5

Linear Time-Invariant Filters

5.0 Introduction

In Section 3.6 we discussed the convolution of two functions $g(\cdot)$ and $h(\cdot)$ defined over the entire real axis $\mathbb{R} = \{t : -\infty < t < \infty\}$ and found that the Fourier transform of their convolution,

$$g * h(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} g(u)h(t-u)\,\mathrm{d}u, \tag{132}$$

was simply the product of $G(\cdot)$ and $H(\cdot)$, our notation for the Fourier transforms of $g(\cdot)$ and $h(\cdot)$. In Section 3.2 we argued that the quantity $|H(f)|^2$ defines an energy spectral density function (SDF) for $h(\cdot)$. Let us now regard the convolution in Equation (132) as a manipulation of a function $h(\cdot)$ that produces a new function $g * h(\cdot)$. The original function $h(\cdot)$ has a distribution of energy with respect to frequency given by its energy SDF, and the new function has a distribution of energy given by

$$|G(f)H(f)|^2 = |G(f)|^2|H(f)|^2.$$

Thus the energy SDF of the new function $g * h(\cdot)$ is multiplicatively related to the energy SDF of the original function $h(\cdot)$ via the Fourier transform of the manipulator $g(\cdot)$.

In this chapter we investigate and extend these ideas through the theory of linear time-invariant (LTI) filters. Our goal is to formalize ways of relating the spectra associated with inputs to an LTI filter to the spectra associated with outputs from the filter. The feature that makes LTI filters so easy to work with is our ability to represent various functions and stationary processes as linear combinations of complex exponentials.

LTI filters play a key role in the spectral analysis of stationary processes. As we shall see, an LTI filter is nothing more than a linear time-invariant transformation of some function of time. If the function of time is a realization of a stationary process, an LTI filter transforms the process into a new process that, under very mild conditions, is also stationary. An important feature of LTI filters is that, given the integrated spectrum of the original process, there is an easy way to determine the integrated spectrum of the new process. The theory of LTI filters thus gives us a powerful means for determining the SDFs of a wide class of stationary processes. We will also make extensive use of LTI filters in Chapter 7 to smooth out the inherent variability in certain estimates of the SDF.

5.1 Basic Theory of LTI Analog Filters

Let us define a continuous parameter *filter* L as a mapping, or association, between an input function $x(\cdot)$ and an output function $y(\cdot)$. Symbolically we write

$$L\{x(\cdot)\} = y(\cdot). \tag{133a}$$

Since we regard both $x(\cdot)$ and $y(\cdot)$ as functions of time $t \in \mathbb{R}$, the qualifier "continuous parameter" is appropriate – in the engineering literature a continuous parameter filter is often called an *analog* filter. In mathematics L is known as a *transformation* or *operator*. It is important to realize that a filter is *not* just an ordinary function: for example, a real-valued function that is defined over $\mathbb{R}$ associates a *point* in $\mathbb{R}$ with another point in $\mathbb{R}$, whereas a filter associates a *function* from some – so far unidentified – abstract space of functions with another function in that same space.

For the remainder of this section we need the following special notation. If α is a real or complex-valued scalar and $x(\cdot)$ is a function, the notation $\alpha x(\cdot)$ refers to the function defined by $\alpha x(t)$ for $t \in \mathbb{R}$. If $x_1(\cdot)$ and $x_2(\cdot)$ are two functions, then $x_1(\cdot) + x_2(\cdot)$ denotes the function defined by $x_1(t) + x_2(t)$. Finally, if τ is a real-valued scalar and $x(\cdot)$ is a function, then $x(\cdot;\tau)$ denotes the function whose value at time t is given by $x(t+\tau)$; i.e.,

$$x(t;\tau) = x(t+\tau), \quad t \in \mathbb{R}.$$

Thus the filter defined by

$$L\{x(\cdot)\} = x(\cdot;\tau)$$

is a shift filter, which takes as input a certain function $x(\cdot)$ and returns a function that is defined by shifting the original function by τ units to the left. For example, if the input to the shift filter is the function defined by $x(t) = \sin(t)$ and $\tau = \pi/2$, the output is the function defined by $x(t;\pi/2) = \cos(t)$.

An analog filter L is called a *linear time-invariant* (LTI) analog filter if it has the following three properties:

[1] Scale preservation:

$$L\{\alpha x(\cdot)\} = \alpha L\{x(\cdot)\};$$

i.e., multiplication of the input by the factor α results in multiplication of the output by α also.

[2] Superposition:

$$L\{x_1(\cdot) + x_2(\cdot)\} = L\{x_1(\cdot)\} + L\{x_2(\cdot)\};$$

i.e., if we define a new function by adding $x_1(\cdot)$ and $x_2(\cdot)$ and if we use it as input to the LTI filter L, the output from L is simply that function defined by adding together the outputs of L when $x_1(\cdot)$ and $x_2(\cdot)$ are separately used as inputs to L.

[3] Time invariance:

$$\text{if } L\{x(\cdot)\} = y(\cdot), \text{ then } L\{x(\cdot;\tau)\} = y(\cdot;\tau); \tag{133b}$$

i.e., if two inputs to the LTI filter are the same except for a shift in time, the outputs will also be the same except for the same shift in time.

Properties [1] and [2] together express the linearity of L:

$$L\{\alpha x_1(\cdot) + \beta x_2(\cdot)\} = \alpha L\{x_1(\cdot)\} + \beta L\{x_2(\cdot)\}.$$

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By induction, it follows that

$$L\left\{\sum_{j=1}^{N} \alpha_j x_j(\cdot)\right\} = \sum_{j=1}^{N} \alpha_j L\{x_j(\cdot)\}. \tag{134}$$

With suitable conditions the above holds "in the limit" so that Equation (134) is valid when the finite summation is replaced by an infinite summation.

Now suppose we take an $L^2(\mathbb{R})$ deterministic function (i.e., one that is square integrable over the entire real axis) or a realization of a stationary process and use it as input to some LTI filter. As we shall see, under mild conditions, the output from this LTI filter is also, respectively, an $L^2(\mathbb{R})$ function or a realization of a different stationary process (defined on a realization by realization basis by the LTI filter). It is interesting that Equations (133b) and (134) are all we need to derive the relationship between the spectra of the input and output to the LTI filter. What follows are the details (this material is based on Koopmans, 1974).

Let the input into the LTI filter be the complex exponential

$$\mathcal{E}_f(t) = \mathrm{e}^{\mathrm{i}2\pi ft}, \quad t \in \mathbb{R},$$

where f is some fixed frequency, and let $y_f(\cdot)$ denote the output function; i.e.,

$$y_f(\cdot) = L\{\mathcal{E}_f(\cdot)\}.$$

The rationale for this approach is simple. If we regard L as a "black box" that accepts an input and transforms it somehow, and if we want to learn something about L, we might feed it simple test functions such as complex exponentials to learn how it reacts. The complex exponentials are of particular interest because all the representations we have examined for deterministic functions and stationary processes involve linear combinations of one kind or another of complex exponentials. Now, by the properties [1] and [3] of an LTI filter, we have for all τ

$$y_f(\cdot;\tau) \overset{[3]}{=} L\{\mathcal{E}_f(\cdot;\tau)\} = L\{\mathrm{e}^{\mathrm{i}2\pi f\tau}\mathcal{E}_f(\cdot)\} \overset{[1]}{=} \mathrm{e}^{\mathrm{i}2\pi f\tau} L\{\mathcal{E}_f(\cdot)\} = \mathrm{e}^{\mathrm{i}2\pi f\tau} y_f(\cdot).$$

This implies that

$$y_f(t;\tau) = y_f(t+\tau) = \mathrm{e}^{\mathrm{i}2\pi f\tau} y_f(t) \ \text{ for all } t \text{ and } \tau.$$

In particular, for $t = 0$ we obtain

$$y_f(\tau) = \mathrm{e}^{\mathrm{i}2\pi f\tau} y_f(0).$$

Since τ can assume any real value, the above implies that

$$y_f(t) = \mathrm{e}^{\mathrm{i}2\pi ft} y_f(0), \ \text{ for all } t\text{; i.e., } \ y_f(\cdot) = y_f(0)\mathcal{E}_f(\cdot).$$

Thus, when the function $\mathcal{E}_f(\cdot)$ is used as input to the LTI filter L, the output is the same function multiplied by some constant, $y_f(0)$, which is independent of time but will depend in general on the frequency f. To keep track of this frequency dependence, define

$$G(f) = y_f(0).$$

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We thus have shown that

$$L\{\mathcal{E}_f(\cdot)\} = G(f)\mathcal{E}_f(\cdot). \tag{135a}$$

In mathematical terms, the complex exponentials (regarded as functions of t with f fixed) would be called the *eigenfunctions* for the LTI filter L, and each $G(f)$ would be called an associated *eigenvalue*. The relationship expressed by Equation (135a) is of fundamental importance: if the input to an LTI filter is a complex exponential, the output is also a complex exponential with the exact same frequency multiplied by $G(f)$. Why is this important? Suppose that $x(\cdot)$ can be represented by

$$x(t) = \sum_f \alpha_f \mathrm{e}^{\mathrm{i}2\pi f t}; \;\text{ i.e., }\; x(\cdot) = \sum_f \alpha_f \mathcal{E}_f(\cdot). \tag{135b}$$

Then Equations (134) and (135a) tell us that the output from the LTI filter is just

$$y(\cdot) = L\{x(\cdot)\} = \sum_f \alpha_f G(f)\mathcal{E}_f(\cdot);$$

i.e., $y(\cdot)$ can be represented by

$$y(t) = \sum_f \alpha_f G(f)\mathrm{e}^{\mathrm{i}2\pi f t}.$$

In particular, the spectral representation theorem for continuous parameter stationary processes says that, if $x(\cdot)$ is a realization of a stationary process $\{X(t)\}$ with zero mean, then

$$x(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t}\,\mathrm{d}Z_x(f), \quad t \in \mathbb{R},$$

where $Z_x(\cdot)$ is the corresponding realization of the orthogonal process $\{Z_X(f)\}$. As a function of t, we can regard the above equation as a special case of Equation (135b) if we equate α_f with the increments $\mathrm{d}Z_x(f)$ (this step requires some justification because we have passed from a finite to an infinite summation). Hence we have

$$y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t} G(f)\,\mathrm{d}Z_x(f) \;\text{ and }\; Y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t} G(f)\,\mathrm{d}Z_X(f).$$

It follows that, under a mild "matching" condition, $\{Y(t)\}$ is a stationary process and hence has a spectral representation given by

$$Y(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t}\,\mathrm{d}Z_Y(f).$$

The matching condition essentially says that $\operatorname{var}\{Y(t)\}$ must be finite. This condition is needed because it is possible to construct an LTI filter that maps a stationary process to a process having infinite variance, which is a problem because our definition of stationarity requires such a process to have finite variance. If we denote the integrated spectra for $\{X(t)\}$ and $\{Y(t)\}$ by $S_X^{(\mathrm{I})}(\cdot)$ and $S_Y^{(\mathrm{I})}(\cdot)$, respectively, the two expressions above for $Y(t)$ – combined with the fact that a stationary process has a unique integrated spectrum – tell us that

$$\mathrm{d}S_Y^{(\mathrm{I})}(f) = E\{|\mathrm{d}Z_Y(f)|^2\} = |G(f)|^2 E\{|\mathrm{d}Z_X(f)|^2\} = |G(f)|^2\,\mathrm{d}S_X^{(\mathrm{I})}(f). \tag{135c}$$

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When SDFs exist for all f,

$$\mathrm{d}S_Y^{(\mathrm{I})}(f) = S_Y(f)\,\mathrm{d}f \ \text{ and } \ \mathrm{d}S_X^{(\mathrm{I})}(f) = S_X(f)\,\mathrm{d}f,$$

so Equation (135c) reduces to

$$S_Y(f) = |G(f)|^2 S_X(f).$$

The function $G(\cdot)$ is called the *transfer function* (or *frequency response function*) of the LTI filter L. The transfer function relates the integrated spectra of the input to – and the output from – an LTI filter in a very simple fashion. In particular, the relationship is independent of time, and power is not transferred from one frequency to another. These facts show the importance of LTI filters within spectral analysis, and, conversely, the usefulness of spectral analysis when LTI transformations are applied to a time series.

In general, $G(\cdot)$ is a complex-valued function, so we can write

$$G(f) = |G(f)|\mathrm{e}^{\mathrm{i}\theta(f)}, \tag{136}$$

where $|G(\cdot)|$ and $\theta(\cdot)$ are called, respectively, the *gain function* and the *phase function* of the LTI filter. Note that $\theta(f) = \arg(G(f))$. The quantities

$$-\frac{1}{2\pi}\cdot\frac{\mathrm{d}\theta(f)}{\mathrm{d}f} \ \text{ and } \ -\theta(f)$$

define the *group delay* and the *phase shift function*.

Equation (135a) gives us a simple rule for computing the transfer function of an LTI filter: if we apply the function $\mathcal{E}_f(\cdot)$ as input to an LTI filter, the coefficient of $\exp(\mathrm{i}2\pi ft)$ in the resulting output from the filter *defines* $G(f)$, the transfer function at frequency f.

The results of this section that pertain to stationary processes can be summarized by the following theorem, which we call the *LTI analog filtering theorem*: if $\{X(t)\}$ is a continuous parameter stationary process with zero mean and integrated spectrum $S_X^{(\mathrm{I})}(\cdot)$ and if L is an LTI analog filter with transfer function $G(\cdot)$ such that the matching condition

$$\int_{-\infty}^{\infty} |G(f)|^2\,\mathrm{d}S_X^{(\mathrm{I})}(f) < \infty$$

holds, then $\{Y(t)\} \stackrel{\mathrm{def}}{=} L\{\{X(t)\}\}$ is a continuous parameter stationary process with zero mean and integrated spectrum $S_Y^{\mathrm{I})}(\cdot)$ such that

$$\mathrm{d}S_Y^{(\mathrm{I})}(f) = |G(f)|^2\,\mathrm{d}S_X^{\mathrm{I})}(f).$$

The proof of this theorem is Exercise [5.1].

