# Macroscopic Balances in Fluid Mechanic

Fluid Mechanics

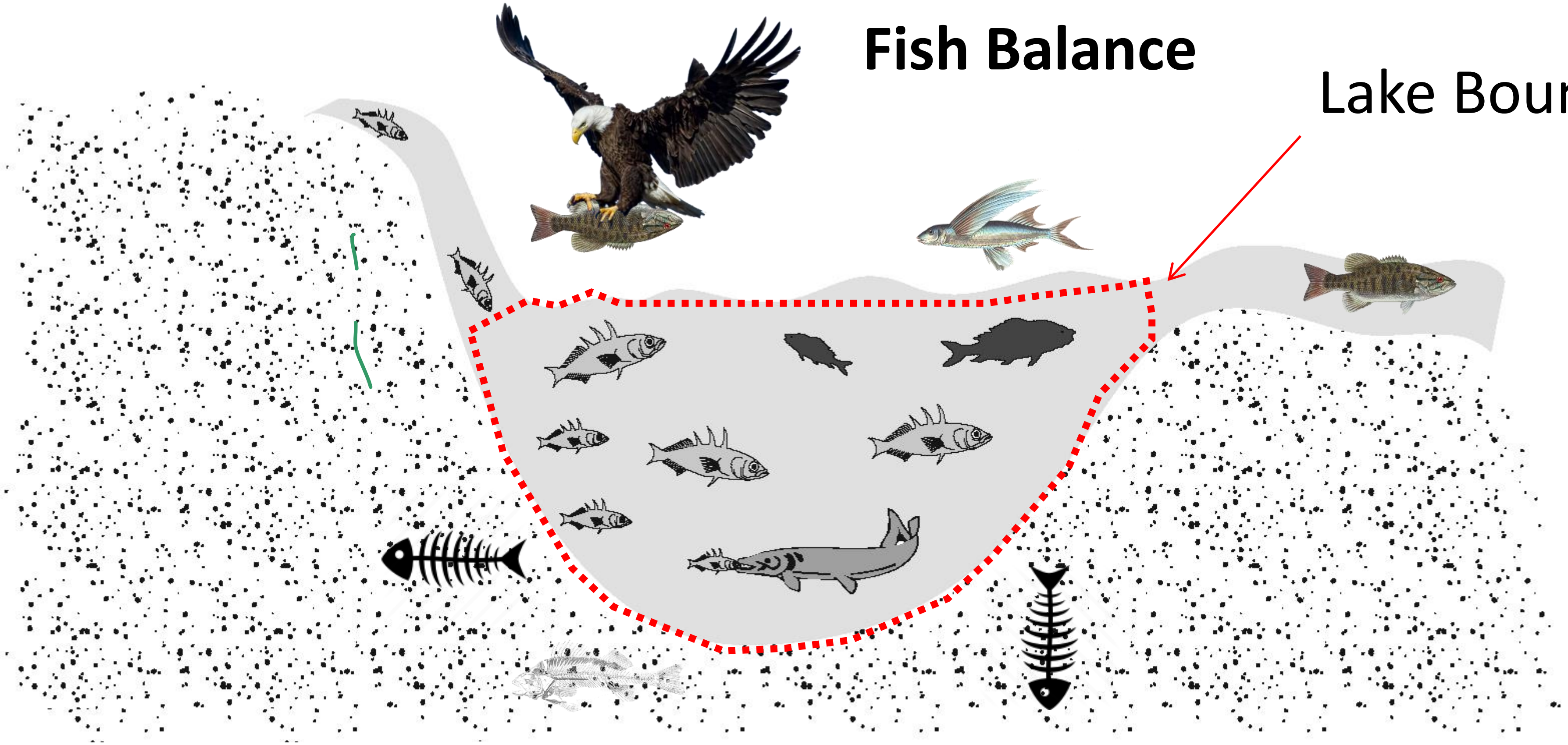
# Macroscopic Conservation Equations.
# (integral form of conservation equations)

Objective: Use of conservation laws in fluid mechanics, in order to analyze devices operating or interacting with fluids

- **Mass** rate equation
- **Linear momentum** rate equation
- **Angular Momentum** rate equation
- **Mechanical Energy** rate equation

This analysis will help to:
I) Quantify the fluid requirements, track materials involved, stored or depleted.
II) Quantify forces and torques involved in the operation and the stability of processes or devices.
III) Quantify energy requirements, energy potentials, and energy losses.

# Are conservation equations exclusive in the field of fluid mechanics ?




*Fish flow rate in– Fish flow rate out + Fish birth rate – Fish death rate = Fish accumulation rate*

Instantaneous

$$\dot{I} - \dot{O} + \dot{B} - \dot{D} = \frac{dN}{dt}$$

Rate conservation Law

Integral form or historical

$$I - O + B - D = \Delta N$$

Balance law

# Any conservation equation takes the form:

$$\boxed{\textit{INPUT} - \textit{OUTPUT} + \textit{GENERATION} - \textit{CONSUMPTION} = \textit{ACCUMULATION}}$$

Positive contributions to the system: Input and Generation

Negative contributions to the system: Output and Consumption

# Two simplified cases of mass conservation equation:

*INPUT – OUTPUT + GENERATION – CONSUMPTION = ACCUMULATION*

I ) If there is no chemical reaction:

Then mass in neither produced nor consumed

***INPUT – OUTPUT = ACCUMULATION***

Then the unbalance of the flow rates will produce accumulation

At steady state, there is no accumulation, so mass conservation is further simplified to:

***INPUT – OUTPUT = 0***

For the continuity equation, this condition is called steady flow

**Analogy:** The conservation equation is valid in different fields like natural sciences, economic sciences, and even social sciences

*Accumulation= Final balance – Initial balance*

*Accumulation= Deposits - Withdrawals + Interest - Fees*

Deposits: what goes in, the inputs (adds to the balance)
Withdrawals: what goes out, the outputs (subtracted from the balance)
Interest generated: gained or produced
Fees: bank charges that consume some of the balance of the account

# Analogy for water within a system:

*Accumulation rate= Input rate – Output rate + Production rate – Consumption rate*

**Input**: water vapor in the incoming air (air flowing into a room), or any other form of water flowing into the room as vapor, liquid or solid (e.g. atomizer)

**Output**: water vapor in the air leaving the room, any other form of water leaving the room, either as a liquid, solid, aerosol, etc.

**Production**: water vapor produced by combustion, or any other chemical reaction or metabolism.

**Consumption**: water reacting, absorbed or adsorbed ($CaCl_2$ + 6 H2O = $CaCl_2$·6H2O)

$$\mathrm{CaSO_4} \cdot \frac{1}{2}\mathrm{H_2O} + \frac{3}{2}\mathrm{H_2O} \rightarrow \mathrm{CaSO_4} \cdot 2\,\mathrm{H_2O}$$

## **Nomenclature of the conservation equation**: It is related with the field of study, but there are equivalences.

*Accumulation*= *Input* - *Output* + *Generation* - *Consumption*

**Accumulation**: Buildup within the system
**Input**: Enters through the system boundaries
**Output**: Leaves through system boundaries
**Generation**: Produced within the system
**Consumption**: consumed within the system

$$\underbrace{Accumulation}_{\text{Storage}} = \underbrace{Input - Output}_{\text{Fluxes crossing surfaces}} + \underbrace{Generation}_{\text{sources}} - \underbrace{Consumption}_{\text{sinks}}$$

The flow is the product of flux by cross sectional area

In fluid mechanics, reservoir engineering and hydraulics, the jargon of “sources” and “sinks”, are not exclusive for reacting systems. They may refer to small portion of the space domain where injection or extraction of fluid is taking place

**Conservation laws**, **Balances**, or **Rate equations** can be set for different extensive properties like:

- Mass
- Moles (Caution !,Stoichiometry may resolve the issue )
- Atoms
- Charge
- Volume ? (Look up !, Caution !)
- Energy
- Mechanical energy
- People, animals or species
- Money
- Heat
- Lineal momentum (force)
- Angular momentum (torque)
- Entropy ? (Look up!, Caution !)
- Exergy

Not conserved

Generation always present

The equations of not conserved properties, can still be used, as long as you include the chemistry (stoichiometry), physics (compressibility) and the thermodynamics ( irreversibility) respectively .

The balance equation can be used to quantify the conserved property, or the rate of the conserved property. As stated before, just need to be aware of the law of dimensional homogeneity.

*Accumulated amount = amount added- amount removed+ generated amount-consumption amount*

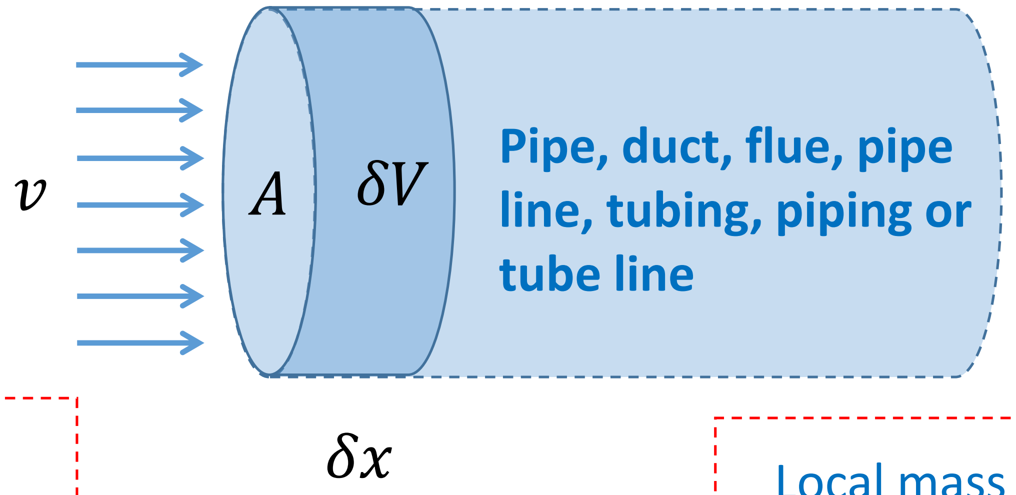
*Accumulation rate= Input rate - Output rate+ Generation rate- Consumption rate*

i.e., these equations may have units of:

| | Mass | Material | Species | Charge | Energy | momentum | Angular momentum |
|---|---|---|---|---|---|---|---|
| **Property** | kg | mol | eq | C | J | kg-m/s | m-kg-m/s |
| **Property Rate** | kg/s | mol/s | eq/s | amp | watts | N | m-N |
| **Name of the conservation law or equation** | Continuity mass rate equation or mass balance | Molar rate equation or molar balance | equivalents rate equation or equivalents balance | Charge balance Electrical current balance | First law energy rate equation or energy balance | Linear momentum rate equation or force balance | angular momentum rate equation or torque balance |

# Mass flow rate

$$\dot{m} = \frac{\delta m}{\delta t} = \rho \frac{\delta V}{\delta t} = \rho A \frac{\delta x}{\delta t} = \rho\, A\, v$$




## Local mass flow rate

Fluid density

Cross sectional area

$$\dot{m} = -\rho\, A\, \underline{n} \cdot \underline{v}$$

Velocity vector

Mass flow rate

Normal unit vector

## Local mass flow rate

$$\dot{m}_i = -\rho_i\, \underline{n}_i \cdot \underline{v}_i\, A_i$$

## Total mass flow rate across an open boundary

$$\dot{m} = -\rho\, \underline{n} \cdot \langle \underline{v} \rangle\, A$$

Average vector velocity

$$\langle \underline{v} \rangle = \frac{\int \underline{v}_i\, dA_i}{A}$$

**Mass flow rate** measures the amount of mass crossing a boundary per unit time

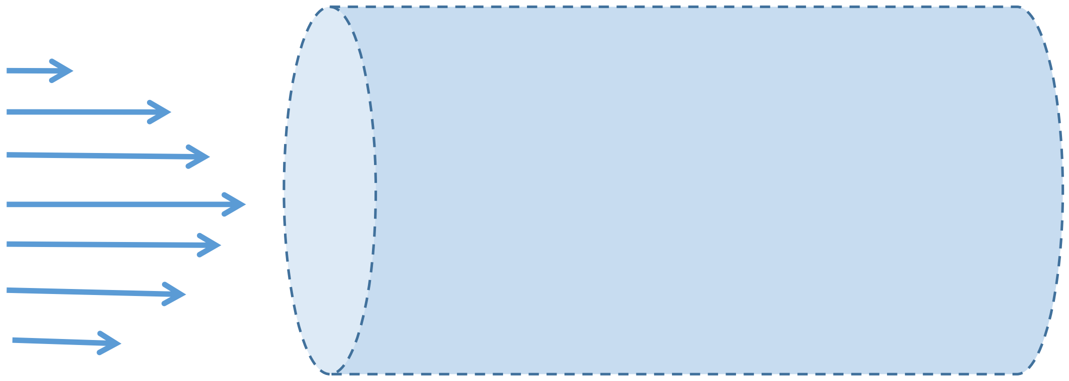
## Net or overall crossing Mass flow rate

$$\dot{m} = -\oiint \rho \underline{n} \cdot \underline{v}\, dA$$

**Disclaimer:** Custom definition of mass flow rate involves absolute value of mass flow rate (i.e. positive value ), but in this course we use both approaches, depending on the type of analysis you are interested in. For algorithmic purposes my recommendation is to use the vector operation form without absolute value (i.e positive value for entering flow rate, and negative value for leaving flow rate)

## Mass flow rate in a non uniform vector field

If velocity vector field is not uniform, you will need the average velocity to quantify mass flow rate.



Net or overall crossing Mass flow rate

$$\dot{m} = -\oiint \rho \underline{n} \cdot \underline{v} \, dA$$

Mass flow rate across a specific surface

$$\dot{m} = -\iint \rho \underline{n} \cdot \underline{v} \, dA$$

Total mass flow rate across an open boundary using average velocity takes the form

$$\dot{m} = -\rho \, \underline{n} \cdot \langle \underline{v} \rangle \, A$$

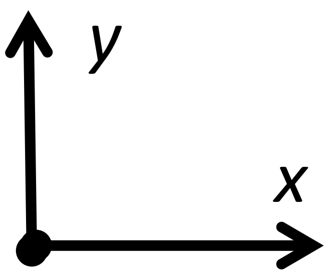
Comparing both equations, we end up with

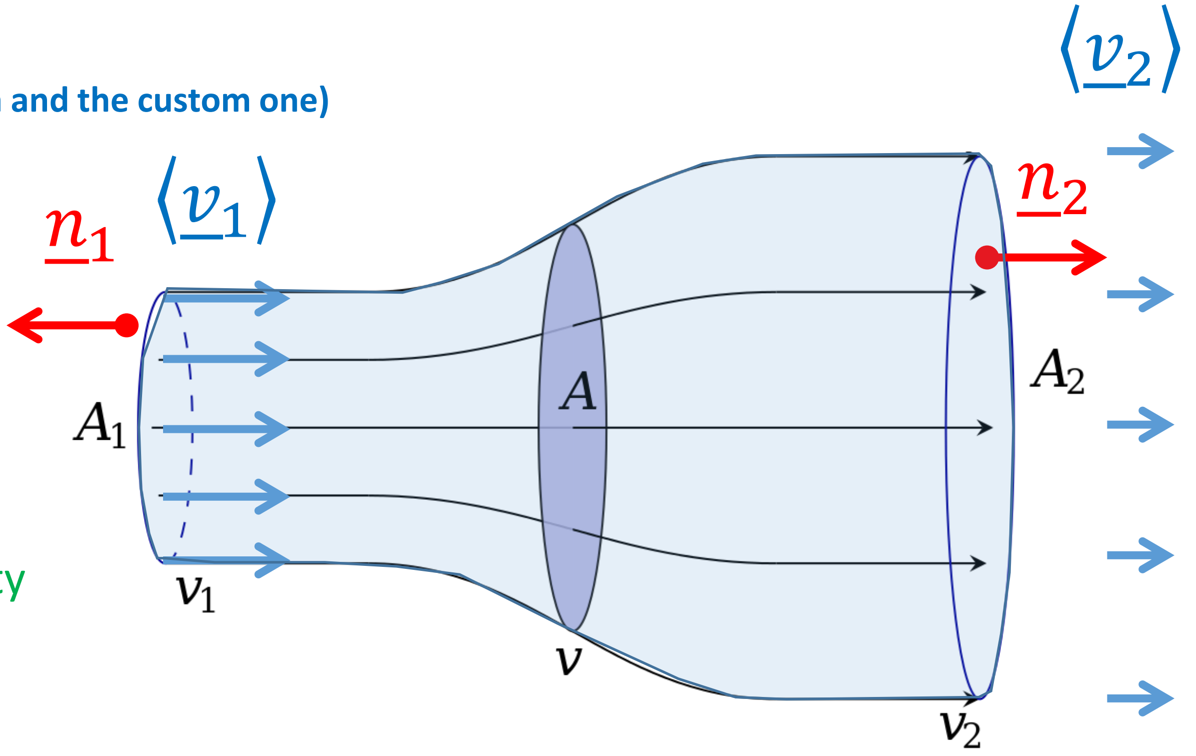
$$\langle \underline{v} \rangle = \frac{1}{A} \iint \underline{v} \, dA$$

Average vector velocity

# Mass flow rates

(find the difference in the vector form and the custom one)




$$\dot{m}_2 = -\rho_2\, \underline{n}_2 \cdot \langle \underline{v}_2 \rangle A_2$$

$$\dot{m}_2 = -\rho_2 \langle v_2 \rangle A_2$$

Flow velocity vector

$$\dot{m}_1 = -\rho_1\, \underline{n}_1 \cdot \langle \underline{v}_1 \rangle A_1$$

$$\dot{m}_1 = \rho_1 \langle v_1 \rangle A_1$$

Fluid speed

Mass balance over the system

$$\frac{dm}{dt} = +\rho_1 \langle v_1 \rangle A_1 - \rho_2 \langle v_2 \rangle A_2$$

$$\langle \underline{v}_1 \rangle = \langle v_1 \rangle\, \hat{\imath} \qquad \langle \underline{v}_2 \rangle = \langle v_2 \rangle\, \hat{\imath}$$

$$\underline{n}_1 = -1\ \hat{\imath} \qquad \underline{n}_2 = +1\ \hat{\imath}$$

$$\frac{dm}{dt} = -\rho_1\, \underline{n}_1 \cdot \langle \underline{v}_1 \rangle A_1 - \rho_2\, \underline{n}_2 \cdot \langle \underline{v}_2 \rangle A_2$$

Mass balance vector form

$$\frac{dm}{dt} = +\dot{m}_1 + \dot{m}_2$$

Mass balance algorithmic form

$$\frac{dm}{dt} = +|\dot{m}_1| - |\dot{m}_2|$$

Mass balance custom form

# Continuity Equation (Conservation of mass)

$$\frac{dm}{dt} = \underbrace{\sum_{i=\text{inputs}} |\dot{m}_i| - \sum_{j=\text{outputs}} |\dot{m}_j|}_{\text{If you use absolute value of mass flow rates}} = \underbrace{\sum_{i=inputs} \left|\rho_i\, \underline{n}_i \cdot \langle \underline{v}_i \rangle A_i\right| - \sum_{j=outputs} \left|\rho_j\, \underline{n}_j \cdot \langle \underline{v}_j \rangle A_j\right|}_{\text{If you decided to split up streams and down streams}}$$

Mass balance custom form

If you use absolute value of mass flow rates

If you decided to split up streams and down streams

$$\frac{dm}{dt} = \sum_{k=i,j} \dot{m}_k = -\sum \rho_k\, \underline{n}_k \cdot \langle \underline{v}_k \rangle A_k$$

System with multiple steams

Generalized form, you include all streams in the summation term

$$\boxed{\dot{m}_i = -\rho_i\, \underline{n}_i \cdot \langle \underline{v}_i \rangle A_i}$$

Mass flow rate for each stream, in this case, flow of stream "i"

# Continuity Equation for moving systems (Conservation of mass)

For a moving object (control volume), you need to use the relative velocity to quantify mass flow rate.

Relative velocity

$$\langle \underline{v}_{kR} \rangle = \langle \underline{v}_k \rangle - \langle \underline{v}_{CV} \rangle$$

Relative Velocity of stream "k" | Velocity of stream "k" | Velocity of control volume

$$\langle \underline{w} \rangle = \langle \underline{v} \rangle - \langle \underline{u} \rangle$$

Relative Velocity of stream "k" | Velocity of stream "k" | Velocity of control volume

$$\frac{dm}{dt} = -\sum \rho_k \, \underline{n}_k \cdot \langle \underline{v}_{kR} \rangle A_k$$

$$\frac{dm}{dt} = -\sum \rho_k \, \underline{n}_k \cdot [\langle \underline{v}_k \rangle - \langle \underline{v}_{CV} \rangle] A_k = -\sum \rho_k \, \underline{n}_k \cdot [\langle \underline{v}_k \rangle - \langle \underline{u}_k \rangle] A_k = -\sum \rho_k \, \underline{n}_k \cdot \langle \underline{w}_k \rangle A_k$$

For a moving control volume

$$\boxed{\dot{m}_i = -\rho_i \, \underline{n}_i \cdot [\langle \underline{v}_i \rangle - \langle \underline{v}_{CV} \rangle] A_i = -\rho_i \, \underline{n}_i \cdot [\langle \underline{v}_i \rangle - \langle \underline{u}_i \rangle] A_i = -\rho_i \, \underline{n}_i \cdot \langle \underline{w}_i \rangle A_i}$$

# Mass flow rate crossing a fixed boundary

$$\dot{m} = -\oint \underline{n} \cdot \rho \underline{v} \, dA = -\oint \underline{n} \cdot \underline{G} \, dA = -\rho \, \underline{n} \cdot \langle \underline{v} \rangle A$$

**Local mass flux** $\underline{G} = \rho \underline{v}$ Mass flux is a vector field

**Average mass flux** $\langle \underline{G} \rangle = \rho \langle \underline{v} \rangle$ Average mass flux is a vector

# Volume flow rate crossing a fixed boundary

$$\dot{V} = -\oint \underline{n} \cdot \underline{v} \, dA = \dot{m}/\rho = -\underline{n} \cdot \langle \underline{v} \rangle A$$

**Local volume flux** $\underline{v}$ Volume flux is a vector field

**Average volume flux** $\langle \underline{v} \rangle$ Average volume flux is a vector

# Mass flow rate

Fluid relative velocity

Velocity of the fluid stream

Velocity of boundary of the control volume where mass exchange is happening

$$\dot{m} = -\oint \underline{n} \cdot \rho \underline{w}\, dA = -\oint \underline{n} \cdot \underline{G}\, dA = -\rho\, \underline{n} \cdot \langle \underline{w} \rangle A$$

$$\langle \underline{w} \rangle = \langle \underline{v} \rangle - \langle \underline{u} \rangle$$

**Local mass flux** $\underline{G} = \rho \underline{w}$ Mass flux is a vector field

**Average mass flux** $\langle \underline{G} \rangle = \rho \langle \underline{w} \rangle$ Average mass flux is a vector

# Volume flow rate

$$\dot{V} = -\oint \underline{n} \cdot \underline{w}\, dA = \dot{m}/\rho = -\underline{n} \cdot \langle \underline{w} \rangle A$$

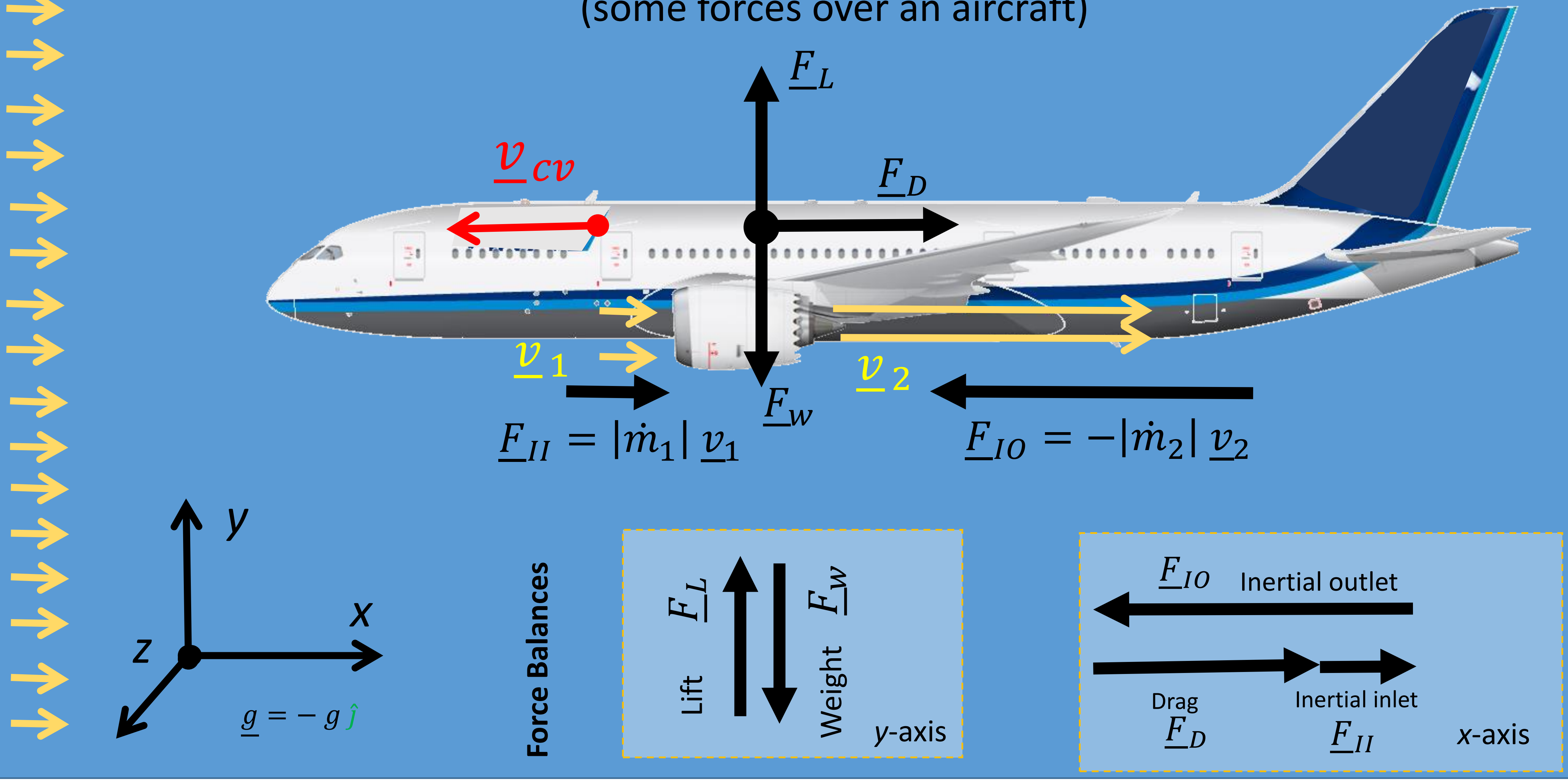
$$\langle \underline{w} \rangle = \langle \underline{v} \rangle - \langle \underline{u} \rangle$$

**Local volume flux** $\underline{w}$ Volume flux is a vector field

**Average volume flux** $\langle \underline{w} \rangle$ Average volume flux is a vector

# Linear momentum rate balance

(some forces over an aircraft)



# Inertial forces

The velocity field in a stream may not be uniform, then the inertial force need to be calculated integrating over the cross sectional area

$$\text{Inertial Force} = -\iint (\rho \underline{n} \cdot \underline{v})\, \underline{v}\, dA$$

If velocity field is unidirectional (in x-direction), then we have

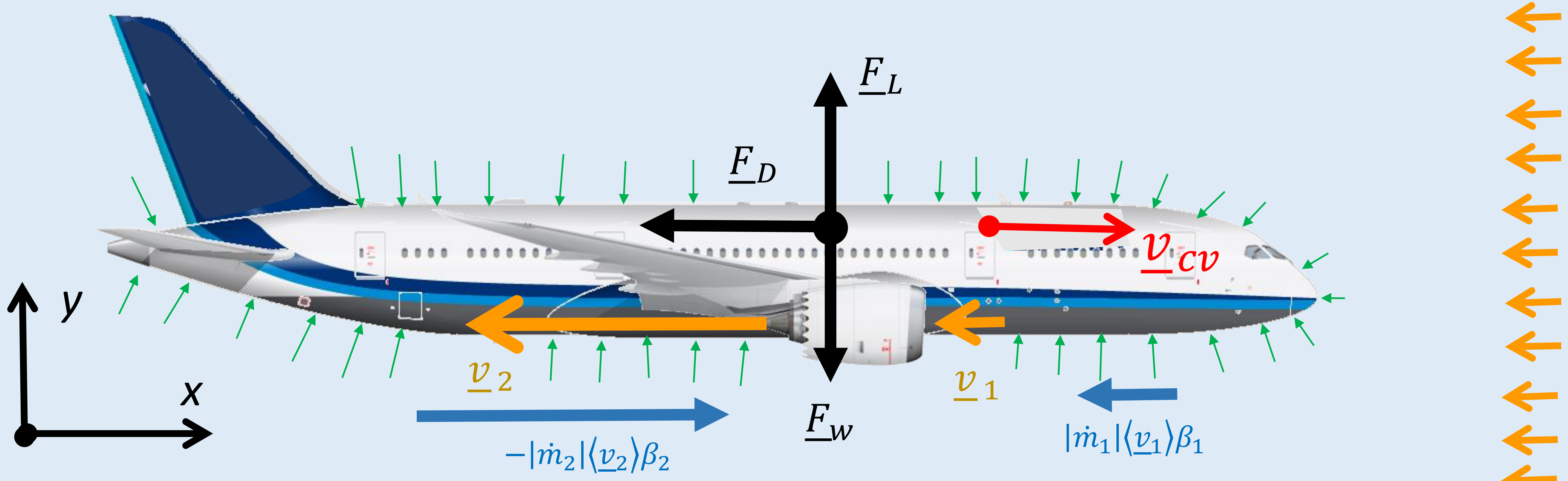
Inertial force correction factor

$$\iint \rho\, v^2\, dA = \beta\, \rho\, \langle v \rangle\, A\, \langle v \rangle = \beta |\dot{m}| \langle v \rangle$$

$$\beta = \frac{\left[\frac{1}{A} \iint v^2\, dA\right]}{\langle v \rangle^2} = \frac{\langle v^2 \rangle}{\langle v \rangle^2}$$

For uniform flow $\beta$=1, for turbulent flow ca 1, and for laminar flow $\alpha$=4/3, under different scenarios like no-Newtonian fluids, or not highly turbulent flow, you need to calculate, measure or estimate this correction factor

# Linear momentum rate balance

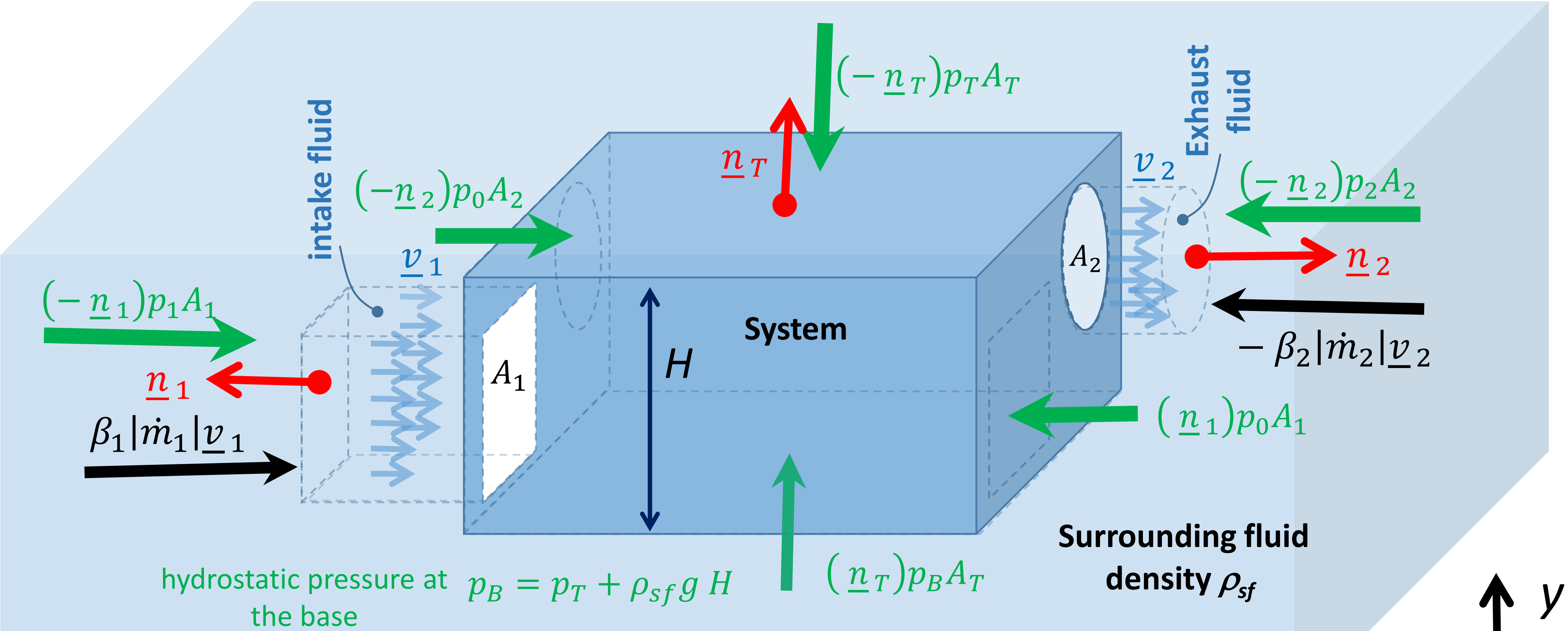



$$\frac{d[m\,\underline{v}]}{dt} = |\dot{m}_1|\langle\underline{v}_1\rangle\beta_1 - |\dot{m}_2|\langle\underline{v}_2\rangle\beta_2 + m\underline{g} + \sum_s p_s A_s(-\underline{n}_s) - \underline{F}_D + \underline{F}_L$$

**If relative motion of control volume and surrounding fluid is present, lift and drag forces are important**

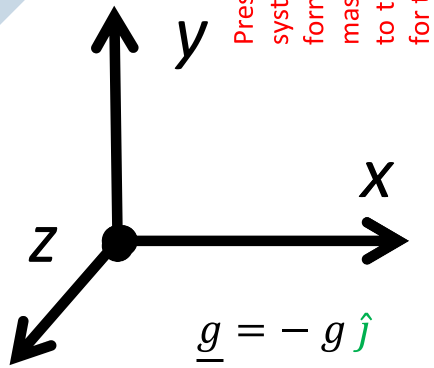
# Linear momentum rate balance

$$\frac{d[m\,\underline{v}]}{dt} = |\dot{m}_1|\langle\underline{v}_1\rangle\beta_1 - |\dot{m}_2|\langle\underline{v}_2\rangle\beta_2 + m\underline{g} + (p_1 - p_0)A_1(-\underline{n}_1) + (p_2 - p_0)A_2(-\underline{n}_2) + (-\underline{n}_T)(p_T - p_B)A_T$$




$$\frac{d[m\,\underline{v}]}{dt} = |\dot{m}_1|\langle\underline{v}_1\rangle\beta_1 - |\dot{m}_2|\langle\underline{v}_2\rangle\beta_2 + m\underline{g} + (p_1 - p_0)A_1(-\underline{n}_1) + (p_2 - p_0)A_2(-\underline{n}_2) - (-\underline{n}_T)(\rho_{sf}\, g\, H)A_T$$

$$\frac{d[m\,\underline{v}]}{dt} = \underbrace{\underbrace{|\dot{m}_1|\langle\underline{v}_1\rangle\beta_1}_{\text{Inlet Inertial Force}} - \underbrace{|\dot{m}_2|\langle\underline{v}_2\rangle\beta_2}_{\text{outlet Inertial Force}}}_{\text{Thrust}} + \underbrace{m\underline{g}}_{\text{Weight}} + \underbrace{(p_1 - p_0)A_1(-\underline{n}_1)}_{\text{Inlet pressure Force}} + \underbrace{(p_2 - p_0)A_2(-\underline{n}_2)}_{\text{Outlet pressure Force}} \underbrace{- \rho_{sf}\,\boldsymbol{V}\,\underline{g}}_{\text{Buoyancy}}$$




Pressure forces in the frontal face and in the rear face of the system are equal , so they cancel each other. The pressure force form the left and right faces on these areas where there is no mass exchange, or these opposite areas of there corresponding to the boundaries where there is mass exchange cancel as well, for this reason they were not included in the force balance.

**If relative motion of control volume and surrounding fluid is zero, lift and drag forces are zero as well**

# Linear momentum rate equation

$$\frac{d[m\,\underline{v}]}{dt} = \underbrace{-\sum[\rho_k\,\underline{n}_k\cdot\langle\underline{v}_k\rangle A_k]\,\langle\underline{v}_k\rangle\beta_k}_{\text{Linear momentum crossing boundaries}} + m\underline{g} + \underbrace{\sum_s p_s A_s(-\underline{n}_s)}_{\text{Pressure over control surface}} - \underline{F}_D \pm \underline{F}_L - \sigma\,L\,\underline{\tau}$$

$$\frac{d[m\,\underline{v}]}{dt} = \sum \dot{m}_k\;\langle\underline{v}_k\rangle\;\beta_k + m\underline{g} + \sum_s p_s A_s(-\underline{n}_s) - \underline{F}_D \pm \underline{F}_L - \sigma\,L\,\underline{\tau}$$

$$\frac{d[m\,\underline{v}]}{dt} = \underbrace{\sum \dot{m}_k\;\langle\underline{v}_k\rangle\;\beta_k}_{\substack{\text{Inertial force over boundaries}\\ \text{with mass flow exchange}}} + V[\underbrace{\rho_{system}}_{\text{Weight}} - \underbrace{\rho_{sf}}_{\text{Buoyancy}}]\underline{g} + \underbrace{\sum_{k=i,j}[p_k - p_o]A_k(-\underline{n}_k)}_{\substack{\text{Pressure force over boundaries}\\ \text{with mass flow exchange}}} - \underbrace{\underline{F}_D}_{\text{Drag}} \pm \underbrace{\underline{F}_L}_{\text{Lift}} - \underbrace{\sigma\,L\,\underline{\tau}}_{\text{Line}}$$

$$\frac{d[m\,\underline{v}]}{dt} = \sum|\dot{m}_i|\,\langle\underline{v}_i\rangle\,\beta_i - \sum|\dot{m}_j|\,\langle\underline{v}_j\rangle\,\beta_j + V[\Delta\rho]\underline{g} + \sum_{k=i,j}[p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_D \pm \underline{F}_L - \sigma\,L\,\underline{\tau}$$

$$\beta_k = \frac{\langle v_k^2\rangle}{\langle v_k\rangle^2} \qquad \langle v_i^n\rangle = \frac{\int v_i^n\,dA_i}{A_i}$$

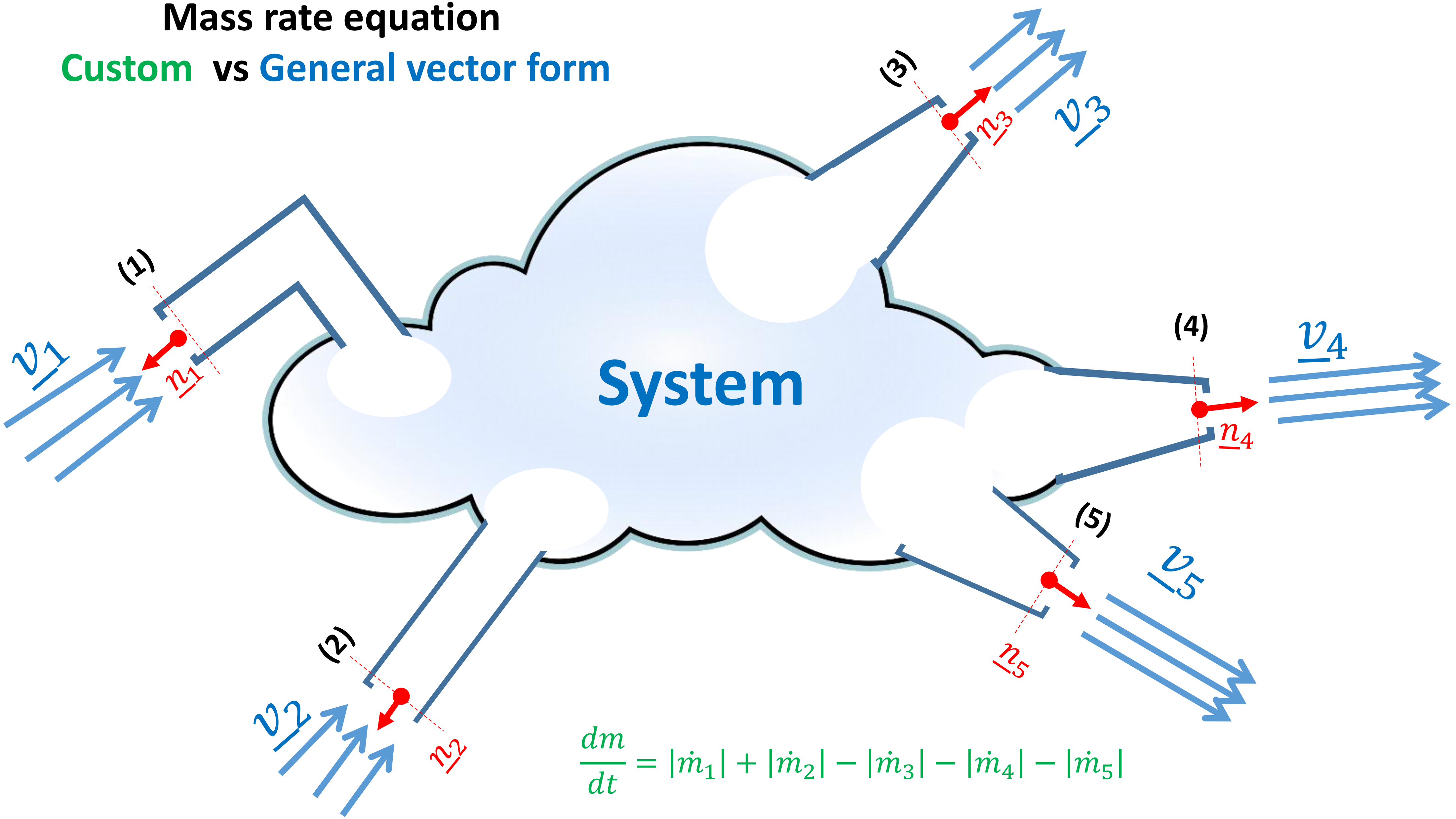
# Macroscopic linear momentum rate equation

$$\frac{d[m\,\underline{v}]}{dt} = -\overbrace{\sum\left[\rho_k\,\underline{n}_k\cdot\langle\underline{v}_k\rangle A_k\right]\,\langle\underline{v}_k\rangle\beta_k}^{\text{Linear momentum transport crossing boundaries}} + \underbrace{m\underline{g}}_{\text{Body Forces}} + \overbrace{\sum_s p_s A_s(-\underline{n}_s)}^{\text{Compression or pressure surface forces over }\textbf{any boundary}} \underbrace{-\,\underline{F}_D \pm \underline{F}_L}_{\text{Shear surface forces}} - \underbrace{\sigma\, L\, \underline{\tau}}_{\text{Line forces}}$$

$$\frac{d[m\,\underline{v}]}{dt} = \underbrace{\sum_{i=\text{in}} |\dot{m}_i|\,\langle\underline{v}_i\rangle\,\beta_i}_{\text{entering inertial force}} - \underbrace{\sum_{j=\text{out}} |\dot{m}_j|\,\langle\underline{v}_j\rangle\,\beta_j}_{\text{Leaving inertial force}} + \underbrace{V[\Delta\rho]\underline{g}}_{\text{Weight and Buoyancy difference}} + \underbrace{\sum_{k=1,2,3,4,5} \overbrace{[p_k - p_o]A_k(-\underline{n}_k)}^{\text{Compression or pressure surface forces over }\textbf{boundaries with mass exchange}}}_{\text{Compression or pressure surface forces over }\textbf{open boundaries}} - \underline{F}_D \pm \underline{F}_L - \sigma\, L\, \underline{\tau}$$

# Mass rate equation

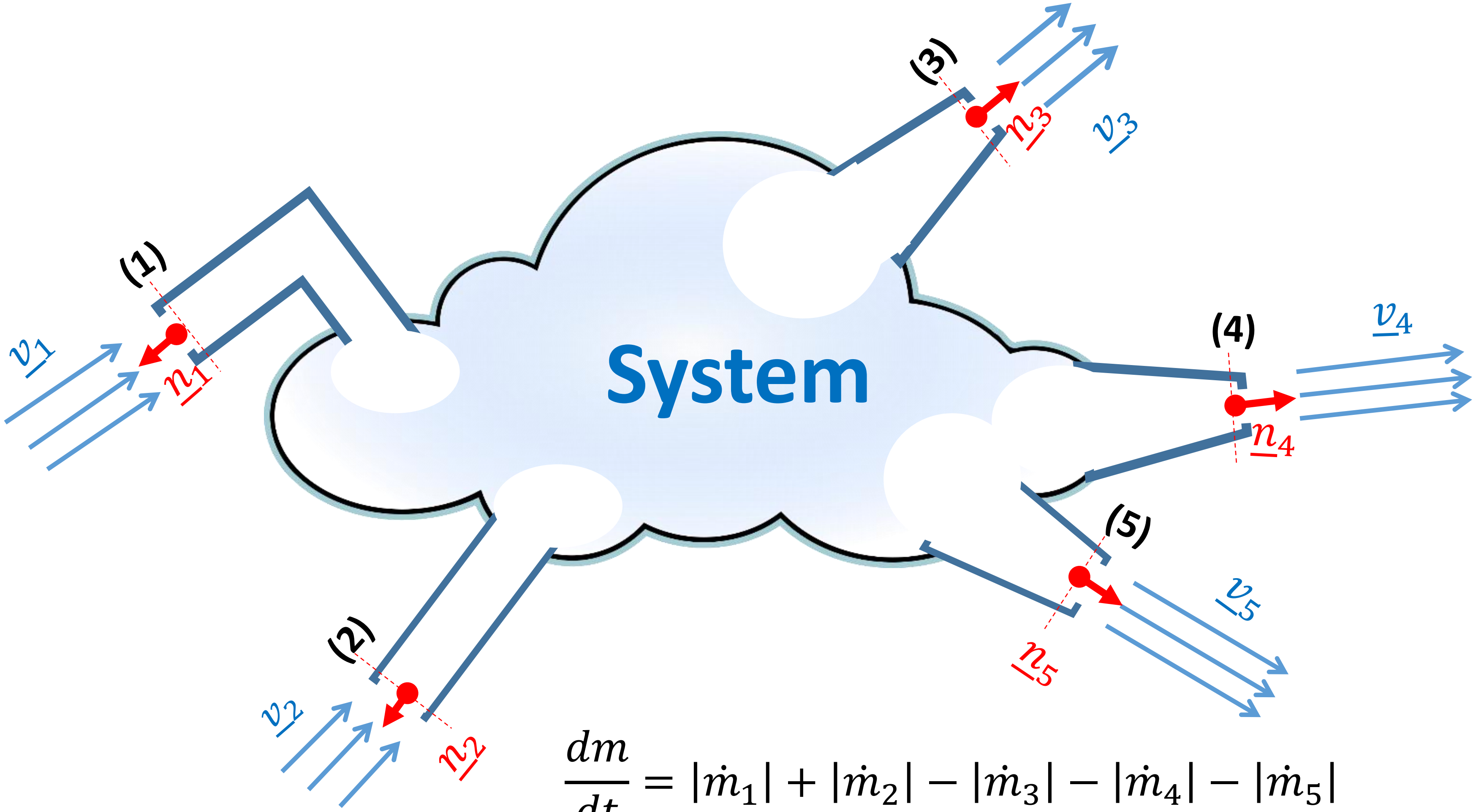
## Custom vs General vector form




$$\frac{dm}{dt} = |\dot{m}_1| + |\dot{m}_2| - |\dot{m}_3| - |\dot{m}_4| - |\dot{m}_5|$$

$$\frac{dm}{dt} = -\rho_1\, \underline{n}_1 \cdot \langle \underline{v}_1 \rangle A_1 - \rho_2\, \underline{n}_2 \cdot \langle \underline{v}_2 \rangle A_2 - \rho_3\, \underline{n}_3 \cdot \langle \underline{v}_3 \rangle A_3 - \rho_4\, \underline{n}_4 \cdot \langle \underline{v}_4 \rangle A_4 - \rho_5\, \underline{n}_5 \cdot \langle \underline{v}_5 \rangle A_5$$

