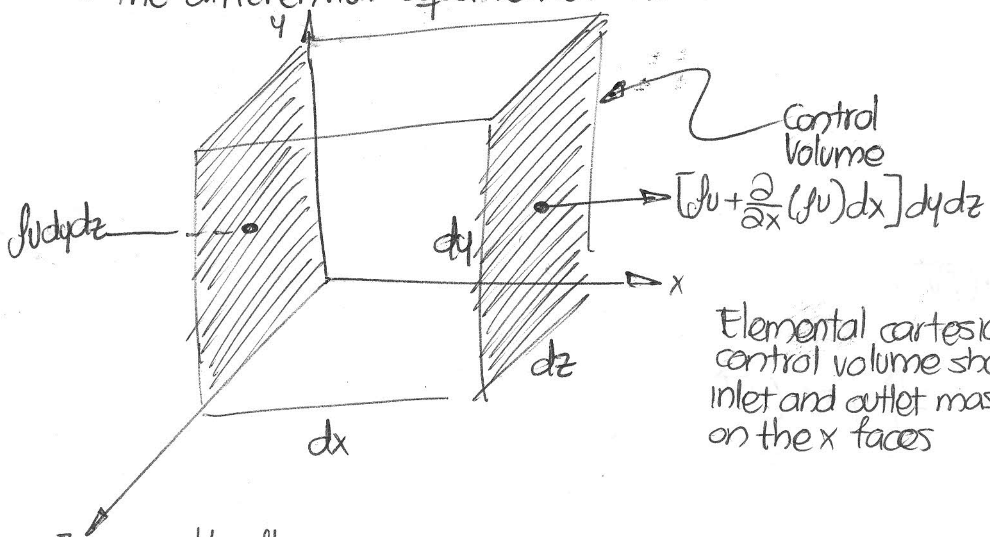
- The differential equation of mass conservation



Elemental cartesian fixed control volume showing the inlet and outlet mass flows on the x faces

Mass flow

$$\int_{CV}\frac{\partial \rho}{\partial t}d\mathcal{V}+\sum_i(\rho_i A_i V_i)_{out}-\sum_i(\rho_i A_i V_i)_{in}=0$$

→ Mass conservation relation

$\left[\frac{kg}{m^3 s}\cdot m^3\right]$ $\left[\frac{kg}{m^3}\cdot m^2\cdot \frac{m}{s}\right]$ $\left[\frac{kg}{m^3}\cdot m^2\cdot \frac{m}{s}\right]$

$$\int_{CV}\frac{\partial \rho}{\partial t}d\mathcal{V}\approx\frac{\partial \rho}{\partial t}dxdydz$$ because the element is so small

Face	Inlet mass flow	Outlet mass flow
x	$\rho u dydz$	$\left[\rho u+\frac{\partial}{\partial x}(\rho u)dx\right]dydz$
y	$\rho v dxdz$	$\left[\rho v+\frac{\partial}{\partial y}(\rho v)dy\right]dxdz$
z	$\rho w dxdy$	$\left[\rho w+\frac{\partial}{\partial z}(\rho w)dz\right]dxdy$

$$\int_{CV}\frac{\partial \rho}{\partial t}d\mathcal{V}+\sum_i(\rho_i A_i V_i)_{out}-\sum_i(\rho_i A_i V_i)_{in}=0$$

$$\frac{\partial \rho}{\partial t}dxdydz+\cancel{\rho u dydz}+\frac{\partial}{\partial x}(\rho u)dxdydz+\cancel{\rho v dxdz}+\frac{\partial}{\partial y}(\rho v)dxdydz+\cancel{\rho w dxdy}+\frac{\partial}{\partial z}(\rho w)dxdydz$$
$$-\cancel{\rho u dydz}-\cancel{\rho v dxdz}-\cancel{\rho w dxdy}=0$$

$$\frac{\partial \rho}{\partial t}dxdydz+\frac{\partial}{\partial x}(\rho u)dxdydz+\frac{\partial}{\partial y}(\rho v)dxdydz+\frac{\partial}{\partial z}(\rho w)dxdydz=0$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0$$

$$\nabla=\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

$$\vec{V}=\hat{i}u+\hat{j}v+\hat{k}w$$

$$\therefore \frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\equiv\nabla\cdot(\rho\vec{V})$$

$$\therefore \frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=\underline{\underline{\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\vec{V})=0}}$$

* Cylindrical Polar Coordinates

$$r=(x^2+y^2)^{1/2}$$

$$\theta=\tan^{-1}\frac{y}{x}$$

$$z=z$$

And for any $\vec{A}(r,\theta,z,t)$

$$\nabla\cdot\vec{A}=\frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(A_\theta)+\frac{\partial}{\partial z}(A_z)$$

