Fluid Mechanics

- Units, Dimensions and Dimensional homogeneity.
- Refresher of vector calculus, nomenclature, relevant mathematical operators in fluid mechanics.
- Fundamentals of forces in fluids , and the relation with physical properties, fields and parameters.
- Coordinate systems and frames of reference

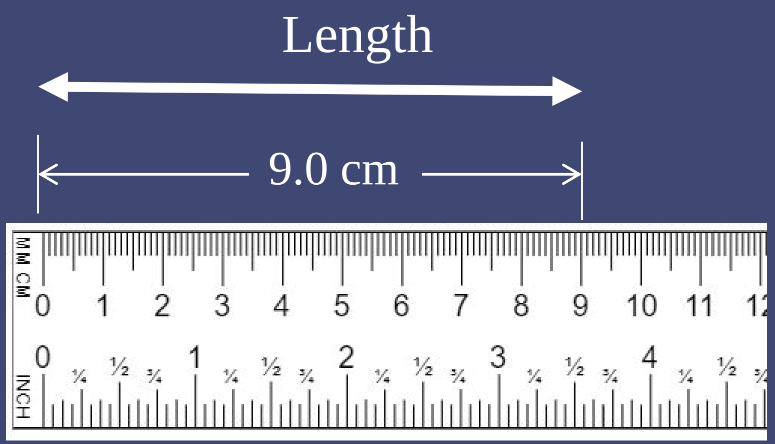
DIMENSIONS AND UNITS

Dimension: A measure of a physical quantity (without numerical values).

Unit: A way to assign a *number* to that dimension.

There are seven **primary dimensions** (also called **fundamental** or **basic dimensions**): mass, length, time, temperature, electric current(*), amount of light, and amount of matter.

All non-primary dimensions can be formed by some combination of the seven primary dimensions.



Dimensions of force {Force} [=] N[=] kg-m/s^2 [=] M-L-Θ^{-2}

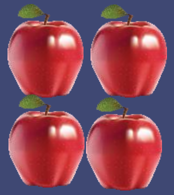
A *dimension* is a measure of a physical quantity without numerical values, while a *unit* is a way to assign a number to the dimension. For example, length is a dimension, but centimeter is a unit.

(*) Some people prefer charge (electrical charge) unit as primary dimension but IUPAP suggest to use current (electrical current) instead, and then coulomb is non primary dimension because can be expressed as the product of current times time, i.e. ampere (second)

UNITS

1 kg_f

1 N



1 lb_f

A body weighing 150 lb_f in Earth will weigh only 57 lb_f on Mars and 25 lb_f over the Moon's surface

How come ?

176# over the Earth at rest

528# at launch

352# far from Earth

$W = mg$ (N)

W weight
m mass
g gravitational acceleration

Units need to be specified

INERTIAL FRAME OF REFERNCE

Body forces measured with spring depend not only of the extent or size of the system, also depend on the intensity of the force field, location and/or frame of reference (inertial or non-inertial)

In science and engineering, properties, parameters, functions, variables and constants may be:

- Scalar, vector or tensor quantities.
- Intensive and extensive.
- May have values that correspond to real, integers, rational, irrational, complex and quaternions numbers, etc.
- If differentiable, they may be exact or inexact.
- Dimensional or dimensionless

Mathematical expression (scalar, vector, tensor, or complex)

- Scalar (time, mass, volume, viscosity of Newtonian fluid, thermal conductivity in a isotropic material, electrical current in D.C., etc.)
- Vector (velocity, force, torque, flux, etc.)
- Tensor (stress, Einstein curvature, thermal conductivity in a non-isotropic material, stiffness, moment of inertia, strain rate, etc.)
- Complex (electrical current in A.C., viscosity in viscoelastic fluids, etc., this may be a construct to easily characterize that property)

Extensivity (Extensive or Intensive)

- Extensive (mass, volume, charge, etc.)
- Intensive (density, temperature, pressure, contact angle, viscosity, surface tension, interfacial tension, etc.)

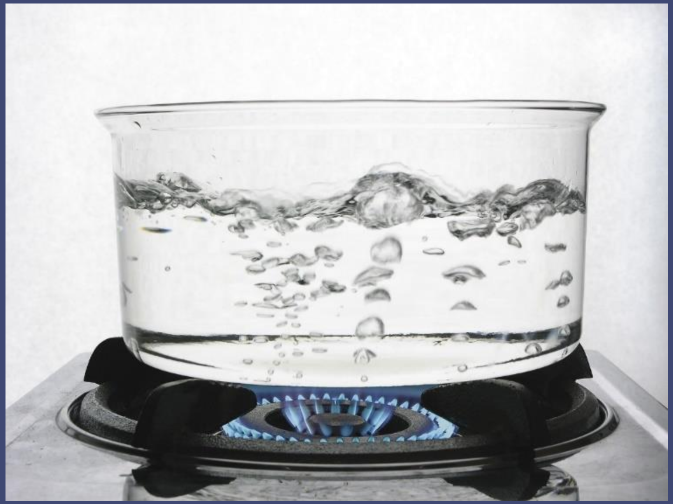
Examples

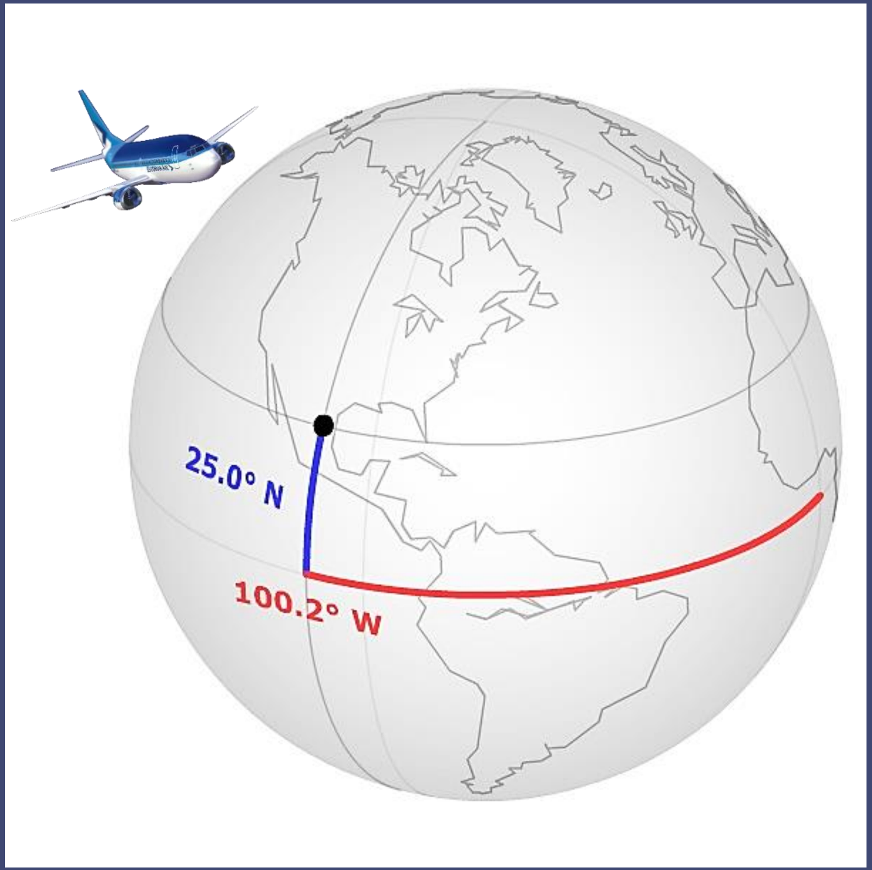
Vector (requires three numbers for quantification)

Monterrey is located 540 m, over the sea level 25°40’ latitude and 100°18’ Longitude

Scalar (requires a single numerical value for quantification)

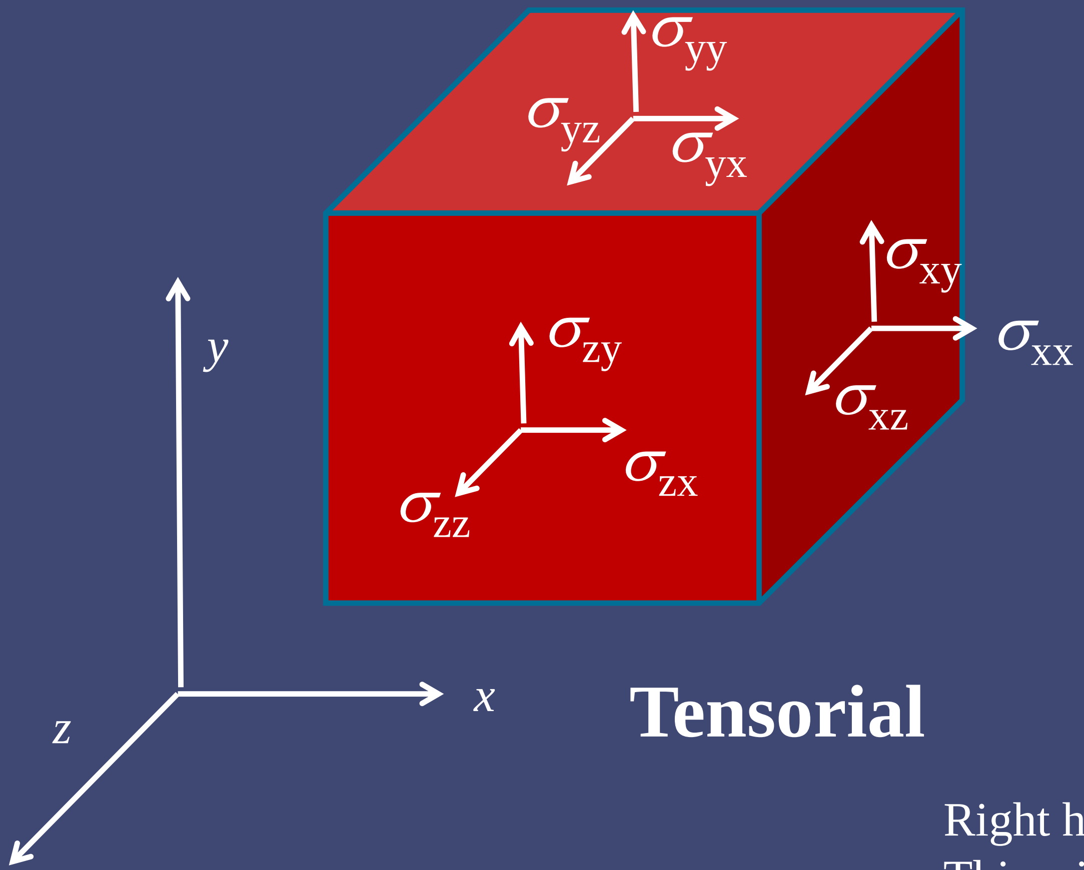
Boiling temperature of water T = 100 °C at sea level

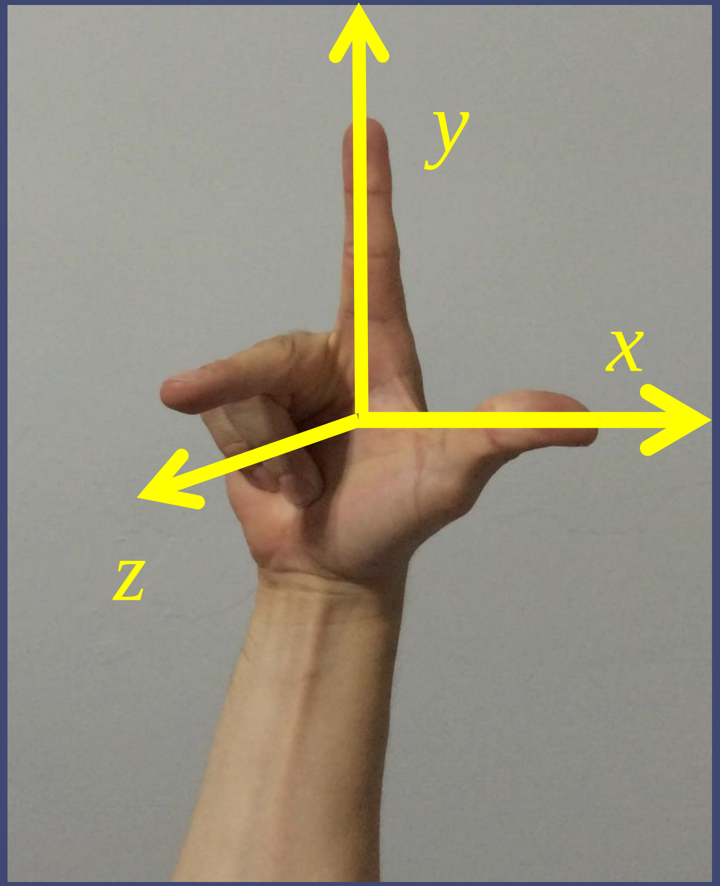




characteristics

Tensor (May require nine numbers to specify its value)

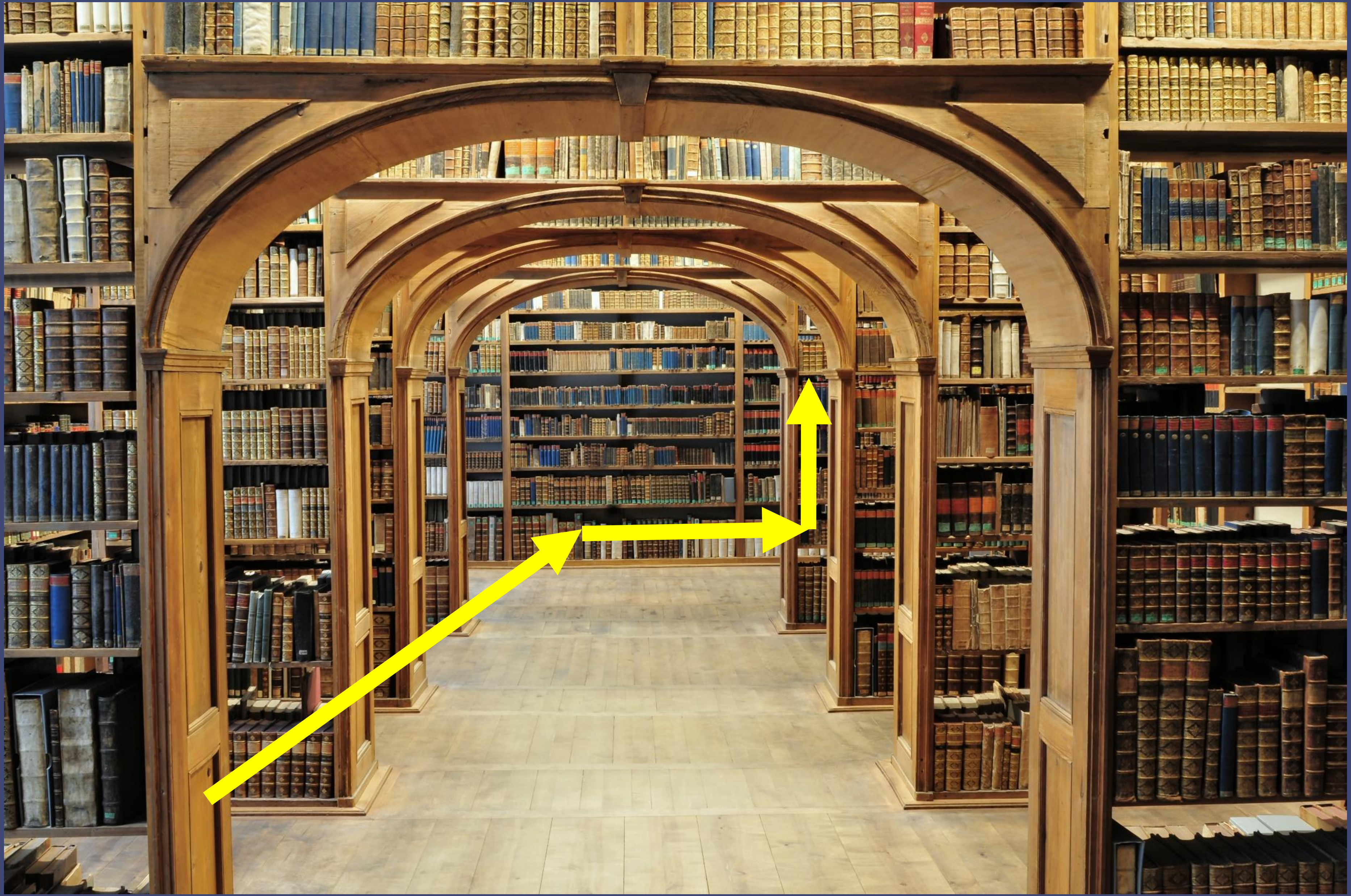




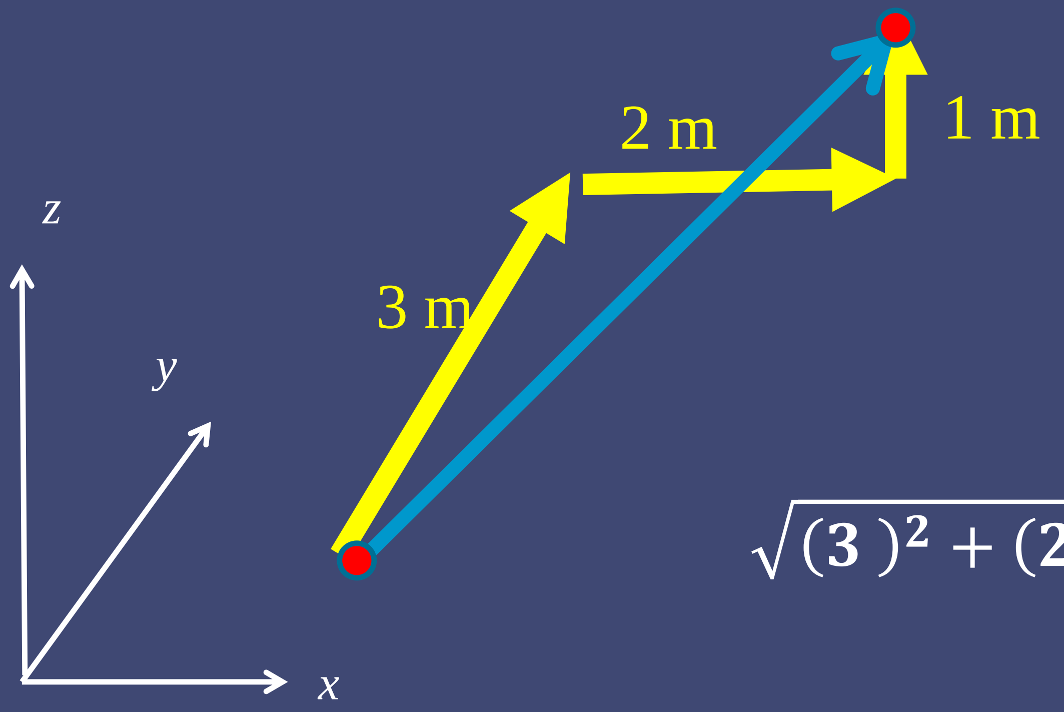
Tensorial

Right hand axes representation.
This will help to visualize vectors and tensors

Vectors



Find a book located 3.74 m away from your original position.
What about if you are told: locate the book located 3 m ahead of your original position, then 2 m to the right and 1 m up



$$\sqrt{(\mathbf{3}\ )^{\mathbf{2}} + (\mathbf{2})^{\mathbf{2}} + (\mathbf{1})^{\mathbf{2}}} = 3.74$$

All non-primary dimensions can be formed by some combination of the seven primary dimensions

Primary dimensions and their associated primary SI

Dimension	Symbol*	SI Unit
Mass	M	kg (kilogram)
Length	L	m (meter)
Time†	T	s (second)
Temperature	Θ	K (kelvin)
Electric current	I	A (ampere)
Amount of light	C	cd (candela)
Amount of matter	N	mol (mole)

Primary dimensions

Quantity	symbol	Unit	Symbol of units	Dimension
Length	l	meter	m	[L]
Mass	m	kilogram	kg	[M]
time	t	second	s	[T]
Electric current	I	ampere	A	[I]
Temperature	T	kelvin	K	[Θ]
Quantity of matter	n	mol	mol	[N]
Luminous intensity	I_v	candela	cd	[C]

Notes: We have seven base units, some authors may include **radian** for plane angles and **stereo radian** for solid angles and list 9 base units, in this course we will not include these units because both are dimensionless (the former is the ratio between two lengths, and the second is the ratio between area and the square of the radius). Some physicists prefer electric charge dimension, and the coulomb as base units, instead of using electric current dimension and ampere as base unit , but we will use the recommendation of IUPAP (International Union of pure and applied physics)

Non-primary dimensions

quantity	symbol	Unit	Symbol of units	Dimension
Area	a	Square meter	m^2	$[L^2]$
Volume	V	Cubic meter	m^3	$[L^3]$
Velocity	$\mathbf{v}$	Meter per second	m/s	$[LT^{-1}]$
Linear momentum	$\mathbf{p}$	Kilogram meter per second	kg m/s	$[MLT^{-1}]$
Force	$\mathbf{F}$	newton	N	$[MLT^{-2}]$
Energy	E	joule	J	$[ML^2T^{-2}]$
Power	P	watt	W	$[ML^2T^{-3}]$
Pressure	p	pascal	Pa	$[ML^{-1}T^{-2}]$
Torque	$\mathbf{T}$	Newton meter	N m	$[ML^2T^{-2}]$

newton=N= kg m/s^2 Joule=J=N m= kg m^2/s^2 watt=J/s=N m/s= kg m^2/s^3

Note: symbols are in italics, some are uppercase, some lowercase, some are bold.

Non-primary dimensions

quantity	symbol	Unit	Customary units	Symbol of units	Dimension
Viscosity	μ	poiseuille	cP (centipoise)	Pa-s	$[ML^{-1}T^{-1}]$
Surface tension	σ	newton/m	Dyne/cm	N/m	$[MT^{-2}]$
Interfacial tension	σ	newton/m	mN/cm	J/m^2	$[MT^{-2}]$
Kinematic viscosity	$\nu = \mu/\rho$	m^2/s	cSt (centistoke)	m^2/s	$[L^2T^{-1}]$
permeability	k	m^2	darcy	m^2	$[L^2]$
porosity	ϕ	m^3/m^3	%	-	[1]
Contact angle	θ	rad	degrees	-	[1]
roughness	ε	m	mm	m	[L]
Molar mass	M	dalton	g/mol	kg/kmol	[M/N]

Note: 100 cP = 1 poise, 1 cP = 1 mPa-s = 1×10^{-3} Pa-s.
1 dyne/cm = 1 mN/m = 1×10^{-3} N/m.
1 St = 1 cm^2/s = 1×10^{-4} m^2/s
1 darcy = $(cP\text{-}cm^3/s)/(cm^2\text{-}atm/cm)$ = 9.8692×10^{-13} m^2

[1] means dimensionless

Non-primary dimensions

Quantity	symbol	Unit	Symbol of units	Dimension
Voltage	V	volt = watt/A	V	$[ML^2T^{-3}I^{-1}]$
Resistance	R	ohm = watt/A^2	Ω	$[ML^2T^{-3}I^{-2}]$
Electric charge	q	coulomb = A s	C	$[I\ T]$
Electric field	$\underline{\mathbf{E}}$	volt/m = N/C	V/m	$[M\ L\ T^{-3}I^{-1}]$
Magnetic field	$\underline{\mathbf{B}}$	tesla=newton /(A m)	T	$[MT^{-2}I^{-1}]$
capacitance	C	farady=A^2 s / W	F	$[M^{-1}\ L^{-2}\ T^4\ I^2]$
Inductance	L	henry = volt s / A	H	$[ML^2T^{-2}I^{-2}]$
Magnetic flux	$\underline{\Phi}$	weber= T m^2	Wb	$[ML^2T^{-2}I^{-1}]$

Note: Energy and electric field are different symbols, Volume and voltage have different symbols.

Derivatives, derivative operators, or difference operators

Quantity	symbol	Unit	Symbol of units	Dimension
Gradient	$\underline{\nabla}$	$1/m = m^{-1}$	$1/m$	$[L^{-1}]$
Laplacean	∇^2	$1/m^2 = m^{-2}$	$1/m^2$	$[L^{-2}]$
Material derivative	$\frac{D}{Dt}$	$1/s = s^{-1}$	$1/s$	$[T^{-1}]$
Time derivative	$\frac{\partial}{\partial t}$	$1/s = s^{-1}$	$1/s$	$[T^{-1}]$
Rate	$\frac{d}{dt}$	$1/s = s^{-1}$	$1/s$	$[T^{-1}]$
Increment	Δ	-	-	[1]

Note: Rate changes or derivatives in engineering usually may be respect to time or respect to space, so these operators must be considered in the dimension analysis.

DIMENSIONAL HOMOGENEITY

The law of dimensional homogeneity: Every additive term in an equation must have the same dimensions.

$$\overbrace{\frac{d(m\,\hat{E})}{dt}}^{\text{W}} = \sum_{in} \dot{m}_i \left(\hat{U}_i + \alpha_i \frac{v_i^2}{2} + g\, z_i + \frac{p_i}{\rho_i}\right) \underbrace{- \sum_{out} \dot{m}_j \left(\hat{H}_j + \alpha_j \frac{v_j^2}{2} + g\, z_j\right)}_{\text{W}} + \underbrace{\dot{Q}}_{\text{W}} - \underbrace{\dot{W}}_{\text{W}}$$

kg (J/kg) / s = W

kg/s	J/kg	m^2/s^2	m-m/s^2	Pa-m^3/kg
	N-m/kg	m^2/s^2	m^2/s^2	N-m^3/m^{2}kg
	kg-m^2/kg-s^2	m^2/s^2	m^2/s^2	kg-m-m/kg-s^2
	m^2/s^2	m^2/s^2	m^2/s^2	m^2/s^2

kg-m /s^2-m/s

N-m/s

J/s

W

additive stands for addition and subtraction

Nondimensionalization of Equations

Nondimensional equation: If we divide each term in the equation by a collection of variables and constants whose product has those same dimensions, the equation is rendered nondimensional.

Normalized equation: If the nondimensional terms in the equation are of order unity (between zero and one), the equation is called normalized (this process may be more complex, but strictly speaking the idea is that the integral of the function within the domain is set to one).

Each term in a nondimensional equation is dimensionless.

Nondimensional parameters: In the process of nondimensionalizing an equation of motion, nondimensional parameters often appear—most of which are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number).

This process is referred to by some authors as inspectional analysis.

The nondimensionalized Bernoulli equation

$$\underbrace{\frac{p}{p_\infty}}_{\{1\}} + \underbrace{\frac{\rho\, v^2}{2\, p_\infty}}_{\{1\}} + \underbrace{\frac{\rho\, g\, z}{p_\infty}}_{\{1\}} = \underbrace{\frac{C}{p_\infty}}_{\{1\}}$$

A *nondimensionalized* form of the Bernoulli equation is formed by dividing each additive term by a pressure (here we use p_∞). Each resulting term is *dimensionless* (dimensions of {1}).

p_∞ is any trivial variable, but if physical meaning is used will be better, e.g. stagnation pressure, saturation pressure, atmospheric pressure, etc.

Equation of motion for a particle in vacuum under gravitational field.

$$\frac{d^2 z}{dt^2} = -g$$

Dimensional Result after integration

$$z = z_0 + w_0\, t - \frac{1}{2} g\, t^2$$

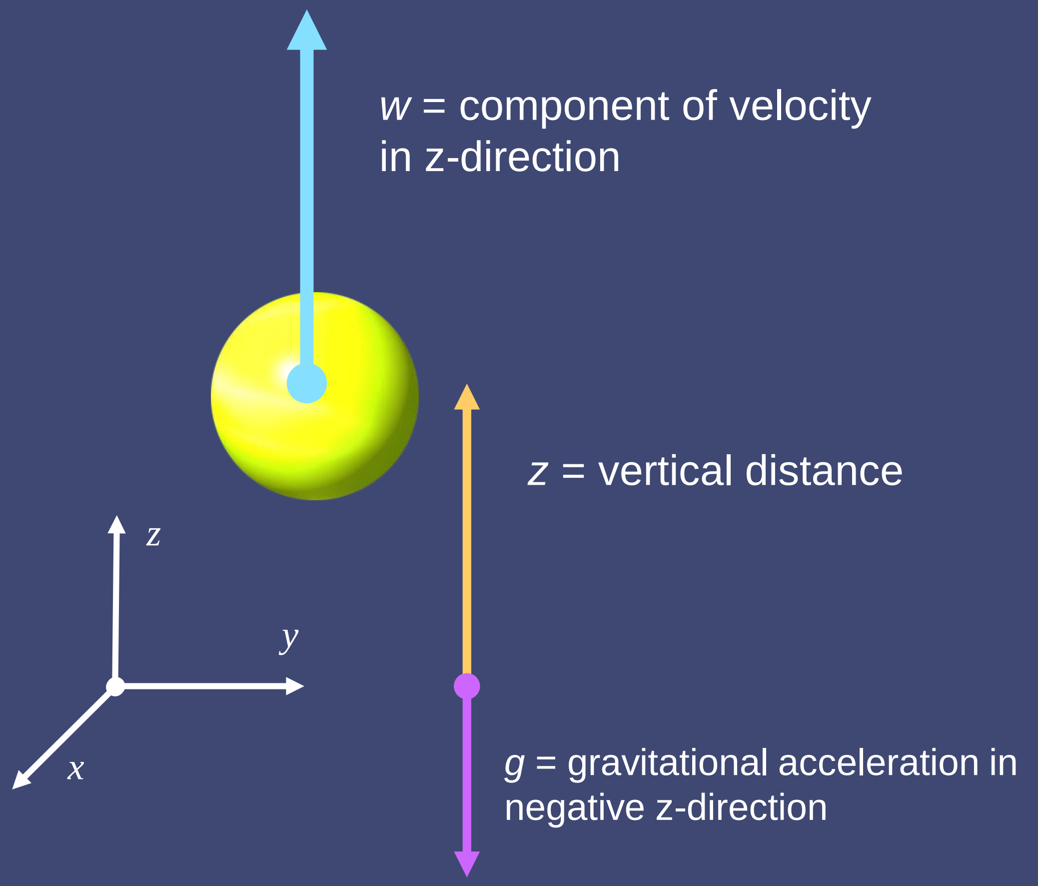
Dimensional variables: Dimensional quantities that change or vary in the problem. Examples: z (dimension of length) and t (dimension of time).

