Differential Equations in Engineering

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For DE and PDE

What is a differential equation ?

Language of Universe

Rate of change

Description of time-varying phenomena ?

Description of varying phenomena in space, in time or in space and time

ODEs contains derivative terms of 1 or more dependent variables with respect to only one independent variable

$$\frac{dy}{dx} = f(x, y) \qquad\qquad F\left(x, y, \frac{dy}{dx}\right) = 0$$

Independent variable, is the variable the derivative is taken respect to. (Appears on the bottom in differential form)

A set of differential equations, can be written in vector form

$$\frac{d\underline{y}}{dx} = \underline{f}(x, \underline{y}) \qquad \begin{cases} \dfrac{dy_1}{dx} = f_1(x, y_1, y_2) \\ \dfrac{dy_2}{dx} = f_2(x, y_1, y_2) \end{cases} \qquad \underline{F}\left(x, \underline{y}, \frac{d\underline{y}}{dx}\right) = 0$$

ODE = Ordinary Differential Equation

PDE=Partial Differential equation

Equations of change, or dynamics of a system that involves more than one independent variable, are modeled by partial differential equations (i.e. PDE)

How ODEs are written (by scientists and engineers)? *

Derivative form
Or Prime
Notation

$$y'' + y' + y = 0$$

Differential form
Or Leibniz
Notation

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

Differential operator
form
Linear operator form

$$\left(D^2 + D + 1\right)y = 0$$

Newton
notation
Dot notation

$$\ddot{y} + \dot{y} + y = 0$$

Preferred when the independent variable is time

Subscript
notation

$$y_{xx} + y_x + y = 0$$

Canonical and
Leibniz notation

$$\frac{dy_2}{dx} + y_2 + y_1 = 0$$

$$\frac{dy_1}{dx} - y_2 = 0$$

Look out $d \neq \delta \neq \partial \neq D \neq \underset{\rightarrow}{\nabla}$

* By mathematicians the particular use of these symbols may not tell anything about the context or application. In Engineering and Science, specially in the context of Fluid Dynamics the symbol "*D*", is reserved for another derivative, called substantial derivative.

How a PDE of a scalar variable is written?

(Scalar equation)

Linear operator form or Gibbs notation

$$\rho\, \hat{c}_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot [k \underline{\nabla} T] - \underline{v} \cdot \underline{\nabla}[\rho\, \hat{c}_p (T - T_o)] + \dot{q}_{Gen} - h a_v [T - T_f]\psi \qquad \underline{v} = v_x \hat{\imath} + v_y \hat{\jmath} + v_z\, \hat{k}$$

Differential form operator form (3-D)

$$\rho\, \hat{c}_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + k\frac{\partial^2 T}{\partial y^2} + k\frac{\partial^2 T}{\partial z^2} - \rho\, \hat{c}_p \left[v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right] + \dot{q}_{Gen} \qquad \underline{v} = (v_x, v_y, v_z)$$

Subscript form (3-D)

$$\rho\, \hat{c}_p T_t = kT_{xx} + kT_{yy} + kT_{zz} - \rho\, \hat{c}_p \left[u\, T_x + v\, T_y + w\, T_z \right] + \dot{q}_{Gen} \qquad \underline{v} = (u, v, w)$$

Einstein Form, or Einstein notation (3-D)

$$\rho\, \hat{c}_p \partial_0 T = k \partial_j \partial_j T - \rho\, \hat{c}_p v_j \partial_j T + \dot{q}_{Gen}$$

Substantial form, consistent with Gibbs (3-D)

$$\rho\, \hat{c}_p \frac{DT}{Dt} = \underline{\nabla} \cdot [k \underline{\nabla} T] + \dot{q}_{Gen}$$

Substantial derivative

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T$$

PDE=Partial differential equation

How a PDE of vector variable is written?

(Vector fields)

Navier-Stokes

Differential form operator form (3-D)

$$\underline{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\,\hat{k}$$

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + v_x\frac{\partial \underline{v}}{\partial x} + v_y\frac{\partial \underline{v}}{\partial y} + v_z\frac{\partial \underline{v}}{\partial z}\right] = \mu\left[\frac{\partial^2 \underline{v}}{\partial x^2} + \frac{\partial^2 \underline{v}}{\partial y^2} + \frac{\partial^2 \underline{v}}{\partial z^2}\right] - \left[\frac{\partial p}{\partial x}\hat{\imath} + \frac{\partial p}{\partial y}\hat{\jmath} + \frac{\partial p}{\partial z}\hat{k}\right] + \rho\left[g_x\hat{\imath} + g_y\hat{\jmath} + g_z\,\hat{k}\right]$$

$$\rho\left[\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v}\right] = \mu\nabla^2\underline{v} - \underline{\nabla}\,p + \rho\,\underline{g}$$

(Gibbs notation)

$$\rho\frac{D\underline{v}}{Dt} = \mu\nabla^2\underline{v} - \underline{\nabla}\,p + \rho\,\underline{g}$$

(Substantial notation)

$$\partial_0 v_i + v_j\partial_j v_i = \frac{\mu}{\rho}\partial_j\partial_j v_i - \frac{1}{\rho}\partial_i p + g_i$$

(Einstein notation)

$$\rho\left[\frac{\partial \underline{v}}{\partial t}+v_x\frac{\partial \underline{v}}{\partial x}+v_y\frac{\partial \underline{v}}{\partial y}+v_z\frac{\partial \underline{v}}{\partial z}\right]=\mu\left[\frac{\partial^2 \underline{v}}{\partial x^2}+\frac{\partial^2 \underline{v}}{\partial y^2}+\frac{\partial^2 \underline{v}}{\partial z^2}\right]-\left[\frac{\partial p}{\partial x}\hat{\imath}+\frac{\partial p}{\partial y}\hat{\jmath}+\frac{\partial p}{\partial z}\hat{k}\right]+\rho\left[g_x\hat{\imath}+g_y\hat{\jmath}+g_z\hat{k}\right]$$

$$\underline{v}=v_x\hat{\imath}+v_y\hat{\jmath}+v_z\,\hat{k}$$

$$\rho\left[\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right]\hat{\imath}=\mu\left[\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right]\hat{\imath}-\left[\frac{\partial p}{\partial x}\hat{\imath}\right]+\rho\left[g_x\hat{\imath}\right]$$

$$\rho\left[\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right]\hat{\jmath}=\mu\left[\frac{\partial^2 v_y}{\partial x^2}+\frac{\partial^2 v_y}{\partial y^2}+\frac{\partial^2 v_y}{\partial z^2}\right]\hat{\jmath}-\left[\frac{\partial p}{\partial y}\hat{\jmath}\right]+\rho\left[g_y\hat{\jmath}\right]$$

$$\rho\left[\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right]\hat{k}=\mu\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right]\hat{k}-\left[\frac{\partial p}{\partial z}\hat{k}\right]+\rho\left[g_z\hat{k}\right]$$

Wave equation: (hyperbolic)

$$\frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2} = \nabla^2 p_1 - \rho_1 \underline{\nabla} \cdot \underline{g} - \left(\underline{g}/c^2\right) \cdot \underline{\nabla} p_1$$

$$\frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2} = \frac{\partial^2 p_1}{\partial^2 x} - \rho_1 \frac{\partial g_x}{\partial x} - (g_x/c^2)\frac{\partial p_1}{\partial x}$$ 1-D

$$\frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2} = \frac{\partial^2 p_1}{\partial^2 x} + \frac{\partial^2 p_1}{\partial^2 y} - \rho_1 \frac{\partial g_x}{\partial x} - \rho_1 \frac{\partial g_y}{\partial y} - (g_x/c^2)\frac{\partial p_1}{\partial x} - \left(g_y/c^2\right)\frac{\partial p_1}{\partial y}$$ 2-D

$$\frac{1}{c^2}\frac{\partial^2 p_1}{\partial t^2} = \frac{\partial^2 p_1}{\partial^2 x} + \frac{\partial^2 p_1}{\partial^2 y} + \frac{\partial^2 p_1}{\partial^2 z} - \rho_1 \frac{\partial g_x}{\partial x} - \rho_1 \frac{\partial g_y}{\partial y} - \rho_1 \frac{\partial g_z}{\partial z} - (g_x/c^2)\frac{\partial p_1}{\partial x} - \left(g_y/c^2\right)\frac{\partial p_1}{\partial y} - \left(g_z/c^2\right)\frac{\partial p_1}{\partial z}$$ 3-D

Using continuity equation, and Cauchy momentum equation find the relationship for speed of sound

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}) = 0$$

$$\frac{\partial (\rho\, \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v}) = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g}$$

Velocity of fluid is negligible, but its gradient. First we recast in substantial derivative form.

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}) = \frac{\partial \rho}{\partial t} + \rho\, \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla}\, \rho = \frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \rho + \rho\, \underline{\nabla} \cdot \underline{v}$$

$$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \rho + \rho\, \underline{\nabla} \cdot \underline{v} = \frac{D\rho}{Dt} + \rho\, \underline{\nabla} \cdot \underline{v} = 0$$

$$\frac{\partial (\rho\, \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v}) = \rho \frac{\partial \underline{v}}{\partial t} + \underline{v} \frac{\partial \rho}{\partial t} + \underline{v}\, \underline{\nabla} \cdot (\rho\, \underline{v}) + \rho\, \underline{v} \cdot \underline{\nabla} \underline{v} = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g}$$

$$\rho \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla} \underline{v} \right] = \rho \frac{D\underline{v}}{Dt} = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g}$$

$$\frac{D\rho}{Dt} + \rho\, \underline{\nabla} \cdot \underline{v} = 0$$

$$\rho \frac{D\underline{v}}{Dt} = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g}$$

From thermodynamics the relationship for entropy is:

$$dS = c_p \frac{dT}{T} - \frac{\beta}{\rho} dp \qquad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \qquad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma}{\rho\, \kappa} \qquad \gamma = \frac{c_p}{c_v} \qquad \kappa = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T$$

For a reversible process

In propagation of sound, viscous effects are negligible, and velocity is negligible, but gradient :

$$\frac{D\rho}{Dt} + \rho\, \underline{\nabla} \cdot \underline{v} \approx \frac{\partial \rho}{\partial t} + \rho\, \underline{\nabla} \cdot \underline{v}$$

$$\rho \frac{D\underline{v}}{Dt} = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g} \approx \rho \frac{\partial \underline{v}}{\partial t} = -\underline{\nabla}\, p + \rho\, \underline{g}$$

$$\frac{\partial \rho}{\partial t} + \rho\, \underline{\nabla} \cdot \underline{v} \approx 0$$

$$\rho \frac{\partial \underline{v}}{\partial t} \approx -\underline{\nabla}\, p + \rho\, \underline{g}$$

At rest

$$\frac{\partial \rho_0}{\partial t} = 0$$

$$-\underline{\nabla}\, p_0 + \rho_0\, \underline{g} = 0$$

Subtracting at wave propagation

$$\frac{\partial [\rho - \rho_0]}{\partial t} + \rho\, \underline{\nabla} \cdot \underline{v} \approx 0$$

$$\rho \frac{\partial \underline{v}}{\partial t} \approx -\underline{\nabla}\, [p - p_0] + [\rho - \rho_0]\, \underline{g}$$

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$$

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{\Delta p}{\Delta \rho}$$

$$\frac{1}{c^2}\frac{\partial [p - p_0]}{\partial t} + \rho\, \underline{\nabla} \cdot \underline{v} \approx 0$$

$$\rho \frac{\partial \underline{v}}{\partial t} \approx -\underline{\nabla}\, [p - p_0] + \frac{1}{c^2}[p - p_0]\, \underline{g}$$

To eliminate velocity, we take the derivative of the first equation respect to time, and we assume physical properties remain more less constant, then..

$$\frac{1}{c^2}\frac{\partial^2 [p - p_0]}{\partial t^2} + \underline{\nabla} \cdot \left[\rho \frac{\partial \underline{v}}{\partial t}\right] \approx 0$$

$$\frac{1}{c^2}\frac{\partial^2 [p - p_0]}{\partial t^2} + \underline{\nabla} \cdot \left[-\underline{\nabla}\, [p - p_0] + \frac{1}{c^2}[p - p_0]\, \underline{g}\right] \approx 0$$

$$\frac{1}{c^2}\frac{\partial^2[p-p_0]}{\partial t^2}+\underline{\nabla}\cdot\left[-\underline{\nabla}\,[p-p_0]+\frac{1}{c^2}[p-p_0]\,\underline{g}\,\right]\approx 0$$

$$\frac{1}{c^2}\frac{\partial^2[p-p_0]}{\partial t^2}-\nabla^2[p-p_0]+\frac{\underline{g}}{c^2}\cdot\underline{\nabla}[p-p_0]+\frac{[p-p_0]}{c^2}\underline{\nabla}\cdot\left[\,\underline{g}\,\right]\approx 0$$

Gravitational field, or body forces do not depend on position,

$$\frac{1}{c^2}\frac{\partial^2[p-p_0]}{\partial t^2}-\nabla^2[p-p_0]\approx 0$$

This equation is known as the wave equation and represents the velocity of propagation of a wave, and c is exactly that, in an nutshell c is the velocity of sound in the media.

General Heat equation (Parabolic)

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot \left[k \underline{\nabla} T\right] - \rho\,\hat{c}_p \underline{v} \cdot \underline{\nabla} T + \dot{q}_{Gen} - \psi a_v\left[h\left(T - T_f\right) + \sigma\varepsilon\left(T^4 - T_{sky}^4\right)\right]$$

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} - \rho\,\hat{c}_p v_x \frac{\partial T}{\partial x} + \dot{q}_{Gen} - a_v\left[h\left(T - T_f\right) + \sigma\varepsilon\left(T^4 - T_{sky}^4\right)\right]$$ 1-D

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} - \rho\,\hat{c}_p \left[v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y}\right] + \dot{q}_{Gen} - a_v\left[h\left(T - T_f\right) + \sigma\varepsilon\left(T^4 - T_{sky}^4\right)\right]$$ 2-D

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + k \frac{\partial^2 T}{\partial y^2} + k \frac{\partial^2 T}{\partial z^2} - \rho\,\hat{c}_p \left[v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right] + \dot{q}_{Gen}$$ 3-D

Advantages of Gibbs notation (3-D)

Heat conduction without advection

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = \underline{\nabla} \cdot \left[k \underline{\nabla} T\right] + \dot{q}_{Gen}$$

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q}_{Gen}$$

Cartesian coordinates

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = \frac{\partial}{r\,\partial r}\left(k\,r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q}_{Gen}$$

Cylindrical coordinates

$$\rho\,\hat{c}_p \frac{\partial T}{\partial t} = \frac{\partial}{r^2\,\partial r}\left(k\,r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right) + \dot{q}_{Gen}$$

Spherical coordinates

PDEs

$$\frac{\partial(\rho\,\varphi)}{\partial t} + \underline{\nabla}\cdot\left(\rho\,\underline{v}\,\varphi\right) = \underline{\nabla}\cdot\left(\Gamma\,\underline{\nabla}\,\varphi\right) + S$$

$$\frac{\partial\left(\rho\,\underline{v}\right)}{\partial t} + \underline{\nabla}\cdot\left(\rho\,\underline{v}\,\underline{v}\right) = \underline{\nabla}\cdot\,\underline{\underline{\sigma}} + \rho\,\underline{g}$$

$$\frac{\partial\left(\rho\,\underline{v}\right)}{\partial t} + \underline{\nabla}\cdot\left(\rho\,\underline{v}\,\underline{v}\right) = \underline{\nabla}\cdot\,\underline{\underline{\tau}} - \underline{\nabla}p + \rho\,\underline{g}$$

$$\partial_0 v_i + v_j \partial_j v_i = \frac{\mu}{\rho}\partial_j\partial_j v_i - \frac{1}{\rho}\partial_i p + g_i$$

Some people may prefer not use index notation, because can be confused by vector notation, or if they use it, they add an extra comma, that is:

dy/dx=y,x $dy/dx = y_{,x}$

Special forms of linear derivative operators

The different differential operators that will be introduced have a physical meaning and is explained, and the form that takes operating in scalars, vectors and tensors is given in the most common coordinate systems.

$$\underline{\nabla} = \vec{\nabla}$$

Nabla – the del operator

$$\underline{\nabla} p = \vec{\nabla} p = \left[\frac{\partial p}{\partial x} \hat{\imath} + \frac{\partial p}{\partial y} \hat{\jmath} + \frac{\partial p}{\partial z} \hat{k} \right]$$

Nabla – An inverted uppercase delta appears in the differential form of many Equations in nature (e.g. the all four Maxwell's Equations). This symbol represents a vector differential operator called "nabla" or del, and its presence instructs you to take the derivatives of the quantity on which the operator is acting

The exact form of those derivatives depends on the symbols following the operator

with $$\underline{\nabla} \cdot (\quad) = \vec{\nabla} \cdot (\quad)$$

signifying divergence

$$\underline{\nabla} \times (\quad) = \vec{\nabla} \times (\quad)$$

indicating curl

and $$\underline{\nabla} (\quad) = \vec{\nabla} (\quad)$$

signifying gradient.

$$\underline{\nabla}(\quad) = \vec{\nabla}(\quad)$$

Is an instruction to take derivatives in three orthogonal directions

$$\underline{\nabla}(\quad) = \vec{\nabla}(\quad) = \left[\frac{\partial(\quad)}{\partial x}\hat{\imath} + \frac{\partial(\quad)}{\partial y}\hat{\jmath} + \frac{\partial(\quad)}{\partial z}\hat{k}\right]$$

For Cartesian coordinates x, y and z

Operates over scalar, vector and tensor variables

The divergence , Del dot $\underline{\nabla} \cdot (\quad) = \vec{\nabla} \cdot (\quad)$

Divergence. James Clerk Maxwell coined the term "convergence" to describe the mathematical operation that measures the rate ay which electric field lines "flow" towards points of negative electrical charge. Few years latter, Oliver Heaviside suggested the term of "divergence" for the same quantity with opposite sign. Thus, positive divergence is associated with the "flow" of electric field lines away from positive charge.

Both flux and divergence deal with the flow of a vector field; flux is defined per unit area, while divergence applies to individual points. In the case of fluid flow, the divergence at any point is a measure of the tendency of the flow vector to diverge from that point. (that is , to carry more material away from it than is brought toward it). Thus points of positive divergence are sources (faucets in situation involving fluid flow, positive electric charge in electrostatics), while points of negative divergence are sinks (drains in fluid flow, negative charge in electrostatics)

$$\underline{\nabla} \cdot (\underline{B}) = \vec{\nabla} \cdot (\vec{B}) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint \underline{n} \cdot \underline{B}\, dA$$

$$\underline{\nabla} \cdot (\underline{B}) = \vec{\nabla} \cdot (\vec{B})$$

The divergence , Del dot

The divergence of a vector field is the scalar measurement of how much of a vector field expands or compress near a point

$$div(\underline{B}) = \underline{\nabla} \cdot (\underline{B}) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint \underline{n} \cdot \underline{B}\, dA$$

Operates over vectors and tensors

$$\frac{\partial(\rho\ )}{\partial t} + \underline{\nabla} \cdot (\rho\ \underline{v}) = 0$$

Accumulation Advection

$$\frac{\partial(\rho\ \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho\ \underline{v}\ \underline{v}) = \underline{\nabla} \cdot \underline{\underline{\sigma}} + \rho\ \underline{g}$$

Accumulation Advection surface forces volume forces

$$\frac{\partial(\rho\hat{E})}{\partial t} + \underline{\nabla} \cdot (\rho\ \underline{v}\ \hat{E}) = \underline{\nabla} \cdot \left[\underline{\underline{\tau}} \cdot \underline{v}\right] - \underline{\nabla} \cdot \left[p\ \underline{\underline{I}} \cdot \underline{v}\right] - \underline{\nabla} \cdot \underline{q} + \dot{q}_G$$

Accumulation Advection Viscous dissipation flowing power conduction generation

$$\oiiint \underline{\nabla} \cdot \underline{B}\ dV = \oiint \underline{n} \cdot \underline{B}\, dA$$

Divergence Theorem, Gauss's Theorem or Ostrogradsky's theorem

$$\frac{\partial\rho}{\partial t} + \underline{\nabla} \cdot (\rho\ \underline{v}) = \frac{\partial\rho}{\partial t} + \rho\ \underline{\nabla} \cdot \underline{v} + \underline{v} \cdot \underline{\nabla}\ \rho = \rho\ \underline{\nabla} \cdot \underline{v} + \frac{\partial\rho}{\partial t} + \underline{v} \cdot \underline{\nabla}\ \rho = \rho\ \underline{\nabla} \cdot \underline{v} + \frac{D\rho}{Dt}$$

Cartesian

$$\underline{\nabla} \cdot \underline{B} = \vec{\nabla} \cdot \left(\vec{B}\right) = \left[\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right]$$

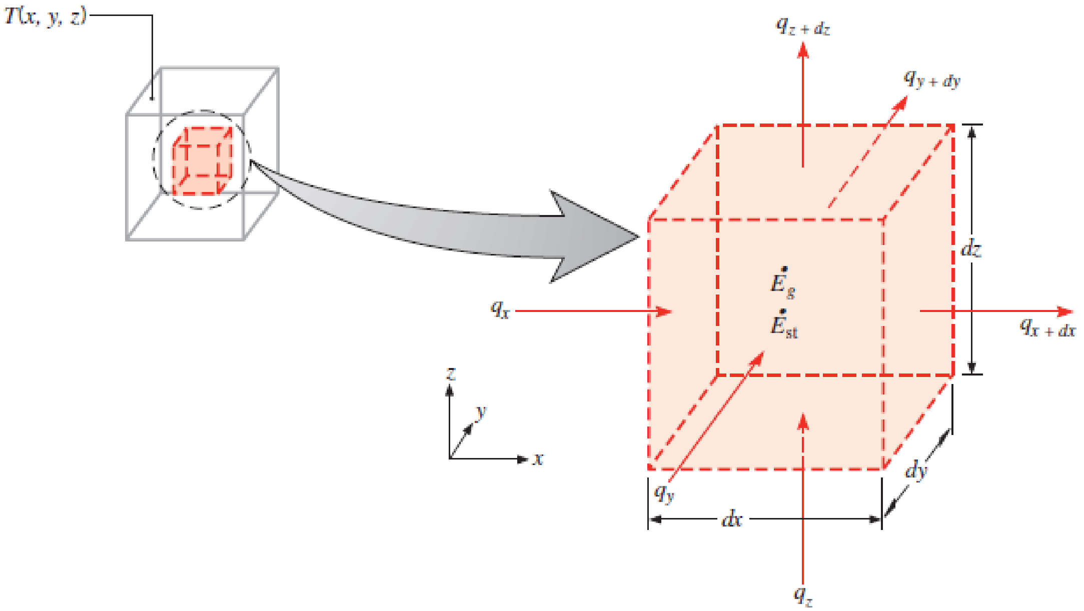
Cylindrical

$$\underline{\nabla} \cdot \underline{B} = \vec{\nabla} \cdot \left(\vec{B}\right) = \left[\frac{1}{r}\frac{\partial [rB_r]}{\partial r} + \frac{1}{r}\frac{\partial B_\phi}{\partial \phi} + \frac{\partial B_z}{\partial z}\right]$$

