Computación Aplicada - Homework 04
Simulation – Basics & Integrals

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February 25, 2019

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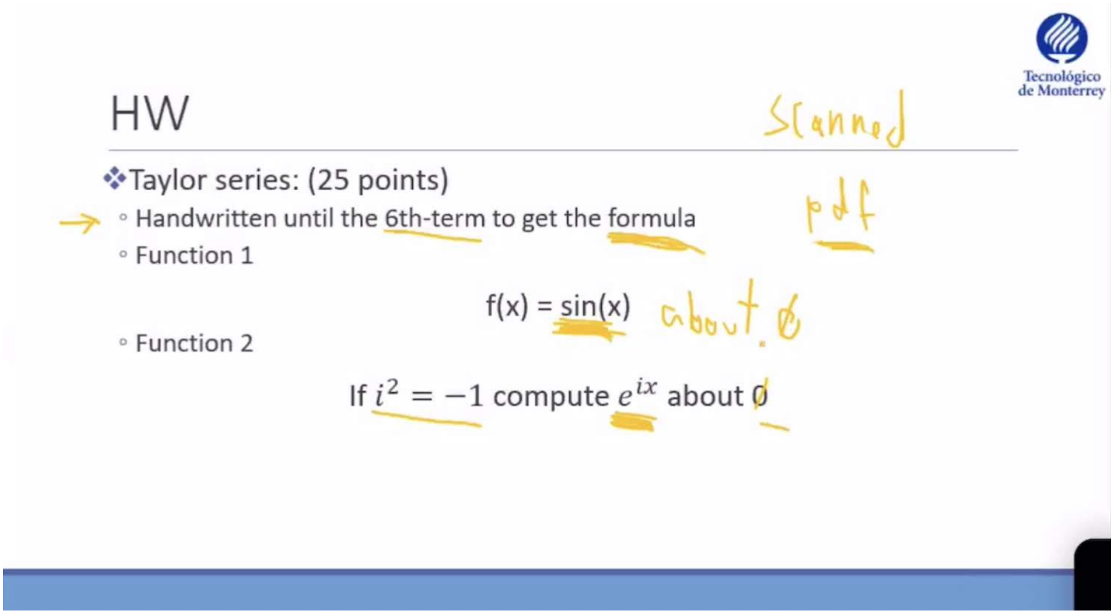

Figure 1: Problem 1 instructions.

1.1 Function 1

Figure 2 shows the “hadwritten” procedure to find the first six terms of the Taylor series of $f(x) = Sin(x)$, centered at 0.

I $\boxed{1}$ $f(x) = Sin(x)$; about 0

$$T(f(x)) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \quad ; \text{ where } \begin{array}{l} c = 0 \\ n = 6 \end{array}$$

$$= f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4 + \frac{1}{5!}f^{(5)}(0)x^5 + \frac{1}{6!}f^{(6)}(0)x^6$$

$$= \cancel{Sin(0)}^{0} + \cancel{Cos(0)}^{1}x - \frac{1}{2}\cancel{Sin(0)}^{0}x^2 - \frac{1}{6}\cancel{Cos(0)}^{1}x^3 + \frac{1}{24}\cancel{Sin(0)}^{0}x^4 + \frac{1}{120}\cancel{Cos(0)}^{1}x^5 - \frac{1}{720}\cancel{Sin(0)}^{0}x^6$$

$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \quad ///$$

Figure 2: Taylor series until the 6th term of Function 1.

1.2 Function 2

Figure 3 shows the “hadwritten” procedure to find the first six terms of the Taylor series of $f(x) = e^{ix}$, centered at 1.

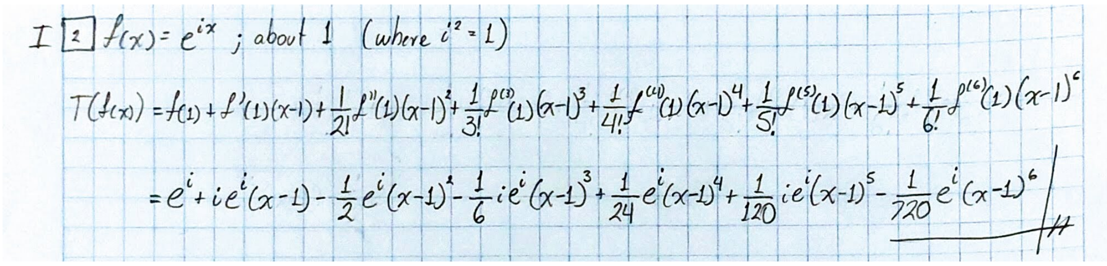

Figure 3: Taylor series until the 6th term of Function 2.

1.3 Let’s plot the approximations

In Listing 1, function *taylorPlot* is implemented to plot the Taylor approximations up to the 2nd, 4th, 6th and 8th terms. With the following properties:

Parameters:

- **f** : function
  Vectorized function of one variable
- **c** : numeric
  point where the series expansion will take place
- **from**, **to** : numeric
  Interval of points to be ploted

Returns:

- **void**

Listing 1: ”The *taylorPlot* function”

```
library(pracma)

taylorPlot <- function(f, c, from, to) {
  x <- seq(from, to, length.out = 100)
  yf <- f(x)

  yp2 <- polyval(taylor(f, c, 2), x)
  yp4 <- polyval(taylor(f, c, 4), x)
  yp6 <- polyval(taylor(f, c, 6), x)
  yp8 <- polyval(taylor(f, c, 8), x)

  plot(
    x,
    yf,
    xlab = "x",
    ylab = "f(x)",
```

```
    type = "l",
    main = ' Taylor Series Approximation of f(x) ',
    col = "black",
    lwd = 2
  )

  lines(x, yp2, col = "#c8e6c9")
  lines(x, yp4, col = "#81c784")
  lines(x, yp6, col = "#4caf50")
  lines(x, yp8, col = "#388e3c")

  legend(
    'topleft',
    inset = .05,
    legend = c("TS 8 terms", "TS 6 terms", "TS 4 terms", "TS 2 terms", "f(x)"),
    col = c('#388e3c', '#4caf50', '#81c784', '#c8e6c9', 'black'),
    lwd = c(1),
    bty = 'n',
    cex = .75
  )
}
```

