d} P(X_i|Y)P(Y)}{\cancel{P(X)}}$$

Constant

predicted rating of item 5 for Alice = 1

- **P(Item5=1|X) = P(X|Item5=1) * P(Item5=1) = 0.125 * 2/4 = 0.0625**
- P(Item5=2|X) = P(X|Item5=2) * P(Item5=2) = 0 * 0 = 0
- P(Item5=3|X) = P(X|Item5=3) * P(Item5=3) = 0 * 1/4 = 0
- P(Item5=4|X) = P(X|Item5=4) * P(Item5=4) = 0 * 1/4 = 0
- P(Item5=5|X) = P(X|Item5=5) * P(Item5=5) = 0 * 0 = 0
- *Note: The data X exclude Alice's ratings*

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Slope One Predictor – 1/3

- Basic Idea: calculate "popularity differential" between items for users.

	Item1	Item5
Alice	2	?

	Item1	Item5
User1	1	2

popularity differential = 2-1 = 1

- Example 1,
  - Predict rating of item 5 for Alice *using Slope One Prediction*:

    = Alice's rating + popular differential

    = 2 + (2-1) = 3

- In reality,
  - There are several items rated by the *target user* (e.g., Alice)
  - There are several users rated the *target item* (e.g., item 5)
- Therefore,
  - The complete calculation of Slope One Prediction is shown in the next slide.

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Slope One Predictor - 2/3

- The average deviation *dev* between the target item *j*, and other items *i* of all **users co-rated both items**,

$$dev_{j,i} = \sum_{(u_j,u_i)\in S_{j,i}(R)} \frac{u_j - u_i}{|S_{j,i}(R)|}$$

Note: *j* is the target item that we want to predict rating for the target user *u*.

u_j: rating that a user *u* gives on item *j*
$S_{j,i}(R)$: no. of users rated both items *i*, *j*

- Predicted rating on the target item *j* for the **target user *u***,

$$pred(u,j) = \frac{\sum_{i\in Relevant(u,j)}(\boxed{dev_{j,i}} + u_i)}{|Relevant(u,j)|}$$

Note: u and u_i in the 2nd and 3rd formula refer to different users.

- Weighted predicted rating on the target item *j* for the **target user *u*** (weighting using no. of users co-rated both items),

$$pred(u,j) = \frac{\sum_{i\in S(u)-\{j\}}(\boxed{dev_{j,i}} + u_i) \times |S_{j,i}(R)|}{\sum_{i\in S(u)-\{j\}}|S_{j,i}(R)|}$$

Slope One Predictor – 3/3

	Item1	Item2	Item3
Alice	2	5	?

	Item1	Item2	Item3
User1	3	2	5
User2	4		3

- $dev_{Item3, Item1} = \Sigma(rating_{Item3} - rating_{Item1})/Num\_users\_rated\_both\_items$

  $= ((5-3) + (3-4)) / 2 = 0.5$

- $dev_{Item3, Item2} = \Sigma(rating_{Item3} - rating_{Item2})/Num\_users\_rated\_both\_items$

  $= (5-2) / 1 = 3$

- pred(Alice, Item3)

  $= [\ \Sigma(dev_{3,i} + rating_{Alice,Item\_i}) * Num\_users\_rated\_both\_items\ ] / \Sigma(Num\_users\_rated\_both\_items)$

  $= [\ \underbrace{((0.5+2)*2)}_{item1} + \underbrace{((3+5)*1)}_{item2}\ ]/ (2+1) = 4.33$

$$pred(u,j) = \frac{\sum_{i \in S(u)-\{j\}} (dev_{j,i} + u_i) \times |S_{j,i}(R)|}{\sum_{i \in S(u)-\{j\}} |S_{j,i}(R)|}$$

Matrix Factorization Methods

- **Basic idea:** derive a set of latent (hidden) factors from the rating patterns and characterize both user and items by such vectors of factors.
- A recommendation for an item *i* is made when the target user and the item *i* are similar wrt these factors.
- Example,
  - Singular Value Decomposition (SVD)
  - Alternating Least Square (ALS)

Additional Resource: ALS http://bugra.github.io/work/notes/2014-04-19/alternating-least-squares-method-for-collaborative-filtering/
Additional Resources: SVD https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

Singular Value Decomposition (SVD)

- **SVD Theorem:**
- A given matrix A can be decomposed into a product of three matrices using linear algebra

$$A = U\Sigma V^T$$

- where,
  - U = left singular vector
  - V = right singular vector
  - Σ = the singular values
- **Key idea:** retain only the most important features (with largest singular values) by taking only the first few columns of U and V^T.

