1 Homogeneous Linear Second-Order ODE's with Constant Coefficients

We have the equation $ay'' + by' + cy = 0$ (1), where a, b, c are constants.
Thereom: If $y_1(x)$ and $y_2(x)$ are any two solutions of (1) and if C_1 and C_2 are arbitraty constants, then $C_1y_1(x) + C_2y_2(x)$ is the general solution of (1).
Consider: $y' + ay = 0$, where a is a constant, we can derive:

$$y' + ay = 0$$
$$y' = -ay$$
$$\frac{dy}{dx} = -ay$$
$$\frac{dy}{y} = -adx$$
$$\int \frac{dy}{y} = \int -adx$$
$$\ln y = -ax + C$$
$$y = Ae^{-ax},\ A = e^C$$

Question: Whether exponentioal solutions exist for homogeneous linear second-order ODE's.
Solution of $ay'' + by' + cy = 0$ (1):
If we try a solution of the form $y = e^{,x},\ m \in \mathbb{R}$ we get $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Now we substitute $y' = me^{mx}$ and $y'' = m^2e^{mx}$ into (1) to get $a(m^2e^{mx}) + b(me^{mx}) + ce^{mx} = 0$ (2) or $e^{mx}(am^2 + bm + c) = 0$. We have $e^{mx} \neq 0$ for all values of x and m, so (2) is 0 when $am^2 + bm + c = 0$. This equation is called the auxillary equation of (1).
We can solve for m with the quadratic formula: $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
We now have three cases: $b^2 - 4ac > 0$, $b^2 - 4ac = 0$, and $b^2 - 4ac < 0$

1.1 Case 1

If $b^2 - 4ac > 0$, then equation (2) has two distinct real solutions m_1 and m_2. Corresponding, equation (1) has two basic solutions: $y_1(x) = e^{m_1x}$ and $y_2(x) = e^{m_2x}$. The complete solution of (1) is: $y(x) = C_1e^{m_1x} + C_2e^{m_2x}$.

1.1.1 Example 1

Given $y'' + 5y' + 6 = 0$, we have the equation $m^2 + 5m + 6 = 0$. We can solve for m.

$$m^2 + 5m + 6 = 0$$
$$(m + 2)(m + 3) = 0$$
$$m_1 = -2,\ m_2 = -3$$

And now solving for y we find that, $y(x) = C_1e^{3x} + C_2e^{-3x}$

1.1.2 Example 2

Given $2y'' - 5y' - 3y = 0$, we have $2m^2 - 5m - 3 = 0$,

$$m = \frac{-(-5) \pm \sqrt{(-5)^2 - 4ac}}{2(2)}$$
$$m_1 = 3,\ m_2 = -\frac{1}{2}$$
$$y(x) = C_1e^{3x} + C_2e^{-\frac{x}{2}}$$