In the remainder of this book we prefer to use the informal notation

$$L\{x(t)\} = y(t) \ \text{ instead of } \ L\{x(\cdot)\} = y(\cdot)$$

to define an LTI analog filter. This is done merely for notational convenience – it allows us to define functions implicitly on a point by point basis without having to come up with an explicit notation for them (as we did for $\mathcal{E}_f(\cdot)$). In all cases our informal notation means that the LTI filter L maps the function defined on a point by point basis by $x(t)$ to the function defined on a point by point basis by $y(t)$. For example, Equation (135a) in this informal notation is

$$L\{\mathrm{e}^{\mathrm{i}2\pi ft}\} = G(f)\mathrm{e}^{\mathrm{i}2\pi ft}.$$

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Comments and Extensions to Section 5.1

[1] A general class of linear transformations of $x(\cdot)$ is given by

$$y(t) = \int_{-\infty}^{\infty} K(t, t')x(t')\,\mathrm{d}t'$$

for some $K(\cdot, \cdot)$ (Bracewell, 2000, p. 210; Champeney, 1987). Under the assumption of time invariance (see Equation (133b)), we have

$$y(t - \tau) = \int_{-\infty}^{\infty} K(t, t')x(t' - \tau)\,\mathrm{d}t'.$$

If we make the change of variable $t'' = t' - \tau$, we get

$$y(t - \tau) = \int_{-\infty}^{\infty} K(t, t'' + \tau)x(t'')\,\mathrm{d}t'',$$

and if we replace $t - \tau$ by t and relabel t'' as t', we have

$$y(t) = \int_{-\infty}^{\infty} K(t + \tau, t' + \tau)x(t')\,\mathrm{d}t'.$$

A comparison of the two expressions for $y(t)$ shows that time invariance implies $K(t, t') = K(t+\tau, t'+\tau)$ for all t and τ. If we let $\tau = -t'$, we have $K(t, t') = K(t - t', 0)$ for all t and t' so that we can write, say, $K(t, t') = g(t - t')$, a function purely of $t - t'$. Hence $y(\cdot)$ can be expressed as a convolution:

$$y(t) = \int_{-\infty}^{\infty} g(t - t')x(t')\,\mathrm{d}t'.$$

Linearity plus time invariance thus *implies* the convolution relationship. However, to be able to include the trivial case $y(t) = x(t)$, we must allow $g(\cdot) = \delta(\cdot)$, the Dirac delta function. If we want to exclude generalized functions like the delta function from the convolution expression, then "linearity plus time invariance" gives a set of filters that are larger than those that can be expressed as a convolution.

[2] The theory of LTI filters also justifies the form in which we have chosen to represent functions and stationary processes, namely, as linear combinations of complex exponentials. For example, suppose $g_p(\cdot)$ is a member of the $L^2(-T/2, T/2)$ class of functions, i.e., the class of all complex-valued continuous time functions such that

$$\int_{-T/2}^{T/2} |g_p(t)|^2\,\mathrm{d}t < \infty.$$

From the results of Section 3.1, we can represent $g_p(\cdot)$ over the interval $[-T/2, T/2]$ as

$$g_p(t) = \sum_{n=-\infty}^{\infty} G_n \mathrm{e}^{\mathrm{i}2\pi f_n t}, \quad \text{where } f_n \stackrel{\text{def}}{=} \frac{n}{T} \text{ and } G_n \stackrel{\text{def}}{=} \frac{1}{T}\int_{-T/2}^{T/2} g_p(t)\mathrm{e}^{-\mathrm{i}2\pi f_n t}\,\mathrm{d}t.$$

Since we can easily extend $g_p(\cdot)$ to the whole real axis as a periodic function with period T, Parseval's theorem

$$\int_{-T/2}^{T/2} |g_p(t)|^2\,\mathrm{d}t = T\sum_{n=-\infty}^{\infty} |G_n|^2$$

allows us to use $|G_n|^2$ to define a discrete power spectrum.

Let us now define

$$\phi_n(t) = \mathrm{e}^{\mathrm{i}2\pi f_n t}/\sqrt{T}, \quad n \in \mathbb{Z}. \tag{137}$$

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We say the collection of functions $\phi_n(\cdot)$ forms an *orthonormal basis* for $L^2(-T/2, T/2)$ because, first,

$$\int_{-T/2}^{T/2} \phi_m(t)\phi_n^*(t)\,\mathrm{d}t = \begin{cases} 0, & m \neq n; \\ 1, & m = n, \end{cases}$$

and, second, any function $g_p(\cdot)$ in $L^2(-T/2, T/2)$ can be written as

$$g_p(t) = \sum_{n=-\infty}^{\infty} \mathcal{G}_n \phi_n(t), \quad \text{where } \mathcal{G}_n \stackrel{\text{def}}{=} \int_{-T/2}^{T/2} g_p(t)\phi_n^*(t)\,\mathrm{d}t. \tag{138a}$$

In this new notation Parseval's theorem becomes

$$\int_{-T/2}^{T/2} |g_p(t)|^2\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |\mathcal{G}_n|^2, \tag{138b}$$

which allows us to use $|\mathcal{G}_n|^2$ (divided by T) to define a discrete power spectrum.

The orthonormal basis for $L^2(-T/2, T/2)$ that we have just defined is not unique. There are many other orthonormal bases that we could define such that both Equations (138a) and (138b) would still hold and $|\mathcal{G}_n|^2$ would thus define a discrete power spectrum with respect to this new basis. The theory of LTI filters tells us there is something special about the orthonormal basis defined by Equation (137). Thus, if we define an LTI filter L as

$$L\{g_p(t)\} = g_p * h_p(t) \stackrel{\text{def}}{=} \frac{1}{T}\int_{-T/2}^{T/2} g_p(u)h_p(t-u)\,\mathrm{d}u,$$

we know that the discrete power spectrum for the output $g_p * h_p(\cdot)$ is

$$\frac{1}{T^2}|\mathcal{G}_n|^2|\mathcal{H}_n|^2, \quad \text{where } \mathcal{H}_n \stackrel{\text{def}}{=} \int_{-T/2}^{T/2} h_p(t)\phi_n^*(t)\,\mathrm{d}t$$

(assuming that $h_p(\cdot)$ is a periodic $L^2(-T/2, T/2)$ function). Such a simple relationship is unique to the basis defined by Equation (137); for any other nontrivially different basis, LTI filtering results in a more complicated relationship between the discrete power spectra of the input and the output.

For example, a nonsinusoidal orthonormal basis that has proved useful in practical applications can be formed from the *Walsh functions* (Beauchamp, 1984). When $T = 1$, these functions assume only the values ± 1. The first nine of one version of these functions are shown in Figure 139. They can be labeled from top to bottom as $W_n(\cdot)$ for $n = 0, 1, \ldots, 8$ – see Beauchamp (1984) for details on how to generate these and higher-order Walsh functions. For Walsh functions, the concept corresponding to frequency is *sequency*, which is defined as half the number of zero crossings over one period (taken to be unity in Figure 139). This is just the number of transitions from ± 1 to ∓ 1 with the convention that the endpoints count as one transition if

$$\lim_{\substack{\delta\to 0\\ \delta>0}} W_n\left(-\tfrac{1}{2}+\delta\right) \neq \lim_{\substack{\delta\to 0\\ \delta>0}} W_n\left(\tfrac{1}{2}-\delta\right).$$

With this definition the sequency of $W_n(\cdot)$ is $\lfloor (n+1)/2 \rfloor$, so each nonzero sequency is associated with exactly two Walsh functions. Figure 139 shows all the Walsh functions for sequencies 0 to 4. By way of comparison, note that each nonzero frequency f_m is associated with exactly two orthogonal sinusoids, namely, $\cos(2\pi f_m t)$ and $\sin(2\pi f_m t)$, and that both sinusoids have sequency m.

If we now redefine

$$\phi_n(t) = \begin{cases} W_{2n}(t)/\sqrt{T}, & n \geq 0; \\ W_{2|n|-1}(t)/\sqrt{T}, & n < 0, \end{cases}$$

it can be shown that Equations (138a) and (138b) still hold. We can thus define a Walsh discrete power spectrum via $|\mathcal{G}_n|^2$. Since $\phi_n(t)$ and $\phi_{-n}(t)$ have sequency n, the $\pm n$ components of this spectrum tell us the contribution to the power from Walsh functions of sequency $|n|$.

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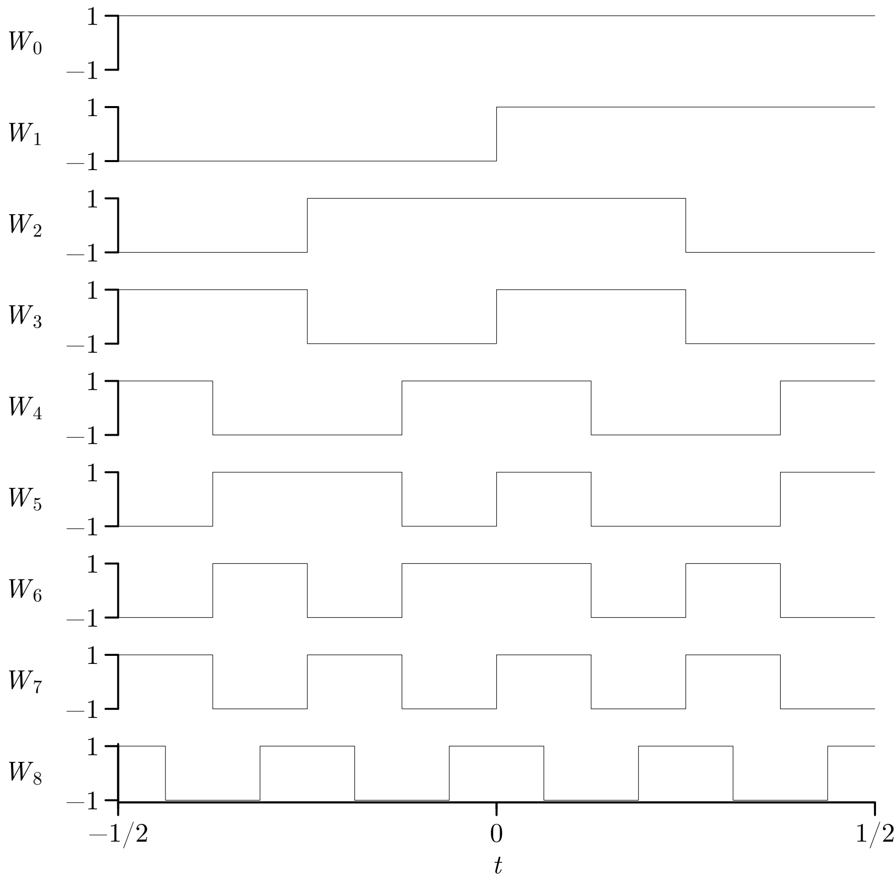

Figure 139 Walsh functions $W_n(\cdot)$ from $n = 0$ (top) to $n = 8$ (bottom). This figure was adapted from Figures 1.4 and 1.7 of Beauchamp (1984).

Now suppose that $g_p(\cdot) = W_2(\cdot)$. Since our periodic function is just the second Walsh function, its Walsh discrete power spectrum is concentrated entirely at sequency 1. If we pass this function through the LTI filter defined by, say,

$$L\{g_p(t)\} = \int_{-1/2}^{1/2} g_p(u)h_p(t-u)\,\mathrm{d}u \stackrel{\text{def}}{=} g_p * h_p(t)$$

(see Section 5.3 for details on this type of filter), where $h_p(\cdot)$ has a period of unity and is defined over $[-1/2, 1/2]$ by

$$h_p(t) = \begin{cases} 4, & |t| \le 1/8; \\ 0, & 1/8 < |t| \le 1/2 \end{cases}$$

(see Figure 140), it is obvious that the Walsh discrete power spectrum for the output from this filter is no longer concentrated just at sequency 1; i.e., it cannot be expressed as just a linear combination of $W_1(\cdot)$ and $W_2(\cdot)$ (see Figure 139). In fact it can be shown that the power is now distributed over an infinite range of sequencies. By contrast, if we pass a band-limited function through an LTI filter, then the output must also be band-limited – the (Fourier) discrete power spectrum for the output from any LTI filter can have nonzero contributions only at those frequencies that are nonzero in the discrete power spectrum of the input (see Figure 141).

Walsh spectral analysis provides some interesting contrasts to the usual Fourier spectral analysis and has proven useful for time series whose values shift suddenly from one level to another. For example,

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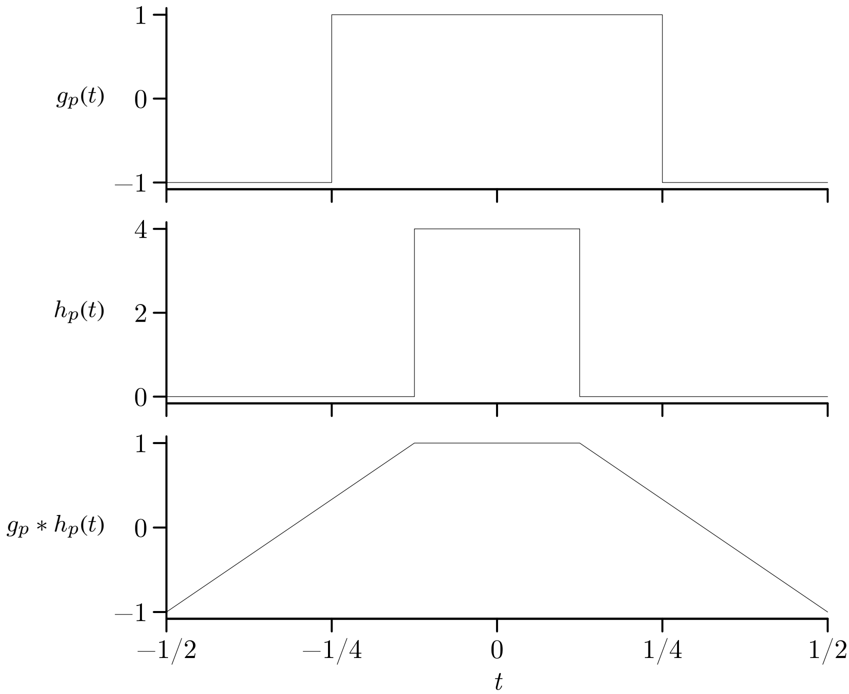

Figure 140 Filtering of $W_2(\cdot)$. When $g_p(\cdot) = W_2(\cdot)$ (top plot) – which has sequency 1 – is convolved with a periodic rectangular smoothing kernel $h_p(\cdot)$ (middle plot), the result is a periodic function $g_p * h_p(\cdot)$ (bottom plot) that cannot be expressed in terms of Walsh functions of sequency 1.

a Fourier power spectrum is invariant with respect to a shift in the time origin of a function, but a Walsh power spectrum need not be; although it is not possible for a function to be both band-limited and time-limited, a function can be both sequency-limited and time-limited; and, if a function has a band-limited Fourier representation, it cannot have a sequency-limited Walsh representation and vice versa. For further details, the reader is referred to Morettin (1981), Beauchamp (1984) and Stoffer (1991); for interesting applications of Walsh spectral analysis, see Stoffer et al. (1988), Lanning and Johnson (1983) and Kowalski et al. (2000).