$f(x) = Sin(x)$ is defined as *f0* and $f(x) = e^{ix}$ is defined as *f1* in Listing 2

Listing 2: "Define *f0* and *f1*"

```
f0 <- function(x) {
  res = sin(x)

  return(res)

}

f1 <- function(x) {
  res = exp(complex(real = 0, imaginary = 1)*x)

  return(res)

}
```

Listing 3 shows the use of function *taylorPlot* to plot the Taylor approximations of *f0* and *f1*. This Listing output-plots are represented within Figures 4 and 5

Listing 3: "Implement *taylorPlot*"

```
taylorPlot(f0, 0, -6.6, 6.6)
taylorPlot(f1, 1, -2*pi, 2*pi)
```

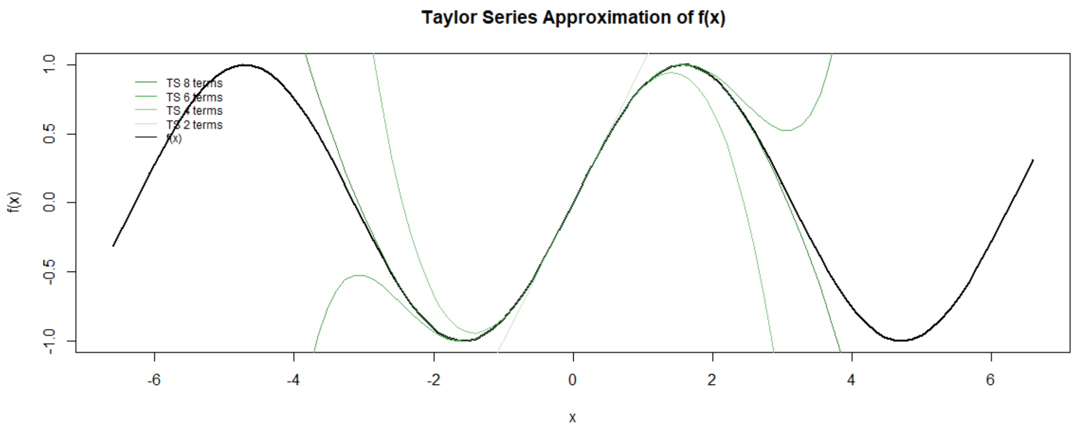

Figure 4: Listing 3 output; $f(x) = Sin(x)$ approximations, centered in 0.

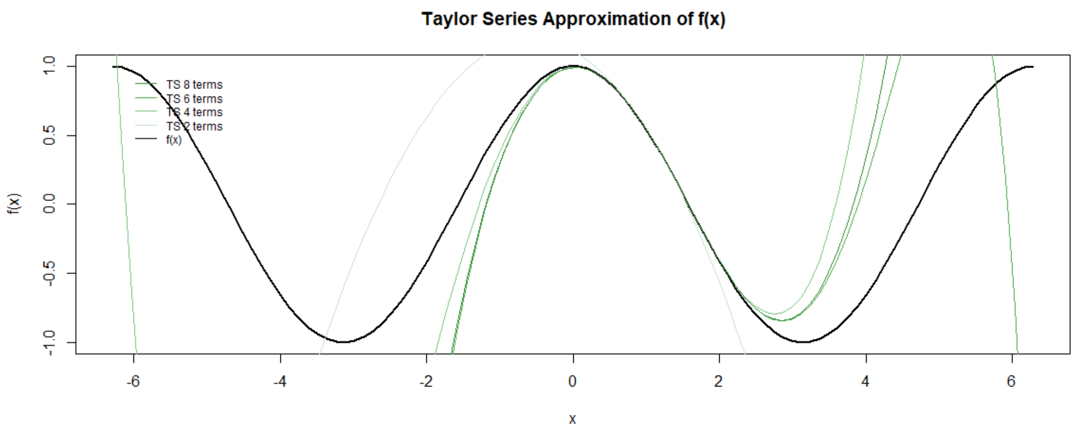

Figure 5: Listing 3 output; $f(x) = e^{ix}$ approximations, centered in 1.

2 Problem II

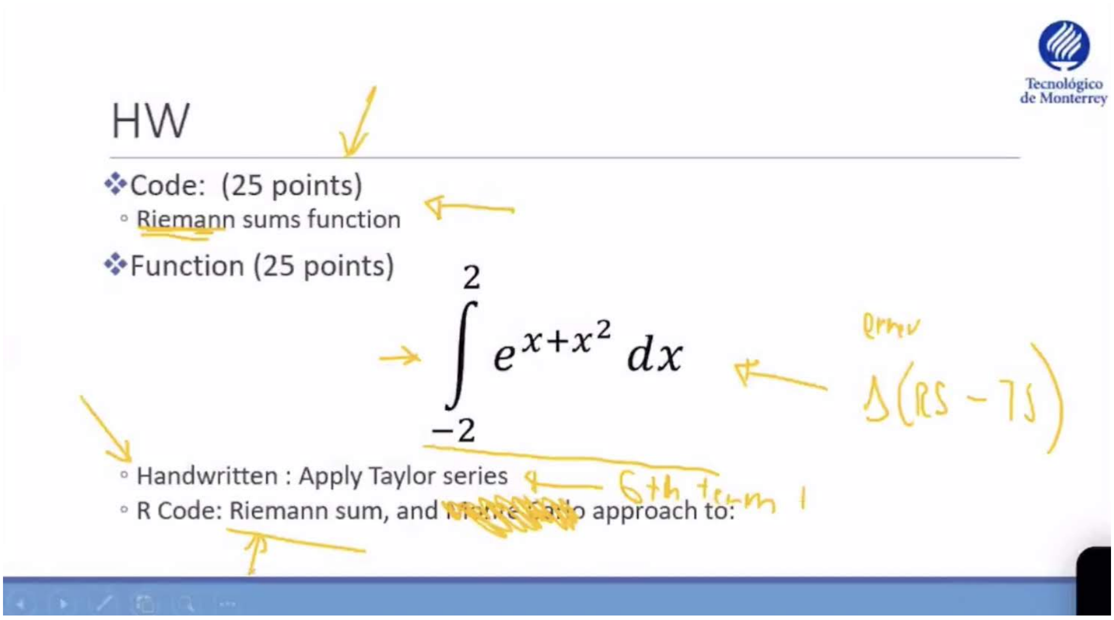

Figure 6: Problem 2 instructions.