Additional Resources: SVD Tutorial
https://datajobs.com/data-science-repo/SVD-Tutorial-[Kirk-Baker].pdf
http://web.mit.edu/be.400/www/SVD/Singular_Value_Decomposition.htm

SVD Example - 1/2

	User1	User2	User3	User4
Item1	3	4	3	1
Item2	1	3	2	6
Item3	2	4	1	5
Item4	3	3	5	2

U2 (Item)	
-0.4312	0.4931
-0.5327	-0.5305
-0.5237	-0.4052
-0.5058	0.5578

V2 (User)	
-0.3593	0.3676
-0.5675	0.0879
-0.4428	0.5686
-0.5938	-0.7305

Σ2	
12.2215	0
0	4.9282

*: The first two columns of U, V and Σ are used here.

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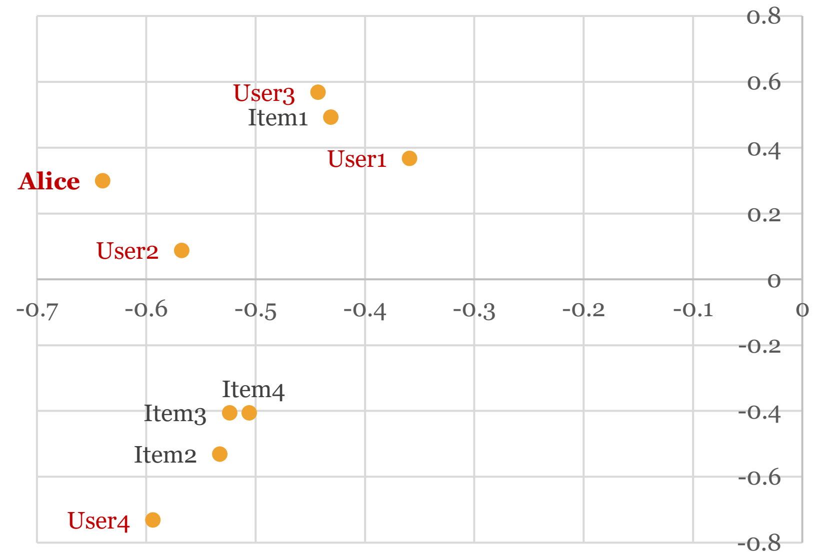
$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

SVD Example - 2/2

- Given Alice's rating vector [5, 3, 4, 4],
  - Calculate Alice's vector in 2D space:
    $\text{Alice}_{2D} = \text{Alice} \times U2 \times \Sigma_2^{-1}$
    $= [-0.64, 0.30]$
  - To suggest items, find the most similar users using the using the 2D matrix (V2), *e.g., cosine sim,* and suggest unseen, high-rating items

A plot of all users (items)' vector in 2D space:



	U2 (Item)	
Item1	-0.4312	0.4931
Item2	-0.5327	-0.5305
Item3	-0.5237	-0.4052
Item4	-0.5058	0.5578

	V2 (User)	
User1	-0.3593	0.3676
User2	-0.5675	0.0879
User3	-0.4428	0.5686
User4	-0.5938	-0.7305

Matrix Inverse: http://mathworld.wolfram.com/MatrixInverse.html

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Demonstration of Full SVD Calculation – 1/11

- Given
  - A rectangular matrix A
- Generate
  - An orthogonal matrix U
  - A diagonal matrix S
  - Transpose of an orthogonal matrix V^T

S.T.

- $A_{mn} = U_{mm}S_{mn}V^T_{nn}$

  where

  $U^TU = I$; the columns of U are orthonormal eignenvector of AA^T,

  $V^TV = I$; the columns of V are orthonormal eignenvector of A^TA, and

  S is a diagonal matrix containing the square roots of eigenvalues from U or V in descending order

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

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Demonstration of Full SVD Calculation – 2/11

Example:

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Step 1) Find U by start with

1.1. Find AA^T

$$A^T = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

Demonstration of Full SVD Calculation – 3/11

1.2. **Find Eigenvalues** of AA^T

- By Solving equation $A\vec{v} = \lambda\vec{v}$

$$\begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

rewrite this as the set of equations

$11x_1 + x_2 = \lambda x_1$ ➔ $(11 - \lambda)x_1 + x_2 = 0$ eq. 1

$x_1 + 11x_2 = \lambda x_2$ ➔ $x_1 + (11 - \lambda)x_2 = 0$ eq. 2

Solve for λ by setting the determinant of the coefficient matrix to zero,

$$\begin{vmatrix} (11 - \lambda) & 1 \\ 1 & (11 - \lambda) \end{vmatrix} = 0$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

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Demonstration of Full SVD Calculation – 4/11