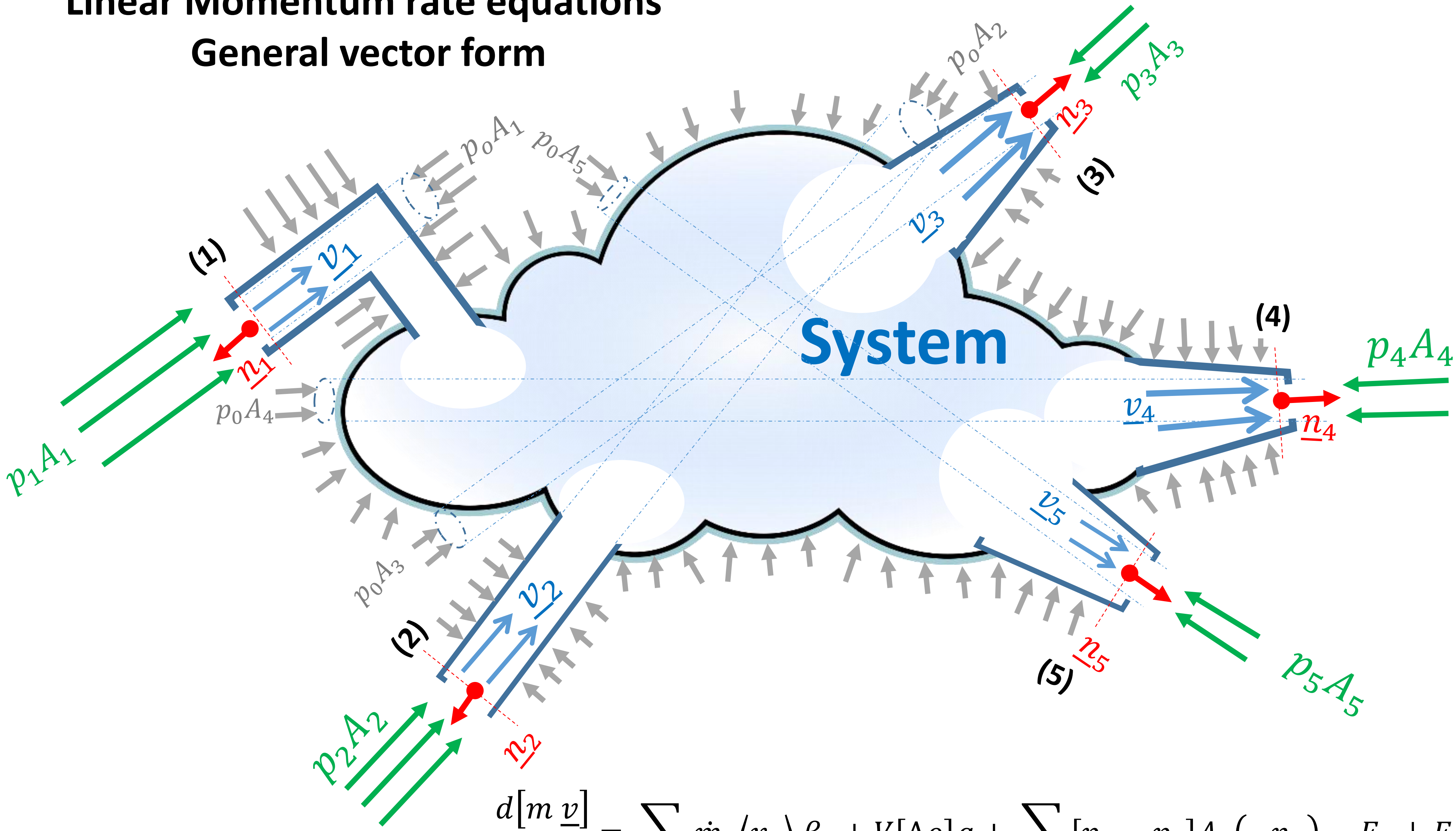
## Mass and Linear Momentum rate equations
## Custom Form




$$\frac{dm}{dt} = |\dot{m}_1| + |\dot{m}_2| - |\dot{m}_3| - |\dot{m}_4| - |\dot{m}_5|$$

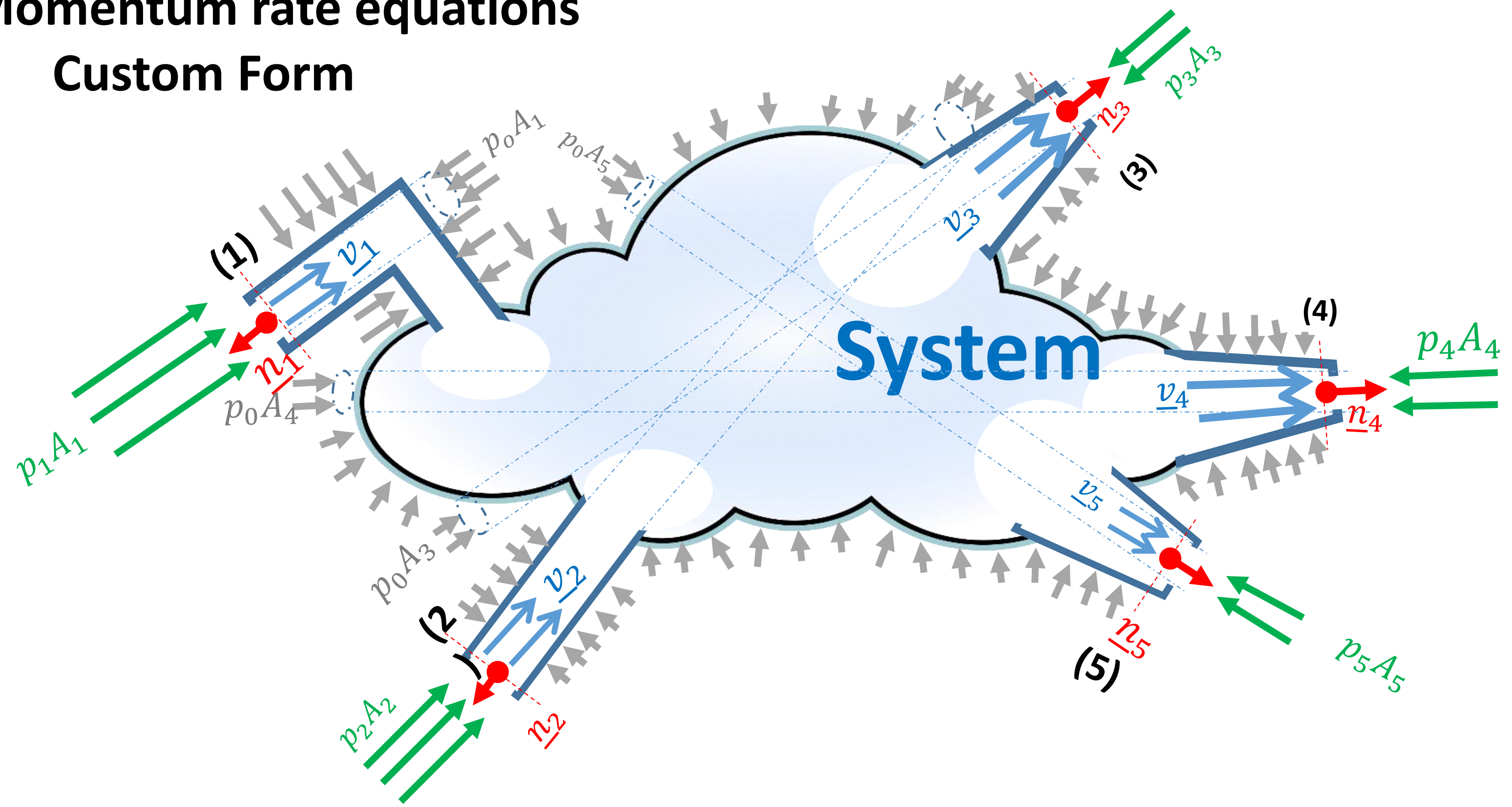
$$\frac{d[m\,\underline{v}]}{dt} = \sum_{i=1,2} |\dot{m}_i|\,\langle \underline{v}_i \rangle\,\beta_i - \sum_{j=3,4,5} |\dot{m}_j|\,\langle \underline{v}_j \rangle\,\beta_j + V[\Delta\rho]\underline{g} + \sum_{k=1,2,3,4,5} [p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_D \pm \underline{F}_L - \sigma\, L\, \underline{\tau}$$

## Linear Momentum rate equations
## General vector form

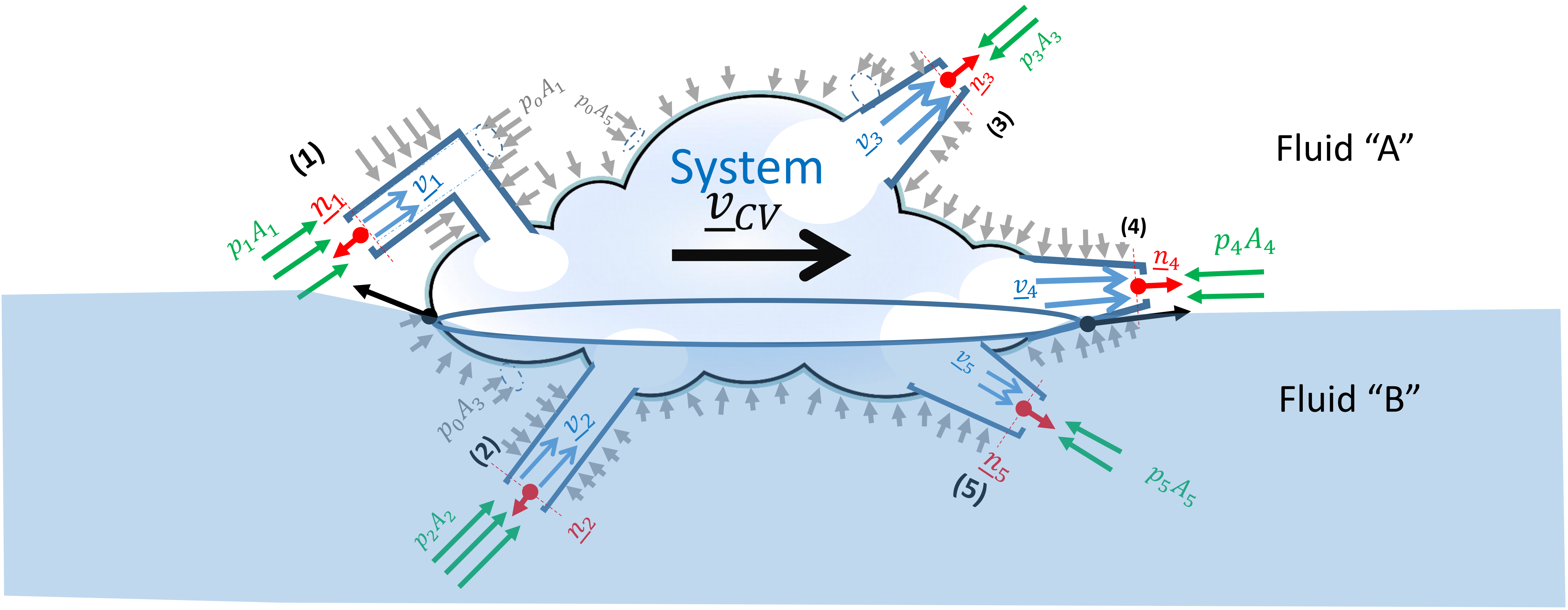



$$\frac{d[m\,\underline{v}]}{dt} = \sum_{k=i,j} \dot{m}_k \langle \underline{v}_k \rangle\, \beta_k + V[\Delta\rho]\underline{g} + \sum_{k=i,j} [p_k - p_o] A_k(-\underline{n}_k) - \underline{F}_D \pm \underline{F}_L - \sigma\, L\, \underline{\tau}$$

# Linear Momentum rate equations
## Custom Form




$$\frac{d[m\,\underline{v}]}{dt} = \sum_{i=1,2} |\dot{m}_i|\,\langle \underline{v}_i \rangle\,\beta_i - \sum_{j=3,4,5} |\dot{m}_j|\,\langle \underline{v}_j \rangle\,\beta_j + V[\Delta\rho]\underline{g} + \sum_{k=1,2,3,4,5} [p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_D \pm \underline{F}_L - \sigma\, L\, \underline{\tau}$$

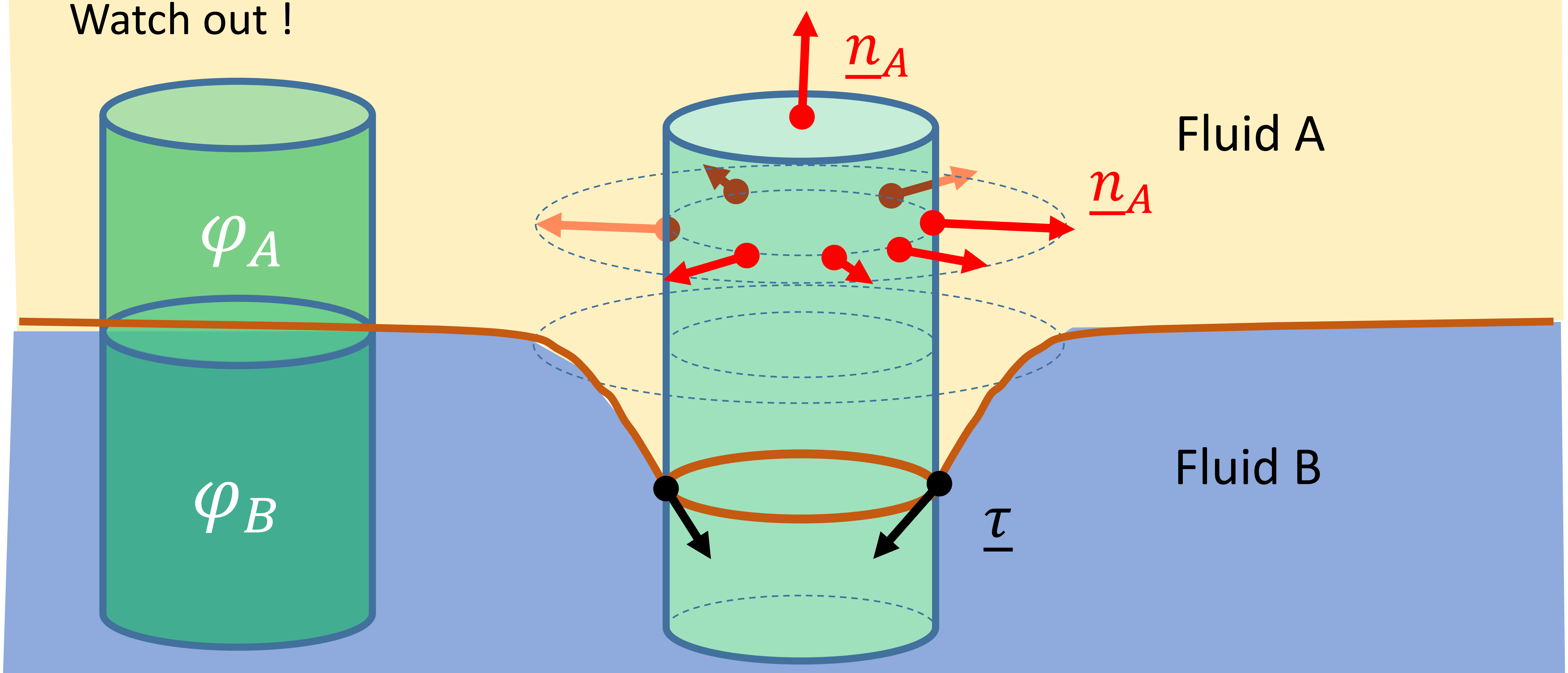



$$\frac{d[m\,\underline{v}]}{dt} = \sum_{i=1,2} |\dot{m}_i|\,\langle \underline{v}_i \rangle\,\beta_i - \sum_{j=3,4,5} |\dot{m}_j|\,\langle \underline{v}_j \rangle\,\beta_j + V[\Delta\rho]\underline{g} + \sum_{k=1,2,3,4,5} [p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_D \pm \underline{F}_L - \sigma\,L\,\underline{\tau}$$

$$\Delta\rho = \rho_{\text{system}} - \rho_{\text{fluid}} \qquad \rho_{\text{fluid}} = \varphi_A\,\rho_A + \varphi_B\,\rho_B$$

$\varphi_A$ Volume fraction of the system embedded in fluid "A", respect to the free surface between fluids "A" and "B"

Watch out !




For systems where line forces are important, is better **not to use** the concept of "**buoyancy**", but use the force caused by pressure over the entire surface of the system. Buoyancy is a valid concept whenever the body is completely submerged within a single phase fluid, or when there is no relative motion of the fluid respect to the object.

## Mechanical energy rate equation

$$\frac{d\left[m\left(\widehat{K}+\widehat{\Phi}\right)\right]}{dt} = \sum \dot{m}_i \left(\frac{p_i}{\rho_i} + \alpha_i \frac{\langle v_i\rangle^2}{2} + g\, z_i\right) - \dot{W}_{\text{out}} + \dot{W}_{\text{in}} - \dot{E}_{\text{loss}}$$

Power transferred by pumps, blowers, fans, compressors, etc.

Pressure or flow energy

Kinetic energy

Potential energy

Power produced by turbines

power dissipated to the environment, or turn into internal energy by friction, viscous effects, irreversible process and so on.

$$\widehat{K} = \alpha \frac{1}{2} \langle \underline{v} \rangle \cdot \langle \underline{v} \rangle$$

$$\widehat{\Phi} = g \langle z \rangle$$

$$\dot{W} = p \frac{dV}{dt} = \frac{p}{\rho}\frac{\rho\, dV}{dt} = \frac{p}{\rho}\dot{m}$$

Kinetic energy correction factor

$$\alpha = \frac{\langle v^3 \rangle}{\langle v \rangle^3}$$

**Energy** can **neither** be **created nor destroyed**; **energy** can only be transferred **or** changed from one form to another or dispersed to the environment.

# Kinetic energy

The velocity field in a stream may not be uniform, then the kinetic energy needs to be calculated integrating over the cross sectional area

$$\text{Kinetic energy flow} = -\iint (\rho \underline{n} \cdot \underline{v}) \frac{1}{2}\, \underline{v} \cdot \underline{v}\, dA$$

If velocity field component is in x-direction we have

Kinetic energy correction factor

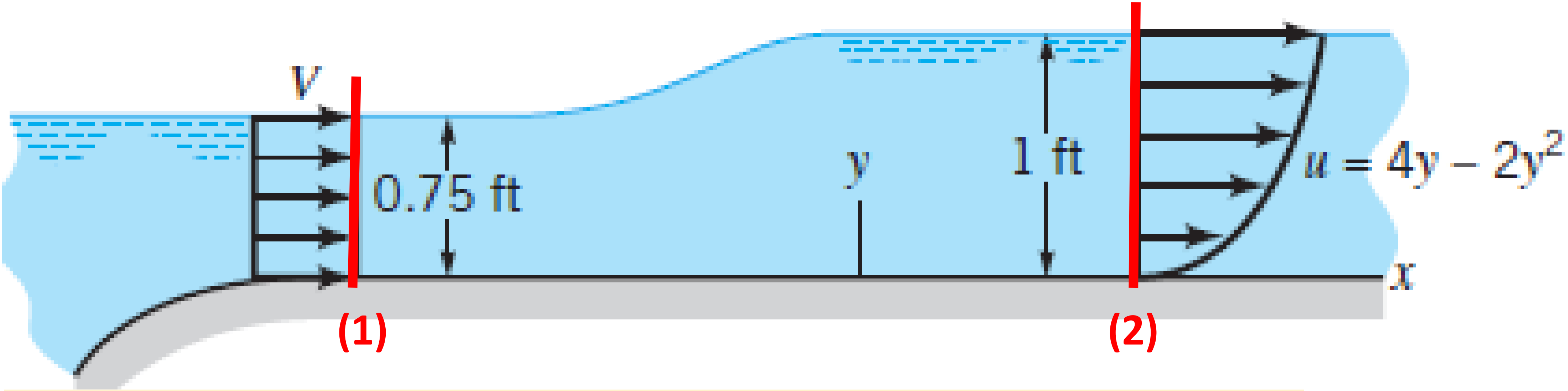
$$\iint \frac{1}{2} \rho\, v^3\, dA = \alpha\, \rho\, \langle v \rangle\, A \left[\frac{1}{2} \langle v \rangle^2\right] = \alpha \frac{1}{2}\, |\dot{m}| \langle v \rangle^2$$

$$\alpha = \frac{\left[\frac{1}{A} \iint v^3\, dA\right]}{\langle v \rangle^3} = \frac{\langle v^3 \rangle}{\langle v \rangle^3}$$

For uniform flow $\alpha$=1, for turbulent flow ca 1, and for laminar flow $\alpha$=2, under different scenarios like no-Newtonian fluids, or not highly turbulent flow, you need to calculate, measure or estimate this correction factor

**Problem PW7.1** As shown in the Figure, at the entrance to a 3-ft-wide channel the velocity distribution is uniform with a velocity ***V***. Further downstream the velocity profile is given by $u = 4y - 2y^2$, where ***u*** is in ft/s and ***y*** is in ft. Determine:

a) The value of ***V***.
b) Calculate the energy loss in this process (specify the units)
c) Speculate if the hydraulic jump is caused by friction and eddy currents or air flowing in the water flow direction.
d) What is the answer if velocity vector field changes to equation given in the footnote.




Velocity profile for part d) $u = u_o\sqrt{(a_o y + b_0 y^2)}$ $u_0$= 4 m/s, $a_0$=4 m$^{-1}$, $b_0$=-6.5 m$^{-2}$

Motivation to solve the previous problem .

- Apply the rule of dimensional homogeneity.
- Use the equation of mass flow rate (continuity equation)  to relate velocity fields at different sections of a stream.
- Use the equation of energy rate (mechanical energy equation), to see the feasibility of a proposed scenario.
- Calculate average velocity of streams.
- Calculate kinetic energy correction factor for velocity fields.

*Problem 5.19*

$W = 3$ [ft] $\cdot \left| 0.3048 \cdot \frac{m}{ft} \right|$

$H_1 = 0.75$ [ft] $\cdot \left| 0.3048 \cdot \frac{m}{ft} \right|$

$H_2 = 1$ [ft] $\cdot \left| 0.3048 \cdot \frac{m}{ft} \right|$

*Physical constants and fluid properties*

$g = 9.81$ [m/s$^2$]

$\rho = 997$ [kg/m$^3$]

*Cross sectional areas*

$A_1 = W \cdot H_1$

$A_2 = W \cdot H_2$

*velocity field*

$u = a \cdot y + b \cdot y^2$

$a = 4$ [1/s]

$b = -2$ [1/ft-s] $\cdot \left| 3.2808399 \cdot \frac{1/m{-}s}{1/ft{-}s} \right|$

*Vector analysis*

$\theta_1 = 0$ [deg] *velocity direction at boundary 1*

$\phi_1 = 180$ [deg] *normal vector at boundary 1*

$\theta_2 = 0$ [deg] *velocity direction at boundary 2*

$\phi_2 = 0$ [deg] *normal vector at boundary 2* *conservation of mass, continuity equation*

$$-\rho \cdot v_1 \cdot A_1 \cdot \left(\cos(\theta_1) \cdot \cos(\phi_1) + \sin(\theta_1) \cdot \sin(\phi_1)\right) - \rho \cdot v_2 \cdot A_2 \cdot \left(\cos(\theta_2) \cdot \cos(\phi_2) + \sin(\theta_2) \cdot \sin(\phi_2)\right) = 0$$

*set the relationship between velocity field and average velocity*

$$v_2 = \frac{\int_{H_0}^{H_2} (u \cdot W)\, dy}{W \cdot (H_2 - H_0)}$$

$$v_1 = v$$

*conservation of energy*

$$\frac{p_1}{\rho} + g \cdot z_1 + \alpha_1 \cdot \frac{v_1^{\,2}}{2} = \frac{p_2}{\rho} + g \cdot z_2 + \alpha_2 \cdot \frac{v_2^{\,2}}{2} + \text{loss}$$

$p_o = 101325$ [Pa]

$$z_1 = \frac{H_1}{2}$$

$$z_2 = \frac{H_2}{2}$$

$$p_1 = p_o + \rho \cdot g \cdot \frac{H_1}{2}$$

$$p_2 = p_o + \rho \cdot g \cdot \frac{H_2}{2}$$

*Assumptions*

$\alpha_1 = 1$ *Uniform velocity field*

$p_o = 101325$ [Pa]

$$u_p = a \cdot y_p + b \cdot y_p^{\,2}$$

$$\alpha_2 = \frac{\int_{H_0}^{H_2} (u_p^{\,3})\, dy_p}{v_2^{\,3} \cdot (H_2 - H_0)}$$

SOLUTION

**Unit Settings: SI C kPa kJ mass deg**

| | |
|---|---|
| $a$ = 4 [1/s] | $\alpha_1$ = 1 |
| $A_1$ = 0.209 [$m^2$] | $A_2$ = 0.2787 [$m^2$] |
| $g$ = 9.81 [$m/s^2$] | $H_0$ = 0 [m] |
| $H_2$ = 0.3048 [m] | loss = -0.7281 [$m^2/s^2$] |
| $\phi_2$ = 0 [deg] | $p_1$ = 102443 [Pa] |
| $p_o$ = 101325 [Pa] | $\rho$ = 997 [$kg/m^3$] |
| $\theta_2$ = 0 [deg] | $u$ = 0.6096 [m/S] |
| $v$ = 0.5419 [m/s] | $v_1$ = 0.5419 [m/s] |
| $W$ = 0.9144 [m] | $y$ = 0.3048 [m] |
| $z_1$ = 0.1143 [m] | $z_2$ = 0.1524 [m] |

Part d)

*Hydraulic jump analysis*

$W = 3\ [ft] \cdot \left| 0.3048 \cdot \frac{m}{ft} \right|$

$H_1 = 0.75\ [ft] \cdot \left| 0.3048 \cdot \frac{m}{ft} \right|$

$H_2 = 1\ [ft] \cdot \left| 0.3048 \cdot \frac{m}{ft} \right|$

$H_0 = 0\ [m]$

*physical constants, and fluid properties*

$g = 9.81\ [m/s^2]$

$\rho = 997\ [kg/m^3]$

*Cross sectional areas*

$A_1 = W \cdot H_1$

$A_2 = W \cdot H_2$

*velocity field*

$u_o = 4\ [m/s]$

$a_o = 4\ [1/m]$

$b_o = -6.5\ [1/m^2]$

$u = u_o \cdot \left( \sqrt{a_o \cdot y + b_o \cdot y^2} \right)$

*vector analysis*

$\theta_1 = 0\ [deg]$

$\phi_1 = 180\ [deg]$

$\theta_2 = 0\ [deg]$

$\phi_2 = 0\ [deg]$

*conservation of mass, continuity equation*

$$-\rho \cdot v_1 \cdot A_1 \cdot \left(\cos\left(\theta_1\right) \cdot \cos\left(\phi_1\right) + \sin\left(\theta_1\right) \cdot \sin\left(\phi_1\right)\right) - \rho \cdot v_2 \cdot A_2 \cdot \left(\cos\left(\theta_2\right) \cdot \cos\left(\phi_2\right) + \sin\left(\theta_2\right) \cdot \sin\left(\phi_2\right)\right) = 0$$

*set the relationship between velocity field and average velocity*

$$v_2 = \frac{\int_{H_0}^{H_2} (u \cdot W)\, dy}{W \cdot (H_2 - H_0)}$$

$$v_1 = v$$

*conservation of energy*

$$\frac{p_1}{\rho} + g \cdot z_1 + \alpha_1 \cdot \frac{v_1^{\,2}}{2} = \frac{p_2}{\rho} + g \cdot z_2 + \alpha_2 \cdot \frac{v_2^{\,2}}{2} + \text{loss}$$

$p_o = 101325$ [Pa]

$$z_1 = \frac{H_1}{2}$$

$$z_2 = \frac{H_2}{2}$$

$$p_1 = p_o + \rho \cdot g \cdot \frac{H_1}{2}$$

$$p_2 = p_o + \rho \cdot g \cdot \frac{H_2}{2}$$

$$\alpha_1 = 1$$

*velocity field*

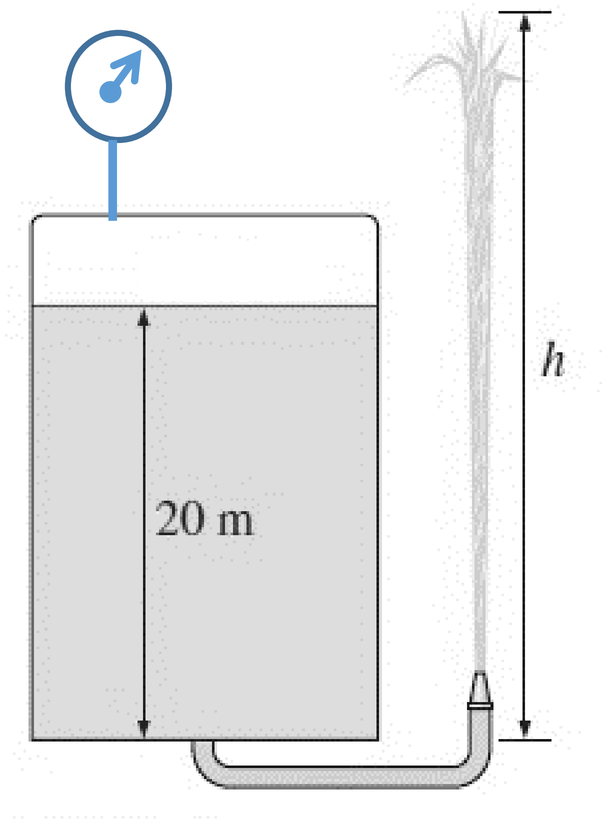
$$u_p = u_o \cdot \left(\sqrt{a_o \cdot y_p + b_o \cdot y_p^{\,2}}\right)$$

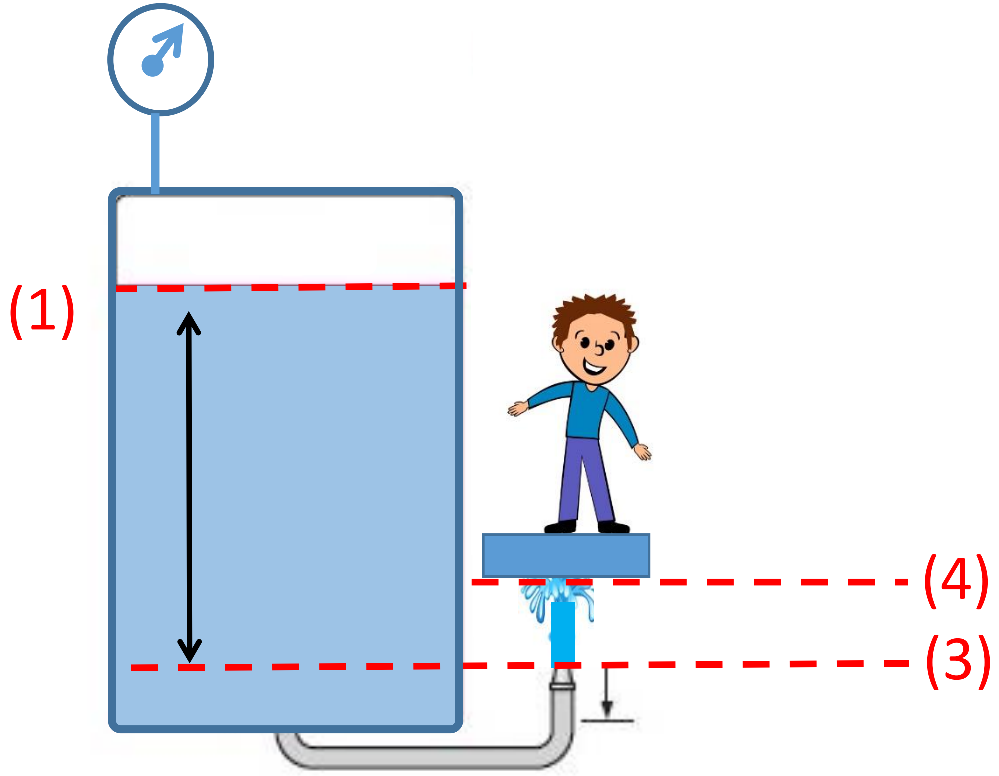
$$\alpha_2 = \frac{\int_{H_0}^{H_2} \left(u_p^{\,3}\right) d\,y_p}{v_2^{\,3} \cdot \left(H_2 - H_0\right)}$$

SOLUTION

| | | |
|---|---|---|
| $a = 4$ [1/s] | $\alpha_1 = 1$ | $\alpha_2 = 1.218$ |
| $A_2 = 0.2787$ [$m^2$] | $a_o = 4$ [1/m] | $b = -6.562$ [1/(s-m)] |
| $g = 9.81$ [$m/s^2$] | $H_0 = 0$ [m] | $H_1 = 0.2286$ [m] |
| $loss = 0.9445$ [$m^2/s^2$] | $\phi_1 = 180$ [deg] | $\phi_2 = 0$ [deg] |
| $p_2 = 102816$ [Pa] | $p_o = 101325$ [Pa] | $\rho = 997$ [$kg/m^3$] |
| $\theta_2 = 0$ [deg] | $u = 3.138$ [m/s] | $u_o = 4$ [m/s] |
| $v = 3.277$ [m/s] | $v_1 = 3.277$ [m/s] | $v_2 = 2.458$ [m/s] |
| $y = 0.3048$ [m] | $y_p = 0.3048$ [m] | $z_1 = 0.1143$ [m] |

**Problem PW7.2** The water level in a tank is 20 m above the ground. A hose is connected to the bottom of the tank, and the nozzle at the end of the hose is pointed straight up. The tank cover is airtight, and the air pressure above the water surface is 2 atm gage. The system is at sea level. A) Determine the maximum height to which the water stream could rise. B) Assuming a boy and flat board with a weight of 70 kg, is standing just above the jet stream. Estimate the height at which the boy will be hovering. The diameter in the nozzle outlet is 1.5-in. C) Discuss or give a strategy to stabilize the kid in the plate to prevent accidents



Motivation to solve the previous problem .

- Use the mechanical energy rate equation (mechanical energy equation), to know the relationship between pressure energy and potential energy to estimate maximum height of a stream.
- Use the mechanical energy rate equation (mechanical energy equation) to relate pressure energy with kinetic energy to estimate mass flow rate of a stream.
- Use the linear momentum rate equation (force balance), to estimate the required inertial force to reach a specific goal.
- Use the mechanical energy rate equation (mechanical energy equation) to obtain a desirable goal of energy requirements.

*Data*

$p_{gage} = 2 \ [atm] \cdot \left| 101325 \cdot \frac{Pa}{atm} \right|$ *Gage pressure*

$p_{atm} = 1 \ [atm] \cdot \left| 101325 \cdot \frac{Pa}{atm} \right|$ *atmospheric pressure*

$p_{gas} = p_{atm} + p_{gage}$ *pressure of the gas*

$D = 1.5 \ [in] \cdot \left| 0.0254 \cdot \frac{m}{in} \right|$ *Diameter of the nozzle*

$m = 70 \ [kg]$ *mass of the board and person*

$H_T = 20 \ [m]$ *tank dept*

*physical constants and properties, not given*

$g = 9.81 \ [m/s^2]$ *gravity*

$\rho = 997 \ [kg/m^3]$ *water density*

*Energy balance*

*Assumptions, constraints and boundary conditions*

$v_1 = 0 \ [m/s]$ *water speed at the surface of the liquid level within the tank*

$v_2 = 0 \ [m/s]$ *at the maximum height the velocity will be zero*

$p_1 = p_{gas}$ *pressure over the surfcae of the tank is the pressure of the gas*

$p_2 = p_{atm}$ *the water jet is open to the atmosphere then the pressure is atmospheric pressure*

$z_1 = H_T$ *the surface of the liquid level is the boundary 1*

$z_2 = h$ *the highest point is located the boundary 2 in scenario a)*

*Mechancial energy equation*

*assumption, no energy losses*

$\alpha_1 = 1$ *uniform velocity at the tank surface*

$\alpha_2 = 1$ *uniform velocity of the water jet*

$$\frac{p_1}{\rho} + \alpha_1 \cdot \frac{v_1^{\ 2}}{2} + g \cdot z_1 = \frac{p_2}{\rho} + \alpha_2 \cdot \frac{v_2^{\ 2}}{2} + g \cdot z_2$$

*Force balance under scenario b)*

*The point 4 is located at the hight reached by the flat board*

$$-m \cdot g + \dot{m} \cdot v_4 = 0$$

$\alpha_3 = 1$ *uniform velocity of the water jet*

*The velocity at the exit of the nozzle is calculated with the eenrgy equation in order to calculate the mass flow rate*

$$\frac{p_1}{\rho} + \alpha_1 \cdot \frac{v_1^{\,2}}{2} + g \cdot z_1 = \frac{p_3}{\rho} + \alpha_3 \cdot \frac{v_3^{\,2}}{2} + g \cdot z_3$$

$p_3 = p_{atm}$

$z_3 = 0$ [m]

$\dot{m} = \rho \cdot v_3 \cdot A_3$ *mass flow rate, required in the linear momentum balance in y-axis*

$A_3 = \pi \cdot \frac{D^2}{4}$ *cross sectional area at the nozzle exit*

$\alpha_4 = 1$ *uniform velocity of the water jet*

*The point 4 is located at the hight reached by the flat board, then the energy equation will give us the location of the force balancing the weight*

$$\frac{p_1}{\rho} + \alpha_1 \cdot \frac{V_1^{\,2}}{2} + g \cdot z_1 = \frac{p_4}{\rho} + \alpha_4 \cdot \frac{V_4^{\,2}}{2} + g \cdot z_4$$

$$p_4 = p_{atm}$$

*continuity equation, required to solve for the area of the water jet, to get the diemater of the jet hitting the board*

$$\dot{m} = \rho \cdot V_4 \cdot A_4$$

$$A_4 = \pi \cdot \frac{D_4^{\,2}}{4}$$

$$D_3 = D$$

SOLUTION

**Unit Settings: SI C kPa kJ mass deg**

$\alpha_1 = 1$

$\alpha_4 = 1$

$h = 40.72$ [m]

$\dot{m} = 32.13$ [kg/s]

$p_{gas} = 303975$ [Pa]

$\alpha_2 = 1$

$D = 0.0381$ [m]

$H_T = 20$ [m]

$p_{atm} = 101325$ [Pa]

$\rho = 997$ [kg/m$^3$]

$\alpha_3 = 1$

$g = 9.81$ [m/s$^2$]

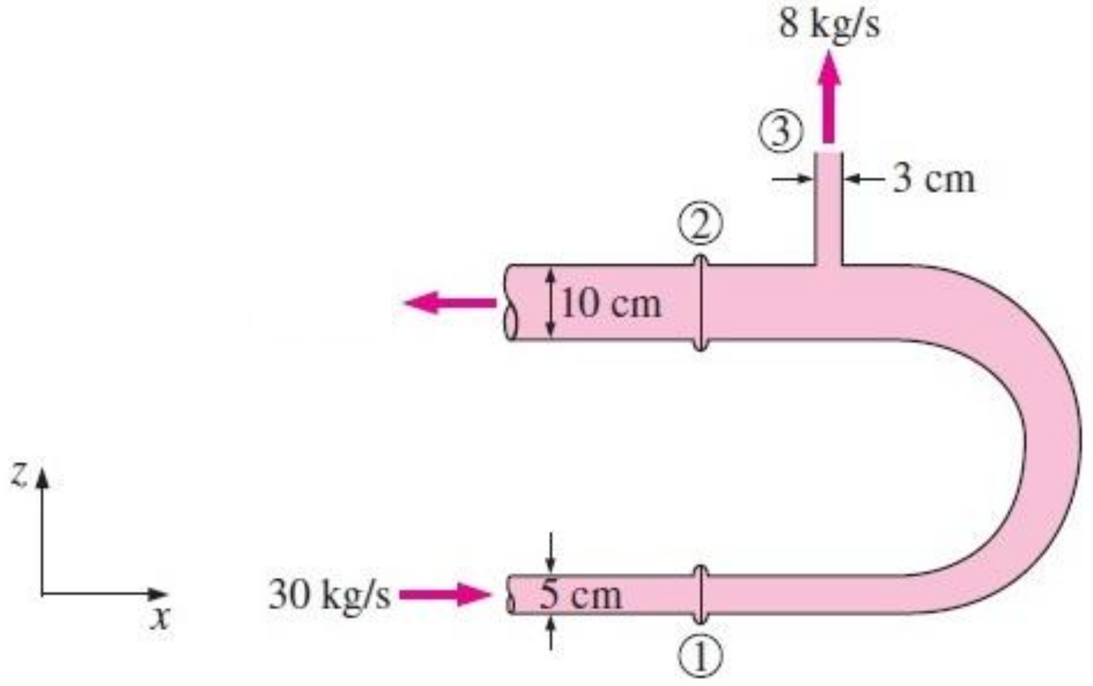
$m = 70$ [kg]

$p_{gage} = 202650$ [Pa]

**Arrays Table: Main**

| | $z_i$ [m] | $V_i$ [m/s] | $p_i$ [Pa] | $A_i$ [m$^2$] | $D_i$ [m] |
|---|---|---|---|---|---|
| 1 | 20 | 0 | 303975 | | |
| 2 | 40.72 | 0 | 101325 | | |
| 3 | 0 | 28.27 | 101325 | 0.00114 | 0.0381 |
| 4 | 17.44 | 21.37 | 101325 | 0.001508 | 0.04381 |

**Problem PW7.3** Water is flowing into a discharging from a pipe U-section as shown in the figure. At flange (1), the total absolute pressure is 200 kPa, and 30 kg/s flows into the pipe. At flange (2), the total pressure is 150 kPa. At location (3), 8 kg/s of water discharges to the atmosphere, which is at 100 kPa. Determine the total x- and z-forces at the two flanges connecting the pipe. Discuss the significance of gravity force for this problem. Take the momentum-flux correction factor to be 1.03.



Motivation to solve the previous problem .

- Use the definition of volume and mass flow rate to estimate velocity of the stream.
- Use the mass rate equation (continuity equation), to quantify missing mass flow rates under steady flow conditions.
- Use the linear momentum rate equation (force balance), to estimate the required force to keep a device in place or in operating conditions.

*Data*

$p_1 = 200000$ [Pa]

$p_2 = 150000$ [Pa]

$p_3 = 100000$ [Pa]

$p_o = p_3$

$\dot{m}_1 = 30$ [kg/s]

$\dot{m}_3 = -8$ [kg/s]

*mass balance*

$$\sum_{i=1}^{3} (\dot{m}_i) = 0$$

*diameter of the corss sectional area of each stream, direction of velocity and unit normal vector*

$D_1 = 0.05$ [m] $\quad \theta_1 = 0$ [deg] $\quad \phi_1 = 180$ [deg]

$D_3 = 0.03$ [m] $\quad \theta_3 = 90$ [deg] $\quad \phi_3 = 90$ [deg]

$D_2 = 0.1$ [m] $\quad \theta_2 = 180$ [deg] $\quad \phi_2 = 180$ [deg]

*Correction factor for inertial forces, cross sectional area, and mass flow rates*

$\beta_i = 1.03$ (for $i = 1$ to $3$)

$$A_i = \pi \cdot \frac{D_i^2}{4} \quad \text{(for } i = 1 \text{ to } 3\text{)}$$

$$\dot{m}_i = -\rho \cdot v_i \cdot A_i \cdot (\mathbf{cos}(\theta_i) \cdot \mathbf{cos}(\phi_i) + \mathbf{sin}(\theta_i) \cdot \mathbf{sin}(\phi_i)) \quad \text{(for } i = 1 \text{ to } 3\text{)}$$

*liquid properties*

$\rho = 997$ [kg/m$^3$]

*force balances*

$$F_{I,x} + F_{P,x} + F_{E,x} = 0$$

*Inertial Forces x-axis*

$$F_{I,x} = \sum_{i=1}^{3} \left(\beta_i \, \dot{m}_i \;\; v_i \cos(\theta_i)\right)$$

*Pressure forces in x-axis*

$$F_{P,x} = -\sum_{i=1}^{3} \left((p_i - p_o) A_i \cos(\phi_i)\right)$$

*Inertial Forces y-axis*

$$F_{I,z} = \sum_{i=1}^{3} \left(\beta_i \, \dot{m}_i \;\; v_i \sin(\theta_i)\right)$$

*Pressure forces in y-axis*

$$F_{P,z} = -\sum_{i=1}^{3} \left((p_i - p_o) A_i \sin(\phi_i)\right)$$

$$F_{I,z} + F_{P,z} + F_{E,z} = 0$$

SOLUTION

**Unit Settings: SI C kPa kJ mass deg**

$F_{E,x}$ = -1126 [N]
$F_{E,z}$ = 93.54 [N]
$F_{I,x}$ = 537.2 [N]
$F_{I,z}$ = -93.54 [N]
$F_{P,x}$ = 589 [N]
$F_{P,z}$ = 0.0000509 [N]
$p_o$ = 100000 [Pa]
$\rho$ = 997 [kg/m$^3$]

Arrays Table: Main

| | $\dot{m}_i$ [kg/s] | $p_i$ [Pa] | $A_i$ [m2] | $D_i$ [m] | $v_i$ [m/s] | $\phi_i$ [deg] | $\theta_i$ [deg] | $\beta_i$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 30 | 200000 | 0.001963 | 0.05 | 15.32 | 180 | 0 | 1.03 |
| 2 | -22 | 150000 | 0.007854 | 0.1 | 2.81 | 180 | 180 | 1.03 |
| 3 | -8 | 100000 | 0.0007069 | 0.03 | 11.35 | 90 | 90 | 1.03 |

**Problem W7.4** Balloons are often filled with helium gas because it weights only about one-seventh of what air weights under identical conditions. The buoyancy force, which can be expressed as $\boldsymbol{F}_b=\rho_{air}\, g\, V_{balloon}$, will push the balloon upward.

a) If the balloon has a diameter of 12 m and carries two people, 70 kg each, determine the acceleration of the balloon when it is first released. Assume the density of air is $\rho$ = 1.16 kg/m$^3$, and neglect the weight of the ropes and cage.

b) Calculate the height if the balloon, if after 1 h the sensors are measuring a pressure of 680 Torr. (the balloon was released at sea level where the pressure is ca 760 Torr)

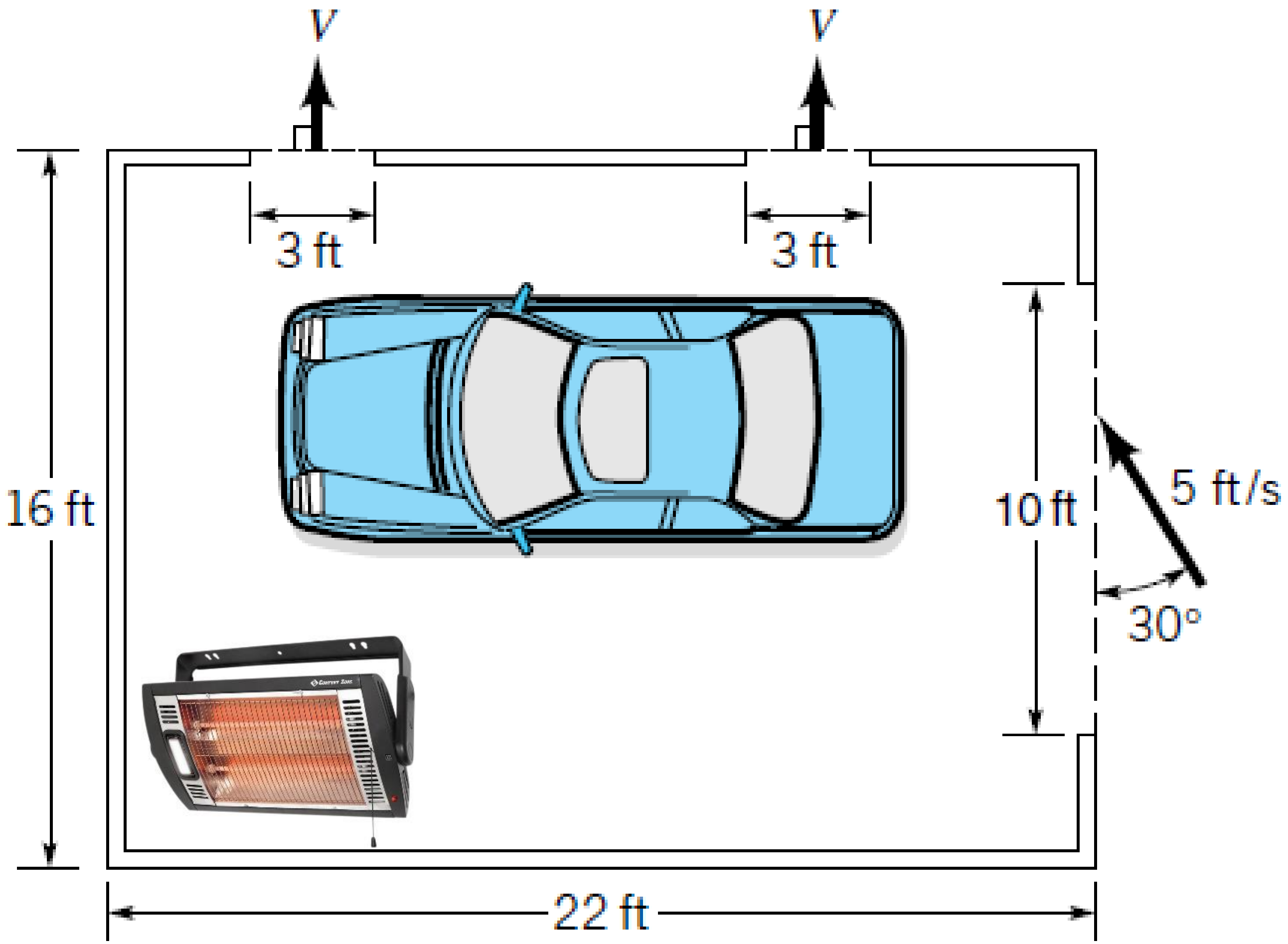
c) Calculate the terminal velocity of the balloon, if the drag force is expressed in the form:

$$F_d = C_D A \left(\frac{1}{2}\right) \rho_{air} v^2$$

For a perfect sphere $C_D$=0.44, and A is the projected area Of the sphere normal to the velocity

**Problem W7.5** The wind blows through a 7 ft × 10 Ft garage door opening with a speed of 5 ft/s and 32°F, as shown in the figure.

(a) Determine the average speed, *V*, of the air through the two 3 ft × 4 ft opening in the windows, if a heater located inside the garage keeps an average temperature of 70°F. All the walls, ceiling and floor are insulated.