5.2 Basic Theory of LTI Digital Filters

In the previous section we defined an analog (or continuous parameter) filter as a transformation that maps a function of time to another such function. A parallel theory exists for a transformation that associates a sequence with another sequence – such a transformation is referred to as a discrete parameter filter or *digital* filter. The theory of linear time-invariant digital filters closely parallels that of LTI analog filters, so we only sketch the key points for sequences in this section.

A digital filter L that transforms an input sequence $\{x_t\}$ into an output sequence $\{y_t\}$ is called a linear time-invariant digital filter if it has the following three properties:

[1] Scale preservation:

$$L\{\{\alpha x_t\}\} = \alpha L\{\{x_t\}\}.$$

[2] Superposition:

$$L\{\{x_{1,t} + x_{2,t}\}\} = L\{\{x_{1,t}\}\} + L\{\{x_{2,t}\}\}.$$

[3] Time invariance:

$$\text{if } L\{\{x_t\}\} = \{y_t\}, \text{ then } L\{\{x_{t+\tau}\}\} = \{y_{t+\tau}\},$$

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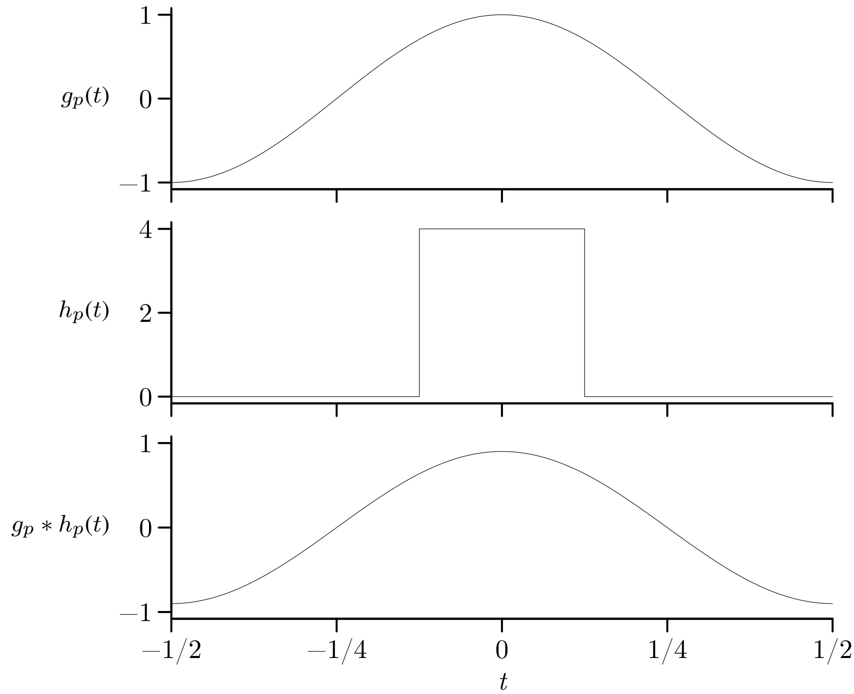

Figure 141 Filtering of a sinusoid with unit frequency. When $g_p(t) = \cos(2\pi t)$ (top plot) is convolved with the same rectangular smoothing kernel $h_p(\cdot)$ as in Figure 140 (middle plot), the result is $g_p * h_p(t) = \frac{2\sqrt{2}}{\pi}\cos(2\pi t) \doteq 0.9\cos(2\pi t)$ (bottom plot), so the output also involves a sinusoid with unit frequency. This figure and Figure 140 together illustrate that, while filtering can widen the sequency range of an input function, it can at most reduce its frequency range.

where τ is integer-valued and the notation $\{x_{t+\tau}\}$ refers to the sequence whose tth element is $x_{t+\tau}$.

By using $\{\exp(\mathrm{i}2\pi ft)\}$ – a sequence with tth element $\exp(\mathrm{i}2\pi ft)$ – as the input to an LTI digital filter, it follows from the same arguments as before that

$$L\{\{\mathrm{e}^{\mathrm{i}2\pi ft}\}\} = G(f)\{\mathrm{e}^{\mathrm{i}2\pi ft}\},$$

where $G(\cdot)$ is the transfer function (cf. Equation (135a)). Note that $G(\cdot)$ is periodic with a unit period since $\exp(\mathrm{i}2\pi[f+k]t) = \exp(\mathrm{i}2\pi ft)$ for all integers k. The result corresponding to the LTI analog filtering theorem is the following *LTI digital filtering theorem*: if $\{X_t\}$ is a discrete parameter stationary process with zero mean and integrated spectrum $S_X^{(\mathrm{I})}(\cdot)$ and if L is an LTI digital filter with transfer function $G(\cdot)$ such that the matching condition

$$\int_{-1/2}^{1/2} |G(f)|^2 \,\mathrm{d}S_X^{(\mathrm{I})}(f) < \infty$$

holds, then $\{Y_t\} \stackrel{\mathrm{def}}{=} L\{\{X_t\}\}$ is a discrete parameter stationary process with zero mean and integrated spectrum $S_Y^{(\mathrm{I})}(\cdot)$ such that

$$\mathrm{d}S_Y^{(\mathrm{I})}(f) = |G(f)|^2 \,\mathrm{d}S_X^{(\mathrm{I})}(f).$$

As before, we shall find it convenient to use the succinct – but formally incorrect – notation

$$L\{x_t\} = y_t \text{ as shorthand for } L\{\{x_t\}\} = \{y_t\}.$$

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Comments and Extensions to Section 5.2

[1] The development in this section assumes that the sampling interval Δ_{t} associated with the inputs $\{x_t\}$ and $\{X_t\}$ is unity. For general Δ_{t}, the transfer function $G(\cdot)$ for L is a periodic function with a period of $1/\Delta_{\mathrm{t}}$ and is obtained by inputing the sequence $\{\exp(\mathrm{i}2\pi f t\,\Delta_{\mathrm{t}})\}$ rather than $\{\exp(\mathrm{i}2\pi f t)\}$; i.e.,

$$L\{\mathrm{e}^{\mathrm{i}2\pi f t\,\Delta_{\mathrm{t}}}\} = G(f)\mathrm{e}^{\mathrm{i}2\pi f t\,\Delta_{\mathrm{t}}}.$$

Once we know the transfer function over any frequency interval of width $1/\Delta_{\mathrm{t}}$, e.g., $|f| \leq 1/(2\,\Delta_{\mathrm{t}}) = f_{\mathcal{N}}$ (the Nyquist frequency), periodicity give us $G(f)$ for all $f \in \mathbb{R}$.

5.3 Convolution as an LTI Filter

We consider in this section some details about an LTI analog filter L of the following form:

$$L\{X(t)\} = \int_{-\infty}^{\infty} g(u)X(t-u)\,\mathrm{d}u \overset{\mathrm{def}}{=} Y(t) \tag{142}$$

(that this indeed satisfies the properties of an LTI filter is the subject of Exercise [5.2a]). Here the input to the LTI filter is a stationary process $\{X(t)\}$ that, for simplicity, we take to have zero mean and a purely continuous spectrum with associated SDF $S_X(\cdot)$. The output is the stochastic process $\{Y(t)\}$ that results from convolving $\{X(t)\}$ with the real-valued deterministic function $g(\cdot)$. The process $\{Y(t)\}$ is thus formed from an infinite linear combination of members of the process $\{X(t)\}$. The characteristics of the LTI filter are entirely determined by $g(\cdot)$, which – in the analog (continuous parameter) case – is called the *impulse response function* for the following reason. Suppose we let the input to the LTI analog filter in Equation (142) be $\delta(\cdot)$, the Dirac delta function with an infinite spike at the origin. By the properties of that function, we have

$$L\{\delta(t)\} = \int_{-\infty}^{\infty} g(u)\delta(t-u)\,\mathrm{d}u = g(t).$$

Hence, the input of an "impulse" (the delta function) into the LTI filter yields an output that defines the impulse response function.

To find the transfer function for L in Equation (142), we apply the function of t defined by $\exp(\mathrm{i}2\pi f t)$ as input to the LTI filter to get

$$L\{\mathrm{e}^{\mathrm{i}2\pi f t}\} = \int_{-\infty}^{\infty} g(u)\mathrm{e}^{\mathrm{i}2\pi f(t-u)}\,\mathrm{d}u = \mathrm{e}^{\mathrm{i}2\pi f t}\int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}2\pi f u}\,\mathrm{d}u.$$

Hence the transfer function

$$G(f) \overset{\mathrm{def}}{=} \int_{-\infty}^{\infty} g(u)\mathrm{e}^{-\mathrm{i}2\pi f u}\,\mathrm{d}u$$

is just the Fourier transform of the impulse response function $g(\cdot)$. Since $S_Y(\cdot)$ and $S_X(\cdot)$ are related by $S_Y(f) = |G(f)|^2 S_X(f)$, the matching condition says that we must have

$$\int_{-\infty}^{\infty} |G(f)|^2 S_X(f)\,\mathrm{d}f < \infty$$

for $\{Y_t\}$ to have a finite variance as required for a stationary process. If $S_X(\cdot)$ happens to be a bounded function, the matching condition becomes

$$\int_{-\infty}^{\infty} |G(f)|^2\,\mathrm{d}f < \infty.$$

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In this case, Parseval's relationship for deterministic nonperiodic functions with finite energy says that, because $g(\cdot)$ is real-valued, the matching condition is equivalent to

$$\int_{-\infty}^{\infty} g^2(u)\,\mathrm{d}u < \infty$$

(this is also a sufficient condition for the existence of $G(\cdot)$).

The corresponding results for the discrete parameter case are quite similar. Recalling the definition for the convolution of two sequences with sampling interval Δ_t (Equation (99b)), we can write

$$L\{X_t\} = \Delta_\mathrm{t} \sum_{u=-\infty}^{\infty} g_u X_{t-u} = Y_t, \tag{143a}$$

say, where $\{g_u\}$ is a real-valued deterministic sequence called the *impulse response sequence* (note that X_t and Y_t have different units). The right-hand side of the above is a Riemann sum approximation to the integral in Equation (142) with a grid size of Δ_t. If $\{X_t\}$ is a stationary process with a bounded SDF $S_X(\cdot)$ and sampling interval Δ_t and if $\{g_u\}$ is square summable, it follows from Exercise [5.3] that $\{Y_t\}$ is a stationary process with an SDF given by

$$S_Y(f) = |G(f)|^2 S_X(f), \tag{143b}$$

where the transfer function is now obtained by using the sequence $\{\exp(\mathrm{i}2\pi f t\,\Delta_\mathrm{t})\}$ as input to the LTI digital filter:

$$L\{\mathrm{e}^{\mathrm{i}2\pi f t\,\Delta_\mathrm{t}}\} = \Delta_\mathrm{t} \sum_{u=-\infty}^{\infty} g_u \mathrm{e}^{\mathrm{i}2\pi f (t-u)\,\Delta_\mathrm{t}} = \mathrm{e}^{\mathrm{i}2\pi f t\,\Delta_\mathrm{t}} G(f),$$

with

$$G(f) = \Delta_\mathrm{t} \sum_{u=-\infty}^{\infty} g_u \mathrm{e}^{-\mathrm{i}2\pi f u\,\Delta_\mathrm{t}} \tag{143c}$$

being the Fourier transform of $\{g_u\}$ as defined by Equation (74a). If the SDF for $\{X_t\}$ is not bounded, then the matching condition

$$\int_{-f_\mathcal{N}}^{f_\mathcal{N}} |G(f)|^2 S_X(f)\,\mathrm{d}f < \infty$$

is required for $\{Y_t\}$ to be a stationary process.

If, as is commonly done is practice, we choose to define the digital filter as

$$L\{X_t\} = \sum_{u=-\infty}^{\infty} g_u X_{t-u} = Y_t, \tag{143d}$$

i.e., we drop the Δ_t so that that X_t and Y_t have the same units, then the transfer function becomes

$$G(f) = \sum_{u=-\infty}^{\infty} g_u \mathrm{e}^{-\mathrm{i}2\pi f u\,\Delta_\mathrm{t}} \tag{143e}$$

and is now merely proportional to the Fourier transform of $\{g_u\}$ unless Δ_t happens to be unity (see Exercise [5.3]). The relationship $S_Y(f) = |G(f)|^2 S_X(f)$ continues to hold.

5.4 Determination of SDFs by LTI Digital Filtering

Let us now determine the SDFs of some discrete parameter stationary processes using the theory described at the end of the previous section (we assume $\Delta_{\mathrm{t}} = 1$ for convenience).