Nondimensional (or dimensionless) variables: Quantities that change or vary in the problem, but have no dimensions. Example: Angle of rotation, measured in degrees or radians, dimensionless units, or any normalized variable.

Dimensional constant: Gravitational constant g, while dimensional, remains constant.

Parameters: Refer to the combined set of dimensional variables, nondimensional variables, and dimensional constants in the problem.

Pure constants: The constant 1/2 and the exponent 2 in equation. Other common examples of pure constants are π and e.



Object falling in a vacuum. Vertical velocity is drawn positively, so $w < 0$ for a falling object.

Conservation of dimensions

For any equation to be valid, every term in the equation must have the same physical character, i.e. the same net dimensions (and consequently the same units in any consistent system of units). This is known as the law of conservation of dimensions.

Example: The vertical elevation (z) and the horizontal distance (x) at any time for a projectile fired from a gun

$$z = a\,x + b\,x^2$$

Find the units of a and b

Answer: a is dimensionless and b has units of inverse of length. a [=] 1, b [=] 1/m

PREFIXES

Specific weight γ: The weight of a unit volume of a substance. (trivial name), it's the gravitational body force, or force per unit volume

$$\underline{F}\,/V = \underline{\gamma} \;= \rho\, \underline{g}$$

Specific volume, **specific** heat capacity, **specific** energy, specific weight (is the weight density) and specific gravity (is a density ratio)

Be careful with the adjectives in science !

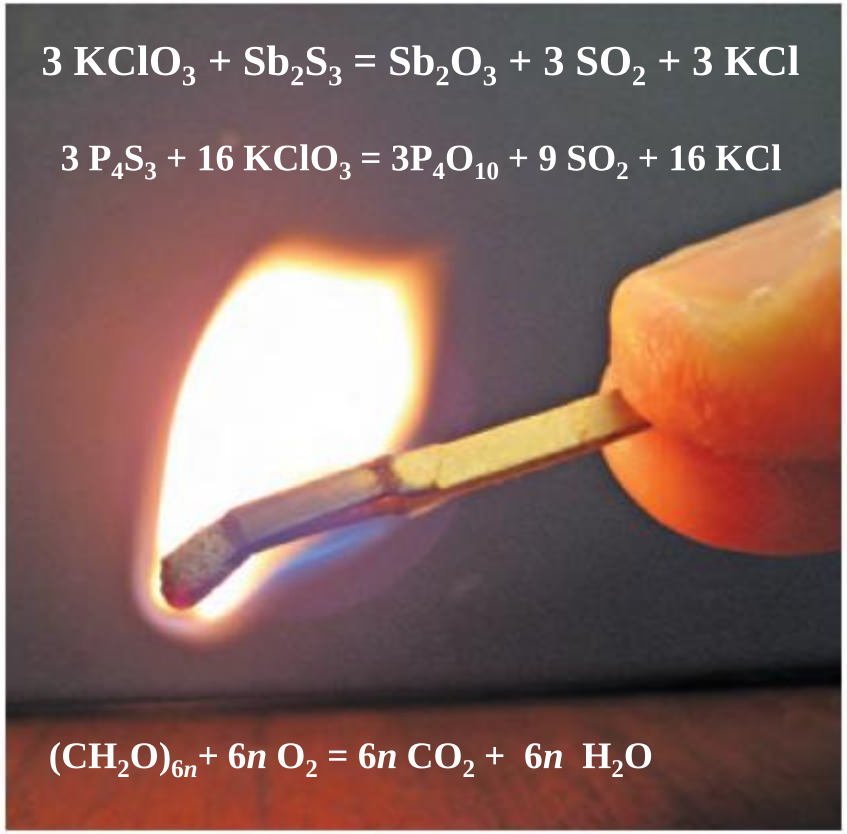
$$\hat{V} = \frac{V}{m} \qquad \hat{c}_p = \frac{Cp}{m} \qquad \hat{E} = \frac{E}{m}$$

$$\gamma = \frac{\rho\, g\, V}{V} \qquad s.g. = \frac{\rho}{\rho_w(4°\mathrm{C})}$$

Be careful with prefixes in SI units

1 MW= 1 000 000 W
1 MPa= 1 000 000 Pa

1 MBTU = 1000 BTU



A typical match yields about one Btu (or one kJ) of energy if completely burned.

Challenge: How much heat rate is generated if it takes 50 s
To complete the burning

CONCERTED vs NO CONCERTED ENERGY

Problem Verify if the equations given (The three equations are used to estimate the velocity of a rising bubble within a immiscible fluid, these correspond to three different flow regimes) obey the rule of dimensional homogeneity. One case will be analyzed and later all of them using a different approach

Spherical regime

$$v = \frac{g\, d_p^2 \Delta\rho}{6\,\mu_L}\left[\frac{1+\mu_G/\mu_L}{2+3\,\mu_G/\mu_L}\right]$$

Ellipsoidal regime 0.25< *Bo* <40

$$v^2 = \frac{2.14\,\sigma}{\rho_L\, d_p} + 0.505\, g\, d_p$$

Spherical cap regime *Bo*>40 and Re>1.2

$$v = \frac{2}{3}\sqrt{\frac{g d_p \Delta\rho}{2\rho_L}}$$

Analysis for the ellipsoidal regime

First step, check the units of each variable.

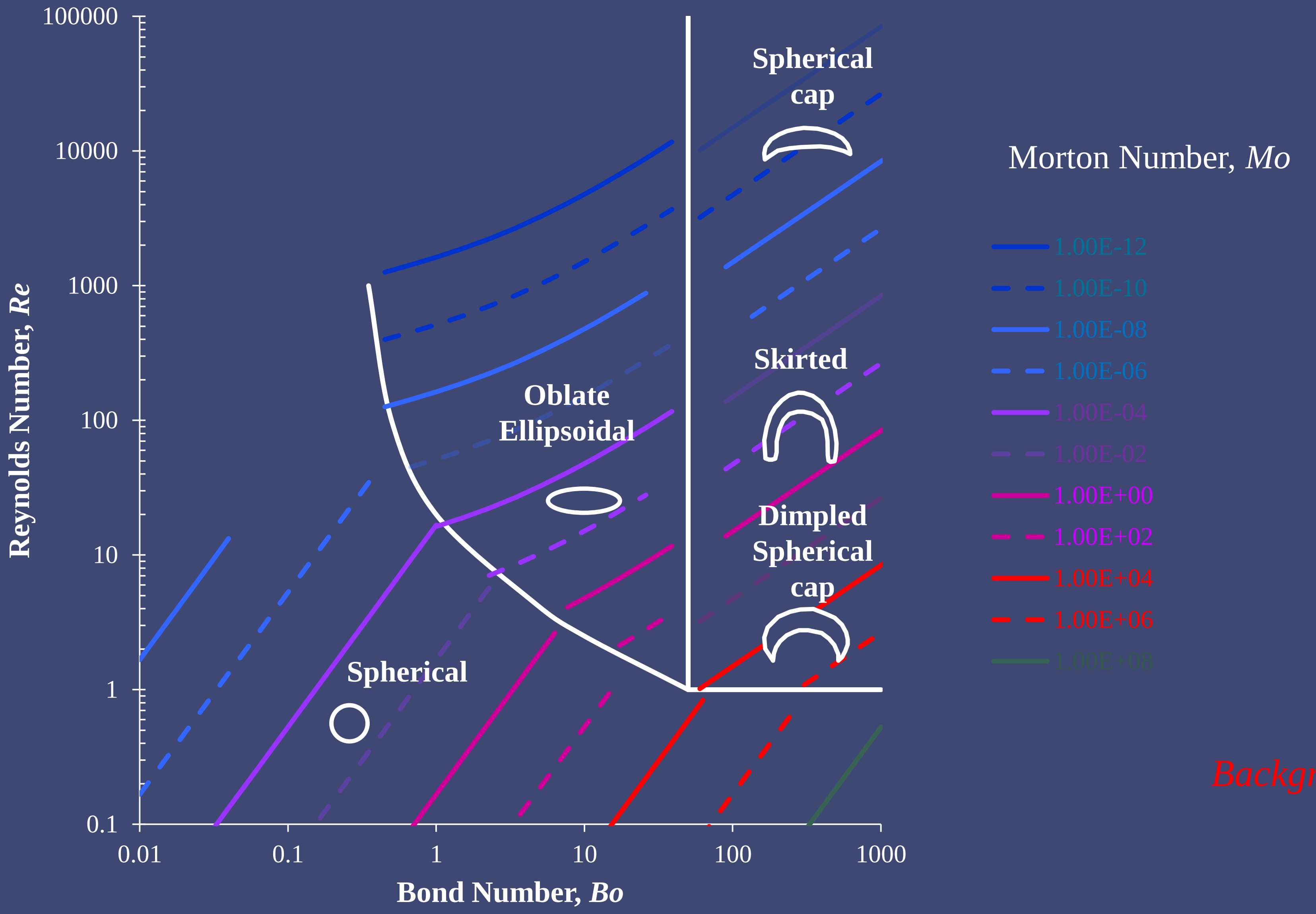
v = velocity [m/s]
σ = surface tension [N/m]
ρ_L = Density of liquid [kg/m^3]
g = gravitational field [m/s^2]
d_p = bubble diameter [m]

Second step, write the derived units in terms of base units:
Force units in base units is …[N] = [kg m / s^2]

Last step, make sure each term of the equation has the same units, in this case [m^2/s^2]

$$\left[\frac{\mathrm{m}}{\mathrm{s}}\right]^2 = \frac{\left[\frac{\mathrm{kg\ m}}{\mathrm{m\ s^2}}\right]}{\left[\frac{\mathrm{kg}}{\mathrm{m^3}}\right][\mathrm{m}]} + \left[\frac{\mathrm{m}}{\mathrm{s^2}}\right][\mathrm{m}]$$

This slide is used to state the context of the previous problem, and can b omitted



Background 3

Adapted from Luz Amaya-Bower and Taehum Lee, Computers & Fluids 39 (2010) 1191-1207

Another approach to verify that the equation obeys the rule of dimensional homogeneity, is to reformulate the equation in terms of dimensionless numbers.

Some of these dimensionless numbers are given in the following list, some other ones are not listed, but even those/these other numbers can be recast in terms of known dimensionless numbers .

Recast the previous equations in dimensionless form
And use any of the following dimensionless numbers

$Bo = \dfrac{g\ D^2 \Delta\rho}{\sigma}$	Bond or Eötvös
$Mo = \dfrac{g\ \mu_L^4 \Delta\rho}{\sigma^3 \rho_L^2}$	Morton
$Re = \dfrac{\rho_L\ v\ D}{\mu_L}$	Reynolds
$Fr = \dfrac{\rho_L\ v^2}{g\ D\ \Delta\rho}$	Froude
$We = \dfrac{D\ \rho_L\ v^2}{\sigma}$	Weber
$Ca = \dfrac{\mu_L\ v}{\sigma}$	Capillary
$Ar = \dfrac{g\ D^3\ (\Delta\rho/\rho_L)}{(\mu_L/\rho_L)^2}$	Archimedes

Relationship between
Dimensionless numbers

$$Bo = \frac{We}{Fr}$$

$$Ar^2 = \frac{Bo^3}{Mo}$$

$$Ar = \frac{Re^2}{Fr}$$

$$Re^4\ Mo = Bo\ We^2$$

$$Ri = \frac{Ar}{Re^2}$$

Richardson

Usually the properties of the continuous phase are used (in this case liquid or denser phase)

$Fo = \dfrac{t\ \mu_L}{\rho_L L^2}$	Fourier
$La = \dfrac{\rho_L \sigma\ L}{\mu_L^2}$	Laplace

All these dimensionless numbers will be studied later

Another easy approach to verify dimensional homogeneity is to recast in dimensionless form each equation. If only dimensionless terms involved, then obeys the rule.

Spherical regime

$$v = \frac{g\, d_p^2 \Delta\rho}{6\, \mu_L}\left[\frac{1 + \mu_G/\mu_L}{2 + 3\,\mu_G/\mu_L}\right]$$

$$Re = \frac{Ar}{6}\left[\frac{1 + \mu_G/\mu_L}{2 + 3\,\mu_G/\mu_L}\right]$$

Ellipsoidal regime 0.25<*Bo*<40

$$v^2 = \frac{2.14\,\sigma}{\rho_L\, d_p} + 0.505\, g\, d_p$$

$$1 = \frac{2.14}{We} + 0.505\,\frac{\rho_L}{Fr\,\Delta\rho}$$

Spherical cap regime *Bo*>40 and Re>1.2

$$v = \frac{2}{3}\sqrt{\frac{g d_p \Delta\rho}{2\rho_L}}$$

$$Fr = \frac{2}{9}$$

Note: Any ratio of variables of the same dimensions are dimensionless, like ratio of viscosities, densities, and so on…

Problem

Govier and Aziz (1972) Reported that Allen (1900) derived the following equation for falling spheres within a fluid for Reynolds number between 1 and 1000

$$V = 0.2\left[\frac{(\rho_p - \rho_L)g}{\rho_L}\right]^{0.72}\left[\frac{d_p^{1.18}}{\left(\frac{\mu}{\rho_L}\right)^{0.45}}\right]$$

Demonstrate that if the previous equation represents the experimental data, it should be written in the form:

$$V = 0.2\left[\frac{d_p(\rho_p - \rho_L)g}{\rho_L}\right]^{1/2}\left[\frac{d_p(\rho_p - \rho_L)g d_p^3}{\left(\frac{\mu}{\rho_L}\right)^2}\right]^{0.225} \Rightarrow \sqrt{Fr} = 0.2[Ar]^{0.225}$$

This problem is created using equations and data given by Baha Abulnaga (Slurry Systems Handbook, McGraw-Hill, 2002)

Note: As a result, you can conclude that the equation follows the rule of dimensional homogeneity

Problem

A researcher tells you that the minimum stress within a fluid to start moving sediments in slurries can be calculated with the equation

$$\tau = 1.21\rho_m \left[\frac{g(\rho_p - \rho_L)\mu}{\rho_L^2}\right]^{2/3}$$

a) Prove that at least the equation satisfy the rule of dimensional homogeneity

b) Prove that they are equivalent

$$\tau = 1.21\rho_m \left[\frac{g(\rho_p - \rho_L)\mu}{\rho_L^2}\right]^{2/3}$$

$$\Downarrow$$

$$Eu\,(Re)^2 = 1.21[Ar]^{2/3}$$

Dimensionless numbers

$$Ar = \frac{g(\rho_p - \rho_L)d_p^3}{\left(\frac{\mu}{\rho_L}\right)^2}$$ Archimedes

$$Re = \frac{\rho_L\, v\, d_p}{\mu}$$ Reynolds

$$Eu = \frac{\tau}{\rho_m v^2}$$ Euler

Challenge

An special device to measure flow rate consists of a beam, in which deformation may occur, this deformation may be calculated with the equation:

$$y = -\frac{\left(\rho_B H^2 g + \frac{11}{10}\rho_F v^2 H\right)}{24\,E\,I}\left(L^3 x - 2Lx^3 + x^4\right)$$

y = Deformation [=] m
x= axial position ($0 \leq x \leq L$) [=] m
E = Young's Modulus [=] Pa
I = Area Inertia (Moment of area) [=] m^4
H = Beam thickness [=] m
L = Length of the beam [=] m
v = velocity of the fluid [=] m/s
ρ_B = Density of the sensor [=] kg/m^3
ρ_F = Density of the fluid [=] kg/m^3
g = gravity [=] m/s^2

a) Prove that at least the equation satisfy the rule of dimensional homogeneity

Some SI and USCS Units

$$1 \text{ lbm} = 0.45359 \text{ kg}$$

$$1 \text{ ft} = 0.3048 \text{ m}$$

$$\text{Force} = (\text{Mass})(\text{Acceleration})$$

$$F = ma$$

$$1 \text{ N} = 1 \text{ kg}\cdot\text{m/s}^2$$

$$1 \text{ lbf} = 32.174 \text{ lbm}\cdot\text{ft/s}^2$$

Work = Force × Distance

1 J = 1 N·m

1 cal = 4.1868 J

1 Btu = 1.0551

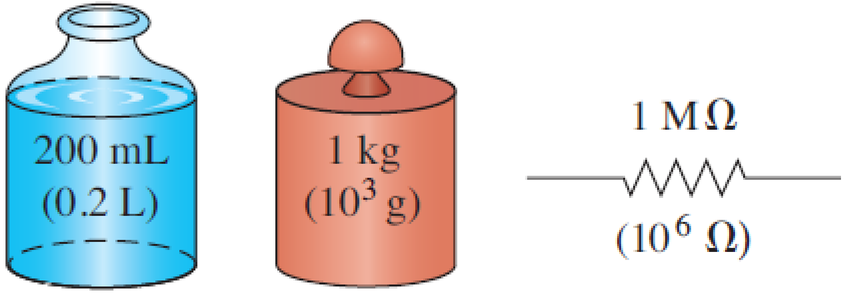


FIGURE 1–6

The SI unit prefixes are used in all branches of engineering.

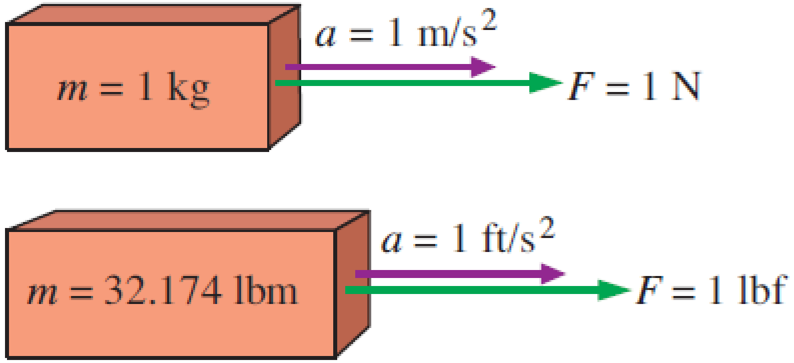


FIGURE 1–7

The definition of the force units.

TABLE 1–2

Standard prefixes in SI units

Multiple	Prefix
10^{24}	yotta, Y
10^{21}	zetta, Z
10^{18}	exa, E
10^{15}	peta, P
10^{12}	tera, T
10^{9}	giga, G
10^{6}	mega, M
10^{3}	kilo, k
10^{2}	hecto, h
10^{1}	deka, da
10^{-1}	deci, d
10^{-2}	centi, c
10^{-3}	milli, m
10^{-6}	micro, μ
10^{-9}	nano, n
10^{-12}	pico, p
10^{-15}	femto, f
10^{-18}	atto, a
10^{-21}	zepto, z
10^{-24}	yocto, y

Some SI and USCS Units

Force

$$\underline{F} = \frac{d(m\underline{v})}{dt}$$

$1\ \mathrm{N} = 1\ \mathrm{kg\ m / s^2}$

$1\ \mathrm{lb_f} = 32.174\ \mathrm{lb_m\text{-}ft/s^2}$

Work

$$W = \int \underline{F} \cdot d\underline{x}$$

$$dW = \underline{F} \cdot d\underline{x}$$

$1\ \mathrm{J} = 1\ \mathrm{N{\cdot}m}$

$1\ \mathrm{cal} = 4.1868\ \mathrm{J}$

$1\ \mathrm{Btu} = 1.0551\ \mathrm{kJ}$

Power

$$\dot{W} = \frac{dW}{dt} = \underline{F} \cdot \frac{d\underline{x}}{dt} = \underline{F} \cdot \underline{v}$$

$$\dot{W} = \frac{dW}{dt} = \underline{T} \cdot \underline{\omega}$$

Power or work rate

1W=1 J/s = 1 N·m/s

1 btu / hour =0.29307107 watts

1 horsepower =745.699872 watts

Torque

$$\underline{T} = \underline{r} \times \underline{F}$$

Torque, **moment**, or **moment of force**

1 N-m = 1 N·m

1 N-m=0.737562 $\mathrm{lb_f}$-ft

Trivial stuff about nomenclature

$$\underline{v} = \frac{d\underline{x}}{dt}$$ velocity

$$\underline{a} = \frac{d\underline{v}}{dt} = \frac{d^2\underline{x}}{dt^2}$$ acceleration

$$\underline{j} = \frac{d\underline{a}}{dt} = \frac{d^2\underline{v}}{dt^2} = \frac{d^3\underline{x}}{dt^3}$$ jerk

$$\underline{s} = \frac{d\underline{j}}{dt} = \frac{d^2\underline{a}}{dt^2} = \frac{d^3\underline{v}}{dt^3} = \frac{d^4\underline{x}}{dt^4}$$ snap or jounce

$$\underline{c} = \frac{d^5\underline{x}}{dt^5}$$ crackle

$$\underline{P} = \frac{d^6\underline{x}}{dt^6}$$ pop

Specific weight γ: The weight of a unit volume of a substance. (trivial name)

$$\gamma = \rho\, g$$

Specific volume, specific heat capacity, specific energy, specific weight and specific gravity

$$\hat{V} = \frac{d(V)}{dm} \qquad \hat{C}_p = \frac{dC_P}{dm} \qquad \hat{E} = \frac{d(E)}{dm} \qquad \gamma = \frac{d(\rho g V)}{dV} \qquad s.g. = \frac{\rho}{\rho_{w,4^o C}}$$

Density, if not modify by unit volume, otherwise use linear or Surface.

e.g. Surface charge density

Density, if not specified for scalar means per unit volume.
For vectors means area flow or flux

e.g. current density (flow/area)

Specific means per unit mass, not any intensive property

Properties: Attributes that are associated with something, character or quality that something has.

Physical property: a characteristic that can be measured or observed without changing the composition of the sample

Mathematical expression in fluid dynamics may refer to scalar, vector or tensor quantities.

Properties or variables in fluid dynamics have extensivity, this means they may be either extensive or intensive properties. The value of an intensive property is not function of the spatial extent, matter content, or mass involved.

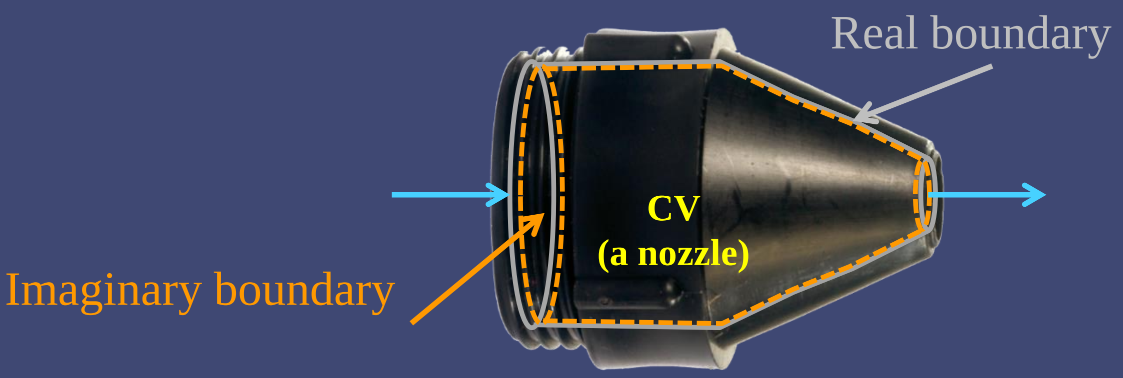
Systems, Control Volumes and Control surfaces

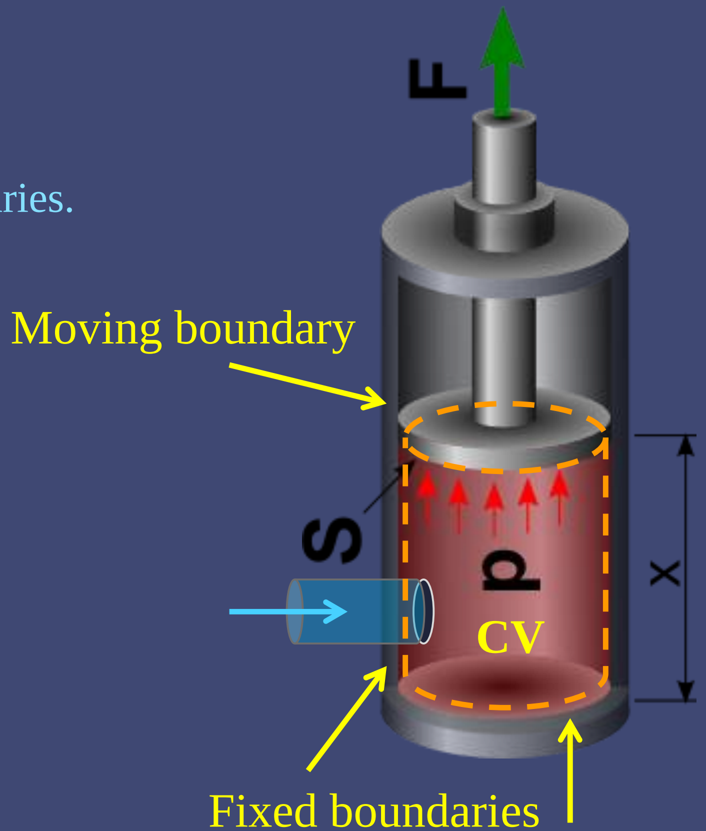
- **System**: A quantity of matter or a region in space chosen for study. Can be an engine, an element of the engine or apparatus, or a collection of elements is a process or an individual unit of a process. The system also may be a collection of sub-systems. Systems may be considered to be *closed* or *open*.
- **Surroundings**: The mass or region outside the system
- **Boundary**: The real or imaginary surface that separates the system from its surroundings. The boundary of a system can be *fixed* or *movable*.
- **Closed system (Control mass):** A fixed amount of mass, and no mass can cross its boundary

- **Open system (control volume)**: A properly selected region in space. It usually encloses a device that involves mass flow such as a compressor, turbine, or nozzle. Both mass and energy can cross the boundary of a control volume.
- **Control surface**: The boundaries of a control volume. It can be real or imaginary.

An open system (a control volume) with one inlet and one exit

A control volume can involve fixed, moving, real, and imaginary boundaries.

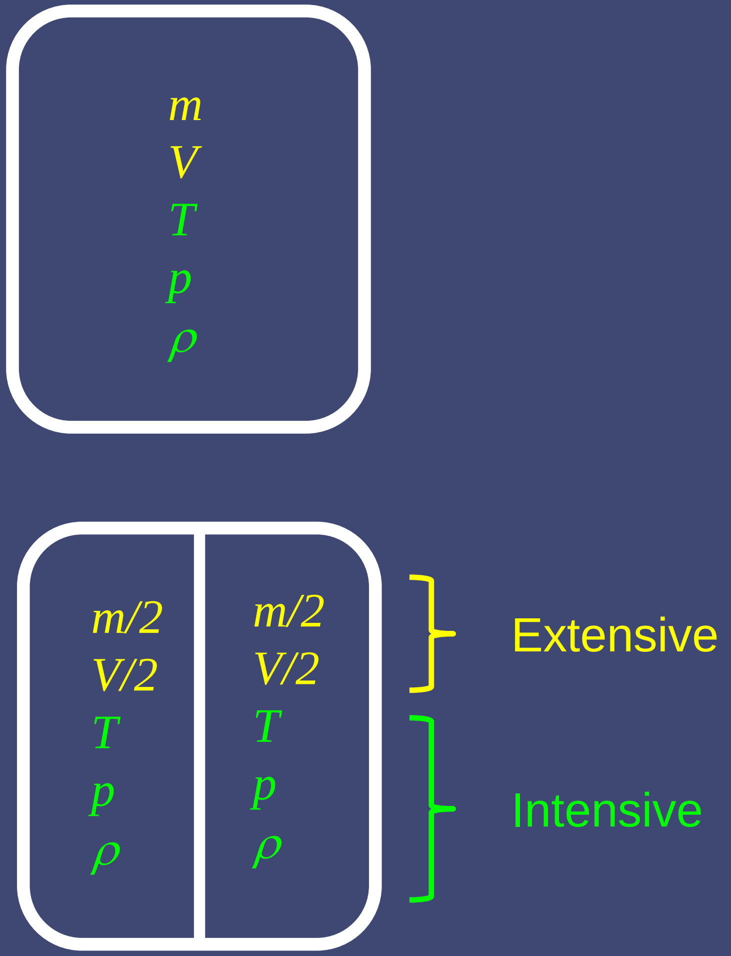




Properties of a System

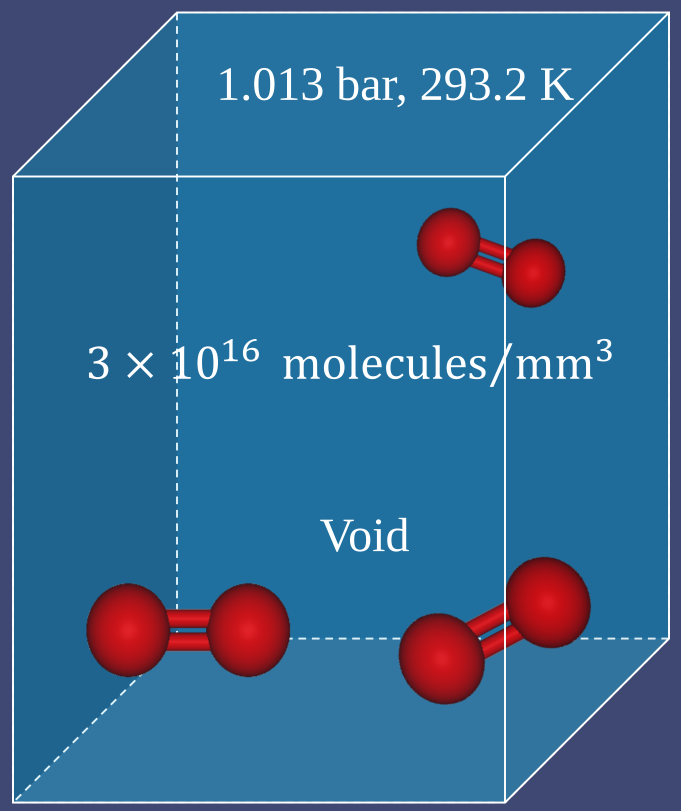
- **Property:** Any characteristic of a system. Some familiar properties are pressure p, temperature T, volume V, and mass m. Properties are considered to be either *intensive* or *extensive*.
- **Intensive properties:** Those that are independent of the mass of a system, such as temperature, pressure, and density.
- **Extensive properties:** Those whose values depend on the size—or extent—of the system.
- **Specific properties:** Extensive properties per unit mass, then becoming intensive properties.

$$v = V/m \qquad u = U/m$$



Continuum

- Matter is made up of atoms that are widely spaced in the gas phase. Yet it is very convenient to disregard the atomic nature of a substance and view it as a continuous, homogeneous matter with no holes, that is, a **continuum**.
- The continuum idealization allows us to treat properties as point functions and to assume the properties vary continually in space with no jump discontinuities.
- This idealization is valid as long as the size of the system we deal with is large relative to the space between the molecules.
- This is the case in practically problems in macro scale.
- In this course we will limit our consideration to substances that can be modeled as a continuum, otherwise will be explicitly stated.