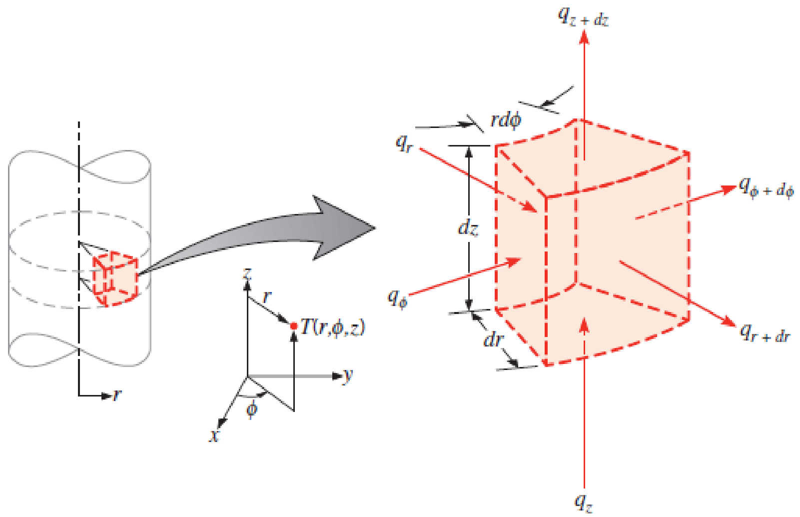
Spherical

$$\underline{\nabla} \cdot \underline{B} = \left[\frac{1}{r^2}\frac{\partial [r^2 B_r]}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial B_\phi}{\partial \phi} + \frac{1}{r \sin\theta}\frac{\partial [\sin\theta B_\theta]}{\partial \theta}\right]$$

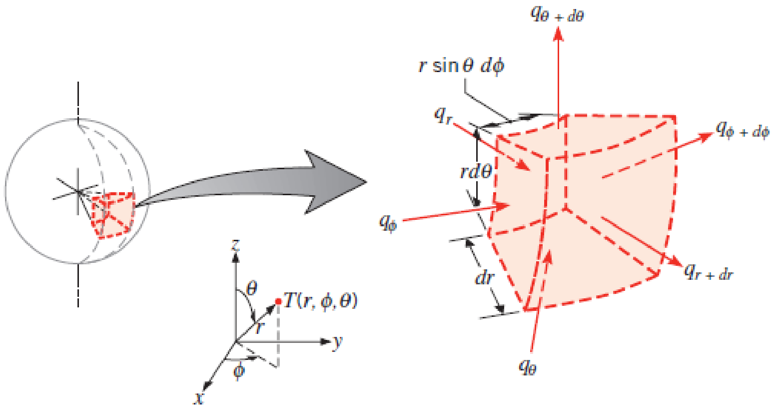
where ϕ represents the azimuthal angle and θ the zenith angle or co-latitude



$$\rho\ \hat{c}_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q}_{Gen}$$



$$\rho\, \hat{c}_p \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(k\, r\, \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \dot{q}_{Gen}$$



$$\rho\, \hat{c}_p \frac{\partial T}{\partial t} = \frac{\partial}{r^2\, \partial r}\left(k\, r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\left(k \frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(k \sin\theta \frac{\partial T}{\partial \theta}\right) + \dot{q}_{Gen}$$

$$\underline{\nabla} \times (\quad) = \vec{\nabla} \times (\quad)$$

The Del cross – the curl

The curl of a vector field is a measure of the field's tendency to circulate about a point- much like a divergence is a measure of the tendency of the field to flow way from a point. Maxwell settled on "curl" after considering several alternatives, including "turn" and "twirl"

Just as the divergence is found by considering the flux through an infinitesimal surface surrounding the point of interest, the curl at specified point may be found by considering the circulation per unit area over an infinitesimal path around that point. The mathematical definition of the curl of a vector field $\underline{B}$ is:

$$\underline{\nabla} \times (\underline{B}) \cdot \underline{n} = \vec{\nabla} \times (\vec{B}) \cdot \vec{n} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint \underline{B} \cdot d\underline{l}$$

$$\underline{\nabla} \times \underline{B} = \vec{\nabla} \times \vec{B}$$

The Del cross – the curl

The curl of a vector field is a vector field. The magnitude measures how much the vector fields rotates about a point.

The direction is determined by the right hand rule and points along the axis of rotation.

$$\underline{\nabla} \times \left(\underline{B}\right) \cdot \underline{n} = \vec{\nabla} \times \left(\vec{B}\right) \cdot \vec{n} = \lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint \underline{B} \cdot d\underline{l}$$

Cartesian

$$\underline{\nabla} \times \underline{B} = \left[\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right]\underline{e_x} + \left[\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right]\underline{e_y} + \left[\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right]\underline{e_z}$$

Cylindrical

$$\underline{\nabla} \times \underline{B} = \left[\frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}\right]\underline{e_r} + \left[\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right]\underline{e_\phi} + \frac{1}{r}\left[\frac{\partial [rB_\phi]}{\partial r} - \frac{\partial B_r}{\partial \phi}\right]\underline{e_z}$$

Spherical

$$\underline{\nabla} \times \underline{B} = \left[\frac{1}{r\sin\theta}\frac{\partial [B_\phi \sin\theta]}{\partial \theta} - \frac{\partial B_\theta}{\partial \phi}\right]\underline{e_r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial B_r}{\partial \phi} - \frac{\partial [rB_\phi]}{\partial r}\right]\underline{e_\theta} + \frac{1}{r}\left[\frac{\partial [rB_\theta]}{\partial r} - \frac{\partial B_r}{\partial \theta}\right]\underline{e_\phi}$$

$$\underline{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\,\hat{k}$$

The gradient

Is an instruction to take derivatives in three orthogonal directions

$$\underline{\nabla}(\quad) = \vec{\nabla}(\quad) = \left[\frac{\partial(\quad)}{\partial x}\hat{\imath} + \frac{\partial(\quad)}{\partial y}\hat{\jmath} + \frac{\partial(\quad)}{\partial z}\hat{k}\right]$$

For Cartesian coordinates *x*, *y* and *z*

Similar to the divergence and the curl, the gradient involves partial derivatives taken in three orthogonal directions. However, whereas the divergence measures the tendency of a vector field to flow away from a point and the curl indicates the circulation of a vector field around the a point, the gradient applies to a scalar fields. Unlike a vector field, a scalar field is specified entirely by its magnitude at various locations.
The information given by the gradient of a scalar field is: Magnitude tells how quickly the field is changing over space, and the direction of the gradient indicates the direction in that the field is changing most quickly with distance.

The gradient $\underline{\nabla}(B) = \vec{\nabla}(B)$

If B is a scalar representing the height of terrain above the sea level, the magnitude of the gradient at any location tells you how steeply the ground is sloped at that location, and the direction of the gradient points uphill along the steepest slope..

Cartesian

$$grad(\psi) = \underline{\nabla}\psi = \vec{\nabla}\psi = \frac{\partial\psi}{\partial x}\underline{e}_x + \frac{\partial\psi}{\partial y}\underline{e}_y + \frac{\partial\psi}{\partial z}\underline{e}_z$$

Cylindrical

$$grad(\psi) = \underline{\nabla}\psi = \vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\phi}\underline{e}_\phi + \frac{\partial\psi}{\partial z}\underline{e}_z$$

Spherical

$$grad(\psi) = \underline{\nabla}\psi = \vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\underline{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\underline{e}_\phi$$

$\underline{\nabla}$ The gradient

The gradient of a scalar function is a vector field. It points into the direction of maximum increase of the function, and the magnitude measures the rate of maximum increase

$$grad(G) = \underline{\nabla} G = \vec{\nabla} G = \frac{\partial G}{\partial x}\underline{\mathrm{e}}_{\mathrm{x}} + \frac{\partial G}{\partial y}\underline{\mathrm{e}}_{\mathrm{y}} + \frac{\partial G}{\partial z}\underline{\mathrm{e}}_{\mathrm{z}}$$

Unit normal vectors are evaluated generally as

$$\underline{n} = \vec{n} = \frac{\underline{\nabla} G}{|\underline{\nabla} G|} = \frac{\vec{\nabla} G}{|\vec{\nabla} G|}$$

Where $\boldsymbol{G}(x,y,z)=0$ defines the location of the surface. The function $\boldsymbol{G}$ is chosen so that $\underline{\boldsymbol{n}}$ points the desired direction.

∇^2 The Laplacean

The laplacean of a scalar field is the divergence of a gradient:

$$\nabla^2\Phi = \underline{\nabla} \cdot \underline{\nabla}\Phi = \vec{\nabla} \cdot \left(\vec{\nabla}\Phi\right)$$

The divergence of a gradient:

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

where φ represents the azimuthal angle and θ the zenith angle or co-latitude

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

where φ represents the azimuthal angle and θ the zenith angle or co-latitude

Even though the selection of *f* as azimuthal and *q* as zenith, in some derivations in this documents the angles will be swapped, and we are doing this on purpose because there is no agreement in the literature of which is which, and you must be acquainted with this common practice. So always you must verify the frame of reference , and use the correct nomenclature.

Check the document under the name **Units** to understand the relationship between the different coordinate systems

Additional symbols

$$\sum_{i=1}^{n} a_i = (a_1) + (a_2) + (a_3) + \cdots + (a_{n-1}) + (a_n)$$

$$\prod_{i=1}^{n} a_i = (a_1)(a_2)(a_3)\cdots(a_{n-1})(a_n)$$

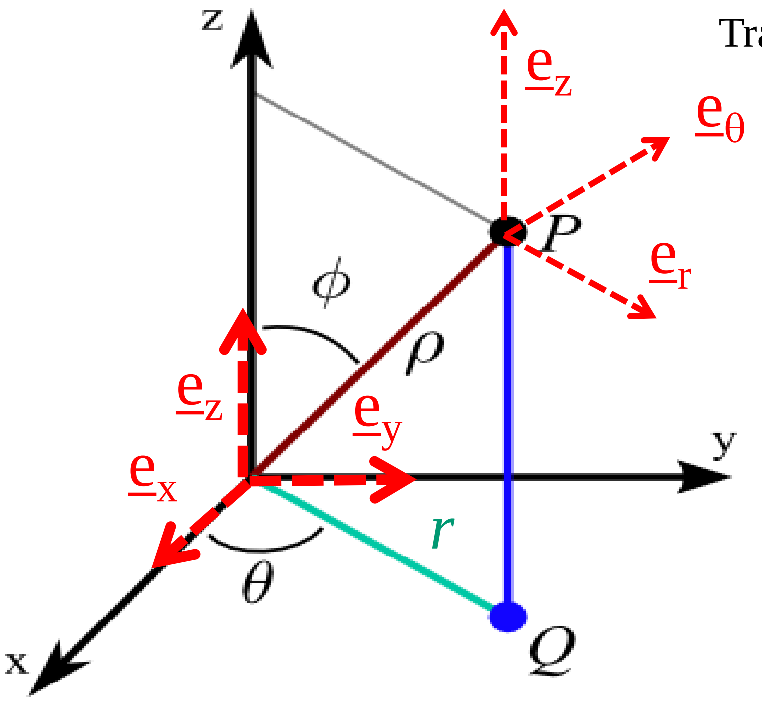
Cylindrical coordinates

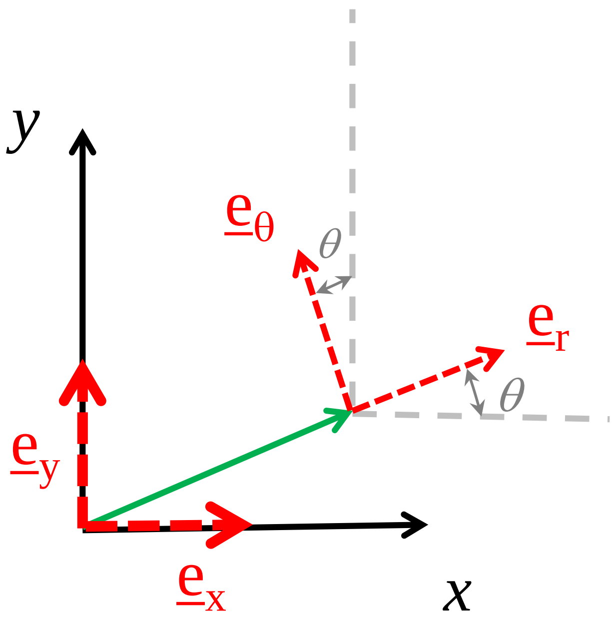
Cylindrical coordinates analysis for scalar, vector and tensors

Transformation of Gradient, Curl, Divergence and Laplacean from Cartesian Coordinates to cylindrical or polar coordinates.

Unit vector conversion

Transformation from polar to Cartesian coordinates





$$\rho^2 = r^2 + z^2 \qquad x = r\,Cos\theta$$

$$r^2 = x^2 + y^2 \qquad y = r\,Sin\theta$$

$$\underline{e}_r = Cos\theta\,\underline{e}_x + Sin\theta\,\underline{e}_y$$

$$\underline{e}_\theta = -Sin\theta\,\underline{e}_x + Cos\theta\,\underline{e}_y$$

$$\left\{\ \underline{e}_r = Cos\theta\,\underline{e}_x + Sin\theta\,\underline{e}_y \right\}\ Sin\ \theta\ \{\ Cos\ \theta\ \}$$

$$\left\{\ \underline{e}_\theta = -Sin\theta\,\underline{e}_x + Cos\theta\,\underline{e}_y \right\}\ Cos\ \theta\ \{-\ Sin\ \theta\}$$

$$\boxed{\underline{e}_x = Cos\theta\,\underline{e}_r - Sin\theta\,\underline{e}_\theta}$$

$$\boxed{\underline{e}_y = Sin\theta\,\underline{e}_r + Cos\theta\,\underline{e}_\theta}$$

Transformation from Cartesian to polar coordinates

From Trigonometry, some relations for ratio of change between polar and Cartesian coordinates

$$r^2 = x^2 + y^2 \qquad 2rdr = 2xdx + 2ydy$$

$$x = r\,Cos\theta \qquad \left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{x}{r} = Cos\theta$$

$$y = r\,Sin\theta \qquad \left(\frac{\partial r}{\partial y}\right)_{x,z} = \frac{y}{r} = Sin\theta$$

$$\theta = ArcTan\left(\frac{y}{x}\right) \qquad \left(\frac{\partial \theta}{\partial x}\right)_{y,z} = \frac{-y/x^2}{1+y^2/x^2} = -\frac{y}{r}\left(\frac{1}{r}\right) = -\left(\frac{Sin\theta}{r}\right)$$

$$\left(\frac{\partial \theta}{\partial y}\right)_{x,z} = \frac{1/x}{1+y^2/x^2} = \frac{x}{r}\left(\frac{1}{r}\right) = \frac{Cos\theta}{r}$$

For Cartesian coordinates (3D)

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z} dx + \left(\frac{\partial f}{\partial y}\right)_{x,z} dy + \left(\frac{\partial f}{\partial z}\right)_{x,y} dz$$

Scalar field change, as function of changes in Cartesian coordinates

For polar coordinates

$$df = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} dr + \left(\frac{\partial f}{\partial \theta}\right)_{r,z} d\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} dz$$

Scalar field change, as function of changes in polar coordinates

Recasting the gradient from Cartesian to cylindrical coordinates

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\right)_{y,z} \underline{e}_x + \left(\frac{\partial f}{\partial y}\right)_{x,z} \underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

Gradient of scalar field

$$df = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} dr + \left(\frac{\partial f}{\partial \theta}\right)_{r,z} d\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} dz$$

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z}\left(\frac{\partial r}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{\partial \theta}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}\left(\frac{\partial z}{\partial x}\right)_{y,z}$$

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} Cos\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(-\frac{Sin\theta}{r}\right) + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}(0)$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z}\left(\frac{\partial r}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{\partial \theta}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}\left(\frac{\partial z}{\partial y}\right)_{x,z}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} Sin\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{Cos\theta}{r}\right) + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}(0)$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\right)_{y,z} \underline{e}_x + \left(\frac{\partial f}{\partial y}\right)_{x,z} \underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z = \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Cos\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(-\frac{Sin\theta}{r}\right)\right]\underline{e}_x + \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Sin\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{Cos\theta}{r}\right)\right]\underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\underline{\nabla} f = \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Cos\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(-\frac{Sin\theta}{r}\right)\right]\left[Cos\theta\, \underline{e}_r - Sin\theta\, \underline{e}_\theta\right] + \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Sin\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{Cos\theta}{r}\right)\right]\left[Sin\theta\, \underline{e}_r + Cos\theta\, \underline{e}_\theta\right] + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\right)_{y,z} \underline{e}_x + \left(\frac{\partial f}{\partial y}\right)_{x,z} \underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} \underline{e}_z$$

Important derivatives of unit vectors

Unit vectors in Cartesian coordinates remain constant, but for cylindrical coordinates, they will depend on the evolution of the flow

$$\frac{d\,\underline{e}_x}{dt} = \frac{d\,\underline{e}_y}{dt} = \frac{d\,\underline{e}_z}{dt} = 0$$

$$\underline{e}_r = Cos\theta\,\underline{e}_x + Sin\theta\,\underline{e}_y$$

$$\frac{d\underline{e}_r}{dt} = -Sin\theta\,\underline{e}_x\frac{d\theta}{dt} + Cos\theta\,\underline{e}_y\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_r}{dt} = \underline{e}_\theta\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_r}{d\theta} = \underline{e}_\theta$$

$$\underline{e}_\theta = -Sin\theta\,\underline{e}_x + Cos\theta\,\underline{e}_y$$

$$\frac{d\underline{e}_\theta}{dt} = -Cos\theta\,\underline{e}_x\frac{d\theta}{dt} - Sin\theta\,\underline{e}_y\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_\theta}{dt} = -\underline{e}_r\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_\theta}{d\theta} = -\underline{e}_r$$

Divergence using dot product of gradient and vector in polar coordinates

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} \underline{e}_z$$

Gradient

$$\underline{a} = a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_z \underline{e}_z$$

$$\underline{e}_r \cdot \underline{e}_r = \underline{e}_z \cdot \underline{e}_z = \underline{e}_\theta \cdot \underline{e}_\theta = 1$$

$$\underline{e}_r \cdot \underline{e}_\theta = \underline{e}_\theta \cdot \underline{e}_z = \underline{e}_z \cdot \underline{e}_r = 0$$

Derivative of the product of two vectors

$$\frac{d(\underline{F}\,\underline{G})}{d\eta} = \frac{d(\underline{F})}{d\eta}\underline{G} + \underline{F}\frac{d(\underline{G})}{d\eta}$$

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \qquad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r$$

$$\frac{\partial(\underline{e}_r \underline{e}_\theta)}{\partial \theta} = \underline{e}_\theta \underline{e}_\theta - \underline{e}_r \underline{e}_r$$

$$\underline{\nabla}(\ ) = \left(\frac{\partial(\ )}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial(\ )}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial(\ )}{\partial z}\right)_{r,\theta} \underline{e}_z$$

$$\underline{\nabla} \cdot \underline{a} = \left(\frac{\partial a_r}{\partial r}\right)_{\theta,z} + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right)_{r,z} + \frac{a_r}{r} + \left(\frac{\partial a_z}{\partial z}\right)_{r,\theta}$$

Divergence

Rotational using cross product of gradient and vector in polar coordinates

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_z \qquad \underline{e}_\theta \times \underline{e}_z = \underline{e}_r \qquad \underline{e}_z \times \underline{e}_r = \underline{e}_\theta$$

$$\underline{e}_r \times \underline{e}_r = \underline{e}_z \times \underline{e}_z = \underline{e}_\theta \times \underline{e}_\theta = 0 \qquad \underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

$$\underline{\nabla} \times \underline{a} = \left[\underline{e}_r \left(\frac{\partial(\;)}{\partial r} \right)_{\theta,z} + \underline{e}_\theta \frac{1}{r} \left(\frac{\partial(\;)}{\partial \theta} \right)_{r,z} + \underline{e}_z \left(\frac{\partial(\;)}{\partial z} \right)_{r,\theta} \right] \times \left[a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_z \underline{e}_z \right]$$

$$\underline{\nabla} \times \underline{a} = \left[\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \right] \underline{e}_r + \left[\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right] \underline{e}_\theta + \left[\frac{\partial a_\theta}{\partial r} - \frac{1}{r} \frac{\partial a_r}{\partial \theta} + \frac{a_\theta}{r} \right] \underline{e}_z$$

Curl of vector $\underline{a}$

Laplacian using dot product of gradient and gradient of scalar field in polar coordinates

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} \underline{e}_z$$

$$\underline{\nabla} \cdot \underline{a} = \left(\frac{\partial a_r}{\partial r}\right)_{\theta,z} + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right)_{r,z} + \frac{a_r}{r} + \left(\frac{\partial a_z}{\partial z}\right)_{r,\theta}$$

$$\underline{\nabla} \cdot \underline{\nabla} f = \nabla^2 f = \left(\frac{\partial^2 f}{\partial r^2}\right)_{\theta,z} + \frac{1}{r}\left(\frac{\partial f}{\partial r}\right)_{\theta,z} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right)_{r,z} + \left(\frac{\partial^2 f}{\partial z^2}\right)_{r,\theta}$$

Divergence of the Gradient is called Laplacian

Divergence of Tensor

$$\underline{\nabla}\cdot\underline{\underline{T}} = \left[\underline{e}_r\left(\frac{\partial(\ )}{\partial r}\right)_{\theta,z} + \underline{e}_\theta\,\frac{1}{r}\left(\frac{\partial(\ )}{\partial \theta}\right)_{r,z} + \underline{e}_z\left(\frac{\partial(\ )}{\partial z}\right)_{r,\theta}\right]\cdot\begin{bmatrix} T_{rr}\,\underline{e}_r\,\underline{e}_r + T_{r\theta}\,\underline{e}_r\,\underline{e}_\theta + T_{rz}\,\underline{e}_r\,\underline{e}_z \\ T_{\theta r}\,\underline{e}_\theta\,\underline{e}_r + T_{\theta\theta}\,\underline{e}_\theta\,\underline{e}_\theta + T_{\theta z}\,\underline{e}_\theta\,\underline{e}_z \\ T_{zr}\,\underline{e}_z\,\underline{e}_r + T_{z\theta}\,\underline{e}_z\,\underline{e}_\theta + T_{zz}\,\underline{e}_z\,\underline{e}_z \end{bmatrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}} = \begin{bmatrix} \dfrac{\partial(T_{rr})}{\partial r}\underline{e}_r + \dfrac{\partial(T_{r\theta})}{\partial r}\underline{e}_\theta + \dfrac{\partial(T_{rz})}{\partial r}\underline{e}_z \\ \dfrac{1}{r}\dfrac{\partial(T_{\theta r})}{\partial \theta}\underline{e}_r + \dfrac{1}{r}\dfrac{\partial(T_{\theta\theta})}{\partial \theta}\underline{e}_\theta + \dfrac{1}{r}\dfrac{\partial(T_{\theta z})}{\partial \theta}\underline{e}_z \\ \dfrac{\partial(T_{zr})}{\partial z}\underline{e}_r + \dfrac{\partial(T_{z\theta})}{\partial z}\underline{e}_\theta + \dfrac{\partial(T_{zz})}{\partial z}\underline{e}_z \end{bmatrix} + \begin{matrix} \dfrac{\underline{e}_\theta}{r}\cdot T_{rr}\,\dfrac{\partial(\underline{e}_r\,\underline{e}_r)}{\partial\theta} + \dfrac{\underline{e}_\theta}{r}\cdot T_{r\theta}\,\dfrac{\partial(\underline{e}_r\,\underline{e}_\theta)}{\partial\theta} + \dfrac{\underline{e}_\theta}{r}\cdot T_{rz}\,\dfrac{\partial(\underline{e}_r\,\underline{e}_z)}{\partial\theta} \\ \dfrac{\underline{e}_\theta}{r}\cdot T_{\theta r}\,\dfrac{\partial(\underline{e}_\theta\,\underline{e}_r)}{\partial\theta} + \dfrac{\underline{e}_\theta}{r}\cdot T_{\theta\theta}\,\dfrac{\partial(\underline{e}_\theta\,\underline{e}_\theta)}{\partial\theta} + \dfrac{\underline{e}_\theta}{r}\cdot T_{\theta z}\,\dfrac{\partial(\underline{e}_\theta\,\underline{e}_z)}{\partial\theta} \end{matrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}} = \begin{bmatrix} \dfrac{\partial(T_{rr})}{\partial r}\underline{e}_r + \dfrac{\partial(T_{r\theta})}{\partial r}\underline{e}_\theta + \dfrac{\partial(T_{rz})}{\partial r}\underline{e}_z \\ \dfrac{1}{r}\dfrac{\partial(T_{\theta r})}{\partial \theta}\underline{e}_r + \dfrac{1}{r}\dfrac{\partial(T_{\theta\theta})}{\partial \theta}\underline{e}_\theta + \dfrac{1}{r}\dfrac{\partial(T_{\theta z})}{\partial \theta}\underline{e}_z \\ \dfrac{\partial(T_{zr})}{\partial z}\underline{e}_r + \dfrac{\partial(T_{z\theta})}{\partial z}\underline{e}_\theta + \dfrac{\partial(T_{zz})}{\partial z}\underline{e}_z \end{bmatrix} + \begin{matrix} \dfrac{\underline{e}_r}{r}T_{rr} + \dfrac{\underline{e}_\theta}{r}T_{r\theta} + \dfrac{\underline{e}_z}{r}T_{rz} \\ \dfrac{\underline{e}_\theta}{r}\cdot T_{\theta r} - \dfrac{\underline{e}_r}{r}\cdot T_{\theta\theta} \end{matrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}}=\begin{bmatrix}\dfrac{\partial(T_{rr})}{\partial r}\underline{e}_r+\dfrac{\partial(T_{r\theta})}{\partial r}\underline{e}_\theta+\dfrac{\partial(T_{rz})}{\partial r}\underline{e}_z\\ \dfrac{1}{r}\dfrac{\partial(T_{\theta r})}{\partial\theta}\underline{e}_r+\dfrac{1}{r}\dfrac{\partial(T_{\theta\theta})}{\partial\theta}\underline{e}_\theta+\dfrac{1}{r}\dfrac{\partial(T_{\theta z})}{\partial\theta}\underline{e}_z\\ \dfrac{\partial(T_{zr})}{\partial z}\underline{e}_r+\dfrac{\partial(T_{z\theta})}{\partial z}\underline{e}_\theta+\dfrac{\partial(T_{zz})}{\partial z}\underline{e}_z\end{bmatrix}+\begin{matrix}\dfrac{\underline{e}_r}{r}T_{rr}+\dfrac{\underline{e}_\theta}{r}T_{r\theta}+\dfrac{\underline{e}_z}{r}T_{rz}\\ \dfrac{\underline{e}_\theta}{r}\cdot T_{\theta r}-\dfrac{\underline{e}_r}{r}\cdot T_{\theta\theta}\end{matrix}$$