In Listing 4, function *riemann_sum* is implemented to compute the Riemann sum of a given function $f(x)$ over an interval $[a, b]$. With the following properties:

Parameters:

- **f** : function
  Vectorized function of one variable
- **a**, **b** : numeric
  Endpoints of the interval [a,b]
- **n** : numeric
  Number of subintervals of equal length in the partition of [a,b]

Returns:

- numeric
  Underestimate and overestimate approximations of the integral given by the Riemann sum.

Listing 4: "The *Riemann* function"

```
riemann_sum <- function(f, a, b, n) {
  # initialize values
  lower.sum <- 0

  upper.sum <- 0

  h <- (b - a) / n
```

```
  # riemann right sum
  for (i in n:1) {
    x <- a + i * h

    lower.sum <- lower.sum + f(x)

  }
  lower.sum <- h * lower.sum


  # riemann left sum
  for (i in 1:n) {
    x <- b - i * h

    upper.sum <- upper.sum + f(x)

  }
  upper.sum <- h * upper.sum


  # let's plot the curve
  integralPlot(
    f = f,
    a = a,
    b = b,
    title = expression(f(x))
  )

  # print/get riemann sum
  cat(sprintf(
    "The true value is between %f and %f.\n",
    as.double(lower.sum),
    as.double(upper.sum)
  ))

  return(c(lower.sum, upper.sum))

}
```

Let's solve the following integral using *riemann_sum*. Listing 5 shows the required commands.

$$\int_{-2}^{2} e^{x+x^2}\ dx$$

Listing 5: "Define the function and implement *riemann_sum*"

```
f4 <- function(x) {
  res = exp(x + x ^ 2)

  return(res)

}

# compute riemann_sum for f4
riemann_sum(f4, -2, 2, 100000)

# let's verify our calualtion using R's function
```

```
integrate(f4, lower = -2, upper = 2)
```

Listing 6: "Listing 5 output"

```
> # compute riemann_sum for f4
> riemann_sum(f4, -2, 2, 100000)
The true value is between 93.170674 and 93.154833.
[1] 93.17067 93.15483

> # let's verify our calualtion using R's function
> integrate(f4, lower = -2, upper = 2)
93.16275 with absolute error < 0.00062
```

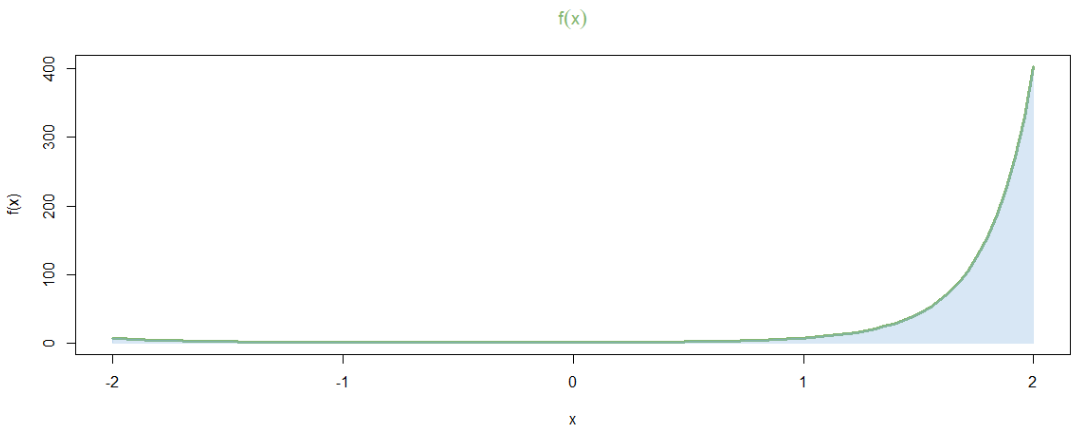

Figure 7: Listing 5 output; $\int_{-2}^{2} e^{x+x^2}\ dx$.

* Appendix A implements the R function to generate plots similar to Figure 7 (, which is used within *riemann_sum*).