Despite the relatively large gaps between molecules, a gas can usually be treated as a continuum because of the very large number of molecules even in an extremely small volume

The most common usage of the adjective specific, refers to an extensive property per unit mass. (in some other instances may be per unit size, i.e. mass, volume, etc.)

Lets take lower case ϕ, as any specific property (most of the Engineering books use lowercase variables to designate specific variables, but I suggest to use a hat over the variable)

$$\phi = \hat{\Phi} = \frac{d\Phi}{dm}$$

If this specific property ϕ is multiplied by the density and velocity, the flux of the variable Φ is obtained

$$\rho\, v\, \phi = \rho\, v\, \hat{\Phi} = \rho\, v \frac{d\Phi}{dm} = \rho\, \frac{dx}{dt}\frac{d\Phi}{dm} = \frac{\rho}{A}\frac{A\, dx}{dt}\frac{d\Phi}{dm} = \frac{1}{A}\frac{dm}{dt}\frac{d\Phi}{dm} = \frac{1}{A}\frac{d\Phi}{dt}$$

Here flux stand for flow per cross sectional area

Physical meaning of different fluxes and expression of flow

Flux	property	Flow symbol	Units of flow	Expression for flow
$\underline{v}$	Volume	$\dot{V}$	m^3/s	$\int \underline{v} \cdot \underline{n}\, dA$
$\rho\, \underline{v}$	mass	$\dot{m}$	kg/s	$\int \rho\, \underline{v} \cdot \underline{n}\, dA$
$\phi\, \rho\, \underline{v}$	Φ	$\dot{\Phi}$	prop/s	$\int \rho\, \phi\, \underline{v} \cdot \underline{n}\, dA$
$\underline{q} = \underline{\dot{Q}}_A$	heat	$\dot{Q}$	J/s	$\int \underline{q} \cdot \underline{n}\, dA = \int \underline{\dot{Q}}_A \cdot \underline{n}\, dA$
$\rho \underline{v}\, \underline{v}$	momentum	$\dot{m}\langle \underline{v} \rangle \beta$	N	$\int \rho \underline{v}\, \underline{v} \cdot \underline{n}\, dA$
$\rho \hat{E}\, \underline{v}$	Energy, E	$\dot{m}\hat{E}$	J/s	$\int \rho \hat{E} \underline{v} \cdot \underline{n}\, dA$

$\phi = \hat{\Phi}$ $\quad \beta = \dfrac{\langle v^2 \rangle}{\langle v \rangle^2}$ $\quad \hat{E} = \hat{U} + gz + \dfrac{\langle v^3 \rangle}{2\langle v \rangle}$

This is expression for flow not flux, in math jargon the term flux and flow are interchangeable

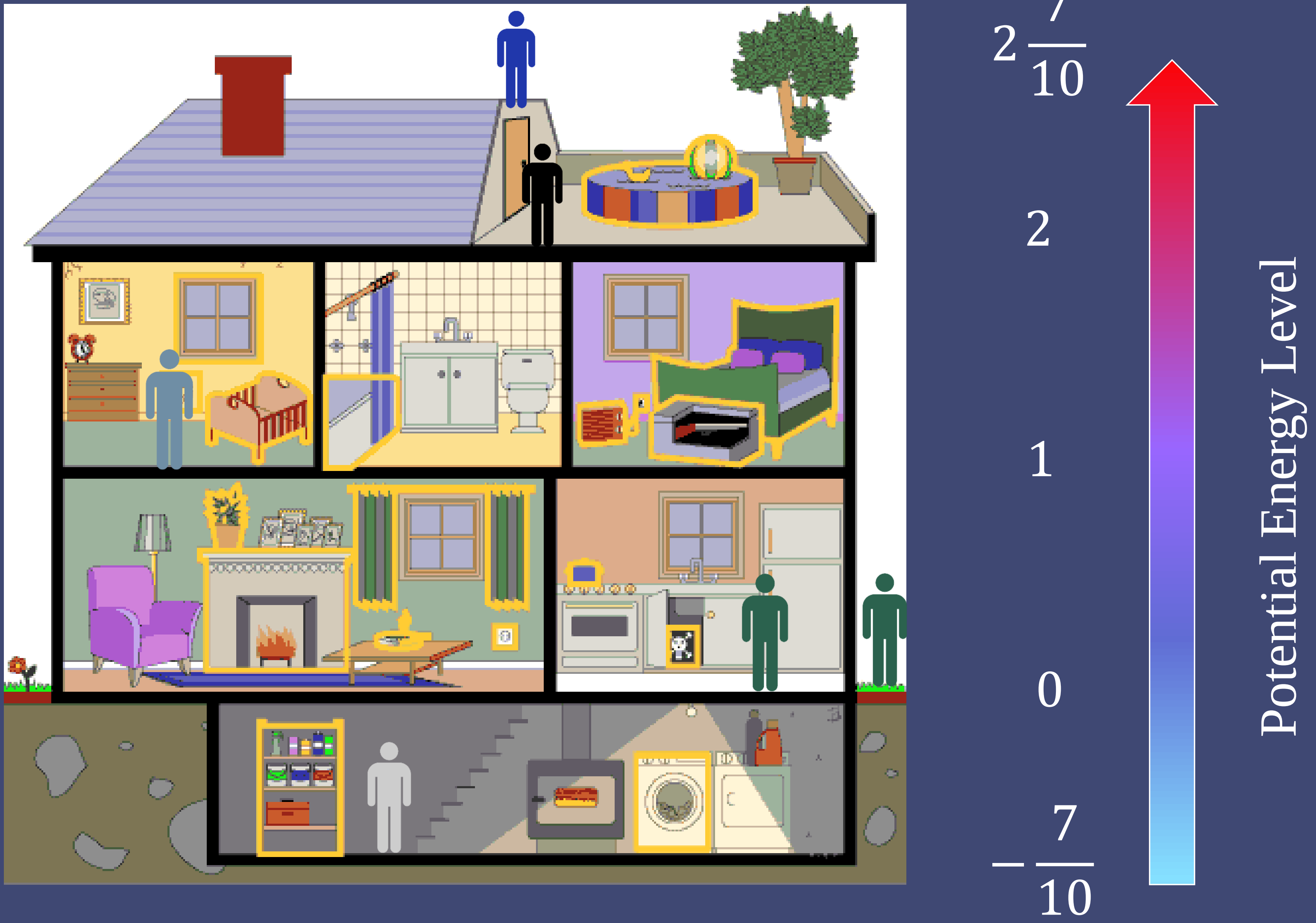
$$\dot{\Phi} = \frac{d\Phi}{dt}$$

$$\hat{\Phi} = \frac{d\Phi}{dm}$$

$$\langle \phi \rangle = \frac{\int \phi\, dA}{\int dA}$$

Frame of reference or reference: Condition or set of properties taken as a reference.

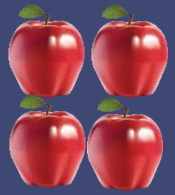
Some properties may take different values depending on the frame of reference, one case already studied in thermodynamics is for instance the enthalpy, where its value depends on the temperature, pressure, aggregation state, phase and of course purity. A negative value is nothing more than knowing that its energy level is below the reference.


$2\frac{7}{10}$
2
1
0
$-\frac{7}{10}$
Potential Energy Level

UNITS

1 kg_f

1 N



1 lb_f

A body weighing 150 lb_f in Earth will weigh only 57 lb_f on Mars and 25 lb_f over the Moon's surface

How come ?

176# over the Earth at rest

528# at launch

352# far from Earth

$$W = mg \qquad (\mathrm{N})$$

W weight
m mass
g gravitational acceleration

Units need to be specified

INERTIAL FRAME OF REFERNCE

Body forces measured with spring depend not only of the extent or size of the system, also depend on the intensity of the force field, location or frame of reference (inertial or non-inertial)

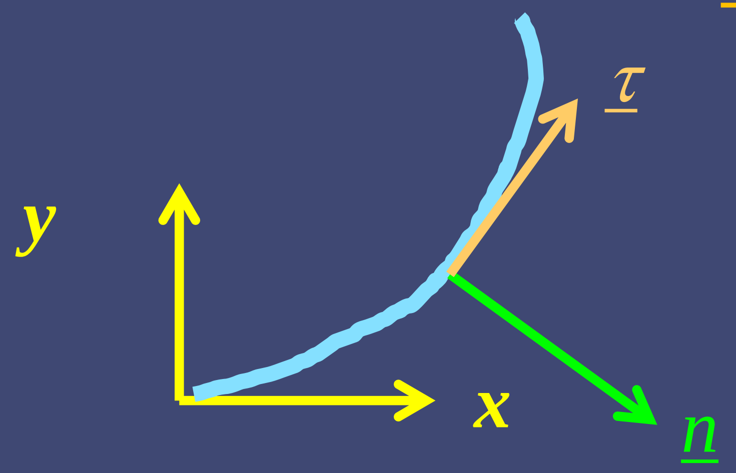
Conservation of dimensions

For any equation to be valid, every term in the equation must have the same physical character, i.e. the same net dimensions (and consequently the same units in any consistent system of units). This is known as the law of conservation of dimensions.

Example: The vertical elevation (z) and the horizontal distance (x) at any time for a projectile fired from a gun

$$z = a\,x + b\,x^2$$

($\underline{n}$, $\underline{\tau}$) Normal unit and tangential unit vectors in a 2D system



$\underline{\hat{\tau}}$

$|\underline{\tau}| = 1$

$|\underline{n}| = 1$

$\underline{\hat{n}}$

Both are vectors of magnitude 1, the first one is tangent to the boundary and the second is perpendicular pointing outward.

Example of vector formulae. General form of constitutive equations in vector form

- ✓ Fourier's Law

(Joseph Fourier-Jean Baptiste Biot, 1822)

$$\underline{q} = \underline{\dot{Q}_A} = \frac{\dot{Q}}{A} = -\underline{\underline{k}} \cdot \underline{\nabla} T$$

- ✓ Newton's Law

(Isaac Newton, 1701)

$$\underline{q} = \underline{\dot{Q}_A} = \frac{\dot{Q}}{A} = h(T - T_\infty)\, \underline{n}$$

Heat Flux

- ✓ Stefan-Boltzmann's Law

(Joseph Stefan 1879, Ludwing Boltzmann 1884)

$$\underline{q} = \underline{\dot{Q}_A} = \frac{\dot{Q}}{A} = \sigma\, \varepsilon\left(T^4 - T_{sky}^4\right) \underline{n}$$

The sign of Convection and Radiation equations will depend of selection of the system

$$\vec{\nabla} = \underline{\nabla}$$ ***Nabla* – the *del* operator**

$$\underline{\nabla} p = \underline{i}\,\frac{\partial p}{\partial x} + \underline{j}\,\frac{\partial p}{\partial y} + \underline{k}\,\frac{\partial p}{\partial z}$$

Nabla – An inverted uppercase delta appears in the differential form of many Equations in nature (e.g. the all four Maxwell’s Equations). This symbol represents a vector differential operator called “*nabla*” or *del*, and its presence instructs you to take the derivatives of the quantity on which the operator is acting

$$\vec{\nabla} = \underline{\nabla}$$

The exact form of those derivatives depends on the symbols following the operator

with $\underline{\nabla} \cdot (\quad)$

signifying divergence

$\underline{\nabla} \times (\quad)$

indicating curl

---

and

$\underline{\nabla}(\quad)$

meaning gradient.

$$\underset{\rightarrow}{\nabla} = \underline{\nabla}$$

$\underline{\nabla}$ Is an instruction to take derivatives in three orthogonal directions

$$\underline{\nabla}(\quad) = \underline{\mathrm{i}}\frac{\partial(\quad)}{\partial x} + \underline{\mathrm{j}}\frac{\partial(\quad)}{\partial y} + \underline{\mathrm{k}}\frac{\partial(\quad)}{\partial z}$$

For Cartesian coordinates *x*, *y* and *z*

$$\underline{\nabla}(\quad) = \hat{\mathrm{e}}_{\mathrm{x}}\frac{\partial(\quad)}{\partial x} + \hat{\mathrm{e}}_{\mathrm{y}}\frac{\partial(\quad)}{\partial y} + \hat{\mathrm{e}}_{\mathrm{z}}\frac{\partial(\quad)}{\partial z}$$

$$\underline{\nabla}(\quad) = \hat{\imath}\frac{\partial(\quad)}{\partial x} + \hat{\jmath}\frac{\partial(\quad)}{\partial y} + \hat{k}\frac{\partial(\quad)}{\partial z}$$

Physicists prefer the hat for the unit vectors, but ChemE’s use hat for specific properties

$$\underline{\nabla} \cdot (\quad)$$

The divergence , *Del* dot

Divergence. James Clerk Maxwell coined the term “convergence” to describe the mathematical operation that measures the rate ay which electric field lines “flow” towards points of negative electrical charge. Few years latter, Oliver Heaviside suggested the term of “divergence” for the same quantity with opposite sign. Thus, positive divergence is associated with the “flow” of electric field lines away from positive charge.

Both flux and divergence deal with the flow of a vector field; flux is defined per unit area, while divergence applies to individual points. In the case of fluid flow, the divergence at any point is a measure of the tendency of the flow vector to diverge from that point. (that is , to carry more material away from it than is brought toward it). Thus points of positive divergence are sources (faucets in situation involving fluid flow, positive electric charge in electrostatics), while points of negative divergence are sinks (drains in fluid flow, negative charge in electrostatics)

$$div(\underline{B}) = \underline{\nabla} \cdot \underline{B} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint \underline{B} \cdot \underline{n} \, dA$$

Conservative equation, to show the typical vector operations and operators

$$\frac{\partial[\rho\hat{E}]}{\partial t} + \underline{\nabla}\cdot[\rho\,\underline{v}\,\hat{E}] = \underline{\nabla}\cdot\left[\underline{\underline{\tau}}\cdot\underline{v}\right] - \underline{\nabla}\cdot\left[p\,\underline{\underline{I}}\cdot\underline{v}\right] - \underline{\nabla}\cdot\underline{q} + \dot{Q}_v$$

$$\frac{\partial[\rho\hat{E}]}{\partial t} + \underline{\nabla}\cdot[\rho\,\underline{v}\,\hat{E}] = \underline{\nabla}\cdot\left[\underline{\underline{\tau}}\cdot\underline{v}\right] - \underline{\nabla}\cdot\left[p\,\underline{\underline{I}}\cdot\underline{v}\right] + \underline{\nabla}\cdot[k\underline{\nabla}T] + \dot{Q}_v$$

Accumulation Advection Viscous dissipation flowing power conduction generation

---

Gauss Divergence Theorem

$$\iiint \underline{\nabla}\cdot\underline{B}\,dV = \oiint \underline{B}\cdot\underline{n}\,dA$$

$$div(\,\underline{B}\,) = \underline{\nabla}\cdot\underline{B} = \lim_{\Delta V\to 0}\frac{1}{\Delta V}\oiint \underline{B}\cdot\underline{n}\,dA$$

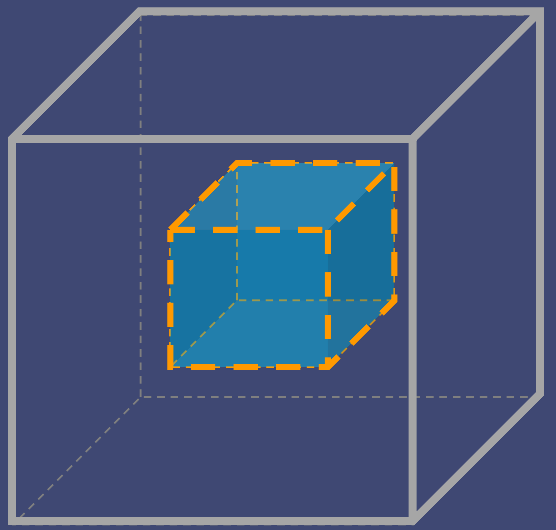
Divergence on different coordinate systems

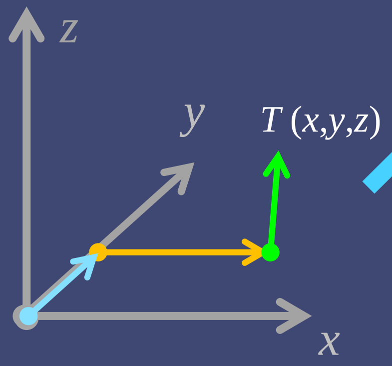
$$\underline{\nabla} \cdot \underline{B} = \frac{\partial(B_x)}{\partial x} + \frac{\partial\left(B_y\right)}{\partial y} + \frac{\partial(B_z)}{\partial z}$$

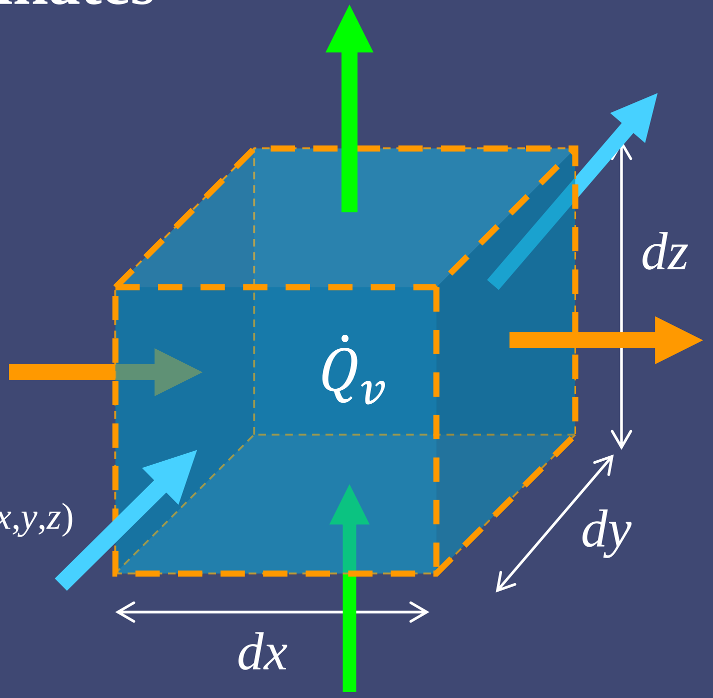
$$\underline{\nabla} \cdot \underline{B} = \frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial\left(B_\phi\right)}{\partial \phi} + \frac{\partial(B_z)}{\partial z}$$

$$\underline{\nabla} \cdot \underline{B} = \frac{1}{r^2}\frac{\partial(r^2 B_r)}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial(B_\theta \sin\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial\left(B_\phi\right)}{\partial \phi}$$

Rectangular coordinates







$$\underline{v} = v_x\ \hat{e}_x + v_z\ \hat{e}_z + v_y\ \hat{e}_y$$

$$d\underline{\ell} = dx\ \hat{e}_x + dz\ \hat{e}_z + dy\ \hat{e}_y$$

$$d\underline{A} = dy\, dz\ \hat{e}_x + dx\, dy\ \hat{e}_z + dz\, dx\ \hat{e}_y$$

$$dV = dx\, dz\, dy$$

$$d\underline{r} = dx\ \hat{e}_x + dz\ \hat{e}_z + dy\ \hat{e}_y$$

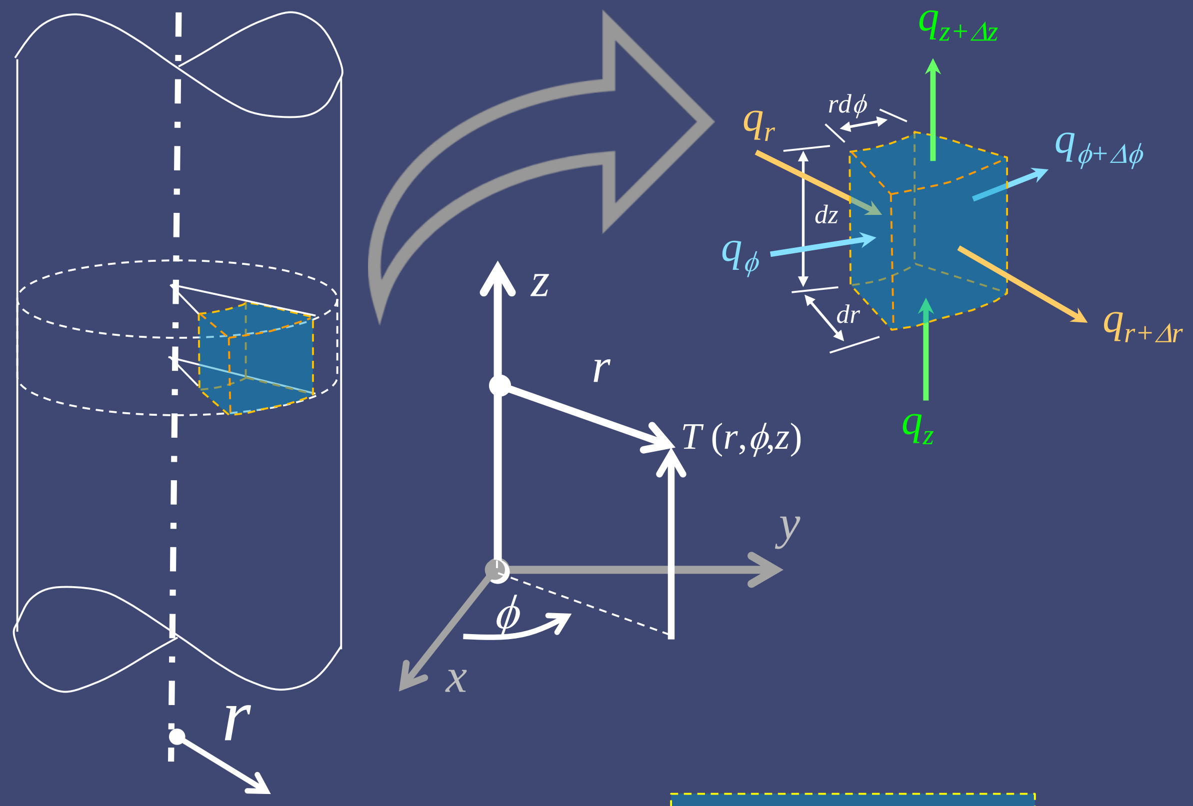
$$\underline{r} = x\ \hat{e}_x + y\ \hat{e}_y + z\ \hat{e}_z$$

$$\underline{r} = x\ \hat{\imath} + y\ \hat{\jmath} + z\ \hat{k}$$

$$\frac{\partial}{\partial x}\left(k\ \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\ \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\ \frac{\partial T}{\partial z}\right) + \dot{Q}_v = \rho c_p \frac{\partial T}{\partial t}$$

$$\rho c_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot \left[k \underline{\nabla} T\right] + \dot{Q}_v$$

Cylindrical coordinates



$$\rho c_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot \left[k \underline{\nabla} T\right] + \dot{Q}_v$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(k\, r\, \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\, \frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\, \frac{\partial T}{\partial z}\right) + \dot{Q}_v = \rho c_p \frac{\partial T}{\partial t}$$

$$\underline{v} = v_r\ \hat{\mathrm{e}}_{\mathrm{r}} + v_z\ \hat{\mathrm{e}}_{\mathrm{z}} + v_\phi\ \hat{\mathrm{e}}_{\phi}$$

$$d\underline{\ell} = dr\ \hat{\mathrm{e}}_{\mathrm{r}} + \ dz\ \hat{\mathrm{e}}_{\mathrm{z}} + r\ d\phi\ \hat{\mathrm{e}}_{\phi}$$

$$d\underline{A} = r\ d\phi\ dz\ \hat{\mathrm{e}}_{\mathrm{r}} + r\ d\phi\ dr\ \hat{\mathrm{e}}_{\mathrm{z}} + dr\ dz\ \hat{\mathrm{e}}_{\phi}$$

$$dV\ = r\ dz\ dr\ d\phi$$

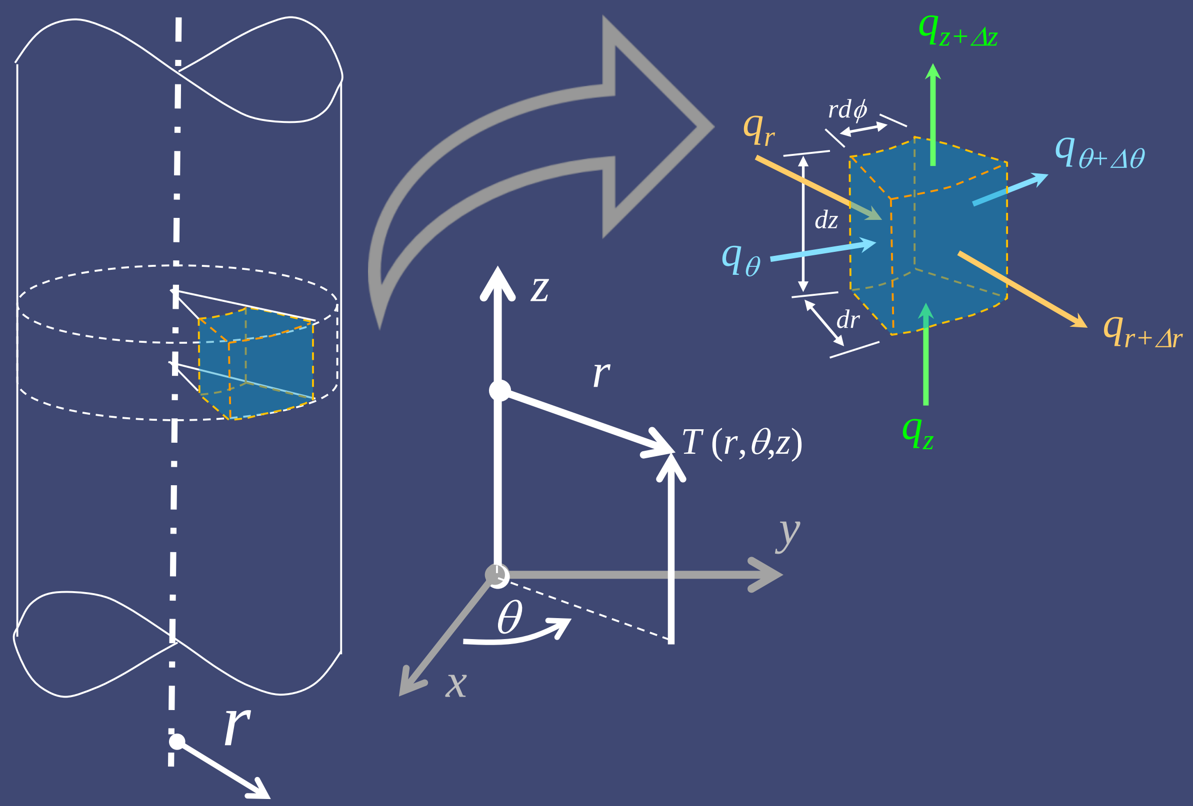
$$\hat{\mathrm{e}}_{\mathrm{r}} = \ \hat{e}_x \cos\phi + \hat{\mathrm{e}}_{\mathrm{y}}\ \sin\phi$$

$$\hat{\mathrm{e}}_{\theta} = -\,\hat{\mathrm{e}}_{\mathrm{x}} \sin\phi + \hat{\mathrm{e}}_{\mathrm{y}}\ \cos\phi$$

$$\frac{\partial\ \hat{\mathrm{e}}_{\mathrm{r}}}{\partial \phi} = \hat{\mathrm{e}}_{\phi}$$

$$\frac{\partial\ \hat{\mathrm{e}}_{\phi}}{\partial \phi} = -\,\hat{\mathrm{e}}_{\mathrm{r}}$$

Cylindrical coordinates



$$\rho c_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot \left[k \underline{\nabla} T\right] + \dot{Q}_v$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(k\, r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(k \frac{\partial T}{\partial \theta}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{Q}_v = \rho c_p \frac{\partial T}{\partial t}$$

$$\underline{v} = v_r\ \hat{\mathrm{e}}_\mathrm{r} + v_z\ \hat{\mathrm{e}}_\mathrm{z} + v_\theta\ \hat{\mathrm{e}}_\theta$$

$$d\underline{\ell} = dr\ \hat{\mathrm{e}}_\mathrm{r} + \ dz\ \hat{\mathrm{e}}_\mathrm{z} + r\ d\theta\ \hat{\mathrm{e}}_\theta$$

$$d\underline{A} = r\ d\phi\ dz\ \hat{\mathrm{e}}_\mathrm{r} + r\ d\theta\ dr\ \hat{\mathrm{e}}_\mathrm{z} + dr\ dz\ \hat{\mathrm{e}}_\theta$$

$$dV = r\ dz\ dr\ d\theta$$

$$\hat{\mathrm{e}}_\mathrm{r} = \ \hat{\mathrm{e}}_\mathrm{x} \cos\theta + \hat{\mathrm{e}}_\mathrm{y} \sin\theta$$

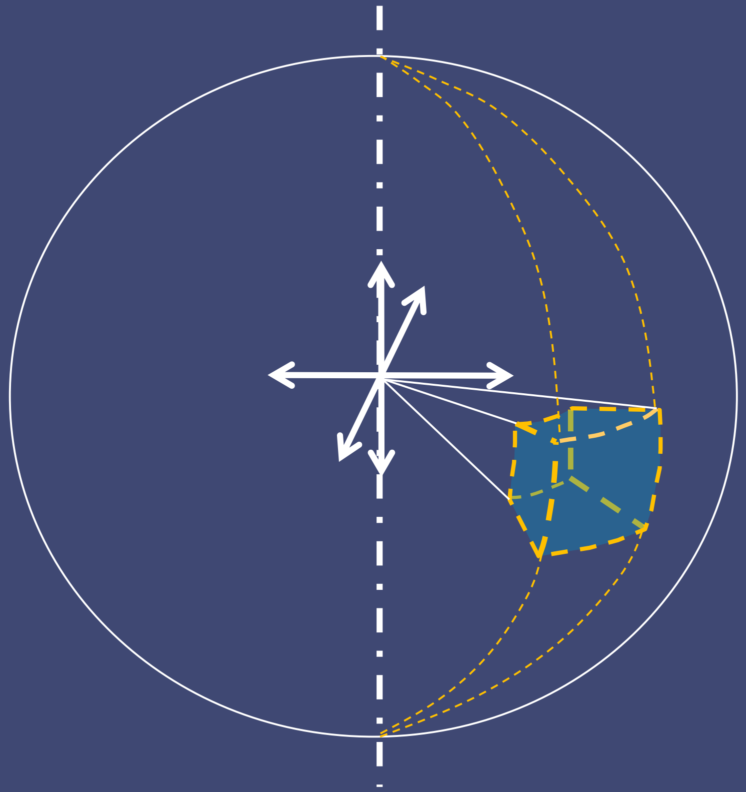
$$\hat{\mathrm{e}}_\theta = -\ \hat{\mathrm{e}}_\mathrm{x} \sin\theta + \hat{\mathrm{e}}_\mathrm{y}\ \cos\theta$$

Disclaimer: people may prefer to use "θ" instead of "ϕ" my advice is to always verify the nomenclature, but this will be trivial specially for cylindrical coordinates.