Moving Average Process

An MA(q) process with zero mean by definition satisfies the equation

$$X_t = \epsilon_t - \sum_{j=1}^{q} \theta_{q,j} \epsilon_{t-j}, \tag{144a}$$

where each $\theta_{q,j}$ is a constant ($\theta_{q,q} \neq 0$) and $\{\epsilon_t\}$ is a white noise process with zero mean and variance σ_ϵ^2 (this is Equation (32a) with $\mu = 0$ and $\theta_{q,0}$ replaced by its defining value of -1). Define the filter L for a sequence $\{y_t\}$ by

$$L\{y_t\} = y_t - \sum_{j=1}^{q} \theta_{q,j} y_{t-j},$$

and note that $X_t = L\{\epsilon_t\}$. It is easy to argue that L is an LTI digital filter. To determine its transfer function, we input the sequence defined by $\exp(\mathrm{i}2\pi ft)$:

$$L\{\mathrm{e}^{\mathrm{i}2\pi ft}\} = \mathrm{e}^{\mathrm{i}2\pi ft} - \sum_{j=1}^{q} \theta_{q,j} \mathrm{e}^{\mathrm{i}2\pi f(t-j)} = \mathrm{e}^{\mathrm{i}2\pi ft} \left(1 - \sum_{j=1}^{q} \theta_{q,j} \mathrm{e}^{-\mathrm{i}2\pi fj} \right),$$

so the transfer function is given by

$$G(f) = 1 - \sum_{j=1}^{q} \theta_{q,j} \mathrm{e}^{-\mathrm{i}2\pi fj}.$$

From the relationships $S_X(f) = |G(f)|^2 S_\epsilon(f)$ and $S_\epsilon(f) = \sigma_\epsilon^2$, we have, for an MA($q$) process,

$$S_X(f) = \sigma_\epsilon^2 \left| 1 - \sum_{j=1}^{q} \theta_{q,j} \mathrm{e}^{-\mathrm{i}2\pi fj} \right|^2. \tag{144b}$$

Autoregressive Process

A stationary AR(p) process with zero mean satisfies the equation

$$X_t = \sum_{j=1}^{p} \phi_{p,j} X_{t-j} + \epsilon_t, \tag{144c}$$

where each $\phi_{p,j}$ is a constant ($\phi_{p,p} \neq 0$) and $\{\epsilon_t\}$ is as in the previous example (see Section 2.6). Define the filter L for a sequence $\{y_t\}$ by

$$L\{y_t\} = y_t - \sum_{j=1}^{p} \phi_{p,j} y_{t-j},$$

and note that $L\{X_t\} = \epsilon_t$. The transfer function of the filter is

$$G(f) = 1 - \sum_{j=1}^{p} \phi_{p,j} \mathrm{e}^{-\mathrm{i}2\pi fj}. \tag{144d}$$

From the relationship $S_\epsilon(f) = |G(f)|^2 S_X(f)$, we conclude that

$$S_X(f) = \frac{\sigma_\epsilon^2}{\left|1 - \sum_{j=1}^{p} \phi_{p,j} \mathrm{e}^{-\mathrm{i}2\pi f j}\right|^2}, \tag{145a}$$

as long as $|G(f)|^2 \neq 0$. In Section 2.6 we mentioned that the $\phi_{p,j}$ coefficients must satisfy certain conditions for $\{X_t\}$ to be stationary. It can be shown (Grenander and Rosenblatt, 1984, pp. 36–8) that a necessary and sufficient condition for the existence of a stationary solution to $\{X_t\}$ of Equation (144c) is that $G(\cdot)$ never vanish; i.e., $G(f) \neq 0$ for any f (for additional discussion, see Section 9.2). In this case, the stationary solution to Equation (144c) is unique and is given by

$$X_t = \int_{-1/2}^{1/2} \frac{\mathrm{e}^{\mathrm{i}2\pi f t}}{G(f)} \, \mathrm{d}Z_\epsilon(f), \tag{145b}$$

where $\{Z_\epsilon(f)\}$ is the orthogonal process in the spectral representation for $\{\epsilon_t\}$.

Autoregressive Moving Average Process

An ARMA(p, q) process with zero mean by definition satisfies the difference equation

$$X_t = \sum_{j=1}^{p} \phi_{p,j} X_{t-j} + \epsilon_t - \sum_{j=1}^{q} \theta_{q,j} \epsilon_{t-j},$$

where $\phi_{p,j}$ and $\theta_{q,j}$ are constants (with $\phi_{p,p} \neq 0$ and $\theta_{q,q} \neq 0$) and $\{\epsilon_t\}$ is as in the previous two examples. Exercise [5.8] is to show that, when this process is stationary, its SDF is given by

$$S_X(f) = \sigma_\epsilon^2 \frac{\left|1 - \sum_{j=1}^{q} \theta_{q,j} \mathrm{e}^{-\mathrm{i}2\pi f j}\right|^2}{\left|1 - \sum_{j=1}^{p} \phi_{p,j} \mathrm{e}^{-\mathrm{i}2\pi f j}\right|^2}. \tag{145c}$$

5.5 Some Filter Terminology

So far our discussion of filters has been rather theoretical and of limited practical use. The remainder of this chapter concentrates on defining some terminology used in filtering theory, developing an interpretation of the spectrum via filters and considering some applications. In particular we shall see that the Slepian sequences introduced in Chapter 3 arise naturally in filter design. The literature on filtering is vast, and hence the discussion given here is not intended to do much more than introduce some ideas of importance to spectral analysis (useful references for filtering theory are Hamming, 1983, and Rabiner and Gold, 1975).

Let us begin by defining the concepts of cascaded filters, low-pass filters, high-pass filters and band-pass filters. A *cascaded filter* is an arrangement of a set of n filters such that the output from the first filter is the input to the second filter and so forth. This notion applies to both analog and digital filters and is important because it is often possible to regard a seemingly complicated filter as a cascaded filter involving n components with easily determined transfer functions. If all n filters are LTI filters and if the input to the system is considered a realization of a stationary process, it is easy to describe the spectral relationship between the input to the first filter and the output from the nth filter. Let $S_j^{(\mathrm{I})}(\cdot)$ be the integrated spectrum of the input to the jth filter, and let $S_{j+1}^{(\mathrm{I})}(\cdot)$ be the integrated spectrum of the output from the jth filter. (We assume that all these spectra exist.) Let $G_j(\cdot)$ be the transfer function of the jth filter. From Equation (135c) we have

$$\mathrm{d}S_{j+1}^{(\mathrm{I})}(f) = |G_j(f)|^2 \, \mathrm{d}S_j^{(\mathrm{I})}(f) \ \text{ for } \ j = 1, 2, \ldots, n.$$

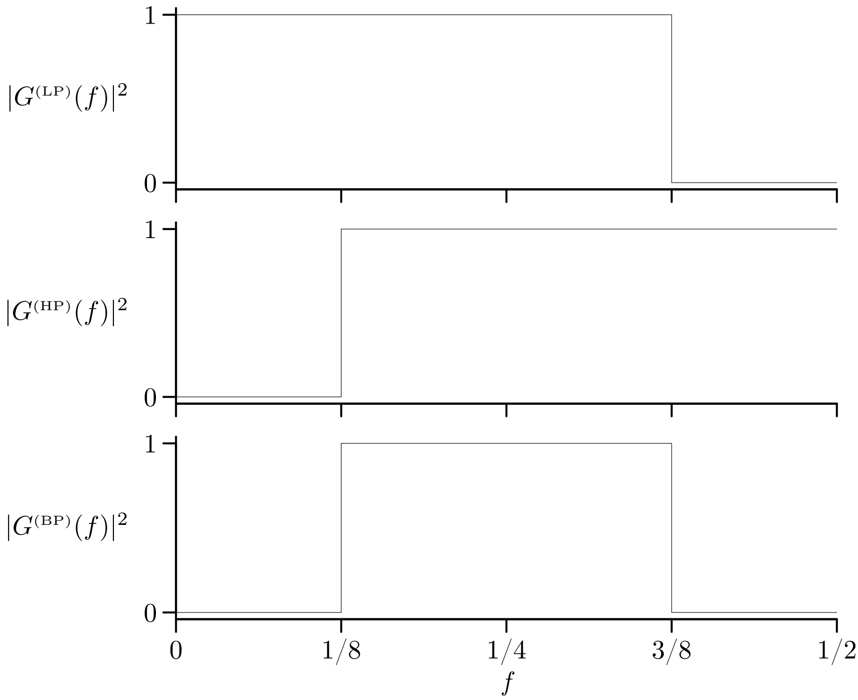

Figure 146 Squared gain functions for an ideal low-pass filter with a cutoff frequency of $f_0 = 3/8$ (upper plot); an ideal high-pass filter that eliminates frequencies whose magnitudes are less than $f_0 = 1/8$ (middle); and an ideal band-pass filter that is formed by cascading the low-pass and high-pass filters together (bottom plot) – note that $|G^{(\mathrm{BP})}(f)|^2 = |G^{(\mathrm{LP})}(f)|^2 |G^{(\mathrm{HP})}(f)|^2$.

The relationship between the integrated spectra of the input and the output to the cascaded filter is thus given by

$$\mathrm{d}S_{n+1}^{(\mathrm{I})}(f) = |G_n(f)|^2 \cdot |G_{n-1}(f)|^2 \cdots |G_1(f)|^2 \, \mathrm{d}S_1^{(\mathrm{I})}(f).$$

Note that the integrated spectrum of the output does not depend on the order in which the filters occur in the cascade (as long as the integrated spectra exist at each step in the cascade).

If the squared gain function of an LTI analog filter has the form

$$|G^{(\mathrm{LP})}(f)|^2 = \begin{cases} 1, & |f| \le f_0; \\ 0, & \text{otherwise}, \end{cases}$$

where $f_0 > 0$, we call the filter an *ideal low-pass filter*, since it passes all components with frequencies lower than f_0, but completely suppresses all components with higher frequencies. Conversely, if the squared gain function has the form

$$|G^{(\mathrm{HP})}(f)|^2 = \begin{cases} 1, & |f| \ge f_0; \\ 0, & \text{otherwise}, \end{cases}$$

we call the filter an *ideal high-pass filter*. Finally, if the squared gain function is of the form

$$|G^{(\mathrm{BP})}(f)|^2 = \begin{cases} 1, & 0 < f_0 \le |f| \le f_1; \\ 0, & \text{otherwise}, \end{cases}$$

where $f_1 > f_0$, we call the associated filter an *ideal band-pass filter*. The effect of this filter is to allow all frequencies whose magnitudes are in the band $[f_0, f_1]$ to pass through unattenuated, but all other frequency components are completely eliminated. As is illustrated in Figure 146, we can create a band-pass filter by cascading appropriate low-pass and high-pass filters together.

Assuming $0 < f_0 < f_1 < 1/2$, the corresponding definitions for LTI digital filters are

$$|G^{(\mathrm{LP})}(f)|^2 = \begin{cases} 1, & |f| \le f_0; \\ 0, & f_0 < |f| \le 1/2; \end{cases} \tag{147a}$$

$$|G^{(\mathrm{HP})}(f)|^2 = \begin{cases} 0, & |f| \le f_0; \\ 1, & f_0 < |f| \le 1/2; \end{cases} \tag{147b}$$

and

$$|G^{(\mathrm{BP})}(f)|^2 = \begin{cases} 0, & |f| < f_0 \text{ or } f_1 < |f| \le 1/2; \\ 1, & f_0 \le |f| \le f_1. \end{cases}$$

For all three filters the squared gain functions are periodic with a period of unity and hence can be determined over $\mathbb{R}$ based upon their definitions over $[-1/2, 1/2]$. In addition to illustrating ideal analog filters, Figure 146 can be taken as an illustration in the digital case as well.

Let us focus now on an LTI digital filter L with impulse response sequence $\{g_u\}$:

$$L\{X_t\} = \sum_{u=-\infty}^{\infty} g_u X_{t-u}.$$

This filter is said to be *causal* if $g_u = 0$ for all $u < 0$ and *acausal* otherwise. If $g_u = 0$ outside a finite range for u, it is called a *finite impulse response* (FIR) filter; otherwise, it is called an *infinite impulse response* (IIR) filter. Ideal low-, high- and band-pass filters are not realizable as FIR filters (see Section 5.8).

5.6 Interpretation of Spectrum via Band-Pass Filtering

We can use ideal band-pass filters to motivate a physical interpretation for the integrated spectrum. Consider the real-valued continuous parameter stationary process $\{X(t)\}$ with integrated spectrum $S_X^{(\mathrm{I})}(\cdot)$. Its variance (a measure of the power for the process) is given by

$$\sigma_X^2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathrm{d}S_X^{(\mathrm{I})}(f) = \int_{-\infty}^{\infty} S_X(f)\,\mathrm{d}f,$$

where the second equality holds in the case where $S_X^{(\mathrm{I})}(\cdot)$ is differentiable with derivative $S_X(\cdot)$ – the SDF of $\{X(t)\}$. Suppose we use this process as input to an ideal band-pass filter with a transfer function that satisfies

$$|G(f)|^2 = \begin{cases} 1, & f' \le |f| \le f' + \Delta_{\mathrm{f}}; \\ 0, & \text{otherwise}, \end{cases}$$

where $f' > 0$ and Δ_{f} is a small positive increment in frequency. Let $\{Y(t)\}$ represent the output from this filter, and let $S_Y^{(\mathrm{I})}(f)$ and $S_Y(f)$ be, respectively, its integrated spectrum and SDF (when the latter exists). Its variance is given by

$$\sigma_Y^2 \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathrm{d}S_Y^{(\mathrm{I})}(f) = \int_{-\infty}^{\infty} S_Y(f)\,\mathrm{d}f.$$

By the LTI analog filtering theorem, the relationship between the spectrum of the output $S_Y^{(\mathrm{I})}(\cdot)$ and the spectrum of the input $S_X^{(\mathrm{I})}(\cdot)$ is

$$\mathrm{d}S_Y^{(\mathrm{I})}(f) = |G(f)|^2\,\mathrm{d}S_X^{(\mathrm{I})}(f),$$

or, in the case where the SDFs exist,

$$S_Y(f) = |G(f)|^2 S_X(f).$$

▷ **Exercise [148]** Show that

$$\sigma_Y^2 = 2\left[S_X^{(\mathrm{I})}(f' + \Delta_\mathrm{f}) - S_X^{(\mathrm{I})}(f')\right]. \tag{148a}$$ ◁

Hence the incremental difference in the integrated spectrum at the frequency f' is simply the variance associated with the output of a narrow band-pass filter bordering that frequency (the factor of 2 is needed due to the two-sided nature of the integrated spectrum). When an SDF exists and does not vary much over $[f', f' + \Delta_\mathrm{f}]$, Equation (148a) yields

$$\sigma_Y^2 \approx 2S_X(f')\Delta_\mathrm{f}.$$

Since the variance of the input to the filter is just $\int_{-\infty}^{\infty} S_X(f')\,\mathrm{d}f'$, the above tells us that $2S_X(f')\Delta_\mathrm{f}$ is approximately the contribution to the variance from frequencies $|f| \in [f', f' + \Delta_\mathrm{f}]$.