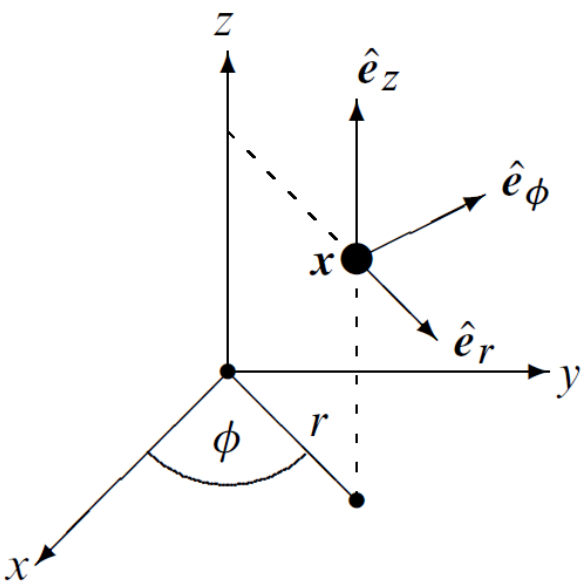
$$\underline{\nabla}\cdot\underline{\underline{T}}=\begin{matrix}\left[\dfrac{\partial(T_{rr})}{\partial r}+\dfrac{1}{r}\dfrac{\partial(T_{\theta r})}{\partial\theta}+\dfrac{\partial(T_{zr})}{\partial z}+\dfrac{1}{r}T_{rr}-\dfrac{1}{r}\cdot T_{\theta\theta}\right]\underline{e}_r\\ \left[\dfrac{\partial(T_{r\theta})}{\partial r}+\dfrac{1}{r}\dfrac{\partial(T_{\theta\theta})}{\partial\theta}+\dfrac{\partial(T_{z\theta})}{\partial z}+\dfrac{1}{r}T_{r\theta}+\dfrac{1}{r}\cdot T_{\theta r}\right]\underline{e}_\theta\\ \left[\dfrac{\partial(T_{rz})}{\partial r}+\dfrac{1}{r}\dfrac{\partial(T_{\theta z})}{\partial\theta}+\dfrac{\partial(T_{zz})}{\partial z}+\dfrac{1}{r}T_{rz}\right]\underline{e}_z\end{matrix}$$

Divergence of Tensor

Operator	**Name**
$\underline{\nabla} f$	Grad of f, Gradient
$\underline{\nabla} \cdot \underline{a}$	Div of a, Divergence
$\underline{\nabla} \times \underline{a}$	Curl of a, rotor or rotational
$\underline{\nabla} \underline{a}$	Dyad of a
$\underline{\nabla} \cdot \underline{\nabla} f = \nabla^2 f$	Laplacian of f
$\underline{\nabla} \cdot \underline{\underline{T}}$	Div of a tensor

$$
\underline{\nabla}\,\underline{a} = \begin{array}{lll}
\underline{e}_r\,\underline{e}_r\,\dfrac{\partial a_r}{\partial r} + & \underline{e}_r\,\underline{e}_\theta\,\dfrac{\partial a_\theta}{\partial r} & +\,\underline{e}_r\,\underline{e}_z\,\dfrac{\partial a_z}{\partial r} \\
+\,\underline{e}_\theta\,\underline{e}_r\left(\dfrac{1}{r}\dfrac{\partial a_r}{\partial \theta} - \dfrac{a_\theta}{r}\right) + & \underline{e}_\theta\,\underline{e}_\theta\left(\dfrac{1}{r}\dfrac{\partial a_\theta}{\partial \theta} - \dfrac{a_\theta}{r}\right) + & \underline{e}_\theta\,\underline{e}_z\left(\dfrac{1}{r}\dfrac{\partial a_z}{\partial \theta}\right) \\
+\,\underline{e}_z\,\underline{e}_r\left(\dfrac{\partial a_r}{\partial z}\right) & +\,\underline{e}_z\,\underline{e}_\theta\left(\dfrac{\partial a_\theta}{\partial z}\right) + & \underline{e}_z\,\underline{e}_z\left(\dfrac{\partial a_z}{\partial z}\right)
\end{array}
$$

If you want to use ϕ instead of θ



$$
\begin{aligned}
\nabla \boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial r} + \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial r} + \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_z \frac{\partial U_z}{\partial r} \\
& + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_r \left(\frac{1}{r} \frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r} \right) + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_\phi \left(\frac{1}{r} \frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r} \right) + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_z \frac{1}{r} \frac{\partial U_z}{\partial \phi} \\
& + \hat{\boldsymbol{e}}_z \hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial z} + \hat{\boldsymbol{e}}_z \hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial z} + \hat{\boldsymbol{e}}_z \hat{\boldsymbol{e}}_z \frac{\partial U_z}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = \hat{\boldsymbol{e}}_r &\left(U_r\frac{\partial U_r}{\partial r} + \frac{U_\phi}{r}\frac{\partial U_r}{\partial \phi} + U_z\frac{\partial U_r}{\partial z} - \frac{U_\phi^2}{r}\right) \\
+ \hat{\boldsymbol{e}}_\phi &\left(U_r\frac{\partial U_\phi}{\partial r} + \frac{U_\phi}{r}\frac{\partial U_\phi}{\partial \phi} + U_z\frac{\partial U_\phi}{\partial z} + \frac{U_r U_\phi}{r}\right) \\
+ \hat{\boldsymbol{e}}_z &\left(U_z\frac{\partial U_z}{\partial r} + \frac{U_\phi}{r}\frac{\partial U_z}{\partial \phi} + U_z\frac{\partial U_z}{\partial z}\right)
\end{aligned}
$$

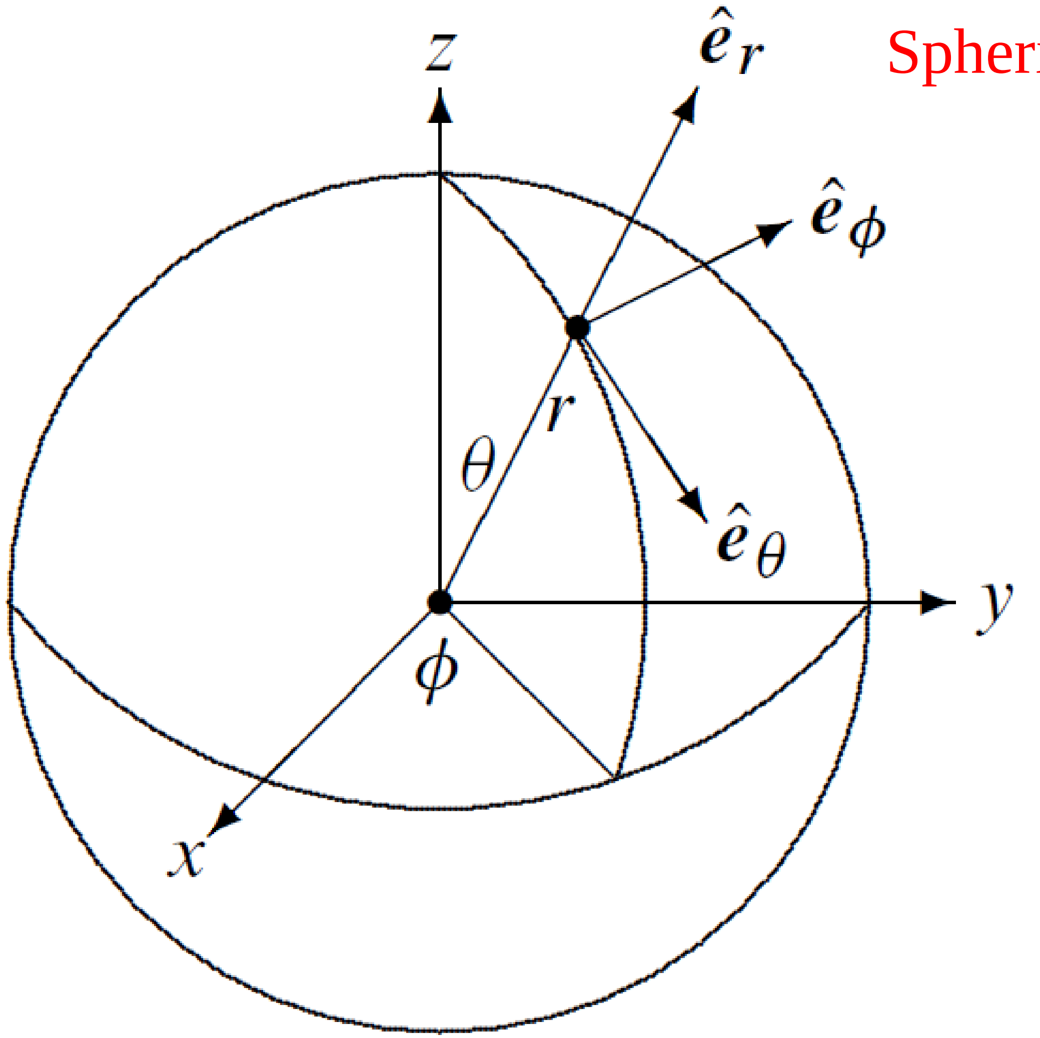
$$
\begin{aligned}
\nabla\cdot\boldsymbol{T} = \hat{\boldsymbol{e}}_r &\left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr}}{r} - \frac{T_{\phi\phi}}{r}\right) \\
+ \hat{\boldsymbol{e}}_\phi &\left(\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{r\phi}}{r} + \frac{T_{\phi r}}{r}\right) \\
+ \hat{\boldsymbol{e}}_z &\left(\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r}\right)
\end{aligned}
$$

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \phi^2} + \frac{\partial^2 S}{\partial z^2} + \frac{1}{r}\frac{\partial S}{\partial r}.$$

$$\begin{aligned}
\nabla^2 \boldsymbol{U} = \hat{\boldsymbol{e}}_r &\left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_r}{\partial \phi^2} + \frac{\partial^2 U_r}{\partial z^2} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{2}{r^2}\frac{\partial U_\phi}{\partial \phi} - \frac{U_r}{r^2}\right) \\
+ \hat{\boldsymbol{e}}_\phi &\left(\frac{\partial^2 U_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \phi^2} + \frac{\partial^2 U_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial U_\phi}{\partial r} + \frac{2}{r^2}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r^2}\right) \\
+ \hat{\boldsymbol{e}}_z &\left(\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_z}{\partial \phi^2} + \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r}\frac{\partial U_z}{\partial r}\right)
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{U}) = \hat{\boldsymbol{e}}_r &\left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U_\phi}{\partial r\partial\phi} + \frac{\partial^2 U_z}{\partial r\partial z} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{1}{r^2}\frac{\partial U_\phi}{\partial\phi} - \frac{U_r}{r^2}\right) \\
+ \hat{\boldsymbol{e}}_\phi &\left(\frac{1}{r}\frac{\partial^2 U_r}{\partial\phi\partial r} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial\phi^2} + \frac{1}{r}\frac{\partial^2 U_z}{\partial\phi\partial z} + \frac{1}{r^2}\frac{\partial U_r}{\partial\phi}\right) \\
+ \hat{\boldsymbol{e}}_z &\left(\frac{\partial^2 U_r}{\partial z\partial r} + \frac{1}{r}\frac{\partial^2 U_\phi}{\partial z\partial\phi} + \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r}\frac{\partial U_r}{\partial z}\right)
\end{aligned}
$$

Spherical coordinates



$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \arccos\frac{z}{\sqrt{x^2 + y^2 + z^2}}, \qquad \phi = \arctan\frac{y}{x}$$

$$\hat{\boldsymbol{e}}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

$$\hat{\boldsymbol{e}}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta),$$

$$\hat{\boldsymbol{e}}_\phi = \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0).$$

$$\boldsymbol{U} = \hat{\boldsymbol{e}}_r U_r + \hat{\boldsymbol{e}}_\theta U_\theta + \hat{\boldsymbol{e}}_\phi U_\phi$$

$$\begin{aligned}
d\boldsymbol{\ell} &= \hat{\boldsymbol{e}}_r\, dr + \hat{\boldsymbol{e}}_\theta\, rd\theta + \hat{\boldsymbol{e}}_\phi\, r\sin\theta\, d\phi, \\
d\boldsymbol{S} &= \hat{\boldsymbol{e}}_r\, r^2 \sin\theta\, d\theta\, d\phi + \hat{\boldsymbol{e}}_\theta r\sin\theta d\phi dr + \hat{\boldsymbol{e}}_\phi\, rdrd\theta, \\
dV &= r^2 \sin\theta\, drd\theta d\phi.
\end{aligned}$$

$$\nabla = \hat{\boldsymbol{e}}_r \nabla_r + \hat{\boldsymbol{e}}_\theta \nabla_\theta + \hat{\boldsymbol{e}}_\phi \nabla_\phi = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi},$$

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial \theta} = \hat{\boldsymbol{e}}_\theta, \qquad \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial \theta} = -\hat{\boldsymbol{e}}_r,$$

$$\frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial \phi} = \cos\theta \, \hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial \hat{\boldsymbol{e}}_r}{\partial \phi} = \sin\theta \, \hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial \hat{\boldsymbol{e}}_\phi}{\partial \phi} = -\sin\theta \, \hat{\boldsymbol{e}}_r - \cos\theta \, \hat{\boldsymbol{e}}_\theta.$$

$$\nabla S = \hat{\boldsymbol{e}}_r \frac{\partial S}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r} \frac{\partial S}{\partial \theta} + \hat{\boldsymbol{e}}_z \frac{1}{r \sin\theta} \frac{\partial S}{\partial \phi},$$

$$\nabla \cdot \boldsymbol{U} = \frac{\partial U_r}{\partial r} + \frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial U_\phi}{\partial \phi} + \frac{2U_r}{r} + \frac{U_\theta}{r \tan\theta}$$

$$
\begin{aligned}
\nabla \times \boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r \left(\frac{1}{r} \frac{\partial U_\theta}{\partial \phi} - \frac{1}{r \sin\theta} \frac{\partial U_\theta}{\partial \theta} + \frac{U_\phi}{r \tan\theta} \right) \\
& + \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r \sin\theta} \frac{\partial U_r}{\partial \phi} - \frac{\partial U_\phi}{\partial r} - \frac{U_\phi}{r} \right) \\
& + \hat{\boldsymbol{e}}_\phi \left(\frac{\partial U_\theta}{\partial r} - \frac{1}{r} \frac{\partial U_r}{\partial \theta} + \frac{U_\theta}{r} \right).
\end{aligned}
$$

$$
\begin{aligned}
\nabla \boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial r} + \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_\theta \frac{\partial U_\theta}{\partial r} + \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial r} \\
& + \hat{\boldsymbol{e}}_\theta \hat{\boldsymbol{e}}_r \left(\frac{1}{r} \frac{\partial U_r}{\partial \theta} - \frac{U_\theta}{r} \right) + \hat{\boldsymbol{e}}_\theta \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} \right) + \hat{\boldsymbol{e}}_\theta \hat{\boldsymbol{e}}_\phi \frac{1}{r} \frac{\partial U_\phi}{\partial \theta} \\
& + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_r \left(\frac{1}{r \sin\theta} \frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r} \right) + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r \sin\theta} \frac{\partial U_\theta}{\partial \phi} - \frac{U_\phi}{r \tan\theta} \right) \\
& + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_\phi \left(\frac{1}{r \sin\theta} \frac{\partial U_\phi}{\partial \phi} + \frac{U_\theta}{r \tan\theta} + \frac{U_r}{r} \right).
\end{aligned}
$$

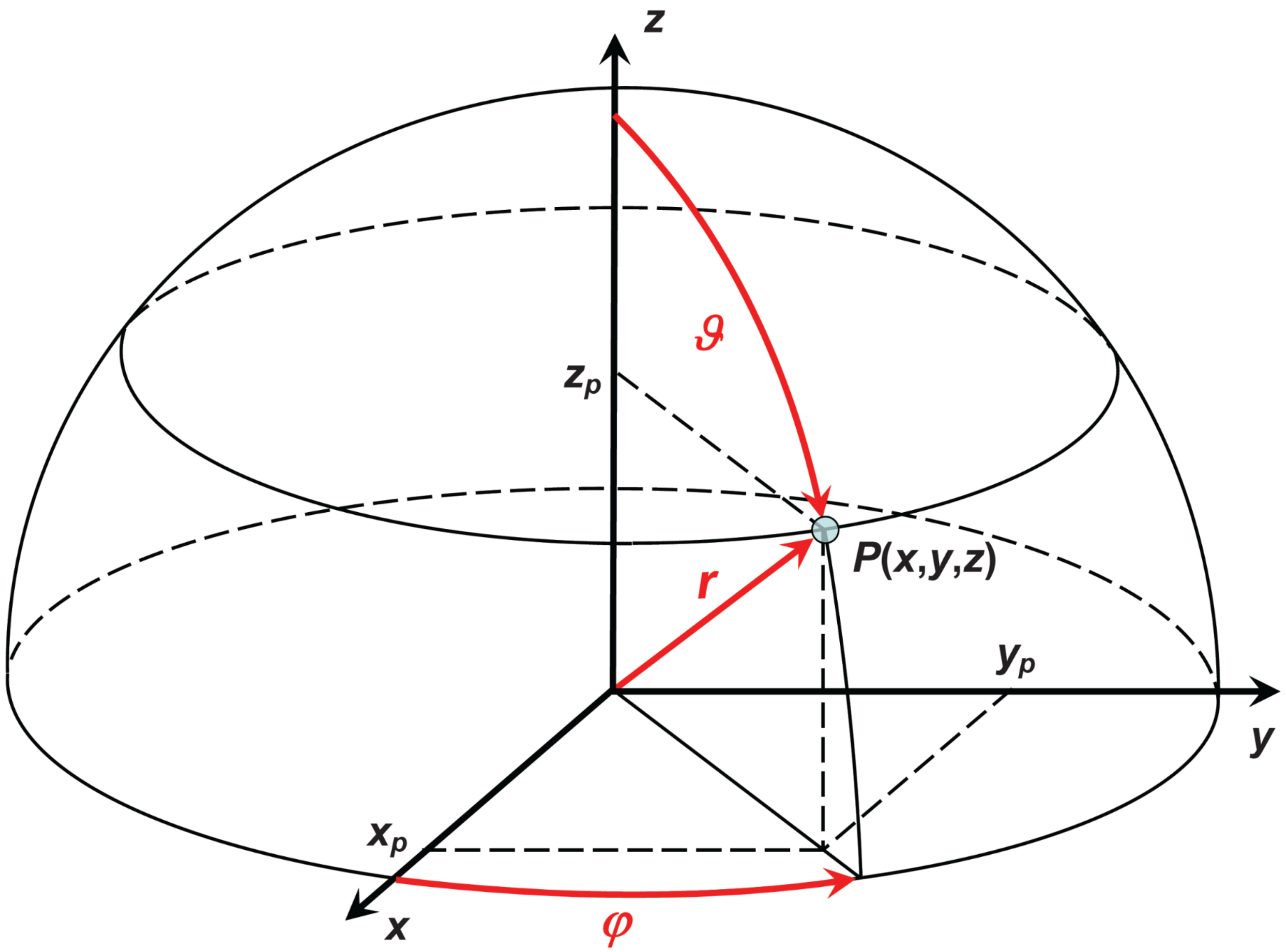
$$
\begin{aligned}
(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r\left(U_r\frac{\partial U_r}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_r}{\partial\theta}+\frac{U_\phi}{r\sin\theta}\frac{\partial U_r}{\partial\phi}-\frac{U_\theta^2}{r}-\frac{U_\phi^2}{r}\right)\\
&+\hat{\boldsymbol{e}}_\theta\left(U_r\frac{\partial U_\theta}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_\theta}{\partial\theta}+\frac{U_\phi}{r\sin\theta}\frac{\partial U_\theta}{\partial\phi}+\frac{U_rU_\theta}{r}-\frac{U_\phi^2}{r\tan\theta}\right)\\
&+\hat{\boldsymbol{e}}_\phi\left(U_r\frac{\partial U_\phi}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_\phi}{\partial\theta}+\frac{U_\phi}{r\sin\theta}\frac{\partial U_\phi}{\partial\phi}+\frac{U_rU_\phi}{r}+\frac{U_\theta U_\phi}{r\tan\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\boldsymbol{T} = {} & \hat{\boldsymbol{e}}_r\left(\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi}+\frac{2T_{rr}}{r}+\frac{T_{\theta r}}{r\tan\theta}-\frac{T_{\theta\theta}}{r}-\frac{T_{\phi\phi}}{r}\right)\\
&+\hat{\boldsymbol{e}}_\theta\left(\frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi}+\frac{2T_{r\theta}}{r}+\frac{T_{\theta r}}{r}+\frac{T_{\theta\theta}}{r\tan\theta}-\frac{T_{\phi\phi}}{r\tan\theta}\right)\\
&+\hat{\boldsymbol{e}}_\phi\left(\frac{\partial \tau_{r\varphi}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{2T_{r\phi}}{r}+\frac{T_{\phi r}}{r}+\frac{T_{\theta\phi}}{r\tan\theta}+\frac{T_{\phi\theta}}{r\tan\theta}\right)
\end{aligned}
$$