Need to consider these identities for differential operators

$$\boxed{\frac{\partial\ \hat{\mathrm{e}}_\mathrm{r}}{\partial \theta} = \hat{\mathrm{e}}_\theta \qquad \frac{\partial\ \hat{\mathrm{e}}_\theta}{\partial \theta} = -\ \hat{\mathrm{e}}_\mathrm{r}}$$

Spherical coordinates

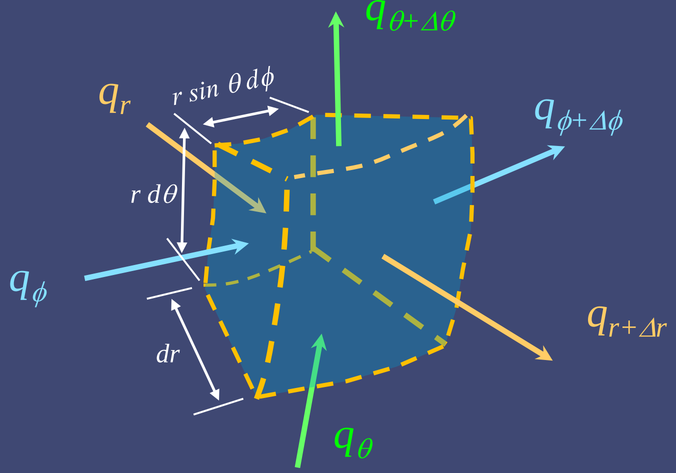


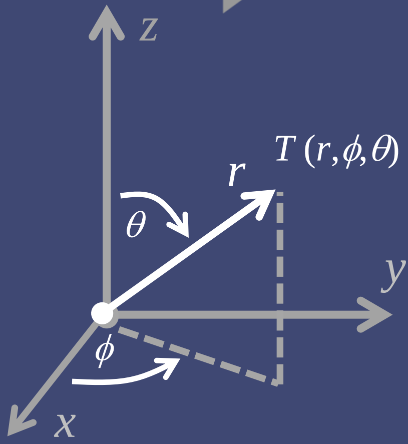
$$\underline{v} = v_r\ \hat{e}_r + v_\theta\ \hat{e}_\theta + v_\phi\ \hat{e}_\phi$$

$$d\underline{\ell} = dr\ \hat{e}_r + r\ d\theta\ \hat{e}_\theta + r\sin\theta\ d\phi\ \hat{e}_\phi$$

$$d\underline{A} = r^2\sin\theta\ d\theta\ d\phi\ \hat{e}_r + r\sin\theta\ d\phi\ dr\ \hat{e}_\theta + r\ dr\ d\theta\ \hat{e}_\phi$$

$$dV = r^2\sin\theta\ dr\ d\theta\ d\phi$$





$$\hat{e}_r = \hat{e}_x\sin\theta\cos\phi + \hat{e}_y\sin\theta\sin\phi + \hat{e}_z\cos\theta$$

$$\hat{e}_\theta = \hat{e}_x\cos\theta\cos\phi + \hat{e}_y\cos\theta\sin\phi - \hat{e}_z\sin\theta$$

$$\hat{e}_\phi = -\hat{e}_x\sin\phi + \hat{e}_y\cos\phi$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(k\, r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(k\frac{\partial T}{\partial\phi}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(k\sin\theta\frac{\partial T}{\partial\theta}\right) + \dot{Q}_v = \rho c_p\frac{\partial T}{\partial t}$$

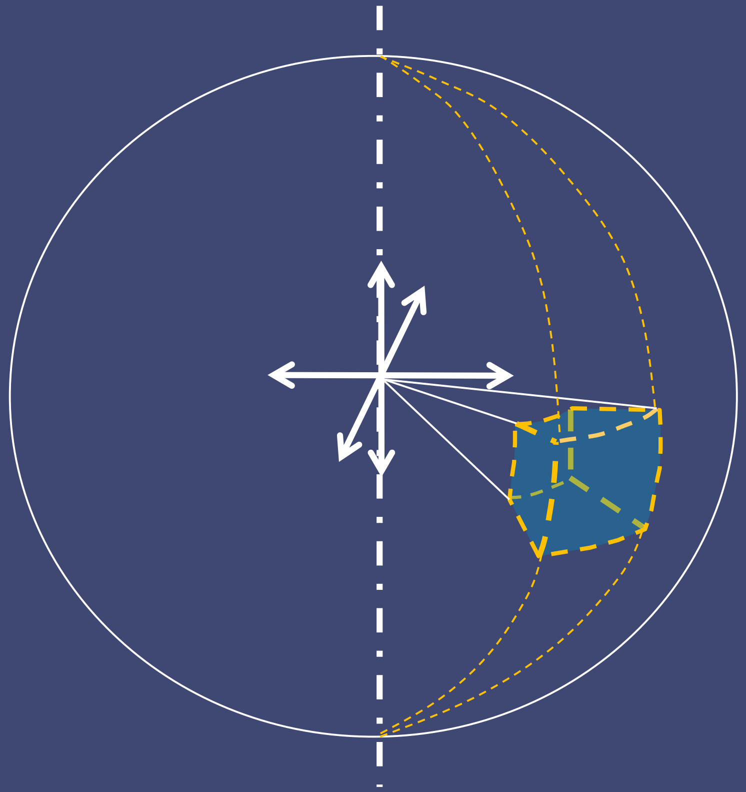
Spherical coordinates

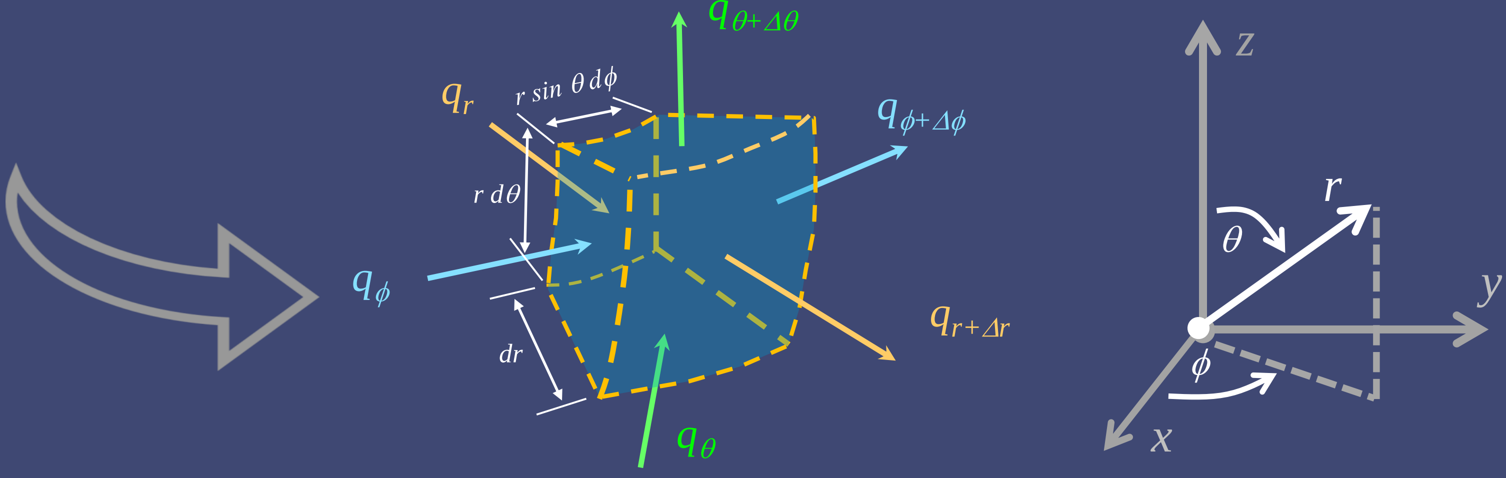
$$\underline{v} = v_r\,\hat{\mathrm{e}}_\mathrm{r} + v_\theta\,\hat{\mathrm{e}}_\theta + v_\phi\,\hat{\mathrm{e}}_\phi$$

$$d\underline{\ell} = dr\,\hat{\mathrm{e}}_\mathrm{r} + r\,d\theta\,\hat{\mathrm{e}}_\theta + r\sin\theta\,d\phi\,\hat{\mathrm{e}}_\phi$$

$$d\underline{A} = r^2\sin\theta\,d\theta\,d\phi\,\hat{\mathrm{e}}_\mathrm{r} + r\sin\theta\,d\phi\,dr\,\hat{\mathrm{e}}_\theta + r\,dr\,d\theta\,\hat{\mathrm{e}}_\phi$$

$$dV = r^2\sin\theta\,dr\,d\theta\,d\phi$$





$$\hat{\mathrm{e}}_\mathrm{r} = \hat{\mathrm{e}}_\mathrm{x}\sin\theta\cos\phi + \hat{\mathrm{e}}_\mathrm{y}\sin\theta\sin\phi + \hat{\mathrm{e}}_\mathrm{z}\cos\theta$$

$$\hat{\mathrm{e}}_\theta = \hat{e}_x\cos\theta\cos\phi + \hat{\mathrm{e}}_\mathrm{y}\cos\theta\sin\phi - \hat{\mathrm{e}}_\mathrm{z}\sin\theta$$

$$\hat{\mathrm{e}}_\phi = -\hat{\mathrm{e}}_\mathrm{x}\sin\phi + \hat{\mathrm{e}}_\mathrm{y}\cos\phi$$

Need to consider these identities for differential operators

$$\frac{\partial\,\hat{\mathrm{e}}_\mathrm{r}}{\partial\theta} = \hat{\mathrm{e}}_\theta \qquad \frac{\partial\,\hat{\mathrm{e}}_\mathrm{r}}{\partial\phi} = \hat{\mathrm{e}}_\phi\sin\theta$$

$$\frac{\partial\,\hat{\mathrm{e}}_\theta}{\partial\theta} = -\,\hat{\mathrm{e}}_\mathrm{r} \qquad \frac{\partial\,\hat{\mathrm{e}}_\theta}{\partial\phi} = \hat{\mathrm{e}}_\phi\cos\theta$$

$$\frac{\partial\,\hat{\mathrm{e}}_\phi}{\partial\theta} = 0 \qquad \frac{\partial\,\hat{\mathrm{e}}_\phi}{\partial\phi} = -\left(\hat{\mathrm{e}}_\mathrm{r}\sin\theta + \hat{\mathrm{e}}_\phi\cos\theta\right)$$

$$\underline{\nabla} \times (\quad)$$

The *Del* cross – the *curl*

The *curl* of a vector field is a measure of the field's tendency to circulate about a point- much like a divergence is a measure of the tendency of the field to flow away from a point. Maxwell settled on "*curl*" after considering several alternatives, including "*turn*" and "*twirl*"

Just as the divergence is found by considering the flux through an infinitesimal surface surrounding the point of interest, the curl at specified point may be found by considering the circulation per unit area over an infinitesimal path around that point. The mathematical definition of the curl of a vector field $\underline{B}$ is:

$$curl(\,\underline{B}\,) \cdot \underline{n} = [\underline{\nabla} \times \underline{B}] \cdot \underline{n} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint \underline{B} \cdot d\underline{\ell}$$

Curl on different coordinate systems

Rectangular

$$\underline{\nabla} \times \underline{B} = \hat{\mathrm{e}}_{\mathrm{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathrm{e}}_{\mathrm{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathrm{e}}_{\mathrm{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

Cylindrical

$$\underline{\nabla} \times \underline{B} = \hat{\mathrm{e}}_{\mathrm{r}}\left(\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right) + \hat{\mathrm{e}}_{\phi}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) + \hat{\mathrm{e}}_{\mathrm{z}}\frac{1}{r}\left(\frac{\partial\left(rB_\phi\right)}{\partial r} - \frac{\partial B_r}{\partial \phi}\right)$$

Spherical

$$\underline{\nabla} \times \underline{B} = \hat{\mathrm{e}}_{\mathrm{r}}\frac{1}{r\sin\theta}\left(\frac{\partial\left(B_\phi \sin\theta\right)}{\partial \theta} - \frac{\partial B_\theta}{\partial \phi}\right) + \hat{\mathrm{e}}_{\theta}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial B_r}{\partial \phi} - \frac{\partial\left(rB_\phi\right)}{\partial r}\right) + \hat{\mathrm{e}}_{\phi}\frac{1}{r}\left(\frac{\partial\left(rB_\theta\right)}{\partial r} - \frac{\partial B_r}{\partial \theta}\right)$$

$\underline{\nabla}$ The gradient

$\underline{\nabla}$ Is an instruction to take derivatives in three orthogonal directions

$$\underline{\nabla}(\quad) = \hat{e}_x \frac{\partial(\quad)}{\partial x} + \hat{e}_y \frac{\partial(\quad)}{\partial y} + \hat{e}_z \frac{\partial(\quad)}{\partial z}$$

For Cartesian coordinates *x*, *y* and *z*

Similar to the divergence and the *curl*, the gradient involves partial derivatives taken in three orthogonal directions. However, whereas the divergence measures the tendency of a vector field to flow away from a point and the curl indicates the circulation of a vector field around the a point, the gradient applies to a scalar fields. Unlike a vector field, a scalar field is specified entirely by its magnitude at various locations.

The information given by the gradient of a scalar field is: Magnitude tells how quickly the field is changing over space, and the direction of the gradient indicates the direction in that the field is changing most quickly with distance.

$\underline{\nabla}$ The gradient

$\underline{\nabla}(\quad)$

If Ψ is a scalar representing the height of terrain above the sea level, the magnitude of the gradient at any location tells you how steeply the ground is sloped at that location, and the direction of the gradient points uphill along the steepest slope..

$$grad(\Psi) = \underline{\nabla}\Psi = \frac{\partial \Psi}{\partial x}\hat{e}_x + \frac{\partial \Psi}{\partial y}\hat{e}_y + \frac{\partial \Psi}{\partial z}\hat{e}_z$$

Rectangular

$$grad(\Psi) = \underline{\nabla}\Psi = \frac{\partial \Psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \Psi}{\partial \phi}\hat{e}_\phi + \frac{\partial \Psi}{\partial z}\hat{e}_z$$

Cylindrical

$$grad(\Psi) = \underline{\nabla}\Psi = \frac{\partial \Psi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \Psi}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial \phi}\hat{e}_\phi$$

Spherical

The location of the unit vector is not trivial, if the gradient (vector operator) operates over a scalar quantity, can be located before or after the derivative, but if operates a vector, you need to keep it before the derivative term.

∇^2 The Laplacian

The Laplacian of a scalar field is:

$$\nabla^2 B = \underline{\nabla} \cdot \underline{\nabla} B$$

The divergence of a gradient:

$$\nabla^2 B = \frac{\partial^2 B}{\partial x^2} + \frac{\partial^2 B}{\partial y^2} + \frac{\partial^2 B}{\partial z^2}$$

Rectangular

$$\nabla^2 B = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 B}{\partial \theta^2} + \frac{\partial^2 B}{\partial z^2}$$

Cylindrical

$$\nabla^2 B = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial B}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial B}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 B}{\partial \phi^2}$$

Spherical

Schey H.M. div, grad, curl, and all that. 4th edition. W.W. Norton, 2005

Gradient of a vector

Rectangular

$$\underline{\nabla}\,\underline{B} = \begin{bmatrix} \dfrac{\partial B_x}{\partial x}\,\hat{\imath}\hat{\imath} & \dfrac{\partial B_y}{\partial x}\,\hat{\imath}\hat{\jmath} & \dfrac{\partial B_z}{\partial x}\,\hat{\imath}\hat{k} \\ \dfrac{\partial B_x}{\partial y}\,\hat{\jmath}\hat{\imath} & \dfrac{\partial B_y}{\partial y}\,\hat{\jmath}\hat{\jmath} & \dfrac{\partial B_z}{\partial y}\,\hat{\jmath}\hat{k} \\ \dfrac{\partial B_x}{\partial z}\,\hat{k}\hat{\imath} & \dfrac{\partial B_y}{\partial z}\,\hat{k}\hat{\jmath} & \dfrac{\partial B_z}{\partial z}\,\hat{k}\hat{k} \end{bmatrix}$$

Cylindrical

$$\underline{\nabla}\,\underline{B} = \begin{bmatrix} \dfrac{\partial B_r}{\partial r}\,\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} & \dfrac{\partial B_\theta}{\partial r}\,\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\theta} & \dfrac{\partial B_z}{\partial r}\,\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{z}} \\ \dfrac{1}{r}\left(\dfrac{\partial B_r}{\partial \theta} - B_\theta\right)\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{r}} & \dfrac{1}{r}\left(\dfrac{\partial B_\theta}{\partial \theta} + B_r\right)\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\theta} & \dfrac{1}{r}\dfrac{\partial B_z}{\partial \theta}\,\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{z}} \\ \dfrac{\partial B_r}{\partial z}\,\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{r}} & \dfrac{\partial B_\theta}{\partial z}\,\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\theta} & \dfrac{\partial B_z}{\partial z}\,\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{z}} \end{bmatrix}$$

Spherical

$$\underline{\nabla}\,\underline{B} = \begin{bmatrix} \dfrac{\partial B_r}{\partial r}\,\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} & \dfrac{\partial B_\theta}{\partial r}\,\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\theta} & \dfrac{\partial B_\phi}{\partial r}\,\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\phi} \\ \dfrac{1}{r}\left(\dfrac{\partial B_r}{\partial \theta} - B_\theta\right)\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{r}} & \dfrac{1}{r}\left(\dfrac{\partial B_\theta}{\partial \theta} + B_r\right)\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\theta} & \dfrac{1}{r}\dfrac{\partial B_\phi}{\partial \theta}\,\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\phi} \\ \dfrac{1}{r\sin\theta}\left(\dfrac{\partial B_r}{\partial \phi} - B_\phi \sin\theta\right)\hat{\mathrm{e}}_{\phi}\hat{\mathrm{e}}_{\mathrm{r}} & \dfrac{1}{r\sin\theta}\left(\dfrac{\partial B_\theta}{\partial \phi} - B_\phi \cos\theta\right)\hat{\mathrm{e}}_{\phi}\hat{\mathrm{e}}_{\theta} & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial B_\phi}{\partial \phi} + \dfrac{B_r}{r} + \dfrac{B_\theta \cot\theta}{r}\right)\hat{\mathrm{e}}_{\phi}\hat{\mathrm{e}}_{\phi} \end{bmatrix}$$

Gradient of a vector

Cartesian

$$\underline{\nabla}\,\underline{B} = \begin{bmatrix} \frac{\partial B_x}{\partial x} & \frac{\partial B_y}{\partial x} & \frac{\partial B_z}{\partial x} \\ \frac{\partial B_x}{\partial y} & \frac{\partial B_y}{\partial y} & \frac{\partial B_z}{\partial y} \\ \frac{\partial B_x}{\partial z} & \frac{\partial B_y}{\partial z} & \frac{\partial B_z}{\partial z} \end{bmatrix}$$

Cylindrical

$$\underline{\nabla}\,\underline{B} = \begin{bmatrix} \frac{\partial B_r}{\partial r} & \frac{\partial B_\theta}{\partial r} & \frac{\partial B_z}{\partial r} \\ \frac{1}{r}\left(\frac{\partial B_r}{\partial \theta} - B_\theta\right) & \frac{1}{r}\left(\frac{\partial B_\theta}{\partial \theta} + B_r\right) & \frac{1}{r}\frac{\partial B_z}{\partial \theta} \\ \frac{\partial B_r}{\partial z} & \frac{\partial B_\theta}{\partial z} & \frac{\partial B_z}{\partial z} \end{bmatrix}$$

Spherical

$$\underline{\nabla}\,\underline{B} = \begin{bmatrix} \frac{\partial B_r}{\partial r} & \frac{\partial B_\theta}{\partial r} & \frac{\partial B_\phi}{\partial r} \\ \frac{1}{r}\left(\frac{\partial B_r}{\partial \theta} - B_\theta\right) & \frac{1}{r}\left(\frac{\partial B_\theta}{\partial \theta} + B_r\right) & \frac{1}{r}\frac{\partial B_\phi}{\partial \theta} \\ \frac{1}{r\sin\theta}\left(\frac{\partial B_r}{\partial \phi} - B_\phi \sin\theta\right) & \frac{1}{r\sin\theta}\left(\frac{\partial B_\theta}{\partial \phi} - B_\phi \cos\theta\right) & \left(\frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi} + \frac{B_r}{r} + \frac{B_\theta \cot\theta}{r}\right) \end{bmatrix}$$

Indices of unit vector are sometimes omitted from vectors and tensors (not recommended if you are not an expert)

Divergence of a Tensor

$$\underline{\nabla} \cdot \underline{\underline{B}}$$

Cartesian

$$\underline{\nabla} \cdot \underline{\underline{B}} = \begin{array}{l} \left[\frac{\partial B_{xx}}{\partial x} + \frac{\partial B_{yx}}{\partial y} + \frac{\partial B_{zx}}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{x}} \\ \left[\frac{\partial B_{xy}}{\partial x} + \frac{\partial B_{yy}}{\partial y} + \frac{\partial B_{zy}}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{y}} \\ \left[\frac{\partial B_{xz}}{\partial x} + \frac{\partial B_{yz}}{\partial y} + \frac{\partial B_{zz}}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{z}} \end{array}$$

Cylindrical

$$\underline{\nabla} \cdot \underline{\underline{B}} = \begin{array}{l} \left[\frac{\partial (B_{rr})}{\partial r} + \frac{1}{r}\frac{\partial (B_{\theta r})}{\partial \theta} + \frac{(B_{rr} - B_{\theta\theta})}{r} + \frac{\partial B_{rz}}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{r}} \\ \left[\frac{\partial (B_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial (B_{\theta\theta})}{\partial \theta} + \frac{(B_{r\theta} + B_{\theta r})}{r} + \frac{\partial B_{z\theta}}{\partial z}\right]\hat{\mathrm{e}}_{\theta} \\ \left[\frac{\partial (B_{rz})}{\partial r} + \frac{1}{r}\frac{\partial (B_{\theta z})}{\partial \theta} + \frac{\partial B_{zz}}{\partial z} + \frac{B_{rz}}{r}\right]\hat{\mathrm{e}}_{\mathrm{z}} \end{array}$$

Spherical

$$\underline{\nabla} \cdot \underline{\underline{B}} = \begin{array}{l} \left[\frac{1}{r^2}\frac{\partial (r^2 B_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (B_{\theta r}\sin\theta)}{\partial \theta} - \frac{(B_{\theta\theta} + B_{\phi\phi})}{r} + \frac{1}{r\sin\theta}\frac{\partial B_{\phi r}}{\partial \phi}\right]\hat{\mathrm{e}}_{\mathrm{r}} \\ \left[\frac{1}{r^3}\frac{\partial (r^3 B_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (B_{\theta\theta}\sin\theta)}{\partial \theta} + \frac{(B_{\theta r} - B_{r\theta} - B_{\phi\phi}\cot\theta)}{r} + \frac{1}{r\sin\theta}\frac{\partial B_{\phi\theta}}{\partial \phi}\right]\hat{\mathrm{e}}_{\theta} \\ \left[\frac{1}{r^3}\frac{\partial (r^3 B_{r\phi})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (B_{\theta\phi}\sin\theta)}{\partial \theta} + \frac{(B_{\phi r} - B_{r\phi} + B_{\phi\theta}\cot\theta)}{r} + \frac{1}{r\sin\theta}\frac{\partial B_{\phi\phi}}{\partial \phi}\right]\hat{\mathrm{e}}_{\phi} \end{array}$$

Laplacian of a Vector

$$\nabla^2 \underline{B} = \underline{\nabla} \cdot \underline{\nabla}\, \underline{B}$$

Cartesian

$$\nabla^2 \underline{B} = \begin{matrix} \left[\dfrac{\partial^2 B_x}{\partial x^2} + \dfrac{\partial^2 B_x}{\partial y^2} + \dfrac{\partial^2 B_x}{\partial z^2}\right] \hat{e}_x \\ \left[\dfrac{\partial^2 B_y}{\partial x^2} + \dfrac{\partial^2 B_y}{\partial y^2} + \dfrac{\partial^2 B_y}{\partial z^2}\right] \hat{e}_y \\ \left[\dfrac{\partial^2 B_z}{\partial x^2} + \dfrac{\partial^2 B_z}{\partial y^2} + \dfrac{\partial^2 B_z}{\partial z^2}\right] \hat{e}_z \end{matrix}$$

Cylindrical

$$\nabla^2 \underline{B} = \begin{matrix} \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial B_r}{\partial r}\right) - \dfrac{B_r}{r^2} + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial B_r}{\partial \theta}\right) - \dfrac{2}{r^2}\left(\dfrac{\partial B_\theta}{\partial \theta}\right) + \dfrac{\partial^2 B_r}{\partial z^2}\right] \hat{e}_r \\ \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial B_\theta}{\partial r}\right) - \dfrac{B_\theta}{r^2} + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial B_\theta}{\partial \theta}\right) + \dfrac{2}{r^2}\left(\dfrac{\partial B_r}{\partial \theta}\right) + \dfrac{\partial^2 B_\theta}{\partial z^2}\right] \hat{e}_\theta \\ \left[\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial B_z}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta}\left(\dfrac{\partial B_z}{\partial \theta}\right) + \dfrac{\partial^2 B_z}{\partial z^2}\right] \hat{e}_z \end{matrix}$$

Spherical

$$\nabla^2 \underline{B} = \begin{matrix} \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial B_r}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial B_r}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 B_r}{\partial \phi^2} - \dfrac{2}{r^2}\left(B_r + \dfrac{\partial B_\theta}{\partial \theta} + B_\theta \cot\theta + \dfrac{1}{\sin\theta}\dfrac{\partial B_\phi}{\partial \phi}\right)\right] \hat{e}_r \\ \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial B_\theta}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial B_\theta}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 B_\theta}{\partial \phi^2} + \dfrac{2}{r^2}\dfrac{\partial B_r}{\partial \theta} - \dfrac{1}{r^2\ \sin^2\theta}\left(B_\theta + 2\cos\theta\dfrac{\partial B_\phi}{\partial \phi}\right)\right] \hat{e}_\theta \\ \left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial B_\phi}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial B_\phi}{\partial \theta}\right) + \dfrac{1}{r^2 \sin^2\theta}\dfrac{\partial^2 B_\phi}{\partial \phi^2} + \dfrac{1}{r^2\ \sin^2\theta}\left(2\sin\theta\dfrac{\partial B_r}{\partial \phi} + 2\cos\theta\dfrac{\partial B_\theta}{\partial \phi} - B_\phi\right)\right] \hat{e}_\phi \end{matrix}$$

Some other useful operators and operations

$$\underline{A}\cdot\underline{\nabla}\,\underline{B}$$

Cartesian

$$\underline{A}\cdot\underline{\nabla}\,\underline{B} = \begin{array}{l} \left[A_x\frac{\partial B_x}{\partial x} + A_y\frac{\partial B_x}{\partial y} + A_z\frac{\partial B_x}{\partial z}\right]\hat{\mathrm{i}} \\ \left[A_x\frac{\partial B_y}{\partial x} + A_y\frac{\partial B_y}{\partial y} + A_z\frac{\partial B_y}{\partial z}\right]\hat{\mathrm{j}} \\ \left[A_x\frac{\partial B_z}{\partial x} + A_y\frac{\partial B_z}{\partial y} + A_z\frac{\partial B_z}{\partial z}\right]\hat{\mathrm{k}} \end{array}$$

Cylindrical

$$\underline{A}\cdot\underline{\nabla}\,\underline{B} = \begin{array}{l} \left[A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\left(\frac{\partial B_r}{\partial \theta} - B_\theta\right) + A_z\frac{\partial B_r}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{r}} \\ \left[A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\left(\frac{\partial B_\theta}{\partial \theta} + B_r\right) + A_z\frac{\partial B_\theta}{\partial z}\right]\hat{\mathrm{e}}_{\theta} \\ \left[A_r\frac{\partial B_z}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_z}{\partial \theta} + A_z\frac{\partial B_z}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{z}} \end{array}$$

Spherical

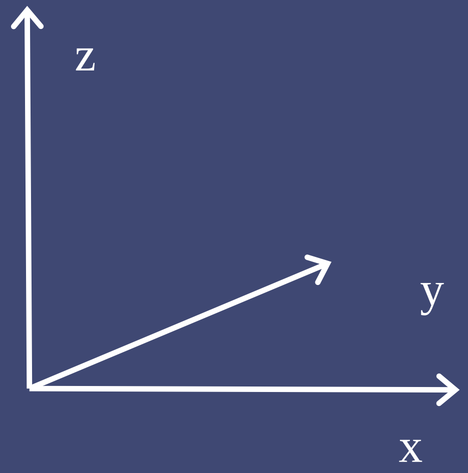
$$\underline{A}\cdot\underline{\nabla}\,\underline{B} = \begin{array}{l} \left[A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\left(\frac{\partial B_r}{\partial \theta} - B_\theta\right) + \frac{A_\phi}{r\sin\theta}\left(\frac{\partial B_r}{\partial \phi} - B_\phi\sin\theta\right)\right]\hat{\mathrm{e}}_{\mathrm{r}} \\ \left[A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\left(\frac{\partial B_\theta}{\partial \theta} + B_r\right) + \frac{A_\phi}{r\sin\theta}\left(\frac{\partial B_\theta}{\partial \phi} - B_\phi\cos\theta\right)\right]\hat{\mathrm{e}}_{\theta} \\ \left[A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial \theta} + \left(\frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi\,B_r}{r} + \frac{A_\phi\,B_\theta\cot\theta}{r}\right)\right]\hat{\mathrm{e}}_{\phi} \end{array}$$

