

Homework 3 Eigenvalue problems

3 Deformaciones elásticas

Dada A en una deformación $y = Ax$, encuentra las direcciones principales y los correspondientes factores de extensión o contracción. Muestra los detalles de tu cálculo.

a) $A = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix}$

$$y = Ax$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} y_1 &= 2.0x_1 + 0.4x_2 \\ y_2 &= 0.4x_1 + 2.0x_2 \end{aligned}$$

> Let's solve the eigenvalue problem...

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$\begin{cases} 2.0x_1 + 0.4x_2 = \lambda x_1 \\ 0.4x_1 + 2.0x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (2.0 - \lambda)x_1 + 0.4x_2 = 0 \\ 0.4x_1 + (2.0 - \lambda)x_2 = 0 \end{cases}$$

so...

$$\det \begin{pmatrix} 2.0 - \lambda & 0.4 \\ 0.4 & 2.0 - \lambda \end{pmatrix} = 0 \quad \text{is the characteristic equation}$$

$$(2.0 - \lambda)^2 - 0.4^2 = 0$$

$$\lambda^2 - 4.0\lambda + 3.84 = 0$$

the eigenvalues are:

$$\lambda_1 = -\frac{8}{5}$$

for $\lambda = \lambda_1$, then

$$\begin{cases} 3.6x_1 + 0.4x_2 = 0 \\ 0.4x_1 + 3.6x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -\frac{12}{5} ; \text{ where } \lambda \text{ is the stretch factor}$$

for $\lambda = \lambda_2$, then

$$\begin{cases} 4.4x_1 + 0.4x_2 = 0 \\ 0.4x_1 + 4.4x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} ; \text{ where } x \text{ is the eigen vector}$$

> The eigenvalues show a contraction, with no direction

Note: The principal directions are the directions of the position vector x for which the direction of the position vector y is the same or exact opposite.

b) $A = \begin{pmatrix} 5 & 2 \\ 2 & 13 \end{pmatrix}$ The eigenvalues show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2 , respectively.

> Let's solve the eigen problem...

$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$\begin{cases} 5x_1 + 2x_2 = \lambda x_1 \\ 2x_1 + 13x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (5-\lambda)x_1 + 2x_2 = 0 \\ 2x_1 + (13-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det \begin{pmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{pmatrix} = 0$$

$$(5-\lambda)(13-\lambda) - 4 = 0$$

$$\lambda^2 - 18\lambda + 61 = 0$$

the eigenvalues are:

$$\lambda_1 = 9 + 2\sqrt{5}$$

$$\lambda_2 = 9 - 2\sqrt{5}$$

for $\lambda = \lambda_1$, then

for $\lambda = \lambda_2$

$$\begin{cases} (-4 - 2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4 - 2\sqrt{5})x_2 = 0 \end{cases}$$

$$\begin{cases} (-4 + 2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4 + 2\sqrt{5})x_2 = 0 \end{cases}$$

$$x_1 = \frac{-(4 - 2\sqrt{5})x_2}{2}$$

$$x_1 = \frac{-(4 + 2\sqrt{5})x_2}{2}$$

$$x_1 = (-2 + \sqrt{5})x_2$$

$$= (-2 - \sqrt{5})x_2$$

$$-2x_2 + 2x_2 = 0$$

$$-2x_2 + 2x_2 = 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-2 + \sqrt{5})x_2 \\ x_2 \end{pmatrix}$$

$$= t \begin{pmatrix} -2 + \sqrt{5} \\ 1 \end{pmatrix}$$

$$x = \begin{pmatrix} -(4 + 2\sqrt{5})x_2 \\ x_2 \end{pmatrix}$$

$$= t \begin{pmatrix} -4 - 2\sqrt{5} \\ 1 \end{pmatrix}$$

> Let's compute the principal directions

$$\theta_1 = \tan^{-1} \left(\frac{x_2}{x_1} \right)$$

$$= \tan^{-1} \left(\frac{1}{-2 + \sqrt{5}} \right)$$

$$= 76^\circ 43' 2.91'' \text{ with the positive } x_1 \text{ direction}$$

$$\theta_2 = \tan^{-1} \left(\frac{1}{-4 - 2\sqrt{5}} \right)$$

$= 173^\circ 16' 5.87''$ with the positive x_1 direction

c) $A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$

> Let's solve the eigen problem...

$$Ax = \lambda x$$

$$\begin{cases} 1.25x_1 + 0.75x_2 = \lambda x_1 \\ 0.75x_1 + 1.25x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (1.25 - \lambda)x_1 + 0.75x_2 = 0 \\ 0.75x_1 + (1.25 - \lambda)x_2 = 0 \end{cases}$$

so...