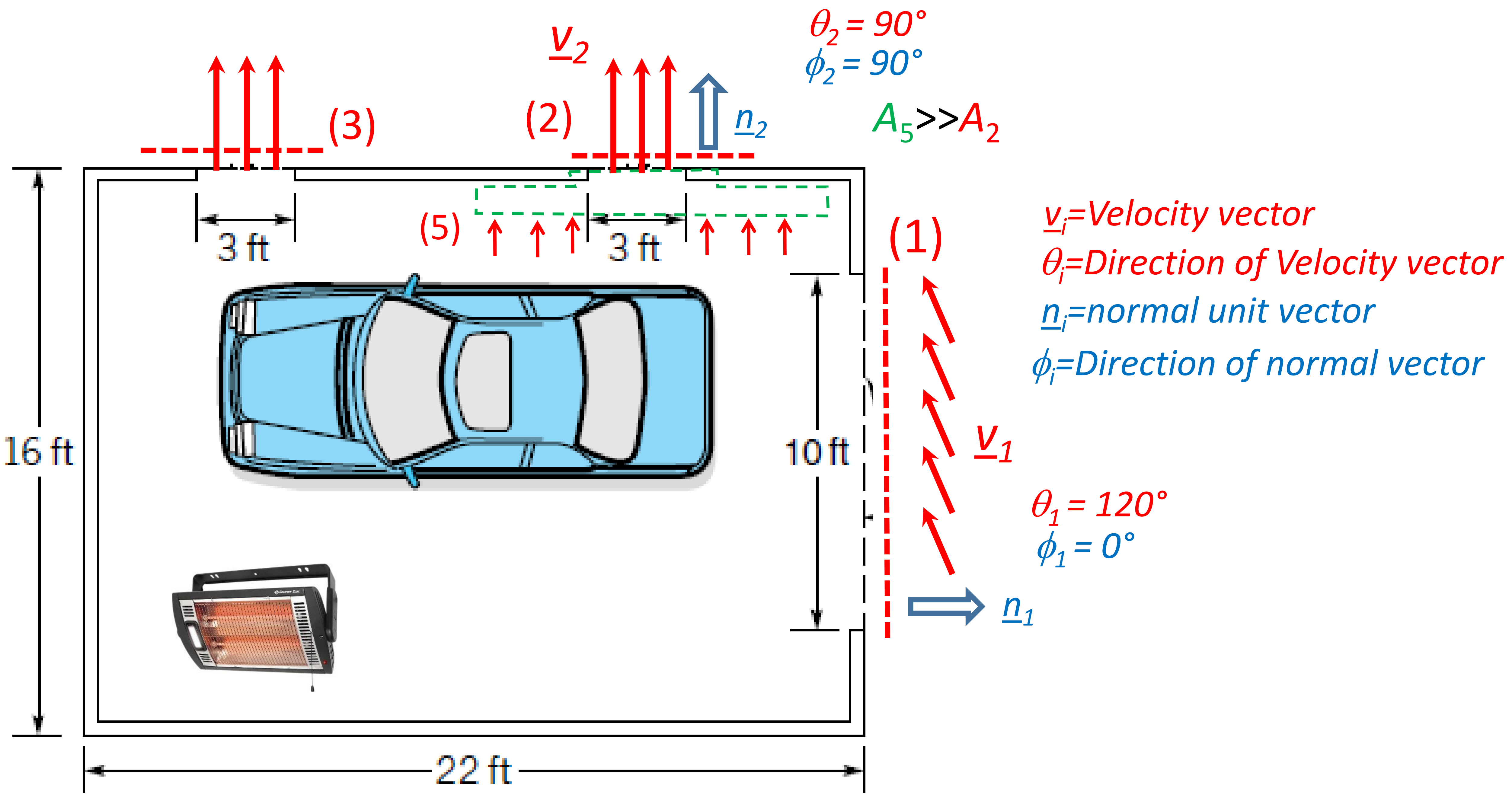

b) Calculate the pressure inside the garage.

c) Calculate the heat rate produced by the heater.

Hint: Air flow can be analyzed as incompressible fluid as long as the velocity satisfies *Ma* < 0.3. Mach Number is the ratio of actual velocity respect to the speed of sound in the media, calculated with the equation:

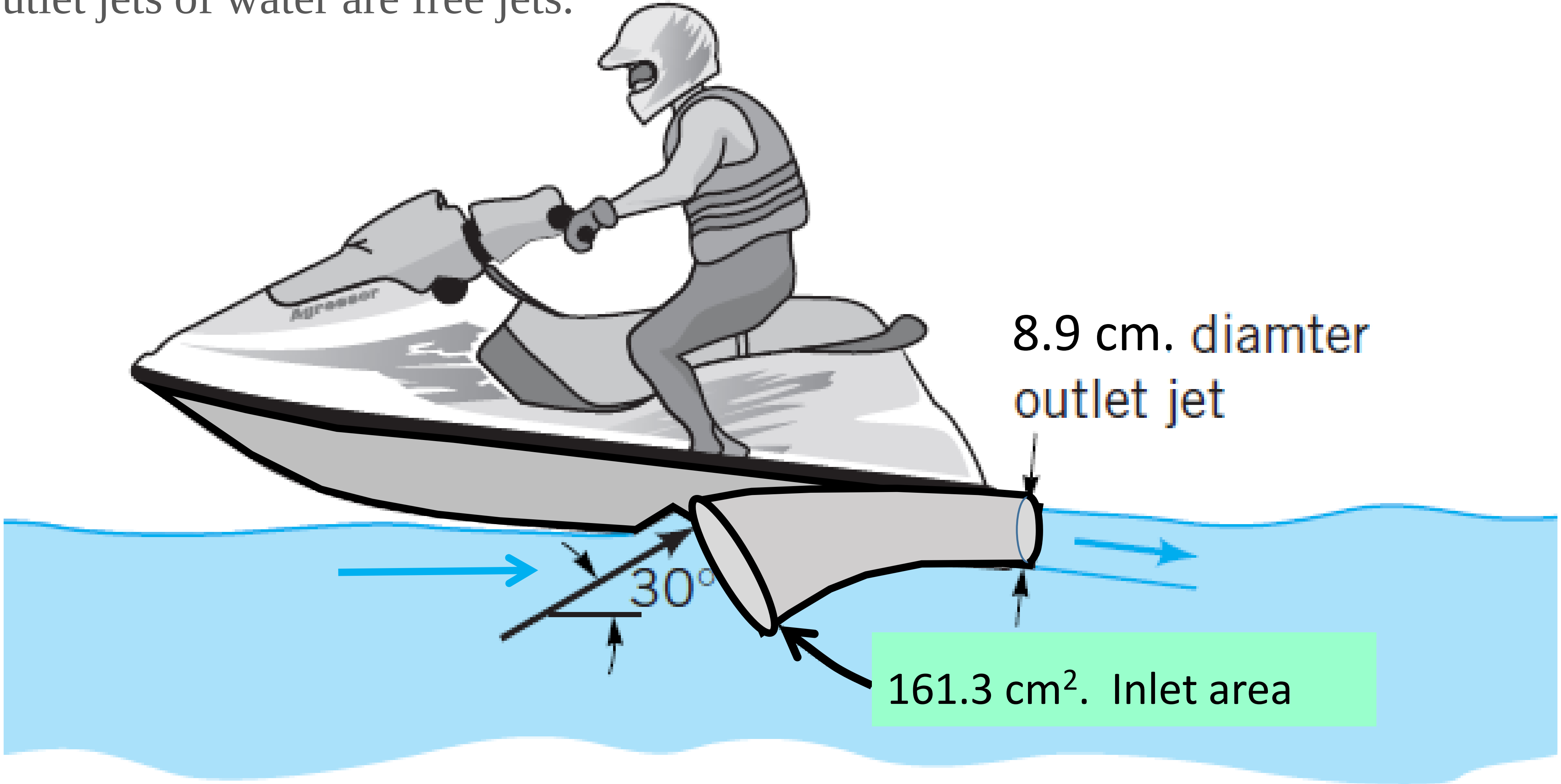
$$Ma = \frac{v}{c} \qquad c \approx \sqrt{\frac{k\,p}{\rho}}$$

*k* = 1.4 For air, which is the specific heat capacity ratio ($k = C_p/C_v$)



$\theta_2 = 90°$
$\phi_2 = 90°$
$\underline{v}_2$
$\underline{n}_2$
$A_5 >> A_2$
(3)
(2)
(5)
(1)
3 ft
3 ft
16 ft
10 ft
22 ft
$\underline{v}_i$=Velocity vector
$\theta_i$=Direction of Velocity vector
$\underline{n}_i$=normal unit vector
$\phi_i$=Direction of normal vector
$\underline{v}_1$
$\theta_1 = 120°$
$\phi_1 = 0°$
$\underline{n}_1$

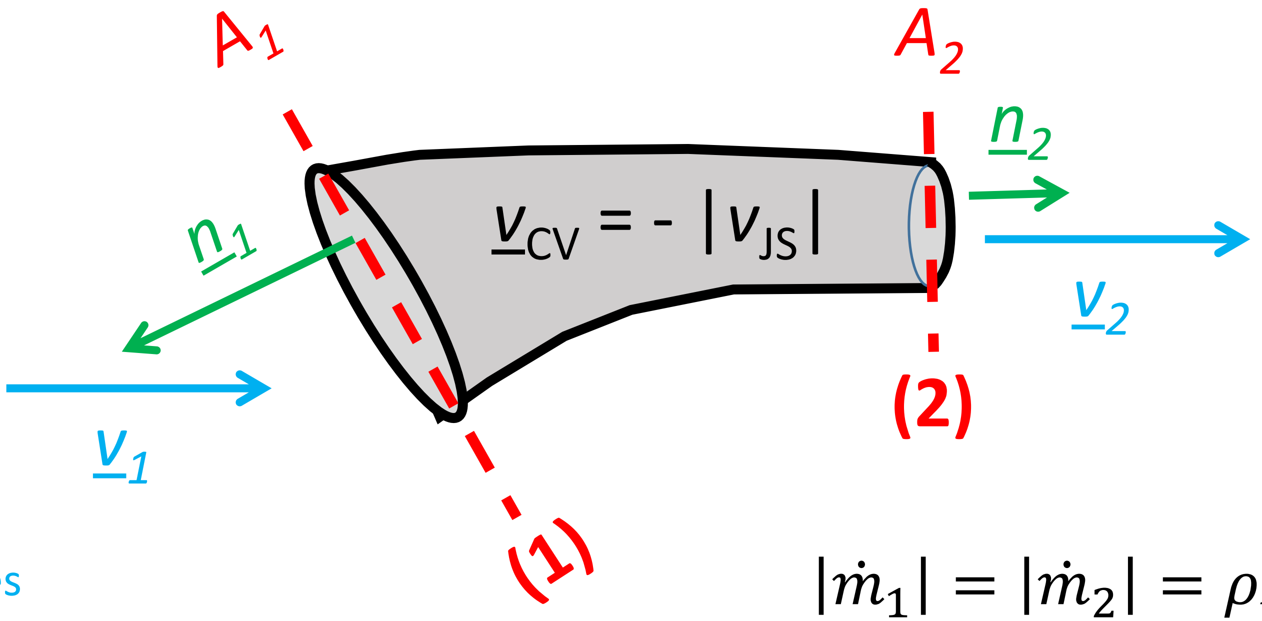
**Problem W7.6** The thrust developed to propel the jet ski shown in the figure is a result of water pumped through the vehicle and exiting as a high-speed jet. For the conditions shown in the figure, what flow rate is needed to produce a 1334.5 N thrust ? Assume the inlet and the outlet jets of water are free jets.



Motivation to solve the previous problem .

- Analysis of control volume under motion, but being the CV an inertial frame of reference.
- Use the mass rate equation (continuity equation), to quantify missing mass flow rates under steady flow conditions, the analysis will be done using vector approach for continuity equation.
- Use the linear momentum rate equation (force balance), to estimate the required force to keep a device in place or in operating conditions.
- This will be solved using the custom approach and the vector approach

# I. Custom approach




Mass balance

$$\frac{dm}{dt} = |\dot{m}_1| - |\dot{m}_2|$$

For steady flow

$$|\dot{m}_1| = |\dot{m}_2| = \rho A_1 \left| \underline{n}_1 \cdot \underline{v}_{r1} \right| = \rho A_2 \left| \underline{n}_2 \cdot \underline{v}_{r2} \right|$$

Relative velocities

$\underline{v}_{r1} = \underline{v}_1 - \underline{v}_{CV}$

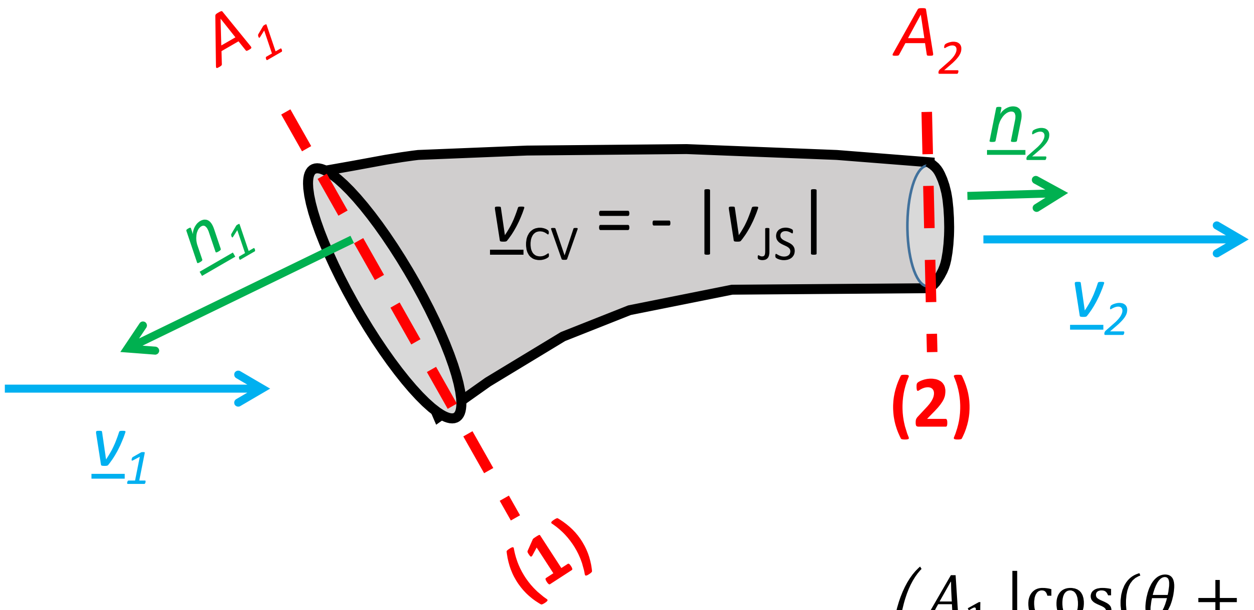
$\underline{v}_{r2} = \underline{v}_2 - \underline{v}_{CV}$

$$\rho A_1 \left| \underline{v}_{r1} \right| |\cos(\theta + \pi)| = \rho A_2 \left| \underline{v}_{r2} \right| |\cos(0)|$$

$$\frac{A_1}{A_2} \frac{|\cos(\theta + \pi)|}{|\cos(0)|} \left| v_{JS} \right| = \left| v_{JS} + v_2 \right|$$

I. Custom approach

Continuity equation in terms of the speed of the Jet Ski, and the velocity of the body of water (lake )



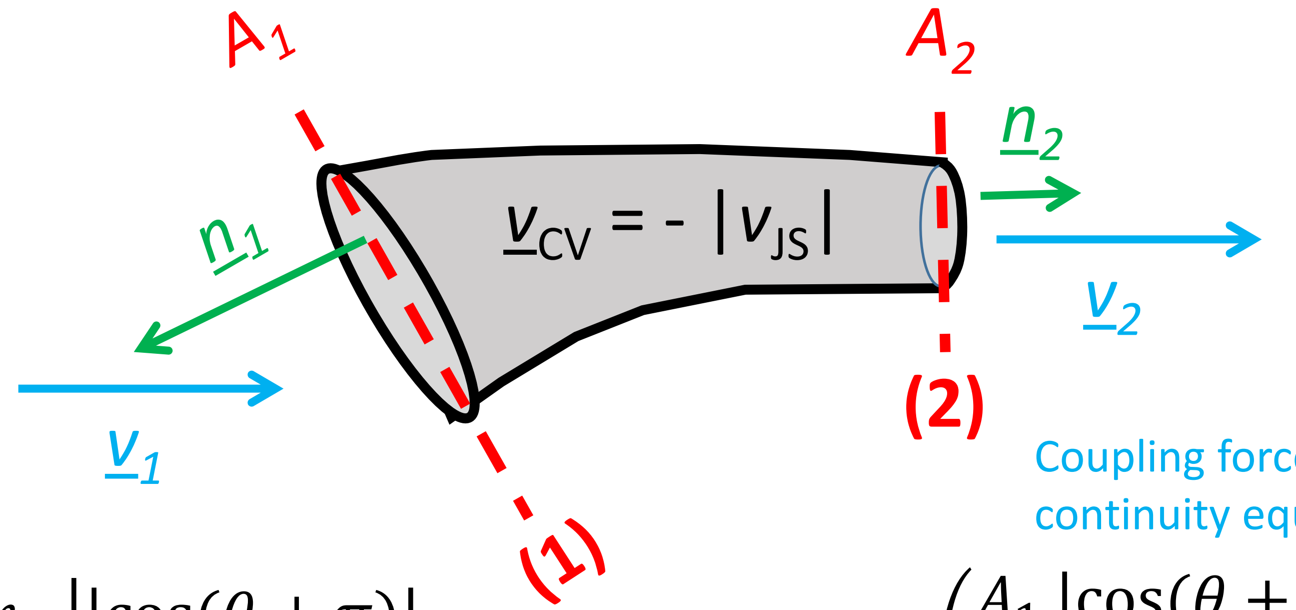

$$\left(\frac{A_1}{A_2}\frac{|\cos(\theta+\pi)|}{|\cos(0)|}-1\right)|v_{JS}| = |v_2|$$

Mass flow rate in terms of the speed of the Jet Ski and the orientation of the intake cross sectional area

$$|\dot{m}_1| = \rho A_1 |\underline{n}_1 \cdot \underline{v}_{r1}| = \rho A_1 |\underline{v}_{r1}||\cos(\theta+\pi)| = \rho A_1 |v_{JS}||\cos(\theta+\pi)|$$

$$\frac{d[m\,\underline{v}]}{dt} = \sum_{i=1} |\dot{m}_i| \langle \underline{v}_i \rangle \beta_i - \sum_{j=2} |\dot{m}_j| \langle \underline{v}_j \rangle \beta_j + m\underline{g} + \sum_{k=1,2} [p_k - p_o] A_k (-\underline{n}_k) + \underline{F}_{ex}$$

# I. Custom approach
# Force Balance




$$|\dot{m}_1| = \rho A_1 |v_{JS}| |\cos(\theta + \pi)|$$

Coupling force balance, mas flow rate and continuity equation

$$\left( \frac{A_1}{A_2} \frac{|\cos(\theta + \pi)|}{|\cos(0)|} - 1 \right) |v_{JS}| = |v_2|$$

$$0 = |\dot{m}_1| \langle v_{1,x} \rangle - |\dot{m}_2| \langle v_{2,x} \rangle + F_{ex,x}$$

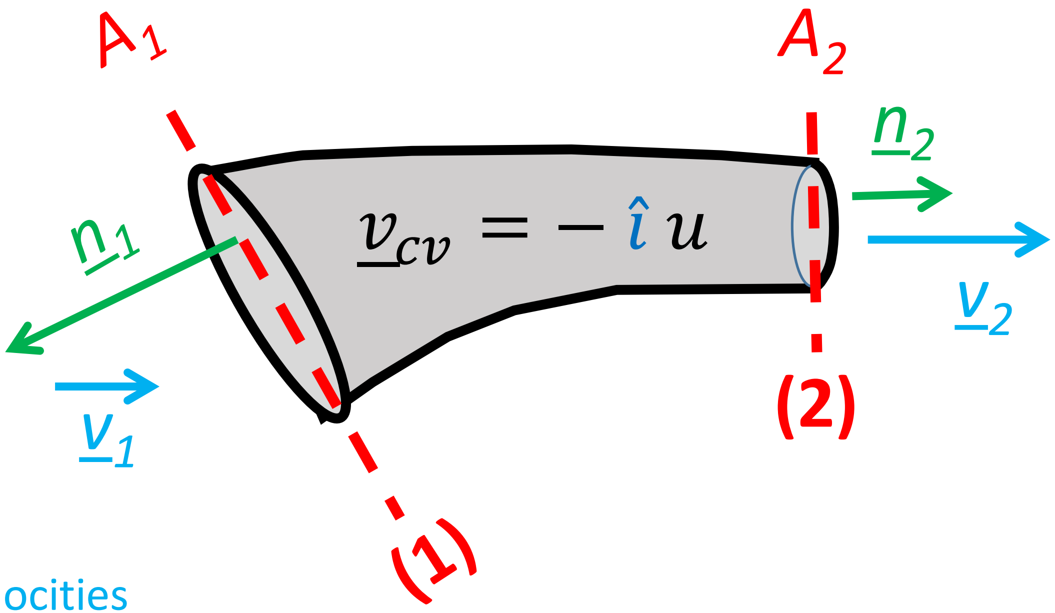
$$F_{ex,x} = \rho A_1 |v_{JS}| |\cos(\theta + \pi)| \left( \langle v_{2,x} \rangle - \langle v_{1,x} \rangle \right)$$

$$F_{ex,x} = |\dot{m}_1| \left( \langle v_{2,x} \rangle - \langle v_{1,x} \rangle \right)$$

$$F_{ex,x} = \rho A_1 |v_{JS}| |\cos(\theta + \pi)| \left( \left( \frac{A_1}{A_2} \frac{|\cos(\theta + \pi)|}{|\cos(0)|} - 1 \right) |v_{JS}| \right)$$

$$\boxed{F_{ex,x} = \rho A_1 |v_{JS}|^2 |\cos(\theta)| \left( \frac{A_1}{A_2} |\cos(\theta)| - 1 \right)}$$

## II. Vector approach




Mass balance $$\frac{dm}{dt} = \dot{m}_1 + \dot{m}_2$$

For steady flow

$$\dot{m}_1 = -\dot{m}_2 = -\rho A_1\,\underline{n}_1 \cdot \underline{v}_{r1} = -\left(-\rho A_2\,\underline{n}_2 \cdot \underline{v}_{r2}\right)$$

$$\dot{m}_1 = -\dot{m}_2 = -\rho A_1\,\underline{n}_1 \cdot \underline{w}_1 = -\left(-\rho A_2\,\underline{n}_2 \cdot \underline{w}_2\right)$$

$$\underline{n}_1 = -\hat{\imath}\cos\theta - \hat{\jmath}\sin\theta$$

$$\underline{n}_2 = +\hat{\imath}$$

$\underline{v}_1 = \hat{\imath}\,0 + \hat{\jmath}\,0$ $\quad$ $\underline{u}_1 = -\hat{\imath}\,u$ $\quad$ $\underline{w}_1 = +\hat{\imath}\,u$

$\underline{v}_2 = +\,\hat{\imath}\,v$ $\quad$ $\underline{u}_2 = -\hat{\imath}\,u$ $\quad$ $\underline{w}_2 = +\,\hat{\imath}\,[v+u]$

Relative velocities

$\underline{v}_{r1} = \underline{v}_1 - \underline{v}_{cv1}$ $\quad$ $\underline{w}_1 = \underline{v}_1 - \underline{u}_1$

$\underline{v}_{r2} = \underline{v}_2 - \underline{v}_{cv2}$ $\quad$ $\underline{w}_2 = \underline{v}_2 - \underline{u}_2$

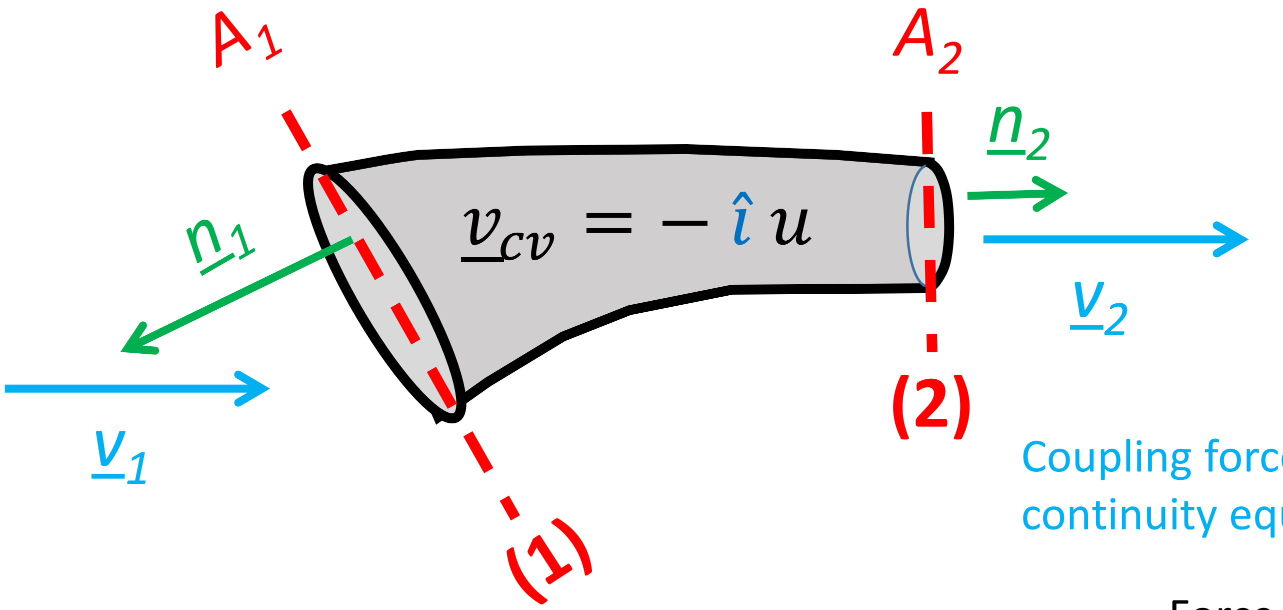
$$\dot{m}_1 = -\dot{m}_2 = -\rho A_1\left(-\hat{\imath}\cos\theta - \hat{\jmath}\sin\theta\right)\cdot\left(\hat{\imath}\,u\right) = -\left(-\rho A_2\,(+\hat{\imath})\cdot\left(\hat{\imath}\,[v+u]\right)\right)$$

$\rho\,A_1\cos\theta\;u = \rho A_2\,[v+u]$ $\qquad$ $v = \left(A_1\cos\theta / A_2 - 1\right)u$ $\qquad$ Relationship between jet ski speed and water jet

$\dot{m}_1 = \rho\,A_1\,u\,\cos\theta$ $\qquad$ Mass flow rate in terms of geometry and speed of the jet ski

$$\frac{d[m\,\underline{v}]}{dt} = \sum_{k=1,2} \dot{m}_k \langle \underline{v}_k \rangle\, \beta_k \;+ m\underline{g} + \sum_{k=1,2} [p_k - p_o] A_k (-\underline{n}_k) + \underline{F_D} \pm \underline{F_L}$$

Force Balance
II. Vector approach




Coupling force balance, mas flow rate and continuity equation

$$\dot{m}_1 = \rho\, A_1\, u\;\cos\theta$$

$$v = (\,A_1 \cos\theta / A_2\; - 1)\; u$$

$$\dot{m}_1 = -\dot{m}_2$$

Force Balance for x-axis

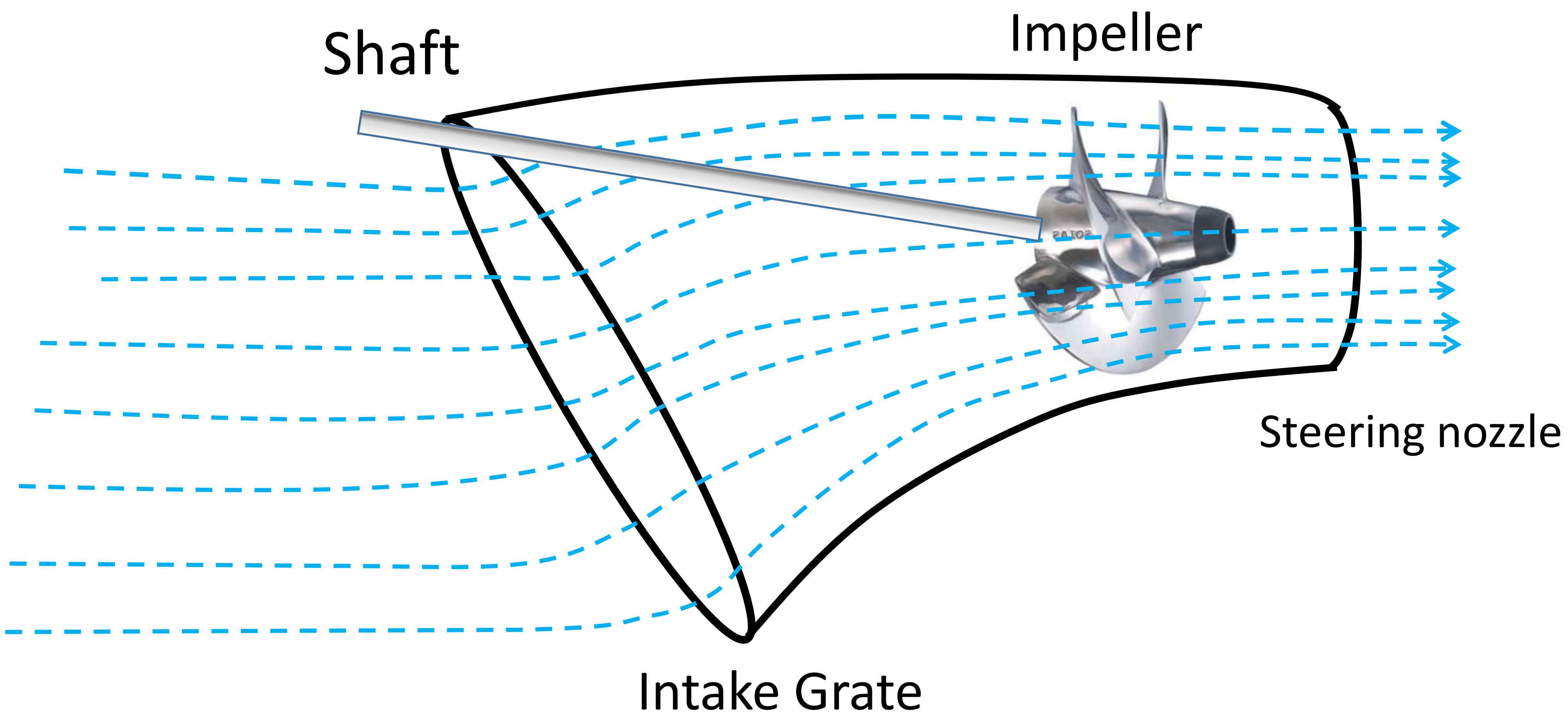
$$\frac{d[m\, v_x]}{dt} = 0 = \dot{m}_1 \langle v_{1,x} \rangle (1) + \dot{m}_2 \langle v_{2,x} \rangle (1) + F_{D,x}$$

$$F_{D,x} = -\dot{m}_2 \langle v_{2,x} \rangle (1)$$

$$F_{D,x} = \;\rho\, A_1\, u\;\cos\theta\;\; v$$

$$\boxed{F_{D,x} = \;\rho\, A_1\, u^2\;\cos\theta\,(\,A_1 \cos\theta / A_2\; - 1)}$$

This force, is the drag force that must be counterbalanced by thrust in order to keep constant speed.


Shaft
Impeller
Steering nozzle
Intake Grate

$$F_x = \rho \cdot A_1 \cdot V_{JS}^2 \cdot \mathrm{Cos}(\theta) \cdot \left( A_1 \cdot \frac{\mathrm{Cos}(\theta)}{A_2} - 1 \right)$$

$$A_1 = 25 \ \left[\mathrm{in}^2\right] \cdot \left| 6.4516 \times 10^{-4} \ \frac{\mathrm{m}^2}{\mathrm{in}^2} \right|$$

$$D_2 = 3.5 \ [\mathrm{in}]$$

$$A_2 = \left( \pi \cdot \frac{D_2^2}{4} \right) \cdot \left| 6.4516 \times 10^{-4} \ \frac{\mathrm{m}^2}{\mathrm{in}^2} \right|$$

$$\theta = 30 \ [\mathrm{deg}]$$

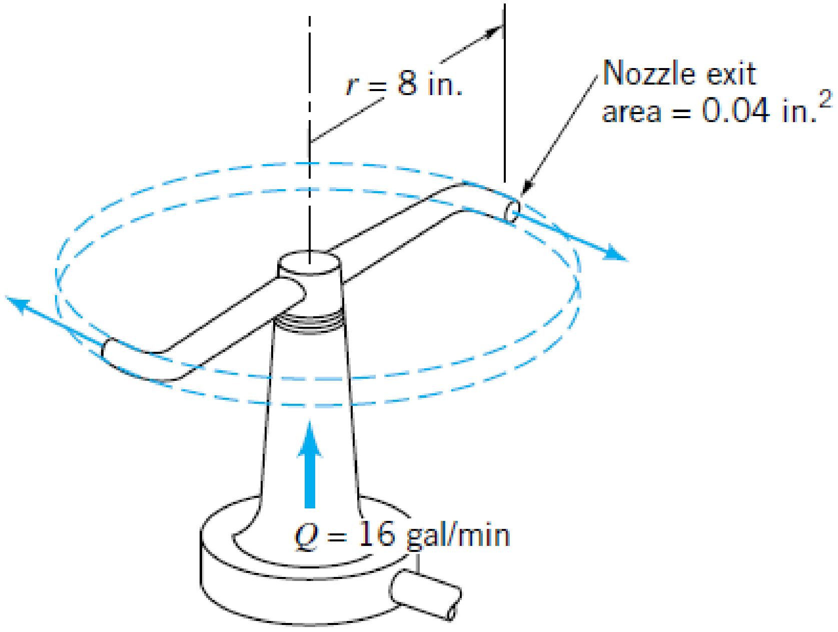
$$\rho = 997 \ \left[\mathrm{kg/m}^3\right]$$

$$F_x = 300 \ [\mathrm{lb_f}] \cdot \left| 4.448222 \ \frac{\mathrm{N}}{\mathrm{lb_f}} \right|$$

$$\dot{m} = \rho \cdot A_1 \cdot V_{JS} \cdot \mathrm{Cos}(\theta)$$

| | |
|---|---|
| $A_1 = 0.01613 \ [\mathrm{m}^2]$ | $A_2 = 0.006207 \ [\mathrm{m}^2]$ |
| $D_2 = 3.5 \ [\mathrm{in}]$ | $F_x = 1334 \ [\mathrm{N}]$ |
| $\dot{m} = 121.9 \ [\mathrm{kg/s}]$ | $\rho = 997 \ [\mathrm{kg/m}^3]$ |
| $\theta = 30 \ [\mathrm{deg}]$ | $V_{JS} = 8.754 \ [\mathrm{m/s}]$ |

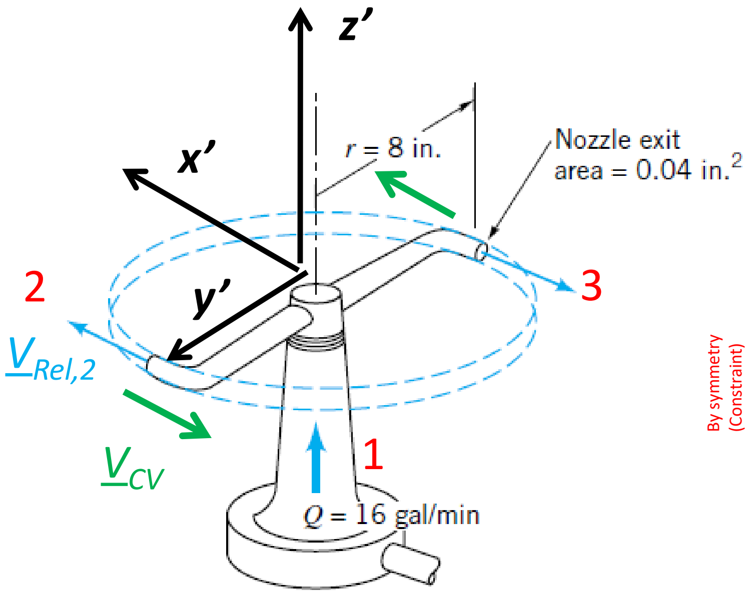
**Problem W7.7** Water enters a rotating lawn sprinkler through its base at the steady rate of 16 gal/min as shown in the figure. The exit cross-sectional area of each of the two nozzles is 0.04 in$^2$, and the flow leaving each nozzle is tangential. The radius from the axis of rotation to the centerline of each nozzle is 8 in. (a) Determine the resisting torque required to hold the sprinkler head stationary. (b) Determine the resisting torque associated with the sprinkler rotating with a constant speed of 500 rev/min. (c) Determine the angular velocity of the sprinkler if no resisting torque is applied. (d) Calculate the power that can be produced by coupling an alternator. (e) Estimate the maximum power that can be generated if an alternator is connected.



Motivation to solve the previous problem .

- Analysis of control volume under motion, but being the CV a non- inertial frame of reference.
- Use the mass rate equation (continuity equation), to quantify missing mass flow rates under steady flow conditions, the analysis will be done using vector approach for continuity equation.
- Use the angular momentum rate equation (torque balance), to estimate the required torque to keep a device in place or in operating conditions.
- Estimate power of a mechanical device by the torque and angular velocity.
- This will be solved using the vector approach.

# Physics of sprinkler (continuity equation)




$$\frac{dm}{dt} = \dot{m}_1 + \dot{m}_2 + \dot{m}_3 = 0$$

No accumulation of mass

By symmetry (Constraint)

| | | | | | | |
|---|---|---|---|---|---|---|
| $\underline{v}_1 = +\hat{\imath}\, v_o$ | $\underline{r}_1 = 0\,\hat{\jmath}$ | $\underline{\omega}_1 = \hat{k}\, 0$ | $\underline{u}_1 = \underline{\omega}_1 \times \underline{r}_1$ | $\underline{u}_1 = +0\,\hat{\imath}$ | $\underline{w}_1 = +\hat{\imath}\, v_o$ | $\underline{n}_1 = -\hat{\imath}$ |
| $\underline{v}_2 = +\hat{\imath}\, v$ | $\underline{r}_2 = +R\,\hat{\jmath}$ | $\underline{\omega}_2 = \hat{k}\,\Omega$ | $\underline{u}_2 = \underline{\omega}_2 \times \underline{r}_2$ | $\underline{u}_2 = -\hat{\imath}\,\Omega\, R$ | $\underline{w}_2 = +\hat{\imath}\,[v + \Omega\, R\,]$ | $\underline{n}_2 = +\hat{\imath}$ |
| $\underline{v}_3 = -\hat{\imath}\, v$ | $\underline{r}_3 = -R\,\hat{\jmath}$ | $\underline{\omega}_3 = \hat{k}\,\Omega$ | $\underline{u}_3 = \underline{\omega}_3 \times \underline{r}_3$ | $\underline{u}_3 = +\hat{\imath}\,\Omega\, R$ | $\underline{w}_3 = -\hat{\imath}\,[v + \Omega\, R\,]$ | $\underline{n}_3 = -\hat{\imath}$ |

$$\dot{m}_2 = -\,\rho\, A_2 \underline{w}_2 \cdot \underline{n}_2 = -\rho\, A_2\,[v + \Omega\, R\,]$$

Continuity equation

$$\dot{m}_3 = -\,\rho\, A_3 \underline{w}_3 \cdot \underline{n}_3 = -\rho\, A_3\,[v + \Omega\, R\,] = \dot{m}_2$$

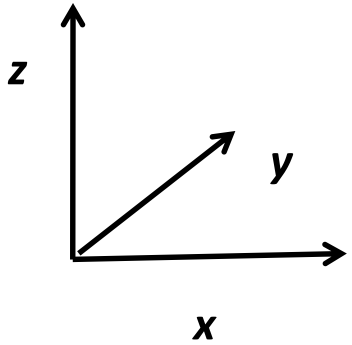
$$\dot{m}_1 + \dot{m}_2 + \dot{m}_3 = \dot{m}_1 + 2\,\dot{m}_2 = 0$$

$$\dot{m}_2 = -\,\dot{m}_1/2 = -\rho\, A_2\,[v + \Omega\, R\,]$$

$$\dot{m}_1/(2\,\rho\, A_2) = \;[v + \Omega\, R\,]$$

$$v = \dot{m}_1/(2\,\rho\, A_2) - \Omega\, R$$

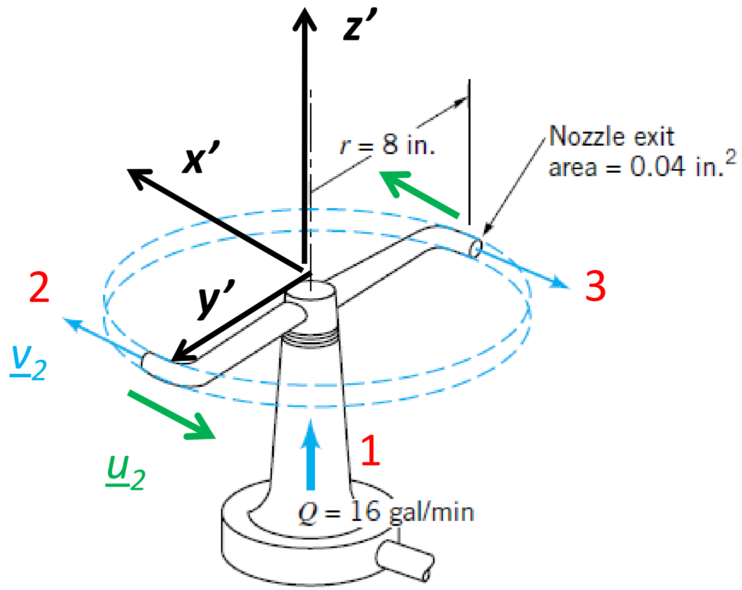
Relationship between total mass flow rate, angular speed and linear fluid speed at the outlet




Axes (*x,y,z*) where the observer is standing, is an inertial frame of reference.
Axes (*x',y',z'*) form a rotating frame of reference in non-inertial

As a matter of fact, control volume velocity is :

# Physics of sprinkler (angular momentum or torque balance)




$$\frac{d\left[\underline{r}\times m\,\underline{v}\right]}{dt}=\dot{m}_1\,\underline{r}_1\times\langle\underline{v}_1\rangle\,\beta_1+\dot{m}_2\,\underline{r}_2\times\langle\underline{v}_2\rangle\,\beta_2+\dot{m}_3\,\underline{r}_3\times\langle\underline{v}_3\rangle\,\beta_3+\underline{r}_{CM}\times m\,\underline{g}-\underline{T}_{ext}-\underline{r}_{CF}\underline{F}_D$$

$$\underline{v}_2=+\hat{\imath}\,v \qquad \underline{v}_3=-\hat{\imath}\,v$$

$$\underline{r}_2=+R\,\hat{\jmath} \qquad \underline{r}_3=-R\,\hat{\jmath}$$

$$\dot{m}_2=-\dot{m}_1/2 \qquad \dot{m}_3=-\dot{m}_1/2$$

$$v=\dot{m}_1/(2\,\rho\,A_2)-\Omega\,R$$

$$\frac{d\left[\underline{r}\times m\,\underline{v}\right]}{dt}=\dot{m}_1\,\underline{r}_1\times\langle\underline{v}_1\rangle\,\beta_1+\dot{m}_2\,\underline{r}_2\times\langle\underline{v}_2\rangle\,\beta_2+\dot{m}_3\,\underline{r}_3\times\langle\underline{v}_3\rangle\,\beta_3+\underline{r}_{CM}\times m\,\underline{g}-\underline{T}_{ext}-\underline{r}_{CF}\underline{F}_D$$

Zero
Steady state

Zero
Zero radius

Zero
Zero radius

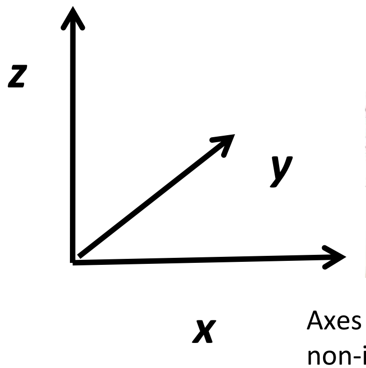
Zero
Negligible viscous effects
Air sprinkler

$$-\underline{T}_{ext}=-\dot{m}_2\,\underline{r}_2\times\langle\underline{v}_2\rangle\,\beta_2-\dot{m}_3\,\underline{r}_3\times\langle\underline{v}_3\rangle\,\beta_3=-\dot{m}_2\left[-2\,\hat{k}\,R\,v\right]=-\hat{k}\,R\,\dot{m}_1[\dot{m}_1/(2\,\rho\,A_2)-\Omega\,R\,]$$

Whenever we have a free jet stream, the assumption of $\beta$ =1 is a good approximation

$$\underline{T}_{ext}=\hat{k}\,R\,\dot{m}_1[\dot{m}_1/(2\,\rho\,A_2)-\Omega\,R\,]$$

$$\dot{W}=\underline{T}_{ext}\cdot\underline{\omega}=\Omega\,R\,\dot{m}_1[\dot{m}_1/(2\,\rho\,A_2)-\Omega\,R\,]$$




Axes (*x,y,z*) where the observer is standing, is an inertial frame of reference. Axes (*x',y',z'*) is a rotating frame of reference, i.e. non-inertial (Located at the top of the sprinkler, or at any point in the spinning section of the sprinkler)

# Main answer of the analysis

$$\dot{m}_1 = \rho \dot{V}_1$$

Mass flow rate in terms of volume flow rate

$$\underline{T}_{ext} = \hat{k}\ R\, \dot{m}_1[\dot{m}_1/(2\,\rho\, A_2) - \Omega\, R\,]$$

Torque in terms of angular speed, flow rate, radius, nozzle cross sectional area and density.

$$\dot{W} = \Omega\, R\, \dot{m}_1[\dot{m}_1/(2\,\rho\, A_2) - \Omega\, R\,]$$

Power obtained with an alternator in terms of angular speed, flow rate, radius, nozzle cross sectional area and density.

