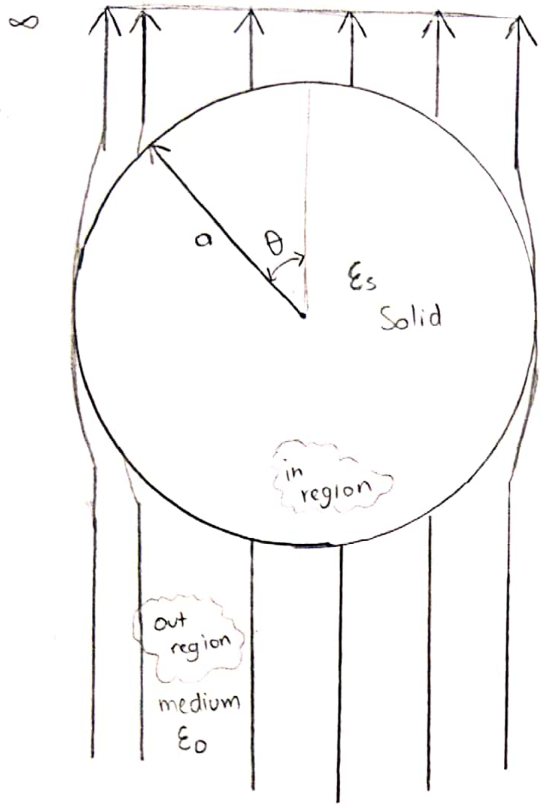

○ Derivation of the effective dipole moment of a dielectric sphere in a dielectric medium.

we know that taking the gradient of the electric potential we get the electrostatic field

$$E = -\nabla V$$

And taking the divergence of the electrostatic field we obtain the Laplace's equation (in the particular case of free space)

$$\nabla^2 V = 0$$

So in the case of the dielectric sphere and the free space surrounding it, Laplace's equation must hold. Solutions to the Laplace equation are called harmonic functions, it is said that such solutions are unique when the potential has the correct values at the boundaries and it satisfies the Laplace equation.

For this problem the boundary conditions are summarized:

(1) V is continuous at $r=a$

$$V_{out}(r=a) = V_{in}(r=a)$$

(2) The normal component of D is continuous at $r=a$

$$\varepsilon_s \left.\frac{\partial V_{in}}{\partial r}\right|_{r=a} = \varepsilon_0 \left.\frac{\partial V_{out}}{\partial r}\right|_{r=a}$$

(3) $$\lim_{r\to\infty} V = -E_0 r \cos\theta$$

We can identify that this phenomena has axial symmetry and therefore is independent to the azimuthal angle φ (since its a sphere, it comes natwally to work in spherical coordinates)

$$\nabla^2 V = 0$$

In spherical coordinates.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \cancelto{0}{\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 V}{\partial\varphi^2}\right)} = 0$$

We will then arrive to the general solution for the Laplace equation for fields possesing axial symmetry

And then apply the already discussed boundary conditions.

$$\frac{1}{r^2}\left[\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)\right]=0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

We already know $V(r,\theta)$, but it is possible to describe V as if it was composed of independent functions that depend only on 'r' and 'θ' respectively. This is commonly done, as arule, and is called 'separation of variables'.

$$V(r,\theta)=R(r)\Theta(\theta)$$

Then the Laplace equation can be seen as:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R\Theta}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial R\Theta}{\partial\theta}\right)=0$$

Since those constants can be taken out as constants:

$$\Theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

This can be further manipulated by dividing by $R\Theta$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

We have changed the '∂' partial derivative to total derivatives to make emphasis that these expressions are only dependent on a single variable.

It is clear by now that:

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)$$

Is independent of r, Then the other term, must also be independent of r, and it is equal to a constant

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=K$$

Then, to be able to cancel out to zero:

$$\frac{1}{\Theta \sin\theta} \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta} \right) = -k$$

Solving for R.