$\therefore$ The general continuity equation

$$\underline{\underline{\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0}}$$

- Steady compressible flow

$$\frac{\partial \rho}{\partial t} \equiv 0 \,,\ \frac{\partial}{\partial t} \equiv 0$$

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \nabla \cdot (\rho \vec{V}) = 0$$

$$\nabla \cdot (\rho \vec{V}) = 0$$

$$\underbrace{\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}\ //$$ • Cartesian

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r \rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0}\ //$$ • Cylindrical

- Incompressible flow

$$\frac{\partial \rho}{\partial t} \approx 0$$

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \nabla \cdot (\rho \vec{V}) = 0$$

$$\nabla \cdot (\rho \vec{V}) = \rho (\nabla \cdot \vec{V}) = 0$$

$$\therefore\ \nabla \cdot \vec{V} = 0$$

$$\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}\ //$$ • Cartesian

$$\underbrace{\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0}\ //$$ • Cylindrical

- When is a given flow incompressible?

$$\nabla \cdot (\rho \vec{V}) \approx \rho (\nabla \cdot \vec{V}) = 0$$

$$\frac{\partial}{\partial x}(\rho u) \approx \rho \frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}(\rho u) \approx \rho \frac{\partial u}{\partial x}$$

this is equivalent to

$$\left| u\frac{\partial \rho}{\partial x} \right| << \left| \rho \frac{\partial u}{\partial x} \right|$$

or

$$\left| \frac{\delta \rho}{\rho} \right| << \left| \frac{\delta V}{V} \right|$$

Speed of sound

$$\delta p = a^2 \delta \rho$$

$$\delta p \approx -\rho V \delta V$$

$\therefore$

$$a^2 \delta \rho = -\rho V \delta V$$

$$\delta \rho = \frac{-\rho V \delta V}{a^2}$$

$$\left| \frac{\delta \rho}{\rho} \right| << \left| \frac{\delta V}{V} \right|$$

$$\frac{\cancel{\rho} V \cancel{\delta V}}{\cancel{\rho} a^2} << \frac{\cancel{\delta V}}{V}$$

$$\frac{V^2}{a^2} << 1$$

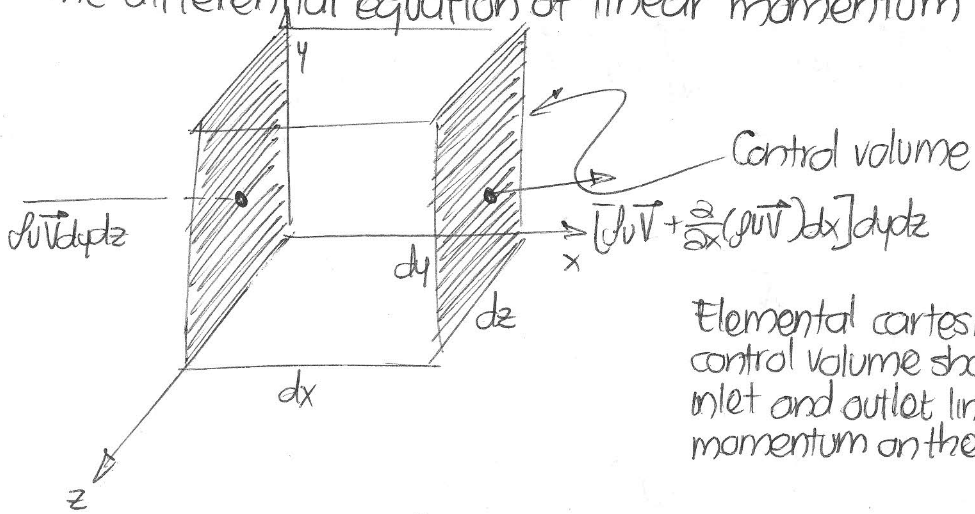
$$Ma = \frac{V}{a}$$

Mach number

$$\therefore Ma^2 << 1$$

$Ma \leq 0.3$ can be considered an incompressible flow

• The differential equation of linear momentum



Elemental cartesian fixed control volume showing the inlet and outlet linear momentum on the x-faces

$$\sum\vec{F} = \frac{\partial}{\partial t}\left(\int_{cv}\vec{V}\rho\, d\forall\right) + \sum_i(\dot{m}_i\vec{V}_i)_{out} - \sum_i(\dot{m}_i\vec{V}_i)_{in} \longrightarrow \text{Linear momentum relation}$$

$$\left[\frac{1}{s}\right]\left[\frac{m}{s}\right]\left[\frac{kg}{m^3}\right][m^3] \quad \left[\frac{kg}{s}\right]\left[\frac{m}{s}\right] \quad \left[\frac{kg}{s}\right]\left[\frac{m}{s}\right]$$