5.7 An Example of LTI Digital Filtering

As a simple example of an LTI digital filter, let us consider the following impulse response sequence:

$$g_u^{(1)} = \begin{cases} 1/2, & u = 0; \\ 1/4, & u = \pm 1; \\ 0, & \text{otherwise.} \end{cases} \tag{148b}$$

This sequence defines an acausal FIR filter with a transfer function given by

$$G^{(1)}(f) = \cos^2(\pi f)$$

for $|f| \le 1/2$ (see Exercise [5.14b]). The squared modulus of this function is shown by the thick curve in the upper plot of Figure 149. Its shape resembles the squared gain function for an ideal low-pass filter in that high-frequency components are attenuated – i.e., reduced in magnitude – in comparison to low-frequency components. The results of applying this filter to some data related to the rotation of the earth are shown in the lower plot. Here the pluses represent the original unfiltered data x_t, and the solid curve represents the filtered data, namely,

$$y_t^{(1)} \stackrel{\mathrm{def}}{=} g_{-1}^{(1)} x_{t+1} + g_0^{(1)} x_t + g_1^{(1)} x_{t-1}.$$

Note that the filtered data have the same shape as the "backbone" of the original data (the low-frequency components) and have less "local variability" (the high-frequency components) than the original data. This is consistent with what we would expect from the shape of the transfer function for the filter.

We now examine the difference between the original data and the filtered data, say,

$$r_t^{(1)} \stackrel{\mathrm{def}}{=} x_t - y_t^{(1)}.$$

This residual series (the dots in the lower plot of Figure 149) can also be expressed as a filtering of our original data because

$$r_t^{(1)} = -g_{-1}^{(1)} x_{t+1} + (1 - g_0^{(1)}) x_t - g_1^{(1)} x_{t-1} = -(x_{t+1} - 2x_t + x_{t-1})/4.$$

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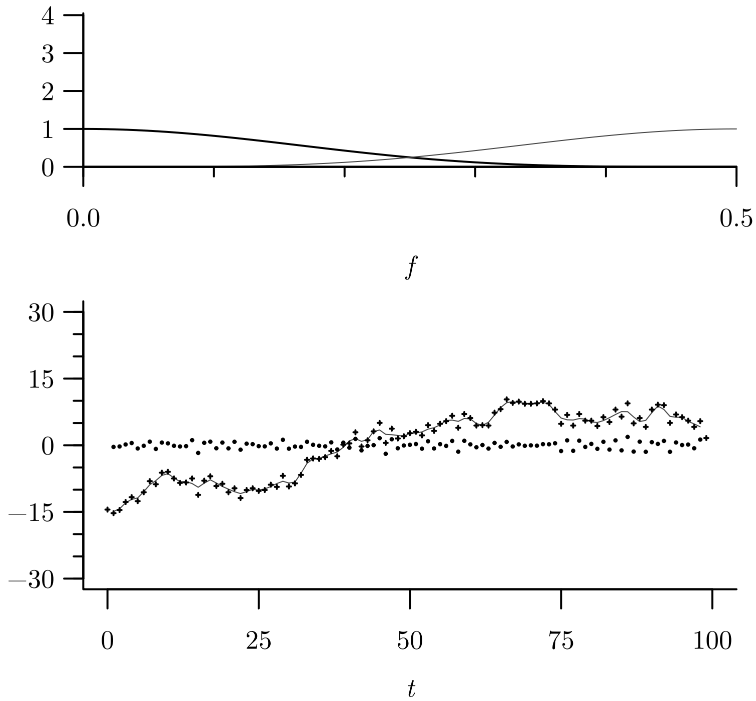

Figure 149 Example of LTI digital filtering. The top plot shows $|G^{(1)}(f)|^2$ versus f (thick curve) and the corresponding quantity for the "residual filter" (thin curve), while the bottom plot shows the unfiltered data (the pluses), the filtered data (solid curve) and the residuals (the dots).

The transfer function for the filter defined by the impulse response sequence $h_0^{(1)} = 1 - g_0^{(1)}$ and $h_u^{(1)} = -g_u^{(1)}$ for $u \neq 0$ is just $\sin^2(\pi f)$ for $|f| \leq 1/2$, and its squared modulus is shown by the thin curve in the upper plot of Figure 149. The shape of this transfer function is that of a nonideal high-pass filter: the filter attenuates the low-frequency components and leaves the high-frequency components relatively unaltered.

In the just-discussed example we were able to construct a useful high-pass filter by subtracting the output of a low-pass filter from its input. This is a useful procedure because it allows us to decompose a time series into two parts, one containing primarily its low-frequency components, and the other its high-frequency components. This decomposition is also possible with other low-pass filters if certain criteria are satisfied. One important property concerns the normalization used for the impulse response sequence of the filter. Note that the *shape* of the squared modulus of a transfer function is unchanged if we multiply each member of the impulse response sequence by a constant. For example, the impulse response sequence defined by $g_u^{(2)} = 2g_u^{(1)}$ has a transfer function given by $2G^{(1)}(\cdot)$. Its squared modulus is shown by the thick curve in the upper plot of Figure 150. The corresponding filtered and residual series are defined, respectively, by

$$y_t^{(2)} = g_{-1}^{(2)} x_{t+1} + g_0^{(2)} x_t + g_1^{(2)} x_{t-1} \text{ and } r_t^{(2)} = x_t - y_t^{(2)}.$$

They are shown by the solid curve and the dots in the lower plot of Figure 150. The filter associated with the residual series, namely,

$$r_t^{(2)} \stackrel{\text{def}}{=} -g_{-1}^{(2)} x_{t+1} + (1 - g_0^{(2)}) x_t - g_1^{(2)} x_{t-1} = -(x_{t+1} + x_{t-1})/2,$$

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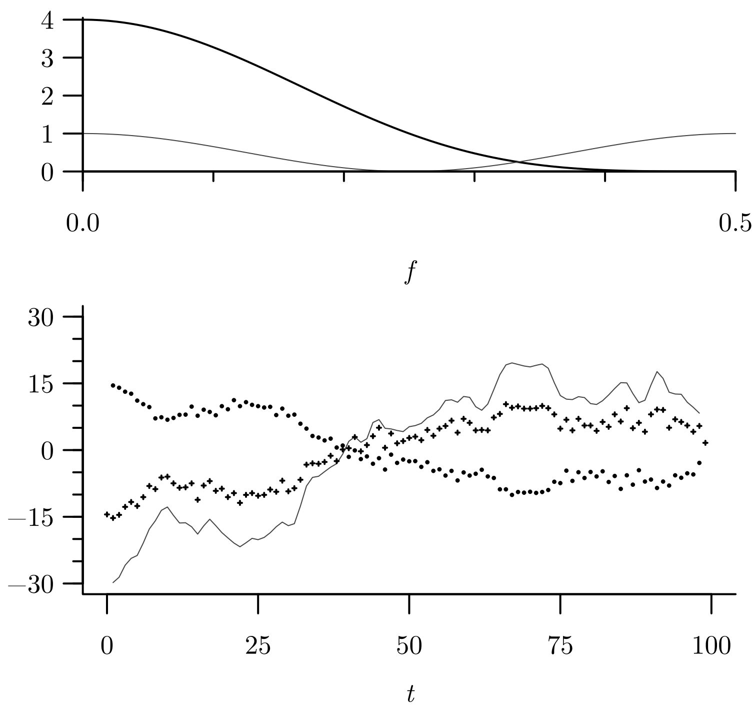

Figure 150 Second example of digital filtering.

has a transfer function whose squared modulus is given by the thin curve in the upper plot. Note that its frequency response characteristics are not at all like those of a high-pass filter and that the resulting residual series is not composed of just high-frequency components.

If we want a low-pass filter whose output traces the backbone of a given time series rather than a rescaled version of the series, the examples of Figures 149 and 150 show that normalization of the impulse response sequence is important. The proper normalization can be specified by insisting that, if the time series is locally smooth around x_t, the output from the filter should be the value x_t itself. We define locally smooth as meaning locally linear. Thus, if

$$x_t = \alpha + \beta t$$

for constants α and β, and if the impulse response sequence $\{g_u\}$ is nonzero only for $|u| \le K$, we want

$$\sum_{u=-K}^{K} g_u x_{t-u} = x_t; \text{ i.e., } \sum_{u=-K}^{K} g_u[\alpha + \beta(t-u)] = \alpha + \beta t. \tag{150}$$

▷ **Exercise [150]** Show that, if the impulse response sequence is symmetric about $u = 0$, i.e., $g_{-u} = g_u$, Equation (150) is satisfied for all α, β and t if

$$\sum_{u=-K}^{K} g_u = 1. \qquad \triangleleft$$

Note that in our examples this normalization holds for the impulse response sequence $\{g_u^{(1)}\}$ but not for $\{g_u^{(2)}\}$.

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If an impulse response sequence is symmetric about $u = 0$, its associated transfer function must be real-valued and is in fact given by

$$G(f) = g_0 + 2\sum_{u=1}^{K} g_u \cos(2\pi f u).$$

If in addition the impulse response sequence sums to unity, then

$$G(0) = g_0 + 2\sum_{u=1}^{K} g_u = 1;$$

hence $|G(0)|^2 = 1$ and $10\log_{10}(|G(0)|^2) = 0$; i.e., the response is 0 dB down at $f = 0$ – another way of stating the requirement $\sum g_u = 1$. If $\{g_u\}$ defines a low-pass filter, $G(\cdot)$ should be of the following form: first, it should be close to 1 for all f less than, say, f_L in magnitude (f_L is called the [nominal] *cutoff frequency* of the filter); and, second, it should be close to 0 for f between, say, f_H and $1/2$ in magnitude (where $f_H > f_L$ but the two are as close as possible). If we now form the residual series

$$r_t = x_t - \sum_{u=-K}^{K} g_u x_{t-u} = \sum_{u=-K}^{K} h_u x_{t-u},$$

where $h_0 = 1 - g_0$ and $h_u = -g_u$ for $|u| \geq 1$, the filter associated with $\{h_u\}$ has a transfer function given by

$$H(f) = h_0 + 2\sum_{u=1}^{K} h_u \cos(2\pi f u) = 1 - g_0 - 2\sum_{u=1}^{K} g_u \cos(2\pi f u) = 1 - G(f).$$

Thus, under the assumptions we have made about $\{g_u\}$, the residual filter $\{h_u\}$ should resemble a high-pass filter: first, it should be close to zero for all f less than f_L in magnitude; and, second, it should be close to one for f between f_H and $1/2$ in magnitude.

To summarize our discussion, two conditions on a low-pass FIR filter are sufficient for its associated residual filter to be a reasonable high-pass filter: first, the impulse response sequence of the filter sums to one; and, second, the sequence is symmetric about $u = 0$. The first condition is essential (as the filter $\{g_u^{(2)}\}$ shows); the second condition was needed for our mathematical development, but it can be circumvented – it is possible to construct examples of asymmetric low-pass filters whose associated residual filter is a reasonable high-pass filter.

Comments and Extensions to Section 5.7

[1] Our example can also be used to point out the strong resemblance between detrending a time series and subjecting it to a high-pass filter. *Trend* can be loosely defined as a "long term change in the mean level" (Chatfield, 2004, p. 12; Granger, 1966). We might argue that the solid curve in the lower plot of Figure 149 represents the trend of the time series, while the dots represent the detrended series. Alternatively, we might fit a least squares line to the series and argue that the fitted regression line is the trend, while the residuals from the regression fit are the detrended data. (As indicated by the results of Exercise [5.18], fitting a regression line is *not* an example of LTI filtering, so unfortunately we cannot use the notion of a transfer function to help us assess the effect of this type of detrending.) This and other forms of detrending act like high-pass filters – detrending effectively removes low-frequency components in a time series (an interesting example of detrending a time series of radar returns from a moving aircraft is given by Alavi and Jenkins, 1965).

5.8 Least Squares Filter Design

In this and the next section we consider some simple effective approaches to approximating an ideal low-pass digital filter (with obvious modifications, these same methods can be used to approximate ideal high-pass and band-pass filters also). The transfer function for an ideal low-pass filter of bandwidth $2W < 1$ is given by

$$G_I(f) \stackrel{\text{def}}{=} \begin{cases} 1, & |f| \leq W; \\ 0, & W < |f| \leq 1/2. \end{cases} \tag{152a}$$

Since $G_I(\cdot)$ is square integrable over the region $[-1/2, 1/2]$, we can periodically extend it outside that region and then appeal to the theory of Section 3.8 to consider it as the Fourier transform of a sequence defined by

$$g_{I,u} = \int_{-1/2}^{1/2} G_I(f) \mathrm{e}^{\mathrm{i}2\pi f u} \, \mathrm{d}f = \int_{-W}^{W} \mathrm{e}^{\mathrm{i}2\pi f u} \, \mathrm{d}f = \begin{cases} 2W, & u = 0; \\ \dfrac{\sin(2\pi W u)}{\pi u}, & u \neq 0. \end{cases} \tag{152b}$$

From this expression we see that the ideal low-pass digital filter defined by Equation (152a) is a symmetric (in the sense that $g_{I,-u} = g_{I,u}$) acausal IIR filter – this limits its usefulness in practical applications and motivates us to look at various approximations. For simplicity, we only consider approximations from within the class of symmetric acausal FIR filters.

Our first approach is called *least squares filter design* and is based upon the following result: if K is a nonnegative integer, then among all functions of the form

$$H_K(f) \stackrel{\text{def}}{=} \sum_{u=-K}^{K} h_u \mathrm{e}^{-\mathrm{i}2\pi f u},$$

the one that minimizes

$$\int_{-1/2}^{1/2} |G_I(f) - H_K(f)|^2 \, \mathrm{d}f$$

is obtained by letting $h_u = g_{I,u}$ for $|u| \leq K$ (see the discussion surrounding Equation (52)). Thus the acausal FIR filter defined by

$$g_{K,u} = \begin{cases} g_{I,u}, & |u| \leq K; \\ 0, & \text{otherwise}, \end{cases}$$

has a transfer function

$$G_K(f) \stackrel{\text{def}}{=} \sum_{u=-K}^{K} g_{K,u} \mathrm{e}^{-\mathrm{i}2\pi f u}$$

that is the best Kth-order approximation – in the least squares sense – to the transfer function of an ideal low-pass digital filter.