(a) Determine the resisting torque required to hold the sprinkler head stationary.

$$\underline{T} = \hat{k}\ R\, \dot{m}_1[\dot{m}_1/(2\,\rho\, A_2)]$$

Set $\Omega$=0

(b) Determine the resisting torque associated with the sprinkler rotating with a constant speed of 500 rev/min.

$$\underline{T} = \hat{k}\ R\, \dot{m}_1[\dot{m}_1/(2\,\rho\, A_2) - \Omega\, R\,]$$

(c) Determine the angular velocity of the sprinkler if no resisting torque is applied.

$$\Omega_{max} = \dot{m}_1/(2\,\rho\, A_2\, R)$$

Set $\underline{T}$=0

(d) Calculate the power that can be produced by coupling an alternator.

$$\dot{W} = \Omega\, R\, \dot{m}_1[\dot{m}_1/(2\,\rho\, A_2) - \Omega\, R\,]$$

(e) Estimate the maximum power that can be generated if an alternator is connected.

$$\dot{W}_{max} = \Omega\, R\, \dot{m}_1[\dot{m}_1/(4\,\rho\, A_2)]$$

$$\dot{W} = R\, \dot{m}_1[\dot{m}_1\Omega\,/(2\,\rho\, A_2) - \Omega^2\, R\,]$$

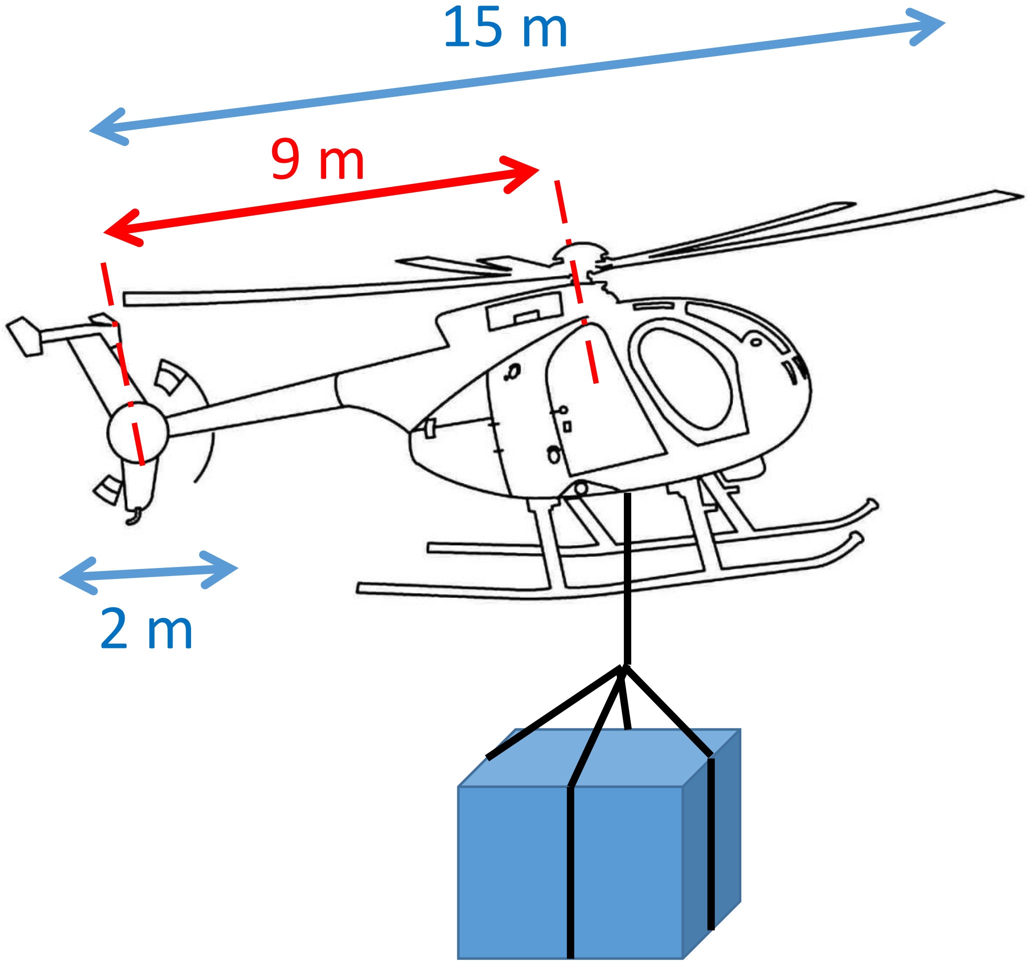
$$\frac{d\dot{W}}{d\Omega} = 0 = R\, \dot{m}_1[\dot{m}_1\,/(2\,\rho\, A_2) - 2\,\Omega\, R\,]$$

$$\Omega_{opt} = \dot{m}_1/(4\,\rho\, A_2\, R)$$

$$\dot{W}_{max} = \Omega\, R\, \dot{m}_1[\dot{m}_1/(4\,\rho\, A_2)]$$

Write power equation as function of $\Omega$, derivate the equation respect to $\Omega$, set to zero and find optimal value of $\Omega$, replace optimal value of $\Omega$ in the power function

**Problem W7.8** An Unloaded helicopter of mass 10,000 kg hovers at sea level while it is being loaded. In the unloaded hover mode, the blades rotate at 400 rpm. The horizontal blades above the helicopter cause a 15-m-diameter air mass to move downward at an average velocity proportional to the overhead blade rotational velocity (rpm). A load of 15,000 kg is loaded onto the helicopter, and the helicopter slowly rises. Determine (a) the volumetric airflow rate downdraft that the helicopter generates during unload hover and the required power input and (b) the rpm of the helicopter blades to hover with the 15,000 kg load and the required power input. Take the density of air of atmospheric air to be 1.18 kg/m$^3$. Assume air approaches the blades from the top through a large area with negligible velocity and air is forced by the blades to move down with a uniform velocity through and imaginary cylinder whose base is the blade span area. (c) How much extra power is needed to prevent the helicopter from spinning,

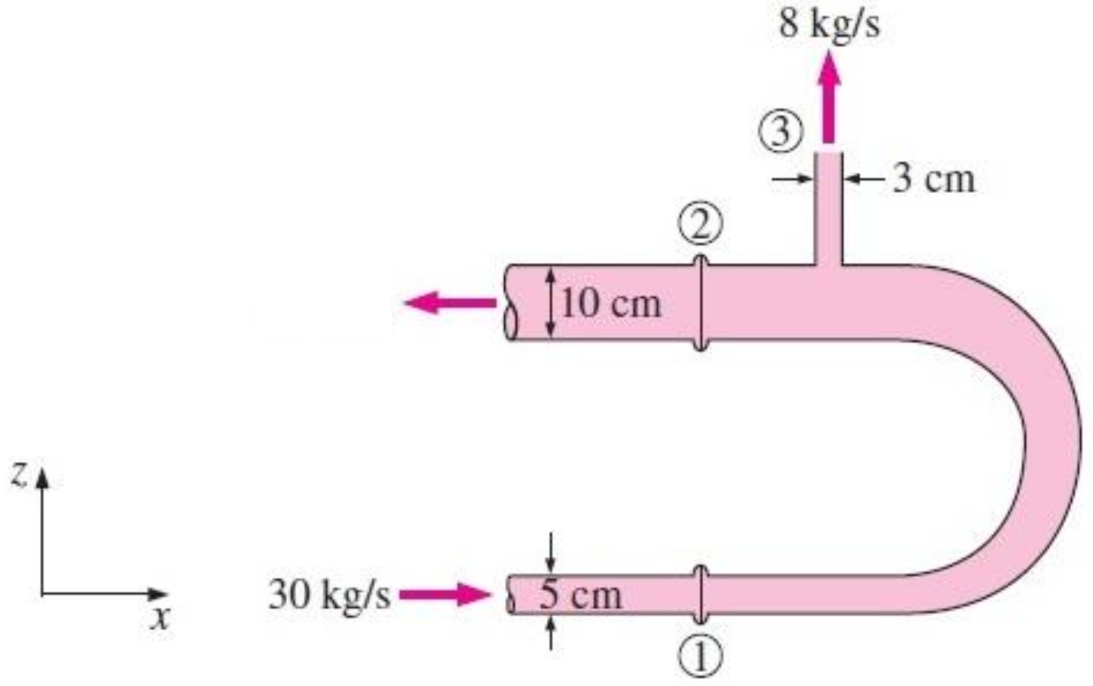



Load 15,000 kg

Motivation to solve the previous problem .

- Analysis of a device (control volume) that requires inertial force to hover, torque to prevent from spinning (i.e. stability), energy to run, and flow rate to reach the goal.
- Use the linear momentum rate equation.
- Use the mass rate equation.
- Use the angular momentum rate equation.
- Use the mechanical energy rate equation.

**Problem PW7.3** Water is flowing into a discharging from a pipe U-section as shown in the figure. At flange (1), the total absolute pressure is 200 kPa, and 30 kg/s flows into the pipe. At flange (2), the total pressure is 150 kPa. At location (3), 8 kg/s of water discharges to the atmosphere, which is at 100 kPa. Determine the total x- and z-forces at the two flanges connecting the pipe. Discuss the significance of gravity force for this problem. Take the momentum-flux correction factor to be 1.03.



$$p_1 = 200000 \ [\mathrm{Pa}]$$

$$p_2 = 150 \ [\mathrm{kPa}] \cdot \left| 1000 \ \frac{\mathrm{Pa}}{\mathrm{kPa}} \right|$$

$$p_3 = 100 \ [\mathrm{kPa}] \cdot \left| 1000 \ \frac{\mathrm{Pa}}{\mathrm{kPa}} \right|$$

$p_o = p_3$ The stream 3 is descharging to atmosphere

$$\dot{m}_1 = 30 \ [\mathrm{kg/s}]$$

$$\dot{m}_3 = 8 \ [\mathrm{kg/s}]$$

Mass balance

$$\dot{m}_1 - \dot{m}_2 - \dot{m}_3 = 0$$

$$D_1 = 5 \times 10^{-2} \ [\mathrm{m}]$$

$$D_3 = 3 \times 10^{-2} \ [\mathrm{m}]$$

$$D_2 = 0.1 \ [\mathrm{m}]$$

$$A_1 = \pi \cdot \frac{D_1^2}{4}$$

$$A_2 = \pi \cdot \frac{D_2^2}{4}$$

$$A_3 = \pi \cdot \frac{D_3^2}{4}$$

$$\rho = 997 \ [\text{kg/m}^3]$$

duplicate $i = 1, \ 3$

$$\dot{m}_i = \rho \cdot v_i \cdot A_i$$

$$\beta_i = 1.03$$

end

$$\theta_1 = 0 \ [\text{deg}]$$

$$\theta_2 = 180 \ [\text{deg}]$$

$$\theta_3 = 90 \ [\text{deg}]$$

$$\phi_1 = 180 \ [\text{deg}]$$

$$\phi_2 = 180 \ [\text{deg}]$$

$$\phi_3 = 90 \ [\text{deg}]$$

Assumption: Gravity is in y-direction

x-axis linear momentum balance

$$\dot{m}_1 \cdot V_1 \cdot \text{Cos}(\theta_1) \cdot \beta_1 - \dot{m}_2 \cdot V_2 \cdot \text{Cos}(\theta_2) \cdot \beta_2 - \dot{m}_3 \cdot V_3 \cdot \text{Cos}(\theta_3) \cdot \beta_3 - (p_1 - p_o) \cdot A_1 \cdot \text{Cos}(\phi_1) - (p_2 - p_o) \cdot A_2 \cdot \text{Cos}(\phi_2) - (p_3 - p_o) \cdot A_3 \cdot \text{Cos}(\phi_3) - F_{ex}$$

z-axis linear momentum balance

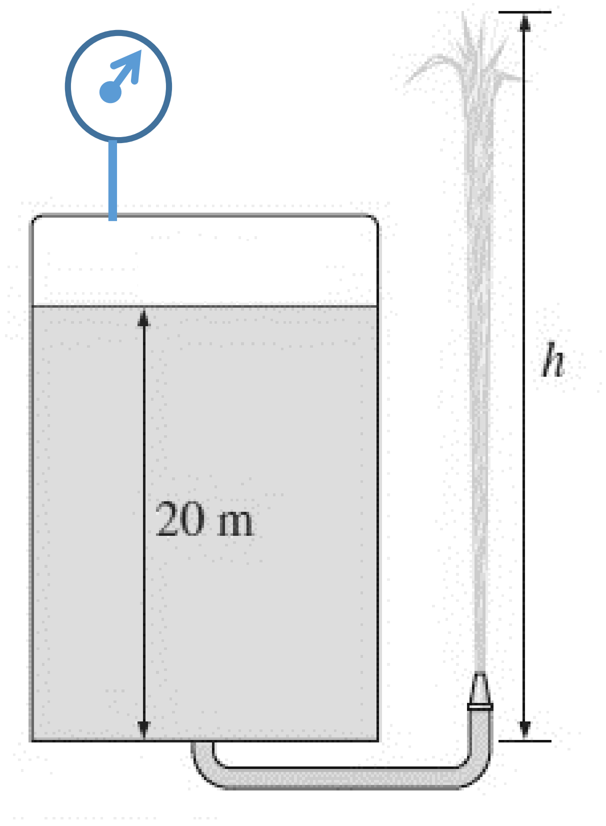
$$\dot{m}_1 \cdot V_1 \cdot \sin(\theta_1) \cdot \beta_1 - \dot{m}_2 \cdot V_2 \cdot \sin(\theta_2) \cdot \beta_2 - \dot{m}_3 \cdot V_3 \cdot \sin(\theta_3) \cdot \beta_3 - (p_1 - p_o) \cdot A_1 \cdot \sin(\phi_1) - (p_2 - p_o) \cdot A_2 \cdot \sin(\phi_2) - (p_3 - p_o) \cdot A_3 \cdot \sin(\phi_3) - F_{ext,z}$$

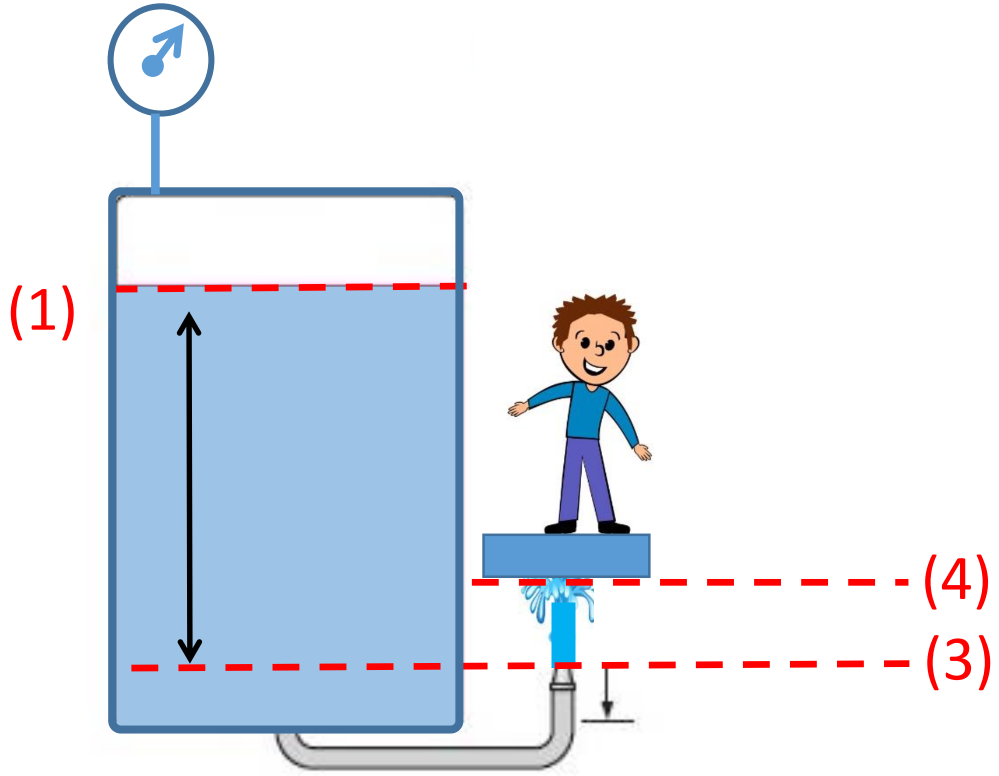
## Solution

$F_{ext,x} = 1126$ [N] $\quad F_{ext,z} = -93.54$ [N]
$p_o = 100000$ [Pa] $\quad \rho = 997$ [$\mathrm{kg/m^3}$]

| Row | $A_i$ [$\mathrm{m^2}$] | $D_i$ [m] | $\dot{m}_i$ [kg/s] | $p_i$ [Pa] | $v_i$ [m/s] | $\beta_i$ | $\phi_i$ [deg] | $\theta_i$ [deg] |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.001963 | 0.05 | 30 | 200000 | 15.32 | 1.03 | 180 | 0 |
| 2 | 0.007854 | 0.1 | 22 | 150000 | 2.81 | 1.03 | 180 | 180 |
| 3 | 0.0007069 | 0.03 | 8 | 100000 | 11.35 | 1.03 | 90 | 90 |

**Problem PW7.2** The water level in a tank is 20 m above the ground. A hose is connected to the bottom of the tank, and the nozzle at the end of the hose is pointed straight up. The tank cover is airtight, and the air pressure above the water surface is 2 atm gage. The system is at sea level. A) Determine the maximum height to which the water stream could rise. B) Assuming a boy and flat board with a weight of 70 kg, is standing just above the jet stream. Estimate the height at which the boy will be hovering. The diameter in the nozzle outlet is 1.5-in. C) Discuss or give a strategy to stabilize the kid in the plate to prevent accidents



## Equations

$$p_o = 100\ [\text{kPa}] \cdot \left|1000\ \frac{\text{Pa}}{\text{kPa}}\right|$$

$$z_1 = 20\ [\text{m}]$$

$$z_2 = h$$

$$p_1 = 200\ [\text{kPa}] \cdot \left|1000\ \frac{\text{Pa}}{\text{kPa}}\right| + \text{p}_\text{o}$$

$$v_1 = 0\ [\text{m/s}]$$

$$\rho = 997\ [\text{kg/m}^3]$$

$$v_2 = 0\ [\text{m/s}]$$

$$g = 9.80665\ [\text{m/s}^2]$$

$$p_2 = p_o$$

$$\frac{p_1}{\rho} + g \cdot z_1 + \frac{v_1^2}{2} = \frac{p_2}{\rho} + g \cdot z_2 + \frac{v_2^2}{2}$$

$$\frac{p_1}{\rho} + g \cdot z_1 + \frac{v_1^2}{2} = \frac{p_3}{\rho} + g \cdot z_3 + \frac{v_3^2}{2}$$

$$p_3 = p_o$$

$$z_3 = 0 \ [\mathrm{m}]$$

$$A_3 = \pi \cdot \frac{D_3^2}{4}$$

$$D_3 = 3.81 \times 10^{-2} \ [\mathrm{m}]$$

$$p_4 = p_3$$

$$\frac{p_3}{\rho} + g \cdot z_3 + \frac{v_3^2}{2} = \frac{p_4}{\rho} + g \cdot z_4 + \frac{v_4^2}{2}$$

Assuming a boy with a flat base with a weight of 70 kg

Is standing just above the jet stream

Estimate the height at which the boy will be hovering

Assuming the diameter in the nozzle outlet is 1.5 in

$$m = 70 \ [\mathrm{kg}]$$

$$-m \cdot g + \dot{m} \cdot V_4 = 0$$

$$\dot{m} = \rho \cdot V_3 \cdot A_3$$

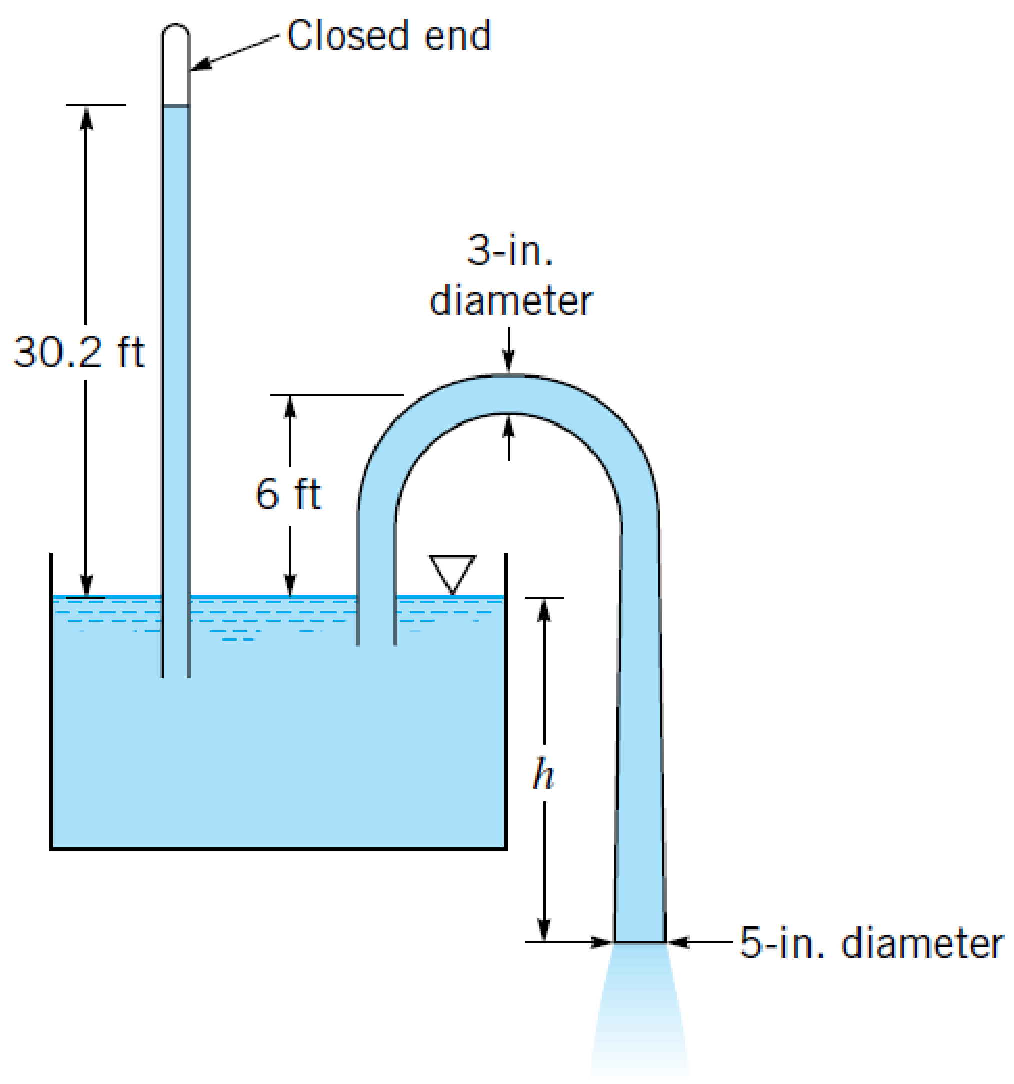
# Solution

$$g = 9.807\ [\mathrm{m/s^2}] \qquad h = 40.46\ [\mathrm{m}]$$
$$m = 70\ [\mathrm{kg}] \qquad \dot{m} = 32.02\ [\mathrm{kg/s}]$$
$$p_o = 100000\ [\mathrm{Pa}] \qquad \rho = 997\ [\mathrm{kg/m^3}]$$

| Row | $p_i$ [Pa] | $v_i$ [m/s] | $z_i$ [m] | $A_i$ [m$^2$] | $D_i$ [m] |
|---|---|---|---|---|---|
| 1 | 300000 | 0 | 20 | | |
| 2 | 100000 | 0 | 40.46 | | |
| 3 | 100000 | 28.17 | 0 | 0.00114 | 0.0381 |
| 4 | 100000 | 21.44 | 17.02 | | |

**Problem W7.11** Water is siphoned from the tank shown. The water barometer indicates a Reading of 30.2 ft. Determine:

a) the flow rate if *h*=2 ft.
b) The maximum value of *h* and the flow rate if you are close to the limit of operation. The limit of operation is the condition when the lower pressure in the fluid line will reach the vapor pressure at one particular position, this condition may produce cavitation.



$$h_w = 30.2\ [\mathrm{ft}] \cdot \left| 0.3048\ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

$$h_c = 6\ [\mathrm{ft}] \cdot \left| 0.3048\ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

$$D_3 = 3\ [\mathrm{in}] \cdot \left| 0.0254\ \frac{\mathrm{m}}{\mathrm{in}} \right|$$

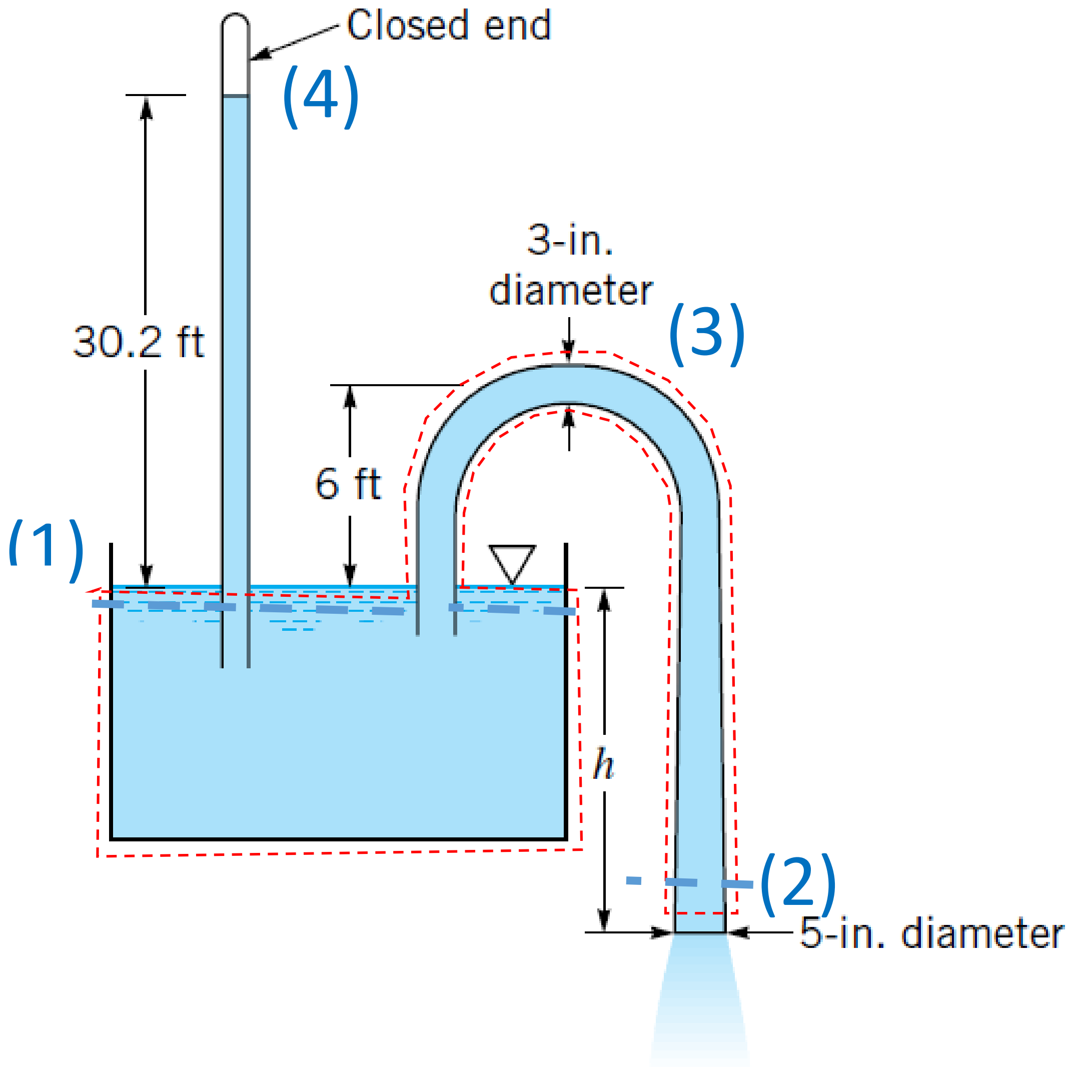
$$D_2 = 5\ [\mathrm{in}] \cdot \left| 0.0254\ \frac{\mathrm{m}}{\mathrm{in}} \right|$$

$D_1 = 1\ [\mathrm{m}]$ Assumption, lets reflect about this assumption

$p_o = 101325\ [\mathrm{Pa}]$ Assumption, sea level

$\rho = 997\ [\mathrm{kg/m^3}]$

$g = 9.81\ [\mathrm{m/s^2}]$



$$p_4 + \rho \cdot g \cdot h_w = p_1$$

$$p_1 = p_o$$

$$p_4 = p_{w,vapor}$$

$$A_1 = \pi \cdot \frac{D_1^2}{4}$$

$$A_2 = \pi \cdot \frac{D_2^2}{4}$$

$$A_3 = \pi \cdot \frac{D_3^2}{4}$$

$$A_1 \cdot V_1 \cdot \rho = A_2 \cdot V_2 \cdot \rho$$

$$A_3 \cdot V_3 \cdot \rho = A_2 \cdot V_2 \cdot \rho$$

$$Z_2 = 0 \ \ [\mathrm{m}]$$

$$z_3 = h + h_c$$

$$Z_1 = h$$

$$z_4 = h + h_w$$

$$p_2 = p_o$$

$$\frac{p_1}{\rho} + \frac{v_1^2}{2} + g \cdot z_1 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g \cdot z_2$$

$$\frac{p_3}{\rho} + \frac{v_3^2}{2} + g \cdot z_3 = \frac{p_2}{\rho} + \frac{v_2^2}{2} + g \cdot z_2$$

Constraint:

$$p_3 = p_{w,vapor}$$

$$\dot{m} = \dot{v} \cdot \rho$$

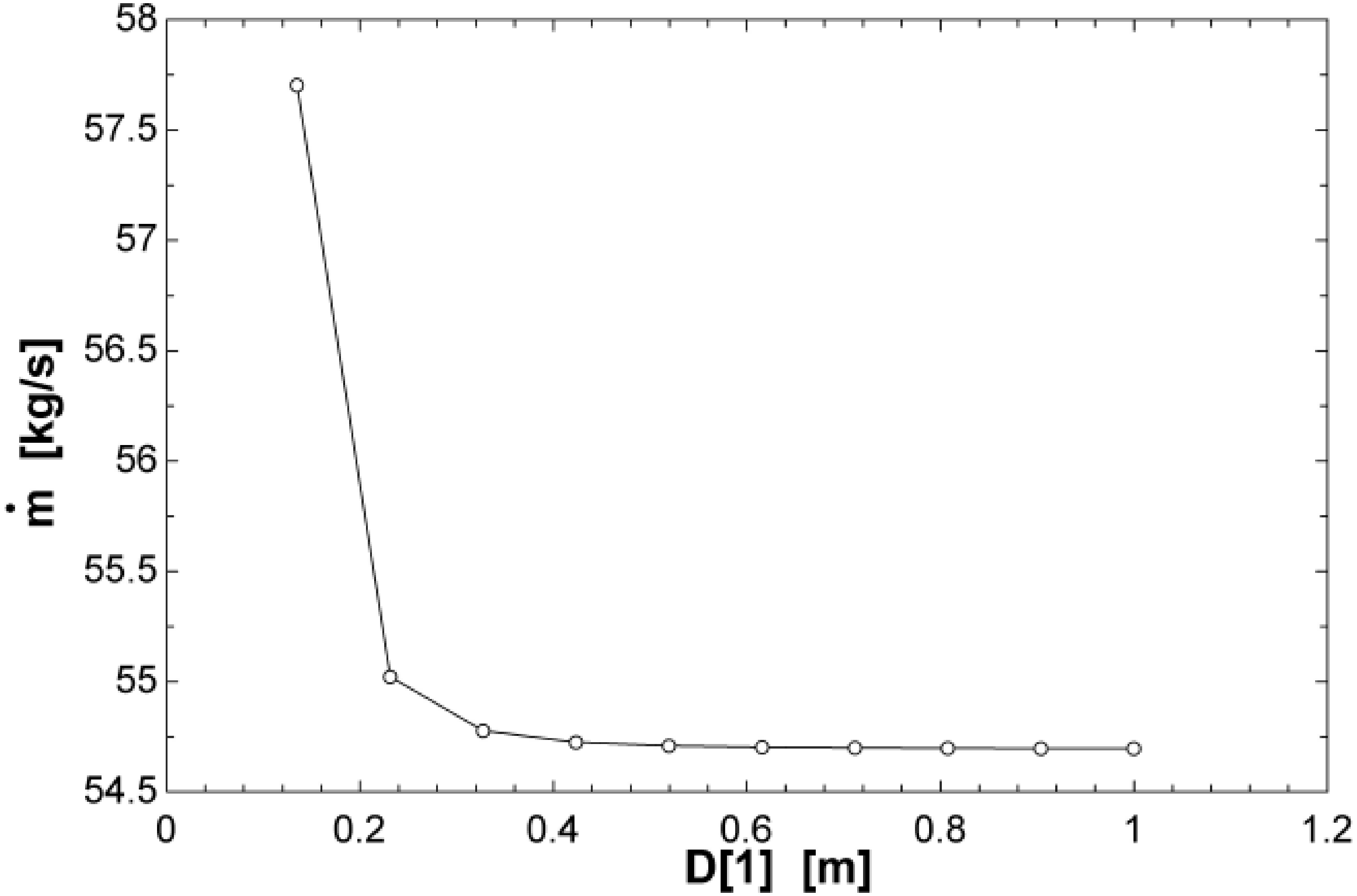
$$\dot{v} = A_2 \cdot V_2$$

## Solution

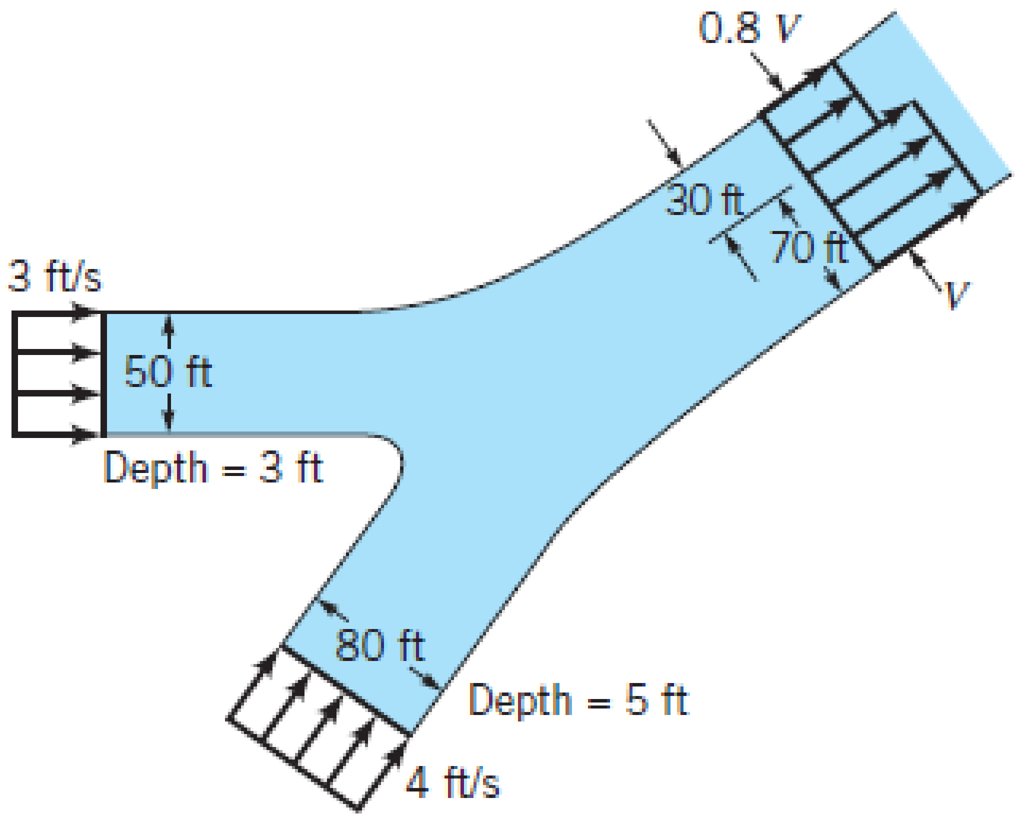
$$g = 9.81 \ [\mathrm{m/s^2}] \qquad h = 0.9557 \ [\mathrm{m}]$$
$$h_c = 1.829 \ [\mathrm{m}] \qquad h_w = 9.205 \ [\mathrm{m}]$$
$$\dot{m} = 54.7 \ [\mathrm{kg/s}] \qquad p_o = 101325 \ [\mathrm{Pa}]$$
$$p_{w,vapor} = 11295 \ [\mathrm{Pa}] \qquad \rho = 997 \ [\mathrm{kg/m^3}]$$
$$\dot{v} = 0.05486 \ [\mathrm{m^3/s}]$$

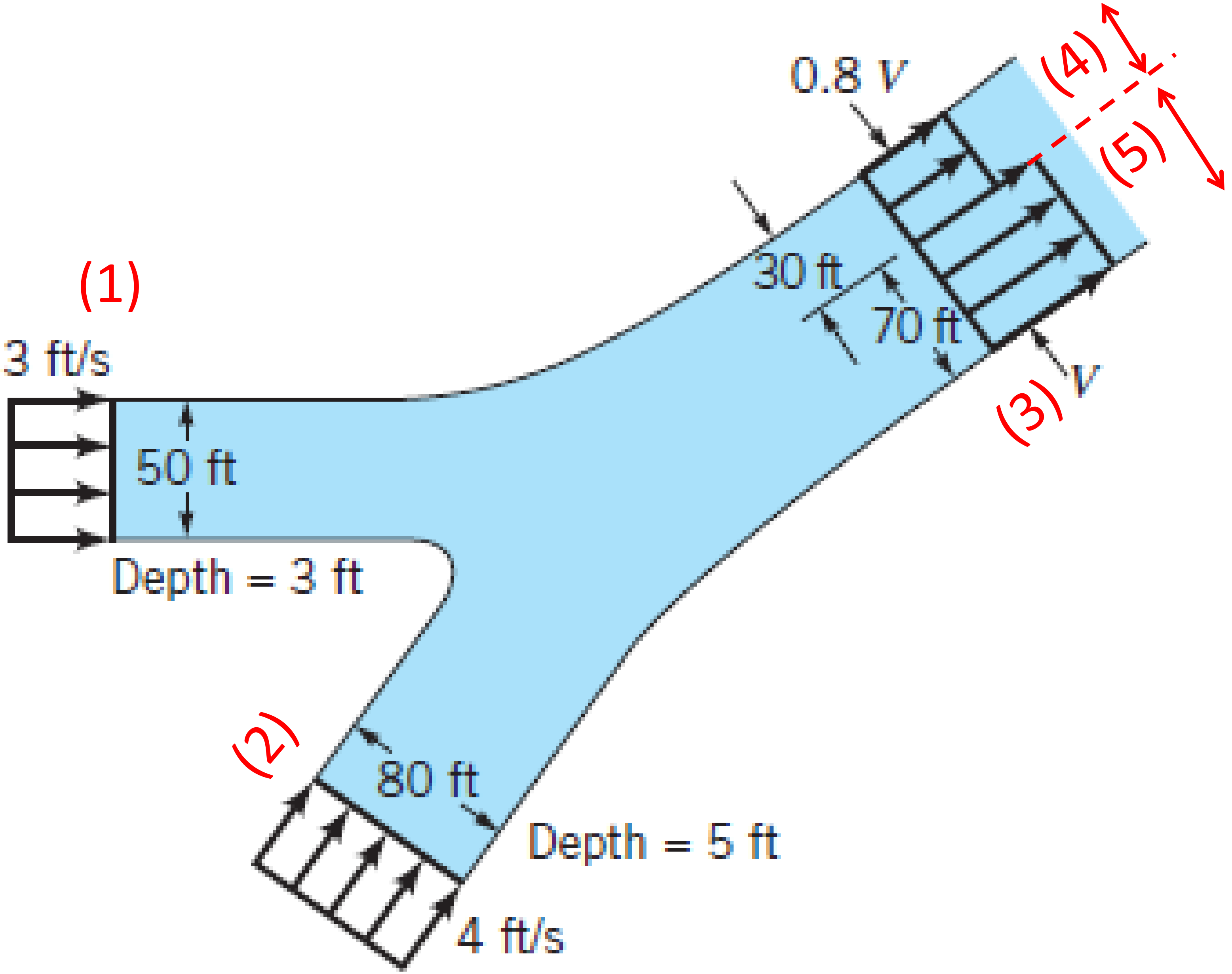
| Row | $D_i$ [m] | $p_i$ [Pa] | $A_i$ [$\mathrm{m^2}$] | $V_i$ [m/s] | $Z_i$ [m] |
|---|---|---|---|---|---|
| 1 | 1 | 101325 | 0.7854 | 0.06985 | 0.9557 |
| 2 | 0.127 | 101325 | 0.01267 | 4.331 | 0 |
| 3 | 0.0762 | 11295 | 0.00456 | 12.03 | 2.785 |
| 4 | | 11295 | | | 10.16 |

What if ?
The problem was solved under the assumption that the diameter of the tank is big, compared with the diameter of the tube. Is this assumption valid ? How much the answer changes, if this assumption is no valid.



**Problem W7.9** Two rivers merge to form a larger river shown in figure. At a location downstream from the junction (before the two streams completely merge), the non-uniform velocity profile is as shown and the depth is 6 ft. Determine the value of *V*





0.8 V
(4)
(5)
30 ft
70 ft
(1)
3 ft/s
V
(3)
50 ft
Depth = 3 ft
(2)
80 ft
Depth = 5 ft
4 ft/s

$V_1 = 3 \; [\text{ft/s}]$

$D_1 = 3 \; [\text{ft}]$

$W_1 = 50 \; [\text{ft}]$

$V_2 = 4 \; [\text{ft/s}]$

$D_2 = 5 \; [\text{ft}]$

$W_2 = 80 \; [\text{ft}]$

$W_4 = 30 \; [\text{ft}]$

$D_4 = 6 \; [\text{ft}]$

$W_5 = 70 \; [\text{ft}]$

$D_5 = D_4$

$V_4 = 0.8 \cdot U$

$V_5 = U$

$$V_1 \cdot A_1 + V_2 \cdot A_2 = V_4 \cdot A_4 + V_5 \cdot A_5$$

Assumption: Incompressible fluid, so volumetic flow rate
balance can be used as a meassure of the mass flow rate

$A_1 = D_1 \cdot W_1$

$A_2 = D_2 \cdot W_2$

$A_4 = D_4 \cdot W_4$

$A_5 = D_5 \cdot W_5$

$V_1 \cdot A_1 + V_2 \cdot A_2 = V_3 \cdot A_3$

$A_3 = (W_4 + w_5) \cdot D_4$

## Solution

$$U = 3.635 \text{ [ft/s]}$$

| Row | $A_i$ [$\text{ft}^2$] | $D_i$ [ft] | $V_i$ [ft/s] | $W_i$ [ft] |
|---|---|---|---|---|
| 1 | 150 | 3 | 3 | 50 |
| 2 | 400 | 5 | 4 | 80 |
| 3 | 600 | | 3.417 | |
| 4 | 180 | 6 | 2.908 | 30 |
| 5 | 420 | 6 | 3.635 | 70 |

**Problem W7.10** A hypodermic syringe is used to apply a vaccine. If the plunger is moved forward at the steady rate of 20 mm/s and if the vaccine leaks past the plunger at 0.1 of the volume flowrate out the needle opening, calculate the average velocity of the needle exit flow. The inside diameters of the syringe and the needle are 20 mm and 0.7 mm

# Equations

$$D_{Syringe} = 20 \times 10^{-3} \ \ [\mathrm{m}]$$

$$D_{Needle} = 0.7 \times 10^{-3} \ \ [\mathrm{m}]$$

$$A_{Syringe} = \pi \cdot \frac{D_{Syringe}^2}{4}$$

$$A_{Needle} = \pi \cdot \frac{D_{Needle}^2}{4}$$

$$dL/dt = -20 \times 10^{-3} \ \ [\mathrm{m/s}]$$ Displacement rate of the piston

$\alpha = 0.1$ Leaked to injected ratio

Assumption: Incompressible fluid

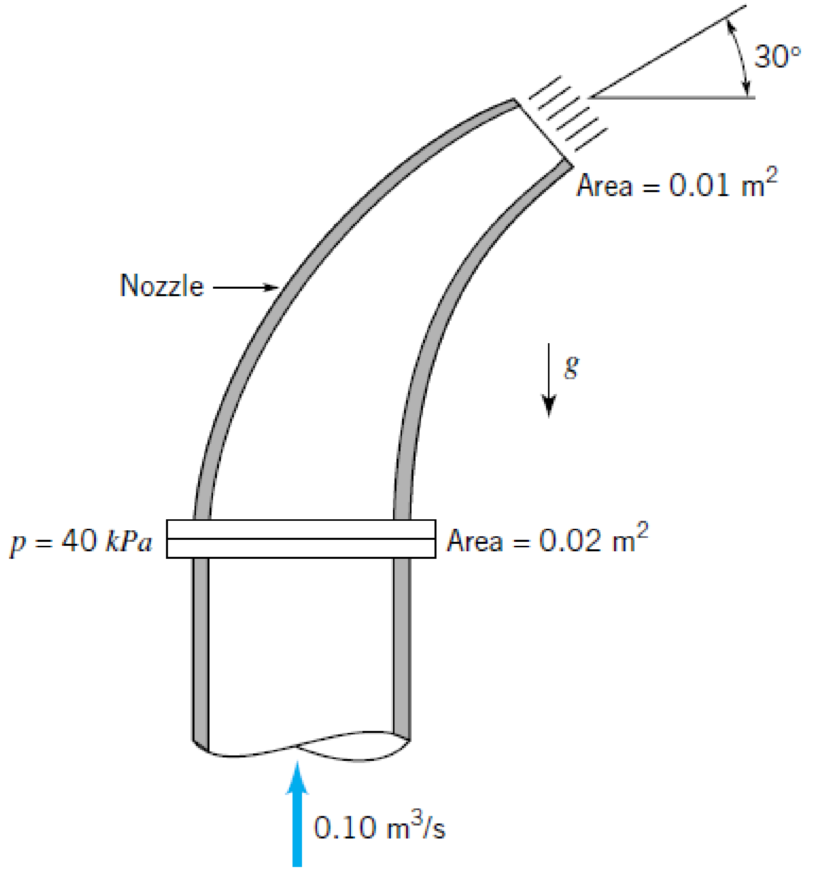
$A_{Syringe} \cdot dLdt = -\alpha \cdot \dot{V}_{Needle} - \dot{V}_{Needle}$ Mass balance in terms of volume

$\dot{V}_{Needle} = U_{Needle} \cdot A_{Needle}$

## Solution

| | |
|---|---|
| $\alpha = 0.1$ | $A_{Needle} = 3.848 \times 10^{-7}$ [m$^2$] |
| $A_{Syringe} = 0.0003142$ [m$^2$] | $dLdt = -0.02$ [m/s] |
| $D_{Needle} = 0.0007$ [m] | $D_{Syringe} = 0.02$ [m] |
| $U_{Needle} = 14.84$ [m/s] | $\dot{V}_{Needle} = 5.712 \times 10^{-6}$ [m$^3$/s] |

**Problem W7.12** A nozzle is attached to a vertical pipe and discharges water into the atmosphere as shown in figure. When the discharge is 0.1 $m^3/s$, the gage pressure at the flange is 40 kPa. Determine the vertical component of the anchoring force required to hold the nozzle in place. The nozzle has a weight of 200 N, and the volume of water in the nozzle is 0.012 $m^3$. Is the anchoring force directed upward or downward ?



$$A_1 = 0.02 \ \ [\mathrm{m}^2]$$

$$A_2 = 0.01 \ \ [\mathrm{m}^2]$$

$$p_{g,1} = 40000 \ \ [\mathrm{Pa}]$$

$$p_{g,2} = 0 \ \ [\mathrm{Pa}]$$

Direction of velocity and normal vectors

$$\theta_1 = 90 \ \ [\mathrm{deg}]$$

$$\phi_1 = 270 \ \ [\mathrm{deg}]$$

$$\theta_2 = 30 \ \ [\mathrm{deg}]$$

$$\phi_2 = 30 \ \ [\mathrm{deg}]$$

$$\dot{V} = 0.1 \ \ [\mathrm{m}^3/\mathrm{s}]$$

$$W_{Nozzle} = 200 \ \ [\mathrm{N}]$$

$$V_{water} = 0.012 \ \ [\mathrm{m}^3]$$

Step 3:Physical constants

$$g = 9.80665 \ [\mathrm{m/s^2}]$$

Step 4: Physical and/or transport properties

$$\rho = 997 \ [\mathrm{kg/m^3}]$$ By heart

$$p_o = 101325 \ [\mathrm{Pa}]$$

$$p_1 = p_{g,1} + p_o$$

$$p_2 = p_{g,2} + p_o$$

$$\dot{m} = \rho \cdot \dot{V}$$

Assumption: Steady Flow: No accumulation

$$\dot{V} = v_1 \cdot A_1$$

$$\dot{V} = v_2 \cdot A_2$$

Force balance y

$$\dot{m}\cdot v_1\cdot \sin\left(\theta_1\right)-\dot{m}\cdot v_2\cdot \sin\left(\theta_2\right)-\left(p_1-p_o\right)\cdot A_1\cdot \sin\left(\phi_1\right)-\left(p_2-p_o\right)\cdot A_2\cdot \sin\left(\phi_2\right)-W_{Nozzle}-\rho\cdot g\cdot V_{water}+F_{ext,y}=0$$

Linear momentun in x-axis

$$\dot{m}\cdot v_1\cdot \mathrm{Cos}(\theta_1)-\dot{m}\cdot v_2\cdot \mathrm{Cos}(\theta_2)-(p_1-p_o)\cdot A_1\cdot \mathrm{Cos}(\phi_1)-(p_2-p_o)\cdot A_2\cdot \mathrm{Cos}(\phi_2)+F_{ext,x}=0$$

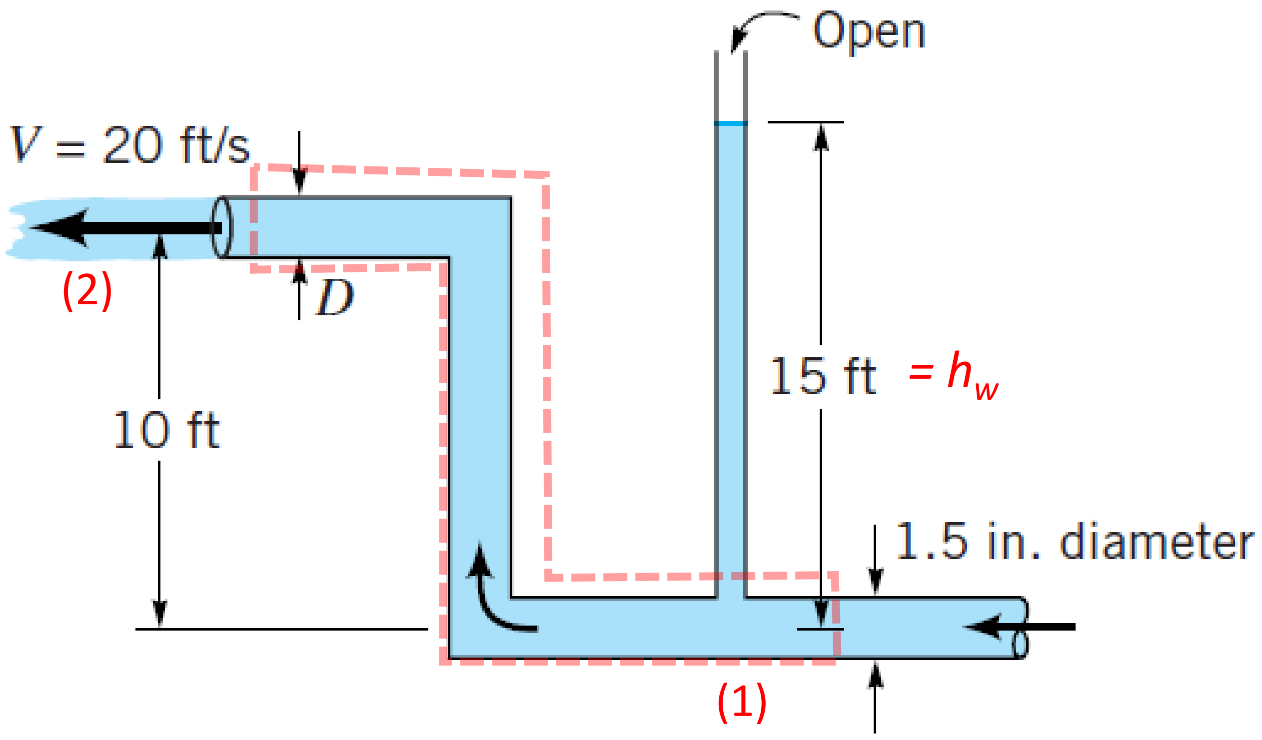
$$F_{ext}=\sqrt{F_{ext,x}^2+F_{ext,y}^2}$$