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 S}{\partial \phi^2} + \frac{2}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2 \tan\theta}\frac{\partial S}{\partial \theta}.$$

$$\begin{aligned}
\nabla^2 \boldsymbol{U} = \hat{\boldsymbol{e}}_r \Bigg(& \frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_r}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_r}{\partial \phi^2} \\
& + \frac{2}{r}\frac{\partial U_r}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{2}{r^2 \sin\theta}\frac{\partial U_\phi}{\partial \phi} - 2\frac{U_r}{r^2} - \frac{U_\theta}{r^2 \tan\theta} \Bigg) \\
+ \hat{\boldsymbol{e}}_\theta \Bigg(& \frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\theta}{\partial \phi^2} \\
& + \frac{2}{r}\frac{\partial U_\theta}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{U_\theta}{r^2 \sin^2\theta} \Bigg) \\
+ \hat{\boldsymbol{e}}_\phi \Bigg(& \frac{\partial^2 U_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\phi}{\partial \phi^2} \\
& + \frac{2}{r}\frac{\partial U_\phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_\phi}{\partial \theta} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{2}{r^2 \sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r^2 \sin^2\theta} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
\nabla(\nabla\cdot\boldsymbol{U}) = \hat{e}_r \Bigg(& \frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U_\theta}{\partial r\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2 U_\phi}{\partial r\partial\phi} \\
& + \frac{2}{r}\frac{\partial U_r}{\partial r} + \frac{\cot\theta}{r}\frac{\partial U_\theta}{\partial r} - \frac{1}{r^2}\frac{\partial U_\theta}{\partial\theta} - \frac{1}{r^2\sin\theta}\frac{\partial U_\phi}{\partial\phi} - \frac{2U_r}{r^2} - \frac{U_\theta}{r^2\tan\theta} \Bigg) \\
+ \hat{e}_\theta \Bigg(& \frac{1}{r}\frac{\partial^2 U_r}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2} + \frac{1}{r^2\sin\theta}\frac{\partial^2 U_\phi}{\partial\theta\partial\phi} \\
& + \frac{2}{r^2}\frac{\partial U_r}{\partial\theta} + \frac{\cot\theta}{r^2}\frac{\partial U_\theta}{\partial\theta} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\phi}{\partial\phi} - \frac{U_\theta}{r^2\sin^2\theta} \Bigg) \\
+ \hat{e}_\phi \Bigg(& \frac{1}{r\sin\theta}\frac{\partial^2 U_r}{\partial\phi\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial^2 U_\theta}{\partial\phi^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U_\phi}{\partial\phi\partial\theta} \\
& + \frac{2}{r^2\sin\theta}\frac{\partial U_r}{\partial\phi} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\theta}{\partial\theta} \Bigg)
\end{aligned}
$$



where φ represents the azimuthal angle and θ the zenith angle or co-latitude

Cylindrical

Cylindrical Coordinates

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\phi = \frac{1}{r}\frac{\partial f}{\partial \phi}; \quad (\nabla f)_z = \frac{\partial f}{\partial z}$$

Cylindrical

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}$$

$$(\nabla \times \mathbf{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(\nabla \times \mathbf{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}$$

Laplacian

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\phi}{\partial \phi} - \frac{A_r}{r^2}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi + \frac{2}{r^2}\frac{\partial A_r}{\partial \phi} - \frac{A_\phi}{r^2}$$

$$(\nabla^2 \mathbf{A})_z = \nabla^2 A_z$$

Cylindrical

Components of $(\mathbf{A} \cdot \nabla)\mathbf{B}$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_r = A_r \frac{\partial B_r}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_r}{\partial \phi} + A_z \frac{\partial B_r}{\partial z} - \frac{A_\phi B_\phi}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_\phi = A_r \frac{\partial B_\phi}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_\phi}{\partial \phi} + A_z \frac{\partial B_\phi}{\partial z} + \frac{A_\phi B_r}{r}$$

$$(\mathbf{A} \cdot \nabla \mathbf{B})_z = A_r \frac{\partial B_z}{\partial r} + \frac{A_\phi}{r} \frac{\partial B_z}{\partial \phi} + A_z \frac{\partial B_z}{\partial z}$$

Cylindrical

Divergence of a tensor

$$(\nabla \cdot T)_r = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rr}) + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} - \frac{T_{\phi\phi}}{r}$$

$$(\nabla \cdot T)_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rT_{r\phi}) + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{\phi r}}{r}$$

$$(\nabla \cdot T)_z = \frac{1}{r}\frac{\partial}{\partial r}(rT_{rz}) + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z}$$

Spherical Coordinates

Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Gradient

$$(\nabla f)_r = \frac{\partial f}{\partial r}; \quad (\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial\theta}; \quad (\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

Spherical

Curl

$$(\nabla \times \mathbf{A})_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial\phi}$$

$$(\nabla \times \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)$$

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial\theta}$$

Spherical

Laplacian

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Spherical

Laplacian of a vector

$$(\nabla^2 \mathbf{A})_r = \nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta} - \frac{2\cot\theta A_\theta}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$(\nabla^2 \mathbf{A})_\theta = \nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} - \frac{A_\theta}{r^2\sin^2\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial \phi}$$

$$(\nabla^2 \mathbf{A})_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial \phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial \phi}$$

Spherical

Components of $(\mathbf{A}\cdot\nabla)\mathbf{B}$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_r = A_r\frac{\partial B_r}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_r}{\partial \theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_r}{\partial \phi} - \frac{A_\theta B_\theta + A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\theta = A_r\frac{\partial B_\theta}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\theta}{\partial \phi} + \frac{A_\theta B_r}{r} - \frac{\cot\theta A_\phi B_\phi}{r}$$

$$(\mathbf{A}\cdot\nabla\mathbf{B})_\phi = A_r\frac{\partial B_\phi}{\partial r} + \frac{A_\theta}{r}\frac{\partial B_\phi}{\partial \theta} + \frac{A_\phi}{r\sin\theta}\frac{\partial B_\phi}{\partial \phi} + \frac{A_\phi B_r}{r} + \frac{\cot\theta A_\phi B_\theta}{r}$$

Spherical

Divergence of a tensor

$$(\nabla \cdot T)_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{rr}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta T_{\theta r}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial\phi} - \frac{T_{\theta\theta}+T_{\phi\phi}}{r}$$

$$(\nabla \cdot T)_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\theta}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta T_{\theta\theta}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial\phi} + \frac{T_{\theta r}}{r} - \frac{\cot\theta T_{\phi\phi}}{r}$$

$$(\nabla \cdot T)_\phi = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_{r\phi}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta T_{\theta\phi}) + \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi} + \frac{T_{\phi r}}{r} + \frac{\cot\theta T_{\phi\theta}}{r}$$

Energy Equation

$$\frac{\partial(\rho\hat{E})}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}\,\hat{E}) = -\underline{\nabla}\cdot[\underline{\underline{\tau}}\cdot\underline{v}] - \underline{\nabla}\cdot[p\,\underline{\underline{I}}\cdot\underline{v}] - \underline{\nabla}\cdot\underline{q} + \dot{q}_G$$

Accumulation **Advection** **Viscous dissipation** **Flow energy** **Conduction** **Generation**

$$\underline{q} = -k\,\underline{\nabla}\,T$$

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + \dot{q}_G$$

$\dot{q}_G = \dot{Q}_{v,Gen}$ Volumetric heat rate generation

COMSOL PDE

$$e_a \frac{\partial^2 u}{\partial t^2} + \ d_a \frac{\partial u}{\partial t} + \boldsymbol{\nabla} \cdot (-c\boldsymbol{\nabla} u - \alpha\, u + \ \gamma) + \ \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u + au = \mathrm{f}$$

PDEPE

$$c\left(x, t, u, \frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = \frac{1}{x^m}\ \frac{\partial}{\partial x}\left(x^m\ f\left(x, t, u, \frac{\partial u}{\partial x}\right)\right) + s\left(x, t, u, \frac{\partial u}{\partial x}\right)$$

Heat Equation

Flow of enthalpy

Coupled convection and radiation

$$\underbrace{\frac{1}{A_c}\frac{\partial}{\partial x}\left[A_c k\frac{\partial T}{\partial x}\right]}_{\textbf{Conduction}} - \underbrace{\frac{1}{A_c}\frac{\partial}{\partial x}\left[A_c\rho\, v_x\hat{c}_p(T-T_o)\right]}_{\textbf{Advection}} + \underbrace{\dot{q}_{Gen}}_{\textbf{Generation}} - \underbrace{h\frac{p}{A_c}\left[T-T_f\right]}_{\textbf{Convection}} = \underbrace{\frac{\partial(\rho\hat{c}_v[T-T_o])}{\partial t}}_{\textbf{Accumulation}}$$

Fourier law heat rate

Heat of reaction or active material

Internal energy or Energy storage

$$\underline{\nabla}\cdot\left[k\underline{\nabla}T\right] - \underline{v}\cdot\underline{\nabla}\left[\rho\,\hat{c}_p(T-T_o)\right] + \dot{q}_{Gen} - ha_v\left[T-T_f\right]\psi = \frac{\partial(\rho\hat{c}_v[T-T_o])}{\partial t}$$

$$\frac{\partial^2\theta}{\partial\eta^2} - Pe\frac{\partial\theta}{\partial\eta} + \beta - Bi\,\gamma\,\theta = \frac{\partial\theta}{\partial\tau}$$

a_v = specific area=area of the interface per unit volume

$\psi = 1$ for 1-D and 2-D flow, $\psi = 0$ for 3-D flow

Comparison of Heat Equation and other applications in Mech. Eng
Under steady state condition

$$-\frac{d}{dx}\left[k\frac{dT}{dx}\right]+h\frac{p}{A_c}T=\dot{q}_{Gen}+h\frac{p}{A_c}\,T_f$$

Heat Conduction

$$-\frac{d}{dx}\left(E\frac{du}{dx}\right)=f(x)$$

Elastic Rod

$$-\frac{d}{dx}\left(T\frac{dW}{dx}\right)+k(x)W=f(x)$$

Cable deflection

$$-\frac{d}{dx}\left(\varepsilon\frac{d\Phi}{dx}\right)=\rho(x)$$

Electrostatics

$$-\frac{d}{dx}\left(\alpha\frac{dU}{dx}\right)+\beta U=f(x)$$

General form

$$\frac{d}{dx}(\tau)+\beta U=f(x)$$

Balance equation (conservation principle)

$$\tau=-\alpha\frac{dU}{dx}$$

Constitutive equation (Physical law)

General form of differential equation for steady state conditions in 1-D

$$-\frac{d}{dx}\left(\alpha \frac{dU}{dx}\right) + \beta U = f(x)$$

General form

$$\frac{d}{dx}(\tau) + \beta U = f(x)$$

Balance equation (conservation principle)

$$\tau = -\alpha \frac{dU}{dx}$$

Constitutive equation (Physical law)

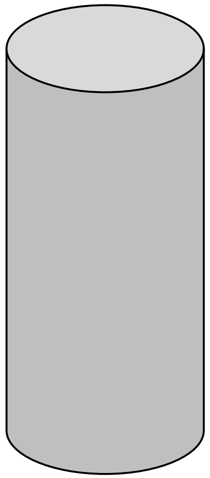
Boundary conditions

Essential	Specify U
Natural	Specify flux τ

Skip this

Problem No.1

Calculate the temperature of the cylinder vs time when is placed inside an oven.



a) $$\rho C_p \frac{dT}{dt} = -h\left(\frac{A}{V}\right)\left(T - T_f\right)$$

b) $$\rho C_p \frac{dT}{dt} = -\left(\frac{A}{V}\right)\left[h\left(T - T_f\right) + \varepsilon\sigma\left(T^4 - T_{wall}^4\right)\right]$$

c) $$\rho C_p \frac{dT}{dt} = -\left(\frac{A}{V}\right)\left[h\left(T - T_{\max} + T_{\min}\sin\left(\frac{2\pi t}{\tau}\right)\right) + \varepsilon\sigma\left(T^4 - T_{wall}^4\right)\right]$$

Data: ρ=700 kg/m^3, C_p=2000 J/kg-K, D=2cm, L=10 cm
σ=5.67e-8 W/m^2-K^4, h=10W/m^2-K, τ=2 h, T_f=353.15 K, T_w=370K
ε=0.9, T_{max}=355, T_{min}=10, T_0=300 K

I) If possible, write your equation in dimensionless form.
II) If you have the tools to solve the most complex scenario, generalize your code starting with the general case.

Skip this

$$\rho C_p \frac{dT}{dt} = -\left(\frac{A}{V}\right)\left[h\left(T - T_{\max} + T_{\min}\sin\left(\frac{2\pi\, t}{\tau}\right)\right) + \varepsilon\sigma\left(T^4 - T_{wall}^4\right)\right]$$

Dimensional equation

$$\frac{dy}{dx} = -\left[y - y_{\max} + y_{\min}\sin\left(\frac{2\pi\, x}{X_p}\right) + \beta\left(y^4 - y_{wall}^4\right)\right]$$

Dimensionless form

Dimensionless numbers

$$x = \frac{t\,h\,A}{V\rho C_p} \qquad \beta = \frac{\varepsilon\sigma T_0^3}{h} \qquad y = \frac{T}{T_0} \qquad y_{\max} = \frac{T_{\max}}{T_0} \qquad y_{\min} = \frac{T_{\min}}{T_0} \qquad y_{wall} = \frac{T_{wall}}{T_0}$$

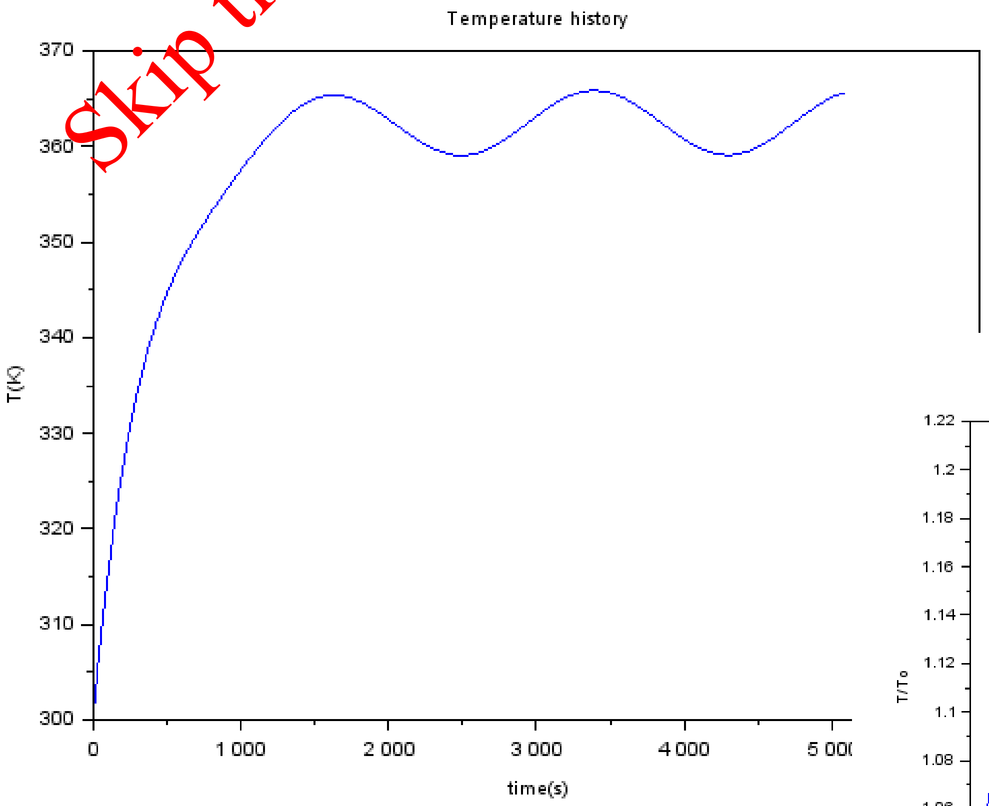
$$X_p = \frac{hA\tau}{\rho C_p V}$$

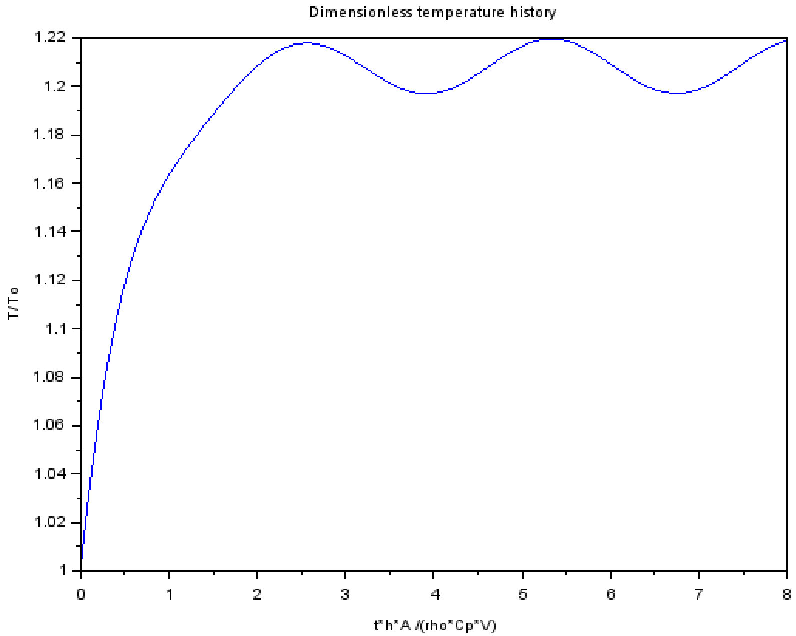
Skip this

```
function [Tout, tout]=TemDyn(id)
 // xspan=[0,0.1,0.25,0.5,0.75,1.0,1.25,1.5,2.0,2.5,3.0,3.5,8]
NM=500
for i=1:NM+1
   x(i)=(i-1)/NM
End
xspan=8.0*x;
xspan=xspan'; Y0=1.0
 [Y]=ode(Y0,0,xspan,TempVar)
 plot(xspan,Y)
xtitle('Dimensionless temperature history','t*h*A
/(rho*Cp*V)','T/To')
 [p]=DEparam(1) rho=p(1);Cp=p(2);D=p(3);L=p(4);sigma=p(5);
h=p(6);tau=p(7);Tf=p(8);Tw=p(9);epsilon=p(10);
Tmax=p(11);Tmin=p(12);T0=p(13);
pi=%pi;
V=pi*D^2*L/4
A=pi*D*L+2*pi*D^2/4
Tout=Y*T0;
tout=xspan*V*rho*Cp/(h*A);
halt('press any key')
 clf
plot(tout,Tout)
xtitle('Temperature history','time(s)','T(K)')
endfunction
```

```
function yprime=TempVar(x, y)
// x is the independent variable // y is the dependent variable
 [p]=DEparam(1)
 rho=p(1);Cp=p(2);D=p(3);L=p(4);sigma=p(5);
h=p(6);tau=p(7);Tf=p(8);Tw=p(9);epsilon=p(10);
Tmax=p(11);Tmin=p(12);T0=p(13);
pi=%pi;
V=pi*D^2*L/4
A=pi*D*L+2*pi*D^2/4
 ymin=Tmin/T0
 ymax=Tmax/T0
ywall=Tw/T0
Xp=h*A*tau/(rho*Cp*V)
beta1=epsilon*sigma*T0^3/h
yprime=y-ymax+ymin*sin(2*pi*x/(Xp))+beta1*(y^4-ywall^4)
yprime=-yprime;
endfunction

function
 [p]=DEparam(id) rho=700 // kg/m3
 Cp=2000 //J/kg-K
 D=2e-2 //m
 L=10e-2 //m
sigma=5.67e-8 // W/m2-K4
h=10 // W/m2-K
tau=0.5*3600 // s
Tf=353.15 //K, Temperature of fluid inside oveb
Tw=370 //K,Temperature of the walls inside the oven
epsilon=0.9 //emissivity of the cylinder
Tmax=355 // K, maximum temperature
Tmin=10 // K, Amplitude of variation of temperature
T0=300 // K, initial temperature of the cylinder
 p=[rho,Cp,D,L,sigma,h,tau,Tf,Tw,epsilon,Tmax,Tmin,T0]
endfunction
```

Skip this


Temperature history
T(K)
370
360
350
340
330
320
310
300
0
1 000
2 000
3 000
4 000
5 000
time(s)


Dimensionless temperature history
T/To
1.22
1.2
1.18
1.16
1.14
1.12
1.1
1.08
1.06
1.04
1.02
1
0
1
2
3
4
5
6
7
8
t*h*A /(rho*Cp*V)

Problem No.2

In 1963,Edward Lorenz developed a simplified mathematical model for atmospheric convection. The model is a system of three ordinary differential equations now known as the Lorenz equations:

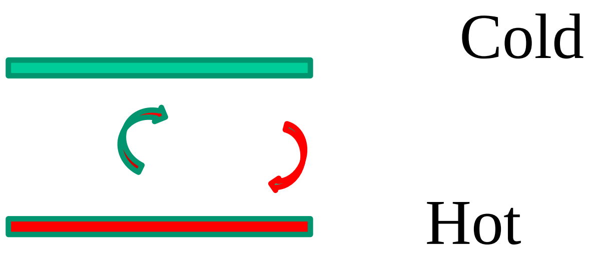
Here *A*, *T* and *D* make up the system state, *t* is time, and *k*, *r* , *b* are the system parameters. The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions and forward osmosis. From a technical standpoint, the Lorenz system is nonlinear, three-dimensional and deterministic. The Lorenz equations have been the subject of at least one book-length study

$$\frac{dA}{dx} = -k(A - T)$$

$$\frac{dT}{dx} = -AD + rA - T$$

$$\frac{dD}{dx} = AT - b\,D$$

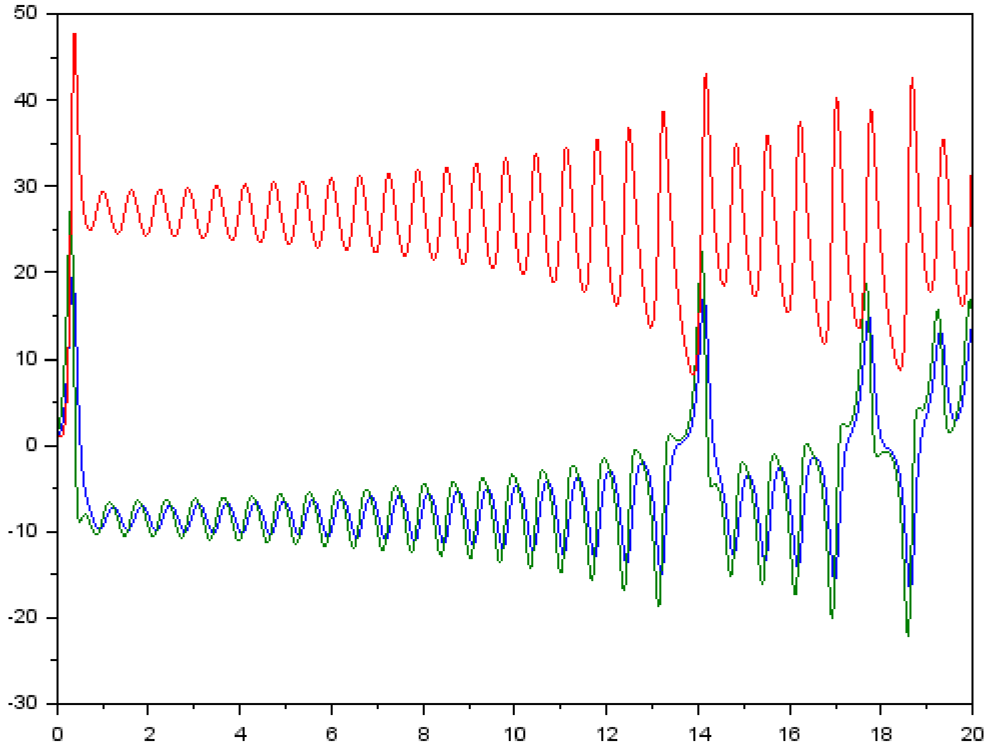
In the Rayleigh-Bernard convection pattern, ***A*** is the amplitude of convection motion; ***T*** is the temperature difference between ascending and descending currents; ***D*** is the deviation of the vertical temperature profile from linearity, and *k*, *r* and *b* are three physical and geometrical parameters with positive values.