$$\frac{\partial}{\partial t}\left(\int_{cv}\vec{V}\rho\, d\forall\right) \approx \frac{\partial}{\partial t}(\vec{V}\rho\, d\forall) = \frac{\partial}{\partial t}(\rho\vec{V})dxdydz$$ because the element is so small

face	Inlet momentum flux	Outlet momentum flux
x	$\rho u\vec{V}dydz$	$[\rho u\vec{V} + \frac{\partial}{\partial x}(\rho u\vec{V})dx]dydz$
y	$\rho v\vec{V}dxdz$	$[\rho v\vec{V} + \frac{\partial}{\partial y}(\rho v\vec{V})dy]dxdz$
z	$\rho w\vec{V}dxdy$	$[\rho w\vec{V} + \frac{\partial}{\partial z}(\rho w\vec{V})dz]dxdy$

$$\sum\vec{F} = \frac{\partial}{\partial t}\left(\int_{cv}\vec{V}\rho\, d\forall\right) + \sum_i(\dot{m}_i\vec{V}_i)_{out} - \sum_i(\dot{m}_i\vec{V}_i)_{in}$$

$$\sum d\vec{F} = \frac{\partial}{\partial t}(\rho\vec{V})dxdydz + \cancel{\rho u\vec{V}dydz} + \frac{\partial}{\partial x}(\rho u\vec{V})dxdydz + \cancel{\rho v\vec{V}dxdz} + \frac{\partial}{\partial y}(\rho v\vec{V})dxdydz$$
$$+ \cancel{\rho w\vec{V}dxdy} + \frac{\partial}{\partial z}(\rho w\vec{V})dxdydz - \cancel{\rho u\vec{V}dydz} - \cancel{\rho v\vec{V}dxdz} - \cancel{\rho w\vec{V}dxdy}$$
$$= \frac{\partial}{\partial t}(\rho\vec{V})dxdydz + \frac{\partial}{\partial x}(\rho u\vec{V})dxdydz + \frac{\partial}{\partial y}(\rho v\vec{V})dxdydz + \frac{\partial}{\partial z}(\rho w\vec{V})dxdydz$$
$$= \underbrace{\left[\frac{\partial}{\partial t}(\rho\vec{V}) + \frac{\partial}{\partial x}(\rho u\vec{V}) + \frac{\partial}{\partial y}(\rho v\vec{V}) + \frac{\partial}{\partial z}(\rho w\vec{V})\right]}_{\rho\frac{\partial}{\partial t}(\vec{V}) + \vec{V}\frac{\partial}{\partial t}(\rho) + \rho u\frac{\partial}{\partial x}(\vec{V}) + \vec{V}\frac{\partial}{\partial x}(\rho u) + \rho v\frac{\partial}{\partial y}(\vec{V}) + \vec{V}\frac{\partial}{\partial y}(\rho v) + \rho w\frac{\partial}{\partial z}(\vec{V}) + \vec{V}\frac{\partial}{\partial z}(\rho w)}dxdydz$$

$$\rho\frac{\partial\vec{V}}{\partial t}+\vec{V}\frac{\partial\rho}{\partial t}+\rho u\frac{\partial\vec{V}}{\partial x}+\vec{V}\frac{\partial}{\partial x}(\rho u)+\rho v\frac{\partial\vec{V}}{\partial y}+\vec{V}\frac{\partial}{\partial y}(\rho v)+\rho w\frac{\partial\vec{V}}{\partial z}+\vec{V}\frac{\partial}{\partial z}(\rho w)$$

$$=\vec{V}\left[\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right]+\rho\left(\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}\right)$$

$$\nabla=\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}$$

$$\rho\vec{V}=\rho u\hat{i}+\rho v\hat{j}+\rho w\hat{k}$$

$$=\vec{V}\underbrace{\cancel{\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{V})\right]}}_{\text{Continuity equation}}^{0}+\rho\left(\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}\right)$$

$$=\rho\left(\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}\right)=\rho\underbrace{\left[\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}\right]}_{\text{total acceleration}}=\rho\frac{d\vec{V}}{dt}$$

$$\vec{a}=\frac{d\vec{V}}{dt}$$

$$\therefore \sum d\vec{F}=\rho\frac{d\vec{V}}{dt}\,dxdydz$$

- Body forces: due to external fields that act on the entire mass within the element
- Surface forces: due to the stresses on the sides of the control surface

• Body forces

$$d\vec{F}_{grav}=\rho\vec{g}\,dxdydz \Rightarrow \left(\frac{d\vec{F}}{d\forall}\right)_{grav}=\rho\vec{g}$$