Additional symbols using during the class

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

$$\prod_{i=1}^{n} a_i = a_1\ a_2\ a_3\ \cdots\ a_n$$

Analogies among heat transfer, mass transfer, electrical current, flow trough porous media and momentum transport (*)



$$\underline{q} = -k\, \underline{\nabla}\, T$$ Fourier's Law

$$\underline{J_A} = -\, \mathcal{D}_{AB}\, \underline{\nabla}\, C_A$$ Fick's Law

$$\underline{j_e} = -\, \sigma_e\, \underline{\nabla}\, \Phi$$ Ohm's Law $\rho_e = R_e(A/L) \quad \sigma_e = (1/\rho_e)$

$$\underline{u} = -\, \frac{K}{\mu}\, \underline{\nabla}\, p$$ Darcy's Law

$$\underline{\underline{\tau}} = 2\,\mu \left[\underline{\underline{\Gamma}} - \frac{1}{3} \underline{\nabla} \cdot \underline{v}\, \underline{\underline{I}} \right]$$ Stoke's Postulate (*)

Some applications of fluid dynamics may use the advantage of analogies, this is, because some of the governing equations are similar compared with heat transfer, mass transfer and applications of electromagnetism. Even momentum transport under specific conditions may use this concept of analogy. Never the less in momentum transport we are dealing with tensors, then caution in the use of the analogy is recommended

Analogies among heat transfer, mass transfer, electrical current, flow trough porous media and momentum transport (*)

Heat rate per unit area
or heat flux [J/(s m^2)]

$$\underline{q} = -k\,\underline{\nabla}\,T$$

k is thermal conductivity [W/m-K],
and T is temperature [K]

Molar rate per unit area
or molar flux [kmol/(s m^2)]

$$\underline{J}_A = -\,\mathcal{D}_{AB}\,\underline{\nabla}\,C_A$$

D_{AB} is diffusivity of component A in phase B [m^2/s],
and C_A is concentration of component A [kmol/m^3]

Charge rate per unit area
or electrical current flux
[C/(s m^2)]

$$\underline{j}_e = -\,\sigma_e\,\underline{\nabla}\,\Phi$$

σ_e is electrical conductivity [1/ohm-m],
and Φ is voltage [V]

Volume rate per unit area
or volume flux
[m^3/(s m^2)] or Darcy velocity

$$\underline{u} = -\,\frac{K}{\mu}\,\underline{\nabla}\,p$$

K is permeability [m^2], μ is viscosity [Pa-s]
and p is pressure [Pa]

Linear momentum rate per unit area
or linear momentum flux
[(kg-m/s)/(s m^2)]

$$\underline{\underline{\tau}} = \mu\left[2\,\underline{\underline{\Gamma}}\right]$$

μ is viscosity [Pa-s] and

$\underline{\underline{\Gamma}}$ is the strain rate tensor

Disclaimer: More complex form of constitutive equation is required, but under specific conditions the previous analogies are valid, the complete form of these equations or alternative forms are given here

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{\Gamma}} + (\kappa - 2\mu/3)\left(\underline{\nabla}\cdot\underline{v}\right)\underline{\underline{I}}$$

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{v} + \left(\underline{\nabla}\,\underline{v}\right)^T\right]$$

μ is shear viscosity [Pa-s] , and
κ is dilatational or bulk viscosity [Pa-s]

$$\underline{u} = -(K/\mu)\left[\underline{\nabla}\,p - \rho\underline{g}\right]$$

Flow across porous media will occur if there is a pressure gradient or gravitational field ($\underline{g}$) in the direction of the flow

Molar rate per unit area or molar flux

Mass rate per unit area or mass flux

$$\underline{J}_A = -\,\mathcal{D}_{AB}\,\underline{\nabla}\,C_A \qquad \underline{J}_A = -C\,\mathcal{D}_{AB}\,\underline{\nabla}\,x_A \qquad \underline{j}_A = -\rho\,\mathcal{D}_{AB}\,\underline{\nabla}\,\omega_A$$

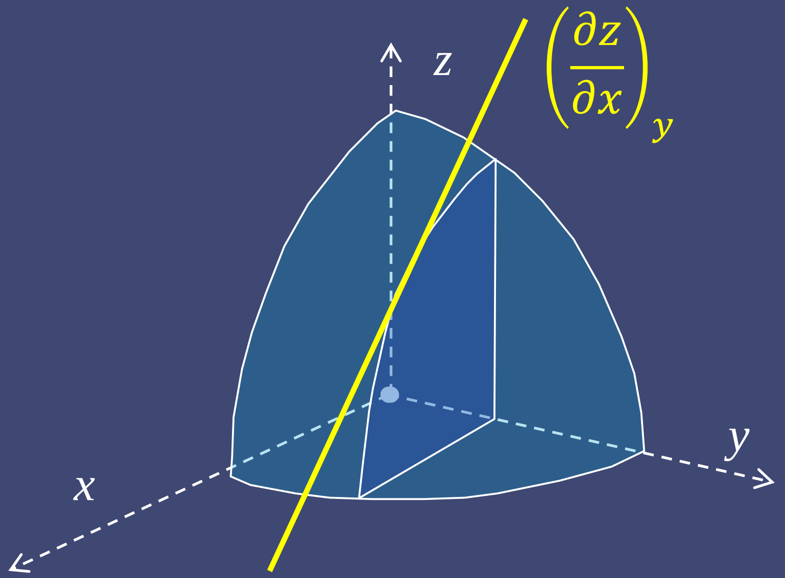
C is molar density [kmol/m^3], ρ is mass density [kg/m^3], x_A is molar fraction of component A, ω_A is mass fraction of component A

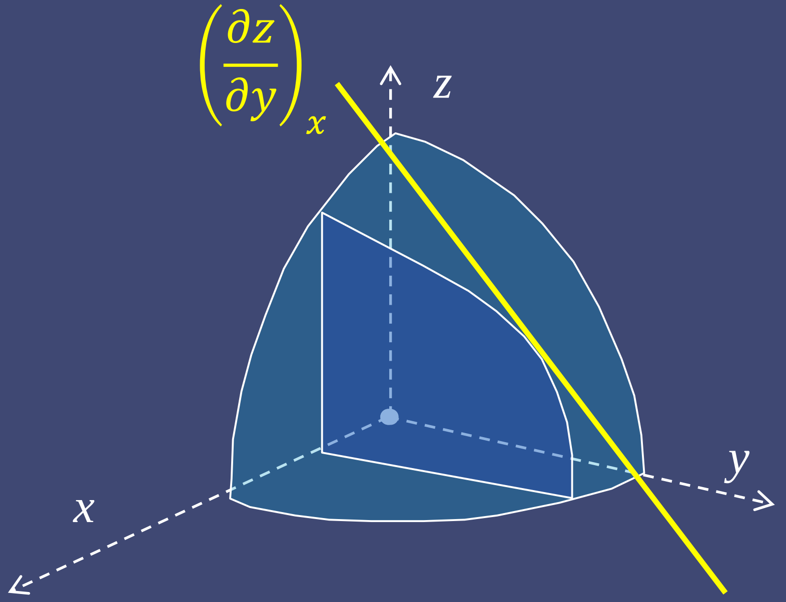
$$\frac{x_i\underline{\nabla}\mu_i}{R\,T} = x_i\underline{\nabla}ln[a_i] = \sum\frac{1}{C\,\mathcal{D}_{ij}}\left(x_i\,\underline{J}_j - x_j\,\underline{J}_i\right)$$

μ_A is molar free energy of component A (Molar partial Gibbs Energy) [J/kmol], a_A is molar activity of component A, R is the gas constant (i.e. R=8314.34 Pa-m^3/kmol-K)

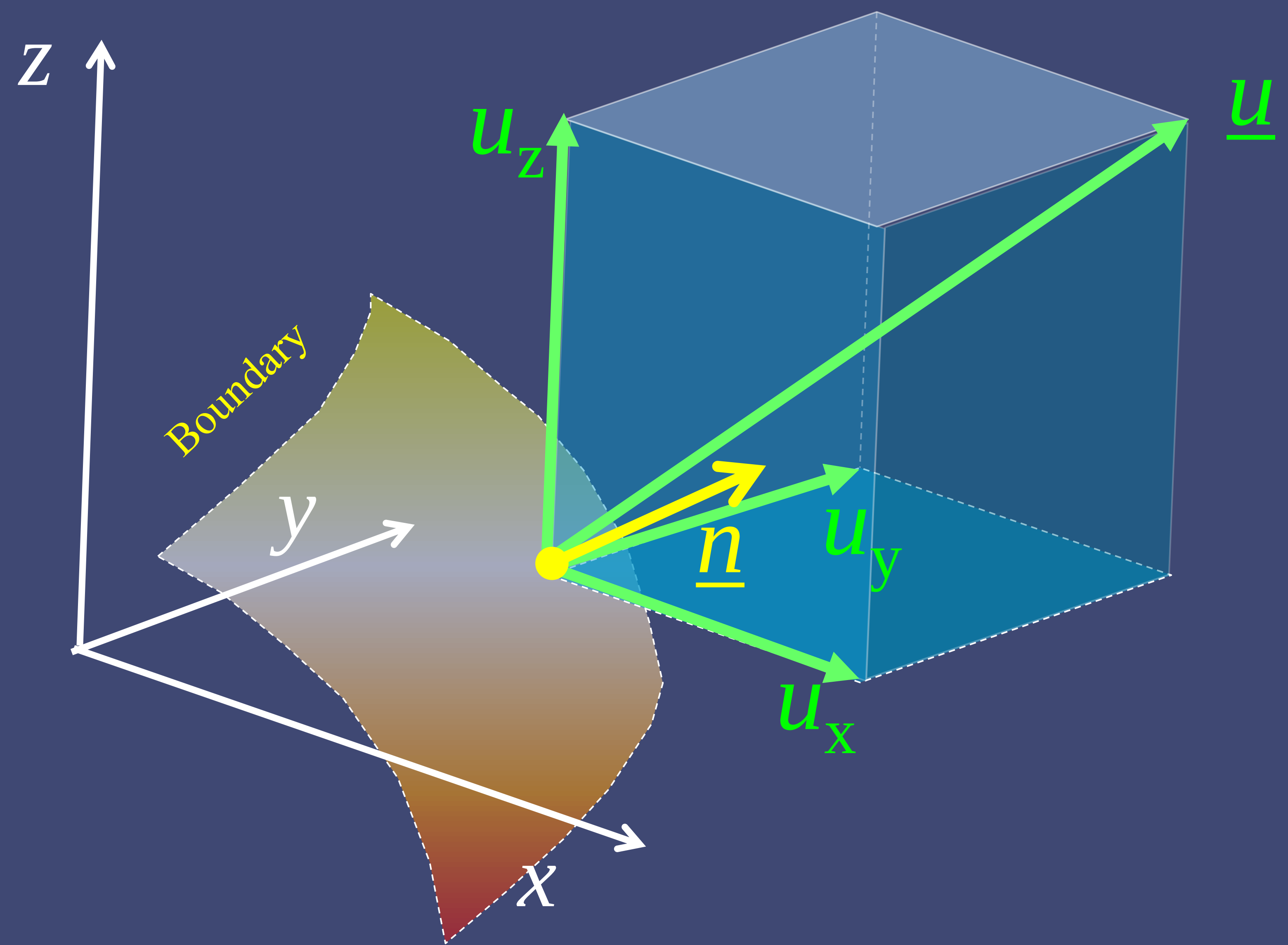
Johnson Richard W. Handbook of Fluid Dynamics, 2nd Edition, 2016. CRC

Partial differential representation in 3-D surface

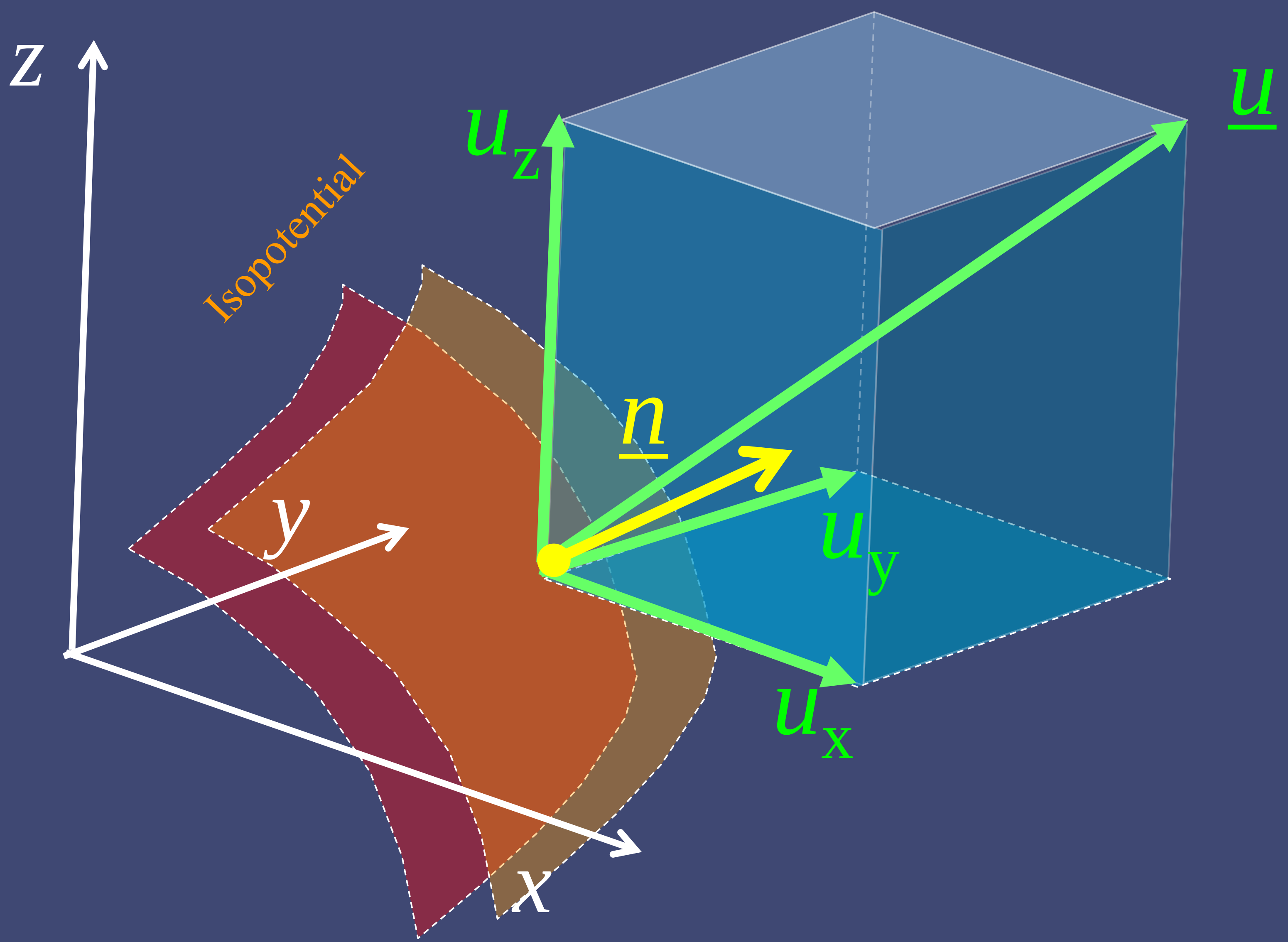




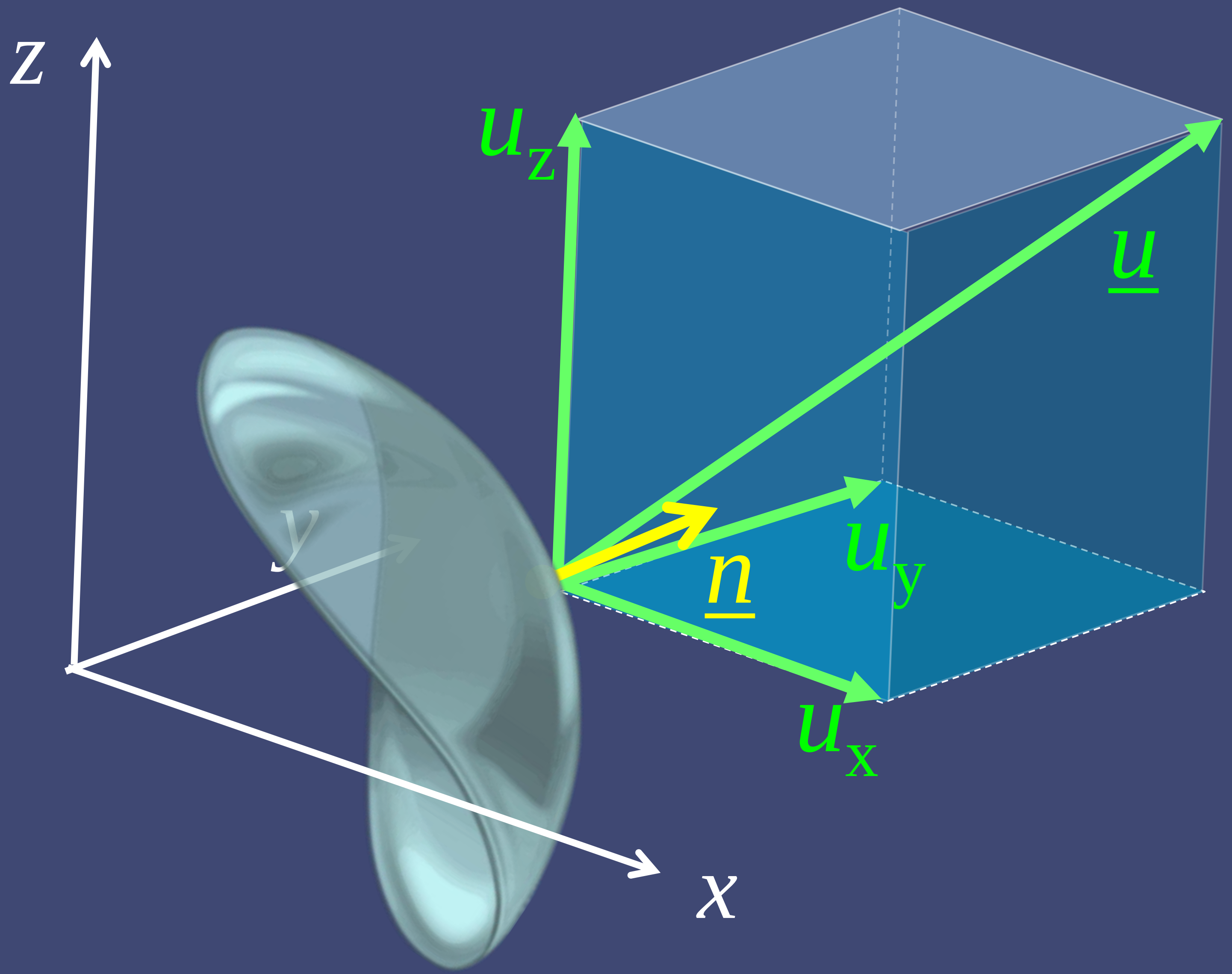
Velocity field across a point at the boundary



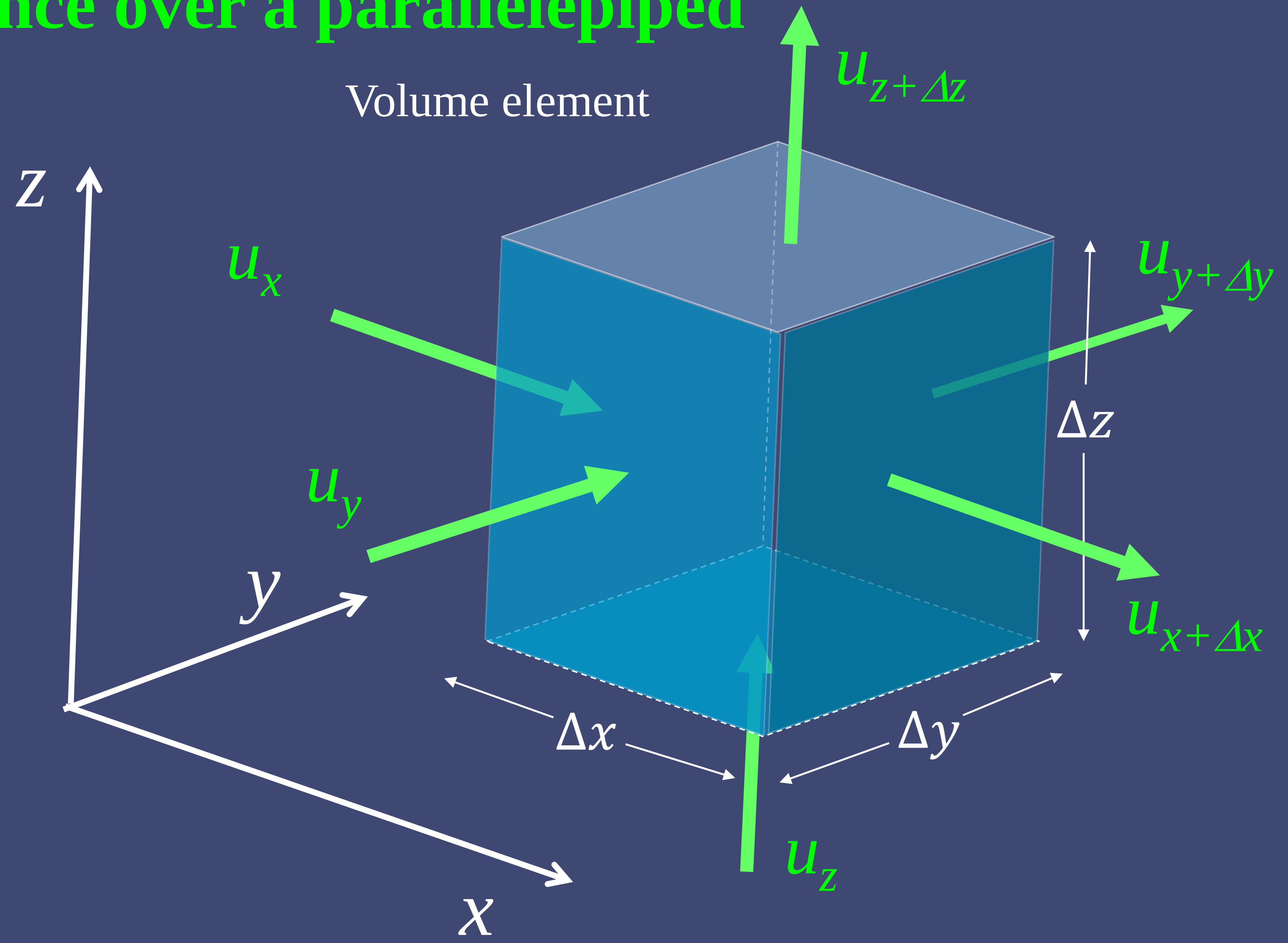
Velocity Field



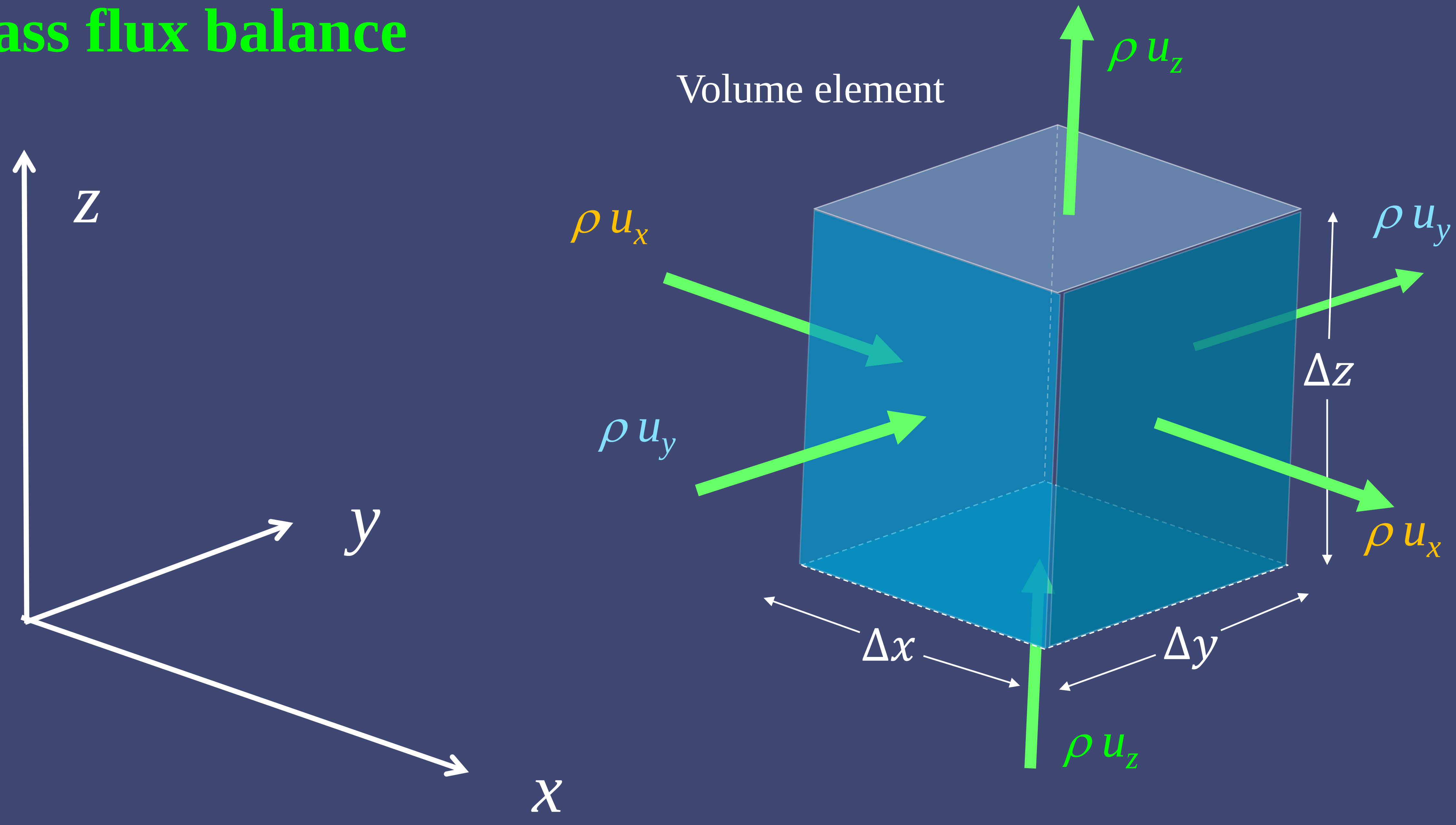
Velocity Field



Volume flux balance over a parallelepiped



mass flux balance

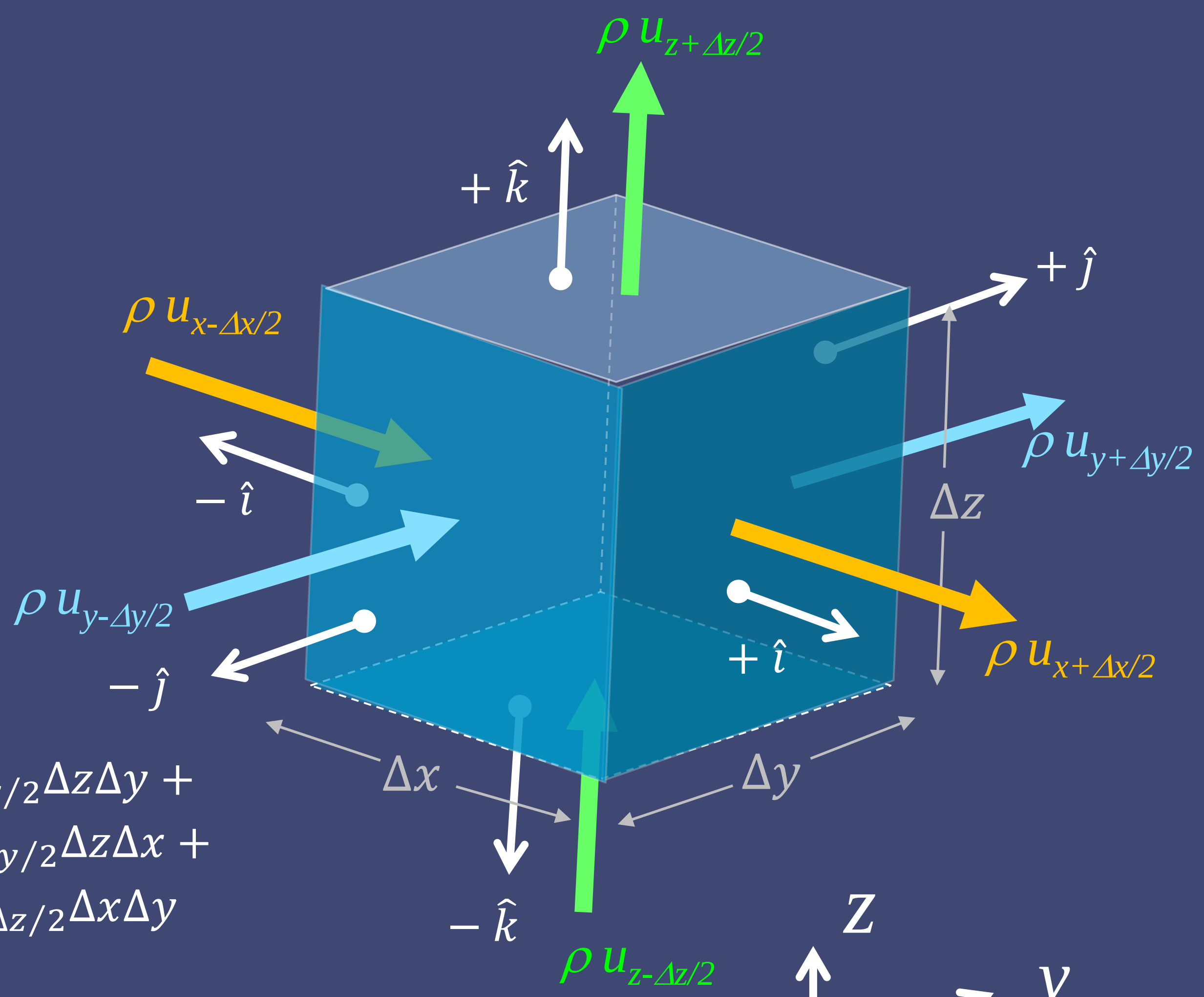


mass flux balance

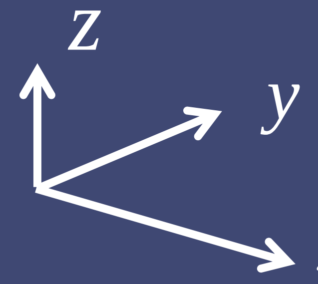
Volume element

$$\frac{\partial[\rho\Delta x\Delta y\Delta z]}{\partial t} = \begin{array}{l} \rho u_x\Delta y\Delta z\Big|_{x-\Delta x/2} - \rho u_x\Delta y\Delta z\Big|_{x+\Delta x/2} + \\ \rho u_y\Delta x\Delta z\Big|_{y-\Delta y/2} - \rho u_y\Delta x\Delta z\Big|_{y+\Delta y/2} + \\ \rho u_z\Delta y\Delta x\Big|_{z-\Delta z/2} - \rho u_z\Delta y\Delta x\Big|_{z+\Delta z/2} \end{array}$$