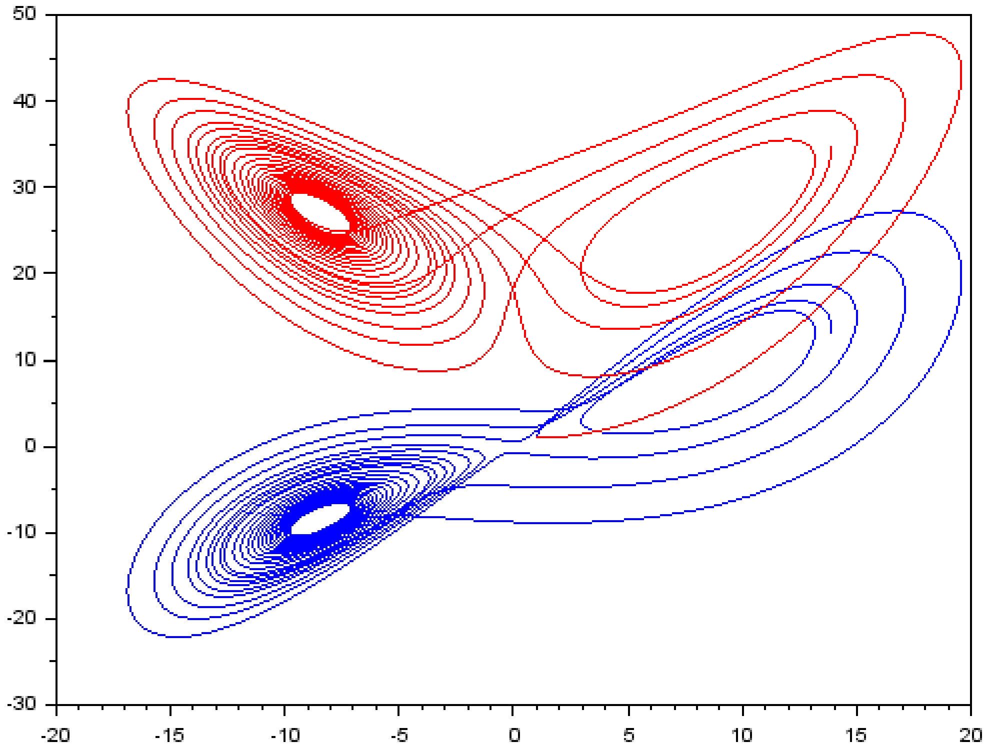
Write a code to solve the Lorentz system, performing computations for *k*=10, *b*=8/3, *A*(0)=*T*(0)=*D*(0)=1, and *r*=0.5,5.0 28.0. Plot and discuss the projections of solution orbit in the ***AT*** and ***AD*** planes , and investigate the effect of time step .

```
function [Y, t]=LorenzP1(id)
// This function is the orchestrator
//This was done to Solve Lorenz Problem,
//but can be used for any other system
//==================================
// dA/dt=-k*(A-T)
// dT/dt=-A*D+r*A-T
// dD/dt=A*T-b*D
// k=10, b=8/3, r=1/2, 5 and 28
// Initial conditions: A(0)=T(0)=D(0)=1
//____________________________________
t0=0; //initial time, or initial independent variable
Y0=[1,1,1]';
//Initial conditions
NP=5000
for i=1:NP
tspan(i)=20*i/NP
end // tspan=times when i need the solution
[Y]=ode("stiff",Y0,t0,tspan,RayBerF,jacRB)
plot(tspan,Y')
t=tspan
halt('press any key')
Clf
 plot(Y(1,:),Y(2,:))
 plot(Y(1,:),Y(3,:),'r')
endfunction
```

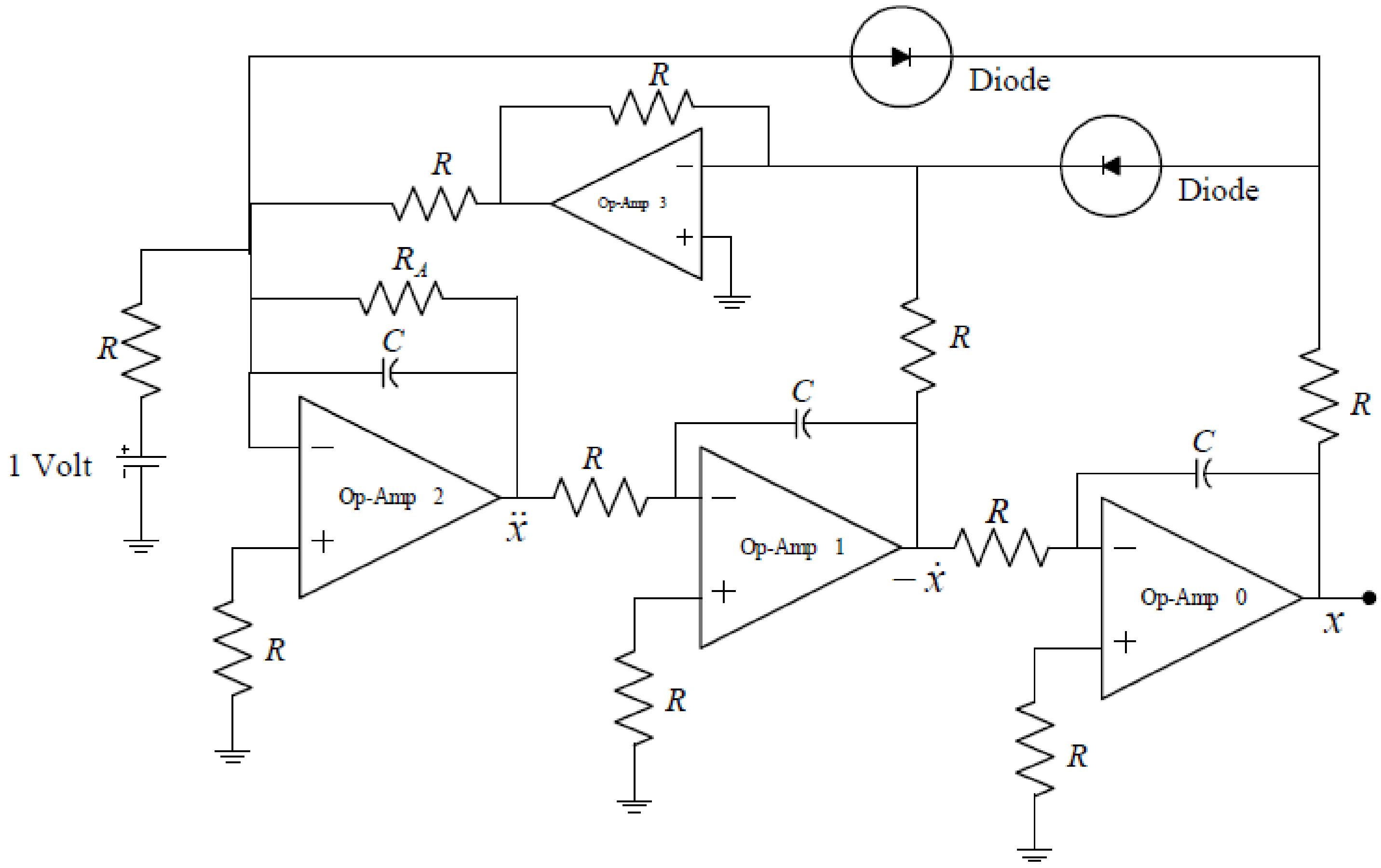
```
function [p]=LorenzP(id) // This function is just to read parameters
k=10;b=8/3;r=28
p=[k,b,r]
endfunction
function [yprime]=RayBerF(x, y)
//x= independent variable, this can be position or time
//y=Dependent variable, remember may be a vector
[p]=LorenzP(1)
k=p(1);b=p(2);r=p(3)
yprime(1)=-k*(y(1)-y(2));
yprime(2)=-y(1)*y(3)+r*y(1)-y(2);
yprime(3)=y(1)*y(2)-b*y(3);
endfunction

function [J]=jacRB(x, y)
[p]=LorenzP(1)
k=p(1);b=p(2);r=p(3)
J(1,1)=-k;J(1,2)=k;J(1,3)=0;
J(2,1)=-y(3)+r;J(2,2)=-1;J(2,3)=-y(1);
J(3,1)=y(2);J(3,2)=y(1);J(3,3)=-b;
endfunction
```



50
40
30
20
10
0
-10
-20
-30
0
2
4
6
8
10
12
14
16
18
20


50
40
30
20
10
0
-10
-20
-30
-20
-15
-10
-5
0
5
10
15
20

Problem No.3



The voltage of the circuit is given by the equation: (Use A=3/5)

$$\dddot{x} + A\ddot{x} + \dot{x} - |x| + 1 = 0$$

For initial conditions of the form: $\ddot{x}(0) = \dot{x}(0) = x(0) = 0$

Integrating from t=0 to 50 s, make a plot of d^2x/dt^2 vs dx/dt, and dx/dt vs x :

High order differential equation

$$\dddot{x} = -\left(A\ddot{x} + \dot{x} - |x| + 1\right)$$

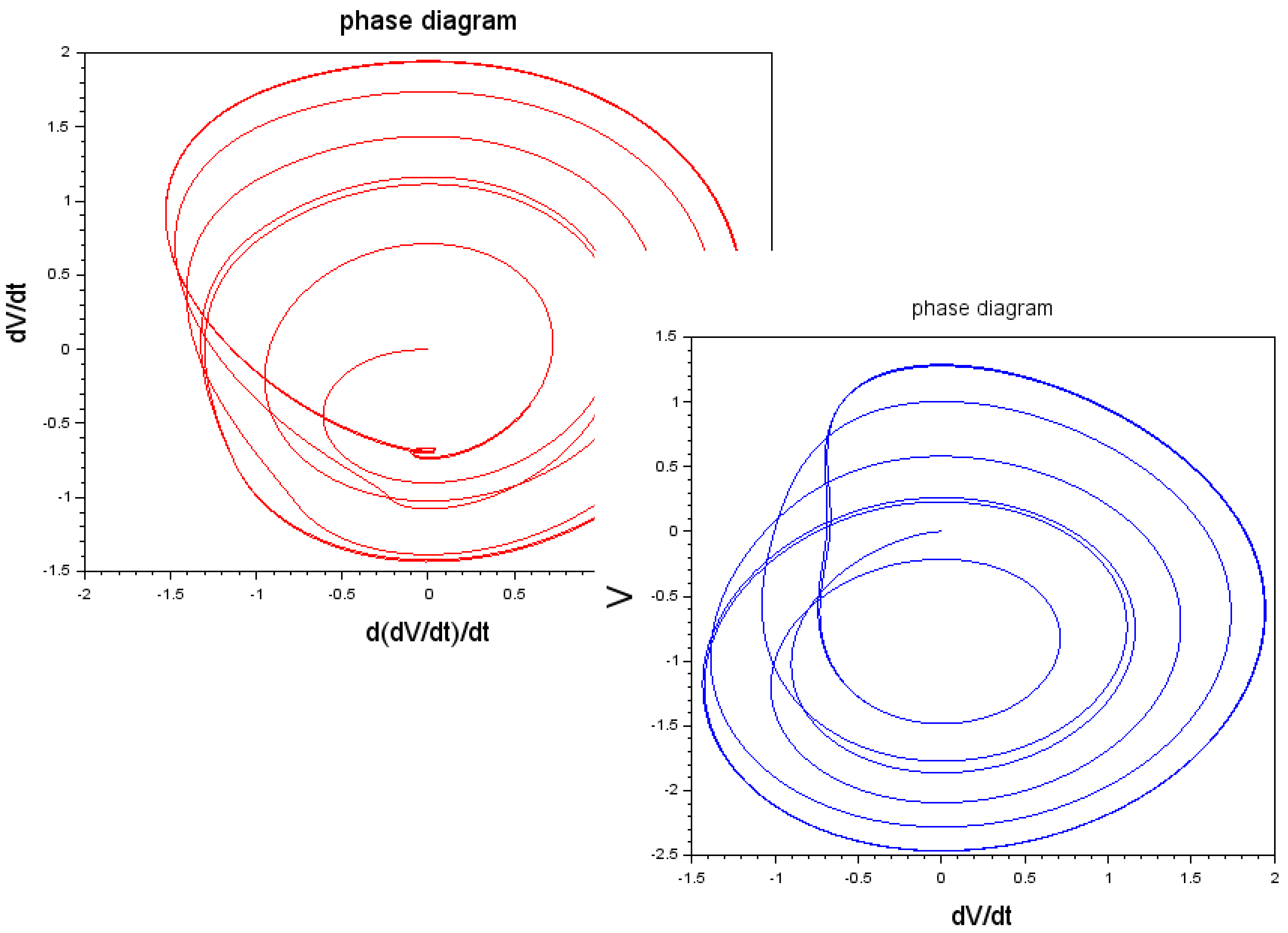
Canonical form

$$\frac{dy_3}{dt} = -\left(Ay_3 + y_2 - |y_1| + 1\right)$$

$$\frac{dy_2}{dt} = y_3$$

$$\frac{dy_1}{dt} = y_2$$

```
function [Y, t]=ProbJerk1(id)
t0=0;
Y0=[0,0,0]'; // Initial conditions
NP=1000
for i=1:NP+1
   tspan(i)=50*(i-1)/NP
end [Y]=ode(Y0,t0,tspan,JerkProb);
plot(Y(3,:),Y(2,:),'r')
xtitle('phase diagram','d(dV/dt)/dt','dV/dt')
halt('press any key to see the next plot')
clf
plot(Y(2,:),Y(1,:))
xtitle('phase diagram','dV/dt','V')
t=tspan;
endfunction

function [yprime]=JerkProb(t, y)
// Original problem
// d3x/dt3+A*d2x/dt2+dx/dt-abs(x)=0
// Caninical form
A=(3/5) ;
yprime(3)=abs(y(1))-1-y(2)-A*y(3)
yprime(2)=y(3)
yprime(1)=y(2)
endfunction
```


phase diagram
dV/dt
d(dV/dt)/dt
phase diagram
V
dV/dt

How to solve a set of differential equations with boundary conditions, not initial conditions?

Boundary value problem

If you have a set of differential equations of the form:

$$
\begin{aligned}
y_1' &= f_1(\underline{y}) \\
y_2' &= f_2(\underline{y}) \\
y_3' &= f_3(\underline{y}) \\
y_4' &= f_4(\underline{y}) \\
y_5' &= f_5(\underline{y})
\end{aligned}
\qquad
\underbrace{\begin{aligned}
y_1(0) &= a_1 \\
y_2(0) &= a_2 \\
y_3(0) &= \zeta_3 \\
y_4(0) &= \zeta_4 \\
y_5(0) &= \zeta_5
\end{aligned}}_{\text{Departure state}}
\qquad
\underbrace{\begin{aligned}
y_1(L) &= \eta_1 \\
y_2(L) &= b_2 \\
y_3(L) &= \eta_3 \\
y_4(L) &= b_4 \\
y_5(L) &= b_5
\end{aligned}}_{\text{Arrival state}}
$$

To solve the set of five differential equations , five boundary conditions are needed. The non overlapping of initial BC and final BC is not obligatory, in this particular example ***a*** is used for known initial boundary conditions, and ***b*** for final BC. $\underline{\zeta}$ is the remaining initial BC conditions unknown, and $\underline{\eta}$ the final BC unknown.

The easiest way to solve the system of differential equations numerically is by using the algorithms knowing initial conditions of all of them. Nevertheless the initial state is not fully known, but a guess can be given to use any technique to solve the ODEs, then after integration the residual for the state variables at the arrival boundary is calculated , and as a final step, the Newton-Raphson technique can be used forcing the residuals to vanish.

$$\frac{d\underline{y}}{dt} = \underline{f}\left(\underline{y}\right)$$

For our system, this is a vectorial equation

$$\frac{dy_1}{dt} = f_1$$

To track how y_1 evolves with initial guess ζ_3, the chain rule can be used

$$\frac{d}{d\zeta_3}\left(\frac{dy_1}{dt}\right) = \frac{\partial f_1}{\partial y_1}\left(\frac{dy_1}{d\zeta_3}\right) + \frac{\partial f_1}{\partial y_2}\left(\frac{dy_2}{d\zeta_3}\right) + \frac{\partial f_1}{\partial y_3}\left(\frac{dy_3}{d\zeta_3}\right) + \frac{\partial f_1}{\partial y_4}\left(\frac{dy_4}{d\zeta_3}\right) + \frac{\partial f_1}{\partial y_5}\left(\frac{dy_5}{d\zeta_3}\right)$$

$$\frac{d}{dt}\left(\frac{dy_1}{d\zeta_3}\right) = \sum \frac{\partial f_1}{\partial y_i}\left(\frac{dy_i}{d\zeta_3}\right)$$

This is done for all the state variables, and for all unknown initial conditions

$$\frac{d}{dt}\left(\frac{dy_1}{d\zeta_3}\right)=\sum\frac{\partial f_1}{\partial y_i}\left(\frac{dy_i}{d\zeta_3}\right) \qquad \frac{d}{dt}\left(\frac{dy_1}{d\zeta_4}\right)=\sum\frac{\partial f_1}{\partial y_i}\left(\frac{dy_i}{d\zeta_4}\right) \qquad \frac{d}{dt}\left(\frac{dy_1}{d\zeta_5}\right)=\sum\frac{\partial f_1}{\partial y_i}\left(\frac{dy_i}{d\zeta_5}\right)$$

$$\frac{d}{dt}\left(\frac{dy_2}{d\zeta_3}\right)=\sum\frac{\partial f_2}{\partial y_i}\left(\frac{dy_i}{d\zeta_3}\right) \qquad \frac{d}{dt}\left(\frac{dy_2}{d\zeta_4}\right)=\sum\frac{\partial f_2}{\partial y_i}\left(\frac{dy_i}{d\zeta_4}\right) \qquad \frac{d}{dt}\left(\frac{dy_2}{d\zeta_5}\right)=\sum\frac{\partial f_2}{\partial y_i}\left(\frac{dy_i}{d\zeta_5}\right)$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{d}{dt}\left(\frac{dy_5}{d\zeta_3}\right)=\sum\frac{\partial f_5}{\partial y_i}\left(\frac{dy_i}{d\zeta_3}\right) \qquad \frac{d}{dt}\left(\frac{dy_5}{d\zeta_4}\right)=\sum\frac{\partial f_5}{\partial y_i}\left(\frac{dy_i}{d\zeta_4}\right) \qquad \frac{d}{dt}\left(\frac{dy_5}{d\zeta_5}\right)=\sum\frac{\partial f_5}{\partial y_i}\left(\frac{dy_i}{d\zeta_5}\right)$$

This is valid for k=3,4 and 5, and can be re casted in matrix form

$$\frac{d}{dt}\left(\frac{d\underline{y}}{d\zeta_k}\right)=\left[\underline{\underline{J}}\right]\left[\frac{d\underline{y}}{d\zeta_k}\right] \qquad \frac{d}{dt}\left(\underline{q}_k\right)=\left[\underline{\underline{J}}\right]\left[\underline{q}_k\right] \qquad \frac{d\left[\underline{\underline{Q}}\right]}{dt}=\left[\underline{\underline{J}}\right]\left[\underline{\underline{Q}}\right]$$

$$\underline{\underline{Q}} = \begin{bmatrix} \frac{dy_1}{d\zeta_3} & \frac{dy_1}{d\zeta_4} & \frac{dy_1}{d\zeta_5} \\ \frac{dy_2}{d\zeta_3} & \frac{dy_2}{d\zeta_4} & \frac{dy_2}{d\zeta_5} \\ \frac{dy_3}{d\zeta_3} & \frac{dy_3}{d\zeta_4} & \frac{dy_3}{d\zeta_5} \\ \frac{dy_4}{d\zeta_3} & \frac{dy_4}{d\zeta_4} & \frac{dy_4}{d\zeta_5} \\ \frac{dy_5}{d\zeta_3} & \frac{dy_5}{d\zeta_4} & \frac{dy_5}{d\zeta_5} \end{bmatrix}$$

$$\left[\frac{d\underline{\underline{Q}}}{dt}\right] = \left[\underline{\underline{J}}\right]\left[\underline{\underline{Q}}\right]$$

[5x3]   [5x5]x[5x3]

The number of differential equations to be solved with coupled with Newton Raphson are:

$$N+N\times N_B=N(1+N_B)$$

N= Number of equations

N_B= Number of boundary conditions

This algorithm is called Shooting method

Matrix $\underline{\underline{Q}}$, tell us how the dependent variables evolve respect the initial guess

$$\underline{\underline{Q}} = \begin{bmatrix} \frac{dy_1}{d\zeta_3} & \frac{dy_1}{d\zeta_4} & \frac{dy_1}{d\zeta_5} \\ \frac{dy_2}{d\zeta_3} & \frac{dy_2}{d\zeta_4} & \frac{dy_2}{d\zeta_5} \\ \frac{dy_3}{d\zeta_3} & \frac{dy_3}{d\zeta_4} & \frac{dy_3}{d\zeta_5} \\ \frac{dy_4}{d\zeta_3} & \frac{dy_4}{d\zeta_4} & \frac{dy_4}{d\zeta_5} \\ \frac{dy_5}{d\zeta_3} & \frac{dy_5}{d\zeta_4} & \frac{dy_5}{d\zeta_5} \end{bmatrix}$$

The boundary conditions at the arrival to be satisfied will be tracked by the residual, this is:

$$R_2 = y_2(L) - b_2$$
$$R_4 = y_4(L) - b_4$$
$$R_5 = y_5(L) - b_5$$

Only three for this problem

$$\underline{R} = \underline{y} - \underline{b}$$

$$R_2 = R_2(\zeta_3, \zeta_4, \zeta_5)$$
$$R_4 = R_4(\zeta_3, \zeta_4, \zeta_5)$$
$$R_5 = R_5(\zeta_3, \zeta_4, \zeta_5)$$

We need the residuals to be equal to zero, then Newton Raphson will be used to update the value of the departure unknown guess until the residual vanishes.

$$R_2 = R_2(\zeta_3, \zeta_4, \zeta_5)$$
$$R_4 = R_4(\zeta_3, \zeta_4, \zeta_5)$$
$$R_5 = R_5(\zeta_3, \zeta_4, \zeta_5)$$

$$\underline{R}(\underline{\zeta}) = \underline{0}$$

$$\underline{\zeta}^{(n+1)} = \underline{\zeta}^{(n)} - \left[\frac{\partial \underline{R}}{\partial \underline{\zeta}}\right]^{-1} \underline{R}$$