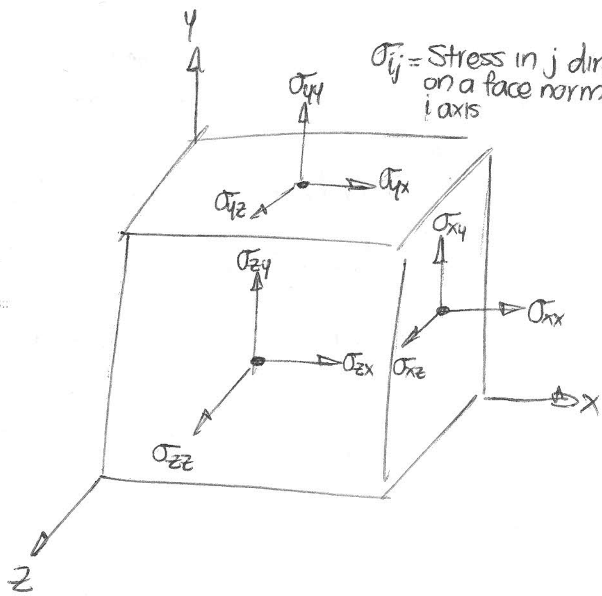
$$\vec{g}=-g\hat{k}$$

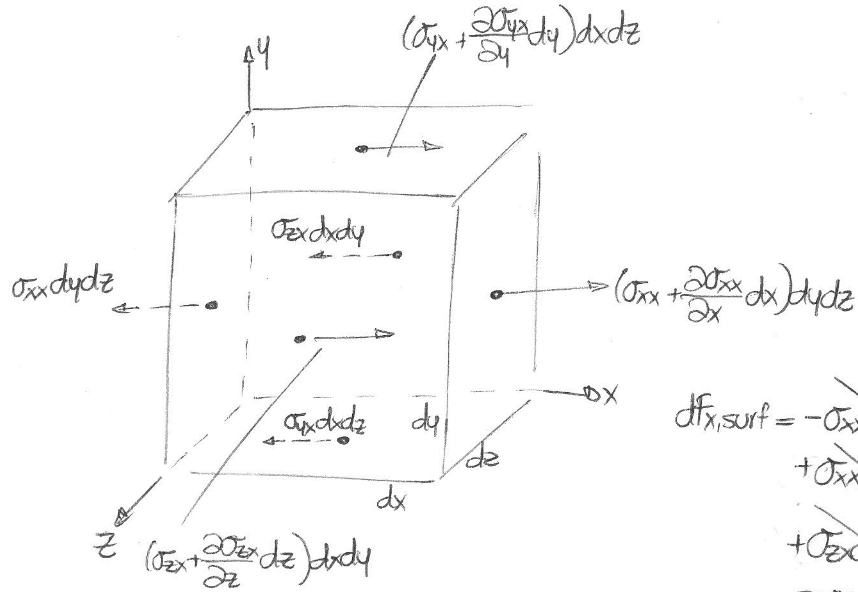
• Surface forces

hydrostatic pressure $\qquad$ viscous stress

$$\overline{\sigma_{ij}}=\begin{vmatrix}-p+\tau_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & -p+\tau_{yy} & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & -p+\tau_{zz}\end{vmatrix}$$

σ_{ij} = Stress in j direction on a face normal to i axis





Elemental cartesian fixed control volume showing the surface forces in the x-direction only

$$
\begin{aligned}
dF_{x,surf} &= -\cancel{\sigma_{xx}dydz} - \cancel{\sigma_{yx}dxdz} - \cancel{\sigma_{zx}dxdy} \\
&\quad + \cancel{\sigma_{xx}dydz} + \frac{\partial \sigma_{xx}}{\partial x}dxdydz + \cancel{\sigma_{yx}dxdy} + \frac{\partial \sigma_{yx}}{\partial y}dxdydz \\
&\quad + \cancel{\sigma_{zx}dxdy} + \frac{\partial \sigma_{zx}}{\partial z}dxdydz \\
&= \frac{\partial \sigma_{xx}}{\partial x}dxdydz + \frac{\partial \sigma_{yx}}{\partial y}dxdydz + \frac{\partial \sigma_{zx}}{\partial z}dxdydz \\
&= \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right]dxdydz \\
&= \left[\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx})\right]d\mathcal{V}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dF_x}{d\mathcal{V}} &= \frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{yx}) + \frac{\partial}{\partial z}(\sigma_{zx}) \\
&= \frac{\partial}{\partial x}(-p + \tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx}) \\
&= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})
\end{aligned}
$$

