Kinza Mushtaq

Numerical Computing

Assignment File

20101013

Chapter # 2 :- ①

Date:

Error Analysis:

True error: Difference between true value and Observed or measured value is known as true error.

Error = True value − approximation

Types of error:

① Absolute error:

Difference between true value and approximation or measured value that must be absolute is known as absolute error.

$E_{abs} = |\text{True value} - \text{approximation}|$

② Relative error:

Relative error is the ratio between absolute error and the true value.

$E_r = |\text{True value} - \text{approximation}| / \text{true value}$

③ Absolute relative approximate error: OR (Percentag relative error):

Relative error when expressed in terms of percentage is known as percentage relative error or Absolute relative approximat error.

$$|E_a| = \frac{|\text{True value} - \text{Observed value}|}{|\text{True value}|} \times 100$$

Why do we use measure errors:

① With the help of measuring error we can out the accuracy and efficiency of numerica method.

② By measuring errors we can use these er as conditional criteria to restrict the number iterations for iterative procedures.

Sources of Errors in Numerical Calculatio

There are atleast two sources of errors in numerical calculations.

① Rounding errors

② Truncation errors.

Rounding errors: Originate from the facts that can only represent numbers using a fixed limited number of significant figures. Thus such as π or $\sqrt{2}$ cannot be represented exactly computer memory. The discrepency introduced limitation is called Round-off error. Even sim addition can result in round-off error. Often have the capacity to represents numbers in tw different precisions, called single and double pre In single precision, 23 bits out of total 32 bit used to represent the significant digits in the On the remaining bits 8 are used to store the and one bit is used to store the sign.

Bits Organization in a Single Precision Number:

When 23 bits represents the significant figures, a single precision number can represent data to about seven decimal places. The 8 bit exponent allow scalling in the range 10^{38} and 10^{-38}. Taken together, single precision numbers can range from $\pm 1.17549435 1 \times 10^{-38}$ to $\pm 3.4028235 \times 10^{38}$. In single precision π can be represented as 3.141593

Due to the limited number of significant digits in single precision, most modern computers use double precision which uses 64 bits with 52 digits used to represent the significant figure. This allows π, for example to be represented as 3.141592653589793, that is 16 digits. The full range of number representation using double precision is $\pm 2.2250738507 2020 \times 10^{-308}$ to $\pm 1.79769 1348623157 \times 10^{38}$.

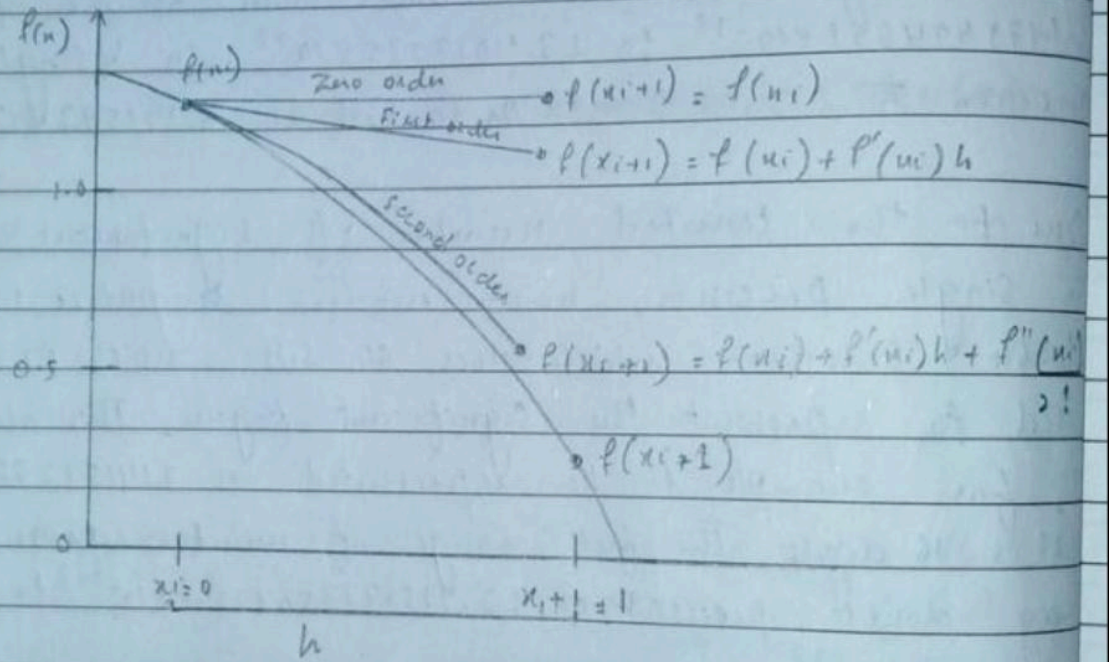
The use of double precisions reduces the effects of rounding error and should be used whenever possible in numerical calculations.

Truncation error: in numerical analysis arises when approximation are used to estimate some quantity often a Tylor Series is used to approximate a solution which is then truncated at different levels. The figure below shows that a function $f(x)$ being approximated by a Tylor series that has been truncated at different levels.

The more terms that are retained in the Tylor series the better the approximation and the smaller the truncation error.

Tylor Series:

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \dots$$



Assignment # 1 5

Date: 21-03-2021

Q What are the absolute and relative errors of the approximation 3.14 to the value π?

Data;

Approximation value of $\pi = 3.142$

Required;

Absolute Error = E_{abs} = ?

Relative Error = E_r = ?

Solution;

Formula;

. For Absolute Error.

$E_{abs} = |\text{True value} - \text{Approximate value}|$

$1.592653 59 \times 10^{-3}$

$E_{abs} = 3.141592654 - 3.14$

$\boxed{E_{abs} \approx 0.001592654}$

Formula:

: For Relative Error.

5.0695751×10^{-4}

$$E_r = \left|\frac{E_{abs}}{\text{True value}}\right| \rightarrow = \left|\frac{0.001592654}{3.141592654}\right|$$

$\boxed{E_r \approx 0.00050695751}$

② A resistor labeled as 240Ω is actually 243.32753 Ω What are the absolute and relative error of the labeled values

Data.

True value of resistor = 243.32753 Ω

Approximate resistance = 240 Ω

Required:

① Approximat error = E_{abs} = ?

② Relative error = E_r = ?

Solution

Formula:

① E_{abs} = |True value − Approximate value|

$= |243.32753 - 240|$

$$\boxed{E_{abs} \approx 3.32753}$$

② Formula:

$$E_r = \frac{E_{abs}}{\text{True value}}$$

$$E_r = \left| \frac{3.32753}{243.32753} \right|$$

$$\boxed{E_r \approx 0.01367510696}$$

Q[3] The voltage in a high-voltage transmission line is stated to be 2.4MV to 2 while the actual voltage may range from 2.1MV to 2.7MV. What is the maximum absolute and relative error of voltage?

Required

Absolute error = E_{abs} = ?

Relative error : E_r = ?

Solution

Formula:

|True value − Approximate value|

① For true value 2.1MV :.

$$E_{abs} = |2.1 - 2.4|$$

$$\boxed{E_{abs} = 0.3MV}$$

② For true value 2.7MV.

$$E_{abs} = |2.7 - 2.4|$$

$$\boxed{E_{abs} = 0.3MV}$$

① Relative Error of 2.1 approximat value

$$E_r = \left|\frac{0.3}{2.1}\right|$$

$$\boxed{E_r \approx 0.14}$$

② Relative error of 2.7 approximate va

$$E_r = \left|\frac{0.3}{2.7}\right|$$

$$\boxed{E_r \approx 0.11}$$

.:Bisection Method:. ③

Q) What is the bisection method and what is it based on?

A) One of the first Numerical methods developed to find out the root of nonlinear equation $f(x)=0$ was the bisection method (also called binary-search method). The method is based on the following theorems.

Theorem:

An equation $f(x)=0$; where $f(x)$: is a real continious function has at least one root between x_l and x_u if $f(x_l)(x_u)<0$ (see figure 1).
Note that if $f(x_l)f(x_u)>0$, there may or may not be any root between x_l and x_u (figure 4) so the Theorem - only. If $f(x_l)f(x_u)<0$, then there may be more than one root between x_l and x_u (figure 4) So the theorem only guarantees one root between x_l and x_u.

Bisection method:,

Since the method is based on finding the root between two points, the method falls under the category of bracketing methods.
Since the root is bracketed between two points x_l and x_u, one can find the mid point, x_m between x_l and x_u. This gives us two new intervals.

1. x_l and x_m and
2. x_m and x_u

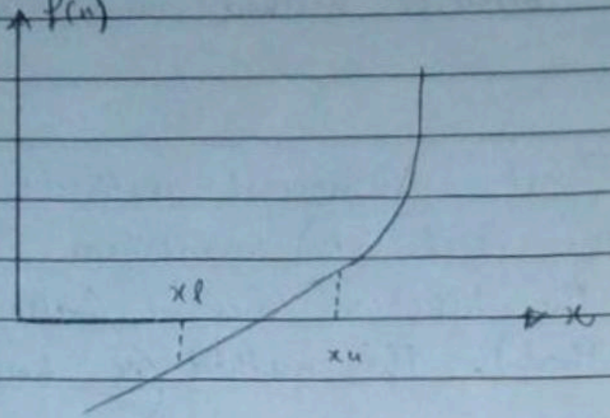


figure 1: At least one root exist between points if the function is real, continious and changes sign.

Alogorith for the Bisection method:

The steps to apply the bisection method to f the root the root of the equation $f(x) = 0$ are

① Choose x_l and x_u as two guesses for the root such that $f(x_l)\,f(x_u) < 0$, in other words, $f(x)$ changes sign between x_l and x_u

② Estimate the root x_m, of the equation $f(x) = 0$ as the mid point between x_l and x_u as

$$x_m = \frac{x_l + x_u}{2}$$

③ Now checking the following

(a) If $f(x_l)\,f(x_m) < 0$ then root lies between x_l and x_m $x_l = x_l$ and $x_u = x_m$

(b) If $f(x_l)\,f(x_m) > 0$ then the root lies between x_m and x_u then $x_l = x_m$ and $x_u = x_u$

(c) If $f(x_l)\,f(x_m) = 0$ then the root ~~lies between~~ is x_m Stop the alogrithm if this is true.

Find absolute relative approximate error as

$$|\epsilon_a| = \left| \frac{x_m^{new} - x_m^{old}}{x_m^{new}} \right| \times 100$$

Where x_m^{new} : estimated root from present iteration
x_m^{old} : estimated root from previous iteration

Compare the absolute relative approximate error $|\epsilon_a|$ with the pre-specified relative error tolerance ϵ_s. If $|\epsilon_a| > \epsilon_s$, then go to step 3, else stop the algorithm. Note one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Accuracy: Refers to how closely measured or calculated value agrees with the true solution/value.

Precision: Refers to how closely measured or calculated values agree with each other when they are repeated.

.: Assignment # 2 :. (11)

Date:

Q Given the function $f(x) = x^2 - \sin(x) - 0.5$ for $[a, b] = [0, 2]$ Find the approximated root using Bisection method. Find out the accuracy of the method using the condition $|Er| < Es = 0.01$.

Solution:

We have

$f(x) = x^2 - \sin(x) - 0.5$

Initial bound $[a, b] = [0, 2]$

Iteration 1:

$f(x_l) = f(0) = (0)^2 - \sin(0) - 0.5$

$f(x_l) = -0.5$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5$

$f(x_u) = f(2) = 4 - 0.90929 - 0.5$

$f(x_u) = f(2) = 2.59071$

$$x_m = \frac{x_l + x_u}{2} = \frac{0+2}{2} = \boxed{1}$$

$f(x_m) = f(1) = (1)^2 - \sin(1) - 0.5$

$f(x_m) = -0.34147$

Iteration 2:

$f(x_l)f(x_m) > 0$; $x_l = x_m$ & $x_u = x_u$

$f(x_l) = f(1) = (1)^2 - \sin(1) - 0.5 = -0.34147$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 = 2.59070$

$$x_m = \frac{x_l + x_u}{2} \rightarrow \frac{1+2}{2} = \boxed{1.5}$$

$f(x_m) = f(1.5) = (1.5)^2 - \sin(1.5) - 0.5 = 0.75250$

$$|Er| = \left|\frac{x_m^n - x_m^o}{x_m^n}\right| \rightarrow = \left|\frac{1.5 - 1}{1.5}\right|$$

$|E_r| = 0.33$

$|E_r| < |E_s| = 0.33 < 0.01$

False

Iteration 3:,

$f(x_l) f(x_m) < 0 \; ; \; x_l = x_l \, , \; x_u = x_m$

$f(x_l) = f(1) = (1)^2 - \sin(1) - 0.5 = -0.34147$

$f(x_u) = f(1.5) = (1.5)^2 - \sin(1.5) - 0.5 = 0.75251$

$x_m = \frac{x_l + x_u}{2} \rightarrow \frac{1 + 1.5}{2} = 1.25$

$f(x_m) = f(1.25) = (1.25)^2 - \sin(1.25) - 0.5 = 0.11352$

$|E_r| = \left| \frac{1.25 - 1.5}{1.25} \right| \rightarrow = 0.2$

$|E_r| < E_s = 0.2 < 0.01$

False

Iteration 4:,

$f(x_l) f(x_m) < 0 \; , \; x_l = x_l \, , \; x_u = x_m$

$f(x_l) = f(1) = (1)^2 - \sin(1) - 0.5 = -0.34147$

$f(x_u) = f(1.25) = (1.25)^2 - \sin(1.25) - 0.5 = 0.11352$

$x_m = \frac{x_l + x_u}{2} \rightarrow \frac{1 + 1.25}{2} = 1.125$

$f(x_m = f(1.125) = (1.125)^2 - \sin(1.125) - 0.5 = -0.13664$

$|E_r| = \left| \frac{1.125 - 1.25}{1.125} \right| \rightarrow = 0.11$

$|E_r| < E_s = 0.11 < 0.01$ False.