For this problem:

$$\begin{bmatrix} \zeta_3^{(n+1)} \\ \zeta_4^{(n+1)} \\ \zeta_5^{(n+1)} \end{bmatrix} = \begin{bmatrix} \zeta_3^{(n)} \\ \zeta_4^{(n)} \\ \zeta_5^{(n)} \end{bmatrix} - \begin{bmatrix} \frac{\partial R_2}{\partial \zeta_3} & \frac{\partial R_2}{\partial \zeta_4} & \frac{\partial R_2}{\partial \zeta_5} \\ \frac{\partial R_4}{\partial \zeta_3} & \frac{\partial R_4}{\partial \zeta_4} & \frac{\partial R_4}{\partial \zeta_4} \\ \frac{\partial R_5}{\partial \zeta_3} & \frac{\partial R_5}{\partial \zeta_4} & \frac{\partial R_5}{\partial \zeta_5} \end{bmatrix}^{-1} \underline{\begin{bmatrix} R_2^{(n)} \\ R_4^{(n)} \\ R_5^{(n)} \end{bmatrix}}$$

$$\begin{bmatrix} \zeta_3^{(n+1)} \\ \zeta_4^{(n+1)} \\ \zeta_5^{(n+1)} \end{bmatrix} = \begin{bmatrix} \zeta_3^{(n)} \\ \zeta_4^{(n)} \\ \zeta_5^{(n)} \end{bmatrix} - \begin{bmatrix} \frac{\partial R_2}{\partial \zeta_3} & \frac{\partial R_2}{\partial \zeta_4} & \frac{\partial R_2}{\partial \zeta_5} \\ \frac{\partial R_4}{\partial \zeta_3} & \frac{\partial R_4}{\partial \zeta_4} & \frac{\partial R_4}{\partial \zeta_4} \\ \frac{\partial R_5}{\partial \zeta_3} & \frac{\partial R_5}{\partial \zeta_4} & \frac{\partial R_5}{\partial \zeta_5} \end{bmatrix}^{-1} \underline{\begin{bmatrix} R_2^{(n)} \\ R_4^{(n)} \\ R_5^{(n)} \end{bmatrix}}$$

The definition of the residuals is used in the Jacobian of the residual:

$$\underline{R} = \underline{y} - \underline{b} \qquad \frac{\partial \underline{R}}{\partial \underline{\zeta}} = \frac{\partial \underline{y}(L)}{\partial \underline{\zeta}}$$

$$\begin{bmatrix} \zeta_3^{(n+1)} \\ \zeta_4^{(n+1)} \\ \zeta_5^{(n+1)} \end{bmatrix} = \begin{bmatrix} \zeta_3^{(n)} \\ \zeta_4^{(n)} \\ \zeta_5^{(n)} \end{bmatrix} - \begin{bmatrix} \frac{\partial y_2(L)}{\partial \zeta_3} & \frac{\partial y_2(L)}{\partial \zeta_4} & \frac{\partial y_2(L)}{\partial \zeta_5} \\ \frac{\partial y_4(L)}{\partial \zeta_3} & \frac{\partial y_4(L)}{\partial \zeta_4} & \frac{\partial y_4(L)}{\partial \zeta_4} \\ \frac{\partial y_5(L)}{\partial \zeta_3} & \frac{\partial y_5(L)}{\partial \zeta_4} & \frac{\partial y_5(L)}{\partial \zeta_5} \end{bmatrix}^{-1} \underline{\begin{bmatrix} R_2^{(n)} \\ R_4^{(n)} \\ R_5^{(n)} \end{bmatrix}}$$

Then the Jacobian of the residual can be extracted from the matrix Q at the end of the integration (i.e. at the arrival boundary)

$$\underline{\underline{Q}} = \begin{bmatrix} \frac{dy_1}{d\zeta_3} & \frac{dy_1}{d\zeta_4} & \frac{dy_1}{d\zeta_5} \\ \frac{dy_2}{d\zeta_3} & \frac{dy_2}{d\zeta_4} & \frac{dy_2}{d\zeta_5} \\ \frac{dy_3}{d\zeta_3} & \frac{dy_3}{d\zeta_4} & \frac{dy_3}{d\zeta_5} \\ \frac{dy_4}{d\zeta_3} & \frac{dy_4}{d\zeta_4} & \frac{dy_4}{d\zeta_5} \\ \frac{dy_5}{d\zeta_3} & \frac{dy_5}{d\zeta_4} & \frac{dy_5}{d\zeta_5} \end{bmatrix}$$

$$\underline{\underline{Q}} = \begin{bmatrix} \frac{dy_1}{d\zeta_3} & \frac{dy_1}{d\zeta_4} & \frac{dy_1}{d\zeta_5} \\ \frac{dy_2}{d\zeta_3} & \frac{dy_2}{d\zeta_4} & \frac{dy_2}{d\zeta_5} \\ \frac{dy_3}{d\zeta_3} & \frac{dy_3}{d\zeta_4} & \frac{dy_3}{d\zeta_5} \\ \frac{dy_4}{d\zeta_3} & \frac{dy_4}{d\zeta_4} & \frac{dy_4}{d\zeta_5} \\ \frac{dy_5}{d\zeta_3} & \frac{dy_5}{d\zeta_4} & \frac{dy_5}{d\zeta_5} \end{bmatrix}$$

At the departure boundary the matrix Q
If filled with the Kronecker delta δ_{ij}

Initial condition or departure condition for Q matrix

$$\underline{\underline{Q}}(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem No.4

Use the shooting method to solve the Blasius equation

$$\frac{1}{2} f\ f'' + f''' = 0$$

$$f(0) = 0$$
$$f'(0) = 0$$
$$f'(5) = 1$$

Canonical form:	Departure boundary:	Arrival boundary:
$\frac{dy_1}{dx} = y_2$	$y_1(0) = 0$	$y_1(5) = \eta_1$
$\frac{dy_2}{dx} = y_3$	$y_2(0) = 0$	$y_2(5) = 1$
$\frac{dy_3}{dx} = -\frac{1}{2} y_1\ y_3$	$y_3(0) = \zeta_3$	$y_3(5) = \eta_3$

$f(0) = 0$	$y_1(0) = a_1$	$y_1(5) = \eta_1$
$f'(0) = 0$	$y_2(0) = a_2$	$y_2(5) = b_2$
$f'(5) = 1$	$y_3(0) = \zeta_3$	$y_3(5) = \eta_3$

$$\underline{\underline{Q}} = \begin{bmatrix} \dfrac{dy_1}{d\zeta_3} \\ \dfrac{dy_2}{d\zeta_3} \\ \dfrac{dy_3}{d\zeta_3} \end{bmatrix}$$

$$\frac{dy_1}{dx} = f_1 = y_2$$

$$\frac{dy_2}{dx} = f_2 = y_3$$

$$\frac{dy_3}{dx} = f_3 = -\frac{1}{2} y_1 \, y_3$$

$$\underline{\underline{Q}} = \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix}$$

$$\frac{d\left[\underline{\underline{Q}}\right]}{dx} = \left[\underline{\underline{J}}\right]\left[\underline{\underline{Q}}\right]$$

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = y_3$$

$$\frac{dy_3}{dx} = -\frac{1}{2} y_1 \, y_3$$

$$\underline{\underline{J}} = \begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \dfrac{\partial f_1}{\partial y_2} & \dfrac{\partial f_1}{\partial y_3} \\ \dfrac{\partial f_2}{\partial y_1} & \dfrac{\partial f_2}{\partial y_2} & \dfrac{\partial f_2}{\partial y_3} \\ \dfrac{\partial f_3}{\partial y_1} & \dfrac{\partial f_3}{\partial y_2} & \dfrac{\partial f_3}{\partial y_3} \end{bmatrix}$$

$$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y_3/2 & 0 & -y_1/2 \end{bmatrix}$$

$$\frac{d\left[\underline{\underline{Q}}\right]}{dx} = \left[\underline{\underline{J}}\right]\left[\underline{\underline{Q}}\right] \qquad \underline{\underline{Q}} = \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix} \qquad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y_3/2 & 0 & -y_1/2 \end{bmatrix}$$

$$\begin{bmatrix} \dfrac{dq_{13}}{dx} \\ \dfrac{dq_{23}}{dx} \\ \dfrac{dq_{33}}{dx} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -y_3/2 & 0 & -y_1/2 \end{bmatrix} \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix}$$

Extra equations needed to guide the search

$$\frac{dq_{13}}{dx} = q_{23}$$

$$\frac{dq_{23}}{dx} = q_{33}$$

$$\frac{dq_{33}}{dx} = -\frac{1}{2} y_3 q_{13} - \frac{1}{2} y_1 q_{33}$$

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = y_3$$

$$\frac{dy_3}{dx} = -\frac{1}{2} y_1\, y_3$$

$$\underline{\underline{Q}} = \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \end{bmatrix} = \begin{bmatrix} \dfrac{dy_1}{d\zeta_3} \\ \dfrac{dy_2}{d\zeta_3} \\ \dfrac{dy_3}{d\zeta_3} \end{bmatrix} \qquad \underline{\underline{Q}}(0) = \begin{bmatrix} q_{13}(0) \\ q_{23}(0) \\ q_{33}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

How to solve it ?

Canonical form:

Initial conditions, or
boundary conditions at the departure

Residual

$$\frac{dy_1}{dx} = y_2 \qquad y_1(0) = 0$$

$$\frac{dy_2}{dx} = y_3 \qquad y_2(0) = 0 \qquad R_2 = y_2(5) - 1$$

$$\frac{dy_3}{dx} = -\frac{1}{2} y_1 \, y_3 \qquad y_3(0) = \zeta_3 = \zeta$$

$$\frac{dy_4}{dx} = y_5 \qquad y_4(0) = 0$$

$$\frac{dy_5}{dx} = y_6 \qquad y_5(0) = 0$$

$$\frac{dy_6}{dx} = -\frac{1}{2} y_3 y_4 - \frac{1}{2} y_1 y_6 \qquad y_6(0) = 1$$

There is only one departure boundary to guess, and one residual to equate to zero

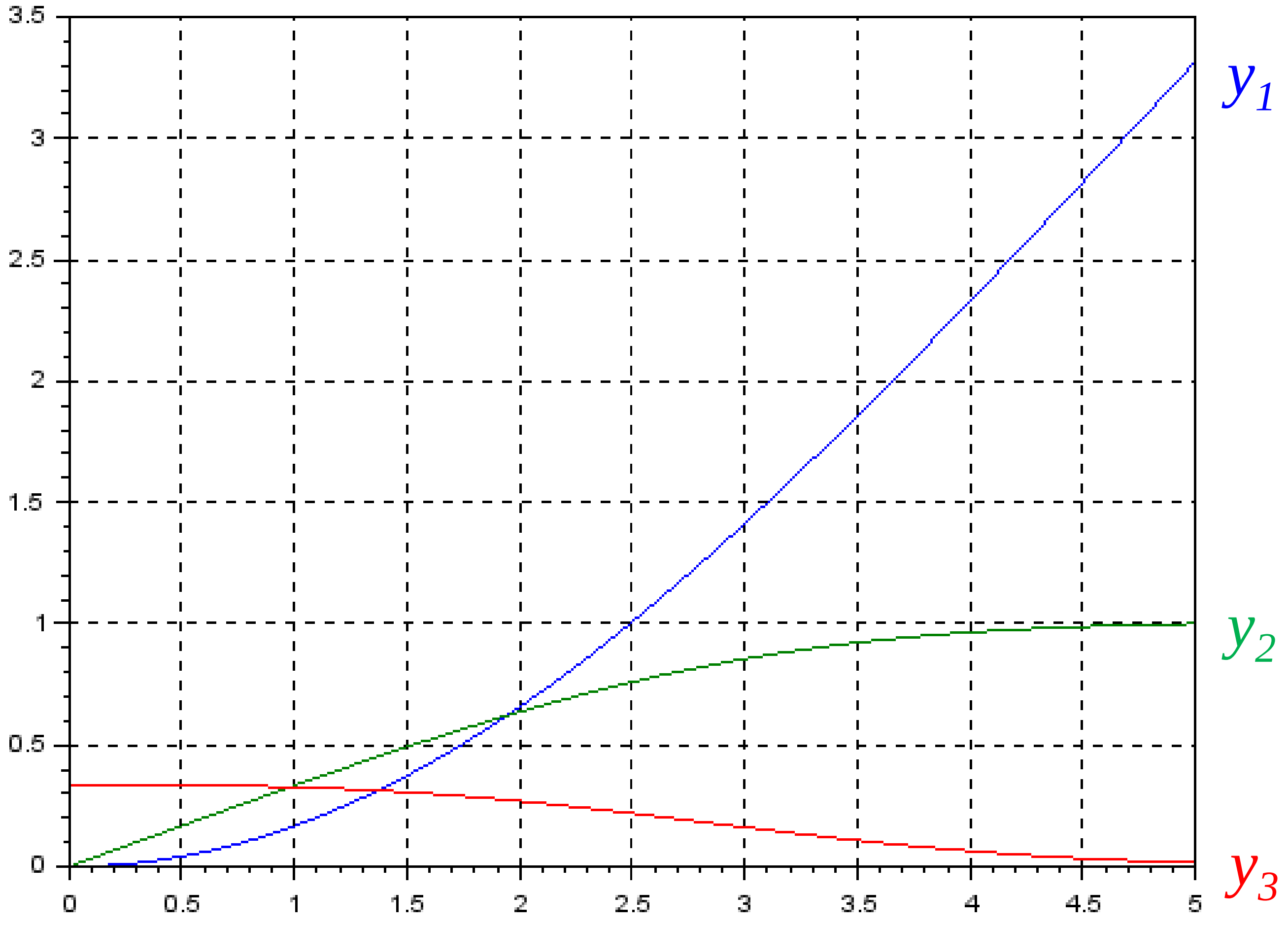
$$R_2 = y_2(5) - 1 = R_2(\zeta_3) \qquad R = R_2 \qquad \zeta = \zeta_3$$

$$\underline{\zeta}^{(n+1)} = \underline{\zeta}^{(n)} - \left[\frac{\partial \underline{R}}{\partial \underline{\zeta}}\right]^{-1} \underline{R}$$

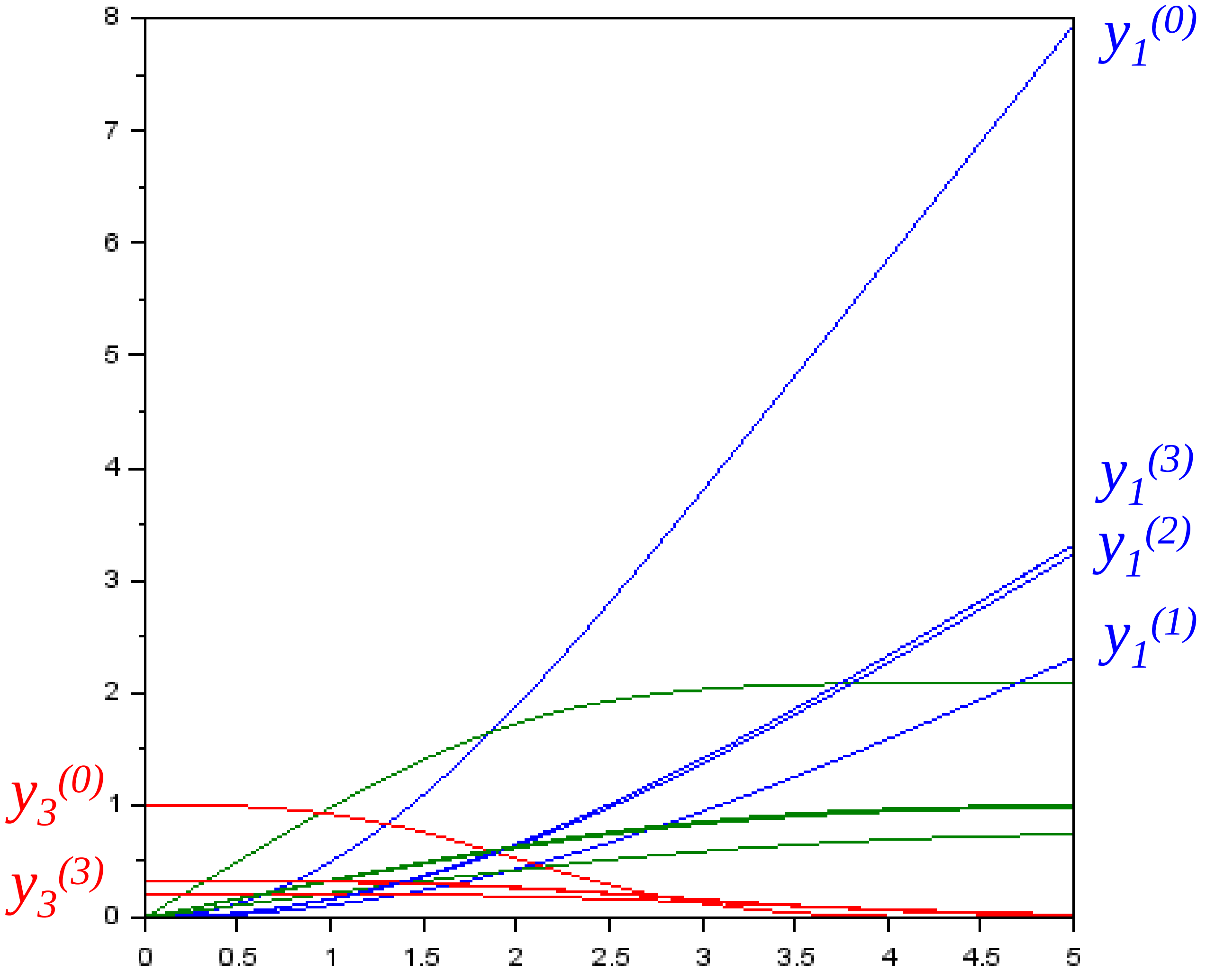
$$\zeta^{(n+1)} = \zeta^{(n)} - \frac{R}{\left(\frac{dR}{d\zeta}\right)} = \zeta^{(n)} - \frac{y_2(5) - 1}{\left(\frac{dy_2}{d\zeta}\right)_{x=5}} \qquad \zeta^{(n+1)} = \zeta^{(n)} - \frac{y_2(5) - 1}{y_5(5)}$$

```
function [Y, res]=FBlasius(id)
 zetha=1
 y0=[0,0,zetha,0,0,1]‘
 t0=0;
 tspan=linspace(0,5)';
 NI=5 //Number of iterations (replace by while loop)
for i=1:NI
        y0=[0,0,zetha,0,0,1]'
        [Y]=ode(y0,t0,tspan,BlasiusDE); plot(tspan,Y(1,:)',tspan,Y(2,:)',tspan,Y(3,:)')
        res=Y(2,$)-1
        zetha=zetha-res/Y(5,$)
end
halt('press any key')
Clf
plot(tspan,Y(1,:)',tspan,Y(2,:)',tspan,Y(3,:)')
 xgrid
 endfunction
 function [yprime]=BlasiusDE(t, y)
// Original problem // f*f''/2+f'''=0 // f(0)=f'(0)=0 and f'(eta=%inf)=0 yprime(1)=y(2)
yprime(2)=y(3)
yprime(3)=-y(1)*y(3)/2
yprime(4)=y(5)
yprime(5)=y(6)
yprime(6)=-y(3)*y(4)/2-y(1)*y(6)/2
endfunction
```

Solution



Iteration process



Explain how this code works

```
function wdot=wsyste(t, w)
 wdot(1) = w(2)
wdot(2) = w(3)
 wdot(3)=-0.5*w(1)*w(3)
endfunction
function [wsol, t]=deal(id)
s = 1
w0 = [0;0;s]
t0 = 0
 t = linspace(0,5)
winit = ode(w0,t0,t,wsyste); //plot(t,winit)
 //halt
 deff('res=fct_3(s)',['w0 = [0;0;s];','w =ode(w0,t0,t,wsyste);','res=w(2,$)-1'])
s = 1
 ssol =fsolve(s,fct_3) // compute solution
 w0 = [0;0;ssol]
 t0 = 0
 t = linspace(0,5)
 wsol = ode(w0,t0,t,wsyste);
 plot(t,wsol)
p = get("hdl"); p.children.thickness = 3
endfunction
```

Problem No.5

Solve the PDE, using techniques to solve ODEs

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial \chi^2} - \beta_1 \left(y - y_f \right) - \beta_2 \left(y^4 - y_w^4 \right)$$

Initial conditions:

$$y(\tau = 0, \chi) = 1$$

Boundary conditions:

$$y(\tau, \chi = 1) = 1$$

$$\left. \frac{\partial y}{\partial \chi} \right|_{\chi = 1} = -\beta_3 \left(y - y_f \right) - \beta_4 \left(y^4 - y_w^4 \right)$$

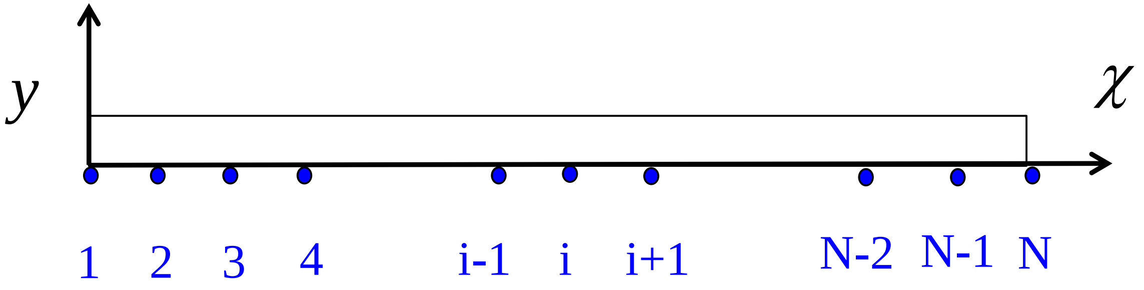
Note: β_1, β_2, β_3, β_4, y_f and y_w are constant parameters

The PDE will be re casted in semi-discrete from.
Discretization will be done in χ domain

Using central finite differences for the internal points

$$\frac{\partial y}{\partial \tau}=\frac{\partial^2 y}{\partial \chi^2}-\beta_1\left(y-y_f\right)-\beta_2\left(y^4-y_w^4\right)$$

$$\frac{dy_i}{d\tau}=\frac{y_{i+1}-2y_i+y_{i-1}}{\left(\Delta\chi\right)^2}-\beta_1\left(y_i-y_f\right)-\beta_2\left(y_i^{\,4}-y_w^4\right)$$



Initial conditions:

$$y\left(\tau=0,\chi\right)=1$$

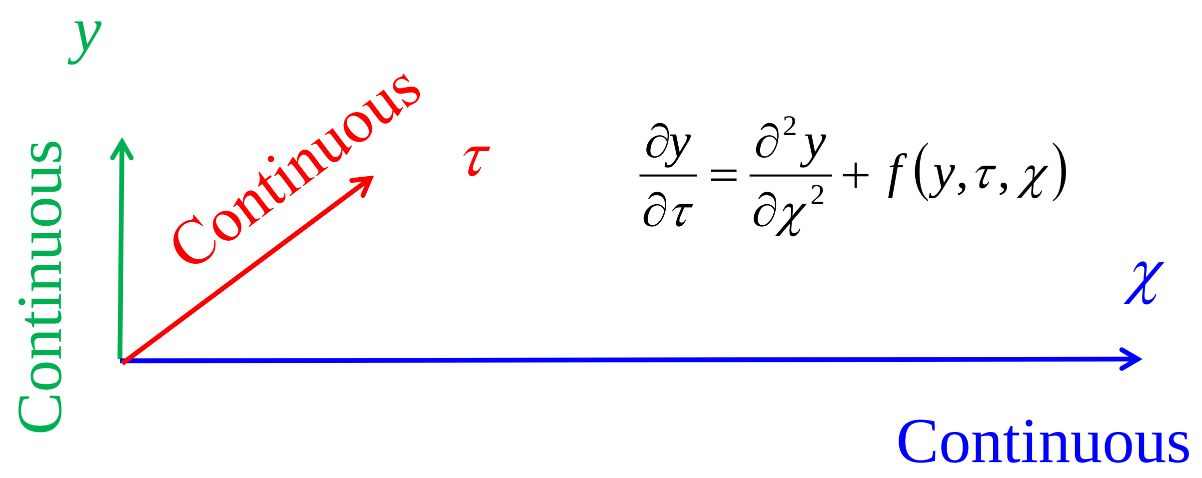
Boundary conditions:

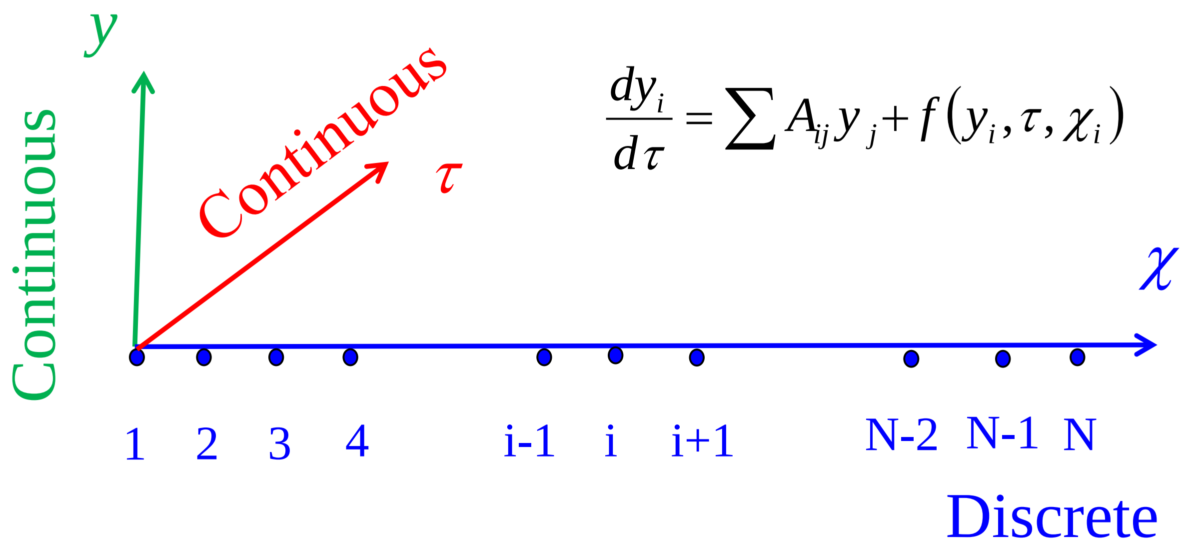
$$y\left(\tau,\chi=1\right)=1$$

$$\left.\frac{\partial y}{\partial \chi}\right|_{\chi=1}=-\beta_3\left(y-y_f\right)-\beta_4\left(y^4-y_w^4\right)$$

Look out: Discretizadion means that the equation are written for a particular point in space then the PDE is transformed into a set of ODEs

Semi discrete approach





Discretization will be done in χ domain not only for the PDE, but for the BC as well

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial \chi^2} - \beta_1\left(y - y_f\right) - \beta_2\left(y^4 - y_w^4\right)$$

Using central finite differences for the internal points

$$\frac{dy_i}{d\tau} = \frac{y_{i+1} - 2y_i + y_{i-1}}{\left(\Delta\chi\right)^2} - \beta_1\left(y_i - y_f\right) - \beta_2\left(y_i^{\,4} - y_w^4\right)$$

Initial conditions:

$$y\left(\tau = 0, \chi\right) = 1$$

$$y_i = 1$$

Boundary conditions:

$$y\left(\tau, \chi = 1\right) = 1$$

$$\frac{dy_1}{d\tau} = 0$$

$$\left.\frac{\partial y}{\partial \chi}\right|_{\chi=1} = -\beta_3\left(y - y_f\right) - \beta_4\left(y^4 - y_w^4\right)$$

$$\frac{y_N - y_{N-1}}{\Delta\chi} = -\beta_3\left(y_N - y_f\right) - \beta_4\left(y_N^{\,4} - y_w^4\right)$$

Note: y_f and y_w are constant parameters not discrete points

Discretization will be done in χ domain not only for the PDE, but for the BC as well

The arrival boundary condition is an algebraic equation, this will be differentiated respect to time to transform it into a differential equation:

$$\left.\frac{\partial y}{\partial \chi}\right|_{\chi=1} = -\beta_3\left(y - y_f\right) - \beta_4\left(y^4 - y_w^4\right)$$

$$\frac{y_N - y_{N-1}}{\Delta\chi} = -\beta_3\left(y_N - y_f\right) - \beta_4\left({y_N}^4 - y_w^4\right)$$

$$\frac{1}{\Delta\chi}\left(\frac{dy_N}{d\tau} - \frac{dy_{N-1}}{d\tau}\right) = -\beta_3\frac{dy_N}{d\tau} - 4\beta_4 {y_N}^3\frac{dy_N}{d\tau}$$

$$\frac{dy_N}{d\tau} = \frac{dy_{N-1}}{d\tau}\left(\frac{1}{1+\beta_3\Delta\chi + 4\beta_4 {y_N}^3\Delta\chi}\right)$$

Note: y_f and y_w are constant parameters not discrete points

The PDE with the BC discretizated in χ domain written as ODEs have the form:

$$\frac{dy_1}{d\tau} = 0$$

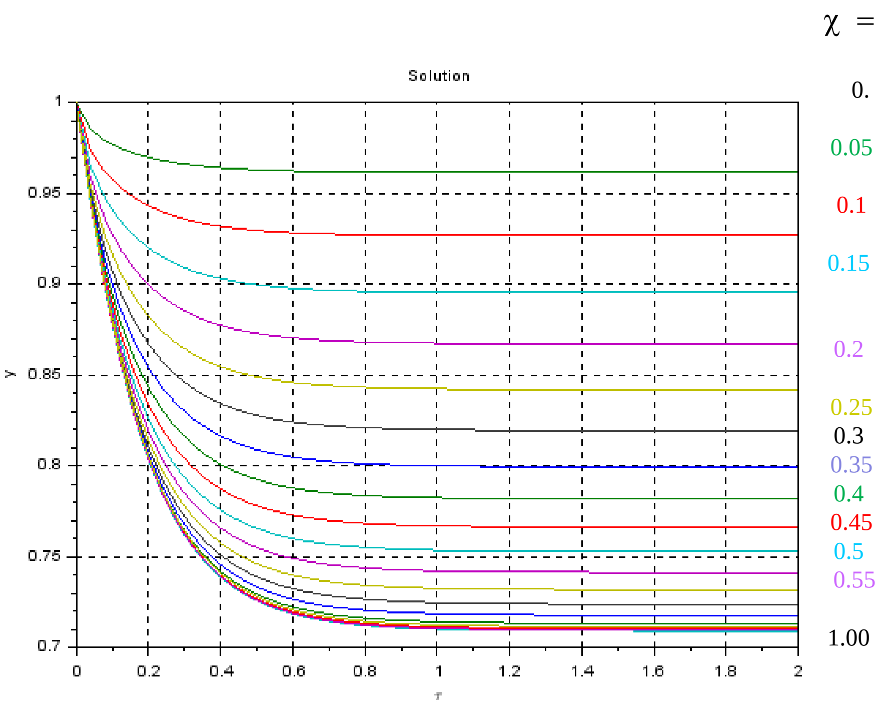
$$\frac{dy_i}{d\tau} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta\chi)^2} - \beta_1 (y_i - y_f) - \beta_2 (y_i^{\ 4} - y_w^4)$$

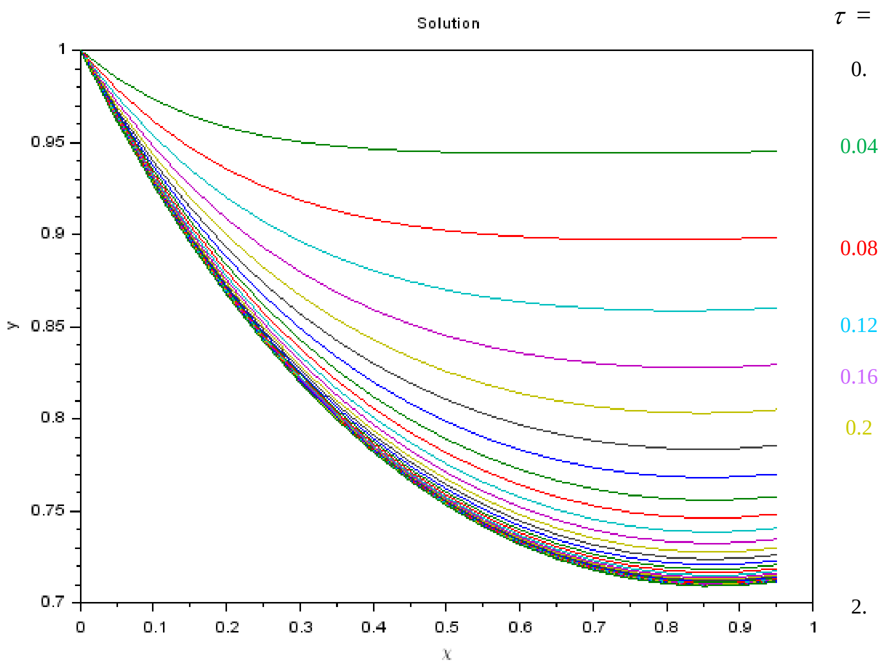
For i=2,3,4,…,N-2,N-1

$$\frac{dy_N}{d\tau} = \frac{dy_{N-1}}{d\tau} \left(\frac{1}{1 + \beta_3 \Delta\chi + 4\beta_4 y_N^{\ 3} \Delta\chi} \right)$$

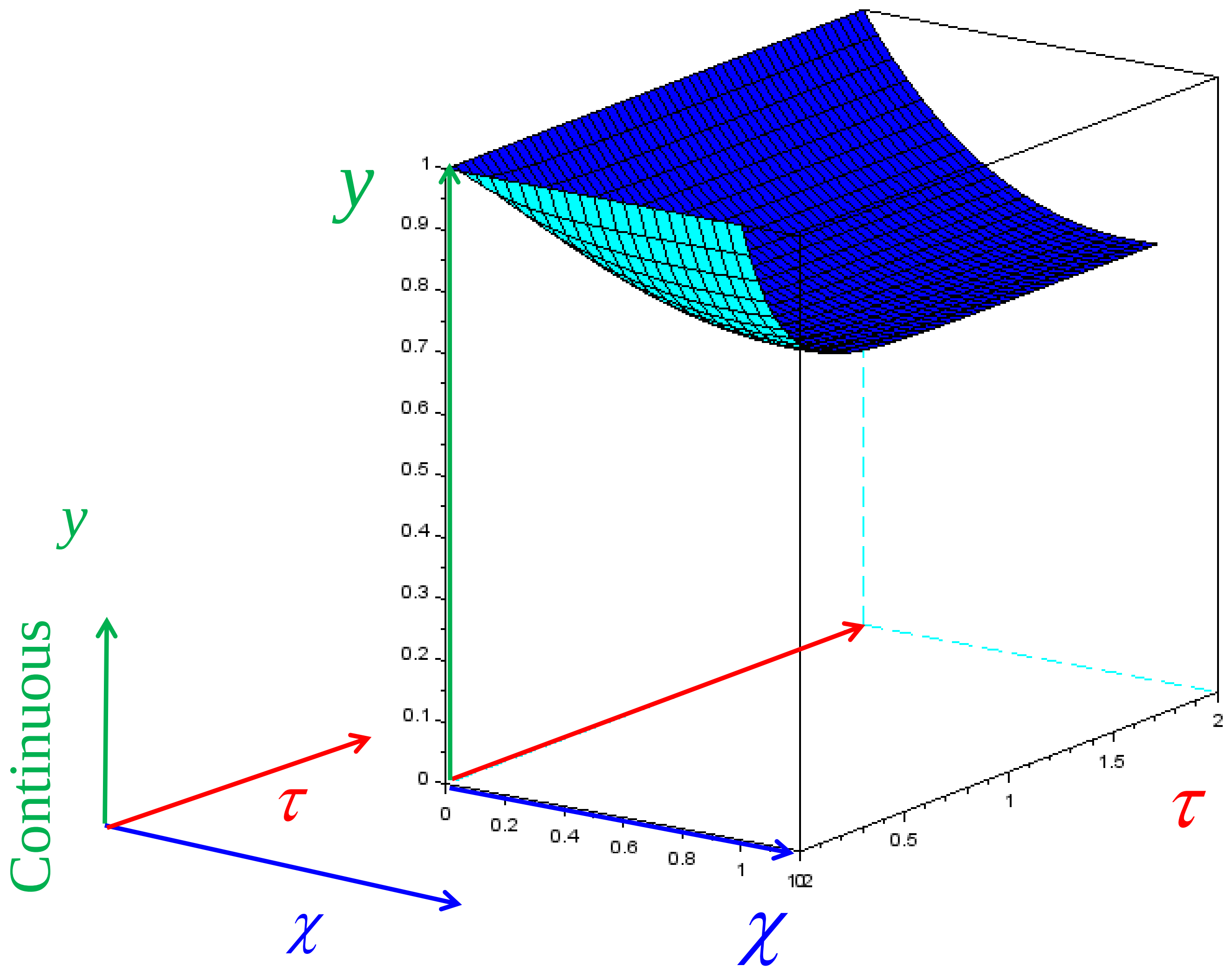
Note: y_f and y_w are constant parameters not discrete points

```
function [Y, t, chi]=Example1()
[par,NP]=Parameters()
 beta1=par(1);beta2=par(2);beta3=par(3);beta4=par(4);yf=par(5); yw=par(6);
 t0=0;
tspan=linspace(0,2,51)‘
 for i=1:NP
 y0(i)=1;
end
[Y]=ode(y0,t0,tspan,ODEexamp1)
 plot(tspan,Y')
 t=tspan;
chi(1)=0
for i=2:NP
 chi(i)=1/NP+chi(i-1);
 end
Xgrid
 xtitle('Solution','$ \tau$','y')
 halt('press any key')
clf // plot(chi,Y(:,$))
plot(chi,Y)
xtitle('Solution','$ \chi$','y')
endfunction
function [yprime]=ODEexamp1(t, y)
[par,NP]=Parameters()
beta1=par(1);beta2=par(2);beta3=par(3);beta4=par(4);yf=par(5); yw=par(6)
deltax=1/(NP-1);
yprime(1)=0.0;
 for i=2:NP-1
 yprime(i)=(y(i+1)-2*y(i)+y(i-1))/deltax^2
yprime(i)=yprime(i)-beta1*(y(i)-yf)-beta2*(y(i)^4-yw^4)
 end
yprime(NP)=yprime(NP-1)/(1+beta3*deltax+4*beta4*deltax*y(NP)^3)
Endfunction
 function [par, NP]=Parameters()
NP=20; beta1=10/4;beta2=10/36;beta3=10/100;beta4=10/900; yf=0.5;yw=0.5
par=[beta1,beta2,beta3,beta4,yf,yw]
endfunction
```


Solution
y
τ
χ =
0.
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
1.00


Solution
y
χ
τ =
0.
0.04
0.08
0.12
0.16
0.2
2.

Solution



plot3d(chi,t,Y)

The problem was originally normalized in χ domain

Hybrid discretization:
Physics of the problem, and finite differences

$$\frac{dy_1}{d\tau} = 0$$

$$\frac{dy_i}{d\tau} = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta\chi)^2} - \beta_1 (y_i - y_f) - \beta_2 (y_i{}^4 - y_w^4)$$

$$\frac{\Delta\chi}{2}\left(\frac{dy_N}{d\tau}\right) = -\left(\frac{y_N - y_{N-1}}{\Delta\chi}\right) - \beta_1\left(\frac{\Delta\chi}{2} + \frac{1}{2\gamma}\right)(y_N - y_f) - \beta_2\left(\frac{\Delta\chi}{2} + \frac{1}{2\gamma}\right)(y_N{}^4 - y_w^4)$$

If $\Delta\chi$ is small enough , both algorithms coincide

Note: β_1=10/4 , β_2=10/36 β_3=10/100 β_4=10/900 y_f = ½ , y_w =1/2 and γ=12

Jacoby Polynomials

$$\int_0^1 x^{\beta}(1-x)^{\alpha} J_j^{(\alpha,\beta)}(x) J_N^{(\alpha,\beta)}(x)dx = 0$$

$$j=0,1,2,3,\ldots,N\text{-}1$$

$$J_j^{(\alpha,\beta)}(x) = \sum_{i=0}^{N}(-1)^{N-i}\gamma_{N,i}x^i = \gamma_{N,0} + \sum_{i=1}^{N}(-1)^{N-i}\gamma_{N,i}x^i$$

$$\gamma_{N,i} = \frac{N!}{i!(N-i)!}\frac{\Gamma(N+i+\alpha+\beta+1)\,\Gamma(\beta+1)}{\Gamma(N+\alpha+\beta+1)\,\Gamma(i+\beta+1)} \qquad \Gamma(t) = \int_0^{\infty} x^{t-1}e^{-x}dx$$

$$dV = x^{a-1}dx$$

$$\int_0^1 \left(1-x^2\right)P_i\left(x^2\right)P_n\left(x^2\right)x^{a-1}dx = 0$$

$$a = \begin{cases} 1 & \text{Slabs} \\ 2 & \text{Cylinders} \\ 3 & \text{Spheres} \end{cases}$$

$$\int_0^1 \left(1-x^2\right)P_i\left(x^2\right)P_n\left(x^2\right)x^{2\left(\frac{a}{2}-1\right)}dx^2 = 0$$

$$z = x^2$$

$$\alpha = 1; \quad \beta = \frac{a}{2} - 1$$

$$\int_0^1 (1-z)P_i(z)P_n(z)z^{\left(\frac{a}{2}-1\right)}dz = 0$$

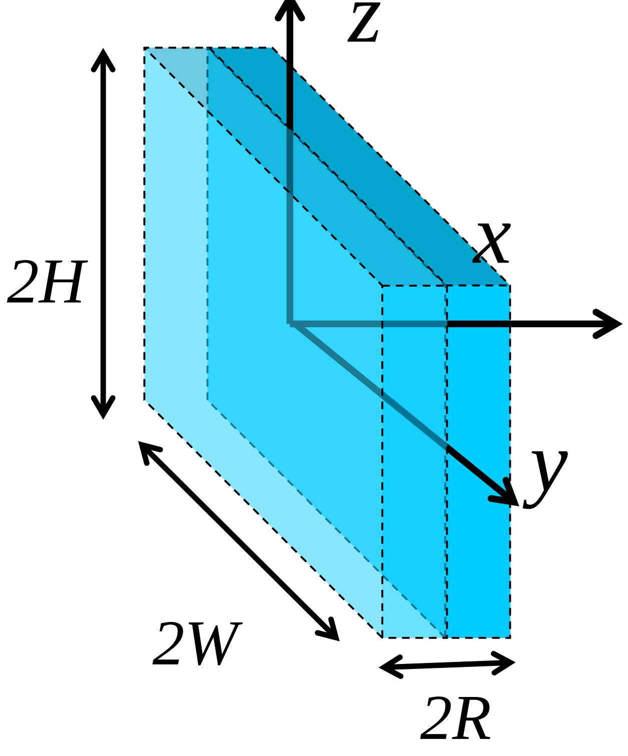
$$\int_0^1 z^{\beta}(1-z)^{\alpha} J_i(z)J_n(z)dz = 0$$

$$a = \begin{cases} 1 & \text{Slabs} \\ 2 & \text{Cylinders} \\ 3 & \text{Spheres} \end{cases}$$

$$dV = WLdx = R^3\left[\frac{WL}{R^2}\right]\left(\chi^{1-1}d\chi\right)$$

$$dV = 2\pi Lrdr = R^3\left[\frac{2\pi L}{R}\right]\left(\chi^{2-1}d\chi\right)$$

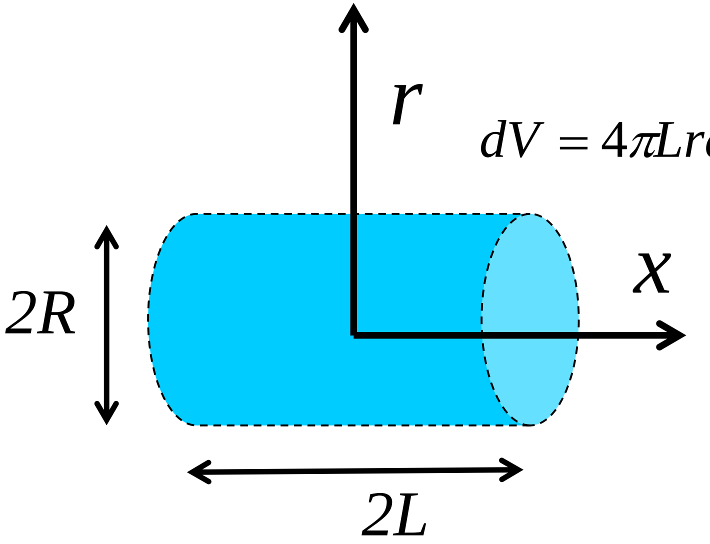
$$dV = 4\pi r^2dr = R^3[4\pi]\left(\chi^{3-1}d\chi\right)$$



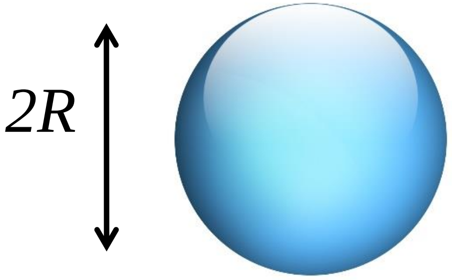
$$dV = 4WHdx = R^3\left[\frac{4WH}{R^2}\right]\left(\chi^{1-1}d\chi\right) \qquad -1<\chi<1$$

$$0<\chi<1$$

$$dV = 4\pi Lrdr = R^3\left[\frac{4\pi L}{R}\right]\left(\chi^{2-1}d\chi\right)$$



$$a = \begin{cases} 1 & \text{Slabs} \\ 2 & \text{Cylinders} \\ 3 & \text{Spheres} \end{cases}$$



$$0<\chi<1$$

$$dV = 4\pi r^2dr = R^3[4\pi]\left(\chi^{3-1}d\chi\right)$$

For 1-D

Cartesian

$$\nabla^2 C = \frac{\partial}{\partial x}\left(\frac{\partial C}{\partial x}\right)$$

Cylindrical

$$\nabla^2 C = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial C}{\partial r}\right)$$

Spherical

$$\nabla^2 C = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial C}{\partial r}\right)$$