Solve for λ by setting the determinant of the coefficient matrix to zero,

$$\begin{vmatrix} (11-\lambda) & 1 \\ 1 & (11-\lambda) \end{vmatrix} = 0$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - bc$$

Equivalent to

$$(11-\lambda)(11-\lambda) - 1 \cdot 1 = 0$$

$$(\lambda - 10)(\lambda - 12) = 0$$

$$\lambda = 10,$$

$$\lambda = 12$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

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Demonstration of Full SVD Calculation – 5/11

1.3. **Find corresponding eigenvectors** of the eigenvalues in 1.2. by substituting values in Eq. 1

For $\lambda = 10$,

$(11 - 10)x_1 + x_2 = 0 \rightarrow x_1 = -x_2$

From several valid values, let's select $x_1 = 1$ *and* $x_2 = -1$

For $\lambda = 12$,

$(11 - 12)x_1 + x_2 = 0 \rightarrow x_1 = x_2$

From several valid values, let's select $x_1 = 1$ *and* $x_2 = 1$

Construct a matrix from the eigenvectors sorted by eigenvalues in descending order: $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

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Demonstration of Full SVD Calculation – 6/11

- 1.4. Apply Gram-Schmidt orthonormalization process to the column vectors to **convert Eigenvectors (matrix) into an orthogonal matrix**.

$$\begin{array}{c} \overrightarrow{v_1} \\ \overrightarrow{v_2} \end{array} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\overrightarrow{u_1} = \frac{\overrightarrow{v_1}}{|\overrightarrow{v_1}|} = \frac{[1,1]}{\sqrt{1^2+1^2}} = \frac{[1,1]}{\sqrt{2}} = [\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$$

Compute $\overrightarrow{w_2} = \overrightarrow{v_2} - \overrightarrow{u_1} \cdot \overrightarrow{v_2} * \overrightarrow{u_1} = [1, -1] - \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] \cdot [1, -1] * \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right] = [1, -1]$

and normalize $\overrightarrow{u_2} = \frac{\overrightarrow{w_2}}{|\overrightarrow{w_2}|} = \left[\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right]$

to give $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

Demonstration of Full SVD Calculation – 7/11

Step 2) Find V by starting with

2.1. Find A^TA

$$A^TA = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

Demonstration of Full SVD Calculation – 8/11

2.2. Find Eigenvalues of A^TA

- By Solving equation $A\vec{v} = \lambda\vec{v}$ (Same as in 1.2.)
  - $\lambda = 0$,
  - $\lambda = 10$,
  - $\lambda = 12$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

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Demonstration of Full SVD Calculation – 9/11

2.3. **Find corresponding eigenvectors** of the eigenvalues in 2.2.

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

2.4. Apply Gram-Schmidt orthonormalization process to the column vectors to **convert Eigenvectors (matrix) into an orthogonal matrix**.

$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{-5}{\sqrt{30}} \end{bmatrix},$$

and calculate

$$V^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

Demonstration of Full SVD Calculation – 10/11

Step 3) Find S by taking the square roots of the non-zero eigenvalues and populate the diagonal with them, putting the largest in s_{11}, the next largest in s_{22} and so on.

- $\lambda = 10$,
- $\lambda = 12$

$$S = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

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Demonstration of Full SVD Calculation – 11/11

- $A_{mn} = U_{mm} S_{mn} V_{nn}^{T}$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Ref: https://davetang.org/file/Singular_Value_Decomposition_Tutorial.pdf

Root Mean Square Error (RMSE)

- **Root mean square error** takes the difference for each observed and predicted value.

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \hat{x}_i)^2}{N}}$$

where

- x_i : actual rating
- $\hat{x}_i$: predicted rating
- N : number of predicted items in a test set.

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Example

UserID	Item ID	Actual Rating	Predicted Rating
1	1	5	4.5
1	2	3	2.5
2	3	4	4
2	4	4	3.5
3	1	5	4.5
3	3	4	4

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \hat{x}_i)^2}{N}}$$

$$= \sqrt{\frac{(5-4.5)^2+(3-2.5)^2+(4-4)^2+(4-3.5)^2+(5-4.5)^2+(4-4)^2}{6}}$$

=0.41

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Further reading

- SVD:
  - https://alyssaq.github.io/2015/singular-value-decomposition-visualisation/
  - https://alyssaq.github.io/2015/20150426-simple-movie-recommender-using-svd/
- ALS:
  - https://towardsdatascience.com/prototyping-a-recommender-system-step-by-step-part-2-alternating-least-square-als-matrix-4a76c58714a1
- SLIM:
  - http://glaros.dtc.umn.edu/gkhome/node/774