Iteration 5:,

$f(x_l) f(x_m) > 0 \; ; \; x_l = x_m \; ; \; x_u = x_u$

$f(x_l) = f(1.125) = (1.125)^2 - \sin(1.125) - 0.5 = -0.13664$

$f(x_u) = f(1.25) = (1.25)^2 - \sin(1.25) - 0.5 = 0.11352$

$x_m = \frac{x_l + x_u}{2} = \frac{1.125 + 1.25}{2} = \boxed{1.1875}$

$f(x_m) = f(1.1875) = (1.1875)^2 - \sin(1.1875) - 0.5$

$f(x_m) = -0.01728$

$|E_r| = \left| \frac{1.1875 - 1.125}{1.1875} \right| = 0.05$

$|E_r| < E_s$, $0.05 < 0.01$

False

Iteration 6:-

$f(x_l) f(x_m) > 0$; $x_l = x_m$; $x_u = x_u$

$f(x_l) = f(1.1875) = (1.1875)^2 - \sin(1.1875) - 0.5 = -0.01728$

$f(x_u) = f(1.25) = (1.25)^2 - \sin(1.25) - 0.5 = 0.11352$

$x_m = \frac{x_l + x_u}{2} \rightarrow \frac{1.1875 + 1.25}{2}$

$\boxed{x_m = 1.21875}$

$f(x_m) = f(1.21875) = (1.21875)^2 - \sin(1.21875) - 0.5$

$f(x_m) = 0.04668$

$|E_r| = \left| \frac{1.21875 - 1.1875}{1.21875} \right| \rightarrow 0.02$

$|E_r| < E_s$; $0.02 < 0.01$ False

Iteration 7:-

$f(x_l) f(x_m) < 0$; $x_l = x_l$; $x_u = x_m$

$f(x_l) = f(1.1875) = (1.1875)^2 - \sin(1.1875) - 0.5$

$f(x_l) = -0.01728$

$f(x_u) = f(1.21875) = (1.21875)^2 - \sin(1.21875) - 0.5$

$f(x_u) = 0.04668$

$x_m = \frac{x_l + x_u}{2} \rightarrow = \frac{1.1875 + 1.21873}{2} = 1.20313$

$f(x_m) = f(1.20313) = (1.20313)^2 - \sin(1.20313) - 0.5 = 0.01434$

$|E_r| = \left| \frac{1.20313 - 1.21875}{1.20313} \right| = 0.01$

$|E_r| < E_s$; $0.01 < 0.01$

False

Iteration 8:-

$f(x_l) \cdot f(x_m) < 0 \; ; \quad x_l = x_l \, , \quad x_u = x_m$

$f(x_l) = f(1.1875) = (1.1875)^2 - \sin(1.1875) - 0.5 = -0.01728$

$f(x_u) = f(1.20313) = (1.20313)^2 - \sin(1.20313) - 0.5 = 0.01434$

$x_m = \frac{x_l + x_u}{2} \rightarrow = \frac{1.1875 + 1.20313}{2} = 1.19513$

$f(x_m) = f(1.19513) = (1.19513)^2 - \sin(1.19513) - 0.5 = -0.00156$

$|E_r| = \left| \frac{1.19513 - 1.20313}{1.19513} \right| = 0.006$

$|E_r| < E_s \; . \; 0.006 < 0.01$

True

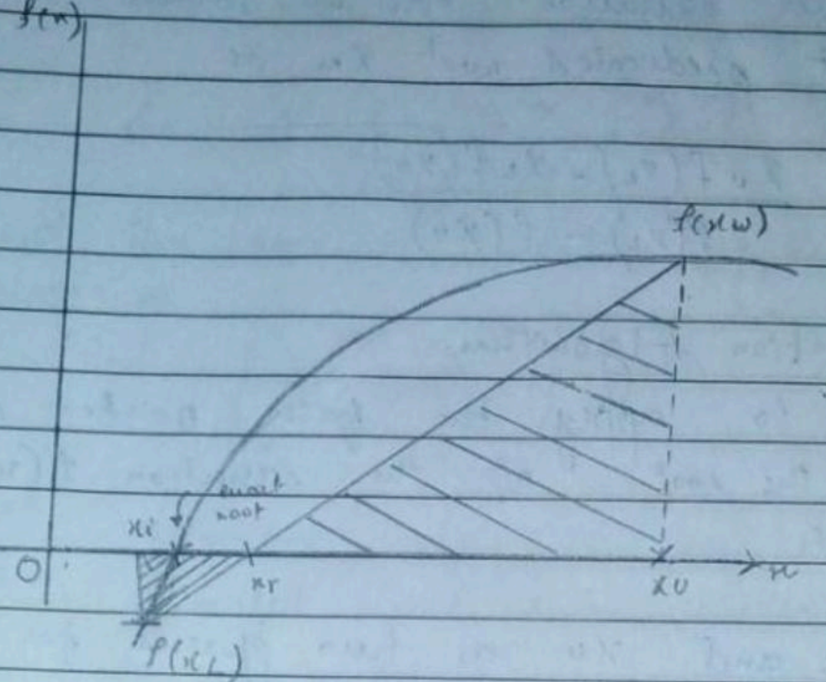
i	x_l	$f(x_l)$	x_u	$f(x_u)$	x_m	$f(x_m)$	$\|E_r\| < 0.01$
1	0	-0.5	2	2.59070	1	-0.34147	
2	1	-0.34147	2	2.59070	1.5	0.75251	False
3	1	-0.34147	1.5	0.75251	1.25	0.11352	False
4	1	-0.34147	1.25	0.11352	1.125	-0.13664	False
5	1.125	-0.13664	1.25	0.11352	1.1875	-0.01728	False
6	1.1875	-0.01728	1.25	0.11352	1.21875	0.04668	False
7	1.1875	-0.01728	1.21875	0.04668	1.20313	0.01434	False
8	1.1875	-0.01728	1.20313	0.01434	1.19513	-0.00156	True

Approximate root of the function $f(x) = x^2 - \sin(x) - 0.5$ using Bisection Method is 1.19531.

Answer.

Date:

False position Method:



Defination:

The false position method takes advantage over Bisection method mathematically by drawing a secant from the function value x_L to the function value x_U, and estimates the root as where it crosses the x-axis.

Derivation:

Consider the above figure.

Based on two similar triangles show in figure, ones get:

$$\frac{0 - f(x_L)}{x_r - x_L} = \frac{0 - f(x_U)}{x_r - x_U} \rightarrow (1)$$

From eqn (1), one obtains.

$$(x_r - x_L) f(x_U) = (x_r - x_U) f(x_L)$$

$$x_U f(x_L) - x_L f(x_U) = x_r \{ f(x_L) - f(x_U) \}$$

Date:

The above equation can be solved to obtain the next predicated root x_m as

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

False position Algorithm:
The steps to apply the false position method to find the root of the equation $f(x) = 0$ are as follows

(1) Choose x_L and x_U as two guesses for the root such that $f(x_L) f(x_U) < 0$, or in other words, $f(x)$ changes sign between x_L and x_U.

(2) Estimate the root, x_r of the equation $f(x) = 0$ a

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

(3) Now check the following.
If $f(x_L) f(x_r) < 0$, then the root lies between x_L and x_r; then $x_L = x_L$ and $x_U = x_r$.

If $f(x_L) f(x_r) > 0$, then the root lies between x_r and x_U; then $x_L = x_r$ and $x_U = x_U$

If $f(x_L) f(x_r) = 0$ then the root is x_r. Stop the algorithm

(4) Find the new estimate of the root

$$x_r = \frac{x_U f(x_L) - x_L f(x_U)}{f(x_L) - f(x_U)}$$

Find the absolute relative approximate error as

$$|\epsilon_a| = \left| \frac{x_r^{new} - x_r^{old}}{x_r^{new}} \right| \times 100$$

x_r^{new} : estimated root from present iteration
x_r^{old} : estimated root from previous iteration.

(5) Compare the absolute relative approximate. *Note One should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, ones needs to terminate the algorithm and notify the user about it.

$|\epsilon_a|$ with pre-specified relative error tolerance. ϵ_s. If $|\epsilon_a| > \epsilon_s$ then go to ? step 3, else stop. the alogrithm.

Q: Given the function $f(x) = x^2 - \sin(x) - 0.5$ for $[a,b] = [0,2]$, Find the approximated root using false position method. Find out the accuracy of the method using the condition $|E_r| < E_s$ where $E_s = 0.01$

Solution:

We have.

$f(x) = x^2 - \sin(x) - 0.5$

Initial bound $[a,b] = [0,2]$

Iteration 1:

$f(x_l) = f(0) = (0)^2 - \sin(0) - 0.5 = -0.5$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 = 2.59070$

$$x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)} = \frac{2(-0.5) - (0)(2.59070)}{-0.5 - 2.59070}$$

$x_r = 0.32355$

$f(x_r) = f(0.32355) = (0.32355)^2 - \sin(0.32355) - 0.5$

$f(x_r) = -0.71325$

Iteration 2:-

$f(x_l) \cdot f(x_u) > 0$, $x_l = x_r$, $x_u = x_u$

$f(x_l) = f(0.32355) = (0.32355)^2 - \sin(0.32355) - 0.5$

$= -0.71325$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 = 2.59070$

$$x_r = \frac{x_u f(x_l) - x_l f(x_u)}{f(x_l) - f(x_u)} = \frac{(2)(-0.71325) - (0.32355)(2.59070)}{-0.71325 - (2.59070)}$$

$x_r = 0.68546$

$f(x_r) = f(0.68546) = (0.68546)^2 - \sin(0.68546) - 0.5$

$f(x_r = -0.66317$

$$|E_r| = \left|\frac{x_r^n - x_r^o}{x_r^n}\right| = \left|\frac{0.68546 - 0.32355}{0.68546}\right|$$

0.52

$|E_r| < E_s$, $0.52 < 0.01$; False

Iteration 3:,

$f(x_l) \cdot f(x_u) > 0$ $x_l = x_r$, $x_u = x_u$

$f(x_l) = f(0.68546) = (0.68546)^2 - \sin(0.68546) - 0.5$

$f(x_l) = -0.66317$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 \rightarrow = 2.59070$

$$x_r = \frac{(2)(-0.66317) - (0.68546)(2.59070)}{(-0.66317) - (2.59070)}$$

$\boxed{x_r = 0.95338}$

$f(x_r) = f(0.95338) = (0.95338)^2 - \sin(0.95338) - 0.5$

$f(x_r) = -0.40645$

$$|E_r| = \left|\frac{0.95338 - 0.68546}{0.95338}\right| = 0.28$$

$|E_r| < E_s$, $0.28 < 0.01$.

False

Iteration 4:,

$f(x_l) \cdot f(x_u) > 0$; $x_l = x_r$, $x_u = x_u$

$f(x_l) = f(0.95338) = (0.95338)^2 - \sin(0.95338) - 0.5$

$f(x_l) = -0.40645$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 \longrightarrow = 2.59070$

$$x_r = \frac{(2)(-0.40645) - (0.95338)(2.59070)}{-0.40645 - 2.59070}$$

$\boxed{x_r = 1.09531}$

$f(x_r) = f(1.09531) = (1.09531)^2 - \sin(1.09531) - 0.5$

$f(x_r) = -0.18937$

$$|E_r| = \left|\frac{1.09531 - 0.9538}{1.09531}\right| = 0.12$$

$|E_r| < E_s$, $0.12 < 0.01$ False

Iteration 5:-

$f(x_l) \cdot f(x_u) > 0; \quad x_l = x_r, \quad x_u = x_u$

$f(x_l) = f(1.09531) = (1.09531)^2 - \sin(1.09531) - 0.5$

$f(x_l) = -0.18937$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 \rightarrow = 2.59070$

$$x_r = \frac{(2)(-0.18937) - (1.09537)(2.59070)}{(-0.18937) - (2.59070)}$$

$\boxed{x_r = 1.15693}$

$f(x_r) = f(1.15693) = (1.15693)^2 - \sin(1.15693) - 0.5$

$f(x_r) = -0.07708$

$$|E_r| = \left| \frac{1.15693 - 1.09531}{1.15693} \right| = 0.05$$

$|E_r| < E_s ; \quad 0.05 < 0.01$

False.

Iteration 6:,

$f(x_l) \cdot f(x_u) > 0; \quad x_l = x_r, \quad x_u = x_u$

$f(x_l) = f(1.15693) = (1.15693)^2 - \sin(1.15693) - 0.5$

$\boxed{f(x_l) = -0.07708}$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 = 2.59070$

$$x_r = \frac{(2)(-0.07708) - (1.15693)(2.59070)}{-0.07708 - 2.59070}$$

$\boxed{x_r = 1.18129}$

$$|E_r| = \left| \frac{1.18129 - 1.15693}{1.18129} \right| = 0.02$$

$|E_r| < E_s \cdot \quad 0.02 < 0.01$

False

Iteration 7:.

$f(x_l) \cdot f(x_u) > 0 : \quad x_l = x_r, \quad x_u = x_u$

$f(x_l) = f(1.18129) = (1.18129)^2 - \sin(1.18129) - 0.5$

$f(x_l) = -0.02965$

$f(x_u) = f(2) = (2)^2 - \sin(2) - 0.5 = 2.59070$

$$x_r = \frac{(2)(-0.02965) - (1.18129)(2.59070)}{(-0.02965) - (2.59070)}$$

$\boxed{x_r = 1.19055}$

$f(x_r) = f(1.19055) = (1.19055)^2 - \sin(1.19055) - 0.5$

$f(x_r) = -0.01115$

$$|E_r| = \left| \frac{1.19055 - 1.18129}{1.19055} \right|$$

$|E_r| = 0.007$

$|E_r| < E_s = 0.007 < 0.01$ True

i	x_l	$f(x_l)$	x_u	$f(x_u)$	x_r	$f(x_r)$	$\lvert E_r\rvert < 0.01$
1	0	-0.5	2	2.59070	0.32355	-0.71325	
2	0.32355	-0.71305	2	2.59070	0.68546	-0.66317	False
3	0.68546	-0.66317	2	2.59070	0.95338	-0.40645	False
4	0.95338	-0.40645	2	2.59070	1.09531	-0.18937	False
5	1.09531	-0.18937	2	2.5970	1.15693	-0.07708	False
6	1.15693	-0.07708	2	2.59070	1.18129	-0.02965	False
7	1.18129	-0.02965	2	2.59070	1.19055	-0.01115	True

Approximate root of the function ~~fx~~ $= x^2 - \sin(x) - 0.5$
$f(x) = x^2 - \sin(x) - 0.5$ using false position method is 1.19055

-Answer

) Findout the efficiency of the two method used in Q_1 & Q_2 and comment on the performance of the two numerical root finding method.

Answer) The root (approximated of the given function $f(x) = x^2 - \sin x - 0.5$ is calculated using bisection method in eight iterations, while using false position method we get approximated root in seven iterations. So, we can say that false position method is much efficient than bisection method.

∴ Assignment # 3.

23 Date:

Q The upward velocity of rocket is given in the table a Determine the value of the velocity at $t = 16$ second using direct method for linear interpolation.

t(sec)	0	10	15	20	22.5	30
V(t) m/s	0	227.04	362.78	517.35	602.97	901.67

when $t = 15$, $V_{(15)} = 362.78$

$t = 20$, $V(20) = 517.35$

we have

$$V(t) = a_0 + a_1 t$$

$$= a_0 + a_1(15) = 362.78 \longrightarrow ①$$

$$= a_0 + a_1(20) = 517.35 \longrightarrow ②$$

$① \Rightarrow 362.78 - a_1 15 = a_0$

OR $a_0 = 362.78 - 15a_1 \longrightarrow ③$

$② \Rightarrow 362.78 - 15a_1 + 20a_1 = 517.35$

$\Rightarrow 362.78 + 5a_1 - 517.35 = 0$

$-154.57 + 5a_1 = 0$

$5a_1 = 154.57$

$a_1 = 154.57/5$

$\boxed{a_1 = 30.914}$

Now for a_0

put the value of a_1 in eqn ③

$\boxed{a_0 = -100.93}$

Now;

$t = 16$

$V_{(16)} = -100.93 + 30.914(16)$

$\boxed{V_{(16)} = 393.694}$ Answer.

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1 (b) Now for Quadratic Interpolation.

Solution.

when

$t = 15$, $V(15) = 362.78$

$t = 20$, $V(20) = 517.35$

$t = 10$, $V(10) = 227.04$

we have $V(t) = a_0 + a_1(t) + a_2(t)^2$

$V(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04 \rightarrow ①$

$V(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78 \rightarrow ②$

$V(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35 \rightarrow ③$

$① \Rightarrow \quad a_0 = 227.04 - a_1(10) - a_2(10)^2 \rightarrow ④$

$② \Rightarrow : \quad a_1 = \dfrac{362.78 - a_2(15)^2 - [227.04 - a_2(10)^2]}{5}$

$a_1 = \dfrac{135.74 - 125a_2}{5} \rightarrow ⑤$

$③ \Rightarrow 227.04 - a_1(10) - a_2(10)^2 + a_1(20) + a_2(20)^2 = 517.35$

$10a_1 + 300a_2 = 290.31$

$10\left[\dfrac{135.74 - 125a_2}{5}\right] + 300a_2 = 290.31$

$1357.4 - 1250a_2 + 1500a_2 = 1451.55$

$250a_2 = 94.15$

$a_2 = \dfrac{94.15}{250}$

$\boxed{a_2 = 0.3766}$

for a_1:

put the value of a_2 in eqn ⑤

$a_1 = \dfrac{135.74 - 125(0.3766)}{5}$

$\boxed{a_1 = 17.733}$

for a_0 put the values of a_1 and a_2 in eqn (4)

$$a_0 = 227.04 - 17.733(10) - 0.3766(10)^2$$

$$\boxed{a_0 = 12.05}$$

At $t = 16$

$$V(16) = 12.05 + 17.733(16) + 0.3766(16)^2$$

$$\boxed{V(16) = 392.1876 \text{ m/s}}$$ Answer.