## Solution

$F_{ext}=989.2$ [N]
$F_{ext,y}=-482.7$ [N]
$\dot{m}=99.7$ [kg/s]
$\rho=997$ $[\mathrm{kg/m^3}]$
$V_{water}=0.012$ $[\mathrm{m^3}]$
$F_{ext,x}=863.4$ [N]
$g=9.807$ $[\mathrm{m/s^2}]$
$p_o=101325$ [Pa]
$\dot{V}=0.1$ $[\mathrm{m^3/s}]$
$W_{Nozzle}=200$ [N]

| Row | $A_i$ $[\mathrm{m^2}]$ | $\phi_i$ [deg] | $p_{g,i}$ [Pa] | $\theta_i$ [deg] | $p_i$ [Pa] | $v_i$ [m/s] |
|---|---|---|---|---|---|---|
| 1 | 0.02 | 270 | 40000 | 90 | 141325 | 5 |
| 2 | 0.01 | 30 | 0 | 30 | 101325 | 10 |

**Problem W7.13** Water flows steadily with negligible viscous effects through the pipe shown in figure. Determine the diameter, *D*, of the pipe at the outlet (a free jet) if the velocity there is 20 ft/s.



$$D_1 = 1.5 \ [\mathrm{in}] \cdot \left| 0.0254 \ \frac{\mathrm{m}}{\mathrm{in}} \right|$$

$$A_1 = \pi \cdot \frac{D_1^2}{4}$$

$$A_2 = \pi \cdot \frac{D_2^2}{4}$$

$$V_2 = 20 \ [\mathrm{ft/s}] \cdot \left| 0.3048 \ \frac{\mathrm{m/s}}{\mathrm{ft/s}} \right|$$

$$V_1 \cdot A_1 = V_2 \cdot A_2$$

$$p_o = 101325 \ [\mathrm{Pa}]$$

$$\rho = 997 \ \left[\mathrm{kg/m^3}\right]$$

$$g = 9.80665 \ \left[\mathrm{m/s^2}\right]$$

$$h_w = 15 \ [\mathrm{ft}] \cdot \left| 0.3048 \ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

$$p_2 = p_o$$

$$p_1 = p_o + \rho \cdot g \cdot h_w$$

$$z_1 = 0 \ [\mathrm{m}]$$

$$z_2 = 10 \ [\mathrm{ft}] \cdot \left| 0.3048 \ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

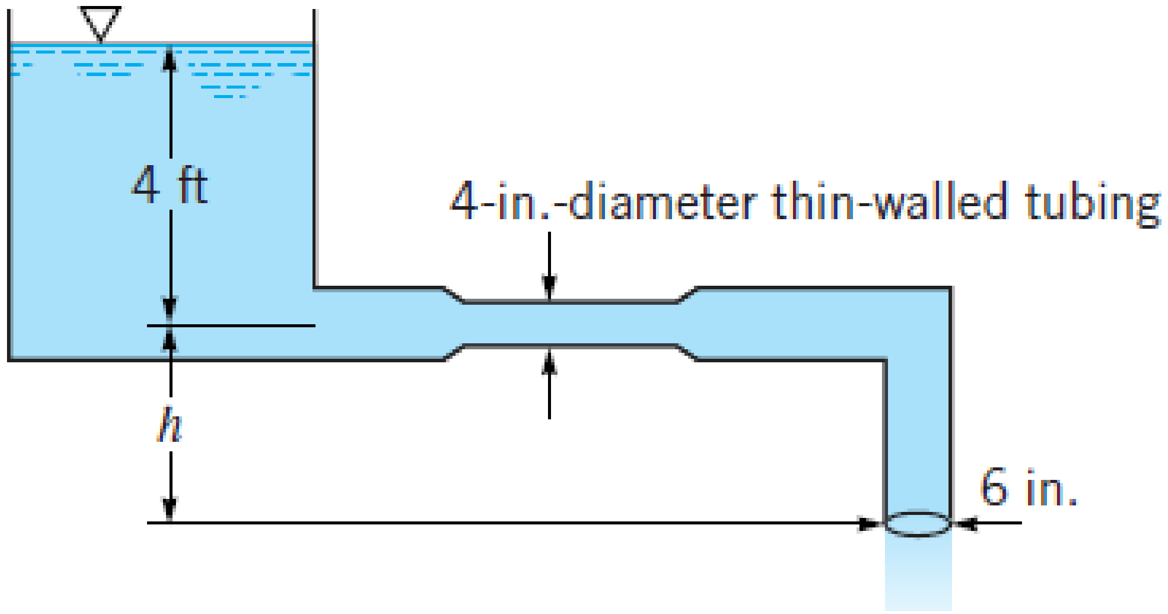
$$\frac{p_1}{\rho} + g \cdot z_1 + \frac{v_1^2}{2} = \frac{p_2}{\rho} + g \cdot z_2 + \frac{v_2^2}{2}$$

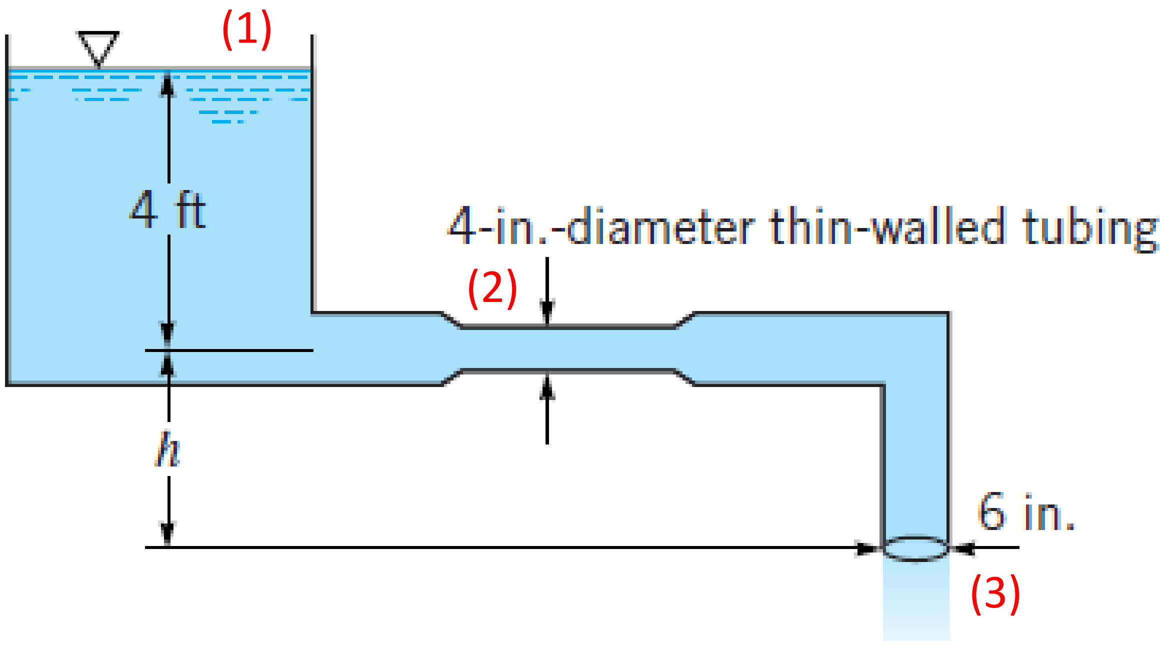
---

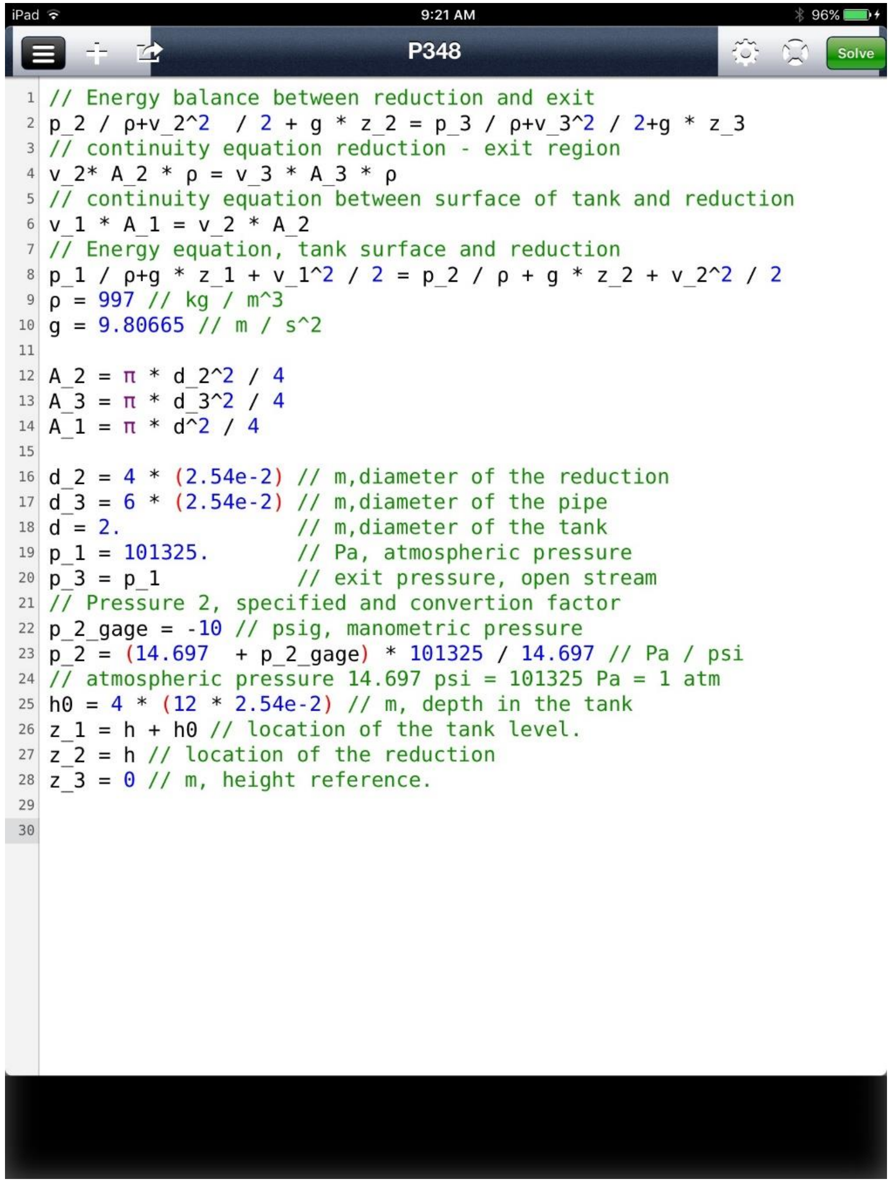
$g = 9.807 \ [\mathrm{m/s^2}]$ $h_w = 4.572 \ [\mathrm{m}]$
$p_o = 101325 \ [\mathrm{Pa}]$ $\rho = 997 \ [\mathrm{kg/m^3}]$

| Row | $A_i$ [$\mathrm{m^2}$] | $D_i$ [m] | $p_i$ [Pa] | $V_i$ [m/s] | $z_i$ [m] |
|---|---|---|---|---|---|
| 1 | 0.00114 | 0.0381 | 146026 | 2.696 | 0 |
| 2 | 0.0005043 | 0.02534 | 101325 | 6.096 | 3.048 |

**Problem W7.14** Water flows steadily with negligible viscous effects through the pipe shown in the figure. It is known that the 4-in.-diameter section of thin-walled tubing will collapse if the pressure within it becomes less than 10 psi below the atmospheric pressure. Determine the maximum value of that ***h*** can have without causing collapse of the tubing.



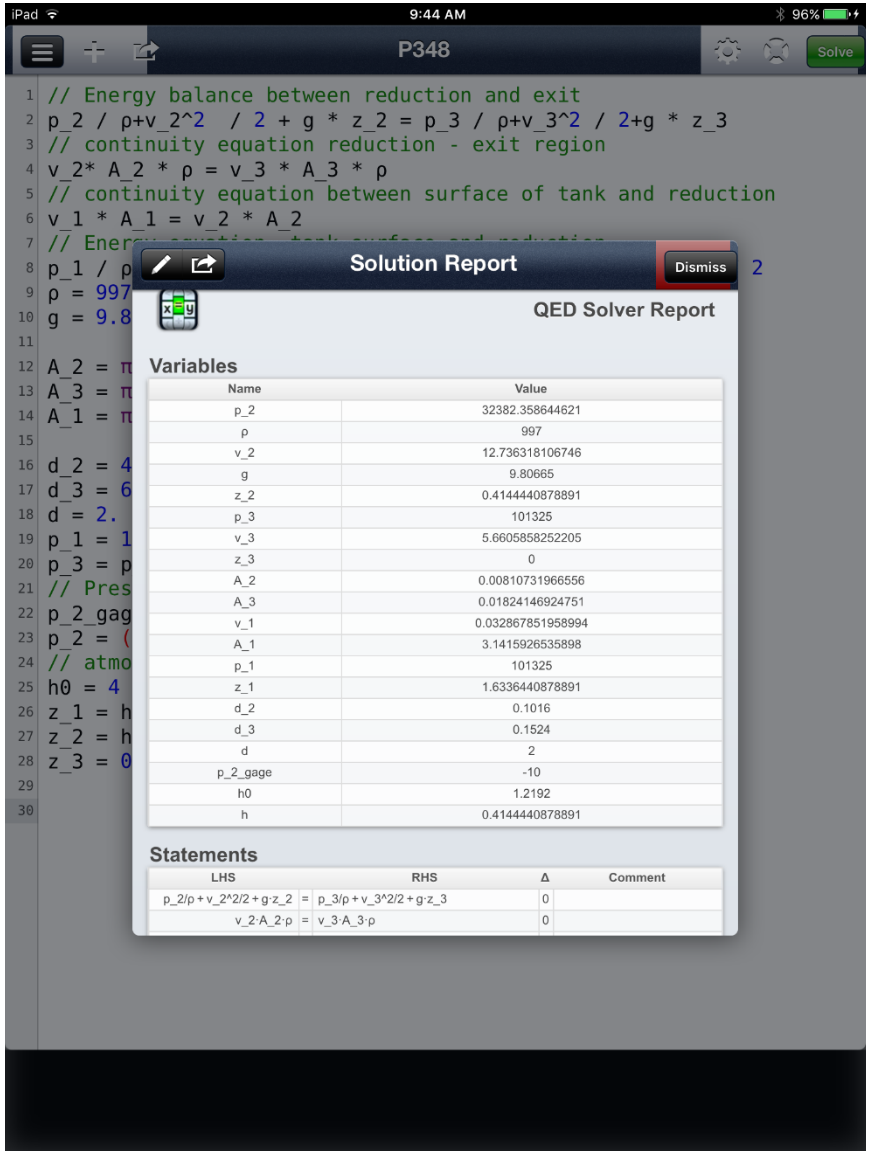

(1)
4 ft
4-in.-diameter thin-walled tubing
(2)
h
6 in.
(3)




```
// Energy balance between reduction and exit
p_2 / ρ+v_2^2  / 2 + g * z_2 = p_3 / ρ+v_3^2 / 2+g * z_3
// continuity equation reduction - exit region
v_2* A_2 * ρ = v_3 * A_3 * ρ
// continuity equation between surface of tank and reduction
v_1 * A_1 = v_2 * A_2
// Energy equation, tank surface and reduction
p_1 / ρ+g * z_1 + v_1^2 / 2 = p_2 / ρ + g * z_2 + v_2^2 / 2
ρ = 997 // kg / m^3
g = 9.80665 // m / s^2

A_2 = π * d_2^2 / 4
A_3 = π * d_3^2 / 4
A_1 = π * d^2 / 4

d_2 = 4 * (2.54e-2) // m,diameter of the reduction
d_3 = 6 * (2.54e-2) // m,diameter of the pipe
d = 2.              // m,diameter of the tank
p_1 = 101325.       // Pa, atmospheric pressure
p_3 = p_1           // exit pressure, open stream
// Pressure 2, specified and convertion factor
p_2_gage = -10 // psig, manometric pressure
p_2 = (14.697  + p_2_gage) * 101325 / 14.697 // Pa / psi
// atmospheric pressure 14.697 psi = 101325 Pa = 1 atm
h0 = 4 * (12 * 2.54e-2) // m, depth in the tank
z_1 = h + h0 // location of the tank level.
z_2 = h // location of the reduction
z_3 = 0 // m, height reference.
```

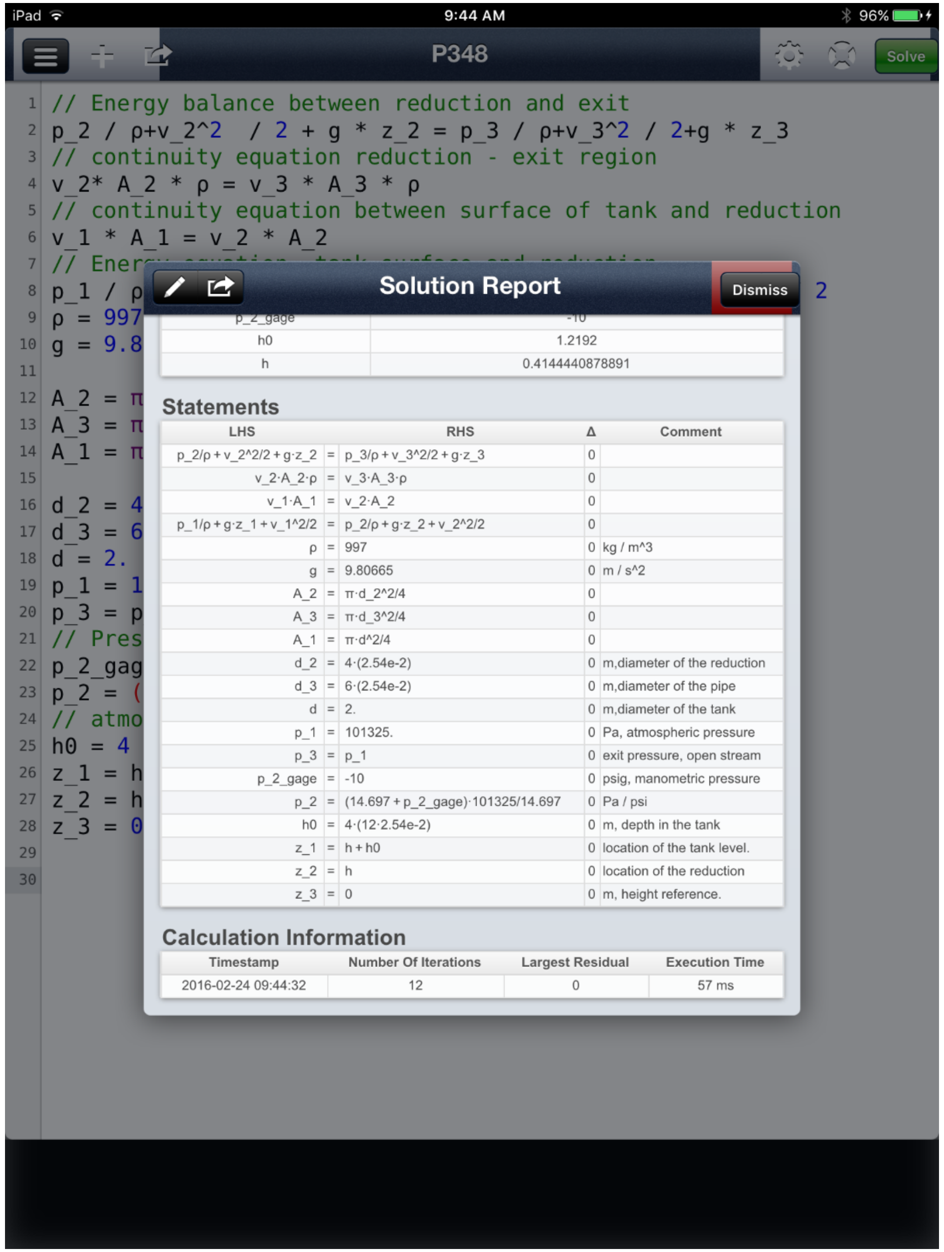



### Variables

| Name | Value |
|---|---|
| p_2 | 32382.358644621 |
| ρ | 997 |
| v_2 | 12.736318106746 |
| g | 9.80665 |
| z_2 | 0.4144440878891 |
| p_3 | 101325 |
| v_3 | 5.6605858252205 |
| z_3 | 0 |
| A_2 | 0.00810731966556 |
| A_3 | 0.01824146924751 |
| v_1 | 0.032867851958994 |
| A_1 | 3.1415926535898 |
| p_1 | 101325 |
| z_1 | 1.6336440878891 |
| d_2 | 0.1016 |
| d_3 | 0.1524 |
| d | 2 |
| p_2_gage | -10 |
| h0 | 1.2192 |
| h | 0.4144440878891 |

### Statements

| LHS | | RHS | Δ | Comment |
|---|---|---|---|---|
| p_2/ρ + v_2^2/2 + g·z_2 | = | p_3/ρ + v_3^2/2 + g·z_3 | 0 | |
| v_2·A_2·ρ | = | v_3·A_3·ρ | 0 | |



iPad 9:44 AM 96%

P348

Solve

```
// Energy balance between reduction and exit
p_2 / ρ+v_2^2  / 2 + g * z_2 = p_3 / ρ+v_3^2 / 2+g * z_3
// continuity equation reduction - exit region
v_2* A_2 * ρ = v_3 * A_3 * ρ
// continuity equation between surface of tank and reduction
v_1 * A_1 = v_2 * A_2
```

**Solution Report**

Dismiss

| | |
|---|---|
| p_2_gage | -10 |
| h0 | 1.2192 |
| h | 0.4144440878891 |

## Statements

| LHS | | RHS | Δ | Comment |
|---|---|---|---|---|
| p_2/ρ + v_2^2/2 + g·z_2 | = | p_3/ρ + v_3^2/2 + g·z_3 | 0 | |
| v_2·A_2·ρ | = | v_3·A_3·ρ | 0 | |
| v_1·A_1 | = | v_2·A_2 | 0 | |
| p_1/ρ + g·z_1 + v_1^2/2 | = | p_2/ρ + g·z_2 + v_2^2/2 | 0 | |
| ρ | = | 997 | 0 | kg / m^3 |
| g | = | 9.80665 | 0 | m / s^2 |
| A_2 | = | π·d_2^2/4 | 0 | |
| A_3 | = | π·d_3^2/4 | 0 | |
| A_1 | = | π·d^2/4 | 0 | |
| d_2 | = | 4·(2.54e-2) | 0 | m,diameter of the reduction |
| d_3 | = | 6·(2.54e-2) | 0 | m,diameter of the pipe |
| d | = | 2. | 0 | m,diameter of the tank |
| p_1 | = | 101325. | 0 | Pa, atmospheric pressure |
| p_3 | = | p_1 | 0 | exit pressure, open stream |
| p_2_gage | = | -10 | 0 | psig, manometric pressure |
| p_2 | = | (14.697 + p_2_gage)·101325/14.697 | 0 | Pa / psi |
| h0 | = | 4·(12·2.54e-2) | 0 | m, depth in the tank |
| z_1 | = | h + h0 | 0 | location of the tank level. |
| z_2 | = | h | 0 | location of the reduction |
| z_3 | = | 0 | 0 | m, height reference. |

## Calculation Information

| Timestamp | Number Of Iterations | Largest Residual | Execution Time |
|---|---|---|---|
| 2016-02-24 09:44:32 | 12 | 0 | 57 ms |

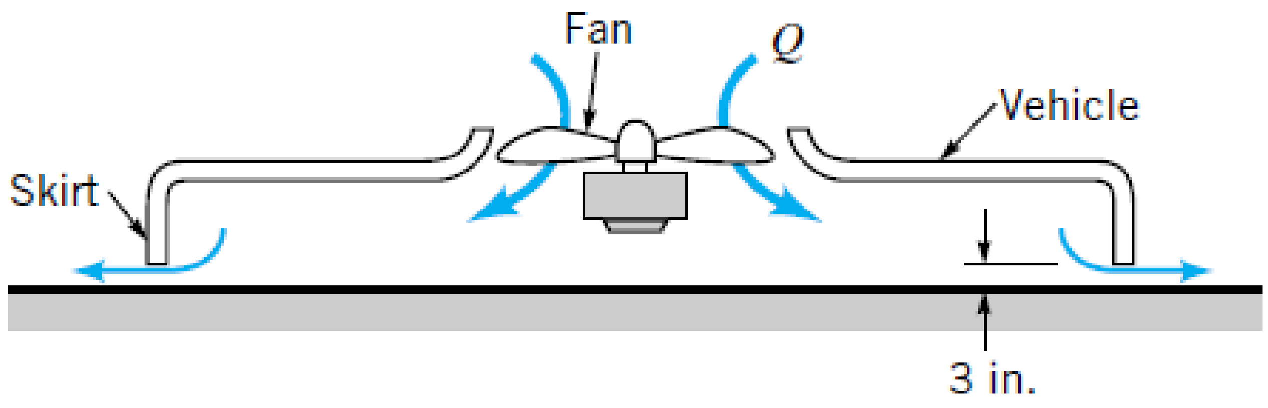
**Problem W7.15** An air cushion vehicle is supported by forcing air into the chamber created by a skirt around the periphery of the vehicle as shown in the figure. The air escapes through the 3-in. clearance between the lower end of the skirt and the ground (or water). Assume the vehicle weights 10,000 lb and is essentially rectangular in shape, 30 by 50 ft. The volume of the chamber is large enough so that the kinetic energy of the air within the chamber is negligible.

a)Determine the flowrate, $\boldsymbol{Q}$, needed to support the vehicle.

b) If the ground clearance were reduced to 2 in., what flowrate would be needed?

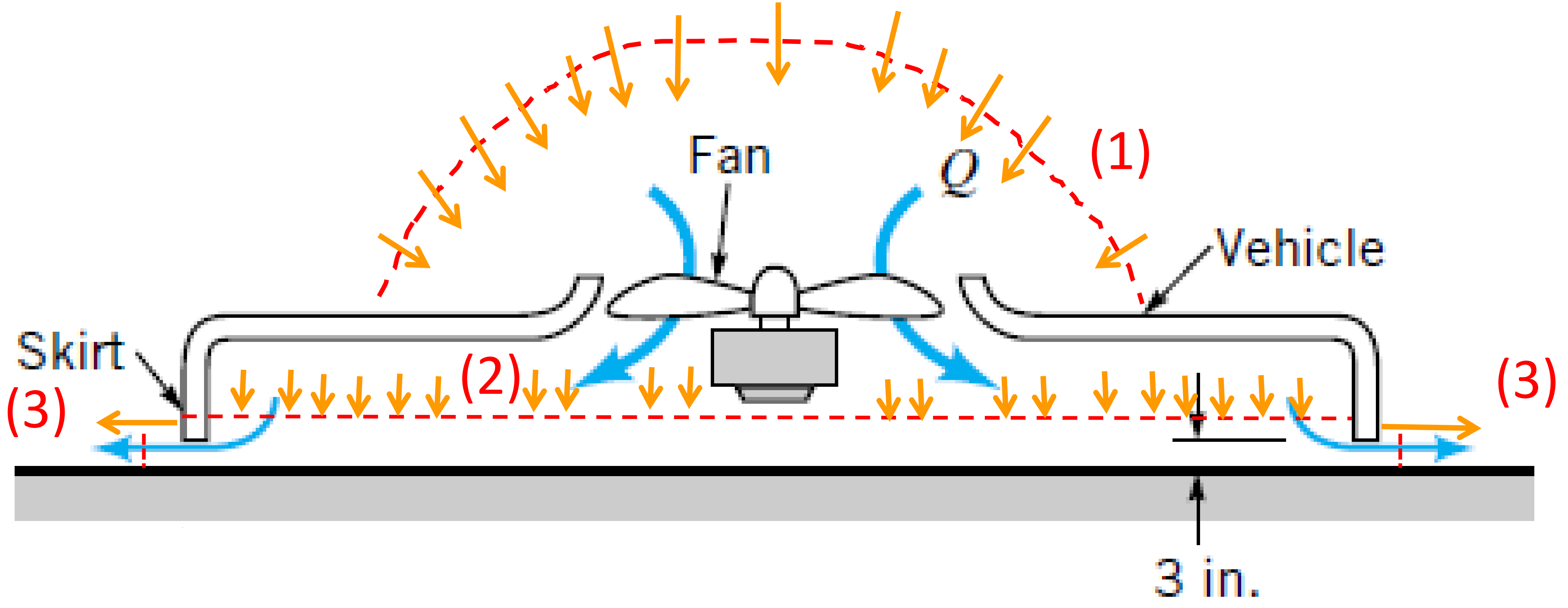
c) If the vehicle weight were reduced to 5,000 lb and the ground clearance maintained at 3 in, what flowrate would be needed ?

d) Calculate the power required by the fan under the three previous scenarios




Some engineers use " $Q$" as a symbol for volume flow rate

$$\dot{V} = Q$$


Fan
Q
(1)
Vehicle
Skirt
(2)
(3)
(3)
3 in.

1. Read problem statement, and collect the information that may be needed.
2. Make a Sketch (Diagram, process flow chart), indicating mass, linear or angular momentum (i.e. forces and torques) and energy interaction, and label each stream and boundaries as well.
3. List Assumptions and Approximations (sometimes they may be inferred by the sketch, but make them explicit) supported by equations if possible (geometric relationships, or fundamental equations).
4. Physical Laws (Fundamental Laws) must be written in full form, and terms can be dropped by the right selection of frame of reference, operating conditions, assumptions, simplifications or constraints.
5. Physical constants should be obtained from a reliable source (knowing this information by heart is always helpful ), geometric relations and formulae must be included as part of your analysis.
6. Physical transport or thermodynamic properties (Thermodynamic relations) should be evaluated, approximated, calculated or obtained from a reliable source.
7. Calculations are done including units. Any algebraic manipulation is recommended in few cases, because limits the step 8, but if needed should be done before using numerical values of constants, properties or variables.
8. Reasoning (Sensitivity analysis, what if), Verification (context), and Discussion should always be part of your answer to any problem, regardless the task requested.

Imagine that the inlet (1) is made for a hemispherical cap of radius *R*, and lets call the velocity upstream $v_R$

The differential of area, in spherical coordinates is:

$$dA = (R d\vartheta)(R\,Sin\vartheta\,d\varphi)$$

The mass flow rate is:

$$\dot{m} = \int_{\vartheta=0}^{\vartheta=\pi/2} \int_{\varphi=0}^{\varphi=2\pi} \rho\, v_R\, R\, d\vartheta\, (R \sin(\vartheta)\, d\varphi) = 2\,\pi\, R^2\, \rho\, v_R$$

(*) The linear momentum in "y-axis":

$$F_{y,in} = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi/2} \left[\rho v_R (R d\vartheta)(R\,Sin\vartheta\,d\varphi) v_R\right] Cos\vartheta = \pi\, R^2 \rho\, v_R^2$$

Kinetic energy flow is:

$$K = \int_{\vartheta=0}^{\vartheta=\pi/2} \int_{\phi=0}^{\phi=2\pi} \left[\rho v_R \left(v_R^2/2\right)(R d\vartheta)(R\,Sin\vartheta\,d\varphi)\right] = \pi\, R^2 \rho\, v_R^3$$

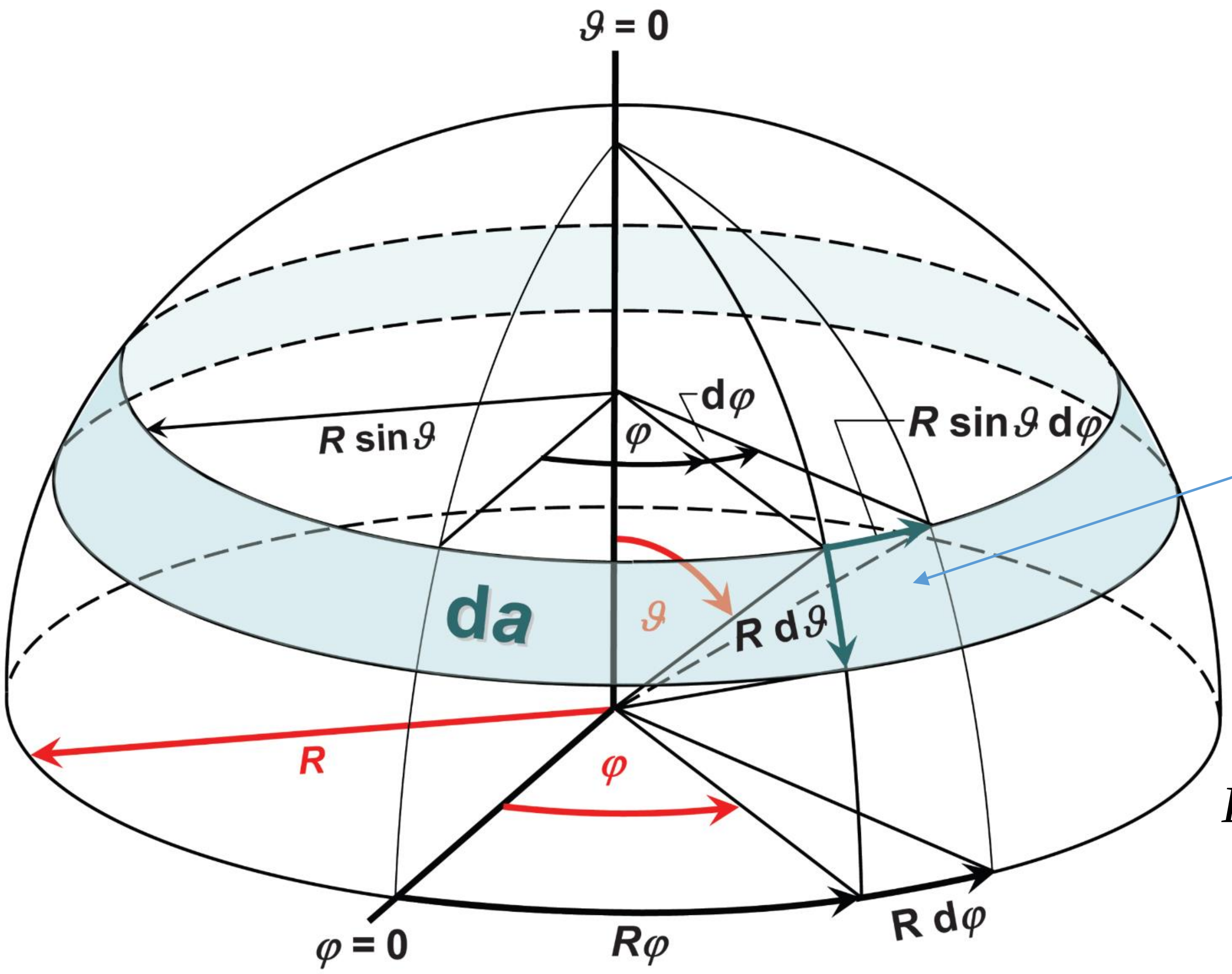
Potential energy flow is:

$$\Phi = \int_{\vartheta=0}^{\vartheta=\pi/2} \int_{\phi=0}^{\phi=2\pi} \left[\rho v_R (g\,R\,Cos\vartheta)(R d\vartheta)(R\,Sin\vartheta\,d\varphi)\right] = \pi\, R^3 \rho\, g\, v_R$$

flow energy flow is:

$$Flow = \int_{\vartheta=0}^{\vartheta=\pi/2} \int_{\phi=0}^{\phi=2\pi} \left[\rho v_R \left(\frac{p}{\rho}\right)(R d\vartheta)(R\,Sin\vartheta\,d\varphi)\right] = 2\,\pi\, R^2 p\, v_R$$

(* this problem is axis-symmetric, then the net forces for *x* and *z* axes are both zero)




Air entering in the hemisphere

$$dA = (R d\vartheta)(R\ Sin\,\vartheta\ d\varphi)$$

$$dF_y = \left[\rho v_1 (R d\vartheta)(R\ Sin\,\vartheta\ d\varphi) v_1\right] Cos\,\vartheta$$

$$F_{y,in} = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi/2} \left[\rho v_1 (R d\vartheta)(R\ Sin\,\vartheta\ d\varphi) v_1\right] Cos\,\vartheta$$

A sensitivity analysis is required to validate this assumption ( Sketching streamlines will help to justify this construct)

$$\rho = 1.18 \ [\mathrm{kg/m^3}]$$

$$p_o = 101325 \ [\mathrm{Pa}]$$

$$L_1 = 30 \ [\mathrm{ft}] \cdot \left| 0.3048 \ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

$$L_2 = 50 \ [\mathrm{ft}] \cdot \left| 0.3048 \ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

$$H = 2 \ [\mathrm{in}] \cdot \left| 0.0254 \ \frac{\mathrm{m}}{\mathrm{in}} \right|$$

Assumption, Radius of hemisphere

$$R = 10 \ [\mathrm{ft}] \cdot \left| 0.3048 \ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

$$A_1 = 2 \cdot \pi \cdot R^2$$

$$A_2 = L_1 \cdot L_2$$

$$A_3 = 2 \cdot (L_1 + L_2) \cdot H$$

$$g = 9.80665 \ [\mathrm{m/s^2}]$$

$$\rho \cdot A_1 \cdot V_1 = \rho \cdot A_2 \cdot V_2$$ Continuity equation sub-system 1-2

$$\rho \cdot A_2 \cdot V_2 = \rho \cdot A_3 \cdot V_3$$ Continuity equation sub-system 2-3

$$p_1 = p_o$$

$$p_3 = p_o$$

Energy equation over the sub-system between boundaries 2-3

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} + g \cdot z_2 = \frac{p_3}{\rho} + \frac{V_3^2}{2} + g \cdot z_3$$

$$z_3 = -H/2$$

$$z_2 = 0 \ \ [\mathrm{m}]$$

$$z_1 = R/2$$ Weight of the air-cushion vehicle

$$W = 5000 \ \ [\mathrm{lb_f}] \cdot \left| 4.448222 \ \frac{\mathrm{N}}{\mathrm{lb_f}} \right|$$

$\theta$ is the angle for direction of velocity , and $\phi$ is the direction of the normal vector

## Force balance in y-axis

$$\phi_2 = 270 \ [\mathrm{deg}]$$

$$\theta_2 = 270 \ [\mathrm{deg}]$$

$$\dot{V} = V_3 \cdot A_3$$

$$\dot{m} = \dot{V} \cdot \rho$$

$$-W - \dot{m} \cdot v_1 \cdot (1/2) - (p_1 - p_o) \cdot A_1 \cdot (1/2) - (p_2 - p_o) \cdot A_2 \cdot \sin(\phi_2) - \dot{m} \cdot V_2 \cdot \sin(\theta_2) = 0$$

$$\frac{d[m\underline{v}]}{dt} = 0 = -\oint \rho \underline{v}(\underline{v} \cdot \underline{n})dA + m\underline{g} + \oint \underline{n}(-p)dA$$

$$-W - \dot{m} \cdot v_1 \cdot (1/2) - (p_1 - p_o) \cdot A_1 \cdot (1/2) - (p_2 - p_o) \cdot A_2 \cdot \sin(\phi_2) - \dot{m} \cdot V_2 \cdot \sin(\theta_2) = 0$$

Energy balance simplified, but coincides after integration

$$\dot{m} \cdot \left( \frac{p_1}{\rho} + \frac{v_1^2}{2} + g \cdot z_1 \right) - \dot{m} \cdot \left( \frac{p_2}{\rho} + \frac{v_2^2}{2} + g \cdot z_2 \right) - \dot{W} = 0$$

$g = 9.807$ [m/s$^2$] $\quad H = 0.0508$ [m]
$L_1 = 9.144$ [m] $\quad L_2 = 15.24$ [m]
$\dot{m} = 48.14$ [kg/s] $\quad p_o = 101325$ [Pa]
$R = 3.048$ [m] $\quad \rho = 1.18$ [kg/m$^3$]
$\dot{V} = 40.79$ [m$^3$/s] $\quad W = 22241$ [N]
$\dot{W} = -5782$ [W]

| Row | $A_i$ [m$^2$] | $p_i$ [Pa] | $V_i$ [m/s] | $z_i$ [m] | $\theta_i$ [deg] | $\phi_i$ [deg] |
|---|---|---|---|---|---|---|
| 1 | 58.37 | 101325 | 0.6988 | 1.524 | -- Variable -- | |
| 2 | 139.4 | 101485 | 0.2927 | 0 | 270 | 270 |
| 3 | 2.477 | 101325 | 16.47 | -0.0254 | | |

## First scenario

### Solution

$g = 9.807$ [m/s$^2$]
$H = 0.0762$ [m]
$L_1 = 9.144$ [m]
$L_2 = 15.24$ [m]
$\dot{m} = 102.1$ [kg/s]
$p_o = 101325$ [Pa]
$R = 3.048$ [m]
$\rho = 1.18$ [kg/m$^3$]
$\dot{V} = 86.54$ [m$^3$/s]
$W = 44482$ [N]
$\dot{W} = -26013$ [W]

### Arrays Table: *Main*

| Row | $A_i$ [m$^2$] | $p_i$ [Pa] | $V_i$ [m/s] | $z_i$ [m] | $\theta_i$ [deg] | $\phi_i$ [deg] |
|---|---|---|---|---|---|---|
| 1 | 58.37 | 101325 | 1.483 | 1.524 | | |
| 2 | 139.4 | 101644 | 0.621 | 0 | 270 | 270 |
| 3 | 3.716 | 101325 | 23.29 | -0.0381 | | |

## Second scenario

### Solution

$g = 9.807$ [m/s$^2$]
$H = 0.0508$ [m]
$L_1 = 9.144$ [m]
$L_2 = 15.24$ [m]
$\dot{m} = 48.14$ [kg/s]
$p_o = 101325$ [Pa]
$R = 3.048$ [m]
$\rho = 1.18$ [kg/m$^3$]
$\dot{V} = 40.79$ [m$^3$/s]
$W = 22241$ [N]
$\dot{W} = -5782$ [W]

### Arrays Table: *Main*

| Row | $A_i$ [m$^2$] | $p_i$ [Pa] | $V_i$ [m/s] | $z_i$ [m] | $\theta_i$ [deg] | $\phi_i$ [deg] |
|---|---|---|---|---|---|---|
| 1 | 58.37 | 101325 | 0.6988 | 1.524 | | |
| 2 | 139.4 | 101485 | 0.2927 | 0 | 270 | 270 |
| 3 | 2.477 | 101325 | 16.47 | -0.0254 | | |

What about if we consider the change in density, Then thermal effects must be considered otherwise you will have Errors in the calculations. Check this Solution and compare results.

**Equations**

$po = 101325 \ [\mathrm{Pa}]$

Assumption 1, standar pressure

$To = 298.15 \ [\mathrm{K}]$

Assumption 2, standar atmosphere

Assumption 3, Isothermal process

$To = T_1$

$T_2 = To$

$T_3 = To$

$Mm = 29 \ [\mathrm{kg/kmol}]$

Air Molecular mass

$R = 8314.34 \ [\mathrm{Pa \cdot m^3/kmol \cdot K}]$

Universal gas constant

$$\rho_1 = po \cdot \frac{Mm}{R \cdot To}$$

Density of air

$g = 9.8067 \ [\mathrm{m/s^2}]$

Gravity

$$m = 10000 \ [\mathrm{lbm}] \cdot \left| 0.453592374 \ \frac{\mathrm{kg}}{\mathrm{lbm}} \right|$$

Mass of the vehicle

$$L_1 = 30 \ [\mathrm{ft}] \cdot \left| 0.3048 \ \frac{\mathrm{m}}{\mathrm{ft}} \right|$$

width

$$L_2 = 50\ [\text{ft}] \cdot \left| 0.3048\ \frac{\text{m}}{\text{ft}} \right|$$

Length

$$H = 3\ [\text{in}] \cdot \left| 0.0254\ \frac{\text{m}}{\text{in}} \right|$$

Clearence

$$A_2 = L_1 \cdot L_2$$

Base area of the vehicle

Force balance in y-axis

$$(p_2 - po) \cdot A_2 = m \cdot g$$

Energy equation

$$p_3 = po$$

$$\rho_3 = \rho_1$$

$$Cv = (5/2) \cdot R/Mm$$

$$Cp - Cv = R/Mm$$

Assumption isothermal process

$$\rho_2 = P_2 \cdot \frac{Mm}{R \cdot T_2}$$

$$\frac{p_2}{\rho_2} + g \cdot z_2 + (1/2) \cdot v_2^2 = \frac{p_3}{\rho_3} + g \cdot z_3 + (1/2) \cdot v_3^2$$

$$\frac{p_4}{\rho_4} + g \cdot z_4 + (1/2) \cdot v_4^2 + Cv \cdot (T_4 - T_5) = \frac{p_5}{\rho_5} + g \cdot z_5 + (1/2) \cdot v_5^2$$

$$p_5 = p_3$$

$$\rho_5 = \rho_3$$

$$T_5 = T_3$$

$$z_4 = z_2$$

$$z_5 = z_3$$

$$(p_4 - po) \cdot A_2 = m \cdot g$$

$$\rho_4 \cdot A_2 \cdot v_4 = \rho_5 \cdot A_3 \cdot V_5$$

$$\rho_4 = P_4 \cdot \frac{Mm}{R \cdot T_4}$$

$$Cp \cdot \ln\left(\frac{T_5}{T_4}\right) - (R/Mm) \cdot \ln\left(\frac{p_5}{p_4}\right) = 0$$

$$\rho_2 \cdot A_2 \cdot v_2 = \rho_3 \cdot A_3 \cdot V_3$$

$$\dot{V}_A = V_5 \cdot A_3$$

$$\dot{m}_A = \dot{V}_A \cdot \rho_5$$

$$\rho_1 \cdot V_1 \cdot A_1 = \rho_2 \cdot V_2 \cdot A_2$$

$$A_1 = 2 \cdot \pi \cdot \frac{D_1^2}{4}$$

$$D_1 = 10 \ \mathrm{[m]}$$

$$Z_1 = H + \frac{\frac{D_1}{2}}{2}$$

$$p_1 = po$$

$$W = m \cdot g$$

$$A_3 = 2 \cdot H \cdot (L_1 + L_2)$$

$$z_3 = H/2$$

$$Z_2 = H$$

$$\dot{V} = V_3 \cdot A_3$$

$$\dot{m} = \dot{V} \cdot \rho_3$$

Energy balance to calculate power

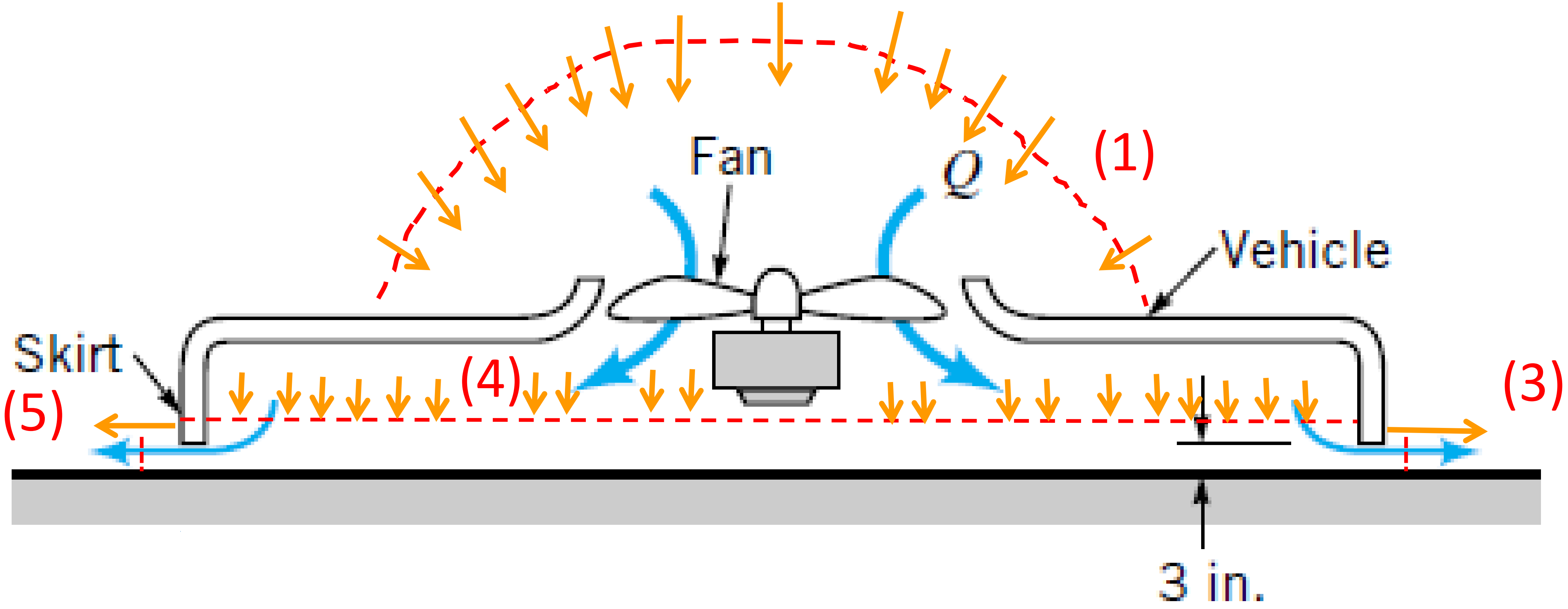
$$\left(\frac{p_1}{\rho_1} + g \cdot z_1 + (1/2) \cdot v_1^2 - \left(\frac{p_2}{\rho_2} + g \cdot z_2 + (1/2) \cdot v_2^2\right)\right) \cdot \dot{m} - \dot{W} = 0$$

$$\left(\frac{p_1}{\rho_1} + g \cdot z_1 + (1/2) \cdot v_1^2 + Cv \cdot (T_1 - T_4) - \left(\frac{p_4}{\rho_4} + g \cdot z_4 + (1/2) \cdot v_4^2\right)\right) \cdot \dot{m}_A - \dot{W}_A = 0$$

## Solution

$$
\begin{array}{ll}
Cp = 1003 \text{ [J/kg-K]} & Cv = 716.8 \text{ [J/kg-K]} \\
g = 9.807 \text{ [m/s}^2\text{]} & H = 0.0762 \text{ [m]} \\
m = 4536 \text{ [kg]} & Mm = 29 \text{ [kg/kmol]} \\
\dot{m} = 3.414 \text{ [kg/s]} & \dot{m}_A = 102.3 \text{ [kg/s]} \\
po = 101325 \text{ [Pa]} & R = 8314 \text{ [Pa-m}^3\text{/kmol-K]} \\
To = 298.2 \text{ [K]} & \dot{V} = 2.88 \text{ [m}^3\text{/s]} \\
\dot{V}_A = 86.29 \text{ [m}^3\text{/s]} & W = 44482 \text{ [N]} \\
\dot{W} = 83.69 \text{ [W]} & \dot{W}_A = -25024 \text{ [W]}
\end{array}
$$

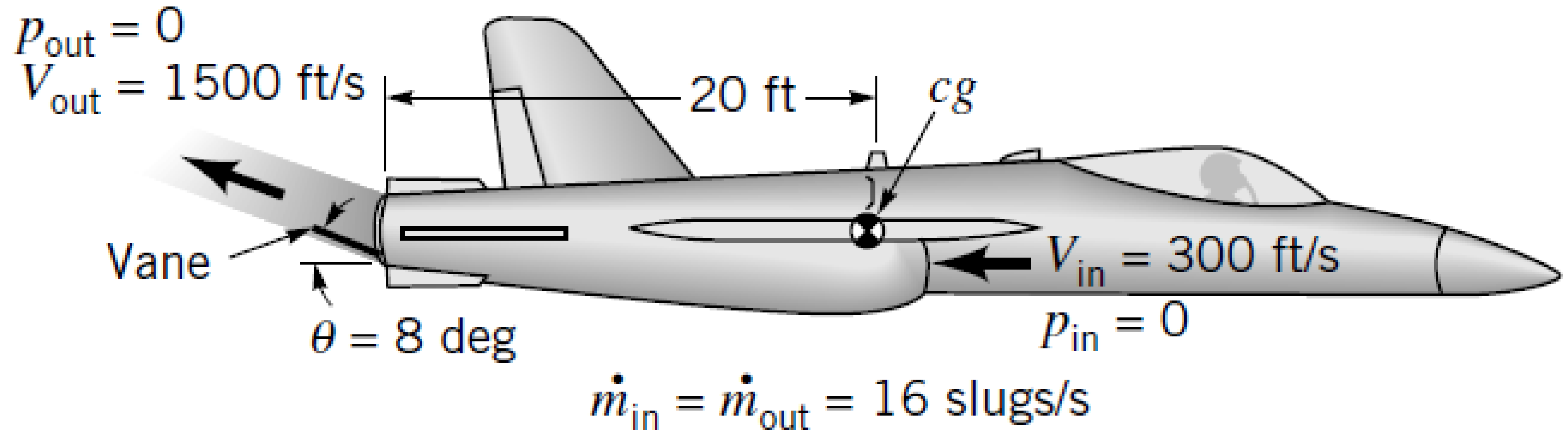
| Row | $\rho_i$ [kg/m$^3$] | $A_i$ [m$^2$] | $L_i$ [m] | $p_i$ [Pa] | $v_i$ [m/s] | $z_i$ [m] | $D_i$ [m] | $T_i$ [K] |
|---|---|---|---|---|---|---|---|---|
| 1 | 1.185 {1.185} | 157.1 {157.1} | 9.144 {9.144} | 101325 {101325} | 0.01833 {0.01833} | 2.576 {2.576} | 10 {10} | 298.2 {298.2} |
| 2 | 1.189 {1.189} | 139.4 {139.4} | 15.24 {15.24} | 101644 {101644} | 0.0206 {0.0206} | 0.0762 {0.0762} | | 298.2 {298.2} |
| 3 | 1.185 {1.185} | 3.716 {3.716} | | 101325 {101325} | 0.7749 {0.7749} | 0.0381 {0.0381} | | 298.2 {298.2} |
| 4 | 1.188 {1.188} | | | 101644 {101644} | 0.6178 {0.6178} | 0.0762 {0.0762} | | 298.4 {298.4} |
| 5 | 1.185 {1.185} | | | 101325 {101325} | 23.22 {23.22} | 0.0381 {0.0381} | | 298.2 {298.2} |

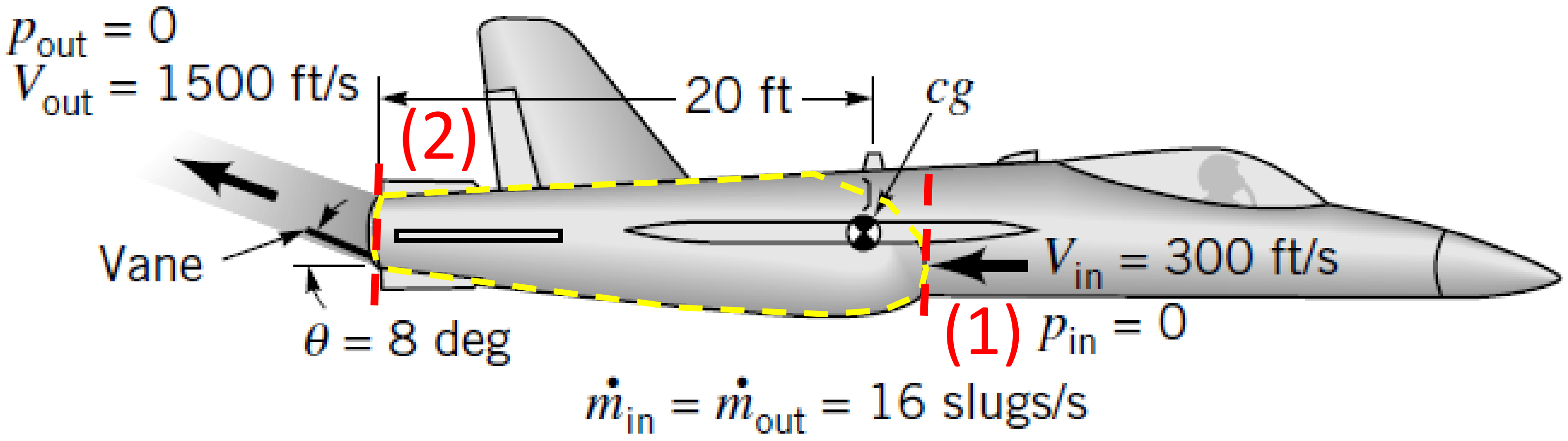



In a nutshell If compressibility effects are considered, Then the thermal effect should be included as well. Otherwise the error in calculations will be important.