First multiply by R.

$$\frac{R}{R} \frac{d}{dr}\left(r^2 \frac{dR}{dr} \right) = kR \therefore \frac{d}{dr}\left(r^2 \frac{dR}{dr} \right) = kR$$

Now, expanding (product rule)

$$r^2 \frac{d^2R}{dr} + 2r \frac{dR}{dr} = kR$$

This equation has the form of an Euler-Cauchy equation:

$$a_n x^n y^{(n)}(x) + a_{n-1} x^{n-1} y^{(n-1)}(x) + \ldots + a_0 y(x) = 0$$

Which has the solution of the form $y(x) = x^m$, in this case $R = r^n$

So $\frac{dR}{dr} = n r^{n-1}$ and $\frac{d^2R}{dr} = (n-1)(n) r^{n-2}$

Substituting the trial solution in the equation:

$$\cancel{r^2}\left[n(n-1) r^{\cancel{n-2}}\right] + 2\cancel{r}(n) r^{\cancel{n-1}} - k r^n = 0$$

$$n(n-1) r^n + 2n r^n - k r^n = 0$$

$$r^n \left[n(n-1) + 2n - k \right] = 0$$

Then,

$$n(n-1) + 2n - k = 0$$

$$n^2 - n + 2n - k = 0 \therefore k = n^2 + n = n(n+1)$$

Setting the constant $k = n(n+1)$

we go back and solve for the equation in a quite similar fashion.

We set the trial solution of the form $R = r^{\lambda}$

$$r^2(\lambda(\lambda-1)r^{\lambda-2}) + 2r(\lambda)r^{\lambda-1} - n(n+1)r^{\lambda} = 0$$

$$r^{\lambda}(\lambda(\lambda-1) + 2\lambda - n(n+1)) = 0$$

Similarly, we assume $r \neq 0$, then

$$\lambda(\lambda-1) + 2\lambda - n^2 - n = 0$$

$$\lambda^2 + \lambda - n^2 - n = 0$$

This can be factored:

$$-(n-\lambda)(n+\lambda+1) = 0$$

Solving for λ:

$$\lambda_1 = n \qquad \lambda_2 = -n-1$$

Then the solution has the form:

$$R = Ar^n + Br^{-(n+1)} \qquad \text{with } K = n(n+1)$$

Now, we can substitute the value of K in the Θ equation:

$$\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -[n(n+1)]$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + n(n+1)\Theta \sin\theta = 0$$

We can manipulate the equation to have a certain familiar form by introducing a change of variable:

$$\mu = \cos\theta$$

Remembering, that for any function (chain rule)

$$\frac{df}{d\theta} = \frac{df}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta \frac{df}{d\mu}$$

and since $\cos^2\theta + \sin^2\theta = 1 \therefore 1-\cos^2\theta = \sin^2\theta$

$$\sin\theta = (1-\cos^2\theta)^{1/2}$$

$$\sin\theta = (1-\mu^2)^{1/2}$$

$$\frac{df}{d\theta} = -(1-\mu^2)^{1/2}\frac{df}{d\mu}$$

Substituting

$$\frac{d}{d\mu}\left[(1-\mu)^2 \frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0$$

Because:

$$\frac{d\Theta}{d\theta} = -\sin\theta \frac{d\Theta}{d\mu} \quad \therefore \quad \frac{d}{d\theta}\left[-\sin^2\theta \frac{d\Theta}{d\mu}\right] = \frac{df}{d\theta}$$

then

$$\frac{df}{d\theta} = -\sin\theta \frac{df}{d\mu} \quad \therefore \quad -\sin\theta \frac{d}{d\mu}\left[-\sin^2\theta \frac{d\Theta}{d\mu}\right] = \sin\theta \frac{d}{d\mu}\left[\sin^2\theta \frac{d\Theta}{d\mu}\right]$$

Everything together

$$\sin\theta\left[\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta\right] = 0$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0$$

and this, is the legendre's equation: (in differential form)

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0$$

When n is an integer, its solutions are the Legendre's Polynomials we've presented in the last point.