(3) For cubic interpolation.

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

Solution,

$t = 10$, $V(10) = 227.04$

$t = 15$, $V(15) = 362.78$

$t = 20$, $V(20) = 517.35$

$t = 22.5$, $V(22.5) = 602.97$

we have

$$V(10) = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3 = 227.04 \rightarrow (1)$$

$$V(15) = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3 = 362.78 \rightarrow (2)$$

$$V(20) = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3 = 517.35 \rightarrow (3)$$

$$V(22.5) = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3 = 602.97 \rightarrow (4)$$

Subtracting eqn (1) from (2)

$$\cancel{a_0} + a_1(15) + a_2(15)^2 + a_3(15)^3 = 362.78$$

$$-\cancel{a_0} + a_1(10) + a_2(10)^2 + a_3(10)^3 = 227.04$$

$$5a_1 + 125a_2 + 2375a_3 = 135.74$$

$$a_1 = \frac{135.74 - 125a_2 - 2375a_3}{5} \rightarrow (5)$$

From eqn ①

$$① \Rightarrow a_0 = -10\left(\frac{135.74 - 125a_2 - 2375a_3}{5}\right) - a_2(10)^2 - a_3(10)^3 + 227.04$$

$$a_0 = -271.48 + 250a_2 + 4750a_3 - 100a_2 - 1000a_3 + 227.04$$

$$a_0 = -44.44 + 150a_2 + 3750a_3 \rightarrow ⑥$$

From eqn ③

$$③ \Rightarrow -44.44 + 150a_2 + 3750a_3 + (542.96 - 500a_2 - 9500a_3) + (20)^2a_2 + (20)^3a_3 = 517.35$$

$$50a_2 = 18.83 - 2250a_3$$

$$a_2 = \frac{18.83 - 2250a_3}{50} \rightarrow ⑦$$

From eqn ④

$$④ \Rightarrow -44.44 + 150a_2 + 3750a_3 + 22.5\left(\frac{135.74 - 125a_2 - 2375a_3}{5}\right)$$

$$+ (22.5)^2a_2 + a_3(22.5)^3 = 602.97$$

$$= -44.44 + 150a_2 + 3570a_3 + 610.83 - 562.5a_2 - 10687.5a_3$$

$$501.76a_2 + 11390.625a_3$$

$$4273.125a_3 = 35.87 - 33.6153 + 4016.7a_3$$

$$4273.125a_3 - 4016.7a_3 = 2.2547$$

$$256.425a_3 = 2.2547$$

$$\boxed{a_3 = 8.792 \times 10^{-3}}$$

put the value of a_3 in eqn ⑦

$$\boxed{a_2 = -0.01904}$$

Date:

for a_0 put the values of a_2 and a_3 in equ (6)

$$\boxed{a_0 = -14.326}$$

for a_1 put the values of a_2 and a_3 in equ (5)

$$\boxed{a_1 = 23.4478}$$

Now;

$t = 16$

$$V_{(16)} = -14.326 + 23.4478(16) + (-0.01904)(16)^2 + 8.792 \times 10^{-2}(16)^3$$

$$\boxed{V_{(16)} = 391.976 \, m/s}$$ Answer

Q4 Using the same question, but solving it through Newton Divided difference for linear interpolation.

$$V(t) = b_0 + b_1(t - t_0) \longrightarrow (1)$$

for b_0; putting $t = t_0$ → in equ (1)

$$V(t_0) = b_0 + b_1(\cancel{t_0} - \cancel{t_0})$$

$$V(t_0) = b_0 + b_1(0)$$

$$V_{(t_0)} = b_0$$

Now for b_1. putting $t = t_1$ in equ (1)

$$V(t_1) = b_0 + b_1(t_1 - t_0)$$

$$\therefore b_0 = V_{(t_0)}$$

$$V_{(t_1)} = V_{(t_0)} + b_1(t_1 - t_0)$$

Date:

$$b_1 = \frac{V(t_1) - V(t_0)}{t_1 - t_0}$$

Now,

$t_0 = 15, \quad V(t_0) = 362.78 \text{ m/s}$

$t_1 = 20, \quad V(t_1) = 517.35 \text{ m/s}$

$b_0 = V(t_0) = 362.78$

$$b_1 = \frac{V(t_1) - V(t_0)}{t_1 - t_0} \rightarrow \frac{517.35 - 362.78}{20 - 15}$$

$b_1 = 30.914$

put the values of b_0, b_1 & t_0 in eqn ①

At $t = 16$

$V(16) = 362.78 + 30.914(16 - 15)$

$\boxed{V(16) = 393.694 \text{ m/s}}$ Answer

Q Using the same question but solving it from Newton divided difference for quadratic Interpolation.

Solution,

$t_0 = 10, \quad V(t_0) = 227.04$

$t_1 = 15, \quad V(t_1) = 362.78$

$t_2 = 20, \quad V(t_1) = 517.35$

where

$V(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$

Now,

$b_0 = V(t_0) = V(10) = 227.04$

$$b_1 = V[t_1, t_0] = \frac{V(t_1) - V(t_0)}{t_1 - t_0}$$

$$b_1 = \frac{V(15) - V(10)}{15 - 10}$$

Date:

$$b_1 = \frac{362.78 - 227.04}{15 - 10}$$

$$\boxed{b_1 = 27.148}$$

$$b_2 = v[t_2, t_1, t_0]$$

$$= \frac{v[t_2, t_1] - v[t_1, t_0]}{t_2 - t_0}$$

Now for

$$v(t_2, t_1) = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

$$= \frac{v(20) - v(15)}{20 - 15}$$

$$= \frac{517.35 - 362.78}{20 - 15}$$

$$v(t_2, t_1) = 30.914$$

Now, $b_2 = \frac{30.914 - 27.148}{20 - 10}$

$$\boxed{b_2 = 0.3766}$$

$$V_2(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1)$$

$$= 227.04 + 27.148(t-10) + 0.3766(t-10)(t-15);$$

At $t = 16$

$$V_2(16) = 227.04 + 27.148(16-10) + 0.3766(16-10)(16-15)$$

$$\boxed{V_2(16) = 392.1876}$$ Answer

Q using the same Question but solving it from Newton divided difference for cubic interpolation

Solution

$t_0 = 10$, $V(t_0) = 227.04$

$t_1 = 15$, $V(t_1) = 362.78$

$t_2 = 20$, $V(t_2) = 517.35$

$t_3 = 22.5$, $V(t_2) = 602.97$

we have.

$$V_3(t) = b_0 + b_1(t-t_0) + b_2(t-t_0)(t-t_1) + b_3(t-t_0)(t-t_1)(t-2) \rightarrow (1)$$

we know that the values of b_0, b_1 and b_2

$b_0 = 227.04$, $b_1 = 27.148$, $b_2 = 0.3766$

Now for b_3:

$$b_3 = v[t_3, t_2, t_1, t_0]$$

$$b_3 = \frac{v[t_3, t_2, t_1] - v[t_2, t_1, t_0]}{t_3 - t_0} \rightarrow (2)$$

$$v[t_3, t_2, t_1] = \frac{v[t_3, t_2] - v[t_2, t_1]}{t_3 - t_1} \rightarrow (3)$$

$$v[t_3, t_2] = \frac{v[t_3] - v[t_2]}{t_3 - t_2} \rightarrow \frac{602.97 - 517.35}{22.50 - 20}$$

$$\boxed{v[t_3, t_2] = 34.248}$$

put the values $v[t_3, t_2]$ & $v[t_2 - t_1]$ in eqn (3)

$$v(t_3, t_2, t_1) \frac{34.248 - 30.914}{22.5 - 15} \rightarrow 0.44453$$

$$v[t_2, t_1, t_0] = 0.3766$$

put the value of $V(t_2, t_1, t_1)$ and (t_2, t_1, t_0) in (1)

$$\boxed{b_3 = 5.4347 \times 10^{-3}}$$

Hence, At $t = 16$ put b_0, b_1, b_2 and b_3 in eqn (1)

we get

$$\boxed{V_{(16)} = 392.06 m/s}$$ Answer

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.: Interpolation:.

Newton divided difference (polynomial Method).

Derivation:

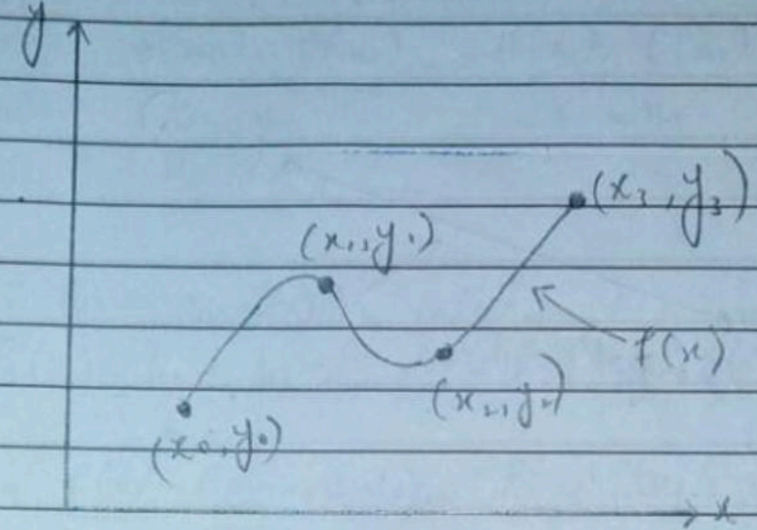


Figure ① Interpolation of discrete data

Linear interpolation:.

Given (x_0, y_0) and (x_1, y_1), fit a linear interpolation interpolant through the data. Noting $y = f(x)$ and

$y_1 = f(x_1) = b_0 + b_1(x - x_0)$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

Since at $x = x_1$,

$$f_1(x x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0$$

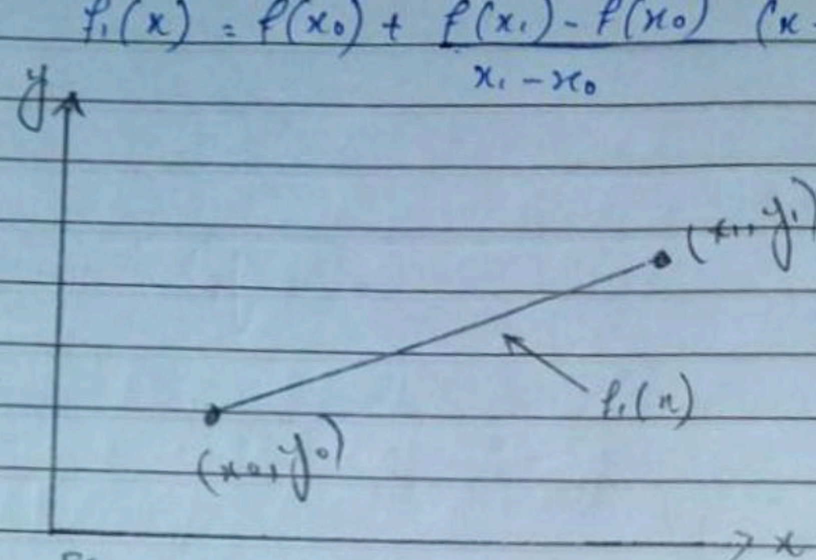


Figure 2 linear interpolation

Quadratic interpolation:-

Given (x_0, y_0), (x_1, y_1) and (x_2, y_2), fit a quadratic interpolant through the data. Noting $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, assume the quadratic interpolant $f_2(x)$ is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$= b_0$$

$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f_2(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0} \times (x - x_0)(x - x_1)$$

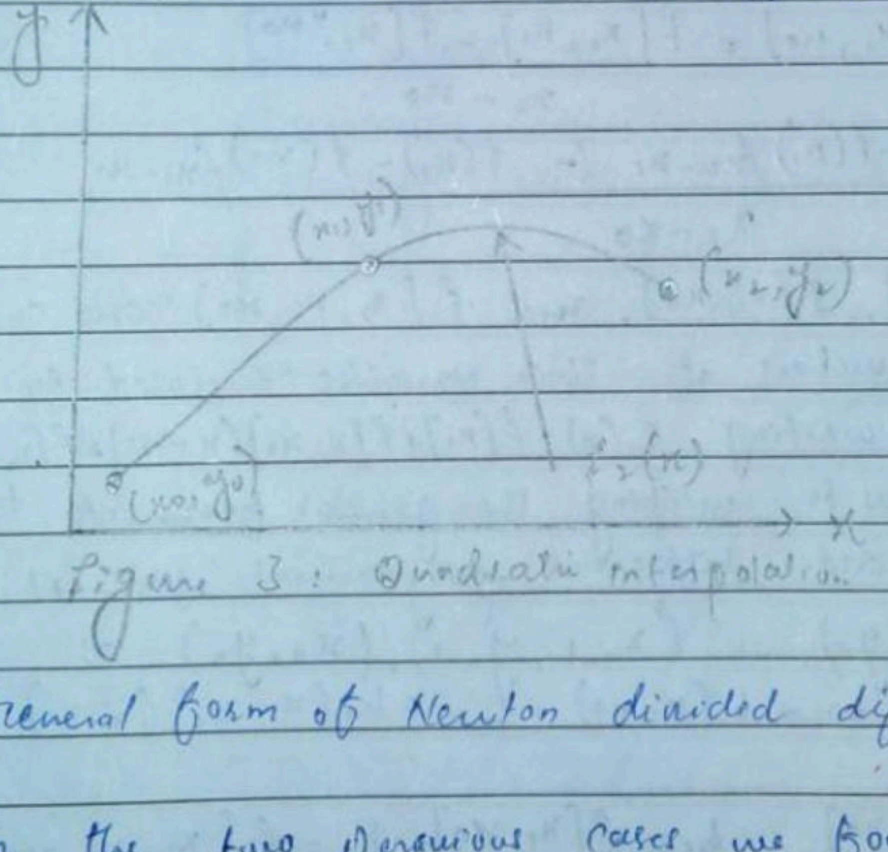


Figure 3: Quadratic interpolation.

General form of Newton divided difference polynomial

In the two previous cases, we found linear and Quadratic interpolants for Newton's divided difference method. Let us revisit the Quadratic polynomial interpolant formula.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Date:

when

$b_0 = f(x_0)$

$b_1 = f(x_1) - f(x_0) / x_1 - x_0$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that b_0, b_1 and b_2 are finite divided differences b_0, b_2 are the second and third divided difference, respectively. we denote the first divided difference by

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

And the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{f(x_2) - f(x_1) / x_2 - x_1 - f(x_1) - f(x_0) / x_1 - x_0}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, and $f[x_2, x_1, x_0]$ are called bracketed functions of their variables enclosed in square brackets., Rewriting $f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$ This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points. $(x_0, y_0), (x_1, y_1), \ldots (x_{n-1}, y_{n-1}), (x_n, y_n)$, as

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})$$

where

$$b_0 = f[x_0], \quad b_1 = f[x_1, x_0], \quad b_2 = f[x_2, x_1, x_0]$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots x_0] \quad b_n = f[x_n, x_{n-1}, \ldots, x_0]$$

where the defination of the m^{th} divided difference is

$$b_m = f[x_m, \ldots, x_0] \rightarrow = \frac{f[x_m, \ldots, x_1] - f[x_{m-1}, \ldots, x_0]}{x_m - x_0}$$

Newton Raphson Method:

Derivation:

The Newton-Raphson method is based on the principle that if the initial guess of the root of $f(x)=0$ is at x_i, then if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x axis is an improved estimate of the root (figure 1).