3 Problem III

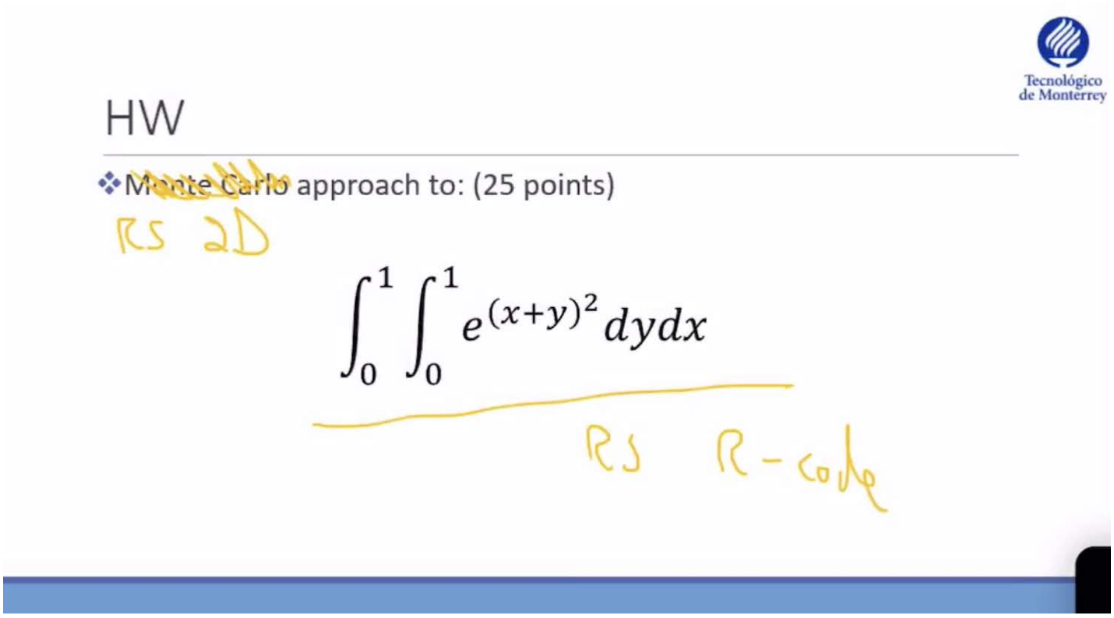

Figure 8: Problem 3 instructions.

In Listing 7, function *riemann_sum_2d* is implemented to compute the Riemann sum of a given function $f(x, y)$ over the intervals $[a, b]$ and $[c, d]$. With the following properties:

Parameters:

- **f** : function
  Vectorized function of one variable
- **a**, **b** : numeric
  Endpoints of the interval [a,b] (inner integral)
- **c**, **d** : numeric
  Endpoints of the interval [c,d] (outer integral)
- **nx** : numeric
  Number of subintervals of equal length in the partition of [a,b]
- **ny** : numeric
  Number of subintervals of equal length in the partition of [c,d]

Returns:

- numeric
  Approximations of the integral given by the Riemann 2D sum.

Listing 7: "The *Riemann 2D* function"

```
riemann_sum_2d <- function(f, a, b, c, d, nx, ny) {
  # initialize values
  dx = (b - a) / nx
  s = 0.0
  x = a

  dy = (d - c) / ny
  y = c

  # riemann 2D sum
  for (i in 1:nx) {
    for (j in 1:ny) {
      x = a + dx / 2 + i * dx
      y = c + dy / 2 + j * dy
      f_i = f(x, y)
      s = s + f_i * dx * dy
    }
  }

  # print/get riemann sum
  cat(sprintf("The␣true␣value␣is␣around␣%f.\n",
              as.double(s)))

  return(s)

}
```

Let's solve the following integral using *riemann_sum_2d*. Listing 8 shows the required commands.

$$\int_0^1 \int_0^1 e^{(x+y)^2} \; dy \; dx$$

Listing 8: "Define the function and implement *riemann_sum_2d*"

```
f5 <- function(x, y) {
  res = exp((x + y) ^ 2)

  return(res)

}

# compute riemann_sum_2d for f5
riemann_sum_2d(f5, 0, 1, 0, 1, 1000, 1000)

# let's verify our calualtion using R's function
integral2(f5, 0,1, 0,1)
```

Listing 9: "Listing 8 output"

```
> # compute riemann_sum_2d for f5
> riemann_sum_2d(f5, 0, 1, 0, 1, 1000, 1000)
The true value is around 4.926310.
[1] 4.92631

> # let's verify our calualtion using R's function
> integral2(f5, 0,1, 0,1)
$Q
[1] 4.899159
$error
[1] 9.974762e-16
```

4 Problem IV

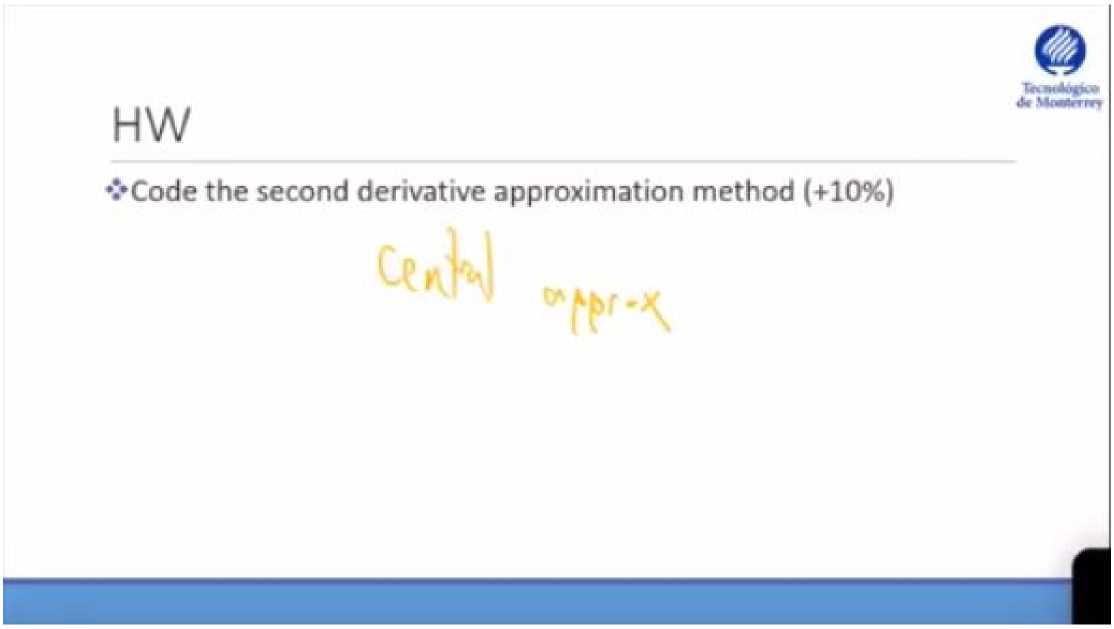

Figure 9: Problem 4 instructions.

In Listing 10, function *derivative* is implemented to compute the derivative of a function.. With the following properties:

Parameters:

- **f** : function $f(x)$
  Vectorized function of one variable
- **h** : numeric
  Let x now change by an amount h. h is the variable that approaches 0

Returns:

- function
  Approximation of the derivative of f, given a step h.