$$\Theta = P_n(\cos\theta)$$

we can use the following property of Legendre's equation

$$n(n+1) = n'(n'+1) \quad \text{if} \quad n' = -(n+1) \quad \text{Since} \quad n(n+1) = -(n+1)(-n-1+1) = -(n+1)(-n)$$

$$n(n+1) = n(n+1)$$

Then

$$P_{-(n+1)}(\cos\theta) = P_n(\cos\theta)$$

Then, for every solution of the laplace equation of the form

$$V = A r^n P_n(\cos\theta)$$

There is another one of the form

$$V = B r^{-(n+1)} P_{-(n+1)}(\cos\theta) = B r^{-(n+1)} P_n(\cos\theta)$$

This look similar to the solution arrived for R

$$R = A r^n + B r^{-(n+1)}$$

So the general solution for the Laplace Equation for fields possesing axial symmetry is

$$\boxed{V = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-(n+1)} \right) P_n(\cos\theta)}$$

We now can continue solving the dielectric sphere problem, since we know the general solution of the Laplace equation.

Let's begin splitting the problem into 2 regions.

a) Inside the sphere. $r \in [-a, a]$

We know the solution has the form

$$V = \sum_{n=0}^{\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n(\cos\theta)$$

With the current form $V \to \infty$ as $r \to 0$, this cannot happen. So we say that for this particular region

$B = 0$ for all n.

Then the solution looks like:

$$V = \sum_{n=0}^{\infty} (A_n r^n) P_n(\cos\theta)$$

b) Outside the sphere $\{ r \in \mathbb{R} \mid r \notin [-a, a] \}$

We know that this Boundary condition must hold:

$$V = -E_o r \cos\theta$$

from

$$V = \sum_{n=0}^{\infty} \left[A_n r^n + \frac{B_n}{r^{n+1}} \right] P_n \cos\theta$$

as $r \to \infty$, we can observe the B_n vanishes as expected, but no other power of r must remain either, the only way to accomplish this B.C. if for

$$A_n = 0 \quad \text{for all} \quad n \neq 1$$

So the solution for this region would look like:

$$V = -E_o r \cos\theta + \sum_{n=0}^{\infty} \frac{B_n}{r^{n+1}} P_n(\cos\theta)$$

	$P_n(\cos\theta)$
$n=0$	1
$n=1$	$\cos\theta$
$n=2$	$\frac{1}{2}(3\cos^2\theta - 1)$

Now we continue with BC (1): V is continuous at $r=a$

That means that:

$$V_{\text{inside region}} = V_{\text{outside region}}$$

$$\sum_{n=0}^{\infty} (A_n r^n) P_n(\cos\theta) = -E_o r \cos\theta + \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta)$$

Because of orthogonality, the $\cos\theta$ terms will all be taken into account for $n=1$, and will not show for any other n. In this way all terms $P_n(\cos\theta)$ can be paired for all n.

So for $n=0$

$$A_o = +\frac{B_o}{r} \quad \text{at } r=a \quad \therefore \quad \boxed{A_o = \frac{B_o}{a}}$$

for $n=1$

$$A_1 r \cancel{\cos\theta} = -E_o r \cancel{\cos\theta} + \frac{B_1}{r^2} \cancel{\cos\theta} \quad \text{at } r=a \quad \therefore \quad \boxed{A_1 a = -E_o a + \frac{B_1}{a^2}}$$

for $n>1$

$$A_n r^n \cancel{P_n(\cos\theta)} = \frac{B_n}{r^{n+1}} \cancel{P_n(\cos\theta)} \quad \text{at } r=a \quad \therefore \quad \boxed{A_n a^n = \frac{B_n}{a^{n+1}}, \quad n>1}$$

Now for B.C. (2): The normal component of D is continuous at r=a

$$\varepsilon_s \left.\frac{\partial V_{in}}{\partial r}\right|_{r=a} = \varepsilon_o \left.\frac{\partial V_{out}}{\partial r}\right|_{r=a}$$

We follow a similar approach.