Using the defination of the slope of a function at $x = x_i$

$$f'(x_i) = \tan\theta \rightarrow \frac{f(x_i) - 0}{x_i - x_{i+1}},$$

Which gives,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow (1)$$

Equation (1) is called the Newton-Raphson formula for solving nonlinear equation of the form $f(x)=0$. So starting with an initial guess, x_i, one can find the next guess, x_{i+1}, by using equation (1). One can repeat this process until one finds the root within a desirable tolerance.

Newton-Raphson Method

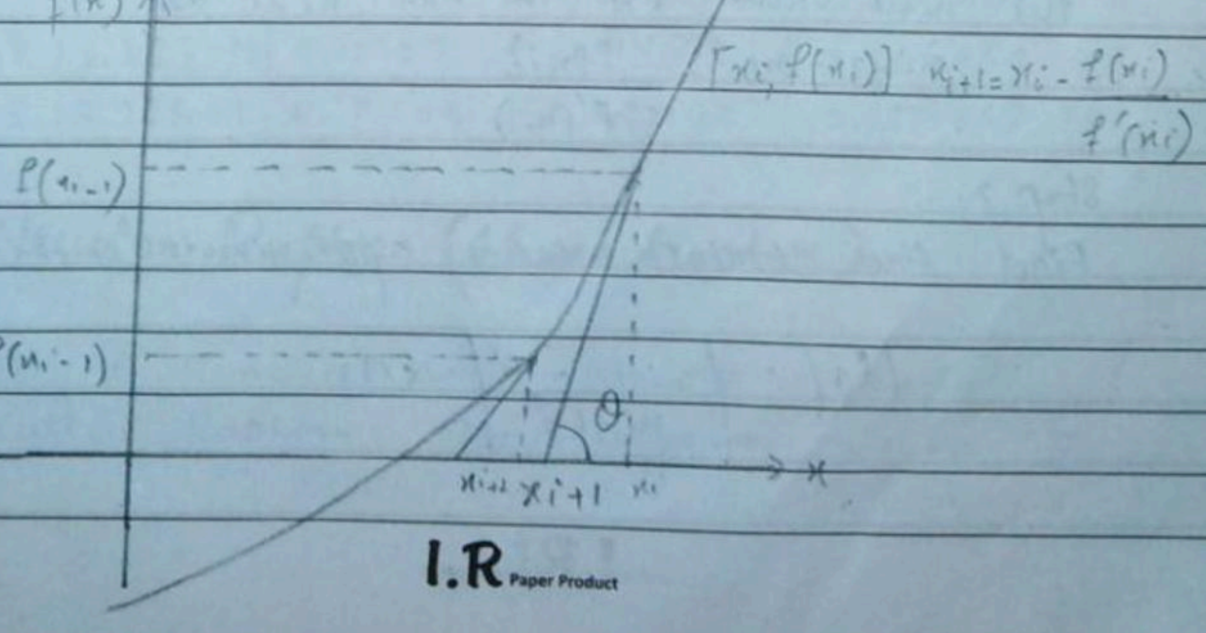
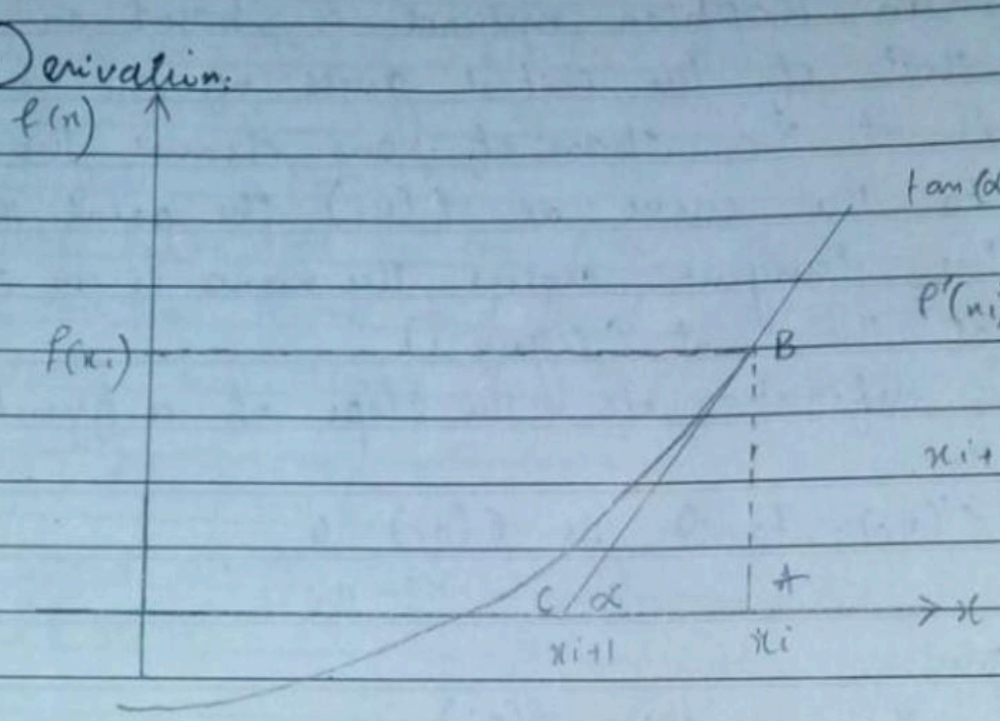


Figure ① Geometrical illustration of the Newton-Raphson method.

Derivation:



$$\tan(\alpha) = \frac{AB}{AC}$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Figure ② derivation of Newton-Raphson method

Algorithm for Newton Raphson Method:

Step 1:

Evaluate $f'(x)$ symbolically.

Step 2:

Use initial guess of the root, x_i, to estimate the new value of the root, x_{i+1} as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Step 3:

Find the absolute relative approximate error $|\epsilon_a|$ as

$$|\epsilon_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

Step 4:-

Compare the absolute relative approximate error with the pre-specified relative error tolerance.

is $|E_a| > E_s$? — Yes → Go to step 2 using new estimate of the root
— No → Stop the alogrithm

Also check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the alogrithm and notify the user.

Q Find out the approximation of the root to the root of the function.

- $f(x) = x^2 - 5$ using Newton Raphson method with $x_i = 1$
- Your approximation must be significant to five decimal places.

S.NO	$f(x) =$	x^{2-5}	Newton-Raphson	
S.NO	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	1	-4	2	3
2	3	4	6	2.333333
3	2.333333	0.444444	4.666667	2.238095
4	2.238095	0.00907	4.47619	2.236069
5	2.236069	411E-06	4.472138	2.236068

Newton Raphson (Characteristics):.

- It's a numerical root finding method.
- Newton Raphson also belongs to category of

Open method.

- This method required only 1 initial bound to start with
- Root will also be located beyond the initial bound.
- It is the fastest numerical root finding method.

Open Method:

- Required at least 1 initial bound.
- Root will be located beyond the initial bound(s)

Assignment # 4:

Date:

Q Use Newton Raphson method to find solution to accurate to within 10^{-4} for the following problems.

Qi $x^3 - 2x^2 - 5 = 0$

Solution. we have

$$f(x) = x^3 - 2x^2 - 5 \rightarrow ①$$

$$f'(x) = 3x^2 - 4x \rightarrow ②$$

$x_0 = 1$

① for $i = 0$

we have general formula of Newton Raphson method.

for $i = 0$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$\boxed{x_1 = -5}$ $\boxed{Ea = 120\%}$

② $i = 1$

$\boxed{x_2 = -3.10526}$. $\boxed{Ea = 61\%}$

③ $i = 2 \rightarrow \boxed{x_3 = -1.79378}$ $\boxed{Ea = 73\%}$

④ $i = 3 \rightarrow \boxed{x_4 = -0.77126}$ $\boxed{Ea = 132\%}$

⑤ $i = 4 \rightarrow \boxed{x_5 = 0.59404}$

⑥ $i = 5 \rightarrow \boxed{x_6 = -3.57758}$

⑦ $i = 6 \rightarrow \boxed{x_7 = -2.12830}$

⑧ $i = 7 \rightarrow \boxed{x_8 = -1.05602}$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	1	-6	-1	-5
2	-5	-180	95	-3.10526
3	-3.10526	-54.22831	41.34903	-1.79379
4	-1.79379	-17.20714	16.82815	-0.77126
5	-0.77126	-6.64848	4.86960	0.59404
6	0.59404	-5.49614	-1.31751	-3.57758
7	-3.57758	-76.88766	52.70745	-2.12830
8	-2.12830	-23.69978	22.10217	-1.05602
9	-1.05602	-8.40798	7.56958	0.05474

(b) $x_0 = 4$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	4	27	32	3.15625
2	3.15625	6.51846	17.26074	2.77860
3	2.77860	1.01131	12.04750	2.69466
4	2.69466	0.04405	11.00493	2.69066
5	2.69066	0.00010	10.95627	2.69065

Approximate root of the function
$f(x) = x^3 - 2x^2 - 5$ with $x_0 = 4$ using Newton Raphson method is 2.69065.
4 is the highly efficient initial bound to approximate the root of
$f(x) = x^3 - 2x^2 - 5$ Answer

Date:

(2) $x - \cos x = 0$

(a) $[0]$ (b) $[\pi/2]$

Solution:

we have $f(x) = x - \cos x$

$f'(x) = 1 + \sin x$

(a)

$x_0 = 0$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	0	-1	1	1
2	1	0.45970	1.84147	0.75036
3	0.75036	0.01892	1.68196	0.73911
4	0.73911	0.00005	1.67363	0.73909
5	0.73909	0.00000	1.67361	0.73909

(b) $x_0 = \pi/2$

i	x_i	$f(x_i)$	$f'(x_i)$	x_{i+1}
1	1.5708	1.5708	2	0.78540
2	0.78540	0.07829	1.70711	0.73954
3	0.73954	0.00075	1.67395	0.73909
4	0.73909	0.0000	1.67361	0.73909

Approximate root of the function $f(x) = x - \cos x$ with $x_0 = \pi/2$ using Newton Raphson method is 0.73909

$\pi/2$ is the Highly efficient bound to approxima[te] the root of $f(x) = x - \cos x$

Ans

∴ Newton Forward difference:

The difference $y_1 - y_0, y_2 - y_1, \ldots\ldots y_n - y_{n-1}$ when denoted by $\Delta y_0, \Delta y_1, \Delta y_2, \ldots\ldots \Delta y_{n-1}$ are $\Delta y_n = y_{n+1} - y_n$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
x_0	y_0			
$x_1 = x_0 + h$	y_1	$\Delta y_0 = y_1 - y_0$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
$x_2 = x_0 + 2h$	y_2	$\Delta y_1 = y_2 - y_1$		
$x_3 = x_0 + 3h$	y_3	[illegible]		
		$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	

∴ Newton Backword difference:

We consider values y_1, y_2, y_3 & y_4 of the function $y = f(x)$ tabulated at equally spaced points x_1, x_2, x_3 & x_4. The backword difference table along with tabulated points will look like

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0		$\nabla^2 y_1 =$	
x_1	y_1	$\nabla y_0 = y_1 - y_0$	$\nabla y_1 - \nabla y_0$	
x_2	y_2	$\nabla y_1 = y_2 - y_1$	$\nabla^2 y_2 =$	$\nabla^3 y_2 = \nabla^2 y_1 - \nabla^2 y_0$
x_3	y_3	$\nabla y_2 = y_3 - y_2$	$\nabla y_2 - \nabla y_1$	

:Assignment NO # 5:, (43)

Date:

Newton Backword difference formula:

For interpolating the value of the function $y = f(x)$ near the end of the table of values and to the extrapolate value of the function a short distance forward from y_n, Newton's Backword interpolation formula is used.

Precaution:,

Let $y = f(x)$ be a function which takes on values $f(x_n)$, $f(x_n - h)$, $f(x_n - 2h)$, ... $f(x_0)$ corresponding to equispaced values x_n, $x_n - h$, $x_n - 2h$, ..., x_0.

Suppose we wish to evaluate the function $f(x)$ at $(x_n + ph)$ where p is any real number Then we have the shift operator E such that

$$f(x_n + ph) = E^p(f(x_n)) = (E^{-1})^{-p} f(x_n) = (1 - \nabla)^{-p} f(x_n)$$

Example

Find $f(15)$ Given the following table values?

x	10	20	30	40	50
$y = f(x)$	46	66	81	93	101

10	46				
		$\Delta y_1 = 20$			
20	66		$\Delta^2 y = -5$		

		$\Delta y_2 = 16$		$\Delta^3 y_1 = 2$	
30	81		$\Delta^2 y_1 = -3$		$\Delta^4 y_1 = -3$
		$\Delta y_2 = 12$		$\Delta^3 y_2 = -1$	
40	93		$\Delta^2 y_2 = -3$		
		$\Delta y_1 = 8$			
50	101		,		

for $P = (x - x_0)h = (15-10)/10$

$\boxed{0.5}$

$$f(15) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

$$f(15) = 46 + (0.5)(20) + \frac{(0.5)(0.5-1)(-5)}{2} + \frac{(0.5)(0.5-1)(0.5-2)(2)}{6} + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)(-3)}{24}$$

$f(15) = 56.86719$ approximate

Answer.

Question for Backword difference:

Example: for the following table of the values estimate $f(7.5)$.

x	$y = f(x)$	Δy_n	$\Delta^2 y_n$	$\Delta^3 y_n$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y_n$	$\Delta^7 y_n$
1	1							
		7						
2	8		12					
		19		-4				
3	27		8		30			
		27		26		-60		

4	64	61	34	-4		-60		
5	125	91	30	6	-30	40	100	-150
6	216	127	36	6	10	-10	-50	
7	343	169	42		0			
8	512							

$x = 7.5, \; x_n = 8, \; h = x_1 - x_0 = 2-1, \; y_n = 512$

$\Delta y_n = 169, \; \Delta^2 y_n = 42, \; \Delta^3 y_n = 6, \; \Delta^4 y = 0, \; \Delta^5 y_n = -10$

$\Delta^6 y_n = -50, \; \Delta y^7 = -150$

$$P = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5$$

By Newton Backword formula we get

$$y(7.5) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \nabla^3 y_n \frac{P(P+1)(P+2)}{3!} + \nabla^4 y \frac{P(P+1)(P+2)(P+3)}{4!} + \ldots \frac{P(P+1)\ldots(P+n-1)}{n!}\nabla^n y_n$$

$$y(7.5) = (512) + (-0.5)(169) + \frac{(-0.5)(-0.5-1)(42)}{2!} + \frac{(-0.5)(-0.5+1)(-0.5+2)(6)}{3!} + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(0)}{4!}$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)(-10)}{5!} +$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)(-0.5+5)(-50)}{6!} + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)(-0.5+4)(-0.5+5)(-0.5+6)(-150)}{7!}$$

$(7.5) = 512 - 84.5 - 5.25 - 0.375 - 0 + 0.2734375 + 1.025391 + 2.416922$

$\boxed{f(7.5) = 425.59082 \text{ approximate}}$

Answer

-: Assignment NO #6 :.

(47) Date:

Q Using difference formula, consider the table below and find out f(3).

x	2	4	9	10
fx	4	56	98	120

Solution.