Listing 10: "The *Riemann 2D* function"

```
derivative <- function(f, h) {
  return(function(x) {
    (f(x + h) - f(x)) / (h)
  })
}
```

Let's solve the following double derivative to test our function using *derivative*. Listing 11 shows the required commands.

$$\frac{d^2}{dy^2}x^4Sin(x)$$

Listing 11: "Define the function and implement *derivative*"

```
f6 <- function(x) {
  res = x^4*sin(x)

  return(res)

}

# df1 = d/dx(x^4 sin(x))
#     = x^3 (4 sin(x) + x cos(x))
df1 <- derivative(f6, 0.0001)

# df2 = d/dx(x^3 (4 sin(x) + x cos(x)))
#     = x^2 (8 x cos(x) - (x^2 - 12) sin(x))
df2 <- derivative(df1, 0.0001)

# Let's evaluate x=pi in the second derivative df2
df2(pi)

# Let's use the eval() funtion to verify our solution. The value should
    be around df2(pi)
eval({ x <- pi; x^2*(8*x*cos(x) - (x^2 - 12)*sin(x))})
```

Listing 12: "Listing 11 output"

```
> # Let's evaluate x=pi in the second derivative df2
> df2(pi)
[1] -248.076
>
> # Let's use the eval() funtion to verify our solution. The value should
    be around df2(pi)
> eval({ x <- pi; x^2*(8*x*cos(x) - (x^2 - 12)*sin(x))})
[1] -248.0502
```

A *integralPlot* function

```r
# Plotting the Areas under Curves ######################################
integralPlot <- function(f,
                         a,
                         b,
                         from = a,
                         to = b,
                         title = NULL) {
  # Plot the area under a function over the interval [a,b] between [from, to].
  #
  # Parameters
  # ----------
  # f : function
  # funtion to be ploted
  # a , b : numeric
  # Endpoints of the integral interval [a, b]
  # from , to : numeric (optional)
  # Endpoints of the plot in the x-axis [x min, x max]
  # title : expression
  # title of the plot
  #
  # Returns
  # -------
  # void

  x <- seq(from, to, length.out = 100) # input continuum
  y <- f(x) # output

  # plot the curve
  plot(
    x,
    y,
    xlim = c(from, to),
    ylim = c(ifelse(min(y) < 0, min(y), 0), max(y)),
    xlab = "x",
    ylab = "f(x)",
    main = title,
    col.main = "#86B875",
    type = "l",
    lwd = 3,
    col = "#86B875"
  )

  # area under the curve
  x <- seq(a, b, length.out = 100)
  y <- f(x)
  polygon(
    c(x, b, a, a),
    c(y, 0, 0, f(a)),
    border = adjustcolor("#7DB0DD", alpha.f = 0.3),
    col = adjustcolor("#7DB0DD", alpha.f = 0.3)
  )
}
```

B Full R Script

```r
#***********************************************************************
#* AUTHOR(S) :
#*      Bruno Gonzalez Soria          (A01169284)
#*      Antonio Osamu Katagiri Tanaka (A01212611)
#*
#* FILENAME :
#*      Homework4.R
#*
#* DESCRIPTION :
#*      Simulations (Ene 19 Gpo 1)
#*      Homework 4
#*
#* NOTES :
#*      - https://www.math.ubc.ca/~pwalls/math-python/integration/riemann-sums/
#*      - https://activecalculus.org/multi/S-11-1-Double-Integrals-Rectangles.html
#*      - http://math.colgate.edu/faculty/valente/math113/supplements/section151handout.pdf
#*      - http://hplgit.github.io/Programming-for-Computations/pub/p4c/p4c-sphinx-Python/._pylight004.html
#*      - https://rstudio-pubs-static.s3.amazonaws.com/131664_1858eec97df54c9b8d5edcd8b22e5818.html
#*
#* START DATE :
#*      21 Feb 2019
#***********************************************************************

# Install required libraries
#install.packages('pracma', dependencies=TRUE);

# Plotting the Areas under Curves ######################################
integralPlot <- function(f,
                         a,
                         b,
                         from = a,
                         to = b,
                         title = NULL) {
  # Plot the area under a function over the interval [a,b] between [from, to].
  #
  # Parameters
  # ----------
  # f : function
  # funtion to be ploted
  # a , b : numeric
  # Endpoints of the integral interval [a, b]
  # from , to : numeric (optional)
  # Endpoints of the plot in the x-axis [x min, x max]
  # title : expression
  # title of the plot
  #
  # Returns
  # -------
  # void

  x <- seq(from, to, length.out = 100) # input continuum
```

```r
  y <- f(x) # output

  # plot the curve
  plot(
    x,
    y,
    xlim = c(from, to),
    ylim = c(ifelse(min(y) < 0, min(y), 0), max(y)),
    xlab = "x",
    ylab = "f(x)",
    main = title,
    col.main = "#86B875",
    type = "l",
    lwd = 3,
    col = "#86B875"
  )

  # area under the curve
  x <- seq(a, b, length.out = 100)
  y <- f(x)
  polygon(
    c(x, b, a, a),
    c(y, 0, 0, f(a)),
    border = adjustcolor("#7DB0DD", alpha.f = 0.3),
    col = adjustcolor("#7DB0DD", alpha.f = 0.3)
  )
}

##########################################################################
# PART 1 #################################################################
# TAYLOR SERIES

library(pracma)

taylorPlot <- function(f, c, from, to) {
  # Plot the Taylor approximations up to the 2nd, 4th, 6th and 8th terms
  #
  # Parameters
  # ----------
  # f : function
  # Vectorized function of one variable
  # c : numeric
  # point where the series expansion will take place
  # from, to : numeric
  # Interval of points to be ploted
  #
  # Returns
  # -------
  # void

  x <- seq(from, to, length.out = 100)
  yf <- f(x)

  yp2 <- polyval(taylor(f, c, 2), x)
  yp4 <- polyval(taylor(f, c, 4), x)
  yp6 <- polyval(taylor(f, c, 6), x)
  yp8 <- polyval(taylor(f, c, 8), x)

  plot(
    x,
```

```
    yf,
    xlab = "x",
    ylab = "f(x)",
    type = "l",
    main = ' Taylor Series Approximation of f(x) ',
    col = "black",
    lwd = 2
  )

  lines(x, yp2, col = "#c8e6c9")
  lines(x, yp4, col = "#81c784")
  lines(x, yp6, col = "#4caf50")
  lines(x, yp8, col = "#388e3c")

  legend(
    'topleft',
    inset = .05,
    legend = c("TS 8 terms", "TS 6 terms", "TS 4 terms", "TS 2 terms", "f(x)"),
    col = c('#388e3c', '#4caf50', '#81c784', '#c8e6c9', 'black'),
    lwd = c(1),
    bty = 'n',
    cex = .75
  )
}

# -----

f0 <- function(x) {
  res = sin(x)

  return(res)

}

f1 <- function(x) {
  res = exp(complex(real = 0, imaginary = 1)*x)

  return(res)

}

# -----

taylorPlot(f0, 0, -6.6, 6.6)
taylorPlot(f1, 1, -2*pi, 2*pi)


###########################################################################
# PART 2 ##################################################################
# RIEMANN SUMS FUNCTION

riemann_sum <- function(f, a, b, n) {
  # Compute the Riemann sum of f(x) over the interval [a,b].
  #
  # Parameters
  # ----------
  # f : function
  # Vectorized function of one variable
  # a , b : numeric
```

```
  # Endpoints of the interval [a,b]
  # n : numeric
  # Number of subintervals of equal length in the partition of [a,b]
  #
  # Returns
  # -------
  # numeric
  # Underestimate and overestimate approximations of the integral given
    by the
  # Riemann sum.

  # initialize values
  lower.sum <- 0

  upper.sum <- 0

  h <- (b - a) / n


  # riemann right sum
  for (i in n:1) {
    x <- a + i * h

    lower.sum <- lower.sum + f(x)

  }
  lower.sum <- h * lower.sum


  # riemann left sum
  for (i in 1:n) {
    x <- b - i * h

    upper.sum <- upper.sum + f(x)

  }
  upper.sum <- h * upper.sum


  # let's plot the curve
  integralPlot(
    f = f,
    a = a,
    b = b,
    title = expression(f(x))
  )

  # print/get riemann sum
  cat(sprintf(
    "The true value is between %f and %f.\n",
    as.double(lower.sum),
    as.double(upper.sum)
  ))

  return(c(lower.sum, upper.sum))

}

# -----