$\therefore$

$$
\frac{dF_y}{dV} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})
$$

$$
\frac{dF_z}{dV} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})
$$

$$
\begin{aligned}
\left(\frac{d\vec{F}}{d\mathcal{V}}\right)_{surf} &= \frac{dF_x}{d\mathcal{V}}\hat{i} + \frac{dF_y}{d\mathcal{V}}\hat{j} + \frac{dF_z}{d\mathcal{V}}\hat{k} \\
&= -\frac{\partial p}{\partial x}\hat{i} + \left[\frac{\partial}{\partial x}(\tau_{xx}) + \frac{\partial}{\partial y}(\tau_{yx}) + \frac{\partial}{\partial z}(\tau_{zx})\right]\hat{i} - \frac{\partial p}{\partial y}\hat{j} + \left[\frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})\right]\hat{j} \\
&\quad - \frac{\partial p}{\partial z}\hat{k} + \left[\frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})\right]\hat{k}
\end{aligned}
$$

$$-\left(\frac{\partial p}{\partial x}\hat{i}+\frac{\partial p}{\partial y}\hat{j}+\frac{\partial p}{\partial z}\hat{k}\right)=-\nabla p$$

$$\frac{\partial}{\partial x}(\tau_{xx})+\frac{\partial}{\partial y}(\tau_{yx})+\frac{\partial}{\partial z}(\tau_{zx})=\nabla\cdot\vec{\tau}_{ix}$$

$$\frac{\partial}{\partial x}(\tau_{xy})+\frac{\partial}{\partial y}(\tau_{yy})+\frac{\partial}{\partial z}(\tau_{zy})=\nabla\cdot\vec{\tau}_{iy}$$

$$\frac{\partial}{\partial x}(\tau_{xz})+\frac{\partial}{\partial y}(\tau_{yz})+\frac{\partial}{\partial z}(\tau_{zz})=\nabla\cdot\vec{\tau}_{iz}$$

$$\therefore \left(\frac{dF}{d\mathcal{V}}\right)_{viscous}=(\nabla\cdot\vec{\tau}_{ix})\hat{i}+(\nabla\cdot\vec{\tau}_{iy})\hat{j}+(\nabla\cdot\vec{\tau}_{iz})\hat{k}$$

$$=\nabla\cdot\overline{\tau_{ij}}, \quad \overline{\tau_{ij}}=\begin{bmatrix}\tau_{xx} & \tau_{yx} & \tau_{zx}\\ \tau_{xy} & \tau_{yy} & \tau_{zy}\\ \tau_{xz} & \tau_{yz} & \tau_{zz}\end{bmatrix}$$

$$\sum d\vec{F}=\rho\frac{d\vec{V}}{dt}dxdydz=\rho\frac{d\vec{V}}{dt}d\mathcal{V}$$

$$\frac{\sum d\vec{F}}{d\mathcal{V}}=\rho\frac{d\vec{V}}{dt}$$

$$\rho\vec{g}-\nabla p+\nabla\cdot\overline{\tau_{ij}}=\rho\frac{d\vec{V}}{dt}$$

$$d\vec{V}=\frac{\partial\vec{V}}{\partial t}+u\frac{\partial\vec{V}}{\partial x}+v\frac{\partial\vec{V}}{\partial y}+w\frac{\partial\vec{V}}{\partial z}$$
$$=\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}$$

Gravity force per unit volume + Pressure force per unit volume + Viscous force per unit volume = Density x acceleration

- x component

$$\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)$$

- y component

$$\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}=\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)$$

- z component

$$\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\tau_{zz}}{\partial z}=\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)$$

• Inviscid flow: Euler's equation

In frictionless flow $\tau_{ij}=0$

$$\therefore \ \rho\vec{g}-\nabla p=\rho\frac{d\vec{V}}{dt}$$

• Newtonian fluid: Navier-Stokes equations

$$\tau_{xx}=2\mu\frac{\partial u}{\partial x} \qquad \tau_{yy}=2\mu\frac{\partial v}{\partial y} \qquad \tau_{zz}=2\mu\frac{\partial w}{\partial z} \qquad \tau \rightarrow \text{viscous stress}$$

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) \qquad \tau_{xz}=\tau_{zx}=\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

$$\tau_{yz}=\tau_{zy}=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

• x component

$$\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}+\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial w}{\partial x}+\mu\frac{\partial u}{\partial z}\right)=\rho\frac{du}{dt}$$

$$\rho g_x-\frac{\partial p}{\partial x}+2\mu\frac{\partial^2 u}{\partial x^2}+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial}{\partial y}\frac{\partial v}{\partial x}+\mu\frac{\partial}{\partial z}\frac{\partial w}{\partial x}+\mu\frac{\partial^2 u}{\partial z^2}=\rho\frac{du}{dt}$$

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)+\frac{\partial^2 u}{\partial y^2}+\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right)+\frac{\partial^2 u}{\partial z^2}+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial z}\right)\right]=\rho\frac{du}{dt}$$