Numerical divided difference method to find solution.

x	y	1'st Order	2nd Order	3rd Order
2	4			
4	56	26		
9	98	84	-2.5143	
10	120	22	2.2667	0.5976

1'st Order:

① $= \frac{(56-4)}{4-2} \rightarrow 26$

② $= \frac{98-56}{9-4} = 42/5 \rightarrow 8.4$

③ $\frac{120-98}{10-9} = 22$

2nd Order.

① $\frac{8.4-2,6}{9-2} = \frac{-17.6}{7}$

-2.5143

(2) $\dfrac{22-8.4}{10-4} = \dfrac{13.6}{6}$

2.2667 Ans

3^{rd} Order:

$$\frac{2.2667-(-2.5143)}{10-2}$$

$$= \frac{4.781}{8} = 0.5976$$

Newton divided difference interpolation formula is

$$f(x) = y_0 + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f[x_0, x_1, x_2, x_3]$$

$$f(5) = 4 + (5-2)\times 26 + (5-2)(5-4)(-2.5143) + (5-2)(5-4)(5-9)(0.5976)$$

$$= 4 + (3)(26) + 3(1)(-2.5143) + (3)(1)(-4)(0.5976)$$

$$f(5) = 4 + 78 + (-7.5429) + (-7.1712)$$

$$= 4 + 78 - 7.5429 - 7.1712$$

$$\boxed{f(5) = 67.2859}$$ Answer.

Date:

Secant Method (derivation):

The Secant method can also be derived from geometry as shown in figure 1. Taking two initial guesses x_{i-1} and x_i. One draws a straight line between $f(x_i)$ and $f(x_{i-1})$ passing through the x-axis at x_{i+1}. ABE and DCF are similar triangles.

$$\frac{AB}{AE} = \frac{DC}{DE}$$

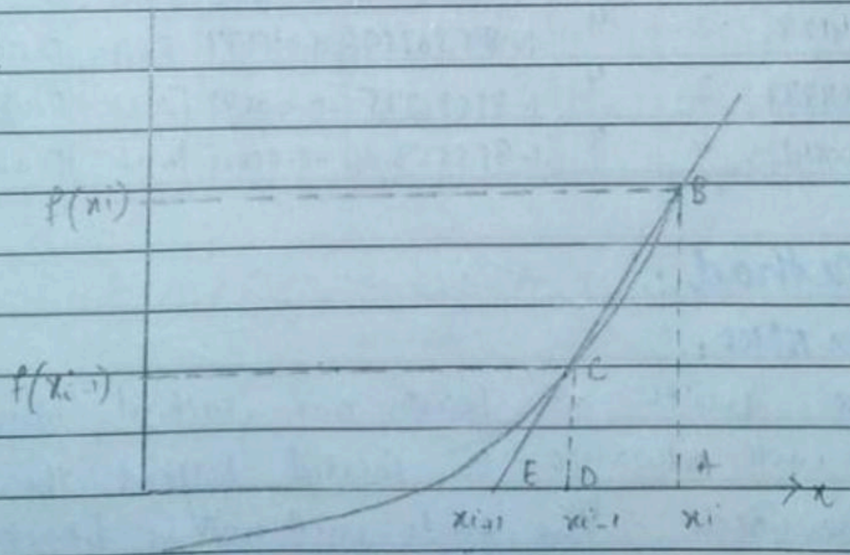
$$\frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

On rearranging the secant method is given as

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i+1})}{f(x_i) - f(x_{i-1})}$$

$$x_r = x_u - f(x_u)(x_u - x_l) / f(x_u) - f(x_l)$$

Q Find



Q Find out the root of the function $f(x) = x^4 - x - 10$ using secant method with [1, 2] initial bounds.

(b) Stop the iterative procedure when the following conditions get satisfied.

(i) $|f(x_r)| < E_s$

(ii) $E_{abs} < E_s$

(iii) $E_r < E_s$ where $E_s = 0.001$

(c) Comment on the efficiency of the following conditions to approximate the root of the given function. $x_l = 1$, $x_u = 2$

A C B D E X

			x^4-x-10		Secant Method				
S.No	x_0	$f(x_0)$	x_1	$f(x_1)$	x_2	$f(x_2)$	error<0.001	E_{abs}<0.001	E_r<0.001
1	1	-10	2	4	1.71428571	-3.07788	False		
2	1.714286	-3.07788	2	4	1.83853125	-0.4128	False	False	False
3	1.838531	-0.4128	2	4	1.85363596	-0.04777	False	False	False
4	1.853636	-0.04777	2	4	1.85536335	-0.00543	False	False	True
5	1.855363	-0.00543	2	4	1.85555944	-0.00062	True	True	True

Open method :.

. Characteristics :.

1. It works with at least one initial bound.
2. The root will always be located beyond the bound(s) or in other words the roots will not be bracketed b/w the bounds.

Secant Method :-

1. Belongs to the open method category.
2. It takes two initial bound with

Date:

: Derivation Of Lagrange Interpolation Polynomial :

we have

$$f(x) = a_0 + a_1 x + a_2 x^2 \cdots a_n x^n \rightarrow ①$$

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Can be arranged as

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \rightarrow ②$$

The lagranges interpolating polynomial can be derived directly from Newton's formulation. Here we are doing only for first order case. For example the first divided difference

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \rightarrow (A)$$

Can be reformulated as

$$f[x_1, x_0] = \frac{f(x_1)}{x_1 - x_0} + \frac{f(x_0)}{x_0 - x_1} \rightarrow (B)$$

which is reference to as the symmetric form. Substituting eqn (B) into eqn ②

we get

$$f_1(x) = f(x_0) + \frac{(x - x_0)}{x_1 - x_0} f(x_1) + \frac{x - x_0}{x_0 - x_1} f(x_0)$$

Finally grouping similar terms and simplifying the Lagrange form.

$$\boxed{f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)}$$

The Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x)\, f(x)$$

where n in $f_n(x)$ stands for the n^{th} order polynomial that approximates the function $y = f(x)$ is given at $n+1$ data points as $(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

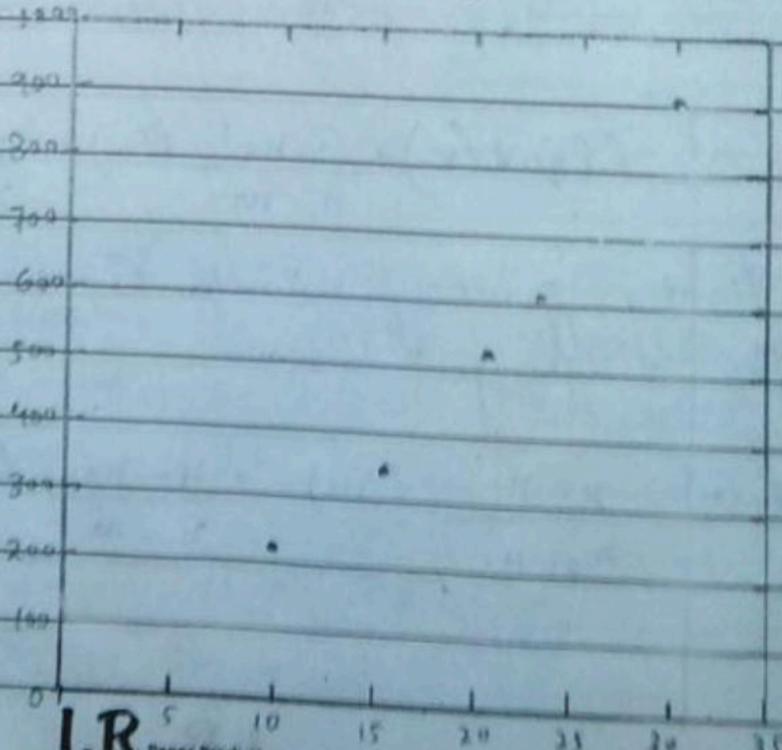
$L_i(x)$ is a weighting function that includes the product of $n-1$ terms with terms of $j=1$ omitted. The application of Lagrangian interpolation will be clarified using an example.

Example:

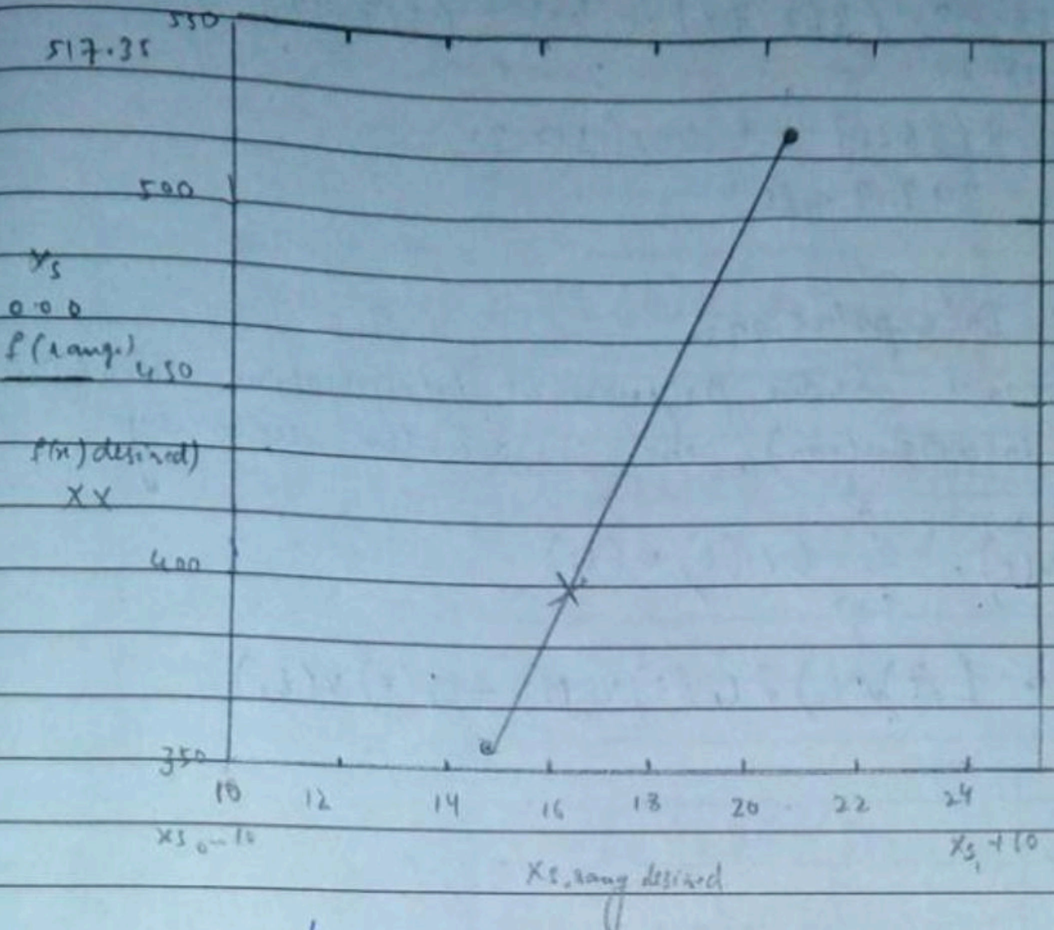
The upward velocity of a rocket is given as a function of time in table 1. Find the velocity at $t=16$ seconds using the Lagrangian method for linear interpolation.

Table velocity as a function of time

$t(s)$	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



Linear Interpolation:



$$V(t) = \sum_{i=0}^{1} L_i(t)\, v(t_i)$$

$$= L_0(t)\, v(t_0) + L_1(t)\, v(t_1)$$

$t_0 = 15,\ v(t_0) = 362.78$

$t_1 = 20,\ v(t_1) = 517.35$

Linear interpolation: (Cond)

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{1} \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} V(t_1) = \frac{t - 20}{15 - 20}(362.78) + \frac{t - 15}{20 - 15}(517.35)$$

Date:

$$V(16) = \frac{16-20}{15-20}(362.78) + \frac{16-15}{20-15}(517.35)$$

$$= 0.8(362.78) + 0.2(517.35)$$

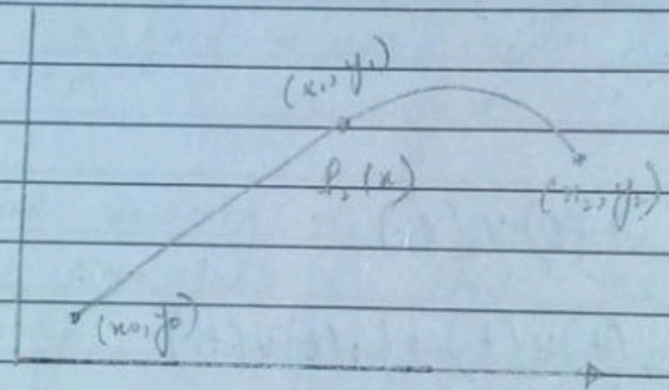
$$\boxed{= 393.7 \text{ m/s}}$$

Quadratic Interpolation:-

For the second order polynomial interpolation also (called quadratic interpolation), we choose the velocity given by

$$V(t) = \sum_{i=0}^{2} L_i(t)\, v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2)$$



Solve for the same question mentioned as above.

Quadratic interpolation (Contd)

$t_0 = 10,\ V(t_0) = 227.04$
$t_1 = 15,\ V(t_1) = 362.78$
$t_2 = 20,\ V(t_2) = 517.35$

$$L_0(t) = \prod_{\substack{j=0 \\ j\neq 0}}^{2} \frac{t-t_j}{t_0-t_j} = \left[\frac{t-t_1}{t_0-t_1}\right]\left[\frac{t-t_2}{t_0-t_2}\right]$$

Date:

$$L_1(t) = \prod_{\substack{j=0 \\ j\neq 1}}^{2} \frac{t - t_j}{t_1 - t_j} = \left[\frac{t - t_0}{t_1 - t_0}\right]\left[\frac{t - t_2}{t_1 - t_2}\right]$$

$$L_2(t) = \prod_{\substack{j=0 \\ j\neq 2}}^{2} \frac{t - t_j}{t_2 - t_j} = \left[\frac{t - t_0}{t_2 - t_0}\right]\left[\frac{t - t_1}{t_2 - t_1}\right]$$

$$v(t) = \left[\frac{t - t_1}{t_0 - t_1}\right]\left[\frac{t - t_2}{t_0 - t_2}\right] v(t_0) + \left[\frac{t - t_0}{t_1 - t_0}\right]\left[\frac{t - t_2}{t_1 - t_2}\right] v(t_1) + \left[\frac{t - t_0}{t_2 - t_0}\right]\left[\frac{t - t_1}{t_2 - t_1}\right] v(t_2)$$

$$v(16) = \left[\frac{16 - 15}{10 - 15}\right]\left[\frac{16 - 20}{10 - 20}\right](227.04) + \left[\frac{16 - 10}{15 - 10}\right]\left[\frac{16 - 20}{15 - 20}\right](362.78) +$$

$$\left[\frac{16 - 10}{20 - 10}\right]\left[\frac{16 - 15}{20 - 15}\right](517.35)$$

$$= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(517.35)$$

$$\boxed{= 392.19 \, m/s}$$

The absolute relative approximate error $|E_a|$ obtained between the result from the first and second order polynomial is

$$|E_a| = \left|\frac{392.19 - 393.70}{392.19}\right| \times 100$$

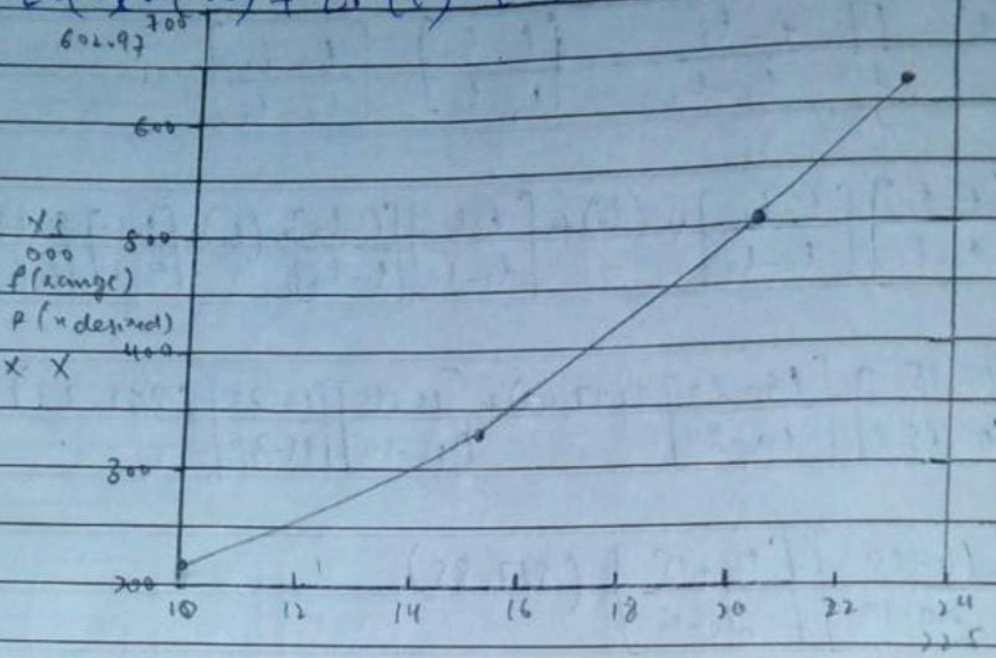
$$\boxed{= 0.38410\%}$$

Cubic Interpolation:

For the third order polynomial (also called cubic interpolation) we choose the velocity given by

$$V(t) = \sum_{i=0}^{3} L_i(t)\, v(t_i)$$

$$L_0(t)\, v(t_0) + L_1(t)\, v(t_1) + L_2(t)\, v(t_2) + L_3(t)\, v(t_3)$$



Example:.