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot[\rho\,\underline{u}] = 0$$



$$\frac{\partial\rho}{\partial t}\Delta x\Delta y\Delta z = \begin{array}{l} -\rho\underline{u}_{x-\Delta x/2}\cdot\hat{n}_{x-\Delta x/2}\Delta z\Delta y - \rho\underline{u}_{x+\Delta x/2}\cdot\hat{n}_{x+\Delta x/2}\Delta z\Delta y + \\ -\rho\underline{u}_{y-\Delta y/2}\cdot\hat{n}_{y-\Delta y/2}\Delta z\Delta x - \rho\underline{u}_{y+\Delta y/2}\cdot\hat{n}_{y+\Delta y/2}\Delta z\Delta x + \\ -\rho\underline{u}_{z-\Delta z/2}\cdot\hat{n}_{z-\Delta z/2}\Delta x\Delta y - \rho\underline{u}_{z+\Delta z/2}\cdot\hat{n}_{z+\Delta z/2}\Delta x\Delta y \end{array}$$



$$\iiint\frac{\partial\rho}{\partial t}dV = -\oiint \underline{n}\cdot[\rho\underline{u}]dA$$

$$\iiint\underline{\nabla}\cdot[\rho\,\underline{u}]dV = \oiint\underline{n}\cdot[\rho\underline{u}]dA$$

$$\iiint \underline{\nabla} \cdot [\, \rho\, \underline{u}\,] dV = \oint\!\!\!\!\iint \underline{n} \cdot [\rho \underline{u}] dA$$

The product of density and velocity is also known ad mass flux vector, so any flux can be analyzed this way, to relate surface integrals to volume integrals in the well known Gauss's divergence theorem. This is.

$$\iiint \underline{\nabla} \cdot \underline{\psi}\, dV = \oint\!\!\!\!\iint \underline{n} \cdot \underline{\psi}\, dA$$

mass flux balance

Continuity equation $\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot [\rho\, \underline{u}] = 0$

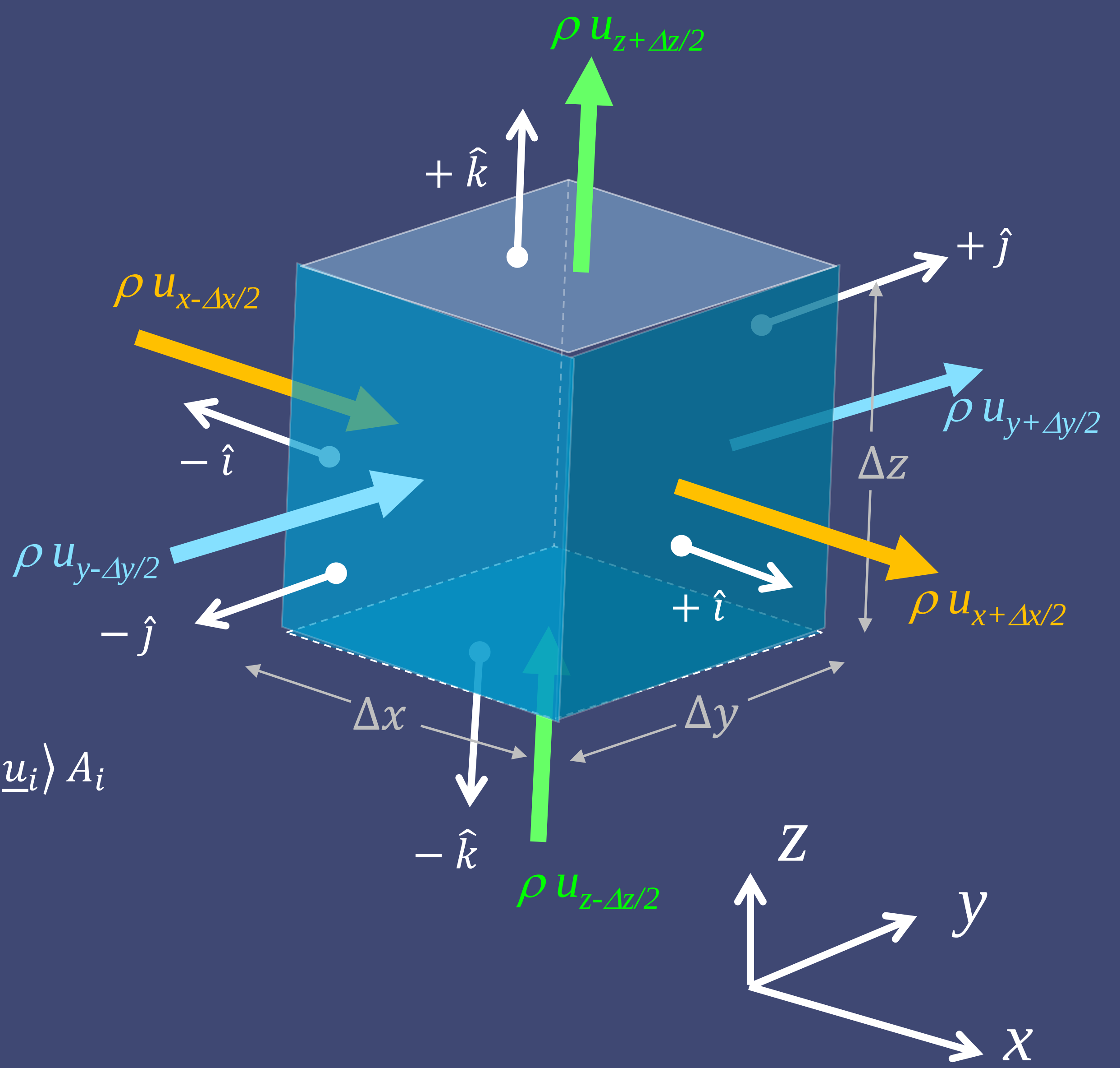
Divergence theorem $\iiint \underline{\nabla} \cdot [\rho\, \underline{u}]dV = \oiint \underline{n} \cdot [\rho \underline{u}]dA$

Over a Control volume $\frac{\partial}{\partial t} \iiint \rho\, dV = -\oiint \underline{n} \cdot [\rho \underline{u}]dA$

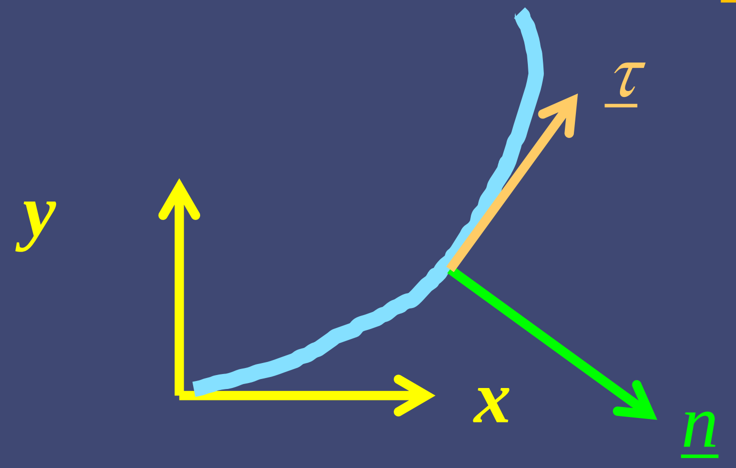
Macroscopic continuity equation $\frac{dm}{dt} = -\oiint \underline{n} \cdot [\rho \underline{u}]dA = -\sum \rho\, \underline{n}_i \cdot \langle \underline{u}_i \rangle A_i$

Average velocity $\langle \underline{u}_i \rangle A_i = \int \underline{u}\, dA$

Volume element



($\underline{n}$, $\underline{\tau}$) Normal unit and tangential unit vectors in a 2D system



$\hat{\underline{\tau}}$

$|\underline{\tau}| = 1$

$|\underline{n}| = 1$

$\hat{\underline{n}}$

Both are vectors of magnitude 1, the first one is tangent to the boundary and the second is perpendicular pointing outward.

Forces classification

Forces

a) Line forces or surface tension force

b) Surface forces

c) Body forces

This force has a magnitude that depends on the size (i.e. length) of the contact line (i.e. the boundary where three phases meet)

This force has a magnitude that depends on the size (i.e. area) of the boundary (i.e. the area surrounding a portion of fluid volume to analyze)

This force has a magnitude that depends on the volume of the fluid (i.e. volume size)

Examples

$\underline{F_L} = \sigma\, L\, \underline{\tau}$

$\underline{F_p} = p\, A\, \underline{n} = \underline{n} \cdot \underline{\underline{I}}\, p\, A$

$\underline{F_w} = \rho\, V\, \underline{g}$

$\underline{\hat{\tau}} = \cos\varphi\; \hat{\imath} + \sin\varphi\; \hat{\jmath}$ θ = contact angle $\phi = \phi(\theta)$ = function of the contact angle

ϕ = is the angle of the tangent line between fluids interface respect to the horizontal (to quantify the direction of the force at the contact line)

$\underline{F_L} = \sigma\, L\, \underline{\hat{\tau}}$ $\underline{F_p} = p\, A\, \underline{\hat{n}} = \underline{\hat{n}} \cdot \underline{\underline{I}}\, p\, A$

The unit vectors usually indicate the direction of the force. $\boldsymbol{n}$ is normal to the surface, and $\boldsymbol{\tau}$ is the tangent fluid-fluid interface in contact with the solid or a third phase

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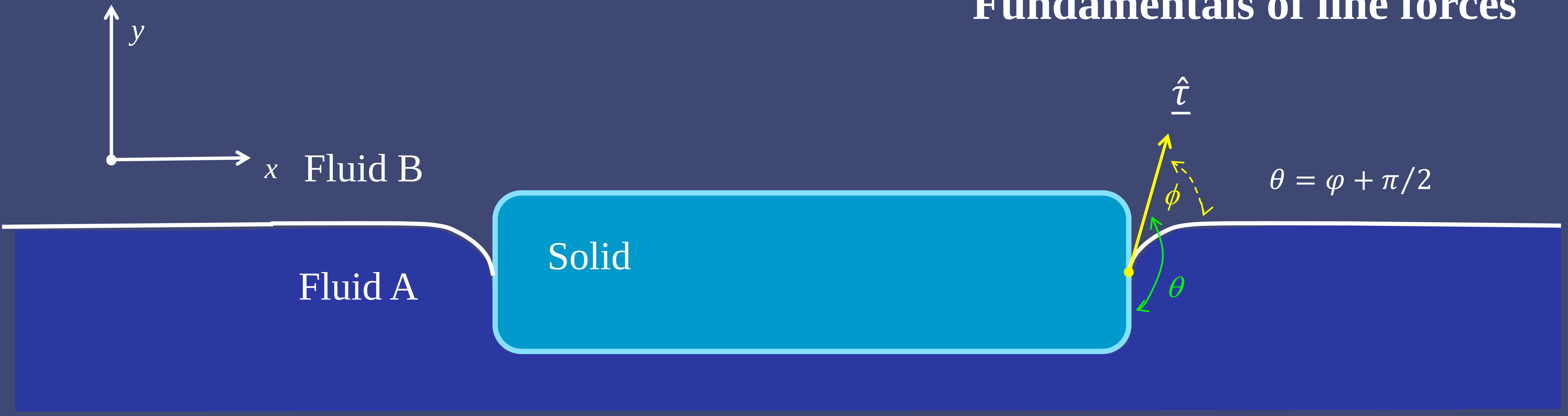
(*) Hauke G. *An Introduction to Fluid Mechanics and Transport Phenomena*. Springer 2008

* This classification was inspired by the work of Hauke

Fundamentals of line forces



How to relate the contact angle with the unit tangent vector (i.e. the unit vector that gives the direction of the force) ?
By Trigonometry. The contact angle is the angle formed by the interfaces having in common the contact line (contact line is the line where the three phases meet), in this case tangent line of solid-fluid A, and the tangent line of fluid A-fluid B (i.e. θ). In this case the solid interface is vertical, so by using a 2-D Cartesian frame of reference with gravitational field in " – *y*" direction, then the relationships are as shown:

$$\hat{\underline{\tau}} = \hat{\imath}\cos\varphi + \hat{\jmath}\sin\varphi$$

$$\theta = \varphi + \pi/2$$

$$\hat{\underline{\tau}} = \hat{\imath}\cos[\theta - \pi/2] + \hat{\jmath}\sin[\theta - \pi/2]$$
$$\hat{\underline{\tau}} = \hat{\imath}\cos[\pi/2 - \theta] - \hat{\jmath}\sin[\pi/2 - \theta]$$
$$\hat{\underline{\tau}} = \hat{\imath}\sin\theta - \hat{\jmath}\cos\theta$$

Fundamentals of line forces

Surface tension goes beyond quantifying the **line forces**, can be used to quantify **pressure difference across interfaces** and also as **energy** required **to increase surface** between interfaces

Line force

$$\underline{F}_L = \sigma \, L \, \underline{\tau}$$

Unit tangent vector

Surface tension

Contact line length

Pressure difference across a curved interface

$$\Delta p = p_A - p_B = \sigma \, [1/r_1 + 1/r_2]$$

The two principal curvature radii

Internal Energy

$$dU = T \, dS \; - p \; dV + \sigma \; da + \; \Psi \; dq + \underline{B} \; \cdot d\underline{I} + \sum \bar{G}_i dn_i \qquad (*)$$

Absolute Temperature

Entropy

Absolute Pressure

Volume

Surface tension

Area of the interface

Electrical potential

Electrical charge

Magnetic field

Magnetic moment

number of moles of species “i”

Molar partial Gibbs energy or chemical potential

(*) Dill Ken A, Bromberg S. *Molecular Driving Forces*, 2nd edition. Garland Science, 2011

Fundamentals of surface forces

Viscosity is a property that relates viscous forces per unit area or **viscous stress** with the velocity of deformation, or **strain rate**. The nature of stress as a **tensor** has to be considered, and by the way the strain rate is a tensor by nature.

$$\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, A$$

Viscous surface force

Unit normal vector

Viscous stress tensor

Area

For a 2-D unidirectional flow

$$\tau_{yx} = \mu \frac{\partial v_x}{\partial y}$$

Viscous stress tensor

Viscosity

Strain rate tensor

$$\underline{\underline{\tau}}_{yx} = \mu \underbrace{\left[\hat{j}\,\frac{\partial}{\partial y}\right]}_{\text{Vector nature of gradient}} \underbrace{[v_x\, \hat{i}\,]}_{\text{Vector nature of velocity}} = \underbrace{\hat{j}\hat{i}\left[\mu \frac{\partial v_x}{\partial y}\right]}_{\text{tensor nature of strain rate}}$$

Force per unit area

$$\boxed{\underline{F_v}/A = \underline{S_v} = \underline{n} \cdot \mu\, 2\, \underline{\underline{\Gamma}}}$$

For a 3-D flow, the strain rate tensor has to be taken into consideration with all the components

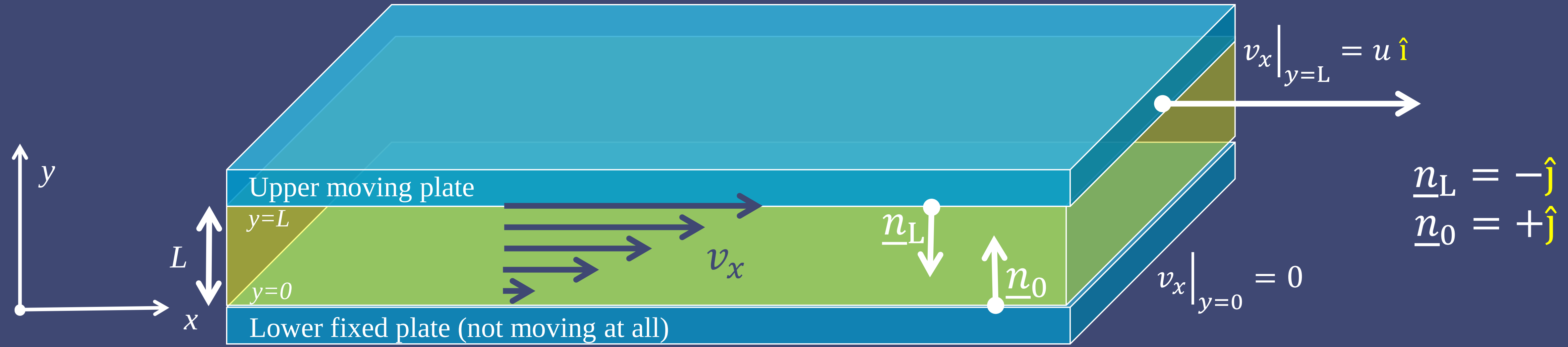
Viscous Surface forces $\underline{F_v} = \oint \underline{n} \cdot \mu \left[2\, \underline{\underline{\Gamma}}\right] dA$

Strain rate tensor $\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\, \underline{V} + \left(\underline{\nabla}\, \underline{V}\right)^T\right]$

Fundamentals of surface forces

Viscosity is a property that relates viscous forces per unit area or **viscous stress** with the velocity of deformation, or **strain rate**. The nature of stress as a **tensor** has to be considered, and by the way the strain rate is a tensor by nature.

For a 2-D unidirectional flow within a thin fluid layer $\underline{\underline{\tau}}_{yx} = \hat{j}\hat{i}\left[\mu\frac{\partial v_x}{\partial y}\right] \approx \hat{j}\hat{i}\,\mu\frac{\Delta v_x}{\Delta y} \approx \hat{j}\hat{i}\,\mu\frac{u}{L}$



The forces at the boundaries are respectively .

$\underline{F}_0 = \underline{n} \cdot \underline{\underline{\tau}}\,A \approx [+\hat{j}] \cdot \left[\hat{j}\hat{i}\,\mu\frac{u}{L}\right] A = \hat{i}\,A\,\mu\frac{u}{L}$ Force exerted by the fluid to the lower plate.

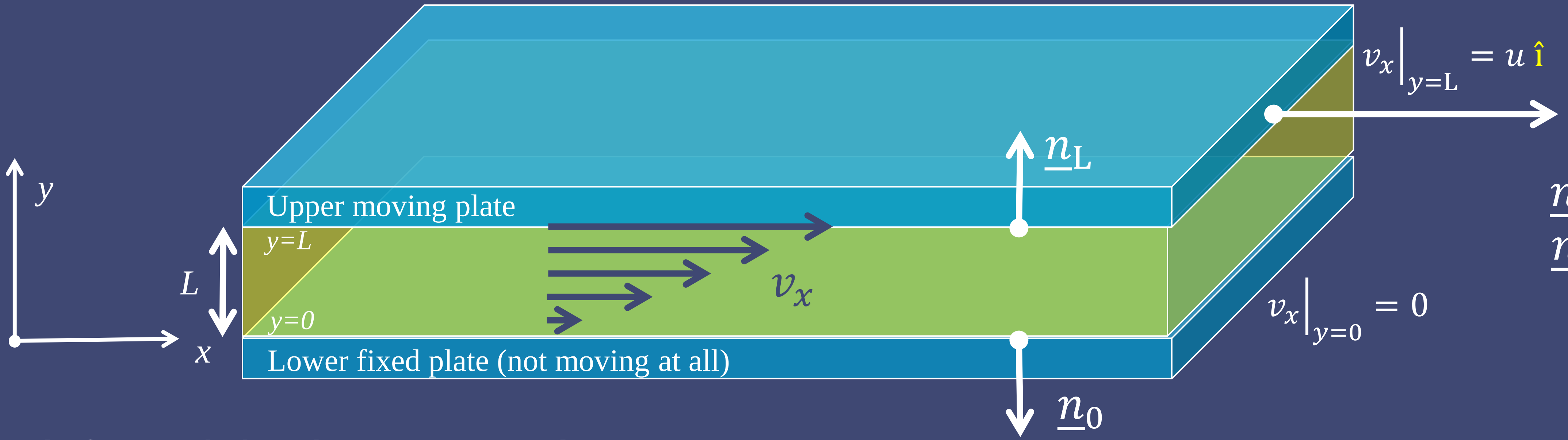
$\underline{F}_L = \underline{n} \cdot \underline{\underline{\tau}}\,A \approx [-\hat{j}] \cdot \left[\hat{j}\hat{i}\,\mu\frac{u}{L}\right] A = -\hat{i}\,A\,\mu\frac{u}{L}$ Force exerted by the fluid to the upper plate.

Systems: The plates

Fundamentals of surface forces

Viscosity is a property that relates viscous forces per unit area or **viscous stress** with the velocity of deformation, or **strain rate**. The nature of stress as a **tensor** has to be considered, and by the way the strain rate is a tensor by nature.

For a 2-D unidirectional flow within a thin fluid layer $\underline{\underline{\tau}}_{yx} = \hat{j}\hat{i}\left[\mu \frac{\partial v_x}{\partial y}\right] \approx \hat{j}\hat{i}\,\mu \frac{\Delta v_x}{\Delta y} \approx \hat{j}\hat{i}\,\mu \frac{u}{L}$



$\underline{n}_{\mathrm{L}} = +\hat{j}$
$\underline{n}_0 = -\hat{j}$

The forces at the boundaries are respectively .

$\underline{F}_0 = \underline{n} \cdot \underline{\underline{\tau}}\, A \approx [-\hat{j}] \cdot \left[\hat{j}\hat{i}\,\mu \frac{u}{L}\right] A = -\hat{i}\, A\, \mu \frac{u}{L}$ Force exerted by the lower plate to the fluid.

$\underline{F}_L = \underline{n} \cdot \underline{\underline{\tau}}\, A \approx [+\hat{j}] \cdot \left[\hat{j}\hat{i}\,\mu \frac{u}{L}\right] A = \hat{i}\, A\, \mu \frac{u}{L}$ Force exerted by the upper plate to the fluid.

System: The fluid

Fundamentals of surface forces

To quantify and characterize the force acting, it is important to define your system. In the previous example we define the fluid as system, when we were willing to estimate stress over the fluid by the plate and the normal vector was considering the control surface enclosing the fluid, then the normal vector points always outwards the surface. On the contrary, if you are willing to estimate the stress exerted by the fluid to the surface, then the solid is my system, in this case the normal vector points outward the solid boundary. With this reasoning in mind, you can tell by the sign of the resulting product the direction of the force.

Fundamentals of surface forces

Pressure is a scalar**, Force** is a vector, and **Stress** is a tensor**.** Even that the pressure and stress have the same units, the nature of each property has to be taken into consideration. See each definition.

$$\underline{F} = \underline{n}\cdot\underline{\underline{\tau}}\,A$$

Stress force
Unit normal vector
Stress tensor
Area

$$\underline{F} = \underline{n}\,p\,A$$

Pressure force
Unit normal vector
Pressure
Area

$$\underline{n}\,p = \underline{n}\cdot\overbrace{\left[\,\underline{\underline{I}}\,p\,\right]}^{\text{Pressure tensor}}$$

Unit normal vector
Pressure
Identity tensor

Force per unit area

$$\boxed{\underline{F_v}/A = \underline{S_v} = \underline{n}\cdot\mu\,2\,\underline{\underline{\Gamma}}}$$

Fundamentals of body forces

Weight is force, is a vector**.** Its magnitude depends on the volume, the fluid properties and the magnitude of the field producing the force. The direction is the same direction as the field vector producing the force.

$$\underline{F} = \rho\, V\, \underline{g}$$

Body force

Mass density

Volume

Gravitational force field

$$\underline{F} = \rho_e\, V\, \underline{E}$$

Body force

Charge density

Volume

Electrical force field

If you are not familiar with tensors from the physical point of view, here is an example of a tensor quantity you must be acquainted with

$$\underline{\underline{I}}_m = \int \rho \left[\underline{r} \cdot \underline{r}\, \underline{\underline{I}} - \underline{r}\,\underline{r} \right] dV$$

Type of forces in fluids

Body Forces:

Gravity $\underline{F_w} = \rho\, V\, \underline{g}$

Electromagnetic $\underline{F_e} = \rho_e\, V\, \underline{E}$

Centrifugal $\underline{F_c} = \rho\, V\, \underline{\Omega} \times \underline{R} \times \underline{\Omega}$

Coriolis $\underline{F_c} = \rho\, V\, [2\, \underline{v} \times \underline{\Omega}]$

Surface Forces:

Viscous stresses $\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, A$

Pressure stresses $\underline{F_p} = p\, A\, \underline{n} = \underline{n} \cdot \underline{\underline{I}}\, p\, A$

Buoyancy (construct resulting from pressure stress)

Line Forces:

Interfacial tension

Surface tension

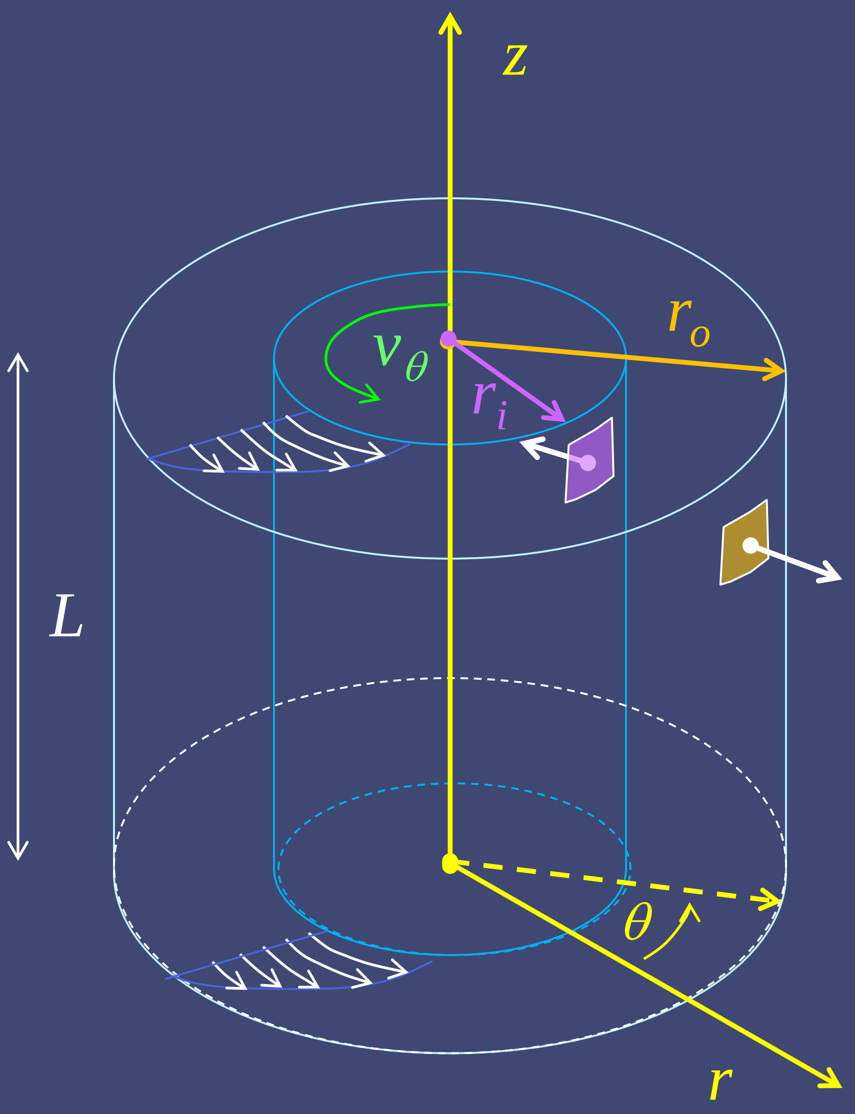
$\underline{F_v} = \sigma\, L\, \underline{\tau}$

Note: Some of these called body forces sometimes are called fictitious forces (for instance Coriolis), and some forces can be quantified as body forces, but they are surface forces like buoyancy (buoyancy is a construct that can be used to quantify forces as weight of displaced fluid, as long as the fluid and the body within the fluid, are at rest and completely immersed in the single phase).

Why tensors are useful ?

In the following example you will se the importance of the stress labeled as tensor, even though the surface over which is happening is the same, the resulting force acts in different direction as a result of the direction of the dot product of the normal vector and the strain rate tensor.

