Definition (multiplication of matrices). Let $A = [a_{ij}]$ be an $n \times r$ matrix and $B = [b_{ij}]$ be an $r \times m$ matrix. Then the matrix product AB is the $n \times m$ matrix whose i, j element is:

$$\sum_{k=1}^{r} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{ir} b_{rj}$$

Remark 1. The product AB is defined if and only if the number of columns of A is equal to the number of rows of B.

Remark 2. Even though AB may be defined, BA need not to be. For example, suppose A has size 2×3 and B has size 3×5. Then AB is defined (it has size 2×5) while BA is not. (why?)

Some examples

Suppose $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$.

1. Compute AB.

   Solution:

$$AB = \begin{bmatrix} 1 \cdot 1 + 3 \cdot 2 & 1 \cdot 1 + 3 \cdot 1 & 1 \cdot 3 + 3 \cdot 4 \\ 2 \cdot 1 + 5 \cdot 2 & 2 \cdot 1 + 5 \cdot 1 & 2 \cdot 3 + 5 \cdot 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 & 15 \\ 12 & 7 & 26 \end{bmatrix}$$

   Therefore

$$\boxed{AB = \begin{bmatrix} 7 & 4 & 15 \\ 12 & 7 & 26 \end{bmatrix}}$$

2. What about BA?

Motivation behind the product of matrices

The motivation comes from the theory of systems of equations. Consider the system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m = b_2 \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m = b_n \end{cases}$$

Define $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$; $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

Now we compute AX:

$$AX = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

so we can write the original system in matrix form as $AX = B$. **Thus the task reduces to finding the matrix X given A and B. We will soon learn matrix methods to solve systems $AX = B$!**

Zero matrix and the identity matrix

Definition (zero matrix). The zero matrix of size $n \times m$ is a matrix whose entries are all equal to 0. The zero matrix of size $n \times m$, where n and m are any positive integers, is denoted by $O_{n,m}$.

Examples. $O_{2,2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $O_{3,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Remark. If we multiply any matrix by the zero matrix (provided the product is defined) we obtain the zero matrix.

Definition (identity matrix). Let n be any positive integer. The identity matrix of size n is a square matrix with ones on the main diagonal and zeros elsewhere. It is denoted by I_n.

Example. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $I_6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Remark. If A is any matrix of size $m \times n$, then $I_m A = A I_m = A$.

Powers of a matrix and diagonal matrices

Definition (power of a matrix). The power A^n of a matrix A, for n a positive integer, is defined as the matrix product of n copies of A:

$$A^n = \underbrace{A \cdot A \cdot A \cdots A}_{n\text{-times}}$$

Definition (diagonal matrices). A diagonal matrix is a square matrix in which the entries outside the main diagonal are all zero.

Examples. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \pi^2 \end{bmatrix}$; $\begin{bmatrix} sinx & 0 & 0 & 0 \\ 0 & cosx & 0 & 0 \\ 0 & 0 & \frac{1}{x^2+1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Problems

1. Compute (by hand) the product AB where $A = \begin{bmatrix} \frac{1}{2} & 4 \\ 5 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 & 7 & -1 \\ -1 & 1 & 16 & \sqrt{2} \end{bmatrix}$; then verify your answer with Mathematica.
2. Let $C = \begin{bmatrix} 11 & 3 \\ -21 & 42 \end{bmatrix}$. Compute C^2 (by hand) and verify your answer with Mathematica.
3. (Tricky! ☺) Think how to use the *ConstantArray* command to generate the following matrix P:

$$P = \begin{bmatrix} \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \\ \pi & \pi & \pi & \pi \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 1 & 5 \\ -\frac{1}{3} & 6 \end{bmatrix}$ and let $C = A^8$. Find c_{21}.