As an example, Figure 153a shows $|G_K(f)|^2$ versus f for $W = 0.1$ and $K = 32$. This figure points out a potential problem with least squares filters, namely, ripples in the region where $G_I(\cdot)$ has a discontinuity – in this example, this occurs at $W = 0.1$ (indicated by the dotted vertical line). These ripples are due to the Gibbs phenomenon and constitute a form of leakage in the region where $G_I(f) = 0$. We introduced these ideas in Section 3.8 and discussed there the use of convergence factors $\{c_u\}$ to reduce the ripples. In the present context these factors would define a new impulse response sequence given by, say,

$$g_{K,u}^{(\mathrm{C})} \stackrel{\text{def}}{=} c_u g_{K,u} = \begin{cases} c_u g_{I,u}, & |u| \leq K; \\ 0, & \text{otherwise}, \end{cases}$$

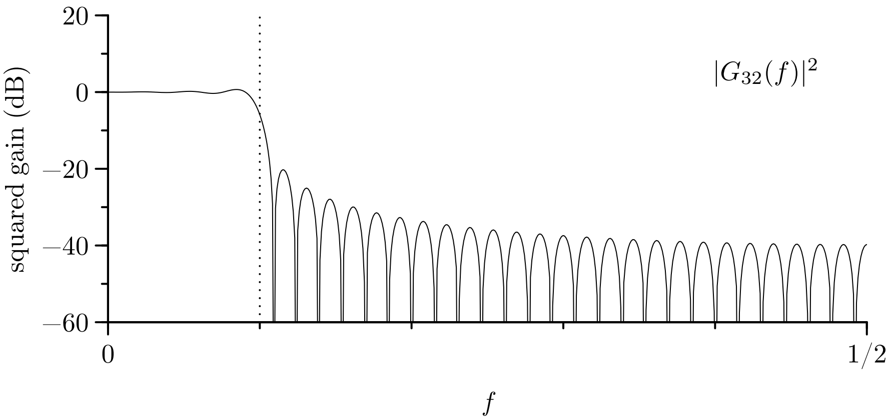

Figure 153a Squared modulus of $G_{32}(\cdot)$, the squared gain function for the least squares approximation of order $K = 32$ to an ideal low-pass filter with pass-band $[-0.1, 0.1]$. The dotted vertical line marks the cutoff frequency $W = 0.1$.

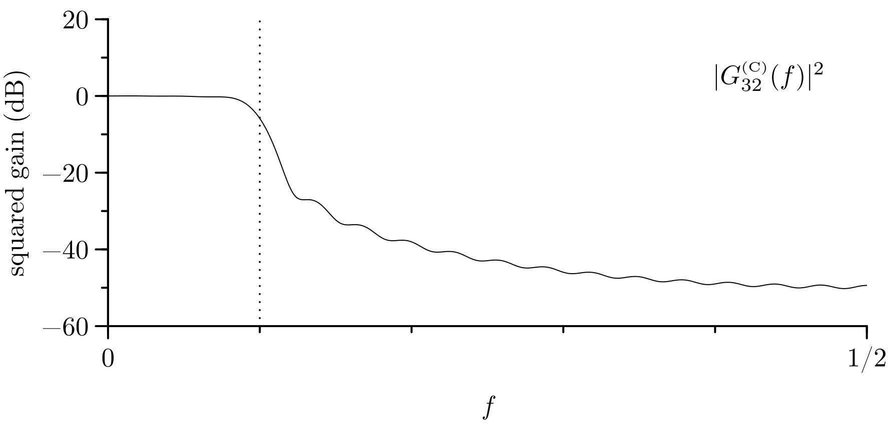

Figure 153b Squared modulus of $G^{(C)}_{32}(\cdot)$ with triangular convergence factors – compare this with $G_{32}(\cdot)$ in Figure 153a.

with a corresponding transfer function given by

$$G^{(C)}_K(f) \stackrel{\text{def}}{=} \sum_{u=-K}^{K} g^{(C)}_{K,u} \mathrm{e}^{-\mathrm{i}2\pi f u}.$$

For example, the use of Cesàro sums yields triangular convergence factors:

$$c_u = \begin{cases} \gamma\left(1 - \dfrac{|u|}{K+1}\right), & |u| \le K; \\ 0, & \text{otherwise;} \end{cases}$$

here γ is a gain factor that allows us to, say, force the normalization $\sum g^{(C)}_{K,u} = G^{(C)}_K(0) = 1$ (see the discussion in the previous section). Figure 153b shows $|G^{(C)}_K(f)|^2$ versus f for the same values of W and K as in Figure 153a. Note that the ripples are substantially reduced in $|G^{(C)}_K(\cdot)|^2$ but also that the discontinuity at $W = 0.1$ is now less accurately rendered than in $|G_K(\cdot)|^2$.

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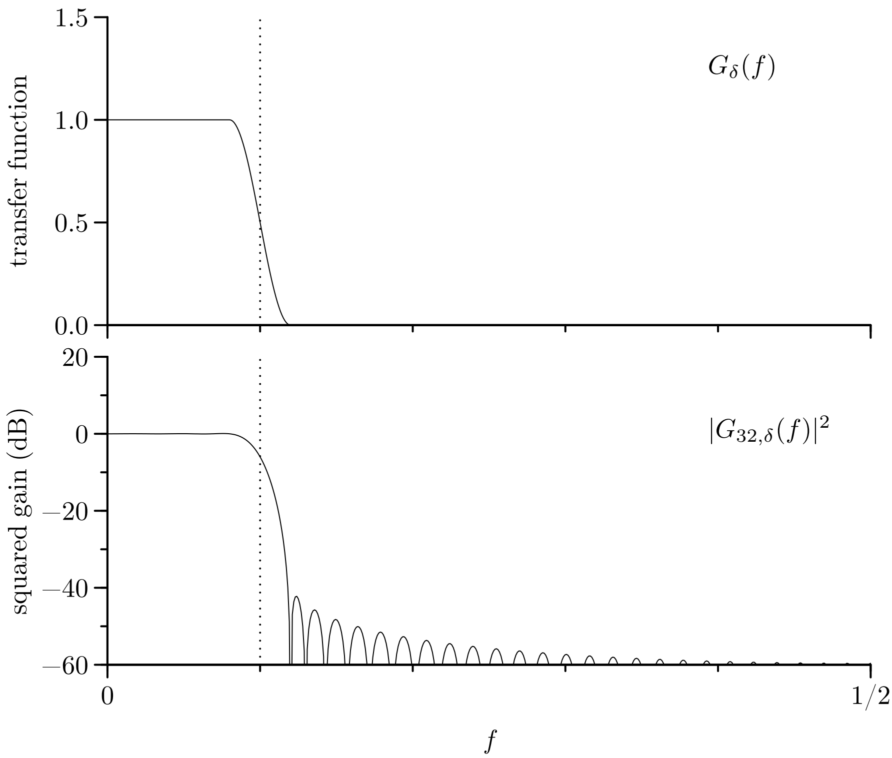

Figure 154 $G_\delta(\cdot)$ with $W = 0.1$ (indicated by the dotted vertical lines) and $\delta = 0.02$ (top plot) and the squared modulus of its approximation $G_{32,\delta}(\cdot)$ (bottom plot).

Let us note the following interesting interpretation of convergence factors (Bloomfield, 2000). Since we can write

$$G_K^{(C)}(f) = \int_{-1/2}^{1/2} G_I(f')C_K(f - f')\,\mathrm{d}f' \text{ for } C_K(f) \stackrel{\text{def}}{=} \sum_{u=-K}^{K} c_u \mathrm{e}^{-\mathrm{i}2\pi f u},$$

$G_K^{(C)}(\cdot)$ is a "smoothed" version of $G_I(\cdot)$ (see Section 3.6). Because $G_I(\cdot)$ is already smooth everywhere except at the discontinuities at $\pm W$, the main effect of convolving it with the smoothing kernel $C_K(\cdot)$ is to eliminate these discontinuities, but we could just as well do this directly. For example, instead of seeking least squares approximations to $G_I(\cdot)$ itself, we could look at such approximations to the following smoothed version of $G_I(\cdot)$:

$$G_\delta(f) \stackrel{\text{def}}{=} \begin{cases} 1, & |f| \le W - \delta; \\ \frac{1}{2}\left[1 + \cos\left(\pi \frac{|f| - W + \delta}{2\delta}\right)\right], & W - \delta < |f| \le W + \delta; \\ 0, & W + \delta < |f| \le 1/2 \end{cases} \tag{154}$$

under the constraints $0 < \delta < W$ and $W + \delta < 1/2$ (see the top plot of Figure 154 for the case $W = 0.1$ and $\delta = 0.02$). This yields

$$G_{K,\delta}(f) \stackrel{\text{def}}{=} \sum_{u=-K}^{K} g_{\delta,u} \mathrm{e}^{-\mathrm{i}2\pi f u} \text{ with } g_{\delta,u} \stackrel{\text{def}}{=} \int_{-1/2}^{1/2} G_\delta(f) \mathrm{e}^{\mathrm{i}2\pi f u}\,\mathrm{d}f$$

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as an approximation to $G_\delta(\cdot)$ (calculation of $g_{\delta,u}$ is the subject of Exercise [5.20]). The squared modulus of the order $K = 32$ approximation for the case depicted in the top plot of Figure 154 is shown in the lower plot there – this should be compared with Figures 153a and 153b. Note in particular that the sidelobes of $G_{K,\delta}(\cdot)$ are considerably lower than those of the other two examples.

5.9 Use of Slepian Sequences in Low-Pass Filter Design

Here we consider a symmetric acausal FIR filter of width $2K + 1$ that approximates an ideal low-pass filter as well as possible in the following sense: among all filters $\{g_u\}$ such that $g_u = 0$ for $|u| > K$ and normalized such that $\sum g_u = 1$, we want the filter whose transfer function

$$G_K(f) \stackrel{\text{def}}{=} \sum_{u=-K}^{K} g_u e^{-i2\pi f u}$$

is as concentrated as possible in the range $|f| \le W$. This is very close to the index-limited concentration problem we considered in Section 3.10. We again use the measure of concentration

$$\begin{aligned}\beta^2(W) &\stackrel{\text{def}}{=} \int_{-W}^{W} |G_K(f)|^2 \,\mathrm{d}f \Big/ \int_{-1/2}^{1/2} |G_K(f)|^2 \,\mathrm{d}f \\ &= \sum_{u'=-K}^{K} \sum_{u=-K}^{K} g_u^* \frac{\sin\left(2\pi W(u'-u)\right)}{\pi(u'-u)} g_{u'} \Big/ \sum_{u=-K}^{K} |g_u|^2\end{aligned}$$

(see Equation (88a)). Following the approach of Section 3.10, the solution to this concentration problem is any eigenvector associated with the largest eigenvalue of the following set of matrix eigenvalue equations:

$$\sum_{u'=-K}^{K} \frac{\sin\left(2\pi W(u'-u)\right)}{\pi(u'-u)} g_{u'} = \lambda g_u, \quad u = -K, \ldots, K. \tag{155}$$

Let $\lambda_0(W)$ denote this eigenvalue, and let

$$\tilde{\boldsymbol{v}}_0(W) \stackrel{\text{def}}{=} [\tilde{v}_{0,-K}(W), \ldots, \tilde{v}_{0,0}(W), \ldots, \tilde{v}_{0,K}(W)]^T$$

be the corresponding eigenvector normalized so that its elements sum to unity. The elements of this eigenvector are renormalized and reindexed portions of the zeroth-order *Slepian sequence*, which we denoted as $\{v_{0,t}(N, W)\}$ in Section 3.10 (Slepian sequences are also known as discrete prolate spheroidal sequences). If we let $\boldsymbol{A}$ be the $(2K+1)\times(2K+1)$ matrix whose (u', u)th element is $\sin\left[2\pi W(u'-u)\right]/[\pi(u'-u)]$ (defined to be $2W$ when $u' = u$), then the set of $2K + 1$ equations in Equation (155) can be written as $\boldsymbol{A}\tilde{\boldsymbol{v}}_0(W) = \lambda_0(W)\tilde{\boldsymbol{v}}_0(W)$, and the concentration measure is

$$\beta^2(W) = \frac{\tilde{\boldsymbol{v}}_0^T(W)\boldsymbol{A}\,\tilde{\boldsymbol{v}}_0(W)}{\tilde{\boldsymbol{v}}_0^T(W)\,\tilde{\boldsymbol{v}}_0(W)} = \frac{\lambda_0(W)\,\tilde{\boldsymbol{v}}_0^T(W)\,\tilde{\boldsymbol{v}}_0(W)}{\tilde{\boldsymbol{v}}_0^T(W)\,\tilde{\boldsymbol{v}}_0(W)} = \lambda_0(W).$$

The solid curves in Figure 156a show the squared magnitude of the resulting transfer function $G_K(\cdot)$ for $W = 0.1$ and the cases $K = 8$ (thin curve) and 32 (thick). The dotted vertical line indicates the location of the cutoff frequency $W = 0.1$. This plot shows that each $G_K(\cdot)$ captures one important aspect of the ideal low-pass digital filter – a small amount of energy in the frequency range $W < |f| \le 1/2$. However, both fail miserably in another

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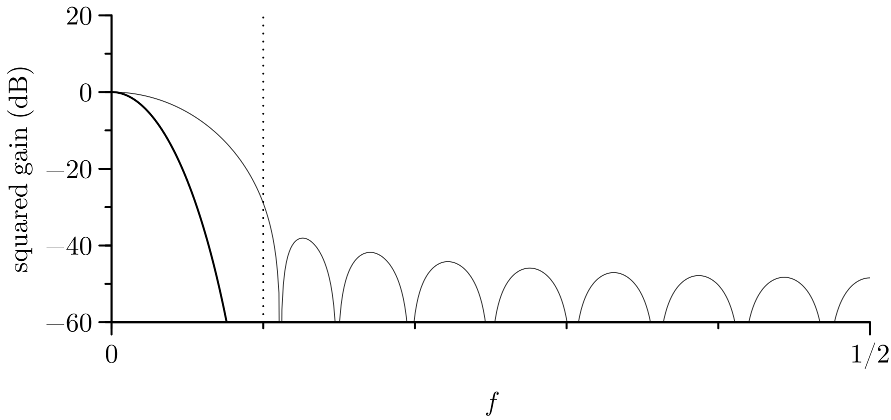

Figure 156a Squared gain functions corresponding to $\tilde{\boldsymbol{v}}_0(W)$ with $W = 0.1$ and $K = 8$ (thin curve) and 32 (thick curve). The dotted vertical line marks the frequency W.