Viscous forces over the wall of a spinning rod (ω) within a concentric fixed hollow cylinder filled with fluid



$$\underline{F_v} = \oint \underline{n} \cdot \mu\, 2\, \underline{\underline{\Gamma}}\; dA \qquad \underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{V} + \left(\underline{\nabla}\,\underline{V}\right)^T\right]$$

$$\underline{S_v} = \underline{n} \cdot \mu\, 2\, \underline{\underline{\Gamma}}$$

Viscous Surface forces

$$\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, A \qquad A = 2\,\pi\, r_i\, L$$

At the inner surface

$$\underline{n_i} = -\underline{\mathrm{e}_\mathrm{r}}$$

$$\underline{\underline{\tau}} = \tau_{r\theta} = \mu \frac{dv_\theta}{dr}\, \underline{\mathrm{e}_\mathrm{r}}\,\underline{\mathrm{e}_\theta}$$

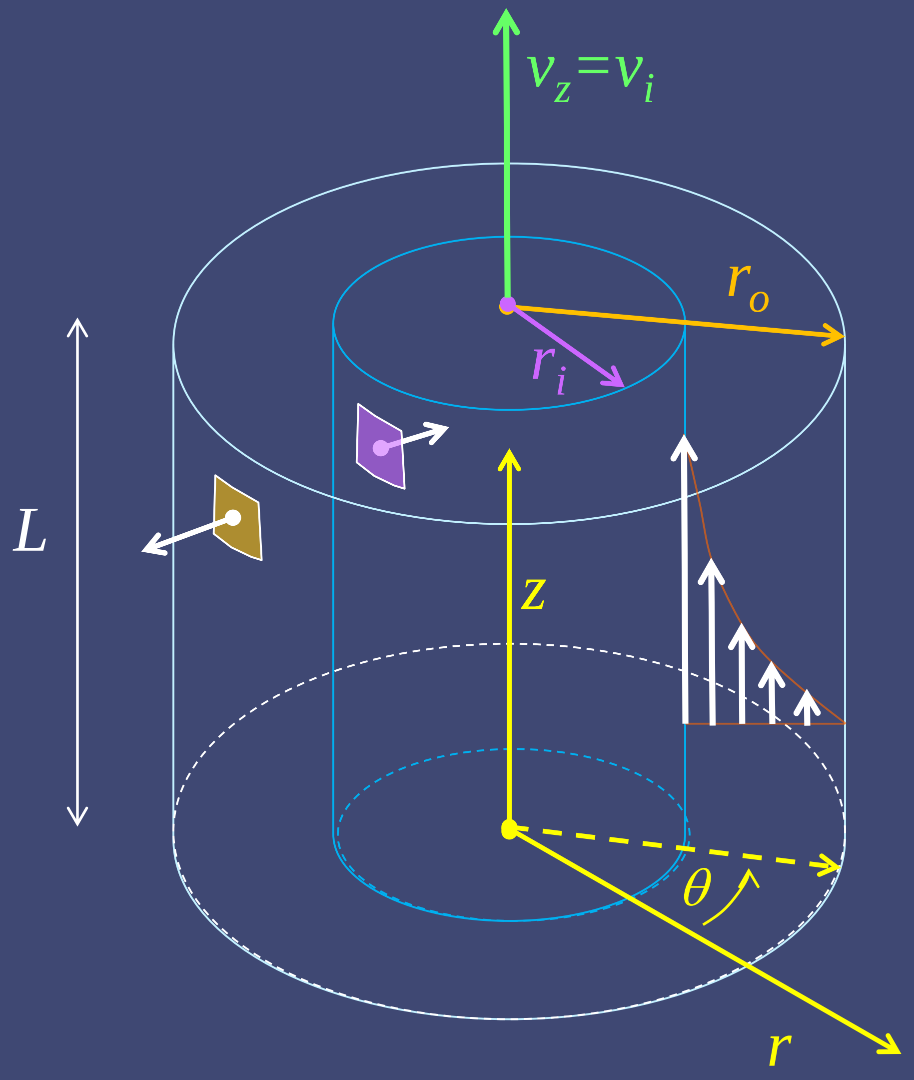
$$\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, A = -2\,\pi\, r_i\, L \left[\mu \frac{dv_\theta}{dr}\right]\left[\underline{\mathrm{e}_\mathrm{r}} \cdot \underline{\mathrm{e}_\mathrm{r}}\,\underline{\mathrm{e}_\theta}\right]$$

$$\underline{F_v} = -2\,\pi\, r_i\, L \left[\mu \frac{dv_\theta}{dr}\right]\left[\underline{\mathrm{e}_\theta}\right] \approx -2\,\pi\, r_i\, L\mu \left[\frac{v_{\theta o} - v_{\theta i}}{r_o - r_i}\right]\left[\underline{\mathrm{e}_\theta}\right]$$

$$\underline{F_v} \approx 2\,\pi\, r_i\, L\mu \left[\frac{\omega\, r_i}{r_o - r_i}\right]\left[\underline{\mathrm{e}_\theta}\right]$$

Viscous Surface force at the inner boundary

Viscous forces over the wall of an axial displacing rod (v) within a concentric fixed hollow cylinder filled with fluid



$$\underline{F_v} = \oint \underline{n} \cdot \mu\, 2\, \underline{\underline{\Gamma}}\, dA \qquad \underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{V} + \left(\underline{\nabla}\,\underline{V}\right)^T\right]$$

$$\underline{S_v} = \underline{n} \cdot \mu\, 2\, \underline{\underline{\Gamma}}$$

Viscous Surface forces

$$\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, A \qquad A = 2\,\pi\, r_i\, L$$

At the inner surface

$$\underline{n_i} = -\underline{\mathrm{e}_r}$$

$$\underline{\underline{\tau}} = \tau_{rz} = \mu \frac{dv_z}{dr}\, \underline{\mathrm{e}_r}\,\underline{\mathrm{e}_z}$$

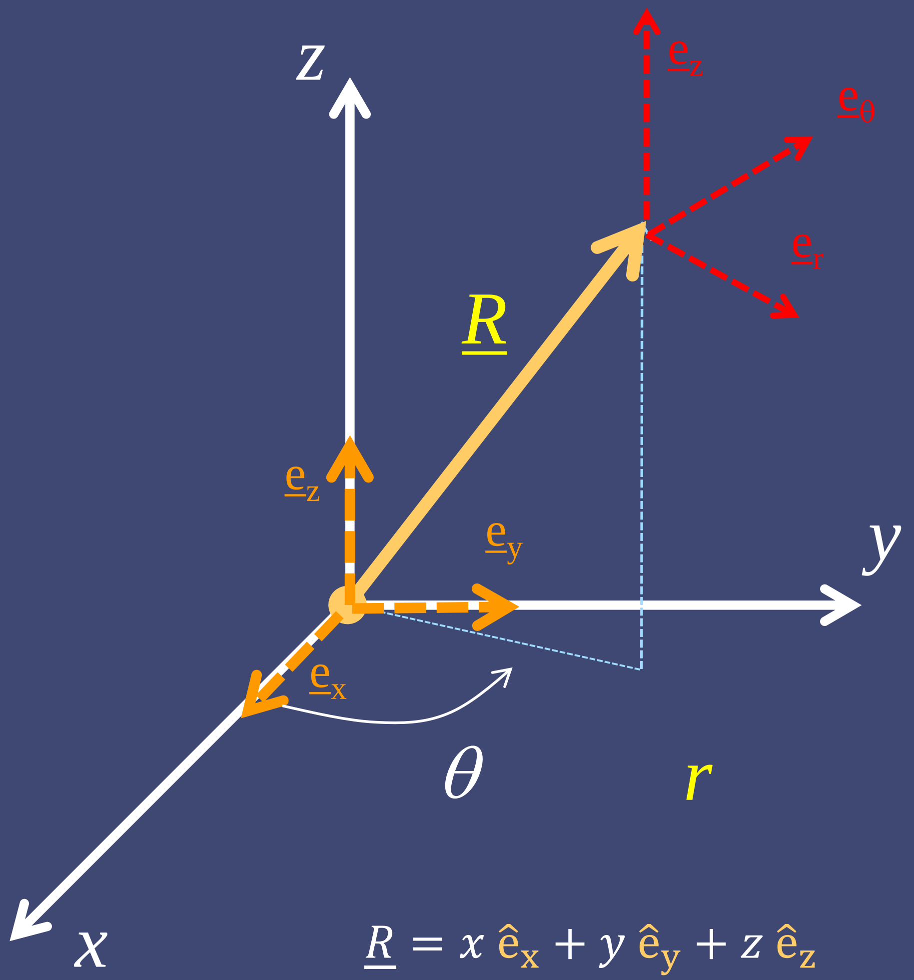
$$\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, A = -2\,\pi\, r_i\, L \left[\mu \frac{dv_z}{dr}\right] \left[\underline{\mathrm{e}_r} \cdot \underline{\mathrm{e}_r}\,\underline{\mathrm{e}_z}\right]$$

$$\underline{F_v} = -2\,\pi\, r_i\, L \left[\mu \frac{dv_z}{dr}\right] \left[\underline{\mathrm{e}_z}\right] \approx -2\,\pi\, r_i\, L \mu \left[\frac{v_{zo} - v_{zi}}{r_o - r_i}\right] \left[\underline{\mathrm{e}_z}\right]$$

$$\underline{F_v} \approx 2\,\pi\, r_i\, L \mu \left[\frac{v_i}{r_o - r_i}\right] \left[\underline{\mathrm{e}_z}\right]$$

Viscous Surface force at the inner boundary

Issues regarding Cartesian and cylindrical coordinates



$$\underline{R} = x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z$$

$$\underline{R} = r\,\hat{e}_r + z\,\hat{e}_z$$

$$\hat{e}_x = \hat{\imath} \qquad \hat{e}_y = \hat{\jmath} \qquad \hat{e}_z = \hat{k}$$

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$$

$$\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$$

$$\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$$

$$\hat{e}_r\cdot\hat{e}_r = \hat{e}_\theta\cdot\hat{e}_\theta = \hat{e}_z\cdot\hat{e}_z = 1$$

$$\hat{e}_r\cdot\hat{e}_\theta = \hat{e}_\theta\cdot\hat{e}_z = \hat{e}_z\cdot\hat{e}_r = 0$$

$$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$$

$$\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$$

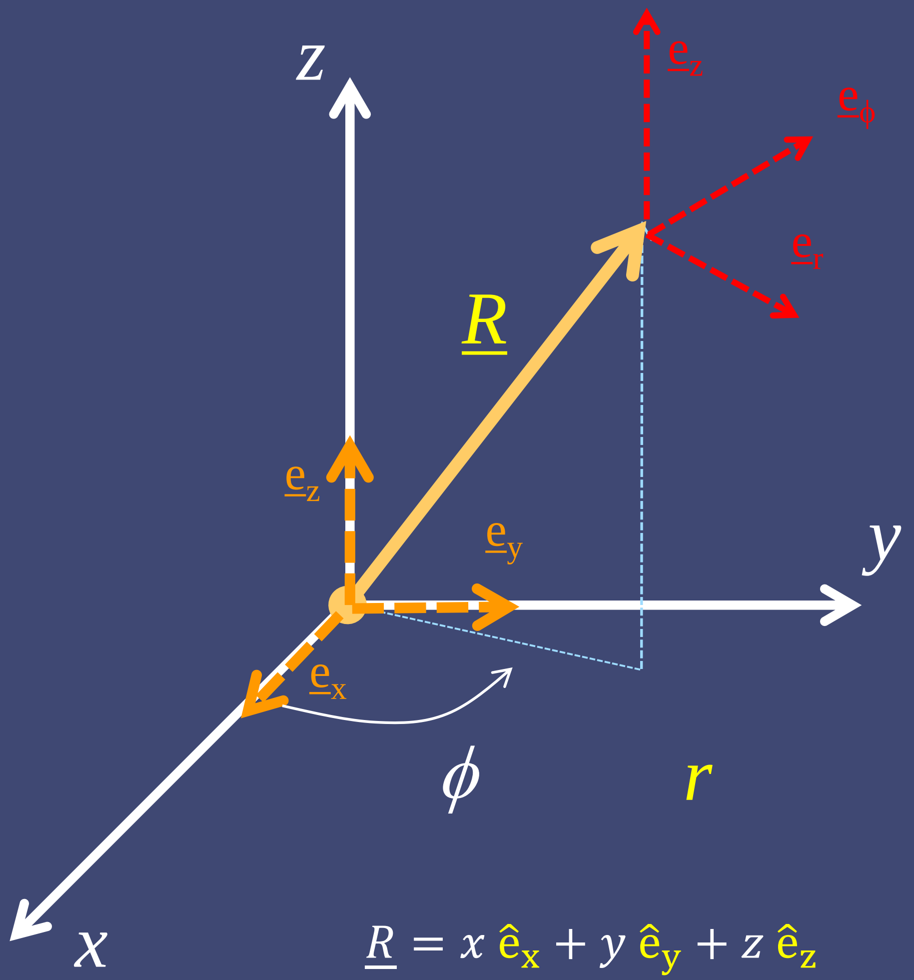
$$\hat{e}_r\times\hat{e}_r = \hat{e}_\theta\times\hat{e}_\theta = \hat{e}_z\times\hat{e}_z = 0$$

$$\hat{\imath}\times\hat{\jmath} = \hat{k} \qquad \hat{e}_r\times\hat{e}_\theta = \hat{e}_z$$

$$\hat{\jmath}\times\hat{k} = \hat{\imath} \qquad \hat{e}_\theta\times\hat{e}_z = \hat{e}_r$$

$$\hat{k}\times\hat{\imath} = \hat{\jmath} \qquad \hat{e}_z\times\hat{e}_r = \hat{e}_\theta$$

Issues regarding Cartesian and cylindrical coordinates



$$\underline{R} = x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z$$

$$\underline{R} = r\,\hat{e}_r + z\,\hat{e}_z$$

$$\hat{e}_x = \hat{\imath} \qquad \hat{e}_y = \hat{\jmath} \qquad \hat{e}_z = \hat{k}$$

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$$

$$\underline{a}\cdot\underline{b} = \underline{b}\cdot\underline{a}$$

$$\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$$

$$\hat{e}_r\cdot\hat{e}_r = \hat{e}_\phi\cdot\hat{e}_\phi = \hat{e}_z\cdot\hat{e}_z = 1$$

$$\hat{e}_r\cdot\hat{e}_\phi = \hat{e}_\phi\cdot\hat{e}_z = \hat{e}_z\cdot\hat{e}_r = 0$$

$$\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$$

$$\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$$

$$\hat{e}_r\times\hat{e}_r = \hat{e}_\phi\times\hat{e}_\phi = \hat{e}_z\times\hat{e}_z = 0$$

$$\hat{\imath}\times\hat{\jmath} = \hat{k} \qquad \hat{e}_r\times\hat{e}_\phi = \hat{e}_z$$

$$\hat{\jmath}\times\hat{k} = \hat{\imath} \qquad \hat{e}_\phi\times\hat{e}_z = \hat{e}_r$$

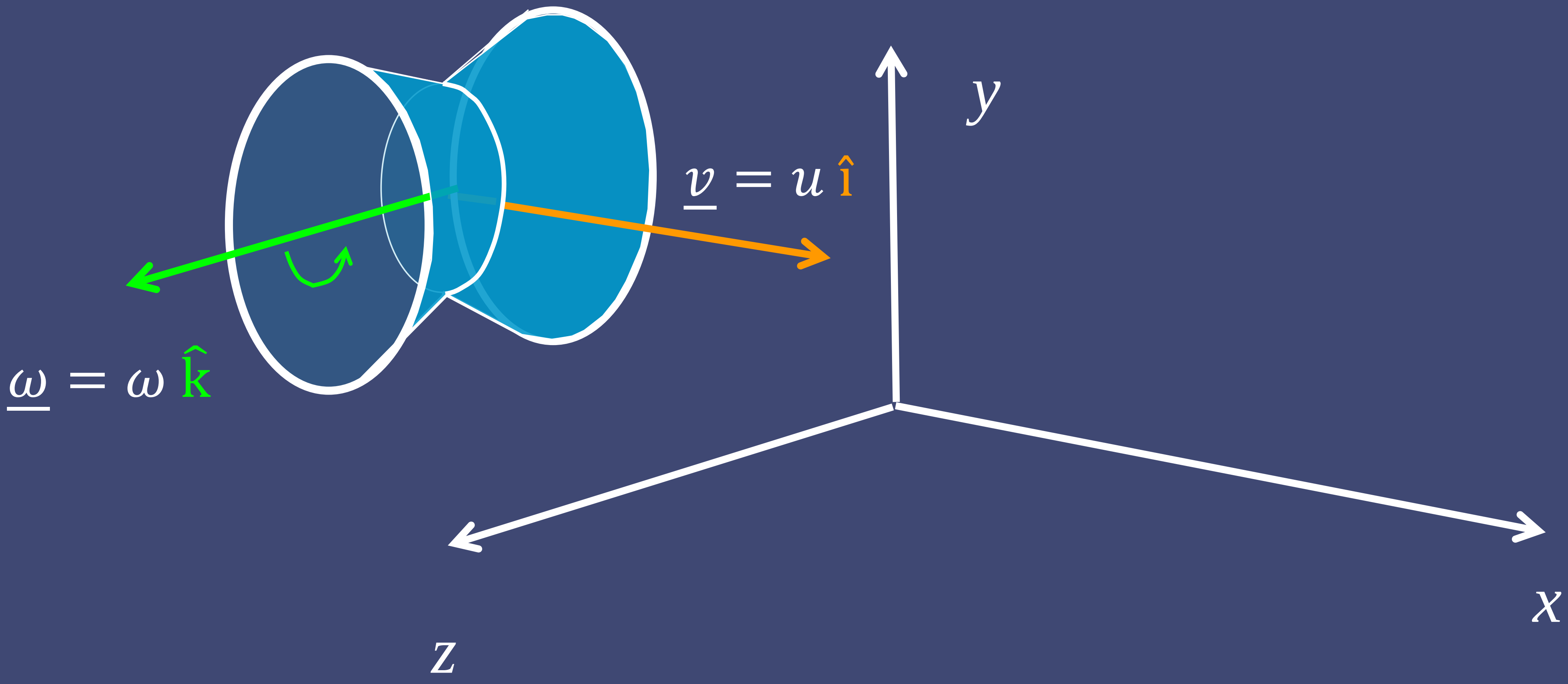
$$\hat{k}\times\hat{\imath} = \hat{\jmath} \qquad \hat{e}_z\times\hat{e}_r = \hat{e}_\phi$$

Challenge: Calculate velocity of the load when just released

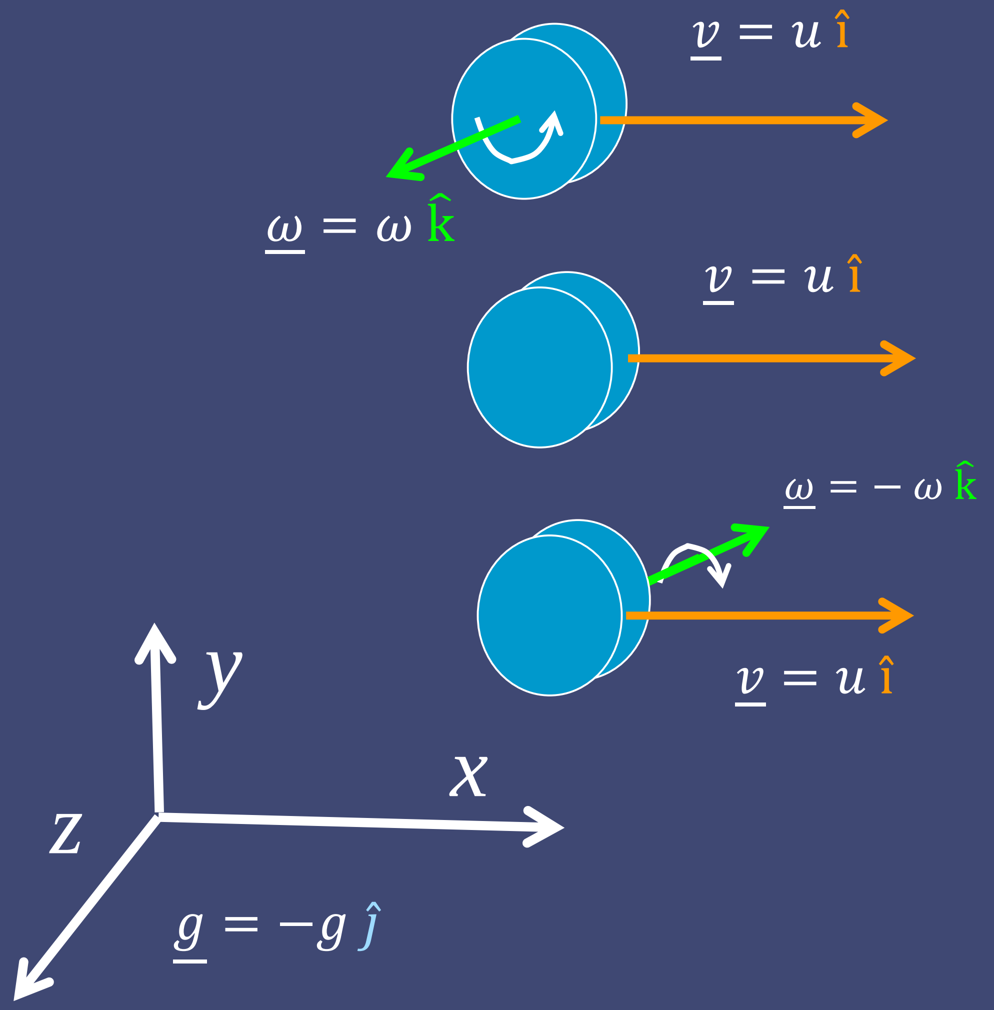
Challenge: Calculate velocity of the load when just released. Assume the inner road has a 1-in I.D. and the gap between the inner and the outer cylinder is 0.5 mm filled with a 0.5 Pa-s wax

Quiz No.1

Make a short clip explaining the different types of forces of each experiments conducted in class (1 experiment per team member). Must include a sketch, description, and a free body diagram to clarify each force.



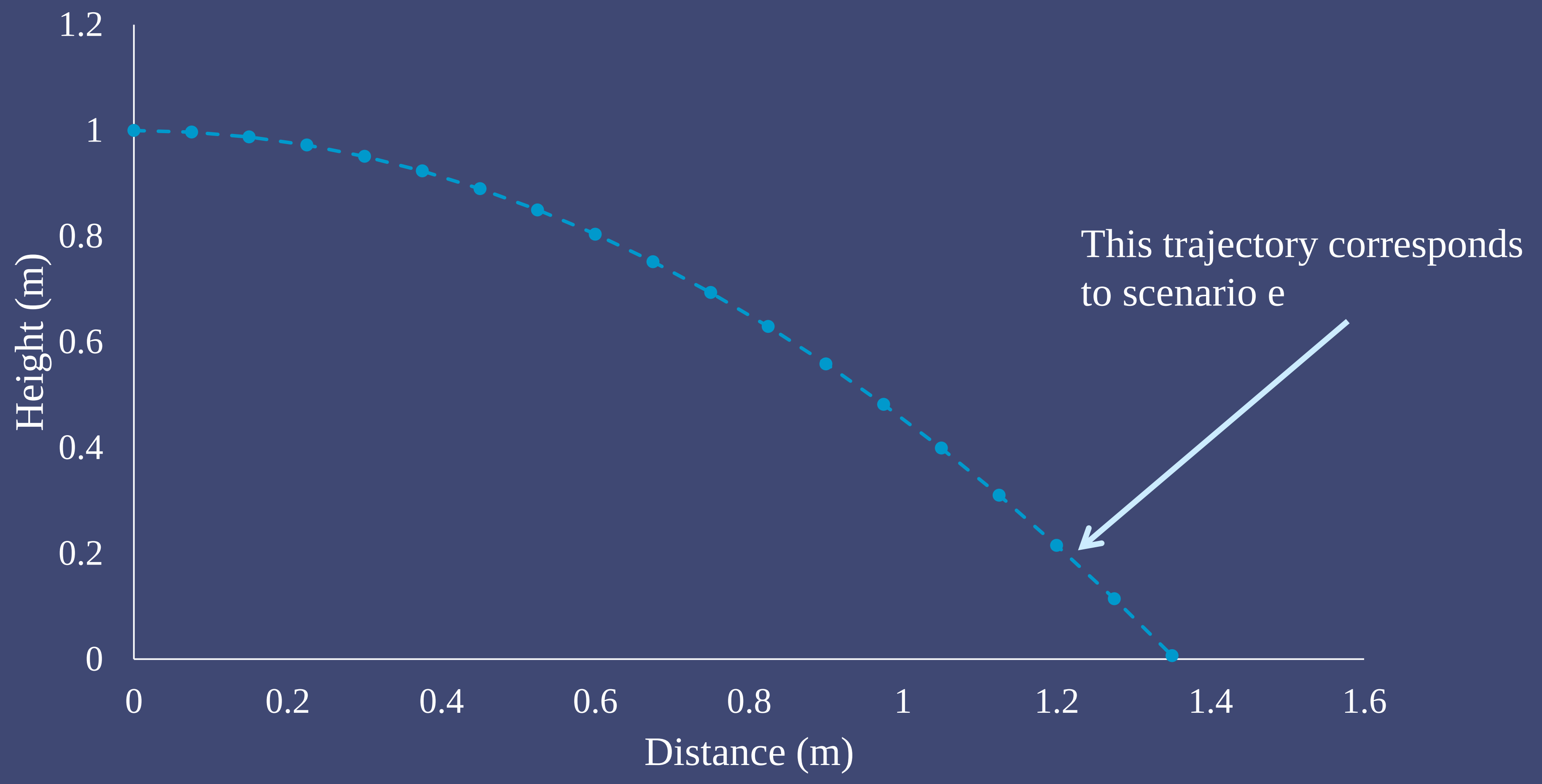
Experiment No.1

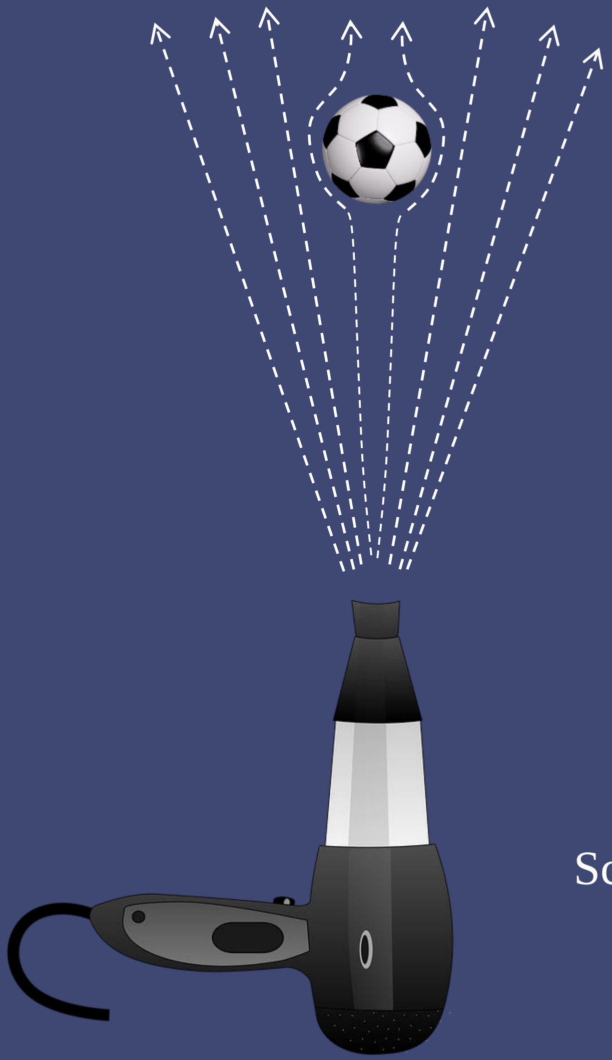


Two plastic cups taped by the base (both bases coincide at the x-y plane) are thrown with linear velocity "u" in x-direction. Make a sketch of the trajectory and also draw a free body diagram indicating the vectors of velocities and forces, and label each type of velocity and force.

Consider five scenarios where gravity field points downward y-axis:

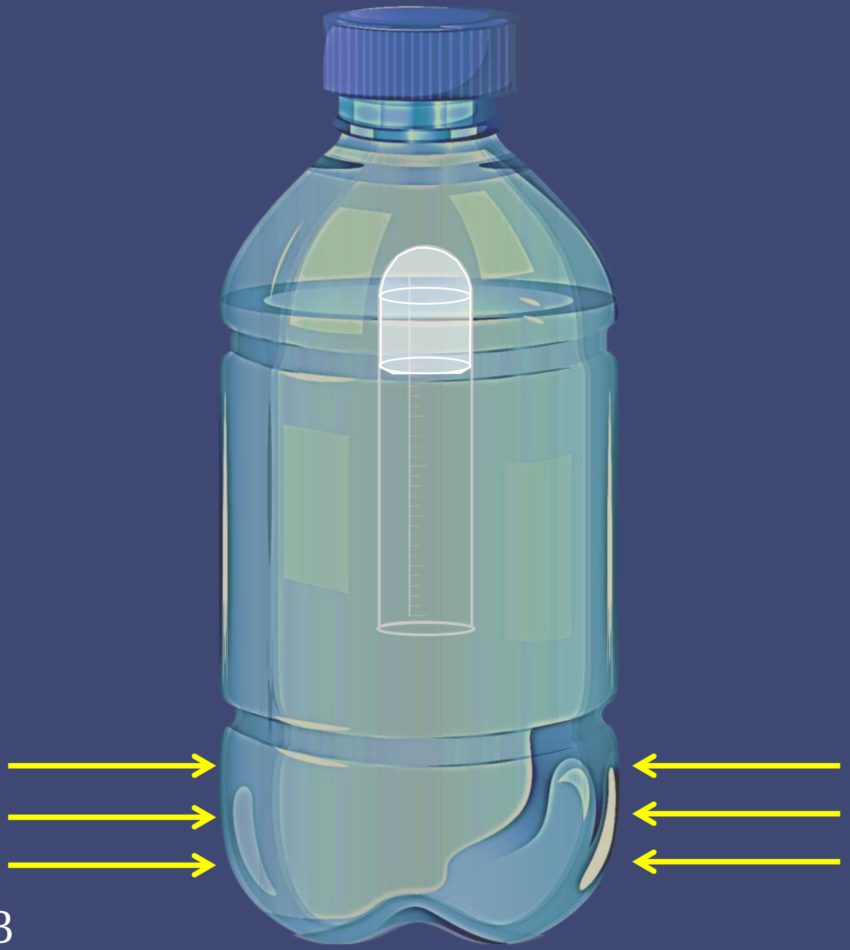
a) Object spins around z-axis (counter clock wise).
b) Object doesn't spin.
c) Object spins around z-axis (clock wise sense).
d) Object is within a vacuum chamber and spins.
e) Object is within a vacuum chamber without angular velocity.


1.2
1
0.8
0.6
0.4
0.2
0
Height (m)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
Distance (m)
This trajectory corresponds to scenario e



Experiment No.2. A blower with a jet stream where a ball keeps its height as long as the air flow jet is uniform (steady) regardless orientation. A) In scenario I, explain why keeps that particular height. B) In scenario II, why keeps its height even then you tilted the orientation of the air stream.

Two Cartesian divers (i.e. test tube partially filled with water) one at the bottom of a water filled bottle, and another floating with a small portion over the water surface are used for two experiments. The first bottle is pressurized at the bottom and the Cartesian diver sinks, while the second when vacuum applied floats. Make a free body diagram and explain the forces over the test tube.



