2 ¿Para qué puntos $P: (x, y, z)$ el gradiente $f = 25x^2 + 9y^2 + 16z^2$ tiene la dirección paralela a una línea que vaya desde P al origen?

- The gradient of a function $f(x, y, z)$ is denoted by grad f or ∇f and is defined as:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla f = (50x, 18y, 32z)$$

$P = (x, y, z) \rightarrow$ point P

$O = (0, 0, 0) \rightarrow$ origin

$\overrightarrow{PO} = (-x, -y, -z) \longrightarrow$ vector from point P to the origin

- ∇f & $\overrightarrow{PO}$ are parallel if they are scalar multiples of one another

$$\frac{\nabla f_i}{\overrightarrow{PO}_i} = \frac{\nabla f_j}{\overrightarrow{PO}_j} = \frac{\nabla f_k}{\overrightarrow{PO}_k}$$, then ∇f & $\overrightarrow{PO}$ are parallel.

$$\frac{50x}{-x} = \frac{18y}{-y} = \frac{32z}{-z} \Rightarrow \underbrace{-50 \neq -18 \neq -32}$$

There are no $P: (x, y, z)$ for which ∇f has a parallel direction to a line that goes from point P to the origin. //

4 En algunos casos simples, se pueden obtener por inspección potencial de un campo vectorial dado. Encuentra un potencial $f(x, y, z)$ para el campo $v = \nabla f$ con $v = [ye^x, e^x, z^2]$

$$\nabla f = [ye^x, e^x, z^2]$$
$$= \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

$$\frac{\partial f}{\partial x} = ye^x \Rightarrow f = ye^x \quad \frac{\partial f}{\partial y} = e^x \Rightarrow f = ye^x \quad \frac{\partial f}{\partial z} = \frac{1}{3}z^3$$

$$\Rightarrow f = ye^x + \frac{z^3}{3} \; //$$

6 Sea el flujo de pozole con vector velocidad $v = x\hat{\imath}$. Muestra que las partículas individuales de pozolito tienen vectores de posición $r(t) = C_1 e^t \hat{\imath} + C_2 \hat{\jmath} + C_3 \hat{k}$, donde C_1, C_2, C_3 son constantes. En $t=0$, las partículas están en un cubo cuyas caras son porciones de los planos $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$. Muestra que en $t=1$, las partículas ocupan el volumen e. ¿Este fluido es compresible o incompresible?

- Let's calculate

$$\operatorname{div} v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$
$$= \frac{\partial}{\partial x} x + 0 + 0$$
$$= 1$$

If
$$v = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = r'(t) \quad \ldots (1)$$

- from (1), let's prove that $r(t)$ is in the form $C_1 e^t \hat{\imath} + C_2 \hat{\jmath} + C_3 \hat{k}$

$$r'(t) = \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = (x, 0, 0) \Rightarrow \frac{\partial x}{\partial t} = x, \quad \frac{\partial y}{\partial t} = 0, \quad \frac{\partial z}{\partial t} = 0$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$x = C_1 e^t$ $\quad$ y is constant $\quad$ z is constant //

- at $t=0$

$$r(0) = (C_1 e^0, C_2 + C_3) = (C_1, C_2, C_3)$$

$\downarrow$

cube of volume $1 \times 1 \times 1 = 1$

- at $t=1$

$$r(1) = (C_1 e^1, C_2, C_3)$$

$\downarrow$

rectangular box of volume $e \times 1 \times 1 = e$

$\Rightarrow$ the volume grew $e-1$ in the x-axis. //

8 Sea v la velocidad de un fluido constante. Para los incisos (a) y (b) siguientes, ¿es un fluido irrotacional? ¿Es incomprensible? Encuentra las líneas de flujo (las trayectorias de las partículas) en cada caso.

- The curl of the vector function v or the vector field given by v is defined by the simbolic determinant

$$\text{curl } v = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{bmatrix}$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

a $v = [\sec(x), \csc(x), 0]$

$$\text{curl } v = \cancel{\left(\frac{\partial 0}{\partial y} - \frac{\partial \csc(x)}{\partial z}\right)}^{0}\hat{i} + \cancel{\left(\frac{\partial \sec(x)}{\partial z} - \frac{\partial 0}{\partial x}\right)}^{0}\hat{j} + \left(\frac{\partial \csc(x)}{\partial x} - \cancel{\frac{\partial \sec(x)}{\partial y}}^{0}\right)\hat{k}$$

$$= [0, 0, -\cot(x)\csc(x)]$$

$\boxed{b}$ $v = [-y, x, \pi]$

$$\text{Curl } v = \left(\cancel{\frac{\partial \pi}{\partial y} - \frac{\partial x}{\partial z}}^{\,0}, \cancel{\frac{\partial(-y)}{\partial z} - \frac{\partial \pi}{\partial x}}^{\,0}, \frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y} \right)$$

$$= (0, 0, 1-(-1))$$

$$= (0, 0, 2) \quad //$$

$\boxed{1}$ La fuerza de un campo electroestático dado por $f(x, y, z)$, tiene la dirección del gradiente. Encuentra ∇f y su valor en el punto P dado.

$\boxed{a}$ $f = \dfrac{x}{x^2+y^2}$, $P: (1, 1)$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) \qquad \frac{u'v - uv'}{v^2} \; ; \left(\frac{u}{v}\right)$$

$$= \left(\frac{-x^2+y^2}{(x^2+y^2)^2}, -\frac{2xy}{(x^2+y^2)^2}, 0 \right)$$

$$\nabla f(1,1) = \left(\frac{-1^2+1^2}{(1^2+1^2)^2}, -\frac{2(1)(1)}{(1^2+1^2)^2}, 0 \right)$$

$$= \left(\frac{1}{2}, -\frac{1}{2}, 0 \right) \quad //$$

$\boxed{b}$ $f = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, $P: (12, 0, 16)$ $\qquad \int nx^{n-1}$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\, 2x$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2y$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, 2z$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= \left(-x(x^2+y^2+z^2)^{-3/2}, -y(x^2+y^2+z^2)^{-3/2}, -z(x^2+y^2+z^2)^{-3/2}\right)$$

$$\nabla f(12,0,16) = \left(-12(12^2+0^2+16^2)^{-3/2}, \cancelto{0}{-0(12^2+0^2+16^2)^{-3/2}}, -16(12^2+0^2+16^2)^{-3/2}\right)$$

$$= \left(-\frac{3}{2000}, 0, -\frac{1}{500}\right)$$