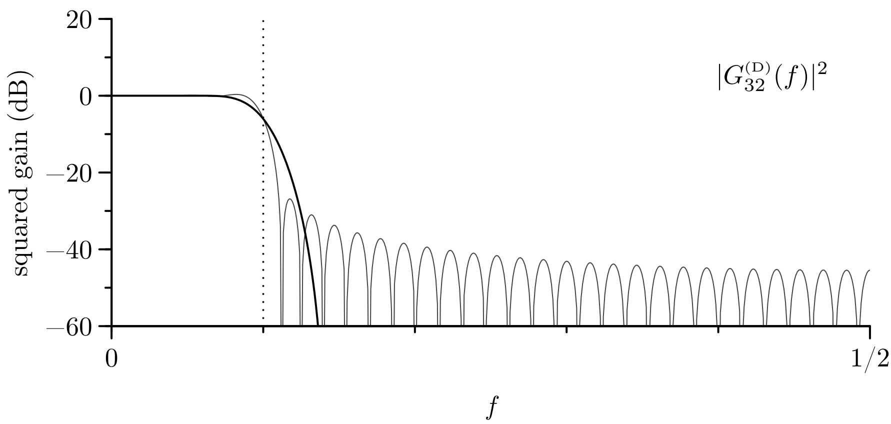

Figure 156b Squared modulus of $G_{32}^{(\mathrm{D})}(\cdot)$ using Slepian sequences as convergence factors for the cases $\delta = 0.04$ (thick curve) and $\delta = 0.01$ (thin). The dotted vertical line marks the position of the cutoff frequency $W = 0.1$.

aspect: the squared modulus of the transfer function of the ideal filter is not anywhere near constant for $0 \le |f| \le W$.

The $(2K+1)\times(2K+1)$ matrix $\boldsymbol{A}$ from which the eigenvectors are derived is a symmetric Toeplitz matrix (see Section 2.4). Makhoul (1981a) has shown that the transfer function for an eigenvector of a symmetric Toeplitz matrix will typically be zero at $2K$ different frequencies f such that $-1/2 < f \le 1/2$ if the eigenvector corresponds to either the maximum or minimum eigenvalue. Thus we expect the transfer function of our FIR filter to have zero response (a notch) at certain frequencies. For example, 8 notches are apparent in the thin solid curve in Figure 156a for $K = 8$, and by symmetry there must be 8 additional notches in the frequency range $[-1/2, 0]$ for a total of $2K = 16$ notches.

A more reasonable use for Slepian sequences in low-pass filter design is as convergence factors (Rabiner and Gold, 1975, sections 3.8 and 3.11). Let us define

$$d_u = \begin{cases} \gamma\tilde{v}_{0,u}(\delta), & |u| \le K; \\ 0, & \text{otherwise,} \end{cases}$$

where, as before, γ is a gain factor; however, note that the cutoff frequency for the Slepian

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sequence is now δ instead of W. We can now define a FIR filter with impulse response sequence

$$g^{(\mathrm{D})}_{K,u} = d_u g_{I,u},$$

where, as in the previous section, $\{g_{I,u}\}$ is the impulse response sequence of an ideal low-pass filter of bandwidth $2W$. The transfer function corresponding to $\{g^{(\mathrm{D})}_{K,u}\}$ is given by

$$G^{(\mathrm{D})}_K(f) = \int_{-1/2}^{1/2} G_I(f') D_K(f - f')\,\mathrm{d}f' \;\text{ for }\; D_K(f) \stackrel{\mathrm{def}}{=} \sum_{u=-K}^{K} d_u \mathrm{e}^{-\mathrm{i}2\pi f u},$$

where $G_I(\cdot)$ is the transfer function given in Equation (152a) for the ideal low-pass digital filter. Because $G_I(\cdot)$ is convolved with $D_K(\cdot)$ to form $G^{(\mathrm{D})}_K(\cdot)$, the discontinuity in $G_I(\cdot)$ is smeared out into a transition band. The width of this transition band is proportional to the width of the central lobe of $D_K(\cdot)$, while the sidelobes of the latter contribute to leakage outside the pass-band. For a given filter width $2K+1$, by adjusting the bandwidth parameter δ for $\{\tilde{v}_{0,u}(\delta)\}$, we can trade off between a small central lobe (its width decreases as δ decreases) and small sidelobes (their heights decrease as δ increases). The Slepian sequence is a natural choice for the convergence factors because the squared modulus of its transfer function is a good approximation to a Dirac delta function.

As an example, Figure 156b shows the squared modulus of $G^{(\mathrm{D})}_K(\cdot)$ for $K = 32$, $W = 0.1$ and the cases $\delta = 0.04$ (thick solid curve) and $\delta = 0.01$ (thin). In each case the gain factor γ was set so that $G^{(\mathrm{D})}_K(0) = 1$. As expected, the transfer function for $\delta = 0.04$ has a wider transition region about $W = 0.1$ than does the one for $\delta = 0.01$, whereas the sidelobe level is much higher in the latter case than in the former. Of course, if we allow ourselves the liberty of increasing K beyond 32, the trade-off between transition region width and sidelobe level becomes easier to manage.

5.10 Exercises

[5.1] Let $\{X(t)\}$ be a stochastically continuous, continuous parameter stationary process with zero mean and spectral representation

$$X(t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t}\,\mathrm{d}Z_X(f),$$

where $\{Z_X(f)\}$ is an orthogonal process such that $E\{|\mathrm{d}Z_X(f)|^2\} = \mathrm{d}S^{(\mathrm{I})}_X(f)$. If $G(\cdot)$ is a complex-valued function such that

$$\int_{-\infty}^{\infty} |G(f)|^2\,\mathrm{d}S^{(\mathrm{I})}_X(f) < \infty, \tag{157}$$

show that

$$Y(t) \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi f t} G(f)\,\mathrm{d}Z_X(f)$$

is a stationary process. Why is the condition shown in Equation (157) necessary?

[5.2] (a) Show that Equation (142) defines an LTI filter.

(b) Show that

$$L\{X(t)\} \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}^p X(t)}{\mathrm{d}t^p}$$

is an LTI filter, and determine its transfer function. This is an example of an LTI analog filter that must make use of Dirac delta functions if it is to be expressed as a convolution.

[5.3] Consider a zero mean stationary process $\{X_t\}$ with a sampling interval of Δ_t, an associated Nyquist frequency $f_{\mathcal{N}} = 1/(2\Delta_t)$, a spectral representation given by

$$X_t = \int_{-f_{\mathcal{N}}}^{f_{\mathcal{N}}} e^{i2\pi f t\,\Delta_t}\, dZ_X(f)$$

and an integrated spectrum $S_X^{(I)}(\cdot)$ specified by $E\{|dZ_X(f)|^2\} = dS_X^{(I)}(f)$. Define

$$Y_t = C \sum_{u=-\infty}^{\infty} g_u X_{t-u},$$

where $C \neq 0$ is a real-valued constant, and $\{g_u\}$ is a real-valued sequence satisfying $\sum g_u^2 < \infty$. Show that $\{Y_t\}$ is a stationary process whose integrated spectrum $S_Y^{(I)}(\cdot)$ is specified by

$$dS_Y^{(I)}(f) = |G(f)|^2\, dS_X^{(I)}(f),$$

where

$$G(f) \stackrel{\text{def}}{=} C \sum_{u=-\infty}^{\infty} g_u e^{-i2\pi f u\,\Delta_t}.$$

(Note that, if $C = \Delta_t$, then $G(\cdot)$ is the Fourier transform of $\{g_u\}$ if we regard that sequence as having a sampling interval of Δ_t, whereas it is just proportional to the transform if C is anything else. Note also that, if $S_X^{(I)}(\cdot)$ is differentiable with derivative $S_X(\cdot)$, then $S_Y^{(I)}(\cdot)$ is also differentiable, and its derivative $S_Y(\cdot)$ is given by $S_Y(f) = |G(f)|^2\, S_X(f)$.)

[5.4] Let $\{X_t\}$ be a discrete parameter stationary process with SDF given by $S_X(\cdot)$ defined over the interval $[-1/2, 1/2]$.

(a) Let the first difference process $\{Y_t\}$ be given by $Y_t \stackrel{\text{def}}{=} X_t - X_{t-1}$. Show that the SDF of $\{Y_t\}$ is given by $S_Y(f) = 4\sin^2(\pi f)\, S_X(f)$. Does a first difference filter resemble a low-pass or high-pass filter?

(b) If a two coefficient filter is now applied to $\{Y_t\}$ to yield $\{Z_t\}$ according to

$$Z_t = aY_t + bY_{t-1}, \quad \text{where } a = \frac{-1+\sqrt{3}}{4} \text{ and } b = \frac{1+\sqrt{3}}{4},$$

show that the SDF of $\{Z_t\}$ is given by

$$S_Z(f) = \sin^2(\pi f)\left[1 + 2\cos^2(\pi f)\right] S_X(f).$$

[5.5] Let $\{X_t\}$ be a discrete parameter stationary process with SDF given by $S_X(\cdot)$ defined over the interval $[-1/2, 1/2]$. With p taken to be a positive integer, show that the SDF for the process $\{Y_t\}$ defined by $Y_t = X_t - \alpha X_{t-p}$ is

$$S_Y(f) = 4\alpha^2 \sin^2(\pi f p)\, S_X(f). \qquad (158)$$

[5.6] Determine the SDF for the process $\{X_t\}$ defined by

$$X_t = \alpha X_{t-d} + \epsilon_t,$$

where $0 < \alpha < 1$; $d \geq 2$ is an integer; and $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ_ϵ^2. At what frequency (or frequencies) does this SDF have its maximum value? What does this suggest about the oscillatory behavior of $\{X_t\}$? (The first part of exercise 4.16, Brockwell and Davis, 1991, is a special case of this exercise.)

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[5.7] Consider the MA(2) process

$$X_t = \epsilon_t - \theta_{2,1}\epsilon_{t-1} - \theta_{2,2}\epsilon_{t-2},$$

where, as usual, $\{\epsilon_t\}$ is a white noise process with mean zero and variance σ_ϵ^2.

(a) Show that the SDF for $\{X_t\}$ is given by

$$S_X(f) = \sigma_\epsilon^2 \left[1 + \theta_{2,1}^2 + \theta_{2,2}^2 - 2\theta_{2,1}\left(1 - \theta_{2,2}\right)\cos(2\pi f) - 2\theta_{2,2}\cos(4\pi f)\right].$$

(b) The autocovariance sequence (ACVS) $\{s_{X,\tau}\}$ for an MA process and its SDF form a Fourier transform pair. Use this relationship to show that the variance of $\{X_t\}$ is given by

$$\sigma_X^2 = \left(1 + \theta_{2,1}^2 + \theta_{2,2}^2\right)\sigma_\epsilon^2$$

and that the lag $\tau = 1$ and 2 members of its ACVS are

$$s_{X,1} = -\theta_{2,1}\left(1 - \theta_{2,2}\right)\sigma_\epsilon^2 \text{ and } s_{X,2} = -\theta_{2,2}\sigma_\epsilon^2.$$

(c) Let $\rho_{X,\tau} = s_{X,\tau}/s_{X,0}$ represent the ACS for $\{X_t\}$ at lag τ. Compute $\rho_{X,\tau}$ at lags $\tau = 1$ and $\tau = 2$ for the cases (i) $\theta_{2,1} = 5/6$ and $\theta_{2,2} = -1/6$ and (ii) $\theta_{2,1} = 5$ and $\theta_{2,2} = -6$. Comment upon the result.

[5.8] Let $\{X_t\}$ be a discrete parameter stationary ARMA(p,q) process, which implies that $\{X_t\}$ satisfies the equation

$$X_t = \sum_{j=1}^{p} \phi_{p,j} X_{t-j} + \epsilon_t - \sum_{j=1}^{q} \theta_{q,j}\epsilon_{t-j},$$

where $\{\epsilon_t\}$ is a white noise process with zero mean and variance σ_ϵ^2. Show that the SDF of this process is given by Equation (145c).

[5.9] (a) Let $X_t = \phi X_{t-1} + \epsilon_t$ be a stationary first-order autoregressive process, where $|\phi| < 1$, and $\{\epsilon_t\}$ is a zero mean white noise process with variance σ_ϵ^2. Let $\{W_t\}$ be a white noise process with zero mean and variance σ_W^2. Suppose that X_t and $W_{t'}$ are uncorrelated for all $t, t' \in \mathbb{Z}$. Let $Y_t \stackrel{\text{def}}{=} X_t + W_t$. Determine the SDF for $\{Y_t\}$.

(b) Determine the SDF for the stationary ARMA(1,1) process $\{U_t\}$ defined by $U_t = \phi U_{t-1} + \eta_t - \theta\eta_{t-1}$, where ϕ is the same as in part (a), and $\{\eta_t\}$ is a zero mean white noise process with variance σ_η^2.

(c) Let $\sigma_\epsilon^2 = \sigma_W^2 = 1$ and $\phi = 1/2$. By equating the SDFs in parts (a) and (b), show that $\{Y_t\}$ in (a) can be written as $\{U_t\}$ in (b), provided $\theta = 1/(2\sigma_\eta^2)$, where σ_η^2 is the solution to the equation

$$\sigma_\eta^2\left(1 + \frac{1}{4\sigma_\eta^4}\right) = \frac{9}{4}.$$

[5.10] Let $\{X_t\}$ be a discrete parameter stationary process with an SDF $S_X(\cdot)$ defined over the interval $[-1/2, 1/2]$. Let

$$Y_t \stackrel{\text{def}}{=} X_t - \frac{1}{2K+1}\sum_{j=-K}^{K} X_{t+j},$$

where $K > 0$ is an integer. Find the SDF for $\{Y_t\}$. Plot this SDF for the cases $K = 4$ and 16 when $\{X_t\}$ is a white noise process.

[5.11] Let $\{X_t\}$ be a discrete parameter stationary process with an SDF $S_X(\cdot)$ defined over the interval $[-1/2, 1/2]$.