Experiment No.3

One yen coin is placed over the surface of water, and the surface barely deforms, make a free body diagram and explain the types of forces.

$D = 20$ mm

$H = 1.5$ mm

Experiment No.3

Fundamentals of line forces

Surface tension goes beyond quantifying the **line forces**, can be used to quantify **pressure difference across interfaces** and also as **energy** required **to increase surface** between interfaces

Line force → $\underline{F}_L = \sigma\, L\, \underline{\tau}$ ← Unit tangent vector

Surface tension   Contact line length

$$p_A - p_B = \sigma\, [1/r_1 + 1/r_2]$$

The two principal curvature radii

$$dU = T\,dS - p\,dV + \sigma\,da + \Psi\,dq + B\,dI + \sum \bar{G}_i dn_i$$

r_1, r_2 = The two principal curvature radii

σ = Surface tension, a = area

Ψ=Electrical potential, q=charge

B=Magnetic field, I= Magnetic moment

U = Internal Energy, T = Absolute Temperature, p= Absolute pressure, V= Volume, S= Entropy

n_i = number of moles of species "i"

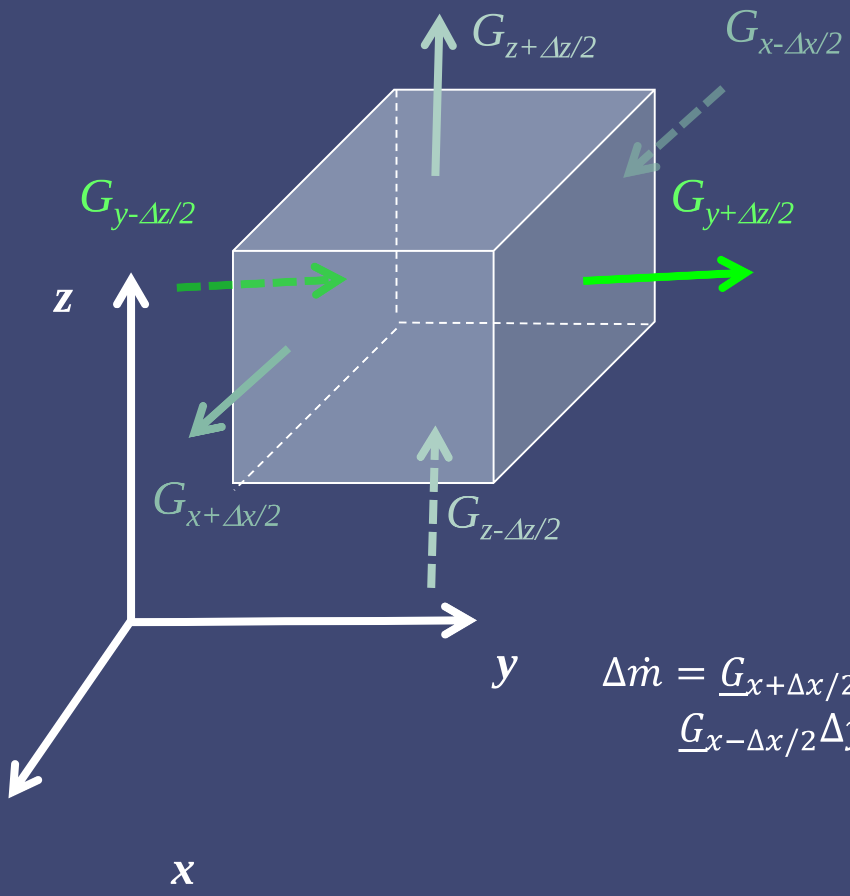
$\bar{G}_i$ = Molar partial Gibbs energy or chemical potential

$$\underline{F}_L = \sigma\, L\, \underline{\hat{t}}$$

Gauss Divergence Theorem

Ostrogradsky's theorem

To calculate mass rate within a porous media



$$\underline{G} = -\frac{\rho\,\mu}{\kappa}\left[\underline{\nabla} p - \rho\,\underline{g}\right]$$

$$\underline{G}_x = -\hat{\imath}\,\frac{\rho\,\mu}{\kappa}\left[\frac{\partial p}{\partial x} - \rho\,g_x\right]$$

$$\underline{G}_y = -\hat{\jmath}\,\frac{\rho\,\mu}{\kappa}\left[\frac{\partial p}{\partial y} - \rho\,g_y\right]$$

$$\underline{G}_z = -\hat{k}\,\frac{\rho\,\mu}{\kappa}\left[\frac{\partial p}{\partial z} - \rho\,g_z\right]$$

$$\Delta\dot{m} = \underline{G}_{x+\Delta x/2}\Delta y\Delta z \cdot \underline{n}_F + \underline{G}_{y+\Delta y/2}\Delta x\Delta z \cdot \underline{n}_E + \underline{G}_{z+\Delta z/2}\Delta y\Delta x \cdot \underline{n}_N + \underline{G}_{x-\Delta x/2}\Delta y\Delta z \cdot \underline{n}_B + \underline{G}_{y-\Delta y/2}\Delta x\Delta z \cdot \underline{n}_W + \underline{G}_{z-\Delta z/2}\Delta y\Delta x \cdot \underline{n}_S$$

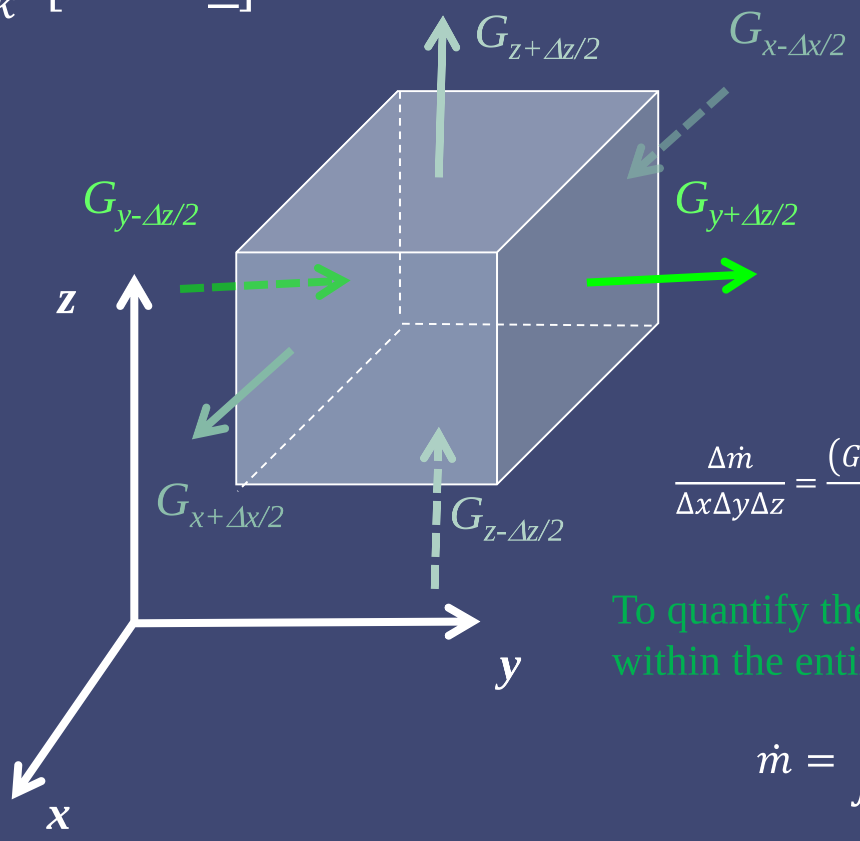
To quantify the total rate of mass crossing the entire system:

$$\dot{m} = \oiint \underline{G}\cdot\underline{n}\,dA = \oiint \underline{n}\cdot\underline{G}\,dA$$

Only 3 faces are shown, but remember we have six faces

To calculate the mass rate by advection

$$\underline{G} = -\frac{\rho\,\mu}{\kappa}\left[\underline{\nabla} p - \rho\,\underline{g}\right]$$



$$\Delta\dot{m} = G_{x+\Delta x/2}\Delta y\Delta z + G_{y+\Delta y/2}\Delta x\Delta z + G_{z+\Delta z/2}\Delta y\Delta x + -G_{x-\Delta x/2}\Delta y\Delta z - G_{y-\Delta y/2}\Delta x\Delta z - G_{z-\Delta z/2}\Delta y\Delta x$$

It can also expressed by unit volume and quantify within the entire volume as:

$$\frac{\Delta\dot{m}}{\Delta x\Delta y\Delta z} = \frac{\left(G_{x+\Delta x/2} - G_{x-\Delta x/2}\right)}{\Delta x} + \frac{\left(G_{y+\Delta y/2} - G_{y-\Delta y/2}\right)}{\Delta y} + \frac{\left(G_{z+\Delta z/2} - G_{z-\Delta z/2}\right)}{\Delta z}$$

To quantify the total amount of mass rate within the entire system:

$$\dot{m} = \iiint \left[\frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G}{\partial z}\right] dV = \iiint \underline{\nabla}\cdot\underline{G}\, dV$$

Comparing with the previous equation the Gauss' Law can be inferred as:

$$\dot{m} = \oiint \underline{G}\cdot\underline{n}\, dA = \oiint \underline{n}\cdot\underline{G}\, dA = \iiint \underline{\nabla}\cdot\underline{G}\, dV$$

Only 3 faces are shown, but remember we have six faces

Gauss Divergence Theorem

Ostrogradsky's theorem

$$\oiint \underline{n} \cdot \underline{\Psi} \, dA = \iiint \underline{\nabla} \cdot \underline{\Psi} \, dV$$

Analysis for cylindrical coordinates

Velocity Gradient tensor in Cylindrical coordinates

$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{e}_r\hat{e}_r & \frac{\partial v_\theta}{\partial r}\hat{e}_r\hat{e}_\theta & \frac{\partial v_z}{\partial r}\hat{e}_r\hat{e}_z \\ \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{e}_\theta\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta\hat{e}_\theta & \frac{1}{r}\frac{\partial v_z}{\partial \theta}\hat{e}_\theta\hat{e}_z \\ \frac{\partial v_r}{\partial z}\hat{e}_z\hat{e}_r & \frac{\partial v_\theta}{\partial z}\hat{e}_z\hat{e}_\theta & \frac{\partial v_z}{\partial z}\hat{e}_z\hat{e}_z \end{bmatrix}$$

Transpose of velocity Gradient tensor

$$[\underline{\nabla}\,\underline{v}]' = \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{e}_r\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{e}_r\hat{e}_\theta & \frac{\partial v_r}{\partial z}\hat{e}_r\hat{e}_z \\ \frac{\partial v_\theta}{\partial r}\hat{e}_\theta\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta\hat{e}_\theta & \frac{\partial v_\theta}{\partial z}\hat{e}_\theta\hat{e}_z \\ \frac{\partial v_z}{\partial r}\hat{e}_z\hat{e}_r & \frac{1}{r}\frac{\partial v_z}{\partial \theta}\hat{e}_z\hat{e}_\theta & \frac{\partial v_z}{\partial z}\hat{e}_z\hat{e}_z \end{bmatrix}$$

Strain rate tensor

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{v} + [\underline{\nabla}\,\underline{v}]'\right] = \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{e}_r\hat{e}_r & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]\hat{e}_r\hat{e}_\theta & \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{e}_r\hat{e}_z \\ \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]\hat{e}_\theta\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta\hat{e}_\theta & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]\hat{e}_\theta\hat{e}_z \\ \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{e}_z\hat{e}_r & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]\hat{e}_z\hat{e}_\theta & \frac{\partial v_z}{\partial z}\hat{e}_z\hat{e}_z \end{bmatrix}$$

Divergence of viscous stress tensor

$$\underline{\nabla}\cdot\underline{\underline{\tau}} = \left[\frac{\partial(\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta r})}{\partial\theta} + \frac{(\tau_{rr}-\tau_{\theta\theta})}{r} + \frac{\partial\tau_{rz}}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{r}} + \left[\frac{\partial(\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta\theta})}{\partial\theta} + \frac{(\tau_{r\theta}+\tau_{\theta r})}{r} + \frac{\partial\tau_{z\theta}}{\partial z}\right]\hat{\mathrm{e}}_{\theta} + \left[\frac{\partial(\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta z})}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r}\right]\hat{\mathrm{e}}_{\mathrm{z}}$$

Velocity divergence

$$\underline{\nabla}\cdot\underline{v} = \frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial(v_\theta)}{\partial\theta} + \frac{\partial(v_z)}{\partial z}$$

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{\Gamma}} + \left[\kappa - \frac{2}{3}\mu\right]\left[\underline{\nabla}\cdot\underline{v}\right]\underline{\underline{I}}$$

Relationship among viscous stress, strain rate tensor, and velocity divergence

μ is shear viscosity, dynamic viscosity.
κ volume viscosity coefficient, bulk viscosity or dilatation viscosity or second coefficient.

Viscous stress tensor

$$\underline{\underline{\tau}} = 2\,\mu\begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)\right]\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\theta} & \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{z}} \\ \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)\right]\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{r}} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right)\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\theta} & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right]\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{z}} \\ \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{r}} & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right]\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\theta} & \frac{\partial v_z}{\partial z}\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{z}} \end{bmatrix} + \left[\kappa - \frac{2}{3}\mu\right]\left[\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial(v_\theta)}{\partial\theta} + \frac{\partial(v_z)}{\partial z}\right]\begin{bmatrix} 1\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} & 0\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\theta} & 0\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{z}} \\ 0\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{r}} & 1\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\theta} & 0\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{z}} \\ 0\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{r}} & 0\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\theta} & 1\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{z}} \end{bmatrix}$$

Understanding Young-Laplace Equation

- Surface Tension to quantify line forces
- Surface Tension to quantify surface energy

Thermodynamics

$$dU = T\,dS - p\,dV + \sigma\,da + \Psi\,dq + B\,dI + \sum \bar{G}_i dn_i$$

$$dU = -p\,dV + \sigma\,da$$

$$dU = -p_i\,dV_i - p_o\,dV_o + \sigma\,da$$

$$dU = -[p_i - p_o]\,dV_i + \sigma\,da$$

$$0 = -[p_i - p_o]\,dV_i + \sigma\,da$$

$$V_i + V_o = c_1$$

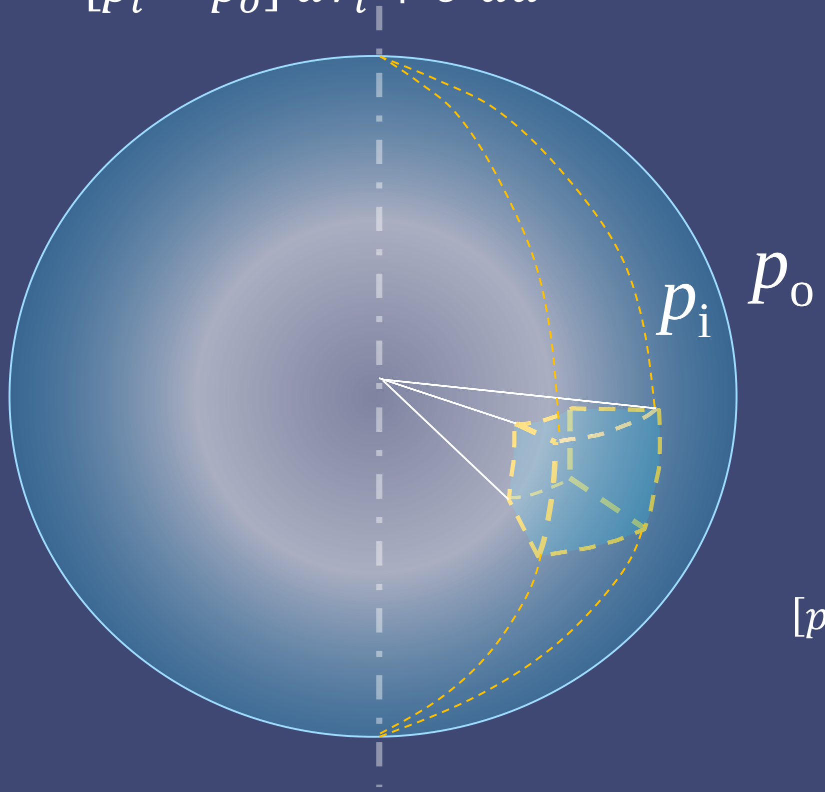
$$dV_i + dV_o = 0$$

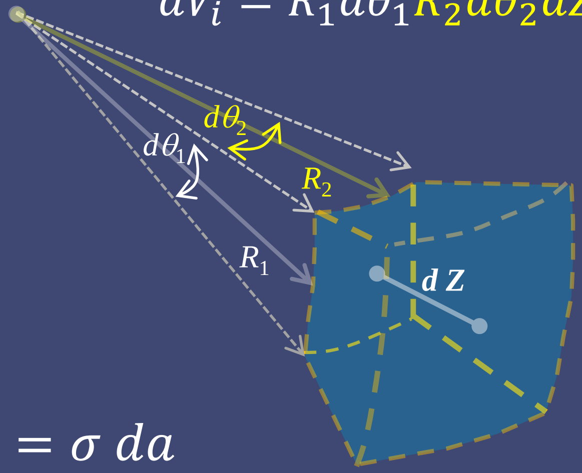
$$a_z = R_1 d\theta_1 R_2 d\theta_2$$

$$a_{z+dz} = (R_1 + dZ)d\theta_1(R_2 + dZ)d\theta_2$$

$$da = a_{z+dz} - a_z = (R_1 dZ + R_2 dZ)\,d\theta_1 d\theta_2$$

$$dV_i = R_1 d\theta_1 R_2 d\theta_2 dZ$$





$$[p_i - p_o]\,dV_i = \sigma\,da$$

$$[p_i - p_o]R_1 d\theta_1 R_2 d\theta_2 dZ = \sigma(R_1 + R_2)\,d\theta_1 d\theta_2 dZ$$

$$[p_i - p_o] = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

Thermodynamics (Energy approach analysis)

$$dU = T\,dS - p\,dV + \sigma\,da + \Psi\,dq + B\,dI + \sum \bar{G}_i dn_i$$

Change in area will require energy, if the energy is constant, a pressure difference across the curved surface will be the result of the curved surface

$$dU = -p\,dV + \sigma\,da$$

Change volume related with surface change

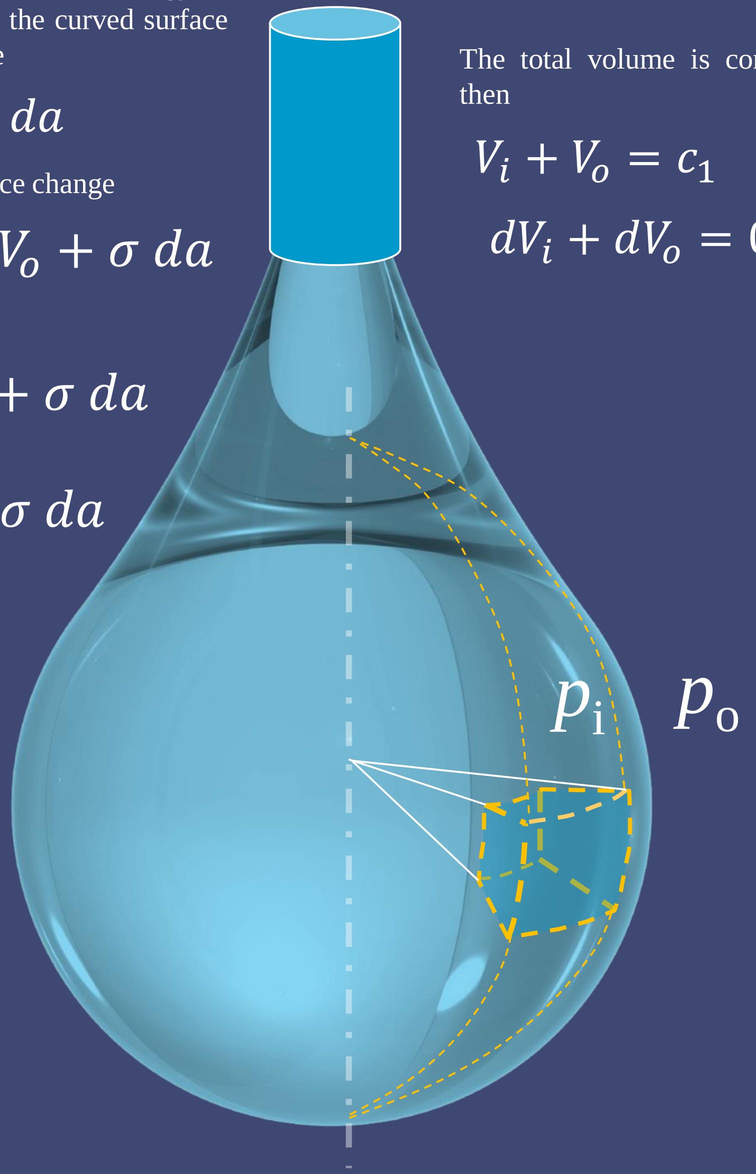
$$dU = -p_i\,dV_i - p_o\,dV_o + \sigma\,da$$

In terms of the fluid within the CV

$$dU = -[p_i - p_o]\,dV_i + \sigma\,da$$

$$0 = -[p_i - p_o]\,dV_i + \sigma\,da$$

Surface tension as energy per unit area



The total volume is constant then

$$V_i + V_o = c_1$$

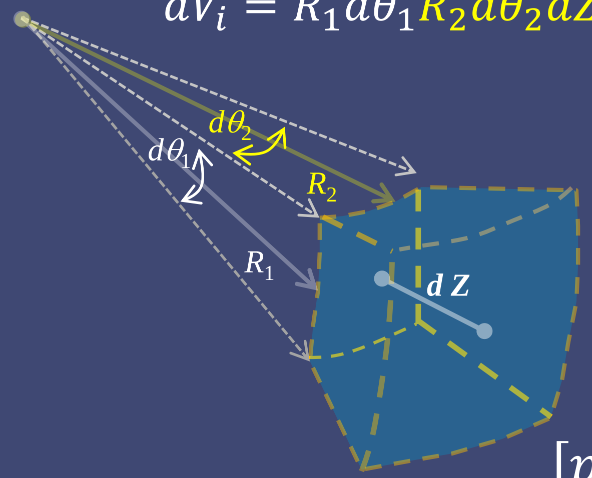
$$dV_i + dV_o = 0$$

$$a_z = R_1 d\theta_1 R_2 d\theta_2$$

$$a_{z+dz} = (R_1 + dZ)d\theta_1 (R_2 + dZ)d\theta_2$$

$$da = a_{z+dz} - a_z = (R_1 dZ + R_2 dZ)\,d\theta_1 d\theta_2$$

$$dV_i = R_1 d\theta_1 R_2 d\theta_2 dZ$$

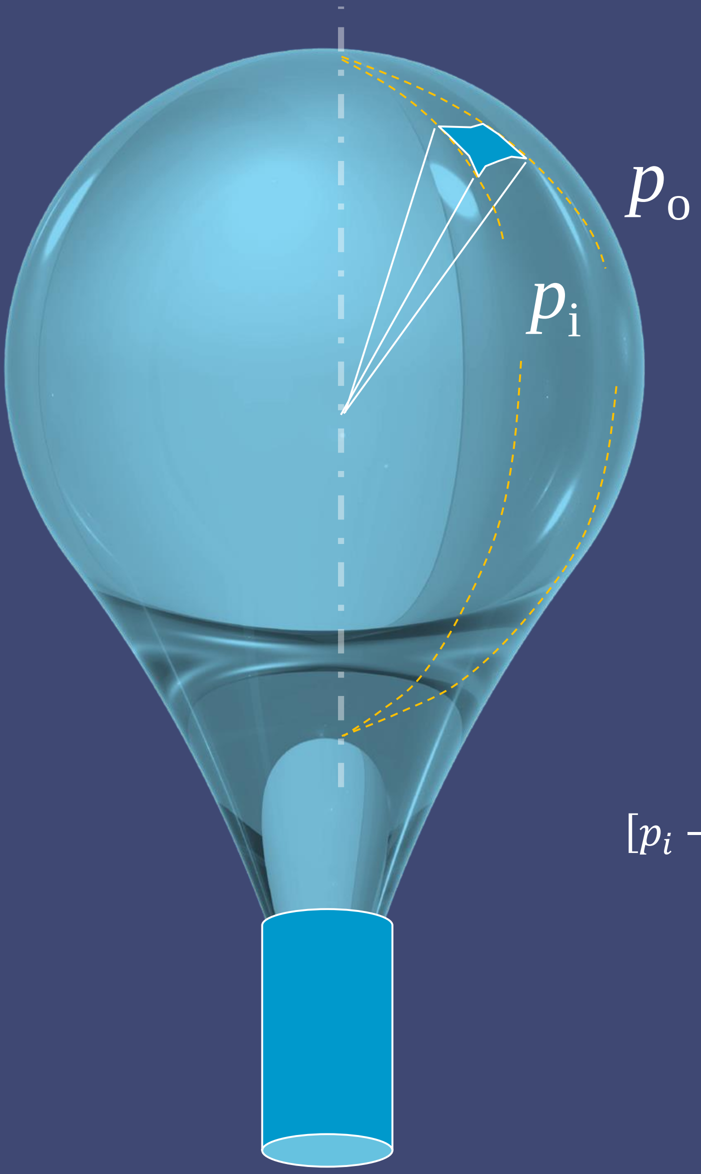


$$[p_i - p_o]\,dV_i = \sigma\,da$$

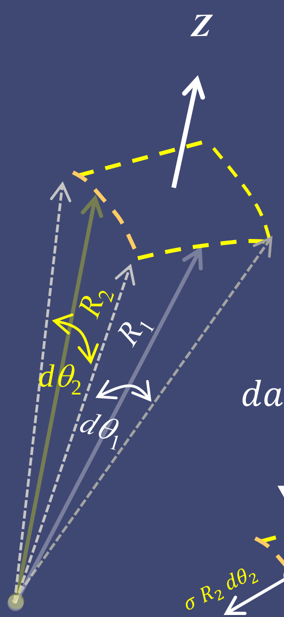
$$[p_i - p_o] R_1 d\theta_1 R_2 d\theta_2 dZ = \sigma (R_1 + R_2)\,d\theta_1 d\theta_2 dZ$$

Young-Laplace equation $\quad [p_i - p_o] = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

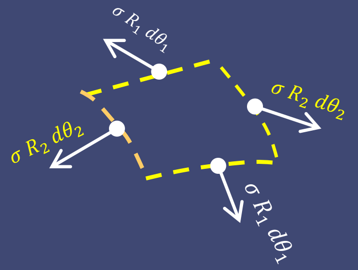
Physics (Force approach analysis)

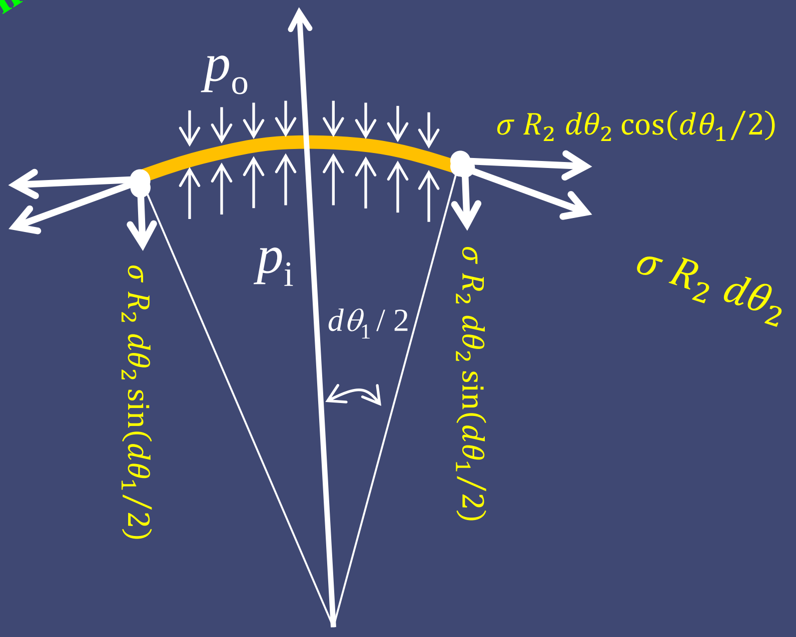


Surface tension as force per unit length



$$da = (R_1 R_2)\, d\theta_1 d\theta_2$$





$$[p_i - p_o] R_1 d\theta_1 R_2 d\theta_2 = \sigma(2\, R_1\, d\theta_1 \sin(d\theta_2/2) + 2\, R_2\, d\theta_2 \sin(d\theta_1/2))$$

$$\lim_{\epsilon \to 0} \sin(\epsilon) = \epsilon$$

$$[p_i - p_o] R_1 d\theta_1 R_2 d\theta_2 = \sigma(\, R_1\, d\theta_1 d\theta_2 + R_2\, d\theta_2 d\theta_1)$$

Young-Laplace equation $\quad [p_i - p_o] = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$

By: Dr. José Luis López Salinas
This material can be used only by the instructor to plan his lecture.
It is by no means, a template nor lecture notes

What did you suppose to have learned ?

Density
Viscosity
Specific gravity
Specific weight
Surface tension
Contact angle
Permeability
Specific heat capacity ratio
Isobaric specific heat capacity
Isochoric specific heart capacity
Molar mass
Coefficient of thermal expansion (Volumetric expansion coefficient, isobaric expansion coefficient)
Compressibility (coefficient of compressibility, isothermal compressibility, inverse of bulk modulus)
Roughness
Permittivity
Kinematic viscosity (momentum diffusivity)
Heat of vaporization
Saturation pressure (vapor pressure)
Capillary pressure
Relative permeability

What did you suppose to have learned or to become acquainted with?

- Density
- Viscosity
- Specific gravity
- Specific weight
- Surface tension
- Contact angle
- Permeability
- Specific heat capacity ratio
- Isobaric specific heat capacity
- Isochoric specific heart capacity
- Molar mass

- Molar mass
- Coefficient of thermal expansion (Volumetric expansion coefficient, isobaric expansion coefficient)
- Compressibility (coefficient of compressibility, isothermal compressibility, inverse of bulk modulus)
- Roughness
- Permittivity
- Kinematic viscosity (momentum diffusivity)
- Heat of vaporization
- Saturation pressure (vapor pressure)
- Capillary pressure
- Relative permeability