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\partial}{\partial x}\underbrace{\cancel{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)}}_{0}\right]=\rho\frac{du}{dt}$$ because of the continuity equation for incompressible flow

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=\rho\frac{du}{dt}$$

• y component

$$\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=\rho\frac{dv}{dt}$$

• z component

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)=\rho\frac{dw}{dt}$$

• The differential equation of energy

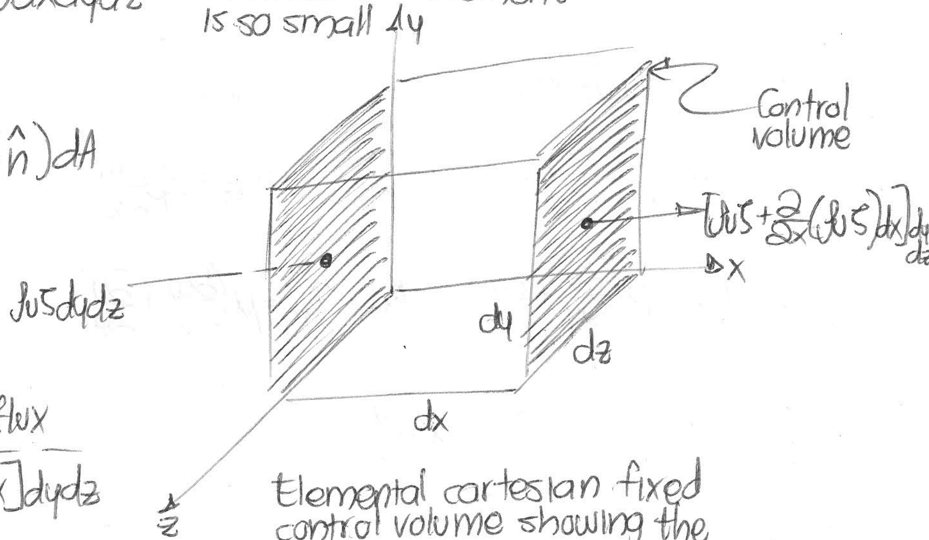
$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\left(\int_{cv} e\rho \, d\mathcal{V}\right) + \int_{cs}\left(e+\frac{p}{\rho}\right)\rho(\vec{V}\cdot\hat{n})\,dA \quad , \quad \dot{W}_s = 0 \longrightarrow \text{Energy relation}$$

$$\left[\frac{1}{s}\right]\left[\frac{J}{kg}\right]\left[\frac{kg}{m^3}\right]\left[m^3\right] \qquad \left(\left[\frac{J}{kg}\right]+\left[\frac{N/m^2}{kg/m^3}\right]\right)\left[\frac{kg}{m^3}\right]\left[\frac{m}{s}\right]\left[m^2\right]$$

$$\frac{\partial}{\partial t}\left(\int_{cv} e\rho \, d\mathcal{V}\right) \approx \frac{\partial}{\partial t}(e\rho \, d\mathcal{V}) = \frac{\partial}{\partial t}(e\rho)\,dx\,dy\,dz$$ because the element is so small

$$\zeta = \frac{p}{\rho} + e$$

$$\int_{cs}\left(e+\frac{p}{\rho}\right)\rho(\vec{V}\cdot\hat{n})\,dA = \int_{cs}\zeta\rho(\vec{V}\cdot\hat{n})\,dA$$



Elemental cartesian fixed control volume showing the inlet and outlet energy flux on the x-faces

Face	Inlet energy flux	Outlet energy flux
x	$\rho u\zeta\,dydz$	$\left[\rho u\zeta + \frac{\partial}{\partial x}(\rho u\zeta)dx\right]dydz$
y	$\rho v\zeta\,dxdz$	$\left[\rho v\zeta + \frac{\partial}{\partial y}(\rho v\zeta)dy\right]dxdz$
z	$\rho w\zeta\,dxdy$	$\left[\rho w\zeta + \frac{\partial}{\partial z}(\rho w\zeta)dz\right]dxdy$

$$\dot{Q} - \dot{W}_v = \frac{\partial}{\partial t}(\rho e)dxdydz + \cancel{\rho u\zeta dydz} + \frac{\partial}{\partial x}(\rho u\zeta)dxdydz + \cancel{\rho v\zeta dxdz} + \frac{\partial}{\partial y}(\rho v\zeta)dxdydz$$
$$+ \cancel{\rho w\zeta dxdy} + \frac{\partial}{\partial z}(\rho w\zeta)dxdydz - \cancel{\rho u\zeta dydz} - \cancel{\rho v\zeta dxdz} - \cancel{\rho w\zeta dxdy}$$
$$= \frac{\partial}{\partial t}(\rho e)dxdydz + \frac{\partial}{\partial x}(\rho u\zeta)dxdydz + \frac{\partial}{\partial y}(\rho v\zeta)dxdydz + \frac{\partial}{\partial z}(\rho w\zeta)dxdydz$$
$$= \left[\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u\zeta) + \frac{\partial}{\partial y}(\rho v\zeta) + \frac{\partial}{\partial z}(\rho w\zeta)\right]dxdydz$$

