1 Homogeneous Linear Second-Order ODE's with Constant Coefficients

We have the equation $ay'' + by' + cy = 0$ (1), where a, b, c are constants.
Thereom: If $y_1(x)$ and $y_2(x)$ are any two solutions of (1) and if C_1 and C_2 are arbitraty constants, then $C_1 y_1(x) + C_2 y_2(x)$ is the general solution of (1).
Consider: $y' + ay = 0$, where a is a constant, we can derive:

$$
\begin{gathered}
y' + ay = 0 \\
y' = -ay \\
\frac{dy}{dx} = -ay \\
\frac{dy}{y} = -adx \\
\int \frac{dy}{y} = \int -adx \\
\ln y = -ax + C \\
y = Ae^{-ax},\ A = e^C
\end{gathered}
$$

Question: Whether exponentioal solutions exist for homogeneous linear second-order ODE's.
Solution of $ay'' + by' + cy = 0$ (1):
If we try a solution of the form $y = e^{,x},\ m \in \mathbb{R}$ we get $y' = me^{mx}$ and $y'' = m^2 e^{mx}$. Now we substitute $y' = me^{mx}$ and $y'' = m^2 e^{mx}$ into (1) to get $a(m^2 e^{mx}) + b(me^{mx}) + ce^{mx} = 0$ (2) or $e^{mx}(am^2 + bm + c) = 0$. We have $e^{mx} \neq 0$ for all values of x and m, so (2) is 0 when $am^2 + bm + c = 0$. This equation is called the auxillary equation of (1).
We can solve for m with the quadratic formula: $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
We now have three cases: $b^2 - 4ac > 0$, $b^2 - 4ac = 0$, and $b^2 - 4ac < 0$

1.1 Case 1

If $b^2 - 4ac > 0$, then equation (2) has two distinct real solutions m_1 and m_2. Corresponding, equation (1) has two basic solutions: $y_1(x) = e^{m_1 x}$ and $y_2(x) = e^{m_2 x}$. The complete solution of (1) is: $y(x) = C_1 e^{m_1 x} + C_2 e^{m_2 x}$.

1.1.1 Example 1

Given $y'' + 5y' + 6 = 0$, we have the equation $m^2 + 5m + 6 = 0$. We can solve for m.

$$
\begin{gathered}
m^2 + 5m + 6 = 0 \\
(m + 2)(m + 3) = 0 \\
m_1 = -2,\ m_2 = -3
\end{gathered}
$$

And now solving for y we find:

$$y(x) = C_1 e^{3x} + C_2 e^{-3x}$$

1.1.2 Example 2

Given $2y'' - 5y' - 3y = 0$, we have $2m^2 - 5m - 3 = 0$,

$$
\begin{gathered}
m = \frac{-(-5) \pm \sqrt{(-5)^2 - 4ac}}{2(2)} \\
m_1 = 3,\ m_2 = -\frac{1}{2} \\
y(x) = C_1 e^{3x} + C_2 e^{-\frac{x}{2}}
\end{gathered}
$$

1.2 Case 2

If $b^2 - 4ac = 0$, then we have one real soution of m. In this case, the basic soultions for (1) are: $y_1(x) = e^{mx},\ y_2(x)xe^{mx}$. The complete soultion is: $y(x) = C_1e^{mx} + C_2xe^{mx}$.

1.2.1 Example 3

Given $y'' + 6y' + 9y = 0$, we have $m^2 + 6m + 9 = 0$. Solving for m:

$$m^2 + 6m + 9 = 0$$

$$(m + 3)^2 = 0$$

$$m = 3$$

Solving for y we get:

$$y(x) = C_1e^{-3x} + C_2xe^{-3x}$$

1.2.2 Example 4

$$y'' - 10y' + 25y = 0''$$

we have:

$$m^2 - 10m + 25 = 0$$

$$(m - 5)^2 = 0$$

$$m = 5$$

Therefore,

$$y(x) = C_1e^{5x} + C_2xe^{5x}$$