```

```r
# let's generate some functions to test our algorithm
f2 <- function(x) {
  res = x

  return(res)

}

f3 <- function(x) {
  res = 4 / (1 + x ^ 2)

  return(res)

}

# -----

riemann_sum(f0, 0, pi / 2, 10)
riemann_sum(f2, 0, 1, 10000) # should be 0.5
riemann_sum(f3, 0, 1, 10000) # should be PI


######################################################################
# PART 3 #############################################################
# Integrate the function f(x)=exp(x+x^2) from -2 to 2, using Rieman sums.
f4 <- function(x) {
  res = exp(x + x ^ 2)

  return(res)

}

# plot Taylor approximations
taylorPlot(f4, 0, -2.3, 1.3)

# compute riemann_sum for f4
riemann_sum(f4, -2, 2, 100000)
# let's verify our calualtion using R's function
integrate(f4, lower = -2, upper = 2)


######################################################################
# PART 4 #############################################################
# RIEMANN SUMS 2D FUNCTION

riemann_sum_2d <- function(f, a, b, c, d, nx, ny) {
  # Compute the Riemann sum of f(x,y) over the intervals [a,b] and [c,d].
  #
  # Parameters
  # ----------
  # f : function
  # Vectorized function of one variable
  # a , b : numeric
  # Endpoints of the interval [a,b] (inner integral)
  # c , d : numeric
  # Endpoints of the interval [c,d] (outer integral)
  # nx : numeric
  # Number of subintervals of equal length in the partition of [a,b]
  # ny : numeric
  # Number of subintervals of equal length in the partition of [c,d]
```

```
  #
  # Returns
  # -------
  # numeric
  # Approximations of the integral given by the Riemann 2D sum.

  # initialize values
  dx = (b - a) / nx
  s = 0.0
  x = a

  dy = (d - c) / ny
  y = c

  # riemann 2D sum
  for (i in 1:nx) {
    for (j in 1:ny) {
      x = a + dx / 2 + i * dx
      y = c + dy / 2 + j * dy
      f_i = f(x, y)
      s = s + f_i * dx * dy
    }
  }

  # print/get riemann sum
  cat(sprintf("The␣true␣value␣is␣around␣%f.\n",
              as.double(s)))

  return(s)

}

# -----

f5 <- function(x, y) {
  res = exp((x + y) ^ 2)

  return(res)

}

# -----

# compute riemann_sum_2d for f5
riemann_sum_2d(f5, 0, 1, 0, 1, 1000, 1000)
# let's verify our calualtion using R's function
integral2(f5, 0,1, 0,1)


#########################################################################
# PART 5 ################################################################
# 2ND DERIVATIVE APPROXIMATION

derivative <- function(f, h) {
  # Compute the Riemann sum of f(x,y) over the intervals [a,b] and [c,d].
  #
  # Parameters
  # ----------
  # f : function f(x)
  # Vectorized function of one variable
```

```
  # h : numeric
  # Let x now change by an amount h. h is the variable that approaches 0
  #
  # Returns
  # -------
  # function
  # Approximations of the derivative of f, given a step h.

  return(function(x) {
    (f(x + h) - f(x)) / (h)
  })
}

f6 <- function(x) {
  res = x^4*sin(x)

  return(res)

}

# df1 = d/dx(x^4 sin(x))
#     = x^3 (4 sin(x) + x cos(x))
df1 <- derivative(f6, 0.0001)

# df2 = d/dx(x^3 (4 sin(x) + x cos(x)))
#     = x^2 (8 x cos(x) - (x^2 - 12) sin(x))
df2 <- derivative(df1, 0.0001)

# Let's evaluate x=pi in the second derivative df2
df2(pi)

# Let's use the eval() funtion to verify our solution. The value should
   be around df2(pi)
eval({ x <- pi; x^2*(8*x*cos(x) - (x^2 - 12)*sin(x))})
```