$$\frac{\partial}{\partial t}(\rho e) + \frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial x}(\cancel{\rho} u p/\cancel{\rho}) + \frac{\partial}{\partial y}(\rho v e) + \frac{\partial}{\partial y}(\cancel{\rho} v p/\cancel{\rho}) + \frac{\partial}{\partial z}(\rho w e) + \frac{\partial}{\partial z}(\cancel{\rho} w p/\cancel{\rho})$$

$$= \rho\frac{\partial e}{\partial t} + e\frac{\partial \rho}{\partial t} + \rho u\frac{\partial e}{\partial x} + e\frac{\partial}{\partial x}(\rho u) + u\frac{\partial p}{\partial x} + p\frac{\partial u}{\partial x} + \rho v\frac{\partial e}{\partial y} + e\frac{\partial}{\partial y}(\rho v) + v\frac{\partial p}{\partial y} + p\frac{\partial v}{\partial y} + \rho w\frac{\partial e}{\partial z} + e\frac{\partial}{\partial z}(\rho w)$$
$$+ w\frac{\partial p}{\partial z} + p\frac{\partial w}{\partial z}$$

$$= \rho\left(\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z}\right) + e\left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$
$$+ u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}$$

- $\frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + w\frac{\partial e}{\partial z} = \underline{\frac{de}{dt}}$
- $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = \cancel{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{V})} = \underline{0}$ (Continuity equation)
- $p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \underline{p(\nabla\cdot\vec{V})}$
- $u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \underline{(\vec{V}\cdot\nabla p)}$

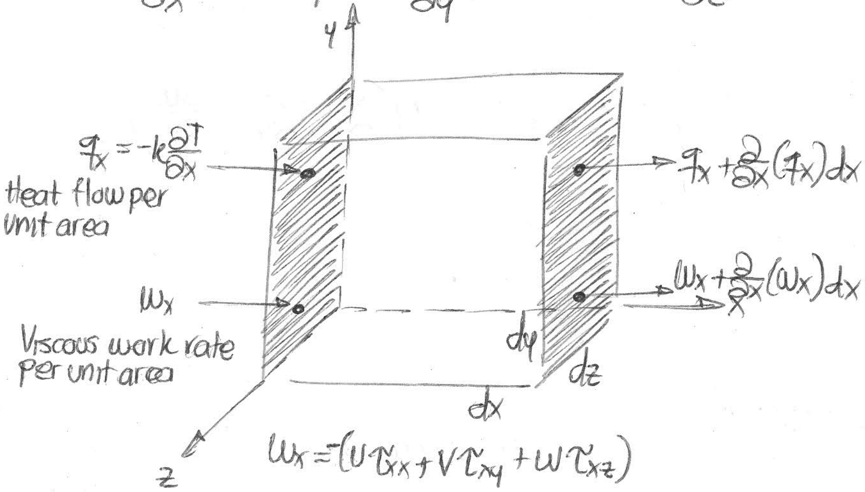
$$= \underline{\rho\frac{de}{dt} + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)}$$

$$\dot{Q} - \dot{W}_v = \left[\rho\frac{de}{dt} + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)\right]dx\,dy\,dz$$

- Thermal conductivity (Fourier's Law)

$$\vec{q} = -k\nabla T$$

$$q_x = -k\frac{\partial T}{\partial x}, \quad q_y = -k\frac{\partial T}{\partial y}, \quad q_z = -k\frac{\partial T}{\partial z}$$



Faces	Inlet heat flux	Outlet heat flux
x	$q_x dydz$	$[q_x + \frac{\partial}{\partial x} q_x dx]dydz$
y	$q_y dxdz$	$[q_y + \frac{\partial}{\partial y} q_y dy]dxdz$
z	$q_z dxdy$	$[q_z + \frac{\partial}{\partial z} q_z dz]dxdy$

$$\dot{Q} = \cancel{q_x dydz} + \cancel{q_y dxdz} + \cancel{q_z dxdy} - \cancel{q_x dydz} - \frac{\partial}{\partial x} q_x dxdydz - \cancel{q_y dxdz} - \frac{\partial}{\partial y} q_y dxdydz$$
$$- \cancel{q_z dxdy} - \frac{\partial}{\partial z} q_z dxdydz$$
$$= -\left[\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_y) + \frac{\partial}{\partial z}(q_z)\right]dxdydz$$
$$= -(\nabla\cdot\vec{q})dxdydz \,/\!/ = [\nabla\cdot(k\nabla T)]dxdydz \,/\!/$$