The only additional assumption needed is to consider that the expansion is reversible, this is no viscous effects , and gradual expansion. Then isoentropic process was considered.

**Problem W7.16** Thrust vector control is a new technique that can be used greatly improve the maneuverability of military fighter aircraft. It consist of using a set of vanes in the exit of a jet engine to deflect the exhaust gases as shown in the figure. (a) Determine the pitching moment (the moment tending to rotate the nose of the aircraft up) about the aircraft's mass center (cg) for the conditions indicated in the figure. (b) By how much the thrust (force along the centerline of the aircraft) reduced for the case indicated compared to normal flight when the exhaust is parallel to the centerline ?





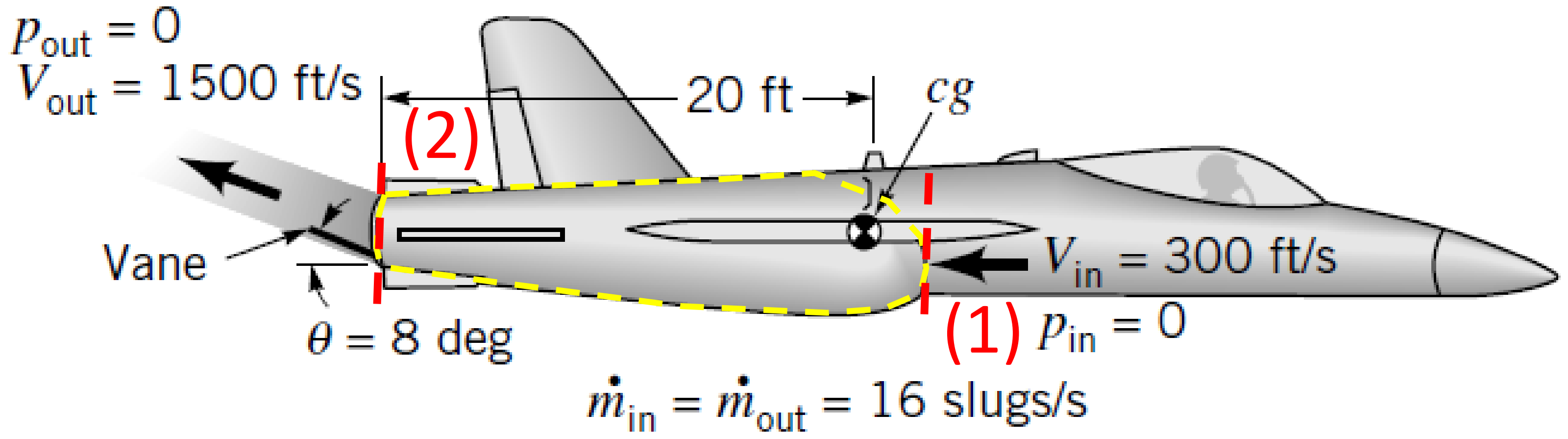

$$\frac{[\underline{r} \times m\, \underline{v}]}{dt} = \underline{r}_1 \times |\dot{m}_1|\, \underline{v}_1 - \underline{r}_2 \times |\dot{m}_2|\, \underline{v}_2 + \underline{r}_{CG} \times m\, \underline{g} \pm \underline{r}_{CG} \times \underline{F}_L - \underline{r}_{CG} \times \underline{F}_D - \underline{r}_1 \times [p_1 - p_0] A_1\, \underline{n}_1 - \underline{r}_2 \times [p_2 - p_0] A_2\, \underline{n}_2$$

$$r_2 = (-6.092\underline{i} + 0\underline{j} + 0\underline{k})$$

$$|\dot{m}_2|\, \underline{v}_2 = (233.5\ \mathrm{kg/s})(457.2\ \mathrm{m/s})\left[\underline{i}\cos(172°) + \underline{j}\sin(172°) + 0\,\underline{k}\right]$$

$$\frac{[\underline{r} \times m\, \underline{v}]}{dt} = -\left[-6.092\,\underline{i} + 0\,\underline{j}\ + 0\,\underline{k}\right](\mathrm{m}) \times (233.5\ \mathrm{kg/s})(457.2\ \mathrm{m/s})\left[\underline{i}\cos(172°) + \underline{j}\sin(172°) + 0\,\underline{k}\right]$$

$$\frac{[\underline{r} \times m\, \underline{v}]}{dt} = -\underline{r}_2 \times |\dot{m}_2|\, \underline{v}_2 = -(-6.092\mathrm{m})(233.5\ \mathrm{kg/s})(457.2\ \mathrm{m/s})[\sin(172°)]\,\underline{k} = [644\,458\ \mathrm{m-N}]\underline{k}$$




$$\frac{[m\,\underline{v}]}{dt} = |\dot{m}_1|\,\underline{v}_1 - |\dot{m}_2|\,\underline{v}_2 + m\,\underline{g} \pm \underline{F}_L - \underline{F}_D - [p_1 - p_0]A_1\,\underline{n}_1 - [p_2 - p_0]A_2\,\underline{n}_2$$

If $\theta$ is 0, and the airplane travels at constant speed, the linear momentum balance in x-axis simplifies to: keep in mind that thrust has to be compensated by drag force…

$$\frac{[m\,v_x]}{dt} = |\dot{m}_1|v_{1,x} - |\dot{m}_2|v_{2,x} - F_{D,x}$$

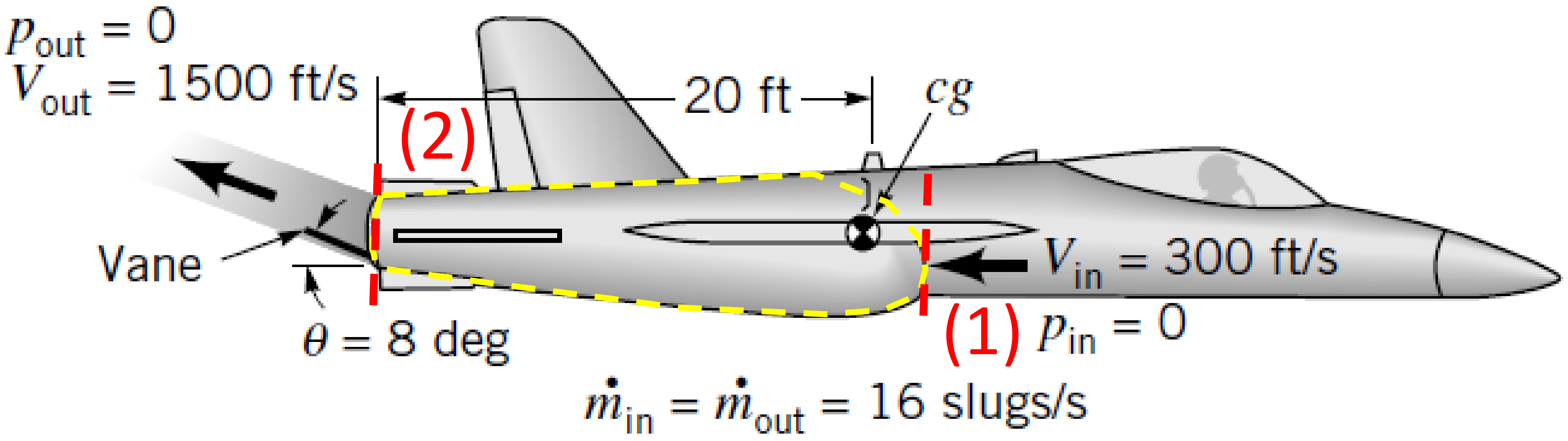
$$F_{D,x} = |\dot{m}_1|(\,v_{1,x} - v_{2,x}\,)$$

$$|\dot{m}_1| = 16\ \text{slugs/s} = 233.5\ \text{kg/s}$$

$$F_{D,x} = |\dot{m}_1|(\,v_{1,x} - v_{2,x}\,) = 85\,406\ \text{N}$$

$$v_{1,x} = -300\,\text{ft/s} = -91.44\,\text{m/s}$$

$$v_{2,x} = -1500\,\text{ft/s} = -457.2\,\text{m/s}$$




$$\frac{[m\,\underline{v}]}{dt} = |\dot{m}_1|\,\underline{v}_1 - |\dot{m}_2|\,\underline{v}_2 + m\,\underline{g} \pm \underline{F}_L - \underline{F}_D - [p_1 - p_0]A_1\,\underline{n}_1 - [p_2 - p_0]A_2\,\underline{n}_2$$

If $\theta$ is 8°, and the airplane travels at constant speed, the linear momentum balance in x-axis simplifies to:
Keep in mind that thrust has to be compensated by drag force…

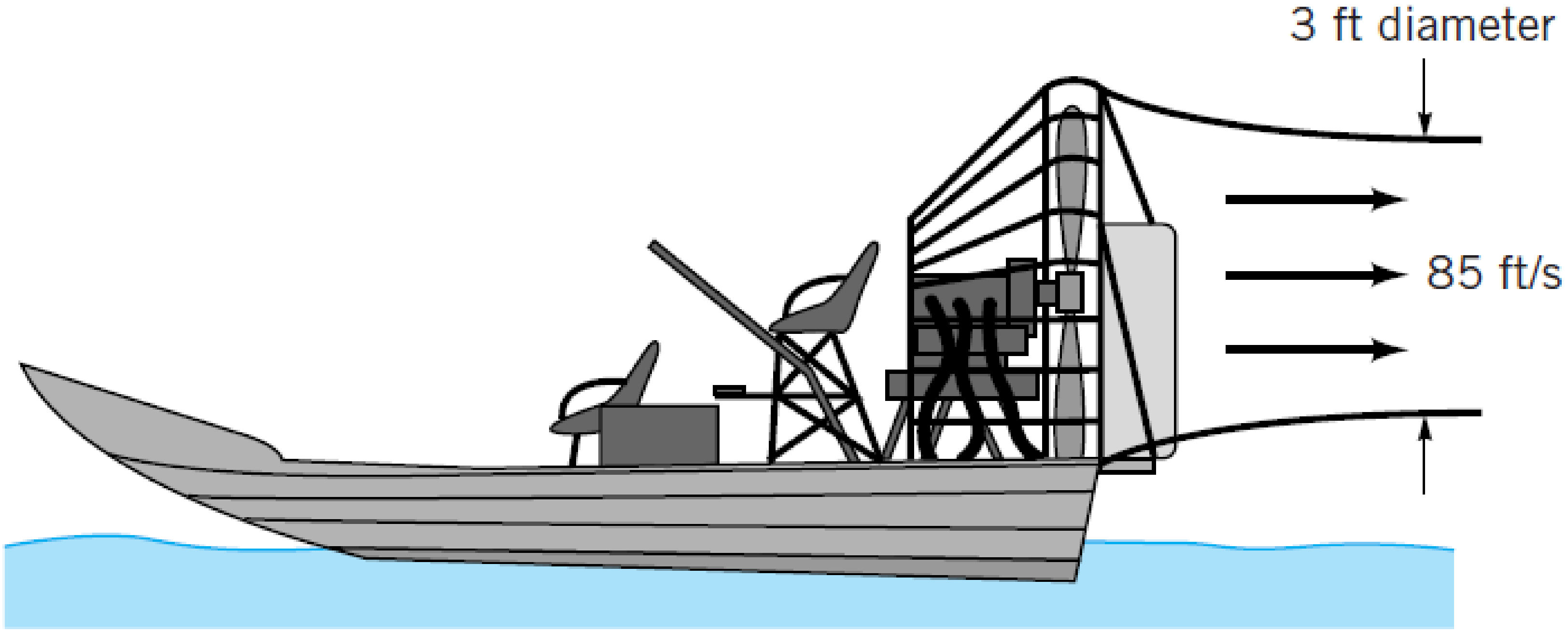
$$\frac{[m\,v_x]}{dt} = |\dot{m}_1|v_{1,x} - |\dot{m}_2|v_{2,x} - F_{D,x} \qquad F_{D,x} = |\dot{m}_1|(\,v_{1,x} - v_{2,x}\,)$$

$$|\dot{m}_1| = 16\ \text{slugs/s} = 233.5\ \text{kg/s}$$

$$F_{D,x} = |\dot{m}_1|(\,v_{1,x} - v_{2,x}\,) = 84\,599.4\ \text{N}$$

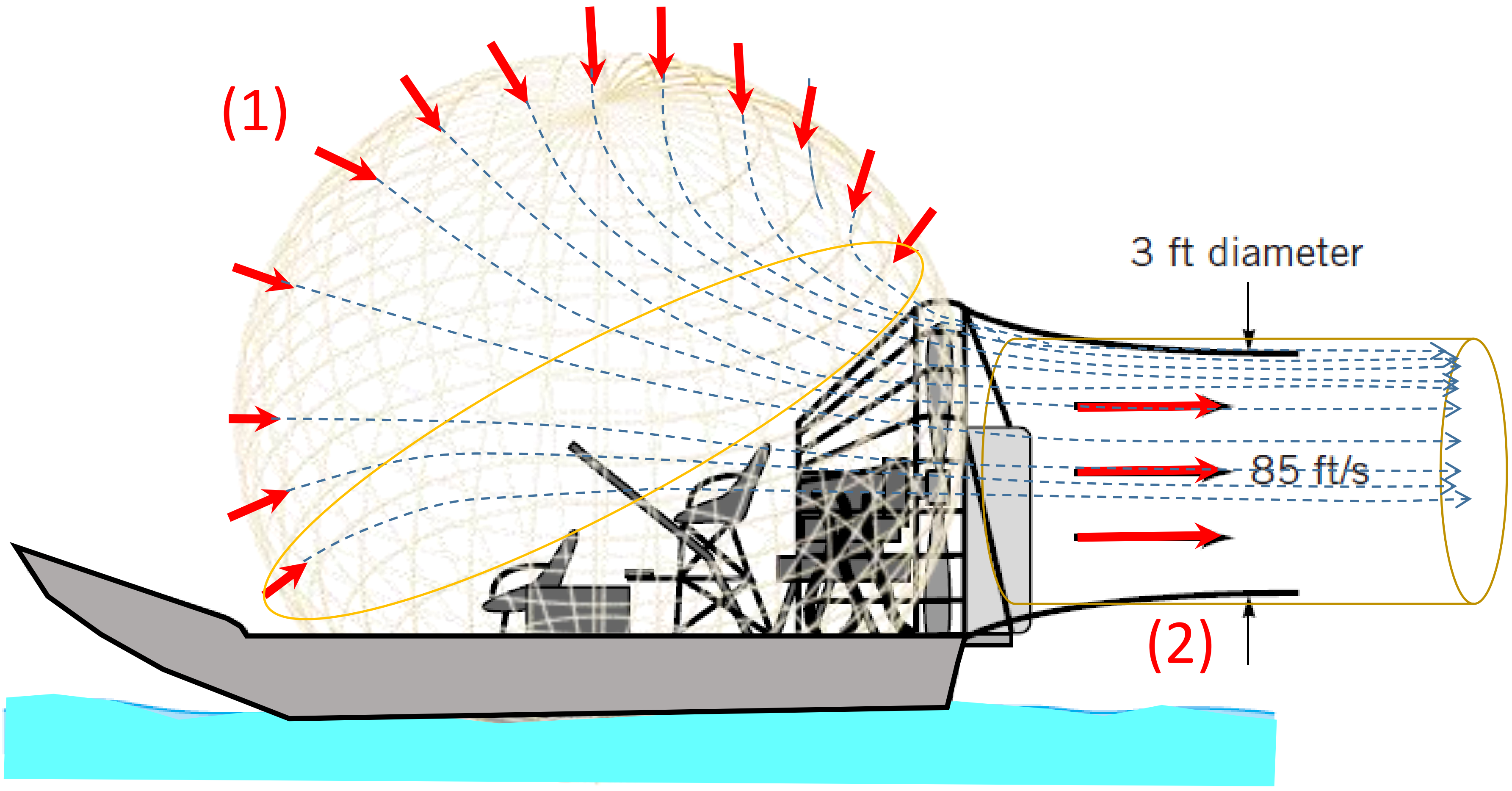
$$v_{1,x} = 91.44\,\text{m/s}\,Cos(180^o) = -91.44\,\text{m/s}$$
$$v_{2,x} = 457.2\,\text{m/s}\,[Cos(172^o)] = -452.75\,\text{N}$$

**Problem W7.17** The propeller on a swamp boat produces a jet of air having a diameter of 3 ft as illustrated in the figure. The ambient air temperature is 80°F, and the axial velocity of the flow is 85 ft/s relative to the boat. What propulsive forces are produced by the propeller when the boat is stationary and when the boat moves forward with a constant velocity of 20 ft/s ?



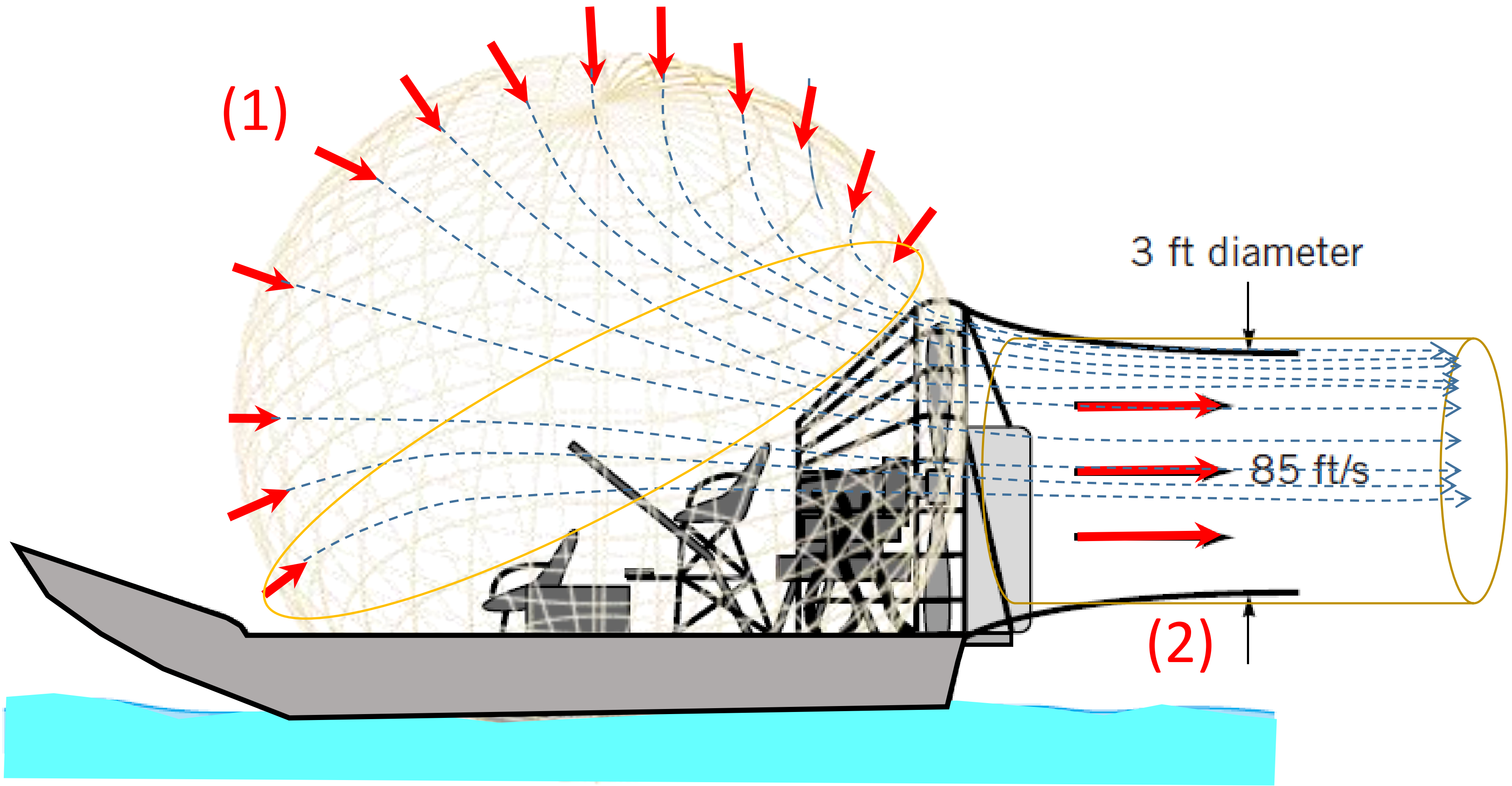
$$\frac{[m\,\underline{v}]}{dt} = |\dot{m}_1|\,\underline{v}_1 - |\dot{m}_2|\,\underline{v}_2 + m\,\underline{g} \pm \underline{F}_L - \underline{F}_D - [p_1 - p_0]A_1\,\underline{n}_1 - [p_2 - p_0]A_2\,\underline{n}_2$$

$$\frac{[m\,\underline{v}]}{dt} = |\dot{m}_1|(\underline{v}_1 - \underline{v}_2) + m\,\underline{g} \pm \underline{F}_L - \underline{F}_{D,A} - \underline{F}_{D,W}$$

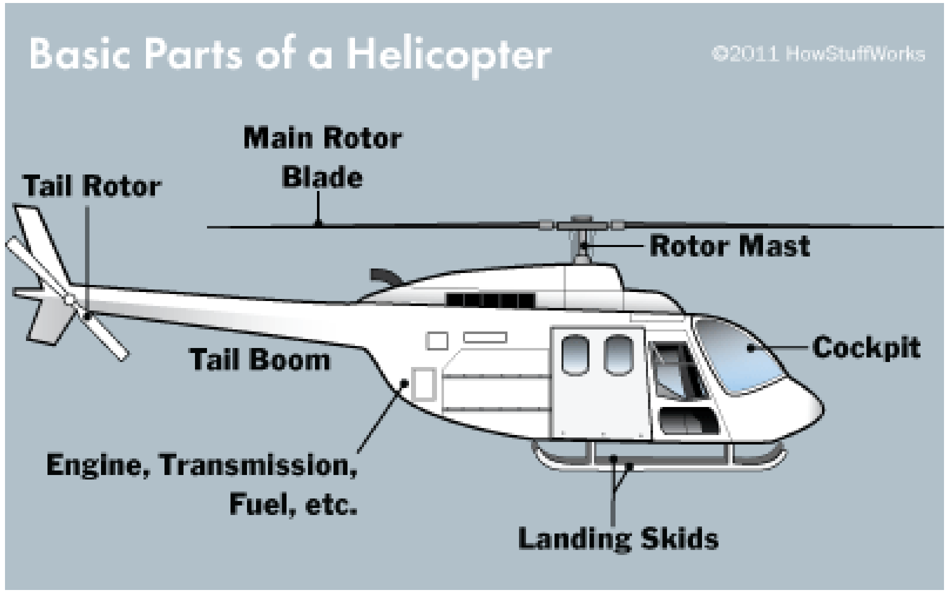


At constant speed, the balance in x-axis is simplified to:

$$0 = |\dot{m}_1|\left(\underline{v}_1 - \underline{v}_2\right) - F_{D,x,A} - F_{D,x,W}$$



If the rear propelled (tail rotor) has a diameter of 1m, what is the velocity needed to prevent the helicopter to start spinning if the rotor tail center is located 8.5 m from the rotor mast



## Equations

Helicopter problem

$R = 7.5 \ [\mathrm{m}]$

Radius of the Main rotor

$m = 25000 \ [\mathrm{kg}]$

Helicopter mass loaded

$m_u = 10000 \ [\mathrm{kg}]$

Helicopter mass unloaded

$\rho = 1.18 \ [\mathrm{kg/m^3}]$

Air density

$R_h = R$

Radius of the hemi-spherical upper section, upstream boundary (hemisphere shaped, radial flow)

$$A = \pi \cdot R^2$$

Area of the downstream flow, axial flow (output)

$$A_h = 4 \cdot \pi \cdot \frac{R_h^2}{2}$$

Area of the hemispherical upstream flow, radial flow (inlet)

$$g = 9.80665 \ \left[\mathrm{m/s^2}\right]$$

Mass balance for air flowing across the propeller

$$\rho \cdot v_1 \cdot A_h = \rho \cdot V_2 \cdot A$$

Force balance, loaded

$$-m \cdot g - \pi \cdot \rho \cdot R_h^2 \cdot V_1^2 + \pi \cdot \rho \cdot R^2 \cdot V_2^2 = 0$$

Force balance, unloaded

$$-m_u \cdot g - \pi \cdot \rho \cdot R_h^2 \cdot V_3^2 + \pi \cdot \rho \cdot R^2 \cdot V_4^2 = 0$$

$$\rho \cdot v_3 \cdot A_h = \rho \cdot v_4 \cdot A$$

$$\dot{V}_u = V_4 \cdot A$$

Volumetric flow rate unloaded

$$\dot{V} = V_2 \cdot A$$

Volume rate loaded

$$\dot{m}_u = \dot{V}_u \cdot \rho$$

mass flow rate unloaded

$$\dot{m} = \dot{V} \cdot \rho$$

mass flow rate loaded

Energy Balance loaded

$$\dot{m} \cdot \left( \frac{V_1^2}{2} - \frac{v_2^2}{2} \right) - \dot{W} = 0$$

Energy Balance unloaded

$$\dot{m}_u \cdot \left( \frac{V_3^2}{2} - \frac{v_4^2}{2} \right) - \dot{W}_u = 0$$

$$RPM_u = 400 \ \text{[rev/min]}$$

$$\Omega_u = RPM_u \cdot \left| 0.104719755 \ \frac{\text{rad/s}}{\text{rev/min}} \right|$$

$$\Omega = RPM \cdot \left| 0.104719755 \ \frac{\text{rad/s}}{\text{rev/min}} \right|$$

$$\dot{W} = \Omega \cdot \tau$$

$$\dot{W}_u = \Omega_u \cdot \tau_u$$

Problem statement says, volumetric flow rate is proportional to rotor velocity

$$\dot{V} = k \cdot \Omega$$

$$\dot{V}_u = k \cdot \Omega_u$$

Volumetric Flow rate number

$$VN = \frac{\dot{V}}{\Omega \cdot R^3}$$

Analogy applies, because problem statement says

Volume flow rate is proportional to RPMs

$$VN_u = \frac{\dot{V}_u}{\Omega_u \cdot R^3}$$

To verify the analogy lets try Power number

$$PN = -\frac{\dot{W}}{\rho \cdot \Omega^3 \cdot R^5}$$

$$PN_u = -\frac{\dot{W}_u}{\rho \cdot \Omega_u^3 \cdot R^5}$$

$$L = 8.5 \ \text{[m]}$$

$$R_T = 1 \ \text{[m]}$$

Radius of the tail propeller

Flowrate in the tail propeller

Angular momentum balance

$$\tau + \tau_T = 0$$

Loaded

$$\tau_u + \tau_{T,u} = 0$$

Loaded

Torque in the tail

$$A_T = \pi \cdot R_T^2$$

$$A_{T,hs} = 2 \cdot \pi \cdot R_T^2$$

$$\tau_T = F_T \cdot L$$

$$\tau_{T,u} = F_{T,u} \cdot L$$

$$\rho \cdot v_5 \cdot A_{T,hs} = \rho \cdot v_6 \cdot A_T$$

Loaded

$$\rho \cdot v_7 \cdot A_{T,hs} = \rho \cdot v_8 \cdot A_T$$

Unloaded

$$\dot{m}_T = \rho \cdot v_6 \cdot A_T$$

$$\dot{m}_{T,u} = \rho \cdot v_8 \cdot A_T$$

$$\dot{V}_T = v_6 \cdot A_T$$

$$\dot{V}_{T,u} = v_8 \cdot A_T$$

$$F_T = \dot{m}_T \cdot (v_6 - v_5)$$

$$F_{T,u} = \dot{m}_{T,u} \cdot (v_8 - v_7)$$

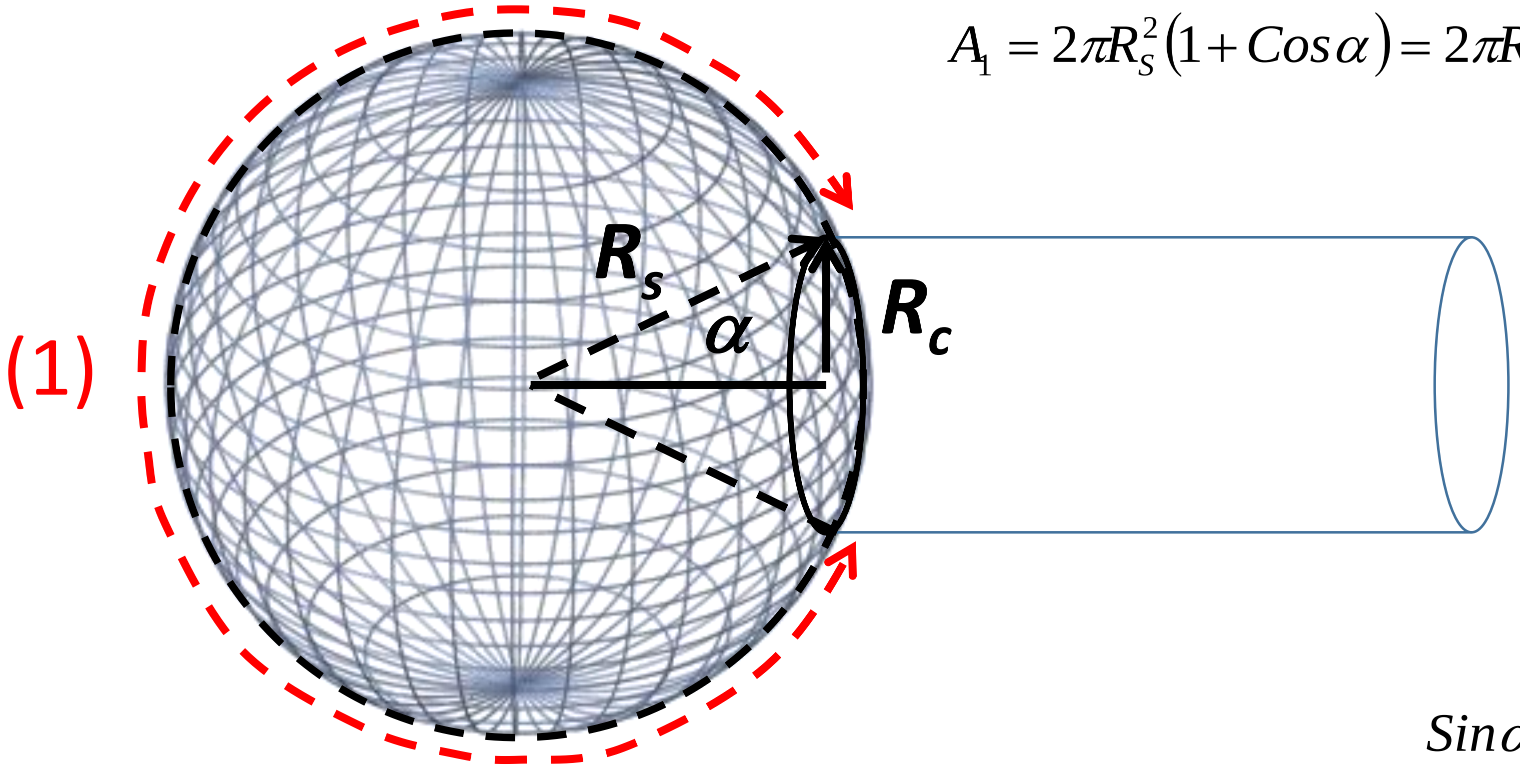
$$\dot{m}_T \cdot \left( \frac{V_5^2}{2} - \frac{v_6^2}{2} \right) - \dot{W}_T = 0$$

$$\dot{m}_{T,u} \cdot \left( \frac{V_7^2}{2} - \frac{v_8^2}{2} \right) - \dot{W}_{T,u} = 0$$

| | |
|---|---|
| $A = 176.7$ [m$^2$] | $A_h = 353.4$ [m$^2$] |
| $A_T = 3.142$ [m$^2$] | $A_{T,hs} = 6.283$ [m$^2$] |
| $F_T = 8621$ [N] | $F_{T,u} = 3449$ [N] |
| $g = 9.807$ [m/s$^2$] | $k = 105.642$ [m$^3$] |
| $L = 8.5$ [m] | $m = 25000$ [kg] |
| $\dot{m} = 8256$ [kg/s] | $\dot{m}_T = 252.8$ [kg/s] |
| $\dot{m}_{T,u} = 159.9$ [kg/s] | $\dot{m}_u = 5222$ [kg/s] |
| $m_u = 10000$ [kg] | $\Omega = 66.23$ [rad/s] |
| $\Omega_u = 41.89$ [rad/s] | $PN = 5.966 \times 10^{-4}$ |
| $PN_u = 5.966 \times 10^{-4}$ | $R = 7.5$ [m] |
| $\rho = 1.18$ [kg/m$^3$] | $RPM = 632.5$ [rev/min] |
| $RPM_u = 400$ [rev/min] | $R_h = 7.5$ [m] |
| $R_T = 1$ [m] | $\tau = -73282$ [N-m] |
| $\tau_T = 73282$ [N-m] | $\tau_{T,u} = 29313$ [N-m] |
| $\tau_u = -29313$ [N-m] | $VN = 0.2504$ |
| $VN_u = 0.2504$ | $\dot{V} = 6997$ [m$^3$/s] |
| $\dot{V}_T = 214.3$ [m$^3$/s] | $\dot{V}_{T,u} = 135.5$ [m$^3$/s] |
| $\dot{V}_u = 4425$ [m$^3$/s] | $\dot{W} = -4.853 \times 10^6$ [W] |
| $\dot{W}_T = -440985$ [W] | $\dot{W}_{T,u} = -111561$ [W] |
| $\dot{W}_u = -1.228 \times 10^6$ [W] | |

$$A_1 = \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi-\alpha} (R_S\, d\vartheta)(R_S\ Sin\vartheta\, d\varphi) = 2\pi R_S^2 (1 + Cos\alpha)$$

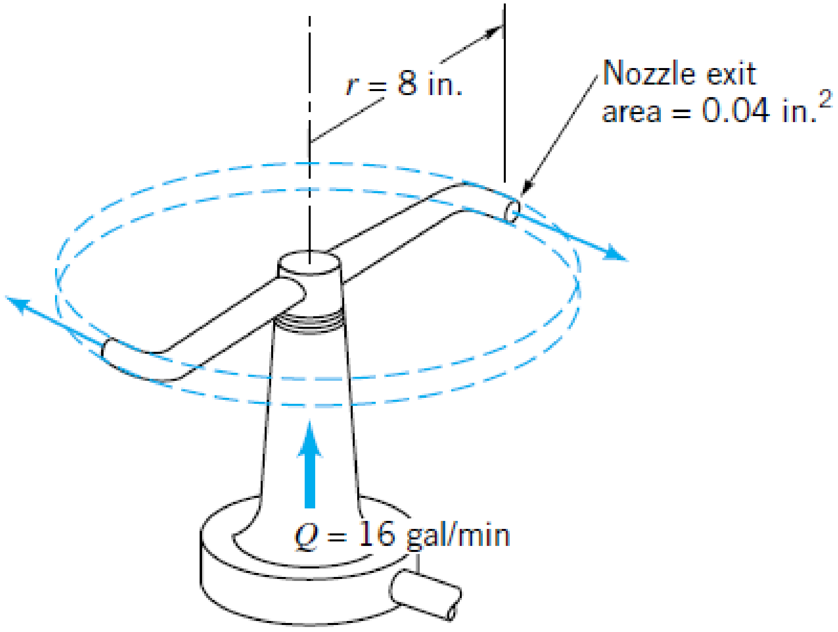
$$A_1 = 2\pi R_S^2 (1 + Cos\alpha) = 2\pi R_S^2 \left(1 + \sqrt{1 - (R_C / R_S)^2}\right)$$

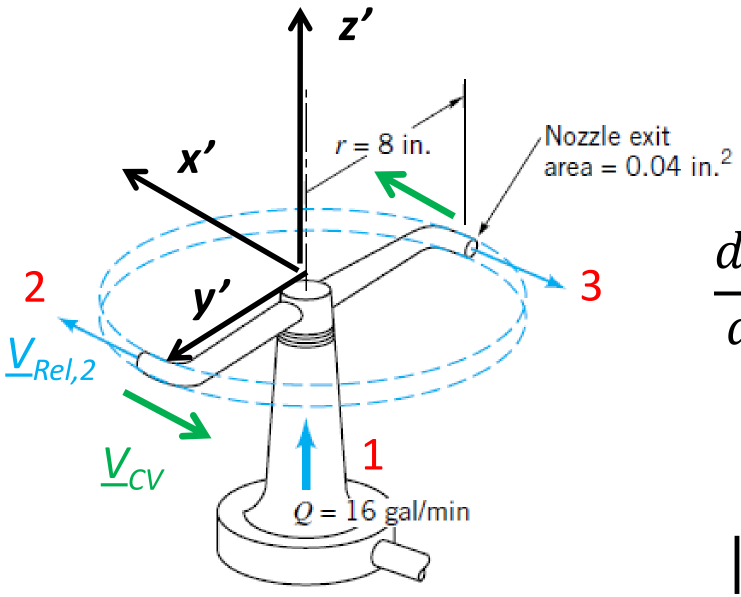



$$Sin\alpha = R_C / R_S$$

$$\alpha = ArcSin(R_C / R_S)$$

**Problem W7.7** Water enters a rotating lawn sprinkler through its base at the steady rate of 16 gal/min as shown in the figure. The exit cross-sectional area of each of the two nozzles is 0.04 in$^2$, and the flow leaving each nozzle is tangential. The radius from the axis of rotation to the centerline of each nozzle is 8 in. (a) Determine the resisting torque required to hold the sprinkler head stationary. (b) Determine the resisting torque associated with the sprinkler rotating with a constant speed of 500 rev/min. (c) Determine the angular velocity of the sprinkler if no resisting torque is applied. (d) Calculate the power that can be produced by coupling an alternator. (e) Estimate the maximum power that can be generated if an alternator is connected.






# Physics of sprinkler (continuity equation)

$$\frac{dm}{dt} = |\dot{m}_{\text{in}}| - |\dot{m}_{\text{out}}| = 0 = |\dot{m}_1| - |\dot{m}_2| - |\dot{m}_3|$$ No accumulation of mass

$$\frac{|\dot{m}_1|}{2} = |\dot{m}_2| = |\dot{m}_3|$$ By symmetry (Constraint)

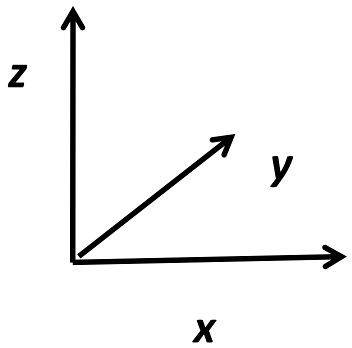
$$|\dot{m}_2| = \rho\, A_2 \underline{v}_{R,2} \cdot \underline{n}_2 = \rho\, A_2 \left(\underline{v}_2 - \underline{v}_{CV}\right) \cdot \underline{n}_2$$ Continuity equation

$$\underline{v}_{CV} = \underline{\omega} \times \underline{R} = -\,\Omega\, R\, \underline{\mathrm{i}}$$

Velocity of the control volume, in this case the sprinkler is rotating so its control volume velocity can be trivial, but the control volume velocity at the exit point of water.