C Full Output Log

```
> #*********************************************************************
> #* AUTHOR(S) :
> #*     Bruno Gonzalez Soria          (A01169284)
> #*     Antonio Osamu Katagiri Tanaka (A01212611)
> #*
> #* FILENAME :
> #*     Homework4.R
> #*
> #* DESCRIPTION :
> #*     Simulations (Ene 19 Gpo 1)
> #*     Homework 4
> #*
> #* NOTES :
> #*     - https://www.math.ubc.ca/~pwalls/math-python/integration/riemann-sums/
> #*     - https://activecalculus.org/multi/S-11-1-Double-Integrals-Rectangles.html
> #*     - http://math.colgate.edu/faculty/valente/math113/supplements/section151handout.pdf
> #*     - http://hplgit.github.io/Programming-for-Computations/pub/p4c/p4c-sphinx-Python/._pylight004.html
> #*     - https://rstudio-pubs-static.s3.amazonaws.com/131664_1858eec97df54c9b8d5edcd8b22e5818.html
> #*
> #* START DATE :
> #*     21 Feb 2019
> #*********************************************************************
>
> # Install required libraries
> #install.packages('pracma', dependencies=TRUE);
>
> # Plotting the Areas under Curves ####################################
> integralPlot <- function(f,
+                          a,
+                          b,
+                          from = a,
+                          to = b,
+                          title = NULL) {
+   # Plot the area under a function over the interval [a,b] between [from,to].
+   #
+   # Parameters
+   # ----------
+   # f : function
+   # funtion to be ploted
+   # a , b : numeric
+   # Endpoints of the integral interval [a, b]
+   # from , to : numeric (optional)
+   # Endpoints of the plot in the x-axis [x min, x max]
+   # title : expression
+   # title of the plot
+   #
+   # Returns
+   # -------
+   # void
+
+   x <- seq(from, to, length.out = 100) # input continuum
```

```
+   y <- f(x) # output
+
+   # plot the curve
+   plot(
+     x,
+     y,
+     xlim = c(from, to),
+     ylim = c(ifelse(min(y) < 0, min(y), 0), max(y)),
+     xlab = "x",
+     ylab = "f(x)",
+     main = title,
+     col.main = "#86B875",
+     type = "l",
+     lwd = 3,
+     col = "#86B875"
+   )
+
+   # area under the curve
+   x <- seq(a, b, length.out = 100)
+   y <- f(x)
+   polygon(
+     c(x, b, a, a),
+     c(y, 0, 0, f(a)),
+     border = adjustcolor("#7DB0DD", alpha.f = 0.3),
+     col = adjustcolor("#7DB0DD", alpha.f = 0.3)
+   )
+ }
>
>
> ######################################################################
> # PART 1 #############################################################
> # TAYLOR SERIES
>
> library(pracma)
>
> taylorPlot <- function(f, c, from, to) {
+   # Plot the Taylor approximations up to the 2nd, 4th, 6th and 8th
    terms
+   #
+   # Parameters
+   # ----------
+   # f : function
+   # Vectorized function of one variable
+   # c : numeric
+   # point where the series expansion will take place
+   # from, to : numeric
+   # Interval of points to be ploted
+   #
+   # Returns
+   # -------
+   # void
+
+   x <- seq(from, to, length.out = 100)
+   yf <- f(x)
+
+   yp2 <- polyval(taylor(f, c, 2), x)
+   yp4 <- polyval(taylor(f, c, 4), x)
+   yp6 <- polyval(taylor(f, c, 6), x)
+   yp8 <- polyval(taylor(f, c, 8), x)
+
```

```
+   plot(
+     x,
+     yf,
+     xlab = "x",
+     ylab = "f(x)",
+     type = "l",
+     main = '␣Taylor␣Series␣Approximation␣of␣f(x)␣',
+     col = "black",
+     lwd = 2
+   )
+
+   lines(x, yp2, col = "#c8e6c9")
+   lines(x, yp4, col = "#81c784")
+   lines(x, yp6, col = "#4caf50")
+   lines(x, yp8, col = "#388e3c")
+
+   legend(
+     'topleft',
+     inset = .05,
+     legend = c("TS␣8␣terms", "TS␣6␣terms", "TS␣4␣terms", "TS␣2␣terms", "f(x)"),
+     col = c('#388e3c', '#4caf50', '#81c784', '#c8e6c9', 'black'),
+     lwd = c(1),
+     bty = 'n',
+     cex = .75
+   )
+ }
>
> # -----
>
> f0 <- function(x) {
+   res = sin(x)
+
+   return(res)
+
+ }
>
> f1 <- function(x) {
+   res = exp(complex(real = 0, imaginary = 1)*x)
+
+   return(res)
+
+ }
>
> # -----
>
> taylorPlot(f0, 0, -6.6, 6.6)
> taylorPlot(f1, 1, -2*pi, 2*pi)
Warning messages:
1: In xy.coords(x, y, xlabel, ylabel, log) :
  imaginary parts discarded in coercion
2: In xy.coords(x, y) : imaginary parts discarded in coercion
3: In xy.coords(x, y) : imaginary parts discarded in coercion
4: In xy.coords(x, y) : imaginary parts discarded in coercion
5: In xy.coords(x, y) : imaginary parts discarded in coercion
>
>
> ########################################################################
> # PART 2 ###############################################################
> # RIEMANN SUMS FUNCTION
```

```
>
> riemann_sum <- function(f, a, b, n) {
+   # Compute the Riemann sum of f(x) over the interval [a,b].
+   #
+   # Parameters
+   # ----------
+   # f : function
+   # Vectorized function of one variable
+   # a , b : numeric
+   # Endpoints of the interval [a,b]
+   # n : numeric
+   # Number of subintervals of equal length in the partition of [a,b]
+   #
+   # Returns
+   # -------