For $n=0$

$$\varepsilon_s \frac{\partial}{\partial r}\left(A_o \right) = \varepsilon_o \frac{\partial}{\partial r}\left(\frac{B_o}{r} \right) \therefore \quad 0 = \varepsilon_o \frac{-B_o}{r^2} \quad \text{at } r=a \therefore \quad 0 = \frac{B_o}{a^2} \quad \therefore \boxed{B_o = 0}$$

for $n=1$

$$\varepsilon_s \frac{\partial}{\partial r}\left(A_1 r \cos\theta \right) = \varepsilon_o \frac{\partial}{\partial r}\left(-E_o r \cos\theta + \frac{B_1}{r^2}\cos\theta \right)$$

$$\varepsilon_s A_1 \cancel{\cos\theta} = \varepsilon_o\left[-E_o \cancel{\cos\theta} - \frac{2B_1}{r^3} \cancel{\cos\theta} \right] \quad \text{at} \quad r=a$$

$$\boxed{\varepsilon_s A_1 + \varepsilon_o E_o = -\frac{\varepsilon_o 2B_1}{a^3}}$$

for $n>1$

$$\varepsilon_s \frac{\partial}{\partial r}\left(A_n r^n \cancel{P_n(\cos\theta)} \right) = \varepsilon_o \frac{\partial}{\partial r}\left(B_n r^{-(n+1)} \cancel{P_n(\cos\theta)} \right)$$

$$\varepsilon_s A_n (n) r^{n-1} = \varepsilon_o B_n [-(n+1)] r^{-n-2} \quad \text{at } r=a$$

$$\left[\varepsilon_s A_n n a^{n-1} = -\varepsilon_o B_n (n+1) a^{-n-2}\right] a^2$$

$$\left[\varepsilon_s A_n n a^{n+1} = -\varepsilon_o B_n (n+1) a^{-n}\right] \times \frac{1}{a^{-n}}$$

$$\boxed{\varepsilon_s A_n n a^{2n+1} = -\varepsilon_o B_n (n+1)}$$

So now we have the following functions:

BC. (1)

$$A_0 = \frac{B_0}{a}$$

$$A_1 a = -E_0 a + \frac{B_1}{a^2}$$

$$A_n a^n = \frac{B_n}{a^{n+1}} \quad , \quad n>1 \longrightarrow A_n a^n (a^{n+1}) = B_n \therefore A_n a^{2n+1} = B_n$$

BC. (2)

$$B_0 = 0$$

$$\varepsilon_s A_1 + \varepsilon_0 E_0 = -\varepsilon_0 \frac{2B_1}{a^3}$$

$$\varepsilon_s A_n n a^{2n+1} = -\varepsilon_0 B_n (n+1), \quad n>1$$

It is clear that

$A_0 = 0$ since $B_0 = 0$

and substituting $A_n a^{2n+1} = B_n$ into last equation of BC. (2)

$$\varepsilon_s n B_n = -\varepsilon_0 B_n (n+1) \quad , \quad n>1$$

Since we dont have any constraints on ε_s such that $\varepsilon_s n = -\varepsilon_0 (n+1)$, we will assume $B_n = 0$ in order to hold the equality. So $B_n = 0$ and $A_n = 0$ for $n>1$.

We are left with

$$A_1 a = -E_0 a + \frac{B_1}{a^2} \quad \text{and} \quad \varepsilon_s A_1 + \varepsilon_0 E_0 = -\varepsilon_0 \frac{2B_1}{a^3}$$

$$A_1 = -E_0 + \frac{B_1}{a^3} \quad \therefore \quad \varepsilon_s \left(-E_0 + \frac{B_1}{a^3}\right) + \varepsilon_0 E_0 = -\varepsilon_0 \frac{2B_1}{a^3}$$

$$-\varepsilon_s E_0 + \frac{\varepsilon_s B_1}{a^3} + \varepsilon_0 E_0 = -\varepsilon_0 \frac{2B_1}{a^3} \quad \therefore \quad E_0 (\varepsilon_0 - \varepsilon_s) = -\frac{B_1}{a^3}(2\varepsilon_0 + \varepsilon_s)$$

$$-B_1 = \frac{E_0 (\varepsilon_0 - \varepsilon_s) a^3}{(2\varepsilon_0 + \varepsilon_s)} \quad \therefore \quad \boxed{B_1 = E_0 a^3 \left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s + 2\varepsilon_0}\right)}$$