$$\det \begin{pmatrix} 1.25 - \lambda & 0.75 \\ 0.75 & 1.25 - \lambda \end{pmatrix} = 0$$

$$\lambda^2 - 2.5\lambda + 1 = 0$$

the eigen values are:

$$\lambda_1 = 2$$

for $\lambda = \lambda_1$, then

$$\begin{cases} -0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 - 0.75x_2 = 0 \end{cases}$$

$$x_1 = x_2$$

$$x = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 0.5$$

for $\lambda = \lambda_2$, then

$$\begin{cases} 0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 + 0.75x_2 = 0 \end{cases}$$

$$x_1 = -x_2$$

$$x = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

> Let's compute the principal directions.

$$\theta_1 = \tan^{-1}\left(\frac{1}{1}\right)$$

$= 45^\circ 0' 0''$ with the positive x_1 direction

$$\theta_2 = \tan^{-1}\left(\frac{1}{-1}\right)$$

$= 135^\circ 0' 0''$ with the positive x_1 direction

The eigen values show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2 , respectively.

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4 Modelos de población.

a) $L = \begin{pmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}$ Note: the Leslie model describes age-specified population growth.

Assumptions:

- i) Let the oldest age attained be 99 years.
- ii) Since the provided matrix L has 3 columns, divide the population into three age classes of 33 years each.

Note: Given a Leslie matrix $L = [l_{jk}]$,

- 1) l_{1k} is the average number of new borns to a single member during the time he/she is in age class k .
- 2) $l_{j,j-1}$ is the fraction of members in age class $j-1$ that will survive and pass into class j .

Note: Proportional change means that we are looking for a distribution vector x such that $Lx = \lambda x$, where λ is the rate of change (growth if $\lambda > 1$, decrease if $\lambda < 1$).

Encuentra el ritmo de crecimiento en el modelo de Leslie con la matriz dada.

▷ Let's find the λ 's

$$\det(L - \lambda I)x = 0$$

$$\det \begin{pmatrix} -\lambda & 9.0 & 5.0 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{pmatrix} = 0$$

$$-0.4 \det \begin{pmatrix} -\lambda & 5.0 \\ 0.4 & -\lambda \end{pmatrix} + (-\lambda) \det \begin{pmatrix} -\lambda & 9.0 \\ 0.4 & -\lambda \end{pmatrix} = 0$$

$$(-0.4)(\lambda^2 - 2) + (-\lambda)(\lambda^2 - 3.6) = 0$$

$$-0.4\lambda^2 + 0.8 - \lambda^3 + 3.6\lambda = 0$$

$$-\lambda^3 - 0.4\lambda^2 + 3.6\lambda + 0.8 = 0$$

$$\lambda_1 = 1.82, \quad \lambda_2 = -0.23, \quad \lambda_3 = -2$$

▷ A positive root is found to be $\lambda = 1.82$

The growth rate will be 1.82 per 33 years.

b) $L = \begin{pmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}$

Assumptions:

- i) Let the oldest age attained be 100 years
- ii) Since the provided matrix L has four columns, let's divide the population into four age classes of 25 years each.

▷ let's find the λ 's

$$\det(L - \lambda I)x = 0$$

$$\begin{aligned} \det \begin{pmatrix} -\lambda & 3.0 & 2.0 & 2.0 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{pmatrix} &= +(-\lambda) \begin{pmatrix} -\lambda & 0 & 0 \\ 0.5 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix} \\ &\quad - (3.0) \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix} \\ &\quad + (2.0) \begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -\lambda \end{pmatrix} \\ &\quad - (2.0) \begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \\ 0 & 0 & 0.1 \end{pmatrix} \\ &= -\lambda \left[+(-\lambda) \begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0 \right] \\ &\quad - 3.0 \left[+(0.5) \begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0 \right] \\ &\quad + 2.0 \left[+(0.5) \begin{pmatrix} 0.5 & 0 \\ 0 & -\lambda \end{pmatrix} - (-\lambda) \begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix} + 0 \right] \\ &\quad - 2.0 \left[+(0.5) \begin{pmatrix} 0.5 & -\lambda \\ 0 & 0.1 \end{pmatrix} - (-\lambda) \begin{pmatrix} 0 & -\lambda \\ 0 & 0.1 \end{pmatrix} + 0 \right] \\ &= -\lambda(-\lambda(\lambda^2)) - 3.0(0.5(\lambda^2)) + 2.0(0.5(0.5\lambda)) \\ &\quad - 2.0(0.5(0.5(0.1))) \\ &= \lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05 \end{aligned}$$

$$\lambda_1 = 1.37 \quad \lambda_2 = -1.03 \quad \lambda_3 = -0.17 + 0.07i \quad \lambda_4 = -0.17 - 0.07i$$

▷ A positive root is found to be $\lambda = 1.37$

The growth rate will be 1.37 per 25 years.