+   # numeric
+   # Underestimate and overestimate approximations of the integral given by the
+   # Riemann sum.
+
+   # initialize values
+   lower.sum <- 0
+
+   upper.sum <- 0
+
+   h <- (b - a) / n
+
+
+   # riemann right sum
+   for (i in n:1) {
+     x <- a + i * h
+
+     lower.sum <- lower.sum + f(x)
+
+   }
+   lower.sum <- h * lower.sum
+
+
+   # riemann left sum
+   for (i in 1:n) {
+     x <- b - i * h
+
+     upper.sum <- upper.sum + f(x)
+
+   }
+   upper.sum <- h * upper.sum
+
+
+   # let's plot the curve
+   integralPlot(
+     f = f,
+     a = a,
+     b = b,
+     title = expression(f(x))
+   )
+
+   # print/get riemann sum
+   cat(sprintf(
+     "The␣true␣value␣is␣between␣%f␣and␣%f.\n",
+     as.double(lower.sum),
```

```
+      as.double(upper.sum)
+    ))
+
+   return(c(lower.sum, upper.sum))
+
+ }
>
> # -----
>
> # let's generate some functions to test our algorithm
> f2 <- function(x) {
+   res = x
+
+   return(res)
+
+ }
>
> f3 <- function(x) {
+   res = 4 / (1 + x ^ 2)
+
+   return(res)
+
+ }
>
> # -----
>
> riemann_sum(f0, 0, pi / 2, 10)
The true value is between 1.076483 and 0.919403.
[1] 1.0764828 0.9194032
> riemann_sum(f2, 0, 1, 10000) # should be 0.5
The true value is between 0.500050 and 0.499950.
[1] 0.50005 0.49995
> riemann_sum(f3, 0, 1, 10000) # should be PI
The true value is between 3.141493 and 3.141693.
[1] 3.141493 3.141693
>
>
> #####################################################################
> # PART 3 ############################################################
> # Integrate the function f(x)=exp(x+x^2) from -2 to 2, using Rieman
    sums.
> f4 <- function(x) {
+   res = exp(x + x ^ 2)
+
+   return(res)
+
+ }
>
> # plot Taylor approximations
> taylorPlot(f4, 0, -2.3, 1.3)
>
> # compute riemann_sum for f4
> riemann_sum(f4, -2, 2, 100000)
The true value is between 93.170674 and 93.154833.
[1] 93.17067 93.15483
> # let's verify our calualtion using R's function
> integrate(f4, lower = -2, upper = 2)
93.16275 with absolute error < 0.00062
>
>
```

```
> ########################################################################
> # PART 4 ###############################################################
> # RIEMANN SUMS 2D FUNCTION
>
> riemann_sum_2d <- function(f, a, b, c, d, nx, ny) {
+   # Compute the derivative of a function.
+   #
+   # Parameters
+   # ----------
+   # f : function
+   # Vectorized function of one variable
+   # a , b : numeric
+   # Endpoints of the interval [a,b] (inner integral)
+   # c , d : numeric
+   # Endpoints of the interval [c,d] (outer integral)
+   # nx : numeric
+   # Number of subintervals of equal length in the partition of [a,b]
+   # ny : numeric
+   # Number of subintervals of equal length in the partition of [c,d]
+   #
+   # Returns
+   # -------
+   # numeric
+   # Approximations of the integral given by the Riemann 2D sum.
+
+   # initialize values
+   dx = (b - a) / nx
+   s = 0.0
+   x = a
+
+   dy = (d - c) / ny
+   y = c
+
+   # riemann 2D sum
+   for (i in 1:nx) {
+     for (j in 1:ny) {
+       x = a + dx / 2 + i * dx
+       y = c + dy / 2 + j * dy
+       f_i = f(x, y)
+       s = s + f_i * dx * dy
+     }
+   }
+
+   # print/get riemann sum
+   cat(sprintf("The true value is around %f.\n",
+               as.double(s)))
+
+   return(s)
+
+ }
>
> # -----
>
> f5 <- function(x, y) {
+   res = exp((x + y) ^ 2)
+
+   return(res)
+
+ }
>
```

```
> # -----
>
> # compute riemann_sum_2d for f5
> riemann_sum_2d(f5, 0, 1, 0, 1, 1000, 1000)
The true value is around 4.926310.
[1] 4.92631
> # let's verify our calualtion using R's function
> integral2(f5, 0,1, 0,1)
$Q
[1] 4.899159

$error
[1] 9.974762e-16

>
>
> ########################################################################
> # PART 5 ###############################################################
> # 2ND DERIVATIVE APPROXIMATION
>
> derivative <- function(f, h) {
+   # Compute the derivative of a function.
+   #
+   # Parameters
+   # ----------
+   # f : function f(x)
+   # Vectorized function of one variable
+   # h : numeric
+   # Let x now change by an amount h. h is the variable that approaches
    0
+   #
+   #
+   # Returns
+   # -------
+   # function
+   # Approximations of the derivative of f, given a step h.
+
+   # return(function(x) {
+   #   (f(x + h) - f(x - h)) / (2 * h)
+   # })
+
+   return(function(x) {
+     (f(x + h) - f(x)) / (h)
+   })
+ }
>
> f6 <- function(x) {
+   res = x^4*sin(x)
+
+   return(res)
+
+ }
>
> # df1 = d/dx(x^4 sin(x))
> #     = x^3 (4 sin(x) + x cos(x))
> df1 <- derivative(f6, 0.0001)
>
> # df2 = d/dx(x^3 (4 sin(x) + x cos(x)))
> #     = x^2 (8 x cos(x) - (x^2 - 12) sin(x))
> df2 <- derivative(df1, 0.0001)
```

```
>
> # Let's evaluate x=pi in the second derivative df2
> df2(pi)
[1] -248.076
>
> # Let's use the eval() funtion to verify our solution. The value should
    be around df2(pi)
> eval({ x <- pi; x^2*(8*x*cos(x) - (x^2 - 12)*sin(x))})
[1] -248.0502
>
```