Gaussian elimination and the reduced row echelon form

Recall that the main motivation of the row echelon form is to **solve linear systems.**

There are two important matrices associated with a linear system:

1. The **coefficient** matrix: this matrix contains the coefficients of the variables.
2. The **augmented** matrix: it consists of the coefficient matrix augmented by an extra column containing the constant terms.

Example. Consider the system

$$\begin{cases} 2x + y - z = 3 \\ x + 5z = 1 \\ -x + 3y - 2z = 0 \end{cases}$$

Then the coefficient matrix is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 5 \\ -1 & 3 & -2 \end{bmatrix}$$

and the augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 0 & 5 & 1 \\ -1 & 3 & -2 & 0 \end{array}\right]$$

When row reduction is applied to the augmented matrix of a system of linear equations, we create an equivalent matrix of a system of linear equations which can be solved by back substitution. This is the idea of the next procedure.

Definition (Gaussian elimination). This is a **3-step** process:

1. Write the augmented matrix of the system of linear equations.
2. Reduce the augmented matrix to row echelon form.
3. Use back substitution to solve the system.

Idea: simply find the row echelon form of the augmented matrix and solve the system by going backwards!

Example 1. Solve the system

$$\begin{cases} 2x_2 + 3x_3 = 8 \\ 2x_1 + 3x_2 + x_3 = 5 \\ x_1 - x_2 - 2x_3 = -5 \end{cases}$$

Step 1 . **Write the augmented matrix**

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 8 \\ 2 & 3 & 1 & 5 \\ 1 & -1 & -2 & -5 \end{array}\right]$$

Step 2 . **Reduce the whole matrix to row echelon form**. First we can interchange rows 1 and 3 (so we can have 1 as a leading entry of the first row); hence we apply $R_1 \leftrightarrow R_3$. This gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 2 & 3 & 1 & 5 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

Now we apply $R_2 - 2R_1 \to R_2$.

We obtain the matrix

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 5 & 5 & 15 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

Now the leading entry of the second row is 5, but it is suggested to work with ones instead. Applying $\frac{1}{5}R_2$ gives

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 2 & 3 & 8 \end{array}\right]$$

Finally, in place of 2 we must have 0, so we apply $R_3 - 2R_2 \to R_3$. We obtain

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Notice that the above matrix is now in row echelon form.

Step 3 . **Use back substitution**. The row echelon form of the previous matrix is

$$\begin{array}{cccc} x & y & z & \text{constants} \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

1. The last equation gives $z = 2$.
2. The second equation gives $y + z = 3$. Since $z = 2$ then we obtain $y + 2 = 3$ and therefore $y = 1$.
3. Finally, the first equation implies that $x - y - 2z = -5$. Substituting $y = 1$ and $z = 2$ gives $x - 1 - 4 = -5$ and hence $x = 0$.

Conclusion. The solution is given by the vector $\boxed{(x, y, z) = (0, 1, 2)}$

Example 2. Solve the system

$$\begin{cases} w - x - y + 2z = 1 \\ 2w - 2x - y + 3z = 3 \\ -w + x - y = -3 \end{cases}$$

Step 1 . **Write the augmented matrix**

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right]$$

Step 2 . **Reduce to row echelon form (exercise!)**. One choice of a row echelon form is

$$\begin{array}{cccc|c} w & x & y & z & \textit{constants} \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

From here we get $y - z = 1$ and $w - x - y + 2z = 1$. How many solutions exist?

Answer: infinitely many!

We solve for the variables corresponding to the **leading entries** (from the bottom to top).

1. From $y - z = 1$ we obtain $y = 1 + z$.
2. From $w - x - y + 2z = 1$ we obtain $w = x + y + 1 - 2z$.

The variables which are **not** equal to the leading variables are called free variables (why?).

In this case, the **free variables are** x **and** z. Since they can be free we set $x = s$ and $z = t$ where $s, t \in \mathbb{R}$.

It follows that $y = 1 + t$ and $w = x + y + 1 - 2z = s + 1 + t + 1 - 2t = s + 2 - t$.

Therefore the general solution is:

$$\boxed{(w, x, y, z) = (t, s + 2 - t, s, t + 1)} \text{ where } s, t \in \mathbb{R}.$$

Example 3. Solve the following system

$$\begin{cases} x - 3y + z = 4 \\ -x + 2y - 5z = 3 \\ 5x - 13y + 13z = 8 \end{cases}$$

A little algebra shows that the row echelon form (**exercise!**) is

$$\left[\begin{array}{ccc|c} x & y & z & constants \\ 1 & -3 & 1 & 4 \\ 0 & -1 & -4 & 7 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

The last equation gives $0 \cdot z = 2$, which implies $0 = 2$, an absurd!.

Conclusion. There is **no** solution and we say that the system is **inconsistent**.

Definition (reduced row echelon form). A matrix is in reduced row echelon form if it satisfies the following properties:

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1.
3. Each column containing a leading 1 has zeros everywhere else

Example. Explain why the following matrix is in reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise. Are the following matrices in reduced row echelon form?

1. $$\begin{bmatrix} -1 & 4 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Answers ☺

1. No
2. No
3. Yes

Some comments.

1. A well known theorem in linear algebra states that the reduced row echelon form is **unique** (the proof is not trivial). If you are curious you can read the following paper by Thomas Yuster: `https://www.maa.org/sites/default/files/Yuster19807.pdf`.
2. Therefore it makes sense to talk about **the** reduced row echelon form of a matrix.
3. One can compute in Mathematica the reduced row echelon form by using the command *RowReduce*.
4. In Mathematica, one can test mathematical equality by using double $=$ sign and the output will be Boolean: True or False.

Question. Using this, how can you check in Mathematica whether a given matrix is in reduced row echelon form or not?

1. Consider the following system:

$$\begin{cases} 2x_2 + 4x_3 = 2 \\ 2x_1 + 4x_2 + 2x_3 = 3 \\ 3x_1 + 3x_2 + x_3 = 1 \end{cases}$$

a. Find, **by hand**, the reduced row echelon form of the augmented matrix.
b. Verify the above answer in Mathematica.
c. Use the reduced row echelon form of the augmented matrix to find the solution.
d. Use *Solve* to verify your answer.

2. Let A be an nxm matrix and let A_{red} be the reduced row echelon form of A. Suppose that $\text{rank}(A) = n$. Is A_{red} a familiar matrix?

3. Consider the following matrix

$$A = \begin{bmatrix} \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi^2 \\ \pi^3 & \pi^3 & \pi^3 \end{bmatrix}$$

1. Find the reduced row echelon form of A; then find the rank of A.
2. (Optional) How can you enter in Mathematica (in one line) the matrix A? (without typing every entry!). *Hint*: Consider the *Table* command.
3. Now generalize the result as follows: let X be the following arbitrary square matrix of size n, where c is any nonzero number. Compute the rank of X.

$$X = \begin{bmatrix} c & c & \cdots & c \\ c^2 & c^2 & \cdots & c^2 \\ \vdots & \vdots & \vdots & \vdots \\ c^n & c^n & \cdots & c^n \end{bmatrix}$$

4. (Harder ☺) For what values(s) of k, if any, will the following system:

$$\begin{cases} x + y + kz = 1 \\ x + ky + z = 1 \\ kx + y + z = -2 \end{cases}$$

have

1. No solution.
2. A unique solution.
3. Infinitely many solutions.

Hint: Find the reduced row echelon form of the augmented matrix, then analyze different cases (beware of division by zero!).