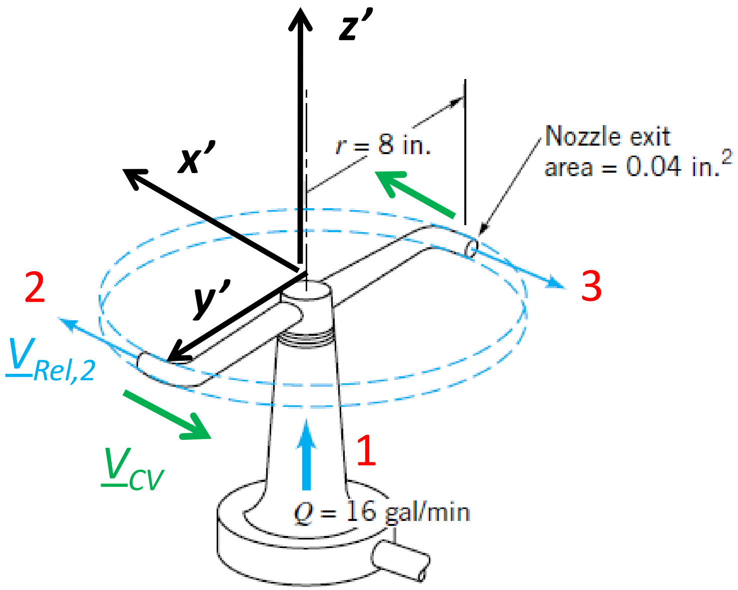
$$\underline{v}_{R,2} = \underline{v} - \underline{v}_{CV}$$

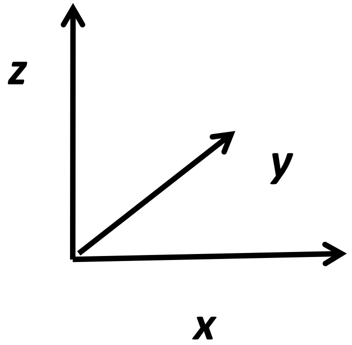
Relative velocity by observer sitting over the nozzle 2
But this frame or reference is non-inertial




Axes (*x,y,z*) where the observer is standing, is an inertial frame of reference.
Axes (*x',y',z'*) form a rotating frame of reference in non-inertial

As a matter of fact, control volume velocity is : $\underline{V}_{CV} = \underline{\omega} \times \underline{R} = (0\,\underline{i} + 0\,\underline{j} + \Omega\,\underline{k}) \times (0\,\underline{i} + R\,\underline{j} + 0\,\underline{k}) = -\,\Omega\, R\, \underline{i} = -\left|\underline{\omega} \times \underline{R}\right|$




# Physics of sprinkler (linear momentum or force balance)

$$\frac{[m\,\underline{v}]}{dt} = |\dot{m}_1|\ \langle \underline{v}_1 \rangle\,\beta_1 - |\dot{m}_2|\,\langle \underline{v}_2 \rangle\,\beta_2 - |\dot{m}_3|\ \langle \underline{v}_3 \rangle\,\beta_3 + m\,\underline{g} - \underline{F}_f - \underline{F}_D$$

$$\underline{F}_{I,2} = |\dot{m}_2|\langle \underline{v}_2 \rangle\,\beta_2$$ Inertial force

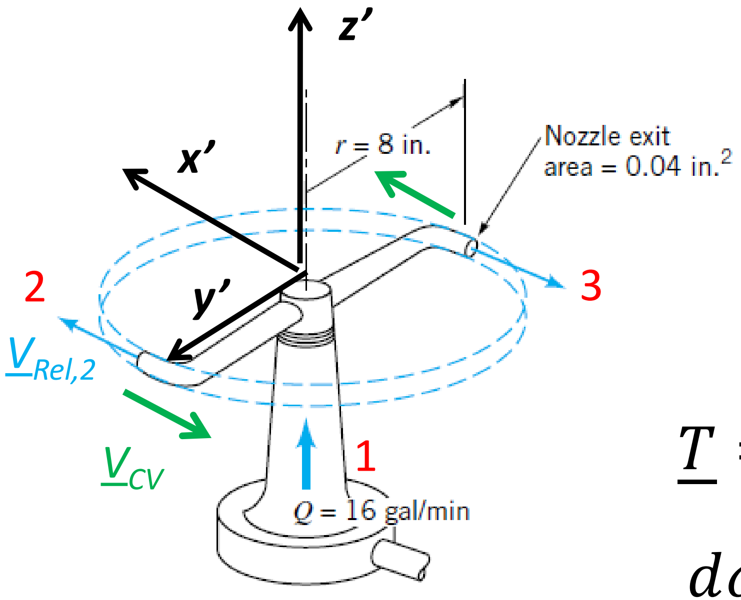
$$\underline{F}_{I,2} = -\underline{F}_{I,3}$$ By symmetry

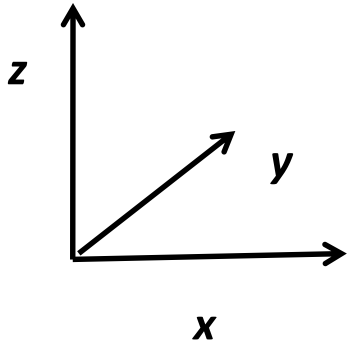
$$\underline{F}_{I,2} = |\dot{m}_2|\langle \underline{v}_{R,2} + \underline{v}_{CV} \rangle\,\beta_2$$

$$\underline{F}_{I,2} = |\dot{m}_2|\langle \underline{v}_{R,2} + \underline{v}_{CV} \rangle\,\beta_2 \approx |\dot{m}_2|\left[\frac{|\dot{m}_2|}{\rho A_2} - \Omega\,R\right]$$

Whenever we have a free jet stream, the assumption of $\beta$ =1 is a good approximation

Axes ($x,y,z$) where the observer is standing, is an inertial frame of reference.
Axes ($x',y',z'$) is a rotating frame of reference, i.e. non-inertial (Located at the top of the sprinkler, or at any point in the spinning section of the sprinkler)




## Physics of sprinkler (angular momentum)

$$\frac{[\underline{r}\times m\,\underline{v}]}{dt} = \frac{[I\,\underline{\omega}]}{dt} = |\dot{m}_1|\underline{r_1}\times\langle\underline{v_1}\rangle\,\beta_1 - |\dot{m}_2|\underline{r_2}\times\langle\underline{v_2}\rangle\,\beta_2 - |\dot{m}_3|\,\underline{r_3}\times\langle\underline{v_3}\rangle\,\beta_3 + \cdots$$

$$\underline{T} = 2\,\underline{r_2}\times|\dot{m}_2|\langle\underline{v_2}\rangle\,\beta_2$$

Total Torque

$$I\,\frac{d\underline{\underline{\omega}}}{dt} = 2\,\underline{r_2}\times|\dot{m}_2|\langle\underline{v_2}\rangle\,\beta_2$$

Assuming no friction nor drag force

$$\underline{L} = \underline{r}\times\underline{p} \qquad \underline{p} = m\underline{v}$$

Axes (*x,y,z*) where the observer is standing, is an inertial frame of reference.
Axes (*x',y',z'*) is a rotating frame of reference i.e. non-inertial

$$\frac{[I\,\underline{\omega}]}{dt} = -(r_2)|\dot{m}_2|\beta_2 v_2\;\underline{j}\times\underline{i} - (r_3)|\dot{m}_3|\beta_3(-v_3)\;\underline{j}\times\underline{i}$$

# Force, Torque and Power

$$\underline{F}_{\mathrm{I},2} = |\dot{m}_2| \langle \underline{v}_{R,2} + \underline{v}_{CV} \rangle \, \beta_2 \approx |\dot{m}_2| \left[ \frac{|\dot{m}_2|}{\rho A_2} - \Omega \, R \right] = \frac{|\dot{m}|}{2} \left[ \frac{|\dot{m}|}{2 \, \rho A_2} - \Omega \, R \right]$$

Force on each branch

$$\underline{T} = 2 \, \underline{r}_2 \times |\dot{m}_2| \langle \underline{v}_2 \rangle \, \beta_2$$

$$\underline{T} = R |\underline{\dot{m}}| \left[ \frac{|\underline{\dot{m}}|}{2 \, \rho A_2} - \Omega \, R \right]$$

Total torque in terms of angular speed and total mas flow rate

$$\dot{W} = \underline{\omega} \cdot \underline{T} = \Omega R |\underline{\dot{m}}| \left[ \frac{|\underline{\dot{m}}|}{2 \, \rho A_2} - \Omega \, R \right]$$

Potential power produced if coupled with a generator

$$\dot{W} = \underline{\omega} \cdot \underline{T} = |\underline{\dot{m}}| \left[ \frac{|\underline{\dot{m}}| \Omega R}{2\,\rho A_2} - (\Omega\, R)^2 \right]$$

From this equation, we can differentiate respect to angular speed in order to find the maximum power.

$$\frac{d\dot{W}}{d\Omega} = \frac{d(\underline{\omega} \cdot \underline{T})}{d\Omega} = |\underline{\dot{m}}| \left[ \frac{|\underline{\dot{m}}| R}{2\,\rho A_2} - 2\,\Omega\, R^2 \right] = 0$$

$$\Omega_{\text{opt}} R = \frac{|\underline{\dot{m}}|}{4\,\rho\, A_2}$$

$$\Omega_{\text{opt}} = \frac{|\underline{\dot{m}}|}{4\, R\, \rho\, A_2}$$

Angular speed to maximize power

$$\dot{W}_{\max} = \frac{|\underline{\dot{m}}|^3}{16(\rho A_2)^2}$$

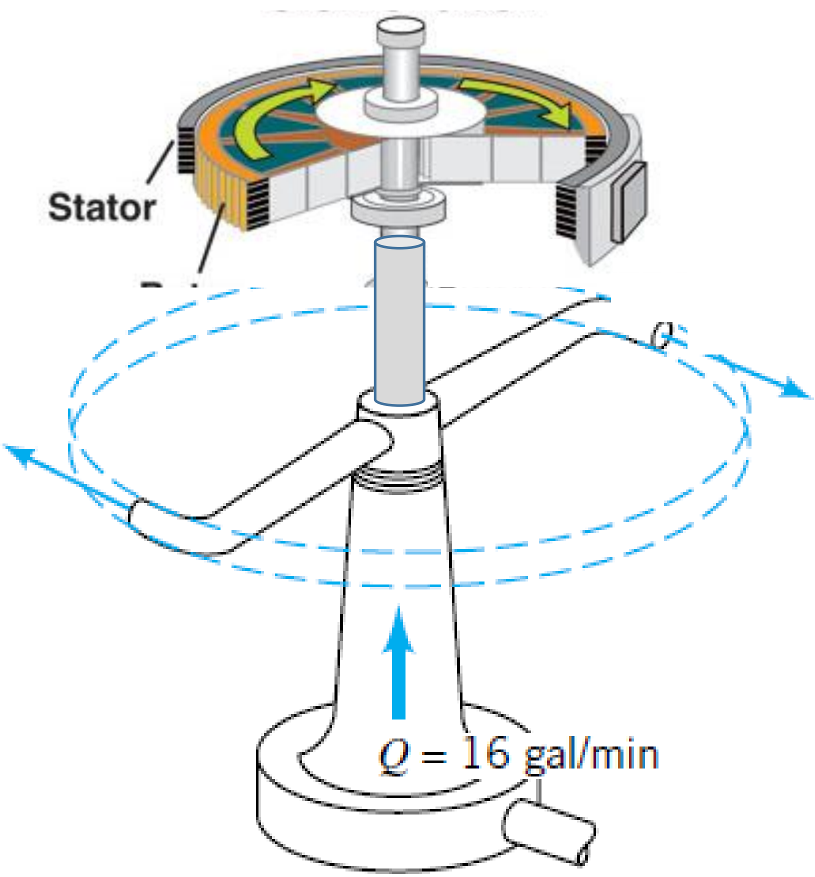
Maximum power capacity

---

$$\dot{W} = 0 = |\underline{\dot{m}}| \left[ \frac{|\underline{\dot{m}}| \Omega_{\max} R}{2\,\rho A_2} - (\Omega_{\max} R)^2 \right]$$

$$\Omega_{\max} = \frac{|\underline{\dot{m}}|}{2\, R\, \rho\, A_2}$$

Maximum angular speed, no power is produced at this speed, because the sprinkler is released from the generator.


Stator
Q = 16 gal/min

# FAQ

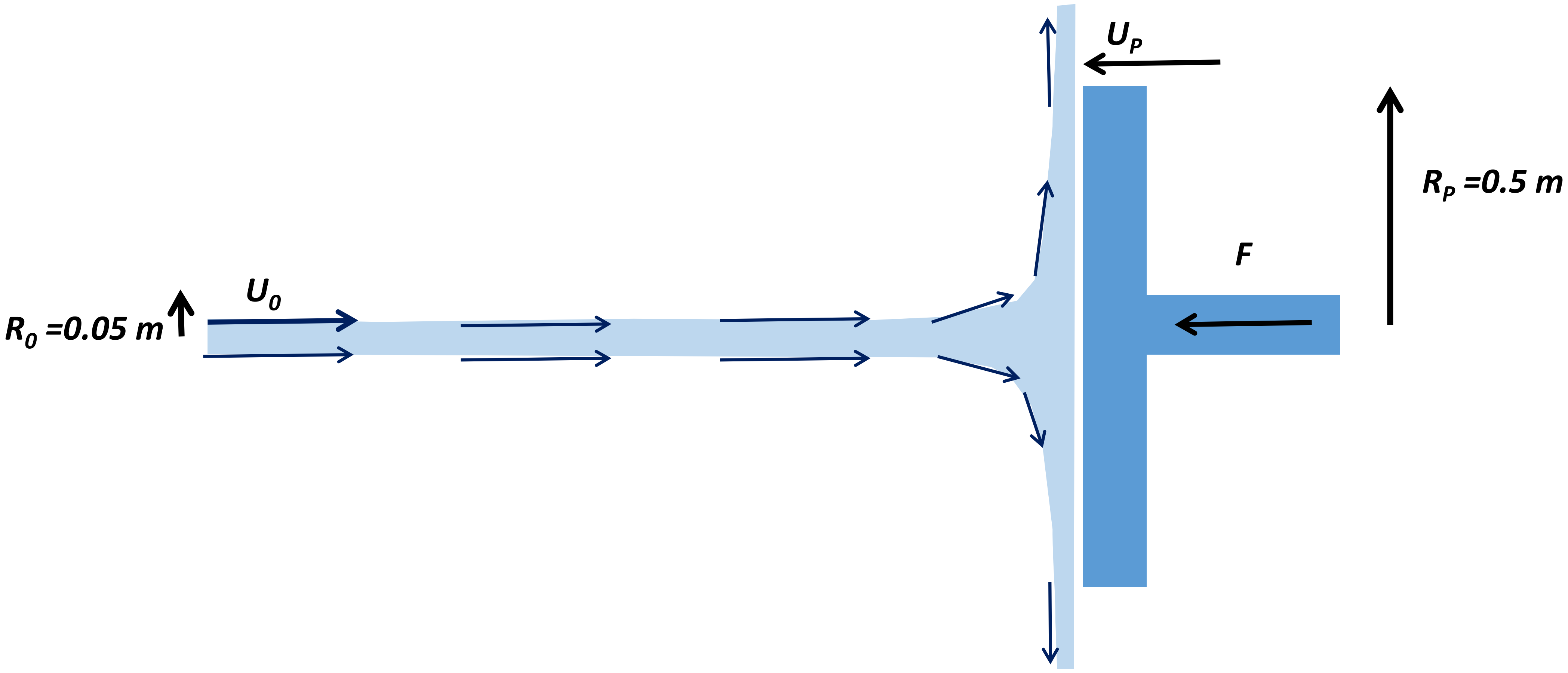
Which is my control volume ?

It is Unsteady state or Quasi-Steady state ?

What is the difference between steady state and steady flow ?

The selection of the control volume is trivial, as long as you follow the laws of the physics, and the approach, depends of the simplicity of the mathematical model, the advice is to use the approach that end up with the simplest one, but even if you select a convoluted approach, as long as you follow the laws of physics you will end up with the correct answer. (lets prove this statement)

P5.100 Calculate Force needed for a moving wall ($U_0$=4 m/s) to maintain a constant velocity ($U_P$=6 m/s) in opposite direction.



# Approach I

Calculate Force needed for a moving wall to maintain a constant velocity

$$\frac{dm}{dt} = \sum_{i=\text{inputs}} |\dot{m}_i| - \sum_{j=\text{outputs}} |\dot{m}_j|$$

$$\frac{dm}{dt} = |\dot{m}_1| - |\dot{m}_2|$$

$$|\dot{m}| = |\dot{m}_1| = |\dot{m}_2|$$

$$|\dot{m}| = |\dot{m}_1| = |\dot{m}_2| = \rho \left| \underline{v}_{R,1} \cdot \underline{n}_1 \right| A$$

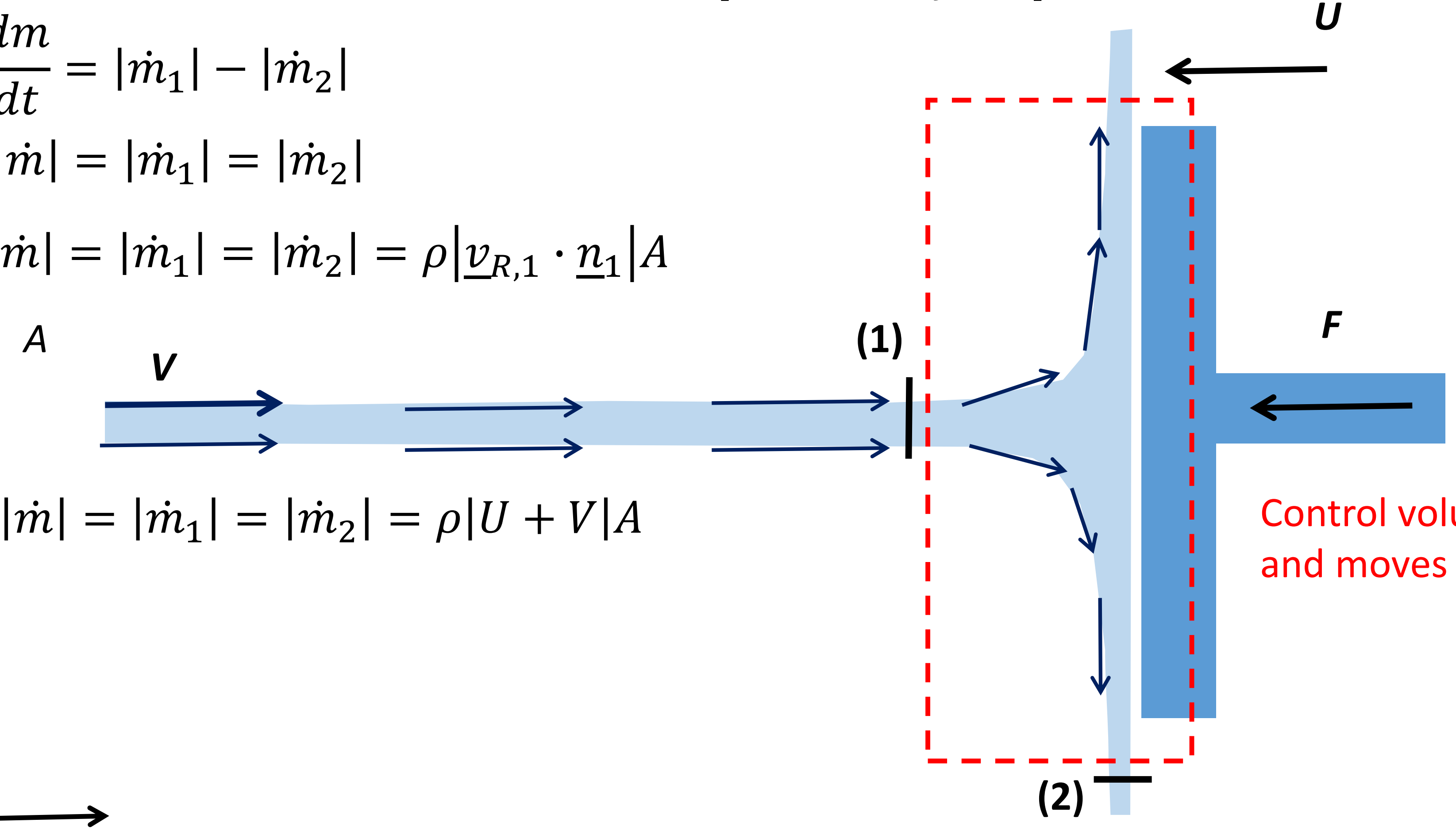
$$\underline{n}_1 = -\,\underline{\mathrm{i}}$$

$$\underline{v}_{CV} = -U\ \underline{\mathrm{i}}$$

$$\underline{v}_1 = V\ \underline{\mathrm{i}}$$

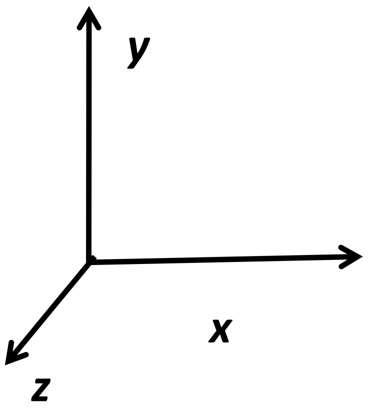
$$\underline{v}_{R,1} = \underline{v}_1 - \underline{v}_{CV}$$

$$\underline{v}_{R,1} = (U+V)\ \underline{\mathrm{i}}$$




$$|\dot{m}| = |\dot{m}_1| = |\dot{m}_2| = \rho |U+V| A$$

Control volume maintains its shape, and moves at constant velocity *-U*



# Approach I

The force balance is evaluated keeping the observer as the frame of reference

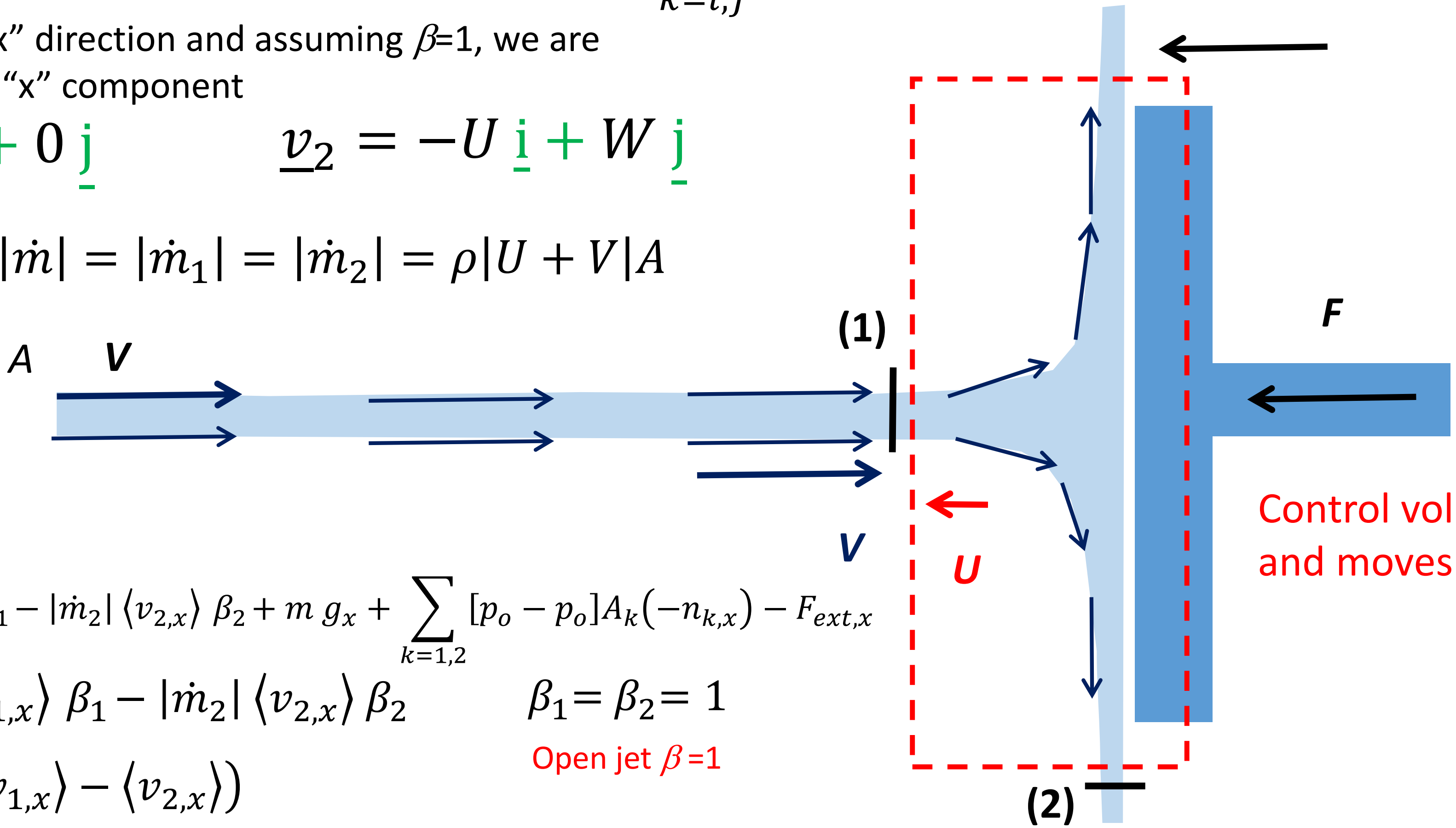
Calculate Force needed for a moving wall to maintain a constant velocity

$$\frac{d[m\,\underline{v}]}{dt} = |\dot{m}_1|\,\langle \underline{v}_1\rangle\,\beta_1 - |\dot{m}_2|\,\langle \underline{v}_2\rangle\,\beta_2 + m\underline{g} + \sum_{k=i,j}[p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_{ext}$$

For a balance in "x" direction and assuming $\beta$=1, we are interested just on "x" component

$$\underline{v}_1 = V\,\underline{\mathrm{i}} + 0\,\underline{\mathrm{j}} \qquad \underline{v}_2 = -U\,\underline{\mathrm{i}} + W\,\underline{\mathrm{j}}$$

$$|\dot{m}| = |\dot{m}_1| = |\dot{m}_2| = \rho|U+V|A$$




Control volume maintains its shape, and moves at constant velocity *-U*

$$\frac{d[mv_x]}{dt} = |\dot{m}_1|\,\langle v_{1,x}\rangle\,\beta_1 - |\dot{m}_2|\,\langle v_{2,x}\rangle\,\beta_2 + m\,g_x + \sum_{k=1,2}[p_o - p_o]A_k(-n_{k,x}) - F_{ext,x}$$

$$F_{ext,x} = |\dot{m}_1|\,\langle v_{1,x}\rangle\,\beta_1 - |\dot{m}_2|\,\langle v_{2,x}\rangle\,\beta_2 \qquad \beta_1 = \beta_2 = 1$$

Open jet $\beta$=1

$$F_{ext,x} = |\dot{m}_1|\big(\langle v_{1,x}\rangle - \langle v_{2,x}\rangle\big)$$

$$F_{ext,x} = |\dot{m}_1|\big(V - (-U)\big) = \rho\,A\,(U+V)^2$$

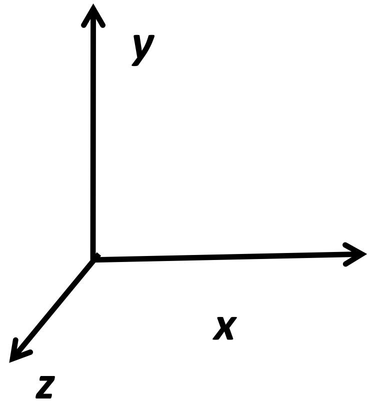
# Approach II

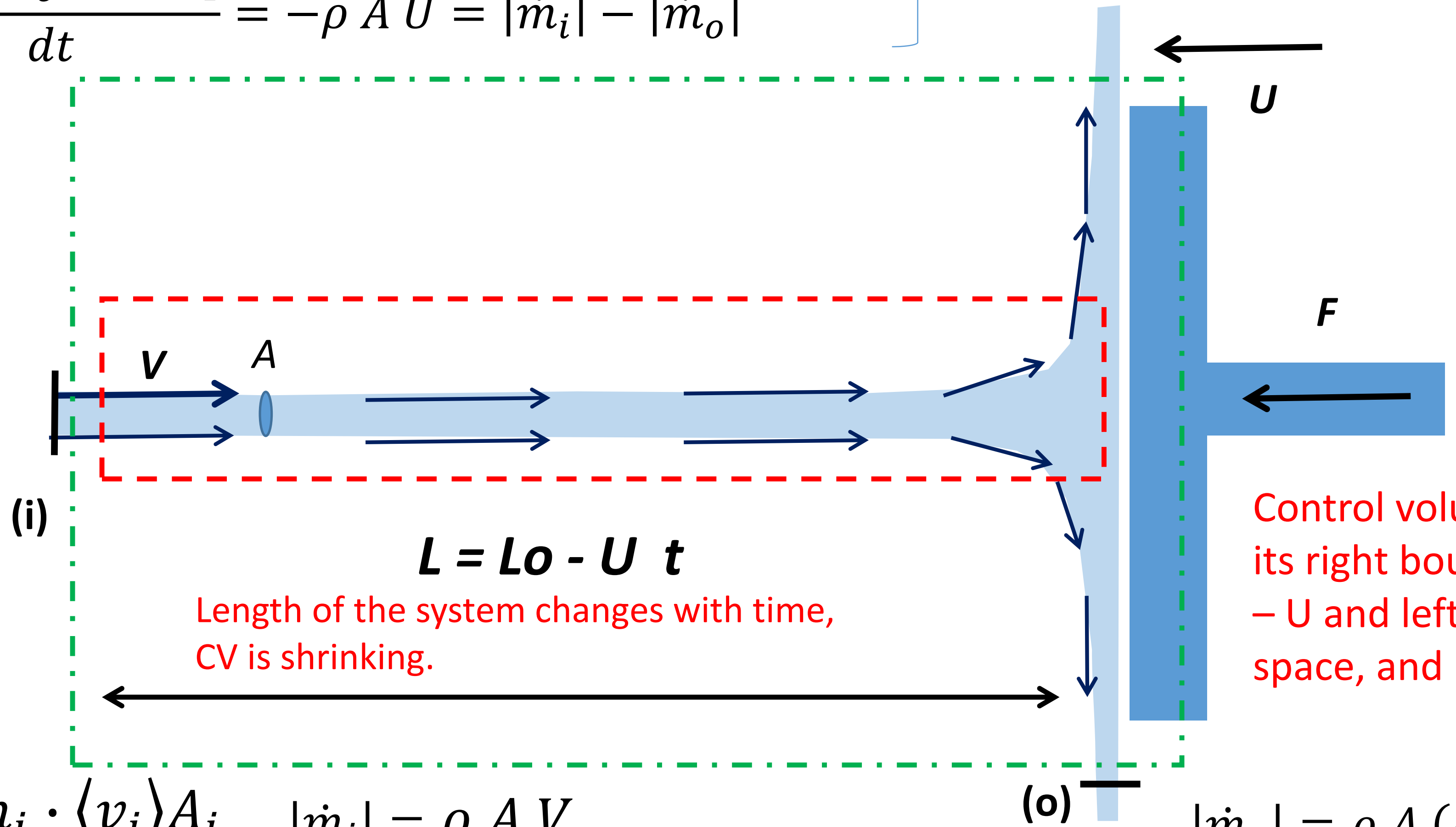
Calculate Force needed for a moving wall to maintain a constant velocity

$$\frac{dm}{dt} = |\dot{m}_i| - |\dot{m}_o|$$

$$\frac{dm}{dt} = \frac{d[\rho A(L_o - U\,t)]}{dt} = -\rho\; A\; U = |\dot{m}_i| - |\dot{m}_o|$$

$$|\dot{m}_o| = \rho\; A\; U + |\dot{m}_i|$$




Influx mass flowrate

$$\underline{n}_i = -\,\underline{\mathrm{i}}$$

$$\underline{v}_i = V\;\underline{\mathrm{i}}$$

Control volume changes in time and its right boundary moves at velocity – U and left boundary is fixed in space, and it is located at the origin.

$$\dot{m}_i = -\rho_i\, \underline{n}_i \cdot \langle \underline{v}_i \rangle A_i \qquad |\dot{m}_i| = \rho\; A\; V$$

$$|\dot{m}_o| = \rho\; A\;(U + V)$$

Recall that a uniformly translation coordinate system is an inertial coordinate system, hence momentum balance may be employed directly, anyhow in this case we used the observed, which is also an inertial frame of reference.

Calculate Force needed for a moving wall to maintain a constant velocity

Approach II

$$\frac{d[m\,\underline{v}]}{dt} = |\dot{m}_1|\,\langle \underline{v}_1\rangle\,\beta_1 - |\dot{m}_2|\,\langle \underline{v}_2\rangle\,\beta_2 + m\underline{g} + \sum_{k=i,j}[p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_{ext}$$

For a balance in "x" direction and assuming $\beta$=1

$$v_x\frac{d[m]}{dt} = |\dot{m}_i|\,\langle v_{1,x}\rangle - |\dot{m}_o|\,\langle v_{2,x}\rangle + -\underline{F}_{ext}$$

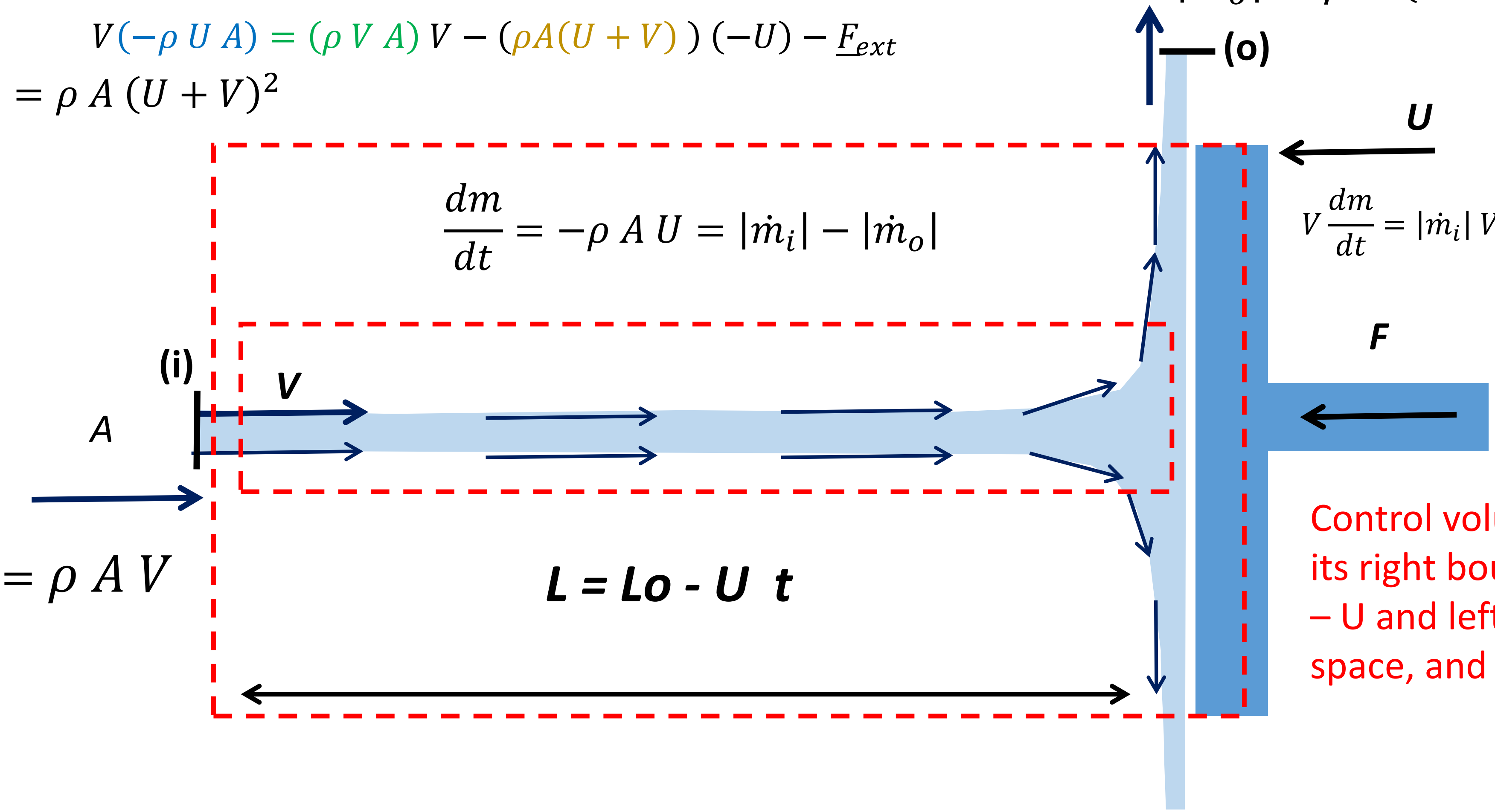
$$V(-\rho\,U\,A) = (\rho\,V\,A)\,V - (\rho A(U+V)\,)\,(-U) - \underline{F}_{ext}$$

$$F_{ext,x} = \rho\,A\,(U+V)^2$$

$$|\dot{m}_o| = \rho\,A\,(U+V)$$

$$\frac{dm}{dt} = -\rho\,A\,U = |\dot{m}_i| - |\dot{m}_o|$$

$$V\frac{dm}{dt} = |\dot{m}_i|\,V + |\dot{m}_o|\,U + -F_{ext,x}$$




$$|\dot{m}_i| = \rho\,A\,V$$

Control volume changes in time and its right boundary moves at velocity – U and left boundary is fixed in space, and it is located at the origin.

# Approach III

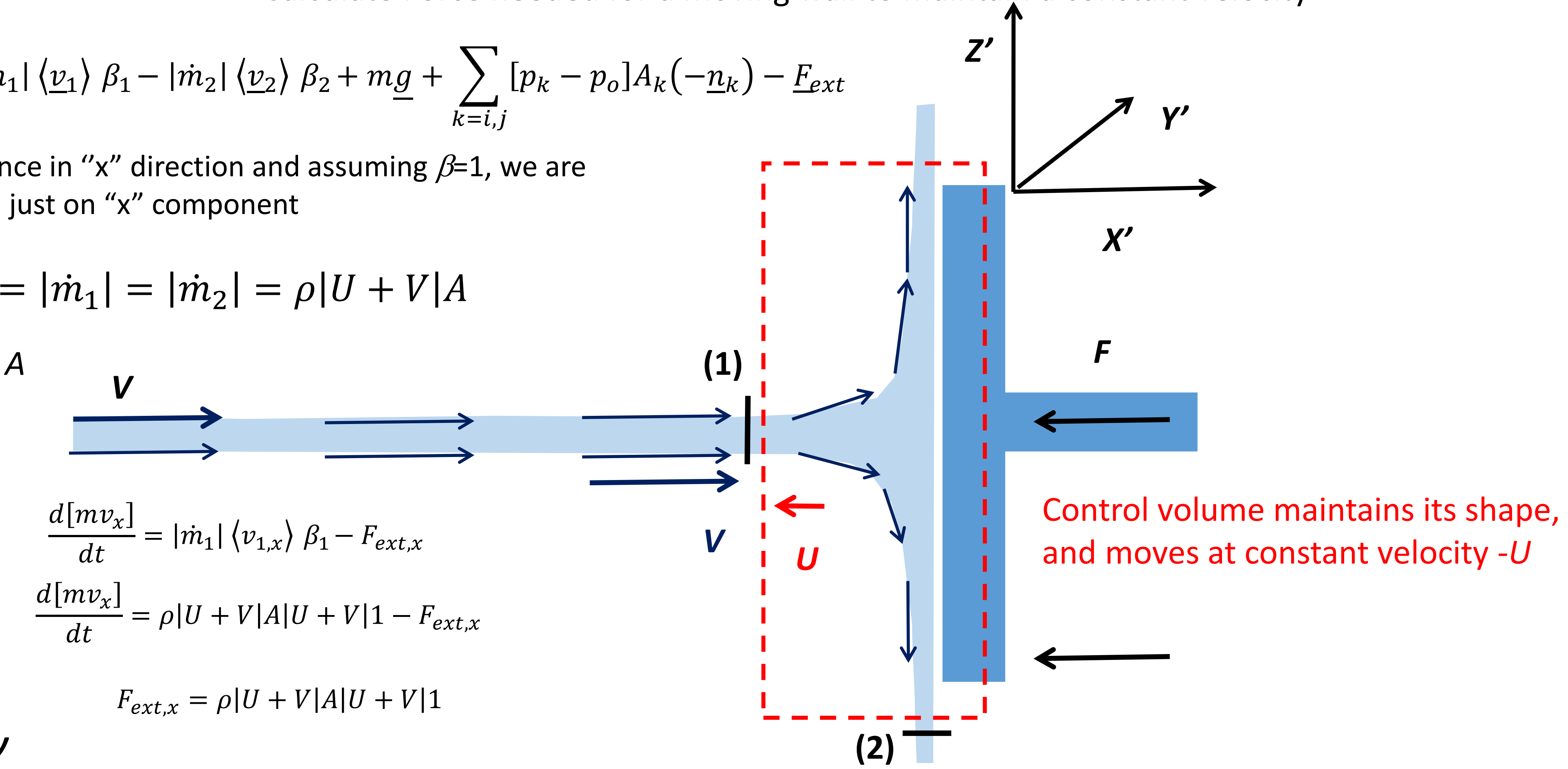
The force balance is evaluated keeping the plate as frame of reference, because is moving at constant velocity and is an inertial frame of reference

Calculate Force needed for a moving wall to maintain a constant velocity

$$\frac{d[m\,\underline{v}]}{dt} = |\dot{m}_1|\,\langle \underline{v}_1 \rangle\,\beta_1 - |\dot{m}_2|\,\langle \underline{v}_2 \rangle\,\beta_2 + m\underline{g} + \sum_{k=i,j} [p_k - p_o]A_k(-\underline{n}_k) - \underline{F}_{ext}$$

For a balance in "x" direction and assuming $\beta$=1, we are interested just on "x" component

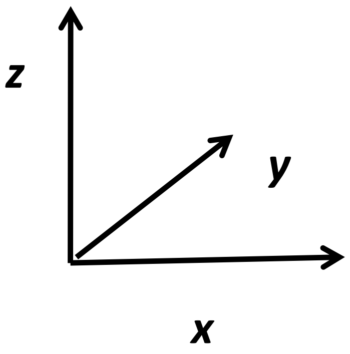
$$|\dot{m}| = |\dot{m}_1| = |\dot{m}_2| = \rho|U+V|A$$




$$\frac{d[mv_x]}{dt} = |\dot{m}_1|\,\langle v_{1,x} \rangle\,\beta_1 - F_{ext,x}$$

$$\frac{d[mv_x]}{dt} = \rho|U+V|A|U+V|1 - F_{ext,x}$$

$$F_{ext,x} = \rho|U+V|A|U+V|1$$



# Approach IV

Calculate Force needed for a moving wall to maintain a constant velocity

$\underline{n}_1 = -\,\underline{\mathrm{i}}$

$\underline{u}_1 = -U\,\underline{\mathrm{i}}$

$\underline{v}_1 = V\,\underline{\mathrm{i}}$

$\underline{w}_1 = \underline{v}_1 - \underline{u}_1$

$\underline{w}_1 = (U+V)\;\underline{\mathrm{i}}$

$\underline{n}_2 = +\,\underline{\mathrm{j}}$

$\underline{u}_2 = -U\,\underline{\mathrm{i}}$

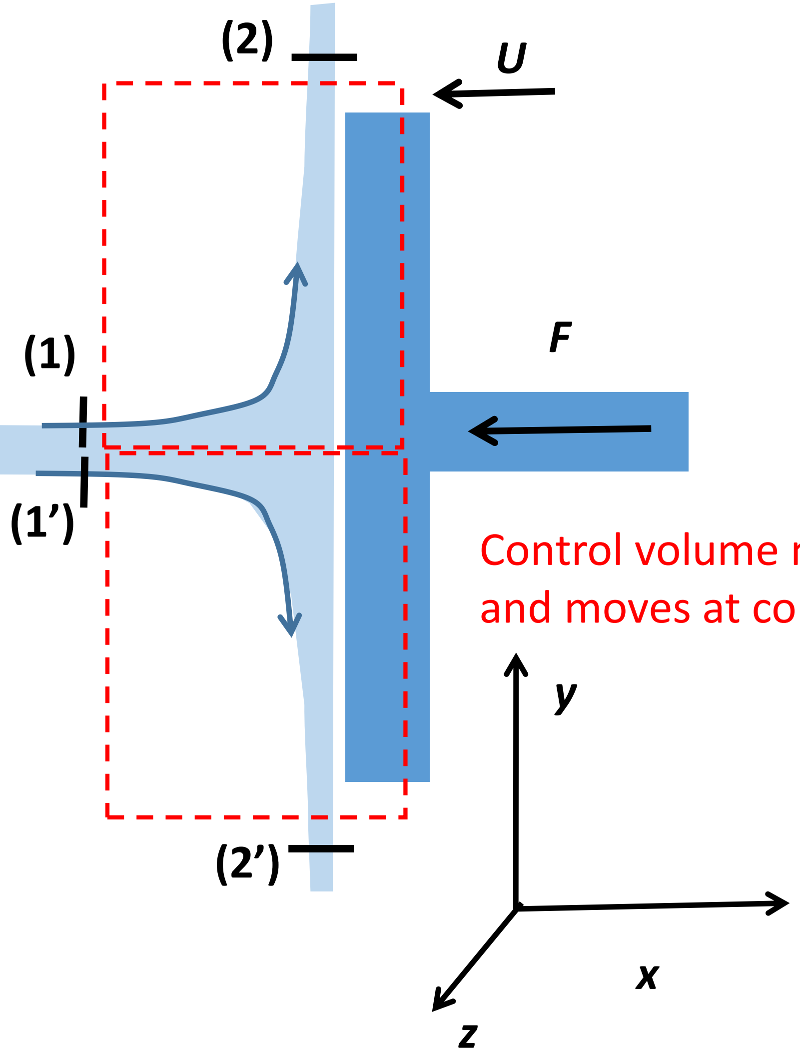
$\underline{v}_2 = -U\,\underline{\mathrm{i}} + (U+V)\;\underline{\mathrm{j}}$

$\underline{w}_2 = \underline{v}_2 - \underline{u}_2$

$\underline{w}_2 = (U+V)\;\underline{\mathrm{j}}$

Calculated with continuity equation

$A_1 = A_2$


Control volume maintains its shape, and moves at constant velocity *-U*

Assumptions:

I. Because of symmetry half of the stream is taken as a subsystem

II. Thickness of the water stream within the subsystem is kept constant ($h = h_1 + h_{1'} = h_2 + h_{2'}$)

III. Uniform velocity field, $\beta_i$=1 $\quad \underline{w}_1 = \langle \underline{w}_1 \rangle$

IV. Diagram is an upper view, gravity is in z-axis

$$\dot{m}_1 = -\rho_1\,\underline{n}_1 \cdot \langle \underline{w}_1 \rangle A_1 = \rho\,(U+V)\,(A/2)$$

$$\dot{m}_1 + \dot{m}_2 = 0 \qquad \dot{m}_2 = -\dot{m}_1 = -\rho_2\,\underline{n}_2 \cdot \langle \underline{w}_2 \rangle A_2$$

$$\underline{w}_2 = (U+V)\;\underline{\mathrm{j}}$$

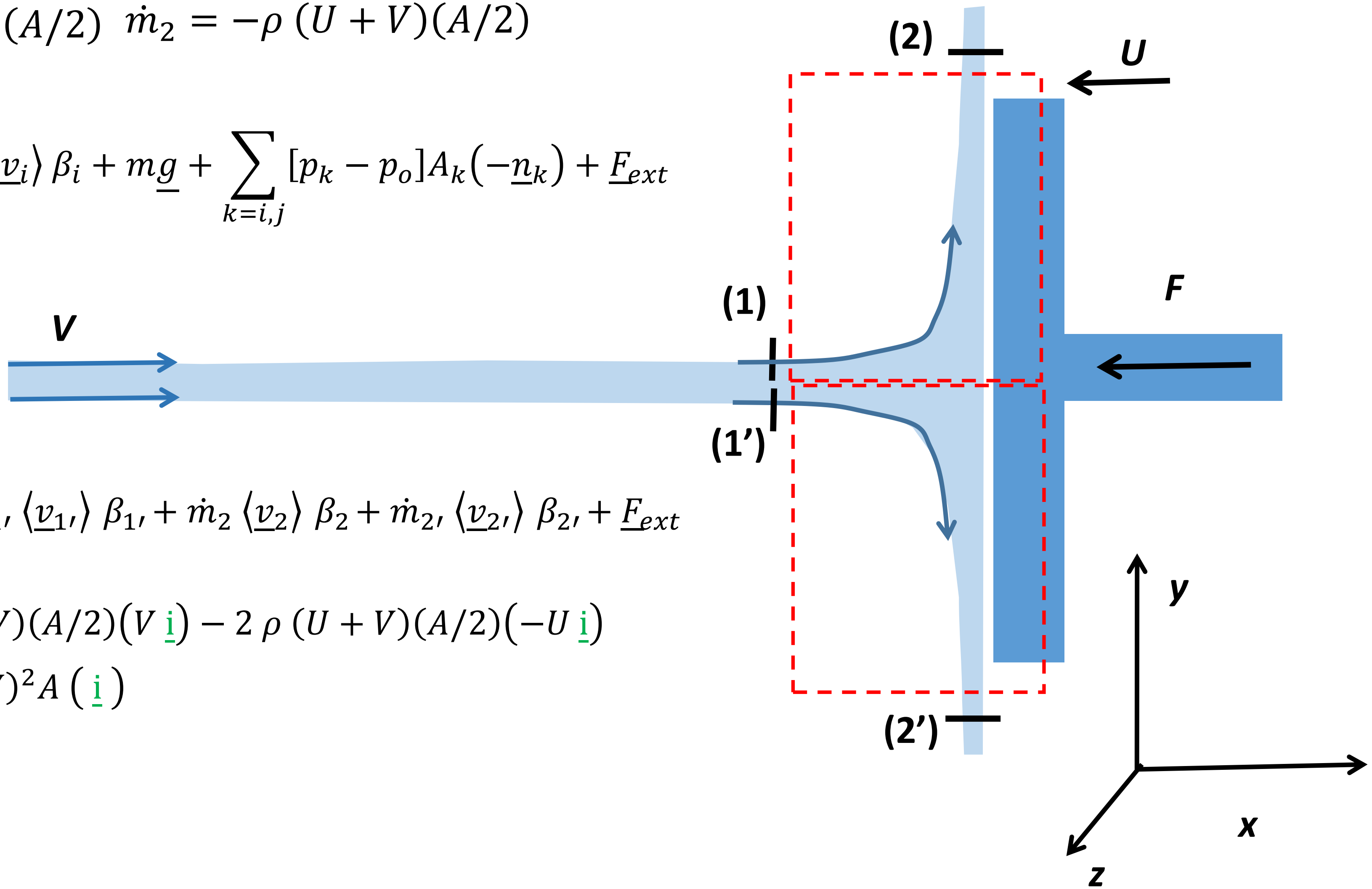
# Approach IV (linear momentum approach)

Calculate Force needed for a moving wall to maintain a constant velocity

$\underline{v}_1 = V\ \underline{\mathrm{i}}$ $\qquad$ $\underline{v}_2 = -U\ \underline{\mathrm{i}} + (U+V)\ \underline{\mathrm{j}}$

$\dot{m}_1 = \rho\,(U+V)(A/2)$ $\qquad$ $\dot{m}_2 = -\rho\,(U+V)(A/2)$

$$\frac{d[m\ \underline{v}]}{dt} = \sum \dot{m}_i \langle \underline{v}_i \rangle\, \beta_i + m\underline{g} + \sum_{k=i,j} [p_k - p_o] A_k (-\underline{n}_k) + \underline{F}_{ext}$$




$$0 = \dot{m}_1 \langle \underline{v}_1 \rangle\, \beta_1 + \dot{m}_{1'} \langle \underline{v}_{1'} \rangle\, \beta_{1'} + \dot{m}_2 \langle \underline{v}_2 \rangle\, \beta_2 + \dot{m}_{2'} \langle \underline{v}_{2'} \rangle\, \beta_{2'} + \underline{F}_{ext}$$

$$-\underline{F}_{ext,x} = 2\,\rho\,(U+V)(A/2)\big(V\ \underline{\mathrm{i}}\big) - 2\,\rho\,(U+V)(A/2)\big(-U\ \underline{\mathrm{i}}\big)$$

$$\underline{F}_{ext,x} = -\,\rho\,(U+V)^2 A\,(\ \underline{\mathrm{i}}\ )$$

$$\underline{F}_{ext,x} = -\underline{\mathrm{i}}\ F$$

$\underline{g} = -g\ \underline{\mathrm{k}}$

# Approach V (energy approach)

Calculate Force needed for a moving wall to maintain a constant velocity

$\underline{v}_1 = V\,\underline{\mathrm{i}}$ $\qquad\qquad$ $\underline{v}_2 = -U\,\underline{\mathrm{i}} + (U+V)\,\underline{\mathrm{j}}$