Faces	Inlet work flux	Outlet work flux
x	$w_x dydz$	$[w_x + \frac{\partial}{\partial x} w_x dx]dydz$
y	$w_y dxdz$	$[w_y + \frac{\partial}{\partial y} w_y dy]dxdz$
z	$w_z dxdy$	$[w_z + \frac{\partial}{\partial z} w_z dz]dxdy$

$$\dot{W}_v = -\left[\frac{\partial}{\partial x}(w_x) + \frac{\partial}{\partial y}(w_y) + \frac{\partial}{\partial z}(w_z)\right]dxdydz$$
$$= +\left[\frac{\partial}{\partial x}(u\tau_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\tau_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\tau_{zz})\right]dxdydz$$
$$= +\left[\frac{\partial}{\partial x}(\vec{V}\cdot\vec{\tau}_{xj}) + \frac{\partial}{\partial y}(\vec{V}\cdot\vec{\tau}_{yj}) + \frac{\partial}{\partial z}(\vec{V}\cdot\vec{\tau}_{zj})\right]dxdydz = +\nabla\cdot(\vec{V}\cdot\vec{\tau}_{ij})dxdydz \,/\!/$$

$$\dot{Q} + \dot{W}_v = \left[\rho \frac{de}{dt} + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)\right] dx\,dy\,dz$$

$$[\nabla\cdot(k\nabla T)]\cancel{dx\,dy\,dz} + \nabla\cdot(\vec{V}\cdot\tau_{ij})\cancel{dx\,dy\,dz} = \left[\rho \frac{de}{dt} + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)\right]\cancel{dx\,dy\,dz}$$

- $e = \hat{u} + \frac{1}{2}V^2 + gz$
- $\nabla\cdot(\vec{V}\cdot\tau_{ij}) \equiv \vec{V}\cdot(\nabla\cdot\tau_{ij}) + \Phi$ ← Viscous-dissipation function

For a newtonian incompressible viscous flow

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right]$$

$$\nabla\cdot(k\nabla T) + \nabla\cdot(\vec{V}\cdot\tau_{ij}) = \rho\frac{de}{dt} + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)$$

$$\nabla\cdot(k\nabla T) + \vec{V}\cdot(\nabla\cdot\tau_{ij}) + \Phi = \rho\frac{de}{dt} + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)$$

$$\nabla\cdot\tau_{ij} = \rho\frac{d\vec{V}}{dt} + \nabla p - \rho\vec{g}$$ from the differential momentum equation

$$\nabla\cdot(k\nabla T) + \vec{V}\cdot\left(\rho\frac{d\vec{V}}{dt} + \nabla p - \rho\vec{g}\right) + \Phi = \rho\frac{d}{dt}\left(\hat{u} + \frac{1}{2}V^2 + gz\right) + p(\nabla\cdot\vec{V}) + (\vec{V}\cdot\nabla p)$$

$$\nabla\cdot(k\nabla T) + \left(\vec{V}\cdot\rho\frac{d\vec{V}}{dt}\right) + \cancel{(\vec{V}\cdot\nabla p)} - (\vec{V}\cdot\rho\vec{g}) + \Phi = \rho\frac{d\hat{u}}{dt} + \frac{1}{2}\rho\frac{dV^2}{dt} + \rho g\frac{dz}{dt} + p(\nabla\cdot\vec{V}) + \cancel{(\vec{V}\cdot\nabla p)}$$

$$\nabla\cdot(k\nabla T) + \left(\vec{V}\cdot\rho\frac{d\vec{V}}{dt}\right) + \cancel{\rho g w} + \Phi = \rho\frac{d\hat{u}}{dt} + \frac{1}{2}\rho\frac{dV^2}{dt} + \cancel{\rho g w} + p(\nabla\cdot\vec{V})$$

$$\nabla\cdot(k\nabla T) + \cancel{\frac{1}{2}\rho\frac{dV^2}{dt}} + \Phi = \rho\frac{d\hat{u}}{dt} + \cancel{\frac{1}{2}\rho\frac{dV^2}{dt}} + p(\nabla\cdot\vec{V})$$

$$\nabla\cdot(k\nabla T) + \Phi = \rho\frac{d\hat{u}}{dt} + p(\nabla\cdot\vec{V})$$

$$\rho\frac{d\hat{u}}{dt} + p(\nabla\cdot\vec{V}) = \nabla\cdot(k\nabla T) + \Phi$$ // general differential energy equation