(a) Let

$$\overline{X}_{K,t} \stackrel{\text{def}}{=} \frac{1}{K}\sum_{k=0}^{K-1} X_{t-k} \text{ and } Y_{K,t} \stackrel{\text{def}}{=} \overline{X}_{K,t} - \overline{X}_{K,t-K}.$$

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Show that $\{Y_{K,t}\}$ is a zero mean stationary process with SDF

$$S_{Y,K}(f) = |G_K(f)|^2 S_X(f) \text{ with } |G_K(f)|^2 = \frac{4\sin^4(K\pi f)}{K^2 \sin^2(\pi f)}.$$

(b) Plot $|G_K(\cdot)|^2$ for $K = 1, 2, 4, 8$ and 16 to see that $|G_K(\cdot)|^2$ is an approximation to the squared magnitude of a transfer function for a band-pass filter with a pass-band defined by $|f| \in [1/(4K), 1/(2K)]$. (This sequence of filters is the basis for a crude method of estimating the SDF called *pilot analysis*. This method was useful as a simple way of computing a rough estimate of the SDF in days before modern computers and the popularization of the fast Fourier transform. For some details, see section 7.3.2 of Jenkins and Watts, 1968.)

[5.12] Let $\{\epsilon_t\}$ be a white noise process with mean zero and variance σ_ϵ^2. Define the stationary processes $\{X_t\}$ and $\{Y_t\}$ by

$$X_t = \tfrac{2}{9}\epsilon_t + \tfrac{5}{9}\epsilon_{t-1} + \tfrac{2}{9}\epsilon_{t-2} \text{ and } Y_t = \tfrac{4}{9}\epsilon_t + \tfrac{4}{9}\epsilon_{t-1} + \tfrac{1}{9}\epsilon_{t-2}.$$

(a) Let $L_X\{\cdot\}$ and $L_Y\{\cdot\}$ be LTI digital filters such that $L_X\{\epsilon_t\} = X_t$ and $L_Y\{\epsilon_t\} = Y_t$. Show that the gain function $|G_X(\cdot)|$ corresponding to $L_X\{\cdot\}$ is given by

$$|G_X(f)| = \tfrac{5}{9} + \tfrac{4}{9}\cos(2\pi f),$$

and find the gain function $|G_Y(\cdot)|$ corresponding to $L_Y\{\cdot\}$. In what respects do the associated transfer functions $G_X(\cdot)$ and $G_Y(\cdot)$ differ?

(b) Compare the SDF of $\{X_t\}$ with the SDF of $\{Y_t\}$.

(c) How do $\{X_t\}$ and $\{Y_t\}$ differ from a moving average process as defined by Equation (32a)? Are there moving average processes that are equivalent to $\{X_t\}$ and $\{Y_t\}$ (i.e., have the same SDFs)?

[5.13] (a) Let $\{X_t\}$ be a real-valued zero mean stationary process with ACVS $\{s_{X,\tau}\}$ and SDF $S_X(\cdot)$ such that $\{s_{X,\tau}\}$ and $S_X(\cdot)$ are a Fourier transform pair (assume $\Delta_t = 1$ for convenience). Define the complex-valued process $\{Z_t\}$ by

$$Z_t = X_t e^{-i2\pi f_0 t},$$

where f_0 is any fixed frequency such that $0 < |f_0| \le 1/2$; i.e., the tth element of the sequence $\{X_t\}$ is multiplied by $\exp(-i2\pi f_0 t)$ to create the tth element of $\{Z_t\}$. Show that $\{Z_t\}$ is a zero mean stationary process with ACVS and SDF given, respectively, by

$$s_{Z,\tau} = s_{X,\tau} e^{-i2\pi f_0 \tau} \text{ and } S_Z(f) = S_X(f_0 + f).$$

Does the transformation from $\{X_t\}$ to $\{Z_t\}$ constitute an LTI filter? (Creation of $\{Z_t\}$ is an example of a technique called *complex demodulation*; see Tukey, 1961, Hasan, 1983, Bloomfield, 2000 and Stoica and Moses, 2005.)

(b) Suppose $f_0 = 1/2$ so that $\exp(-i2\pi f_0 t) = (-1)^t$. Show that we can effectively turn a low-pass filter into a high-pass filter by the following three operations: (i) form $Z_t = (-1)^t X_t$; (ii) pass $\{Z_t\}$ through a low-pass filter to produce, say, $\{Z_t'\}$; and (iii) form $X_t' \stackrel{\text{def}}{=} (-1)^t Z_t'$, where $\{X_t'\}$ can be regarded as a high-pass filtered version of $\{X_t\}$.

(c) Let $\{X_t\}$ and f_0 be as defined in part (a). Define the real-valued process $\{Y_t\}$ by $Y_t = X_t \cos(2\pi f_0 t)$. Is $\{Y_t\}$ a stationary process?

[5.14] Part (b) of the previous exercise demonstrated that, by demodulating the input to a low-pass filter at $f_0 = 1/2$ and then demodulating the output at $f_0 = 1/2$ also, we can effectively turn a low-pass filter into a high-pass filter. This complementary exercise shows that we can achieve the same result by demodulating the filter itself at $f_0 = 1/2$.

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(a) Consider two filters with real-valued impulse response sequences $\{g_u\}$ and $\{h_u\}$ related via

$$h_u = (-1)^u g_u. \tag{161a}$$

Show that the corresponding transfer functions $G(\cdot)$ and $H(\cdot)$ are related by

$$H(f) = G(f - \tfrac{1}{2}). \tag{161b}$$

Use the above to argue that, if $\{g_u\}$ resembles a low-pass filter, then $\{h_u\}$ must resemble a high-pass filter.

(b) As a concrete example, consider the impulse response sequence $\{g_u^{(1)}\}$ of Equation (148b). Show that its transfer function is given by

$$G^{(1)}(f) = \cos^2(\pi f).$$

Define $\{h_u^{(1)}\}$ in terms of $\{g_u^{(1)}\}$ as per Equation (161a), and let $H^{(1)}(\cdot)$ denote its transfer function. Determine the transfer function $H^{(1)}(\cdot)$ for $\{h_u^{(1)}\}$ using the fact that $\{h_u^{(1)}\} \longleftrightarrow H^{(1)}(\cdot)$, and demonstrate that $H^{(1)}(\cdot)$ is indeed related to $G^{(1)}(\cdot)$ as per Equation (161b).

(c) Suppose now that the filters $\{g_u\}$ and $\{h_u\}$ are related by $h_u = (-1)^u g_{-u}$ rather than by Equation (161a). What is the relationship between the transfer functions for $\{g_u\}$ and $\{h_u\}$? If $\{g_u\}$ resembles a low-pass filter, does $\{h_u\}$ still resemble a high-pass filter?

(d) Finally consider the causal FIR filter $\{g_u\}$, where $g_u = 0$ for $u < 0$ and $u \geq L$ for some positive integer L. Define $h_u = (-1)^u g_{L-1-u}$. What is the relationship between the transfer functions for $\{g_u\}$ and $\{h_u\}$? If $\{g_u\}$ resembles a low-pass filter, does $\{h_u\}$ still resemble a high-pass filter?

[5.15] Let $\{X(t)\}$ be a continuous parameter stationary process with SDF $S_X(\cdot)$ defined over $\mathbb{R}$. For $\Delta_{\mathrm{t}} > 0$ let

$$\overline{X}(t) \stackrel{\mathrm{def}}{=} \frac{1}{\Delta_{\mathrm{t}}} \int_{t-\Delta_{\mathrm{t}}}^{t} X(u)\,\mathrm{d}u$$

represent the average value of the process over the interval $[t - \Delta_{\mathrm{t}}, t]$.

(a) Show that $\{\overline{X}(t)\}$ is a stationary process, and find its SDF $S_{\overline{X}}(\cdot)$.

(b) Let $\overline{X}_{\Delta'_{\mathrm{t}},t} \stackrel{\mathrm{def}}{=} \overline{X}(t\,\Delta'_{\mathrm{t}})$, $t \in \mathbb{Z}$, be a discrete parameter process formed by taking samples Δ'_{t} time units apart from $\{\overline{X}(t)\}$. Determine the SDF $S_{\overline{X}_{\Delta'_{\mathrm{t}}}}(\cdot)$ for $\{\overline{X}_{\Delta'_{\mathrm{t}},t}\}$ in terms of $S_X(\cdot)$.

(c) Suppose now that $S_X(f) = C$ for $f \in \mathbb{R}$, where C is a positive constant. Since $S_X(\cdot)$ is flat, we can regard it as the SDF for a continuous parameter version of white noise. As noted in Section 2.7, such a process technically does not exist but is sometimes a convenient fiction. This fiction makes sense in part (a) because $S_{\overline{X}}(\cdot)$ is a well-defined SDF when $S_X(f) = C$ (i.e., as opposed to $S_X(\cdot)$, the integral of $S_{\overline{X}}(\cdot)$ over $\mathbb{R}$ is finite). Show that, if $\Delta'_{\mathrm{t}} = \Delta_{\mathrm{t}}$ in part (b), then the SDF $S_{\overline{X}_{\Delta_{\mathrm{t}}}}(\cdot)$ is flat, i.e., corresponds to the SDF for white noise. Why might we have anticipated this result? Hint: use the fact that

$$\sum_{k=-\infty}^{\infty} (a+k)^{-2} = \pi^2 \operatorname{cosec}^2(\pi a).$$

(d) As a follow-on to part (c), show that, if $\Delta'_{\mathrm{t}} = \Delta_{\mathrm{t}}/2$, then the SDF is given by

$$S_{\overline{X}_{\Delta'_{\mathrm{t}}}}(f) = C\cos^2(\pi f\,\Delta_{\mathrm{t}}/2), \quad |f| \leq f_{\mathcal{N}} \ \text{ where here } \ f_{\mathcal{N}} = 1/\Delta_{\mathrm{t}}.$$

Show that this SDF is that of a first-order moving average process whose unit lag autocorrelation is $1/2$. Why might we have anticipated this result?

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(e) In part (b), let $\Delta_{\mathrm{t}}' = \Delta_{\mathrm{t}}$, and define the discrete parameter process $\overline{Y}_t = \overline{X}_{\Delta_{\mathrm{t}},t} - \overline{X}_{\Delta_{\mathrm{t}},t-1}$; i.e., $\{\overline{Y}_t\}$ is the first difference process corresponding to $\{\overline{X}_{\Delta_{\mathrm{t}},t}\}$ (see Exercise [5.4]). Find the SDF $S_{\overline{Y}_t}(\cdot)$ for $\{\overline{Y}_t\}$.

(f) Let $\overline{Y}(t) = \overline{X}(t) - \overline{X}(t - \Delta_{\mathrm{t}})$, $t \in \mathbb{R}$, define a continuous parameter process representing the difference between average values of $\{X(t)\}$ that are Δ_{t} time units apart. Let $\overline{Y}_t' = \overline{Y}(t\,\Delta_{\mathrm{t}})$, $t \in \mathbb{Z}$, define a discrete parameter process formed by taking samples Δ_{t} time units apart from $\{\overline{Y}(t)\}$. Show that the SDF for $\{\overline{Y}_t'\}$ is the same as that of $\{\overline{Y}_t\}$ of part (e); i.e., we can interchange the order in which we carry out differencing and sampling. (This way of processing $\{X(t)\}$ arises in the study of the timekeeping properties of atomic clocks; see Barnes et al., 1971.)

[5.16] If the action of L on complex exponentials is described by Equation (135a), and if $G(-f) = G^*(f)$ for all f, what does L do to sines and cosines?

[5.17] Suppose that $g_p(\cdot)$ and $h_p(\cdot)$ are two periodic functions with period T such that

$$\int_{-T/2}^{T/2} |g_p(t)|^2 \,\mathrm{d}t < \infty \quad \text{and} \quad \int_{-T/2}^{T/2} |h_p(t)|^2 \,\mathrm{d}t < \infty.$$

Let $\{G_n\}$ and $\{H_n\}$ be their Fourier coefficients as defined by Equation (49b). If we regard $h_p(\cdot)$ as a function of interest and $g_p(\cdot)/T$ as a filter, their convolution

$$g_p * h_p(t) \stackrel{\mathrm{def}}{=} \frac{1}{T}\int_{-T/2}^{T/2} g_p(u) h_p(t-u)\,\mathrm{d}u$$

can be regarded as a filtered version of $h_p(\cdot)$ (the above is Equation (96b)). What plays the role of the transfer function here?

[5.18] For the filter

$$L\{X(t)\} \stackrel{\mathrm{def}}{=} \alpha + \beta X(t),$$

where α and β are arbitrary nonzero constants, consider all three conditions needed for it to be an LTI filter. Which conditions (if any) fail to hold?

[5.19] Suppose that $g(\cdot)$ is an $L^2(\mathbb{R})$ function whose Fourier transform $G(\cdot)$ is also such. The mapping from $g(\cdot)$ to $G(\cdot)$ is an example of an analog filter, but is it an LTI filter?

[5.20] Show that the impulse response sequence $\{g_{\delta,u}\}$ corresponding to $G_\delta(\cdot)$ of Equation (154) is given by

$$g_{\delta,u} = \begin{cases} 2W, & \text{when } u = 0; \\ \delta \sin(\pi W/2\delta), & |u| = 1/4\delta; \\ \dfrac{\sin(2\pi W u)\cos(2\pi\delta u)}{\pi(u - 16u^3\delta^2)}, & \text{otherwise.} \end{cases}$$

Hint: consider the inverse Fourier transform of the following periodic function with a period of unity:

$$D(f) \stackrel{\mathrm{def}}{=} G_\delta(f) - G_I(f) = \begin{cases} 0, & |f| \le W - \delta; \\ \dfrac{1}{2}\left[\cos\left(\pi\dfrac{|f| - W + \delta}{2\delta}\right) - 1\right], & W - \delta < |f| \le W; \\ \dfrac{1}{2}\left[\cos\left(\pi\dfrac{|f| - W + \delta}{2\delta}\right) + 1\right], & W < |f| \le W + \delta; \\ 0, & W + \delta < |f| \le 1/2, \end{cases}$$

where $G_\delta(\cdot)$ and $G_I(\cdot)$ are defined by Equations (154) and (152a).

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