Solve cubic interpolation for the above example.

Cubic Interpolation Condition:.

$t_0 = 10 \quad v(t_0) = 227.04 \qquad t_1 = 15, \quad v(t_1) = 362.78$

$t_2 = 20 \quad v(t_2) = 517.35 \qquad t_3 = 22.5, \quad v(t_3) = 602.97$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^{3} \frac{t - t_j}{t_0 - t_j} = \left[\frac{t - t_1}{t_0 - t_1}\right]\left[\frac{t - t_2}{t_0 - t_2}\right]\left[\frac{t - t_3}{t_0 - t_3}\right]$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^{3} \frac{t - t_j}{t_1 - t_j} = \left[\frac{t - t_0}{t_1 - t_0}\right]\left[\frac{t - t_2}{t_1 - t_2}\right]\left[\frac{t - t_3}{t_1 - t_3}\right]$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^{3} \frac{t - t_j}{t_2 - t_j} = \left[\frac{t - t_0}{t_2 - t_0}\right]\left[\frac{t - t_1}{t_2 - t_1}\right]\left[\frac{t - t_3}{t_2 - t_3}\right]$$

Date:

$$L_3(t) = \prod_{\substack{j=0 \\ j\neq 3}}^{3} \frac{t-t_j}{t_3-t_j} = \left[\frac{t-t_0}{t_3-t_0}\right]\left[\frac{t-t_1}{t_3-t_1}\right]\left[\frac{t-t_2}{t_3-t_2}\right]$$

Cubic Interpolation (contd):

$$v(t) = \left[\frac{t-t_1}{t_0-t_1}\right]\left[\frac{t-t_2}{t_0-t_2}\right]\left[\frac{t-t_3}{t_0-t_3}\right]v(t_0) + \left[\frac{t-t_0}{t_1-t_0}\right]\left[\frac{t-t_2}{t_1-t_2}\right]\left[\frac{t-t_3}{t_1-t_3}\right]v(t_1)$$

$$+ \left[\frac{t-t_0}{t_2-t_0}\right]\left[\frac{t-t_1}{t_2-t_1}\right]\left[\frac{t-t_3}{t_2-t_3}\right]v(t_2) + \left[\frac{t-t_0}{t_3-t_0}\right]\left[\frac{t-t_1}{t_3-t_1}\right]\left[\frac{t-t_2}{t_3-t_2}\right]v(t_3)$$

$$v(16) = \left[\frac{16-15}{10-15}\right]\left[\frac{16-20}{10-15}\right]\left[\frac{16-22.5}{10-22.5}\right](227.04) + \left[\frac{16-10}{15-10}\right]\left[\frac{16-20}{15-20}\right]\left[\frac{16-22.5}{15-22.5}\right](362.78)$$

$$+ \left[\frac{16-10}{20-10}\right]\left[\frac{16-15}{20-15}\right]\left[\frac{16-22.5}{20-22.5}\right](517.35) + \left[\frac{16-10}{22.5-10}\right]\left[\frac{16-15}{22.5-15}\right]\left[\frac{16-20}{22.5-20}\right](602.97)$$

$$= (-0.0416)(227.04) + (0.832)(362.78) + (0.312)(517.35) + (-0.1024)(602.97)$$

The absolute relative approximate error $|\epsilon_a|$, obtained between the results from the first and second order polynomial is

$$|\epsilon_a| = \left|\frac{392.06 - 392.19}{392.06}\right| \times 100$$

$$\boxed{= 0.033269\%}$$ Ans.

Comparison table :-

Order of polynomial	1	2	3
$v(t=16)$ m/s	393.69	392.19	392.06
Absolute relative error	----	0.384%	0.033269 %

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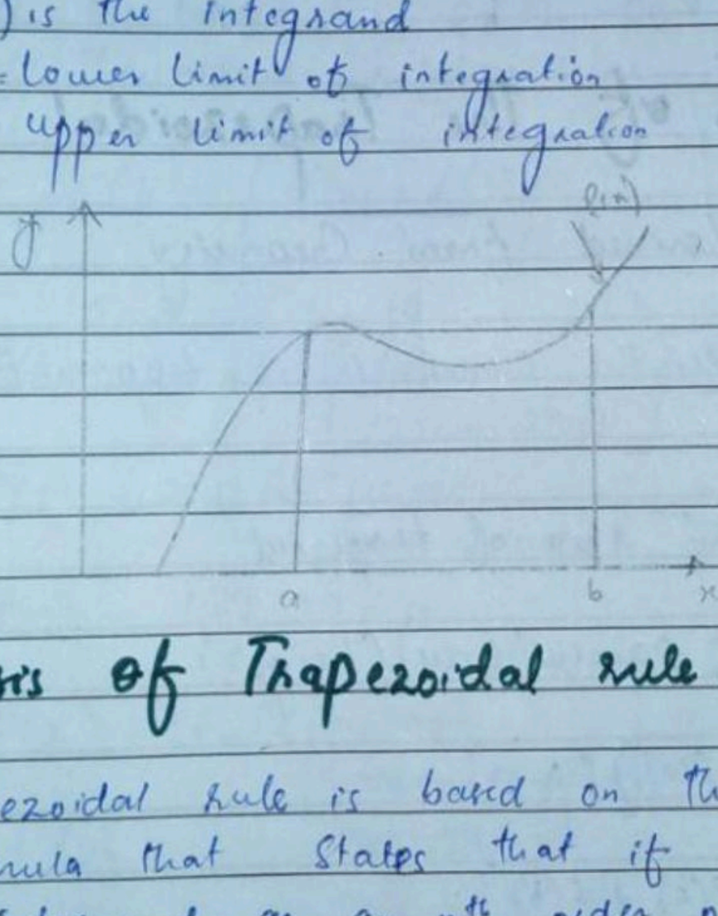
Trapezoidal rule of Integration:

Q What is integration?

The process of measuring the area under the curve of a function plotted on a graph.

$$I = \int_a^b f(x)\,dx$$

Where
$f(x)$ is the integrand
a = lower limit of integration
b = upper limit of integration



Basis of Trapezoidal rule:

Trapezoidal rule is based on the Newton-Cotes formula that states that if one can approximate the integrand as an n^{th} order polynomial

$$I = \int_a^b f(x)\,dx \quad \text{where } f(x) = f_n(x)$$

and $f_n(x) = a_0 + a_1 x + a_2 x + \dots + a_{n-1} x^{n-1} + a_n x^n$

Than the integral of that function is approximated by the integral of that n^{th} order polynomial

$$\int_a^b f(x) = \int_a^b f_n(x)$$

Trapezoidal rule assume that $n=1$ that is The area under the linear polynomial

$$\int_a^b f(x)\,dx = (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$

Derivation of The Trapezoidal Rule:

Method derived from Geometry:

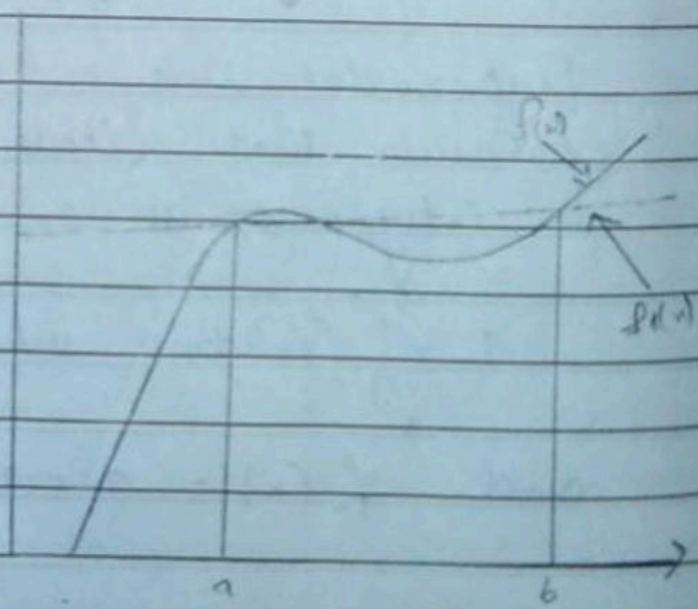
The area under the curve is a trapezoid the integral

$$\int_a^b f(x)\,dx \simeq \text{Area of trapezoid}$$

$$1/2 = (\text{sum of parallel sides})(\text{height})$$

$$1/2\,(f(b) + f(a))\,(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right]$$



Example #1:

The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by.

$$x = \int_8^{30} \left[2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right] dt$$

(a) Using single segment Trapezoidal rule to find the distance covered.
(b) Find the true error, E_t for part (a)
(c) Find the absolute relative error $|E_a|$ for part a.

Solution

(a) $$I \approx (b-a) \left[\frac{f(a) + f(b)}{2} \right]$$

$$a = 8, \quad b = 30$$

$$f(t) = 2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t$$

$$f(8) = 2000 \ln \left[\frac{140000}{140000 - 2100(8)} \right] - 9.8(8)$$

$$f(8) = 177.27 \text{ m/s}$$

Answer.

(b) $$f(30) = 2000 \ln \left[\frac{140000}{140000 - 2100(30)} \right] - 9.8(30)$$

$$\boxed{f(30) = 901.67 \text{ m/s}}$$

$$I = (30-8) \left[\frac{177.27 + 901.67}{2} \right]$$

$$\boxed{I \approx 11868 \text{ m}}$$ Answer

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(b) The exact value of the above integral is

$$x = \int_8^{30} \left[2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right] dt = 11061 \, m$$

E_t = True value − Approximate value
= 11061 − 11868
$\boxed{= 807 \, m}$ — Ans.

(c) The absolute relative true error; would be

$$|\epsilon_t| = \left| \frac{11061 - 11868}{11061} \right| \times 100\%$$

$$\boxed{|\epsilon_t| = 7.2959\%}$$ Answer.

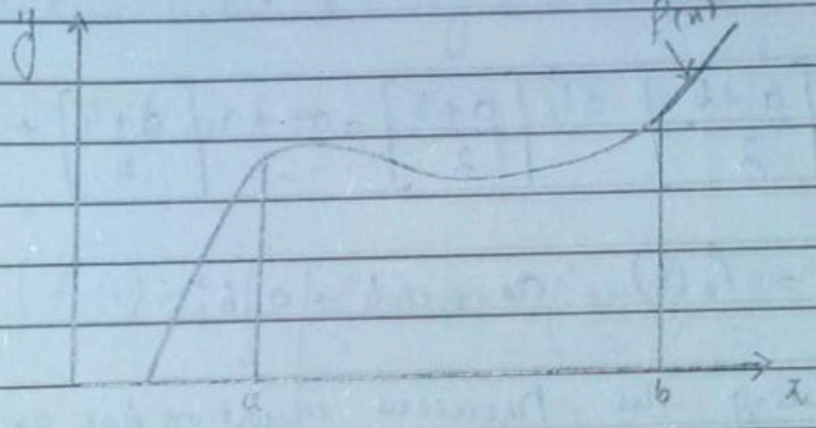
Simpson's 1/3rd rule of Integration:

What is Integration?

The process of measuring the area under a curve

$$I = \int_a^b f(x)\,dx$$

$f(x)$ is the integrated function (integrand)
a = lower limit of integration
b = upper limit of integration



Basis of Simpson's 1/3rd Rule:

Trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial in the interval of integration. Simpson 1/3rd rule is the extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

Hence
$$I = \int_a^b f(x)\,dx \approx \int_a^b f_2(x)\,dx$$

where $f_2(x)$ is a second order polynomial

$$f_2(x) = a_0 + a_1 x + a_2 x^2$$

Choose

$$(a, f(a)), \left[\frac{a+b}{2}, f\left[\frac{a+b}{2}\right]\right], \text{ and } (b, f(b))$$

as three points of the function to evaluate a_0, a_1 and a_2

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left[\frac{a+b}{2}\right] = f_2\left[\frac{a+b}{2}\right] = a_0 + a_1\left[\frac{a+b}{2}\right] + a_2\left[\frac{a+b}{2}\right]^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the previous equations for a_0, a_1 & a_2

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left[a+b/2\right] + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = \frac{af(a) - 4af\left[a+b/2\right] + 3af(b) + 3bf(a) - 4bf\left(a+b/2\right) + bf(a)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left[f(a) - 2f\left(a+b/2\right) + f(b)\right]}{a^2 - 2ab + b^2}$$

$$I \approx \int_a^b f_2(x)\,dx \Rightarrow \int_a^b (a_0 + a_1 x + a_2 x^2)\,dx$$

$$\left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$a_0(b-a) + a_1 \frac{b^2-a^2}{2} + a_2 \frac{b^3-a^3}{3}$$

Substituting values of a_0, a_1, a_2

$$\int_a^b f_2(x)\,dx = \frac{b-a}{6}\left[f(a) + 4f\left[\frac{a+b}{2}\right] + f(b)\right]$$

Since for Simpson's 1/3 rule, the interval $[a, b]$ is broken into 2 segments width

$$h = \frac{b-a}{2}$$

hence

$$\int_a^b f_2(x)\,dx = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Because the above form has 1/3 in its formula. it is called Simpson's 1/3 rule.

Example 1:

The distance covered by a rocket from $t = 8$ to $t = 3$ is given by

$$x = \int_8^{30}\left[2000 \ln\left[\frac{140000}{140000 - 2100t}\right] - 9.8t\right] dt$$

(a) Use Simpson 1/3rd rule to find the approximat value of x

(b) Find the true error E_t = ?

(c) Find the absolute relative error?