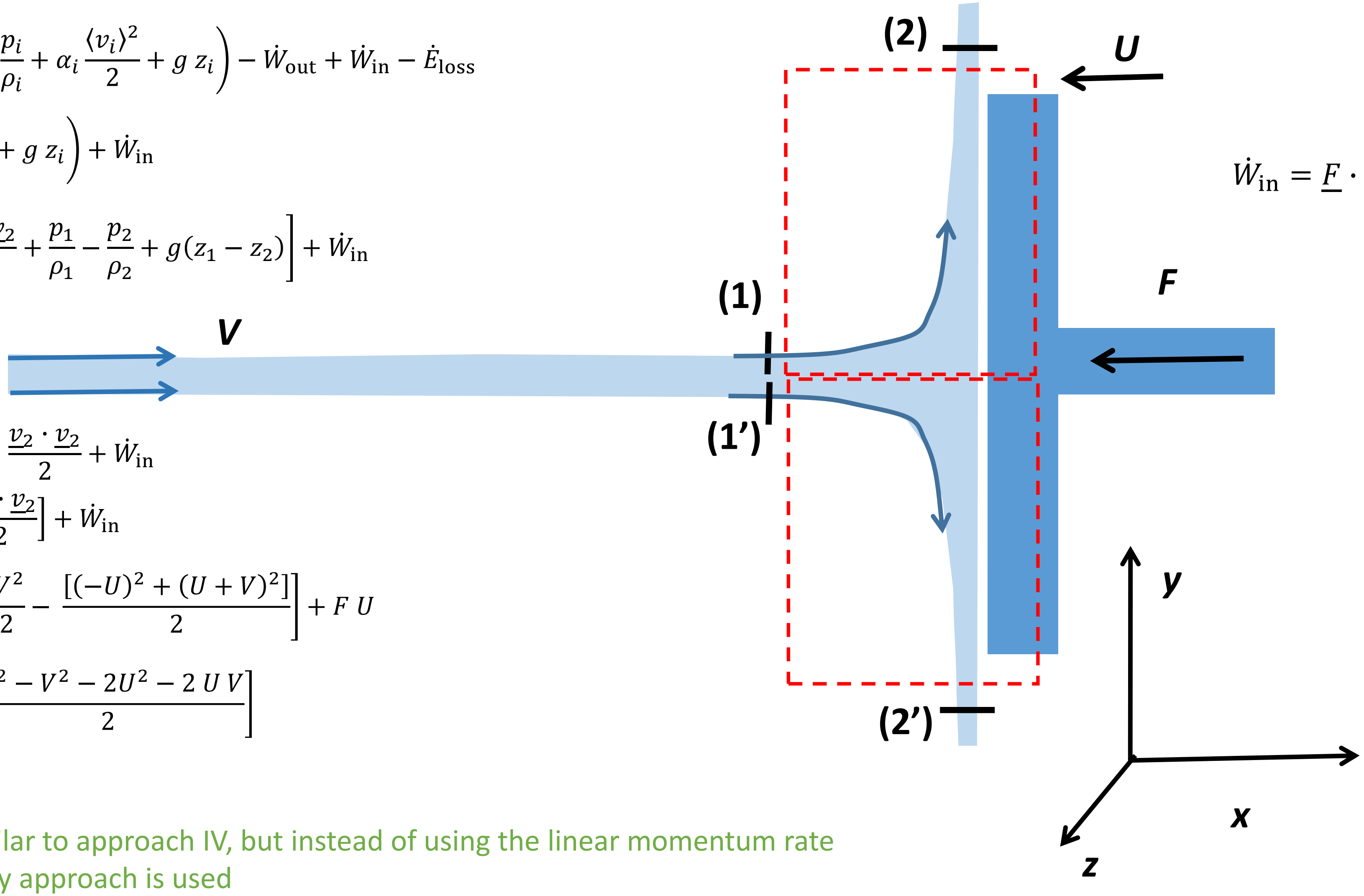
$\dot{m}_1 = \rho\,(U+V)(A/2)$ $\qquad\qquad$ $\dot{m}_2 = -\rho\,(U+V)(A/2)$

$$\frac{d\left[m\left(\widehat{K}+\widehat{\Phi}\right)\right]}{dt} = \sum \dot{m}_i\left(\frac{p_i}{\rho_i} + \alpha_i\frac{\langle v_i\rangle^2}{2} + g\,z_i\right) - \dot{W}_{\mathrm{out}} + \dot{W}_{\mathrm{in}} - \dot{E}_{\mathrm{loss}}$$

$$0 = \sum \dot{m}_i\left(\frac{p_i}{\rho_i} + \alpha_i\frac{\langle v_i\rangle^2}{2} + g\,z_i\right) + \dot{W}_{\mathrm{in}}$$

$$0 = 2\,\dot{m}_1\left[\frac{\underline{v}_1\cdot\underline{v}_1}{2} - \frac{\underline{v}_2\cdot\underline{v}_2}{2} + \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} + g(z_1 - z_2)\right] + \dot{W}_{\mathrm{in}}$$

$$\dot{W}_{\mathrm{in}} = \underline{F}\cdot\underline{U} = \left(-F\,\underline{\mathrm{i}}\right)\cdot\left(-U\,\underline{\mathrm{i}}\right) = F\,U$$




$$0 = 2\,\dot{m}_1\frac{\underline{v}_1\cdot\underline{v}_1}{2} + 2\,\dot{m}_2\frac{\underline{v}_2\cdot\underline{v}_2}{2} + \dot{W}_{\mathrm{in}}$$

$$0 = 2\,\dot{m}_1\left[\frac{\underline{v}_1\cdot\underline{v}_1}{2} - \frac{\underline{v}_2\cdot\underline{v}_2}{2}\right] + \dot{W}_{\mathrm{in}}$$

$$0 = 2\rho\,(U+V)(A/2)\left[\frac{V^2}{2} - \frac{\left[(-U)^2 + (U+V)^2\right]}{2}\right] + F\,U$$

$$-F\,U = \rho\,(U+V)A\left[\frac{V^2 - V^2 - 2U^2 - 2\,U\,V}{2}\right]$$

$$F = \rho\,(U+V)^2 A$$

$\underline{g} = -g\,\underline{\mathrm{k}}$

This approach is similar to approach IV, but instead of using the linear momentum rate equation , the energy approach is used

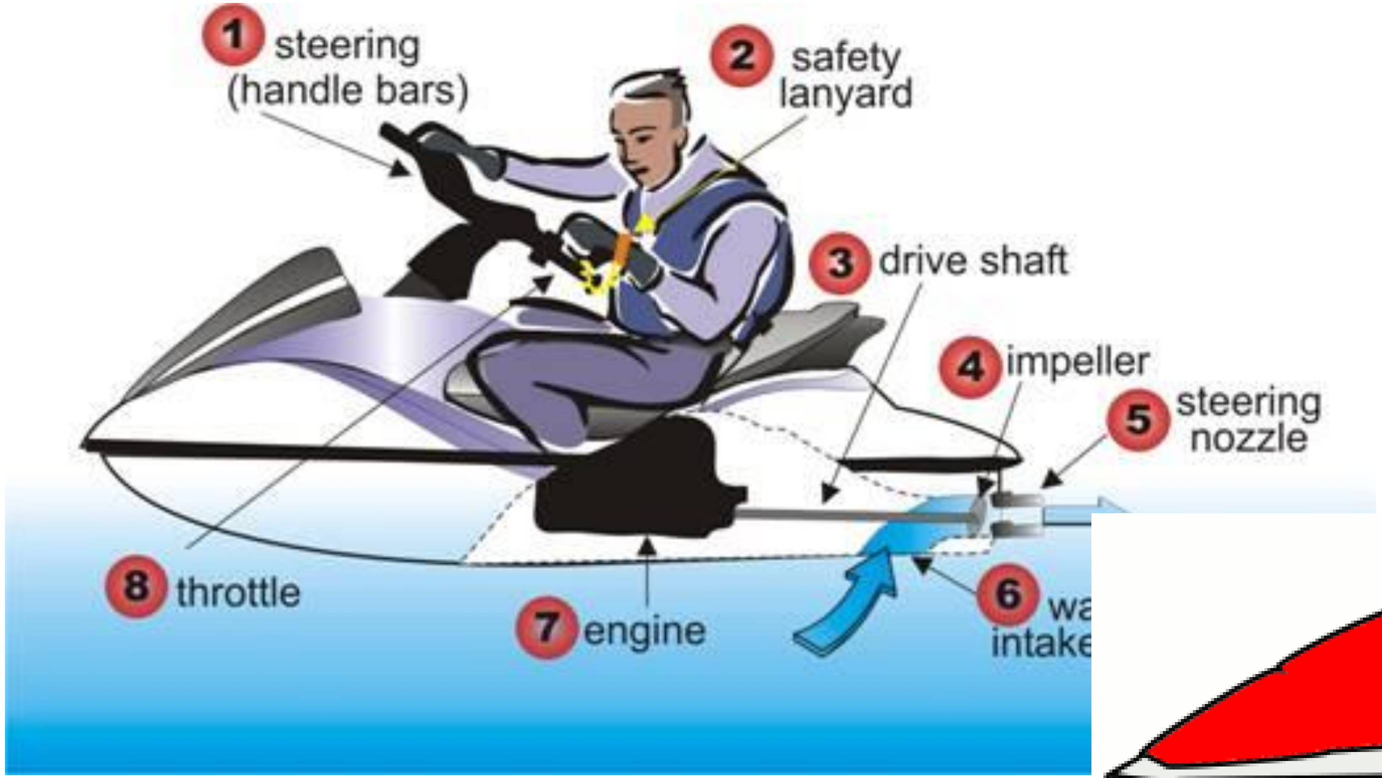
## Conclusion

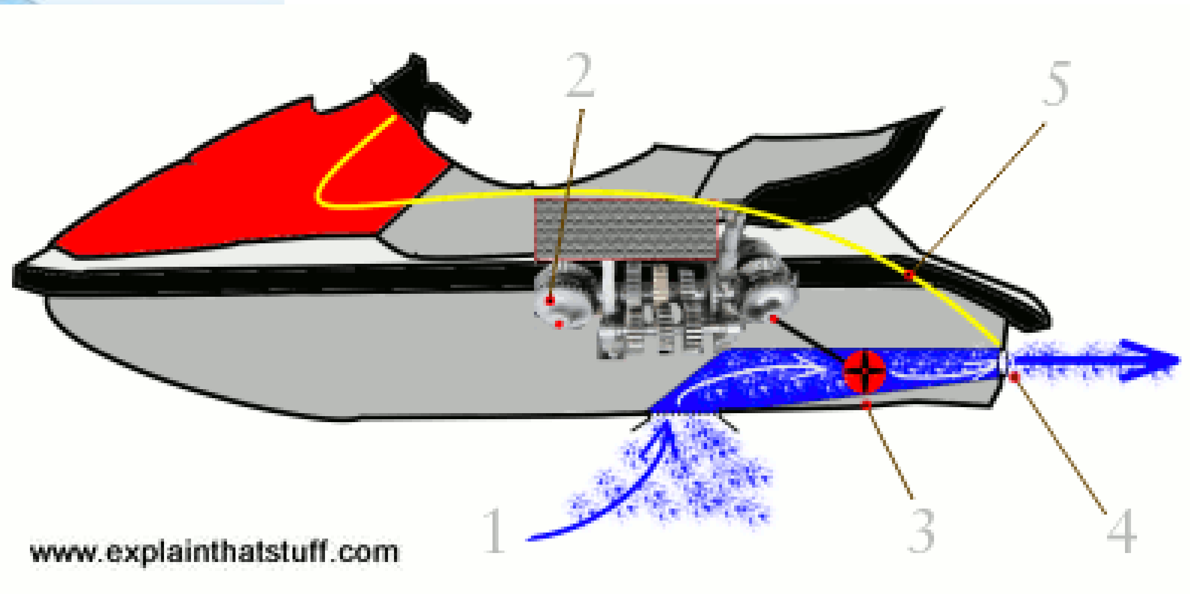
The answer should be the same regardless the selection of the control volume or the frame of reference (FOR), or the approach

## Caution !

If the siting frame of reference over the moving control volume is not an inertial FOR, for the LMRB, AMRB and MERB should be done using an inertial FOR, but the mass flow rate calculation done with the flow rate crossing the moving boundary.

LMRB=linear momentum rate balance, AMRB=Angular momentum rate balance, MERB=Mechanical energy rate balance


1 steering (handle bars)
2 safety lanyard
3 drive shaft
4 impeller
5 steering nozzle
6
7 engine
8 throttle


2
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1
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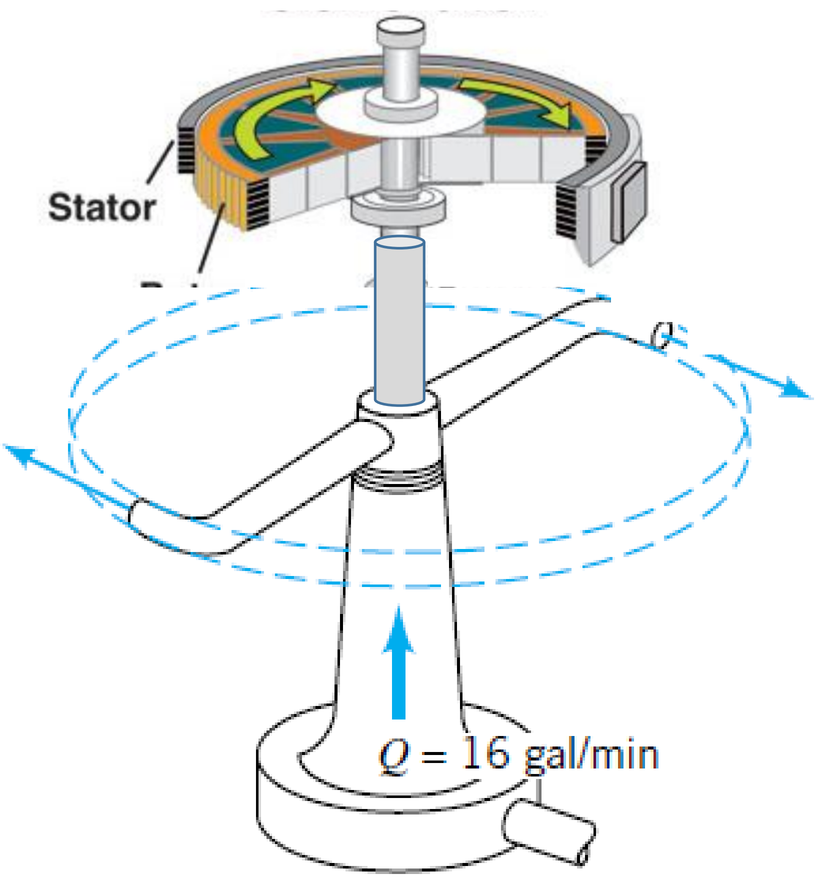
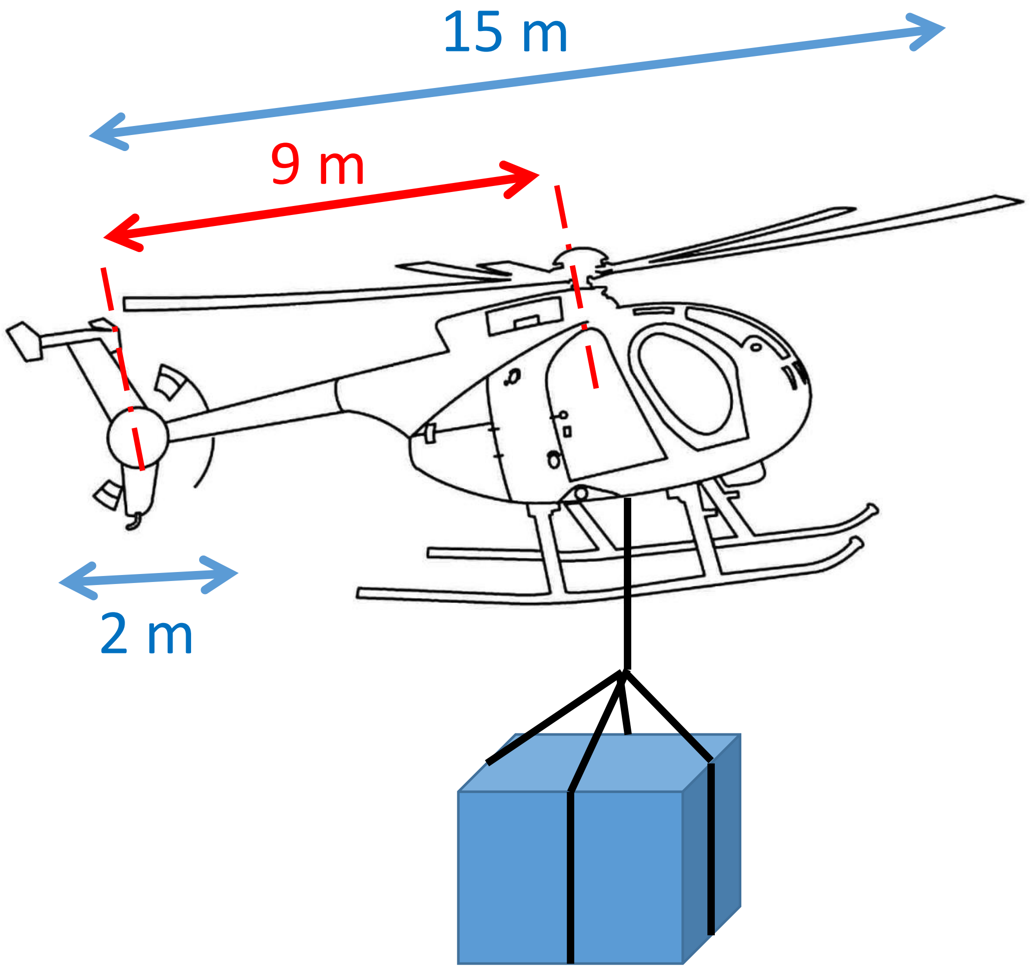
4
www.explainthatstuff.com

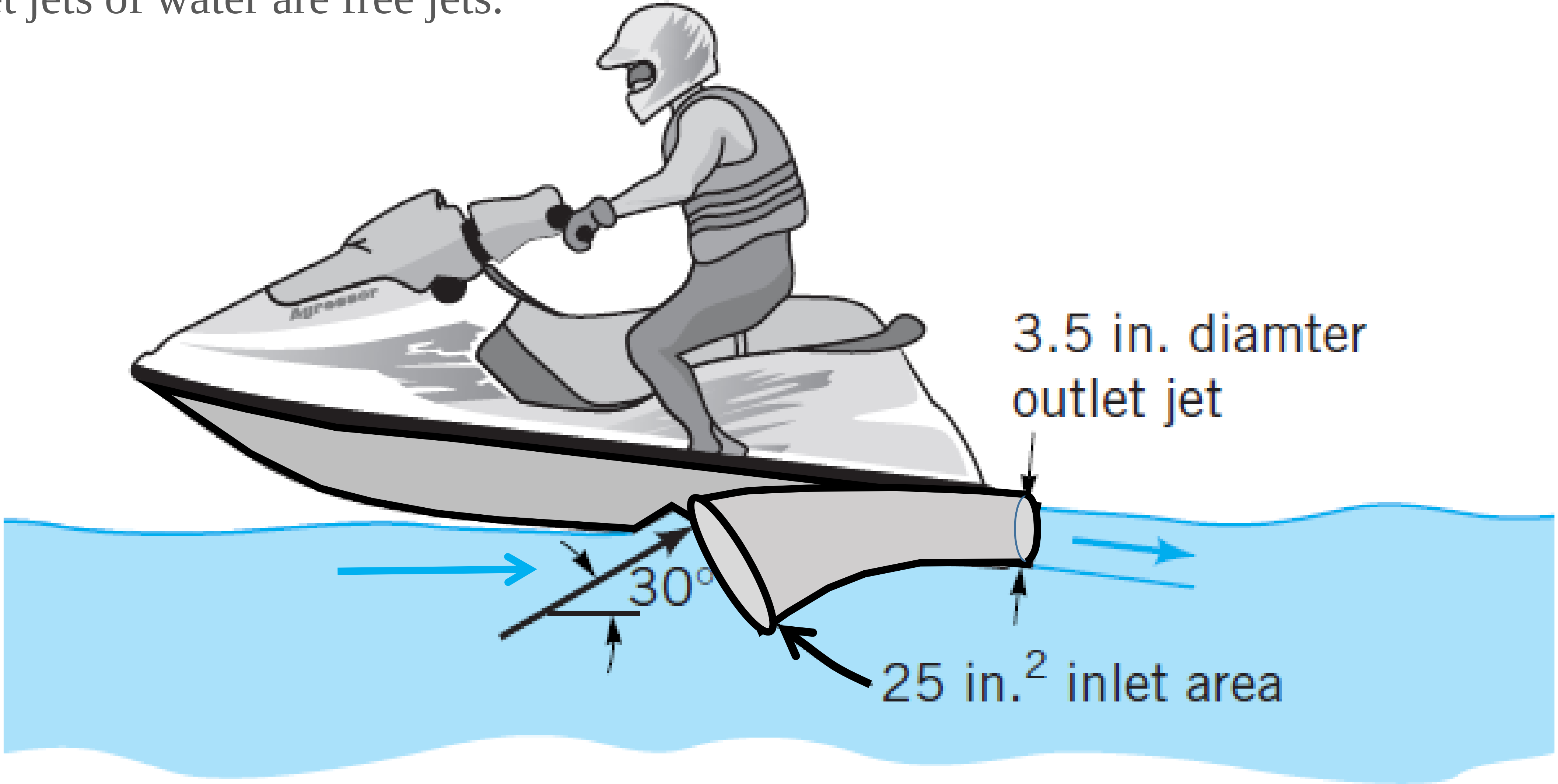

Stator
$Q$ = 16 gal/min

PW6.5 An Unloaded helicopter of mass 10,000 kg hovers at sea level while it is being loaded. In the unloaded hover mode, the blades rotate at 400 rpm. The horizontal blades above the helicopter cause a 15-m-diameter air mass to move downward at an average velocity proportional to the overhead blade rotational velocity (rpm). A load of 15,000 kg is loaded onto the helicopter, and the helicopter slowly rises. Determine (a) the volumetric airflow rate downdraft that the helicopter generates during unload hover and the required power input and (b) the rpm of the helicopter blades to hover with the 15,000 kg load and the required power input. Take the density of air of atmospheric air to be 1.18 kg/m$^3$. Assume air approaches the blades from the top through a large area with negligible velocity and air is forced by the blades to move down with a uniform velocity through and imaginary cylinder whose base is the blade span area. (c) How much extra power is needed to prevent the helicopter from spinning,

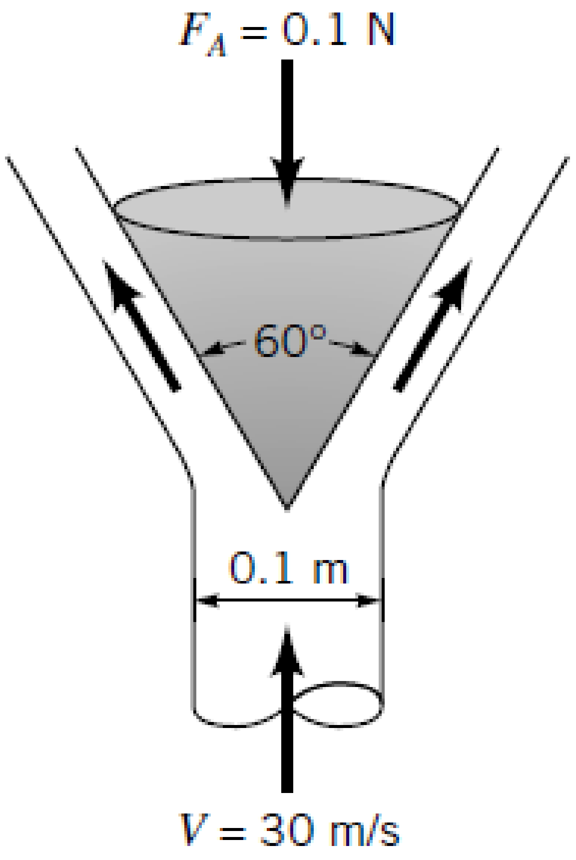



Load 15,000 kg

Problem No. 2 The thrust developed to propel the jet ski shown in the figure is a result of water pumped through the vehicle and exiting as a high-speed jet. For the conditions shown in the figure, what flow rate is needed to produce a 300 lb thrust ? Assume the inlet and the outlet jets of water are free jets.



**Problem E.1** A vertical, circular cross-sectional jet of air strikes a conical deflector as indicated in the figure. A vertical anchoring force of 0.1 N is required to hold the deflector in place. Determine the mass (kg) of the deflector. The magnitude of velocity of the air remains constant.



$V = 30 \ \ [\mathrm{m/s}]$

$\rho = 1.18 \ \ [\mathrm{kg/m^3}]$

$F_A = 0.1 \ \ [\mathrm{N}]$

$D = 0.1 \ \ [\mathrm{m}]$

$$A = \pi \cdot \frac{D^2}{4}$$

$\theta = 60 \ \ [\mathrm{deg}]$

$g = g\#$

## Force balance

$$-F_A - m \cdot g + \dot{m} \cdot V - \dot{m} \cdot V \cdot \sin(\theta) = 0$$

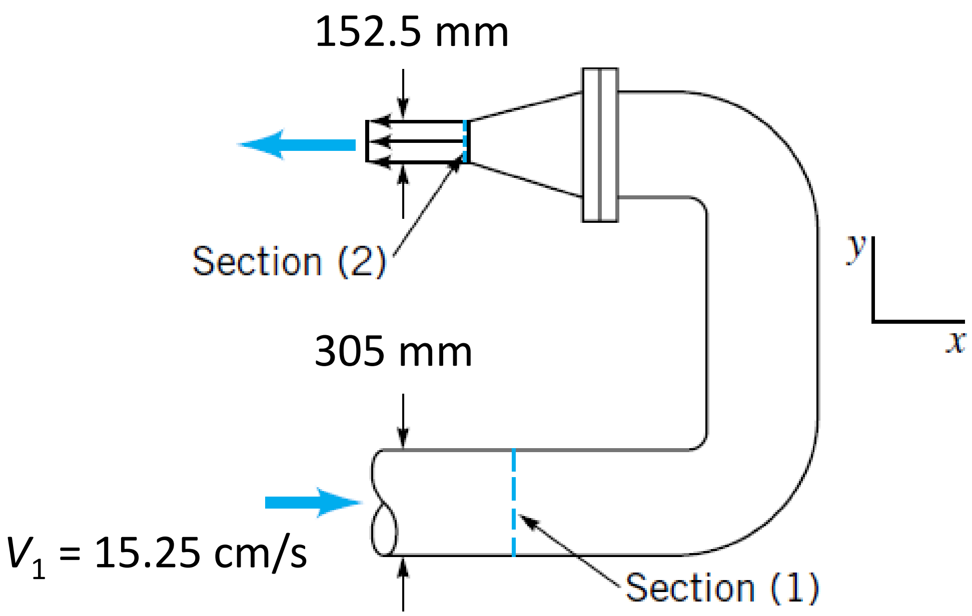
$$\dot{m} = \rho \cdot V \cdot A$$

# Solution

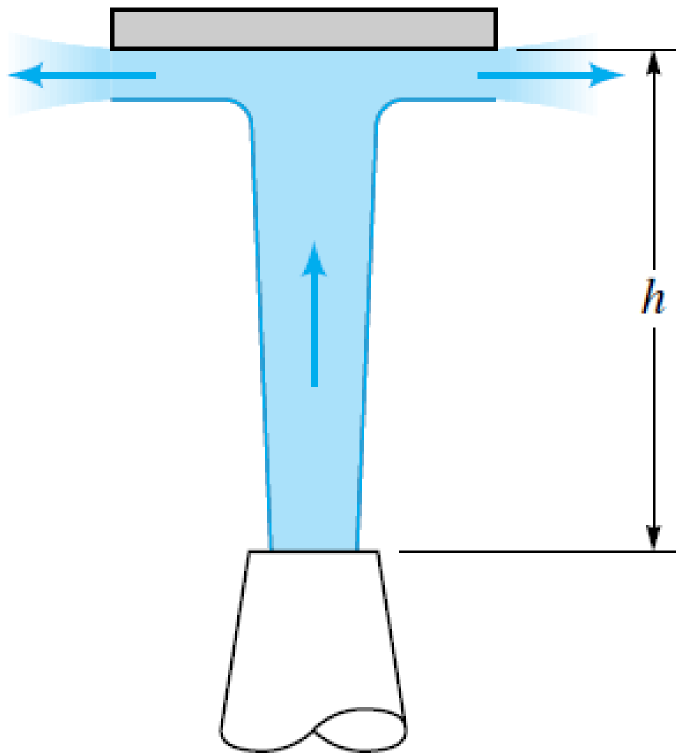
| | |
|---|---|
| $A = 0.007854 \ [\mathrm{m^2}]$ | $D = 0.1 \ [\mathrm{m}]$ |
| $F_A = 0.1 \ [\mathrm{N}]$ | $g = 9.807 \ [\mathrm{m/s^2}]$ |
| $m = 0.1037 \ [\mathrm{kg}]$ | $\dot{m} = 0.278 \ [\mathrm{kg/s}]$ |
| $\rho = 1.18 \ [\mathrm{kg/m^3}]$ | $\theta = 60 \ [\mathrm{deg}]$ |
| $V = 30 \ [\mathrm{m/s}]$ | |

**Problem E.2** For the 180° elbow and nozzle flow shown in the figure, which operates with water ($\rho$=997 kg/m$^3$), determine:

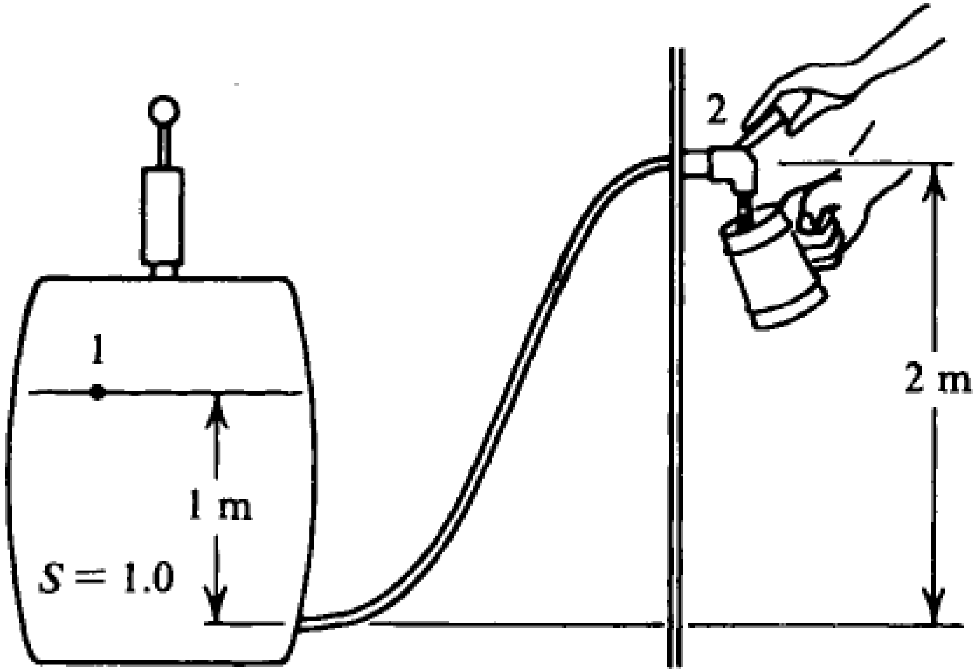
(a) The mass flow rate.
(b) The velocity in section (2)
(c) The pressure at section (1)
(d) The force required to hold the elbow-nozzle system in place. (Section 2 discharges to the atmosphere, and all the system is at ground level)



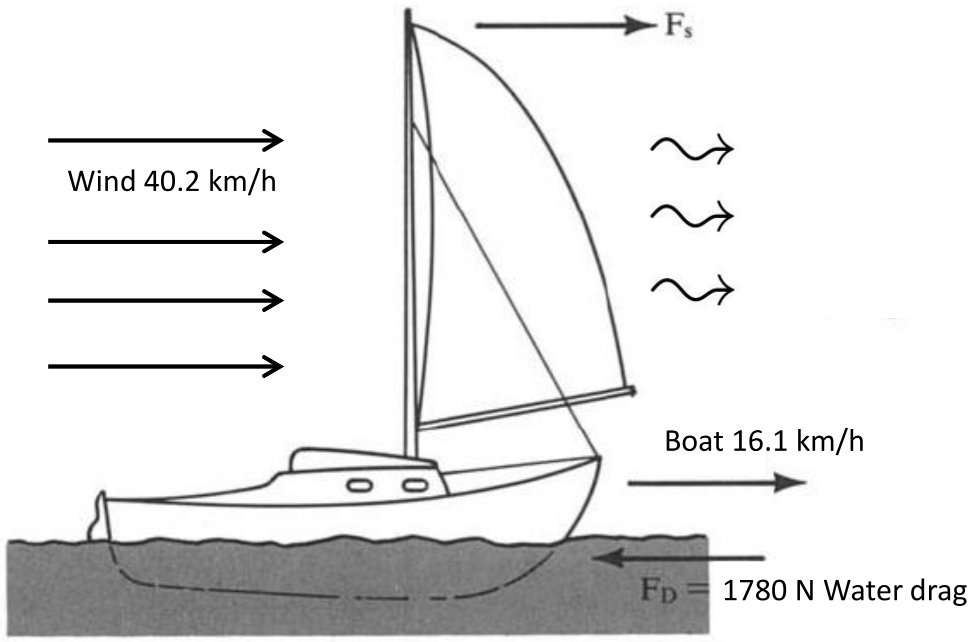
**Problem E.3** A vertical jet of water leaves a nozzle at a speed of 10 m/s and a diameter of 20 mm. It suspends a plate having a mass of 1.5 kg. What is the vertical distance *h* ?

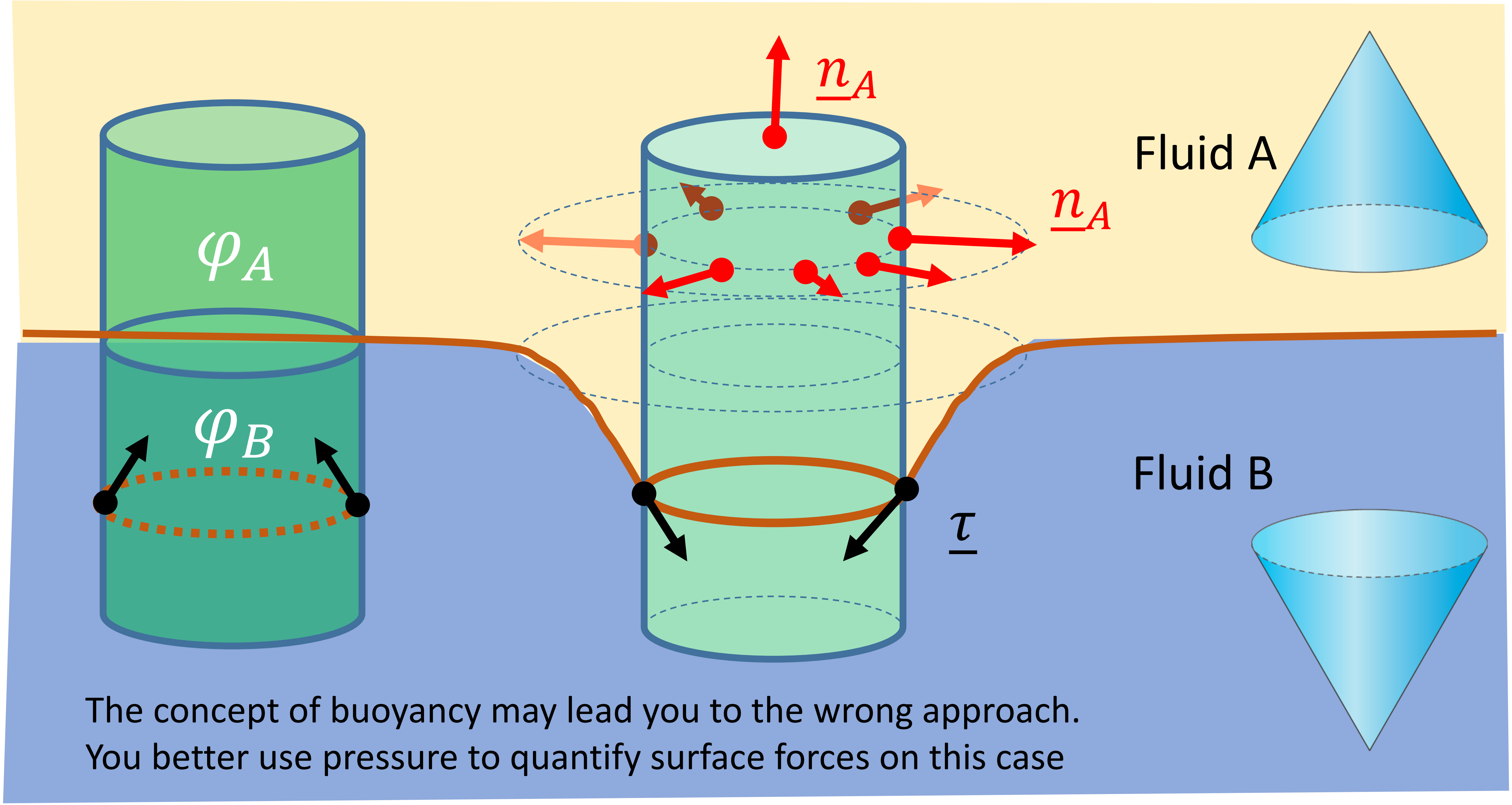


**Problem E.4** As demonstrated by figure, in order to control the energy on beer exiting the tap the exit velocity must be 1 cm/s. If one stroke of the pump adds 100 Pa to the keg, what is the maximum number of pumps that can be added and still have suitable energy ?. The initial pressure in the keg is atmospheric. Assume energy losses of 5 $m^2/s^2$ (i.e. 5.0 J/kg).



**Problem E.5** A Boat is sailing downwind with just its spinnaker up. If the wind speed is 40.2 km/h and the boat is going 16.1 km/h, what is the forward force generated in the spinnaker ? Assume the hull drag is 1780 N at 16.1 km/h. The average velocity of air leaving the spinnaker is 19.3 km/h, and the effective area of the spinnaker is 8.36 $m^2$




$\underline{n}_A$
$\underline{n}_A$
Fluid A
$\varphi_A$
$\varphi_B$
Fluid B
$\underline{\tau}$
The concept of buoyancy may lead you to the wrong approach.
You better use pressure to quantify surface forces on this case

# Conclusion of Macroscopic Conservation Equations.

(integral form of conservation equations, integral over the CV)

Conservation laws in fluid mechanics are used to analyze (in order to  design, control, optimize, evaluate, etc.) processes, devices and/or engines operating or interacting with fluids. These equations are:

- **Mass** rate equation
- **Linear momentum** rate equation
- **Angular Momentum** rate equation
- **Mechanical Energy** rate equation

This analysis with the aid of the conservation rate equations is needed to:

I) Quantify the fluid requirements.

II) Track materials, forces, torques and energy involved, stored or depleted.

III) Quantify forces and torques involved in the operation and the stability of processes or devices.

III) Quantify energy requirements, energy potentials, and energy losses.

The rate equations, can be used…

a) Independent of each other (only one or some)
b) Coupled or simultaneous ( some or all of them)
c) Alternate (one or another)
d) The same set of the equations on different portions of the system (over each subsystem, or component/element/section)

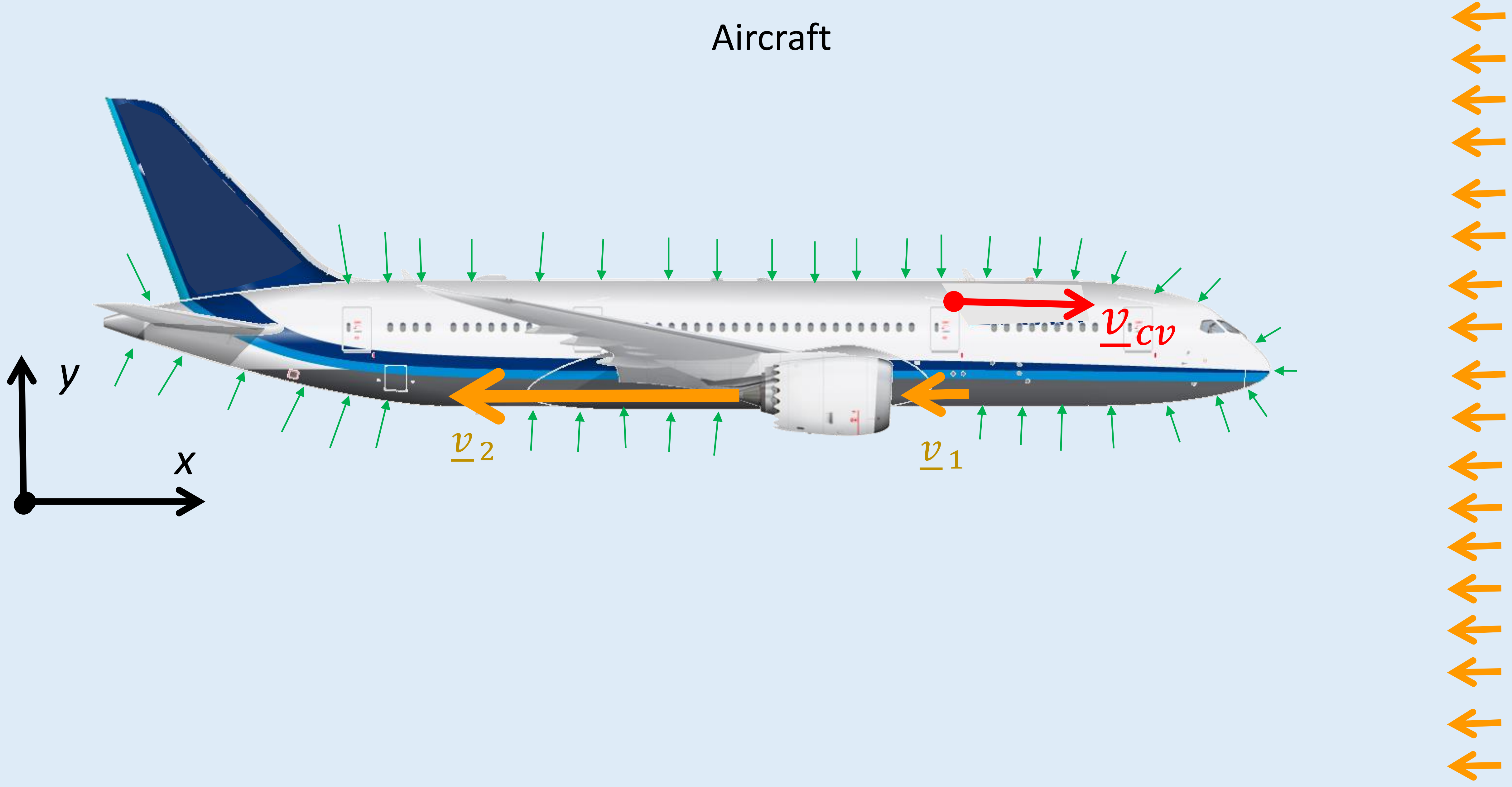
## Quiz:

Make a free body diagram on each device, and identify, label and indicate the direction of all the forces acting over the device, and specify if that force is important or negligible, and its role.

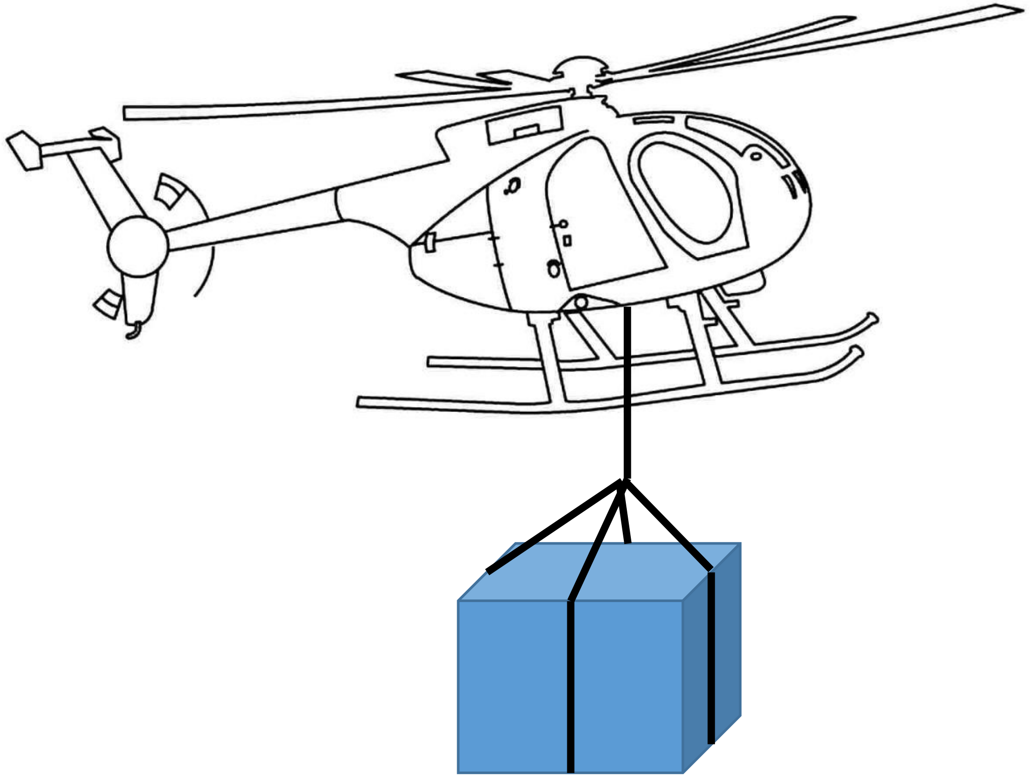
## Note:

If you need an extra sketch feel free to add it, even if you want to include an upper view , frontal view, or side view.

# Aircraft

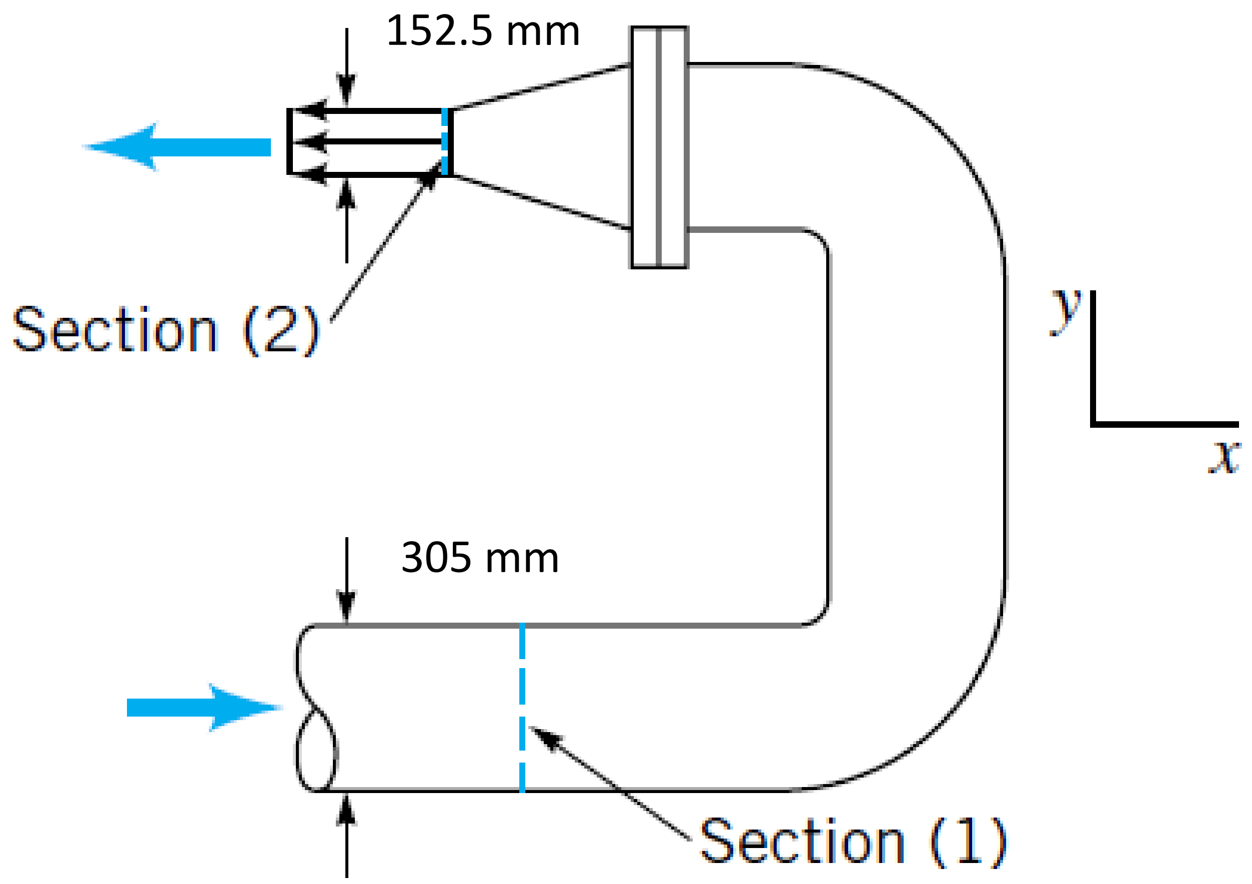


# Helicopter



Blimp

# U. bend



# Hovercraft

Race car

Sailboat

End of the quiz

By: Dr. José Luis López Salinas
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