We can substitute that again to obtain A_1

$$A_1 = -E_0 + \frac{B_1}{a^3} \longrightarrow A_1 = -E_0 + \frac{a^3 E_0}{a^3}\left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s - 2\varepsilon_0}\right)$$

$$A_1 = E_0\left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s + 2\varepsilon_0} - 1\right) = E_0\left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s + 2\varepsilon_0} - \frac{\varepsilon_s + 2\varepsilon_0}{\varepsilon_s + 2\varepsilon_0}\right) = E_0\left(\frac{\varepsilon_s - \varepsilon_s - \varepsilon_0 - 2\varepsilon_0}{\varepsilon_s + 2\varepsilon_0}\right)$$

$$A_1 = -E_0\left(\frac{3\varepsilon_0}{\varepsilon_s + 2\varepsilon_0}\right)$$

So finally, getting everything together

$$\boxed{V_{inside} = A_1 r \cos\Theta = -E_0\left(\frac{3\varepsilon_0}{\varepsilon_s + 2\varepsilon_0}\right) r \cos\Theta}$$

$$V_{outside} = -E_0 r \cos\Theta + B_1 r^{-(2)} \cos(\Theta) = -E_0 r \cos\Theta + \frac{E_0 a^3}{r^2}\left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s - 2\varepsilon_0}\right)\cos\Theta$$

$$\boxed{V_{outside} = -\left[1 - \left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s - 2\varepsilon_0}\right)\frac{a^3}{r^3}\right] E_0 r \cos\Theta}$$

Now we have everything to work out the effective dipole moment.

From previous sections we know

$$V_{dipole} = \frac{1}{4\pi\varepsilon_0} P \left(\frac{\cos\Theta}{r^2}\right)$$

The effective dipole moment can be found in the outside potential

$$V_{outside} = \underbrace{-E_0 r \cos\Theta}_{\text{due to uniform field}} + \underbrace{\frac{E_0 a^3}{r^2}\left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s - 2\varepsilon_0}\right)\cos\Theta}_{\text{effective dipole moment}}$$

$$\frac{1}{4\pi\varepsilon_0} P \left(\frac{\cos\Theta}{\cancel{r^2}}\right) = E_0 a^3 \left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s - 2\varepsilon_0}\right)\left(\frac{\cos\Theta}{\cancel{r^2}}\right)$$

$$\boxed{\vec{P}_{eff} = 4\pi\varepsilon_0 \vec{E}_0 a^3 \left(\frac{\varepsilon_s - \varepsilon_0}{\varepsilon_s - 2\varepsilon_0}\right)}$$

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o Obtain expression for the electric force acting on the dielectric sphere.

We know that

$$\vec{F} = (\vec{P}_{ele} \cdot \nabla)\vec{E}_o$$

and

$$\vec{P}_{ers} = 4\pi \varepsilon_o a^3 \left(\frac{\varepsilon_s - \varepsilon_o}{\varepsilon_s - 2\varepsilon_o}\right) \vec{E}_o$$

Then

$$\vec{F} = 4\pi \varepsilon_o a^3 \left(\frac{\varepsilon_s - \varepsilon_o}{\varepsilon_s - 2\varepsilon_o}\right) \vec{E}_o \cdot \nabla \vec{E}_o$$

We go to the identity

$$\frac{1}{2}\nabla(A \cdot A) = (A \cdot \nabla)A + A \times (\nabla \times A) = A \nabla A$$

and noting that for a static field

$$\nabla \times E = 0$$

Then

$$\frac{1}{2}\nabla(A \cdot A) = (A \cdot \nabla)A + 0 = (A \cdot \nabla)A = \frac{1}{2}\nabla A^2$$

Then $\vec{E}_o \cdot \nabla \vec{E}_o = \frac{1}{2} \nabla \vec{E}_o^2$

$$\boxed{\vec{F} = 2\pi \varepsilon_o a^3 \left(\frac{\varepsilon_s - \varepsilon_o}{\varepsilon_s - 2\varepsilon_o}\right) \nabla \vec{E}_o^2}$$