Solution

(a) $x = \int_{8}^{30} f(t)\,dt$

$x = \left(\frac{b-a}{6}\right)\left[f(a) + 4f\left[\frac{a+b}{2}\right] + f(b)\right]$

$x = \left[\frac{30-8}{6}\right]\left[f(8) + 4f(19) + f(30)\right]$

$x = (22/6)\left[177.2667 + 4(484.7455) + 901.6740\right]$

$\boxed{= 11065.72\,m}$

(b) The exact value of the above integral is

$x = \int_{8}^{30}\left[2000\ln\left[\frac{140000}{140000-2100t}\right] - 9.8t\right]dt$

$= 11061.34\,m$

True error

$E_t = 11061.34 - 11065.72$

$\boxed{E_t = -4.38\,m}$

(c) Absolute relative error.

$|\epsilon_t| = \left|\frac{11061.34 - 11065.72}{11061.34}\right| \times 100\%$

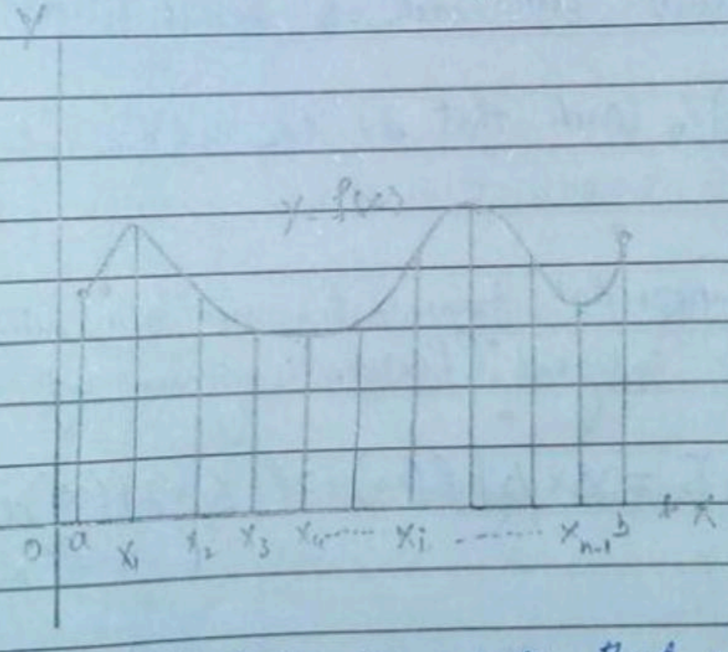
$\boxed{\epsilon_t = 0.0396\%}$ Answer

Date:

① Trapezoidal Rule:

In Calculus "Trapezoida Rule is one of the important integration rules. The name trapezoidal is because when the area under the curve is evaluated than the total area is divided into small trapezoids instead of rectangles. This rule is use for approximating the definite integrals where "it uses the linear approximations of the functions.
The trapezoidal rule is mostly use in the numerical analysis process. To evaluate the definite integrals we can also use Rieman Sums where we use small rectangles to evaluate the area under the curve.

Trapezoidal Rule defination:,



Trapezoidal Rule is a rule that evaluates the area under the curves by dividing the total area into smaller trapezoids rather than using rectangles. This integration works by approximating the region under the graph of a function as a trapezoids, and it calculate the area. This rule takes the average of the left and the right Sum.

The Trapezoidal rules does not give accurate value as Simpson's rule when the underlying function is Smooth. It is because Simpson's rule and Trapezoidal give the approximation value but Simpson's rule results in even more accurate approximation value of the integrals.

Trapezoidal Rule formula:-

Left $f(x)$ be a continuous function on the interval $[a, b]$. Now divide the intervals $[a, b]$ into n equal Subintervals with each of equal width

$\Delta x = (b-a)/n$ such that $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} = b$

Then the trapezoidal formula for area approximating the definite integral $\int_a^b f(x)dx$ is given by

$$\int_a^b f_x(x)\,dx \approx T_n = \Delta x/2\,[f(x_0)+2f(x_1)+2f(x_2)+\cdots 2f(x_{n-1}) + f(x_n)]$$

x_0	x_1	x_2	x_3	x_4	$(8-0)/4 = 2$
0	2	4	6	8	

$f(x_0)\ f(x_1) \cdots\cdots f(x_4)$

-: Assignment No # 7:-

Date:

Example # 1

Approximate the area under the curve $y = f(x)$ between $x = 0$ and $x = 8$ using Trapezoidal rule with $n = 4$ Subintervals. A function $f(x)$ is given in the table of values

x	0	2	4	6	8
f(x)	3	7	11	9	3

Solution

The trapezoidal rule formula for $n = 4$ Subintervals is given as

$$T_4 = (\Delta x/2)[f(x_0) + 2f(x_1) + f(x_4)]$$

here the Subintervals with $\Delta x = 2$

Now Substitute the values from the table, to find the approximation value of the area under the curve

$$A \approx T_4 = (2/2)[3 + 2(7) + 2(11) + 2(9) + 3]$$

$$A \approx T_4 = 3 + 14 + 22 + 18 + 3 = 60$$

Therefore, the approximate value of area under the curve using trapezoidal Rule is 60.

Q Find out $\int_0^8 2x.dx$ using Trapezoidal rule with $n = 4$ intervals.

Solution,

first make the Scale of x

0	2	4	6	8

The trapezoidal rule formula for $n = 4$

Subintervals is given as

$$T_4 = \Delta x/2 \; [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

here $\Delta x = \frac{b-a}{n} = \frac{8-0}{4} = 2$

$$T_4 = \cancel{2}/\cancel{2} \; [0 + 2(2) + 2(4) + 2(6) + 8]$$

$$T_4 = [4 + 8 + 12 + 8]$$

$$\boxed{T_4 = 32} \text{ Answer}$$

Example 2:

Approximate the area under the curve $y = f(x)$ between $x = 4$ and $x = 2$ using Trapezoidal rule with $n = 6$ Subintervals. A function $f(x)$ is given in the table of values.

x	-4	-3	-2	-1	0	1	2
f(x)	0	4	5	3	10	11	2

Solution

The Trapezoidal rule formula for $n = 6$ subintervals is given as.

$$T_6 = (\Delta x/2)[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5)]$$

Here the subintervals width $\Delta x = 1$

Now, substitute the values from the table to find the approximate value of the area under the curve.

$A \approx T_6 = (1/2)\,[0 + 2(4) + 2(5) + 2(3) + 2(10) + 2(11) + 2]$

$A \approx T_6 = (1/2)\,[8 + 10 + 6 + 20 + 22 + 2] = 68/2 = 34$

Therefore the approximate value of area under the curve using Trapezoidal

Q Calculate the integral of the function $f(x) = 2x$ in the interval $(0, 2)$ with $n = 6$? or $\int_0^2 2x$

Solution.

Given $A = 0$, $B = 2$ let $n = 6$

$h = b - a/n = 2 - 0/6 = 2/6 = 1/3$

$x_0 = a = 0$

$x_1 = x_0 + h = 0 + 1/3 = 1/3$

$x_2 = x_1 + h = 1/3 + 1/3 = 2/3$

$x_3 = x_2 + h = 2/3 + 1/3 = 1$

$x_4 = x_3 + h = 1 + 1/3 = 4/3$

$x_5 = x_4 + h = 4/3 + 1/3 = 5/3$

$x_6 = x_5 + h = 5/3 + 1/3 = 2$

$f(x_0) = y_0 = 2(0) = 0$

$f(x_1) = y_1 = 2(1/3) = 2/3$

$f(x_2) = y_2 = 2(2/3) = 4/3$

$f(x_3) = y_3 = 2(1) = 2$

$f(x_4) = y_4 = 2(4/3) = 8/3$

$f(x_5) = y_5 = 2(5/3) = 10/3$

$f(x_6) = y_6 \; 2(2) = 4$

Applying Simpson formula,

$$\int_0^2 2x\,dx = h/3\ [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)]$$

$$= \frac{1/3}{3}\ [0 + 4(2/3) + 2(4/3) + 4(2) + 2(8/3) + 4(10/3) + 4]$$

$$= 1/9\ [0 + 8/3 + 8/3 + 8 + 16/3 + 16/3 + 40/3 + 4]$$

$f(x)dx = 4$ Answer.

Q Performing the same question with above data for Trapezoidal Rule.

Sol; $$\int_0^2 2x\,dx = \frac{h}{2}\ [(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$T_6 = \frac{1/3}{2}\ [(0+4) + 2(2/3 + 4/3 + 2 + 8/3 + 10/3)]$$

$$T_6 = 1/6\ [4 + 2(20)]$$

$$T_6 = 1/6\ (4) \rightarrow T_6 = \cancel{24}^{4}/6$$

$\boxed{T_6 = 4}$ Answer

Q Calculate the analytical result also find out the efficiency of Simpson Rule percentage relative error.

$|E| = |\text{True value} - \text{approximate value}| / \text{True value}$

$|E| = 4 - 4/4$

$|E| = 0$ Answer.

Assignment NO# 8

Simpson 1/3rd rule:

Q Given the integral as $\int_0^3 \frac{1}{1+x^5}\,dx$

Find out the approximated value of given integral using Simpson's 1/3rd rule.

True value = 1.06587.

Solution;

$$\int_0^3 \frac{1}{1+x^5}\,dx$$

First make the scale of x

$$0 \quad \tfrac{1}{2} \quad 1 \quad \tfrac{3}{2} \quad 2 \quad \tfrac{5}{2} \quad 3$$

$$Dx \text{ or } h = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$$

Applying Simpson's formula

$$\int_0^3 \frac{1}{1+x^3}\,dx = \frac{h}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)+4f(x_5)+f(x_6)\right]$$

$$= \frac{1/2}{3}\left[\frac{1}{1+(0)^5}+4\times\frac{1}{1+(0.5)^5}+2\times\frac{1}{1+1^5}+4\times\frac{1}{1+(1.5)^5}+2\times\frac{1}{P\times 2^5}+4\times\frac{1}{1+(2.5)^5}+\frac{1}{1+3^5}\right]$$

$$= \frac{1}{6}\left[1+3.878787+1+0.46545+0.060606+0.040544+0.00041\right]$$

$$= \frac{6.4491}{6}$$

$$\boxed{\int_0^3 \frac{1}{1+x^5}\,dx = 1.074855}$$ Answer,

(b) Find out the accuracy of the approximated value. We need true value to find any sort of error to find the accuracy

True-value = 1.06587

Solution

$$|E_r| = \left| \frac{\text{True value} - \text{Approximate value}}{\text{True value}} \right| \times 100\%$$

$$|E_r| = \frac{1.06587 - 1.074855}{1.06587} \times 100\%$$

$\boxed{|E_r| = 0.843\%}$ Answer.

For Accuracy:

$100\% - 0.843\%$

The accuracy is 99.157% Answer.

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Q What does a matrix look like?

Ans Matrix are everywhere. If you have used spreadsheet such as excel or written numbers in a table, you have a used matrix. Matrices make presentation of number clever and make calculation easier to program. Look at the matrix below about the sales of the tires in a Blowout store given by quarter and make of tires.

$$\begin{array}{l} \\ \text{Firestone} \\ \text{michigan} \\ \text{copper} \end{array} \begin{array}{c} \begin{array}{cccc} Q_1 & Q_2 & Q_3 & Q_4 \end{array} \\ \begin{bmatrix} 25 & 20 & 3 & 2 \\ 5 & 10 & 15 & 25 \\ 6 & 16 & 7 & 27 \end{bmatrix} \end{array}$$

So what is Matrix?

Ans A Matrix is a rectangular array of element. The element can be symbolic expression and number. Matrix (A) is denoted by

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Row i of $[A]$ has n element and is $[a_{i1} \; a_{i2} \ldots a_{in}]$

and column j of $[A]$ has m elements and is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Iterative Methods for Solving Linear System of equations:

Date:

(1) Iterative Solution Method:

- Starts with an initial approximation for the solution
- At each iteration update the x vector by using the system $Ax = b$
- Each iteration involves a matrix-vector product.
- If A is sparse this product efficiency done

Iterative Solution Procedure:

- Write the system $Ax = b$ in equivalent form $x = Ex + f$ like $x = g(x)$ for fixed point iteration.
- Starting with x, generate a sequence of approximation (x^k) iteratively by $x^{k+1} = Ex^k + f$
- Representation of E and f depends on the type of the method used.
- But for every method E and f are obtained from A and B, but in a different way.

Convergence:

- As $k \to \infty$, the sequence (x^k) convergence to the solution vector under some on E matrix.
- The imposes different conditions on A matrix different methods.
- For the same A matrix, one method may converge with the other may diverge
- Therefore for each method the relation b/w A and E should be found to decide on the convergence.

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Jacobi Method:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_n x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$x = \begin{bmatrix} x_1^0 \\ x_2^0 \\ \vdots \\ x_n^0 \end{bmatrix}$$

$$x_1^1 = 1/a_{11}\left(b_1 - a_{12}x_2^0 - \cdots - a_{1n}x_n^0\right)$$
$$x_2^1 = 1/a_{21}\left(b_2 - a_{21}x_1^0 - a_{23}x_3^0 - \cdots a_{2n}x_n^0\right)$$
$$x_{nn}^1 = 1/a_{nn}\left(b_n - a_{n1}x_1^0 - a_{n2}x_2^0 - \cdots - a_{a_{nn-1}}x_{n-1}^0\right)$$

$$x_1^{k+1} = \frac{1}{a_{11}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^k - \sum_{j=1-1}^{n} a_{ij}x_j^k\right].$$

$x^{k+1} = Ex^k + f$ iteration for Jacobi method.

A can be written as $A = C = D + C$ (not decomposition)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_i^{k+1} = \frac{1}{a_{ii}}\left[b_i - \sum_{j=1}^{j-1} a_{ij}x_j^k - \sum_{j=i+1} a_{ij}x_j^k\right]$$

$$x^{k+1} = -D^{-1}(L+U) + D^{-1}b$$
$$E = -D^{-1}(L+U)$$
$$F = D^{-1}b$$

Jacobi Iteration:-

To solve $Ax = b$ given an initial approximation $x^{(0)}$:

Input the number of equations and unknown n; the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i \le n$ of b; the entries XO_i, $1 \le i \le n$ of $XO = x^{(0)}$; tolerance TOL; Maximum numbers of iterations N,

Output: The approximate solution $x_1, \ldots x_n$ or a message that the number of iterations was exceeded

Step ① Set $k = 1$

Step ② while $(k \le N)$ do step 3 - 6.

Step ③ for $i = 1, \ldots, n$

$$\text{Set } x_i = \sum_{\substack{j=1 \\ j \neq 0}}^{n} (a_{ij} XO_j) + b_i \Big/ a_{ii}$$

Step ④ If $\| X - XO \| < TOL$ then output $(x_1 \ldots x_n)$.
(Procedure completed successfully STOP)

Step ⑤ Set $k = k+1$.

Step ⑥ for $i = 1, \ldots, n$ Set $XO_i = x_i$

Step ⑦ Output (Maximum number of iteration exceeded)

(Procedure completed successfully)
STOP.

Assignment NO# 9

Date:

: System of linear equation by Jacobi Method :

Q
$$5x - 2y + 3z = -1$$
$$-3x + 9y + z = 2$$
$$2x - y - 7z = 3$$

continue the iterations until two successive approximations are identical when rounded to three (3) significant digits. Starts with $(x_0, y_0, z_0) = (0, 0, 0)$

Solution

To begin, write the system in the form

$$x = \frac{-1 + 2y - 3z}{5}$$

$$y = \frac{2 + 3x - z}{9}$$

$$z = \frac{-(3 + y - 2x)}{7}$$

$$(x_0, y_0, z_0) = (0, 0, 0)$$

Iteration 1;

$$x = \frac{-1 + 2(0) - 3(0)}{5} = -0.2$$

$$y = \frac{2 + 3(0) - 0}{9} = 0.222$$

$$z = \frac{-(3 + (0) - 2(0))}{7} = -0.42857$$

Iteration 2: $(x_1, y_1, z_1) = (-0.2, 0.222, -0.4285)$

$$x_2 = \frac{-1 + 2(0.222) - 3(-0.42857)}{5} = 0.145942$$

$$y_2 = \frac{2 + 3(-0.2) - (-0.42857)}{9} = 0.203174$$

$$z_2 = \frac{-(3 + (0.222) - 2(-0.2))}{7} = -0.51743$$

Iteration 3: $(x_2, y_2, z_2) = (0.145942, 0.203174, -0.51743)$

$$x_3 = \frac{-1 + 2(0.203174) - 3(-0.51743)}{5} = 0.191728$$

$$y_3 = \frac{2 + 3(0.203174) - (-0.51743)}{9} = 0.3474391$$

$$z_3 = \frac{-(3 + 0.203174 - 2(0.145942))}{7} = -0.4158986$$

Iteration 4: $(x_3, y_3, z_3) = (0.191728, 0.3474391, -0.4158989)$

$$x_4 = \frac{-1 + 2(0.3474391) - 3(-0.4158986)}{5} = 0.188515$$

$$y_4 = \frac{2 + 3(0.191728) - (-0.4158986)}{9} = 0.3323425$$

$$z_4 = \frac{-(3 + 0.3474391 - 2(0.191728))}{7} = -0.4234262$$

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Iteration 5: $(x_4, y_4, z_4) = (0.188515, 0.332425, -0.4234262)$

$$x_5 = \frac{-1 + 2(0.3323425) - 3(-0.4234262)}{5} = 0.186993$$

$$y_5 = \frac{2 + 3(0.188515) - (-0.4234262)}{9} = 0.33211$$

$$z_5 = \frac{-(3 + (0.33234257) - 2(0.188515))}{7} = -0.42219$$

Iteration 6: $(x_5, y_5, z_5) = (0.186993, 0.33211, -0.42219)$

$$x_6 = \frac{-1 + 2(0.33211) - 3(-0.42219)}{5} = 0.186158$$

$$y_6 = \frac{2 + 3(0.186993) - (-0.42219)}{9} = 0.332$$

$$z_6 = \frac{-(3 + (0.33211) - 2(0.186993))}{7} = -0.422$$

$(x_6, y_6, z_6) = (0.186158, 0.332, -0.422)$

	x_n	y_n	z_n
1	0	0	0
2	-0.2	0.222	-0.42857
3	0.145942	0.203174	-0.51743
4	0.191728	0.3474390	-0.415896
5	0.188515	0.33234	-0.4234262
6	0.186993	0.3321	-0.42219
7	0.186158	0.332	-0.422

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True value: $x = 0.186$, $y = 0.331$, $z = -0.422$

For x: $|E_r| = \left|\dfrac{0.186 - 0.186}{0.186}\right| \times 100 = 0\%$

For y: $|E_r| = \left|\dfrac{0.331 - 0.332}{0.331}\right| \times 100 = 0.302\%$

For z: $|E_r| = \left|\dfrac{-0.422 - (-0.422)}{-0.422}\right| \times 100 = 0\%$

Accuracy of x: $100 - 0 = 100\%$ accuracy

Accuracy of y: $100 - 0.302 = 99.698\%$

Accuracy of z: $100 - 0 = 100\%$ Answer.

:. Jacobi Method: Example #2 :.

Consider the following system of equation.

$$10x_1 - x_2 + 2x_3 = 6$$
$$-x_1 + 11x_2 - x_3 + 3x_4 = 25$$
$$2x_1 - x_2 + 10x_3 - x_4 = -11$$
$$3x_2 - x_3 + 8x_4 = 15$$

Solution.

$$x_1 = \frac{6 + x_2 - 2x_3}{10}$$

$$x_2 = \frac{25 + x_1 + x_3 - 3x_4}{11}$$

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$$x_{33} = \frac{-11 - 2(1.047) + (1.716) + (0.885)}{10} = -1.049$$

$$x_{43} = \frac{15 - 3(1.716) + (-0.805)}{8} = 1.131$$

Iteration #4: $(x_{13}, x_{23}, x_{33}, x_{43}) = (0.933, 2.053, -1.049, 1.131)$

$$x_{14} = \frac{6 + 2.053 - 2(-1.049)}{10} = 1.015$$

$$x_{24} = \frac{25 + 0.933 + (-1.049) - 3(1.131)}{11} = 1.954$$

$$x_{34} = \frac{-11 - 2(0.933) + (2.053) + (1.131)}{10} = -0.968$$

$$x_{44} = \frac{15 - 3(2.053) + (-1.049)}{8} = 0.974$$

Iterateration 5: $(x_{14}, x_{24}, x_{34}, x_{44}) = (1.015, 1.954, -0.968, 0.974)$

$$x_{15} = \frac{6 + 1.954 - 2(-0.968)}{10} = 0.989$$

$$x_{25} = \frac{25 + 1.015 + (-0.968) - 3(0.974)}{11} = 2.0114$$

$$x_{35} = \frac{-11 - 2(1.015) + (1.954) + (0.974)}{10} = -1.0102$$

$$x_{45} = \frac{15 - 3(1.954) + (-0.968)}{8} = 1.0213$$

Iteration 6: $(x_{15}, x_{25}, x_{35}, x_{45}) = (0.989, 2.0114, -1.0102, 1.0$

$$x_{16} = \frac{6 + 2.0114 - 2(-1.0102)}{10} = 1.0032$$

$$x_{26} = \frac{25 + 0.989 + (-1.0102) - 3(1.0213)}{11} = 1.9923$$

Date:

$$x_3 = \frac{(-11-2x_1+x_2+x_4)}{10}$$

$$x_4 = \frac{15-3x_2+x_3}{8}$$

Initial bounds are $(x_1, x_2, x_3, x_4) = (0,0,0,0)$

Iteration, 1:

$$x_{11} = \frac{6+(0)-2(0)}{10} = 0.6$$

$$x_{21} \; \frac{25+0+0-3(0)}{11} = 2.27273 \to 2.273$$

$$x_{31} = -11-2(0)+(0)+(0) / 11 \to -1.1$$

$$x_{41} = 15-3(0)+0 / 8 = 1.875$$

Iteration 2:

$(x_{11}, x_{21}, x_{31}, x_{41}) = (0.6, 2.273, -1.1, 1.875)$

$$x_{12} = \frac{6+2.273-2(-1.1)}{10} = 1.0473 \approx 1.047$$

$$x_{22} = \frac{25+0.6+(-1.1)-3(1.875)}{11} = 1.7159 \approx 1.716$$

$$x_{32} = \frac{-11-2(0.6)+(2.273)+(1.875)}{10} = -0.8052$$

$$x_{42} = \frac{15-3(2.273)+(-1.1)}{8} = 0.885$$

Iteration 3: $(x_{12}, x_{22}, x_{32}, x_{42}) (1.047, 1.716, -0.8052, 0.88$

$$x_{13} = \frac{6+1.716-2(-0.805}{10} = 0.9326 \approx 0.933$$

$$x_{23} = \frac{25+1.047+(-0.805)-3(0.885)}{11} = 2.053$$

$$x_{36} = \frac{-11-2(0.989)+(2.0114)+(1.0213)}{10} = -0.995$$

$$x_{46} = \frac{15-3(2.0114)+(-1.0102)}{8} = 0.995$$

Iteration 7: $(x_{16}, x_{26}, x_{36}, x_{46}) = (1.0032, 1.9923, -0.995, 0.995$

$$x_{17} = \frac{6+1.9923-2(-0.995)}{10} = 0.9982$$

$$x_{27} = \frac{25+1.0032+(-0.995)-3(0.995)}{11} = 2.002$$

$$x_{37} = \frac{-11-2(1.0032)+(1.9923)+(0.995)}{10} = -1.002$$

$$x_{47} = \frac{15-3(1.9923)+(-0.995)}{8} = 1.004$$

Iteration 8: $(x_{17}, x_{27}, x_{37}, x_{47})(0.9982, 2.002, -1.002, 1.00$

$$x_{18} = \frac{6+2.002-2(-1.002)}{10} = 1.001$$

$$x_{28} = \frac{25+0.9982+(-1.002)-3(1.004)}{10} = 2.002$$

$$x_{38} = \frac{-11-2(0.9982)+(2.002)+(1.004)}{10} = -1.000$$

$$x_{48} = \frac{15-3(2.002)+(-1.002)}{8} = 0.999 \simeq 1.000$$

Date:

Iteration 9 : $(x_{18}, x_{28}, x_{38}, x_{48}) = (1.001, 2.000, -1.000, 1.000)$

$$x_{19} = \frac{6 + (2.000) - 2(-1.000)}{10} = 1.000$$

$$x_{29} = \frac{25 + (1.001) + (-1.000) - 3(1.000)}{11} = 2.000$$

$$x_{39} = \frac{-11 - 2(2.001) + (1.000)}{10} = -1.000$$

$$x_{49} = \frac{15 - 3(2.000) + (-1.000)}{8} = 1.000$$

$(x_{19}, x_{29}, x_{39}, x_{49}) = (1.000, 2.000, -1.000, 1000)$

S.No	X^1_n	X^2_n	X^3_n	X^4_n
0	0	0	0	0
1	0.6	2.273	-1.1	1.875
2	1.047	1.716	-0.805	0.885
3	0.933	2.053	-1.049	1.131
4	1.015	1.954	-0.968	0.974
5	0.989	2.0114	-1.0102	1.0213
6	1.003	1.9923	-0.995	0.995
7	0.998	2.002	-1.002	1.004
8	1.001	2.000	-1.000	1.000
9	1.000	2.000	-1.000	1.000

Analytical Solution

$x_1 = 1$

$x_2 = 2$

$x_3 = 1$

$x_4 = 1$

Answer.

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Gauss Seidal Iterative Method:

To Solve $Ax = b$ given an initial approximation $x^{(0)}$;
Input the number of equations and unknowns as the entries a_{ij}, $1 \le i, j \le n$ of the matrix A; the entries b_i, $1 \le i, j \le n$ of the b; The entries XO_i, $1 \le i \le n$ of $XO = x^{(0)}$; tolerance TOL; maximum number of iteration N,
Output; The approximate Solution $x_i, \ldots x_n$ or a message that the number of iterations will exceeded

Step 1: Set $k = 1$
Step 2: while ($k \le N$) to step 3-6
Step 3: for $i = 1, \ldots n$

$$\text{set } x_i = -\sum_{j=1}^{i-1} a_{ij} x_j - \sum_{j=i+1} a_{ij} \cdot XO_j + b_i \Big/ a_{ii}$$

Step 4: If $\|x - XO\| \le TOL$ then output $(x_1 \ldots x_n)$;
(Procedure completed successfully)
STOP

Step 5: Set $k = k+1$
Step 6: for $i = 1, \ldots n$ set $XO = x_i$
Step 7 Output (maximum number of iteration exceeded)
(Procedure Completed successfully)
STOP

Comparison:-

Gauss Seidal iteration converge more rapidly

than Jacobi iteration.
Since it uses the latest updates.

But there are some cases that Jacobi iterations goes converge but Gauss Seidal does not.

Gauss Seidal Method:

Convert the following system of linear equation

$$5x - 2y + 3z = -1$$
$$-3x + 9y + z = 2$$
$$2x - y - 7z = 3$$

Continue the iteration until two successive approximations are identical when rounded to three (3) significant digit. Start with $(x_0, y_0, z_0) = (0, 0, 0)$

Solution.

$$x = \frac{-1 + 2y - 3z}{5} \qquad y = \frac{2 + 3x - z}{9}$$

$$z = \frac{-(3 + y - 2x)}{7}$$

Iteration 1: $(x_0, y_0, z_0) = (0, 0, 0)$

$$x_1 = \frac{-1 + 2(0) - 3(0)}{5} = -0.2$$

$$y_1 = \frac{2 + 3(-0.2) - 0}{5} = 0.1556$$

Date:

$$z_1 = \frac{-(3 + (0.1556) - 2(-0.2)}{7} = -0.50794$$

Iteration 2: $(x_1, y_1, z_1) = (-1.2, 0.1556, -0.50794)$

$$x_2 = \frac{-1 + 2(0.1556) - 3(-0.50794)}{5} = 0.16698$$

$$y_2 = \frac{2 + 3(0.16698) - (-0.50794)}{9} = 0.3343$$

$$z_2 = \frac{-(3 + (0.3343) - 2(0.16698)}{7} = -0.42862$$

Iteration 3: $(x_2, y_2, z_2) = (0.16698, 0.3343, -0.4286$

$$x_3 = \frac{-1 + 2(0.3343) - 3(-0.42862)}{5} = 0.19090$$

$$y_3 = \frac{2 + 3(0.19090) - (-0.4862)}{9} = 0.33348$$

$$z_3 = \frac{-(3 + (0.33348) - 2(0.19090)}{7} = -0.42167$$

Iteration 4: $(x_3, y_3, z_3) = (0.19090, 0.33348, -0.421$

$$x_4 = \frac{-1 + 2(0.33348) - 3(-0.42167)}{5} = 0.1863$$

$$y_4 = \frac{2 + 3(0.18639) - (-0.42167)}{9}$$

Date:

$$z_4 = \frac{-(3+(0.3312)-2(0.18639)}{7} = -0.4227$$

Iteration 5: $(x_4, y_4, z_4) = (0.18639, 0.3312, -0.4227)$

$$x_5 = \frac{-1+2(0.3312)-3(-0.4227)}{5} = 0.18606$$

$$y_5 = \frac{2+3(0.18606)-(0.3312)}{9} = 0.3312$$

$$z_5 = \frac{-(3(+(0.3312)+2(0.18606)}{7} = -0.4227$$

$(x_5, y_5, z_5) = (0.18606, 0.3312, -0.4227)$

x_n	y_n	z_n
0	0	0
-0.2	0.1556	-0.50794
0.16698	0.3343	-0.4286
0.19090	0.33348	-0.42167
0.18639	0.3312	-0.4226
0.18606	0.3312	-0.4227

True values: $X = 0.186$, $Y = 0.331$, $Z = -0.422$

For X $|E_r| = \left| \frac{0.186 - 0.18606}{0.186} \right| \times 100 = 100$

For Y $|E_r| = \left| \frac{0.331 - 0.331}{0.331} \right| \times 100 = 100$

For Z $= |E_r| = \left| \frac{-0.422 - 0.422}{0.422} \right| \times 100 = 100$

Answer

Date:

∴ Gauss Elemination Method:.

Q:

$$2x_2 + x_3 = -8$$
$$x_1 - 2x_2 - 3x_3 = 0$$
$$-x_1 + x_2 + 2x_3 = 3$$

Solution

$$\left[\begin{array}{ccc:c} 0 & 2 & 1 & -8 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{array}\right]$$

R_{31}
$$\left[\begin{array}{ccc:c} -1 & 1 & 2 & 3 \\ 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

R_{12}
$$\left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ -1 & 1 & 2 & 3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

R_{2+1}.
$$\left[\begin{array}{ccc:c} 1 & -2 & -3 & 0 \\ 0 & -1 & -1 & 3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

∴ $R_1 + R_2$: $(R_2(-1))$
$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & -8 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$

Date:

$R_3 - 2(R_2)$

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & -8 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$R_2 + R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & -8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

$R_3(-1)$

$$\left[\begin{array}{ccc:c} 1 & 0 & -2 & -8 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$R_1 + 2(R_3)$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_1 = -4$

$x_2 = -5$

$x_3 = 2$

$$X = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$