General

$$\nabla^2 C = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial C}{\partial r}\right)$$

$$n = \begin{cases} 1, \text{ Planar} \\ 2, \text{ Cylindrical} \\ 3, \text{ Spherical} \end{cases}$$

$$\nabla^2 y = \frac{1}{\chi^{n-1}} \frac{\partial}{\partial \chi}\left(\chi^{n-1} \frac{\partial y}{\partial \chi} \right)$$

$$y(\chi) = y(1) + (1 - \chi^2) \sum_{i=1}^{N} a_i P_{i-1}(\chi^2)$$

$$y(\chi) = \sum_{i=1}^{N+1} b_i P_{i-1}(\chi^2) = \sum_{i=1}^{N+1} d_i \chi^{2i-2}$$

$$y(\chi_j) = \sum_{i=1}^{N+1} d_i \chi_j^{\;2i-2} = \sum_{i=1}^{N+1} \chi_j^{\;2i-2} d_i \qquad Q_{ji} = \chi_j^{2i-2}$$

$$\begin{bmatrix} y(\chi_1) \\ y(\chi_2) \\ y(\chi_3) \\ \vdots \\ y(\chi_{N+1}) \end{bmatrix} = \begin{bmatrix} \chi_1^{\;0} & \chi_1^{\;2} & \chi_1^{\;4} & \cdots & \chi_1^{\;2N} \\ \chi_2^{\;0} & \chi_2^{\;2} & \chi_2^{\;4} & \cdots & \chi_2^{\;2N} \\ \chi_3^{\;0} & \chi_3^{\;2} & \chi_3^{\;4} & \cdots & \chi_3^{\;2N} \\ \vdots & \vdots & \vdots & & \vdots \\ \chi_{N+1}^{\;\;0} & \chi_{N+1}^{\;\;2} & \chi_{N+1}^{\;\;4} & \cdots & \chi_{N+1}^{\;\;2N} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N+1} \end{bmatrix} \qquad \underline{y} = \underline{\underline{Q}}\ \underline{d}$$

$$\nabla^2 y = \frac{1}{\chi^{n-1}} \frac{\partial}{\partial \chi}\left(\chi^{n-1} \frac{\partial y}{\partial \chi} \right)$$

$$Q_{ji} = \chi_j^{2i-2} \qquad C_{ji} = (2i-2)\chi_j^{2i-3} \qquad D_{ji} = \nabla^2\left(\chi^{2i-2}\right)\Big|_{x_j}$$

$$D_{ji} = (2i-2)(2i+a-4)\chi_j^{2i-4}$$

$$\underline{y} = \underline{\underline{Q}}\ \underline{d} \qquad \frac{d\underline{y}}{d\chi} = \underline{\underline{C}}\ \underline{d} \qquad \nabla^2 \underline{y} = \underline{\underline{D}}\ \underline{d}$$

$$\underline{d} = \underline{\underline{Q}}^{-1}\underline{y} \qquad \frac{d\underline{y}}{d\chi} = \underline{\underline{C}}\ \underline{\underline{Q}}^{-1}\underline{y} = \underline{\underline{A}}\ \underline{y} \qquad \nabla^2 \underline{y} = \underline{\underline{D}}\ \underline{\underline{Q}}^{-1}\underline{y} = \underline{\underline{B}}\ \underline{y}$$

$$y(\chi_j) = \sum_{i=1}^{N+1} d_i \chi_j^{\;2i-2} = \sum_{i=1}^{N+1} \chi_j^{\;2i-2} d_i$$

If the function y is needed at the center or origin

$$y(0) = \sum_{i=1}^{N+1} d_i \chi^{2i-2} = d_1$$

$$\begin{bmatrix} \chi_1^{\;0} & \chi_1^{\;2} & \chi_1^{\;4} & \cdots & \chi_1^{\;2N} \\ \chi_2^{\;0} & \chi_2^{\;2} & \chi_2^{\;4} & \cdots & \chi_2^{\;2N} \\ \chi_3^{\;0} & \chi_3^{\;2} & \chi_3^{\;4} & \cdots & \chi_3^{\;2N} \\ \vdots & \vdots & \vdots & & \vdots \\ \chi_{N+1}^{\;0} & \chi_{N+1}^{\;2} & \chi_{N+1}^{\;4} & \cdots & \chi_{N+1}^{\;2N} \end{bmatrix}^{-1} \begin{bmatrix} y(\chi_1) \\ y(\chi_2) \\ y(\chi_3) \\ \vdots \\ y(\chi_{N+1}) \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_{N+1} \end{bmatrix} \qquad \underline{d} = \underline{\underline{Q}}^{-1} \underline{y}$$

If integration is needed

$$\int_0^1 G(x)\, x^{a-1} dx = \sum_{i=1}^{N+1} w_i G(x_i)$$

$$f_i = \frac{1}{2i-2+a} \qquad \underline{w} = \left\lfloor \left[\underline{f} \right]^T \left\lfloor \underline{\underline{Q}}^{-1} \right\rfloor \right\rfloor^T$$

Average properties

$$\langle G \rangle = \frac{\int_0^1 G(x)\, x^{a-1} dx}{\int_0^1 x^{a-1} dx} = \frac{\sum_{i=1}^{N+1} w_i G(x_i)}{\sum_{i=1}^{N+1} w_i}$$

Problem No.6

The concentration of oxygen inside a spherical catalyst changes respect to the following equation

$$\frac{\partial C}{\partial \tau} = \nabla^2 C - \alpha\, C^n$$

The concentration changes with respect to the radius and the time

$$C = C(\tau, \chi)$$

χ is the dimensionless or normalized radius $\chi = r/R$

The boundary conditions and initial conditions are:

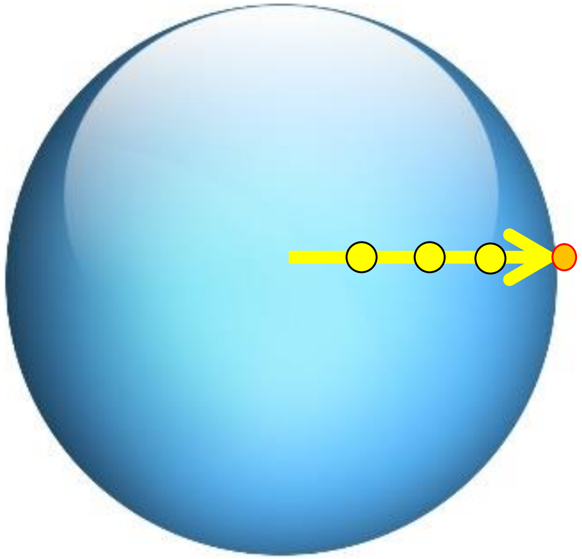
$C(0, \chi) = 0$	Initial concentration inside the catalyst is zero	IC
$\left.\frac{\partial C}{\partial \chi}\right\|_{\chi=0} = 0$	Symmetry condition, no flux or oxygen generation/consumption at the center	BC
$-\left.\frac{\partial C}{\partial \chi}\right\|_{\chi=1} = \beta(C - C_\infty)$	Diffusion of oxygen at the interface equals mass transfer by convection	BC

The spherical catalyst will be discretizated, will include internal points (1,2,3…,N), and a node at the surface as well (N+1)

Differential equation (PDE)

$$\frac{\partial C}{\partial \tau} = \nabla^2 C - \alpha\, C^m$$

$$-\left.\frac{\partial C}{\partial \chi}\right|_{\chi=1} = \beta\left(C - C_\infty\right)$$



Discrete form (PDE → ODE)

$i=1,2,3,\ldots N$

$$\frac{dC_i}{d\tau} = \sum_{j=1}^{N+1} B_{i,j} C_j - \alpha\, C_i^{\,m}$$

$$-\left(\sum_{j=1}^{N+1} A_{N+1,j} C_j\right) = \beta\left(C_{N+1} - C_\infty\right)$$

Note: N is the number of internal points, the point N+1 is located at the surface (i.e. χ=1), and the center is not included in the collocation points, and no need to include because the orthogonal polynomials selected are symmetric polynomials (even) which satisfy the boundary condition at the center.

1. Calculate the concentration of oxygen inside the pellet as function of space and time
2. Calculate the concentration of oxygen at the center of the spherical pellet
3. Calculate the concentration of oxygen at the Surface of the sphere.
4. Calculate the average concentration of oxygen inside the catalyst
5. Calculate the rate of oxygen consumption inside the sphere

Note: Do calculations using $\alpha=\beta=n=1$.
Notice that time has dimensionless form and the radius is normalized.

```
function [YOA, t, chi]=PelletPDE()
 t0=0; tspan=linspace(0,2,100)'
 [par]=Parameters()
 NP=par(1); bet=par(2);alf=par(3);m=par(5);C_inf=par(6);
for i=1:NP
 y0(i)=0.0;
end
[Y]=ode(y0,t0,tspan,diffeqM)
 //[Y]=ode(y0,t0,tspan,ODEexamp1)
 plot(tspan,Y')
t=tspan; al=1.0;be=0.5;ag=3.0;
[xc,A,B,w,Q]=FJcbPlyx(al,be,ag,NP)
 [nrt,nct]=size(t) YT=Y;
// The boundary condition can be applied to calcualte the
//Concentration at the surface (i.e. dependent variable)
// YT is the solution including the point at the surface
for k=1:nrt
 sum3=0;
for i=1:NP
sum3=sum3+A(NP+1,i)*Y(i,k);
 end
YT(NP+1,k)=(bet*C_inf-sum3)/(bet+A(NP+1,NP+1));
end
// Concentration at the center, d are the coefficients of the
 // polynomial for the solution
 d=inv(Q)*YT
 // The solution will be extended to include the concentration
 // at the center
// Y is the solution of the internal points
// YT is the solution including the Surface
 // YAO is the solution , including surface and center
// chi is the dimensionless position , end includes center and surface
chi=[0,xc']';
YOA(1,:)=d(1,:); YOA=[YOA;YT]
 halt
 disp('type any key')
```

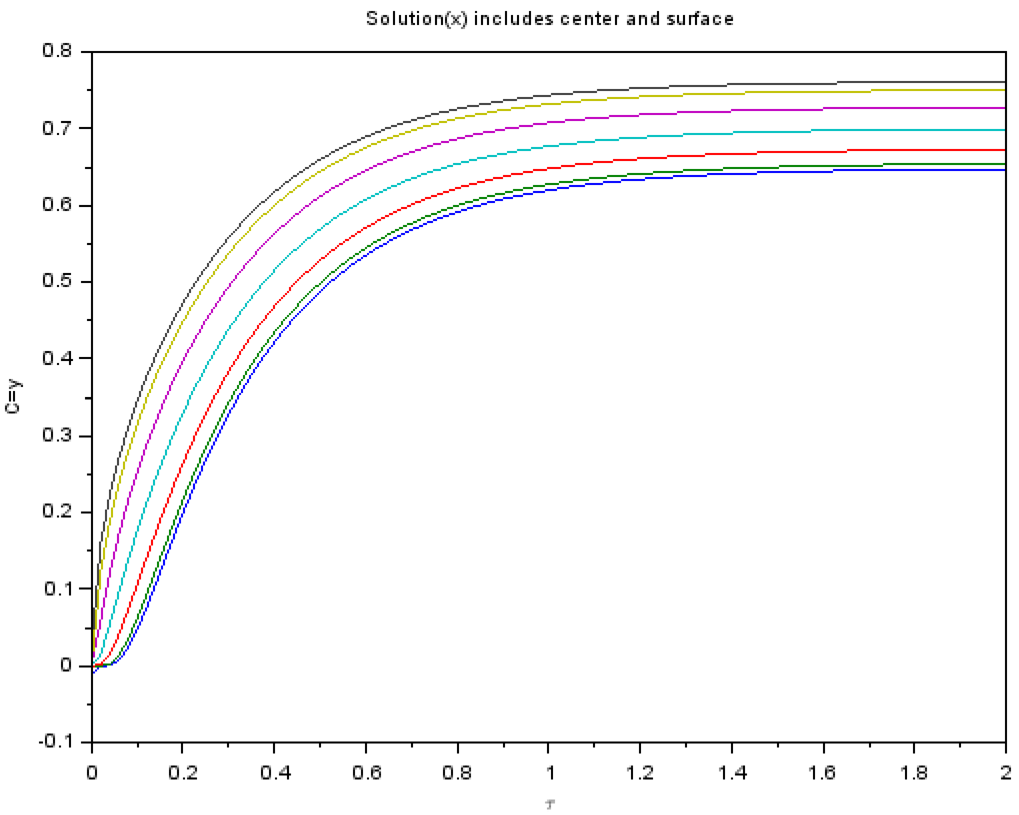
```
plot(tspan,YOA')
xtitle('Solution(x) includes center and surface','$ \tau$','C=y')
 halt
 disp('type any key')
plot(chi,YOA)
 xtitle('Solution(tau) different times','$ \chi$','C=y')
halt
disp('type any key') //plot(tspan,d(1,:))
plot3d(chi,t,YOA)
xtitle('Solution','$ \chi$','$ \tau$','C=y')
for k=1:nrt
sum5=0.0;
for i=1:NP+1
sum5=sum5+w(i)*YT(i,k);
end sum5=sum5/sum(w);
Yave(k)=sum5;
end
halt
disp('any other key, last one')
 plot(t,Yave,'r--o')
 plot(tspan,YOA')
// plot3d(xc,t,YT)
 endfunction
```

```
function [yprime]=diffeqM(t, y)
 [par]=Parameters()
NE=par(1);
for i=1:NE
CP(i)=y(i);
end
 al=1.0;be=0.5;ag=3.0; bet=par(2);alf=par(3);m=par(5);C_inf=par(6);
[xc,A,B,w,Q]=FJcbPlyx(al,be,ag,NE)
su1=0.0
for i=1:NE
su1=su1+A(NE+1,i)*CP(i)
 end su1=su1-bet*C_inf;
CP(NE+1)=-su1/(bet+A(NE+1,NE+1));
for i=1:NE
su2=0.0
for k=1:NE+1
su2=su2+B(i,k)*CP(k)
 end
zprime(i)=su2-alf*CP(i)^m;
end
//Look out , this is a bug of the SciLab not of my code
yprime=real(zprime);
Endfunction
function [par]=Parameters()
N=5; beta1=1.0;alpha=1;m=1;C_inf=1;Nx=4;
 par=[N,beta1,alpha,Nx,m,C_inf];
endfunction
```

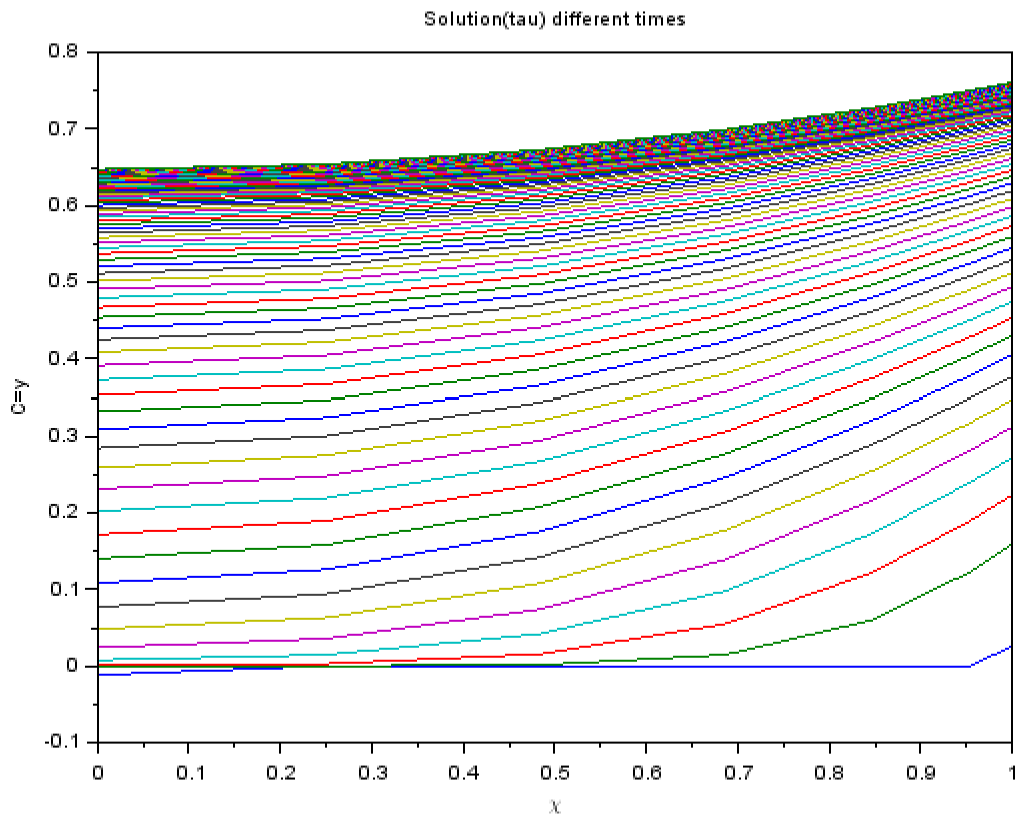
```
// Function to calculate the discretization coefficients using
// Orthogonal Collocation approach
// By: Dr. Jose Luis Lopez Salinas
 // Updated version: 01-05-2015
// Mechanical Engineering Department (ITESM, Campus Monterrey)
// Warning: This is a general function so beta and ag are related
// beta=(ag-2)/2
 // Calculates the coefficients of Jacobi Polynomials
// And also the roots of the polynomial of order N
// Integral (z^beta*(1-z)^alpha)*Ji*Jn*dz=0 , 0<z<1
 // i=0,1,2,...,n-1
 // Ji(z)=Jacobi Polynomial of order "i"
// r are the roots of the polynomial
// J(z)=a0+ap(1)*z+ap(2)*z^2+ap(3)*z^3+...+ap(N)*z^N
 // J(z)=c(1)*z^N+c(2)*z^(N-1)+c(3)*z^(N-2)+...+c(N)*z+c(N+1)
// For the diffusion problem // ag=1,2 or 3 depending of the geometry
 // 1 is for planar, 2 is for cylindrical and 3 for spherical geometry
// alpha=1, and beta=(ag-2)/2 // Then the polynomial has the form:
 // J(x^2)=a0+ap(1)*x^2+ap(2)*x^4+ap(3)*x^6+...+ap(N)*x^(2*N)
// Integral (1-x^2)*Pi(x^2)*Pn(x^2)*x^(ag-1)*dx, 0<x<1
// In this case the differential of volume is
// dV=x^(ag-1)*dx
//=========================================================
// Differential equation of the form
// dC/dt - E Div (Grad(C))=R(C)
 // dC/dt - E Lap (C)=R(C)
// Boundary condition of the form
 // Grad(C)=-Bi(C-Co) at r/R=1
 // Grad(C)=0 at r/R=0, this is already satisfied with symmetrical constrain
 // Div = Divergence operator
// Grad = Gradient
// Lap=Laplacian = Lap()=Div(Grad())
// Sum=Summation
```

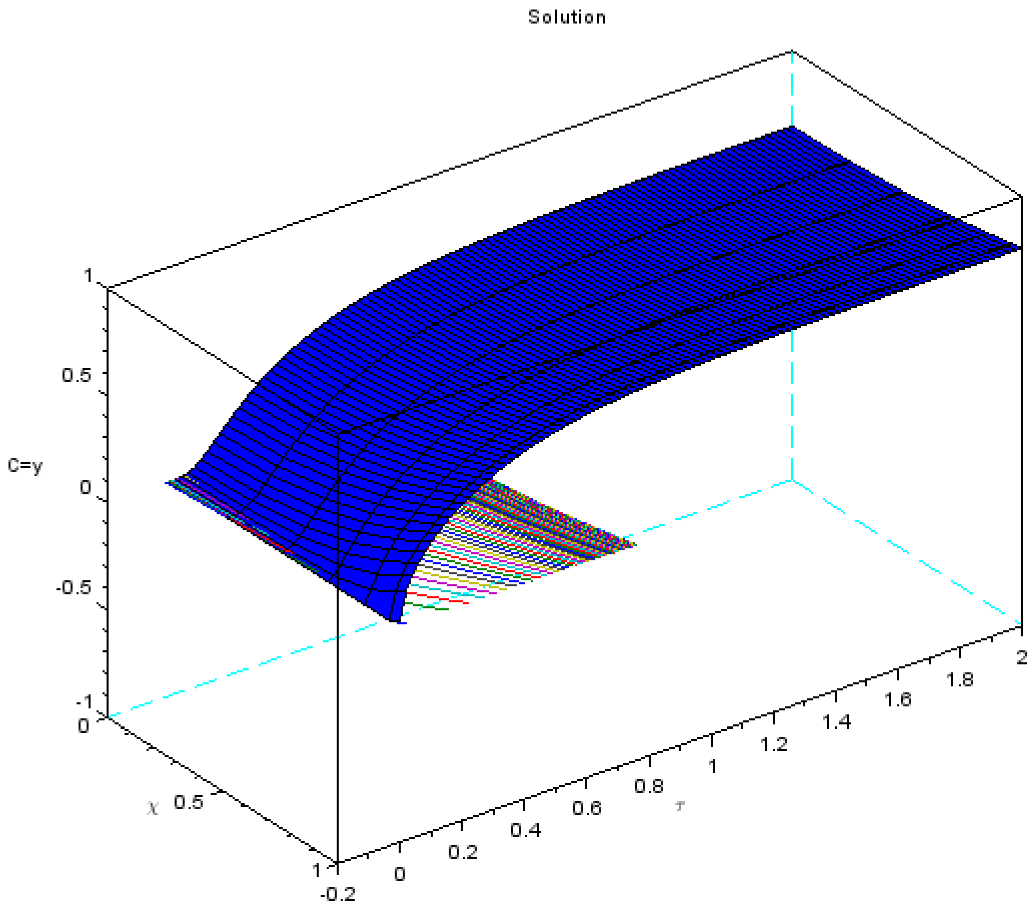
```
// ============================================================
 // These matrices are used to discretize partial differential equations
 // or just differential equations
 // dc/dt - E Div (Grad(c))=R(c)
// dc(j)/dt - E Sum (B(j,i) c(i)) = [Lc^2/(E Co)]R(c(j)) {i=1,2,3...,N+1}
 // B(j,i) = is an element of Matrix B
 // A(j,i) = is an element of Matrix A
 // w(j)=weights in the discretization
// Int (f*x^(ag-1)*dx)=Sum (w(i)*f(x(i)))
// {i=1,2,3...,N+1}, and 0<x<1
// Boundary condition
 // -Grad(c)=Biot*(c-cinf) at r/R=1, this is at the external boundary r/R=1
// After discretization
 // Sum ( A(N+1,i) c(i))= - Biot*(c(N+1)-cinf) {i=1,2,3...,N+1}
// Here Lc=Characteristic length
// E = Diffusion coeffieint
 // Co = Reference molar concentration
// R(c)=Reaction rate in units of , kmol /(m^3-s)
// The rest of differnetial operators are dimensionless
 // t=time*E/(Lc^2)
 // x=r/R
// To calculate dependent variable at the center you can use
// The first term of the column vector callculated as:
 // d=[Q^(-1)]*y
// So Q can be included in the output parameters of the function in the form
// function [c,ap0,ap,r,rx,xc,A,B,w,Q] = FJcbPly( alpha,beta,ag,N ) //
=======================================================
```

```
function [xc, A, B, w, Q]=FJcbPlyx(alpha, beta1, ag, N)
// Function to =====================================================
 a=alpha;b=beta1; gamma0=1.0; gamma1(1)=N*(N+1+a+b)/(b+1);
for i=2:N
gamma1(i)=((N-i+1)*(N+i+a+b)/((i)*(i+b)))*gamma1(i-1);
end
ap0=gamma0*(-1)^(N);
for i=1:N
ap(i)=((-1)^(N-i))*gamma1(i);
end
for i=1:N
c(i)=ap(N+1-i);
end c(N+1)=ap0;
if N==0
ap=0;
 end
 r=roots(c);
rx=sqrt(r);
 [nv,mv]=size(rx);
 // Here the vectors are column vectors
// xc are collocation points
 for i=1:N
 xc(i,1)=rx(N+1-i);
end
xc(N+1,1)=1.0;
x=xc;
for i=1:N+1
 for j=1:N+1
 Q(j,i)=x(j)^(2*i-2);
C(j,i)=(2*i-2)*x(j)^(2*i-3);
D(j,i)=(2*i-2)*(2*i+ag-4)*x(j)^(2*i-4);
end
end
// A is the First Derivative Matrix
 // B is Laplacian Matrix
 // w the weights
A=C*Q^(-1); B=D*Q^(-1);
for i=1:N+1
 f(i)=1.0/(2*i-2+ag);
End
 w=f'*Q^(-1);
 endfunction
```

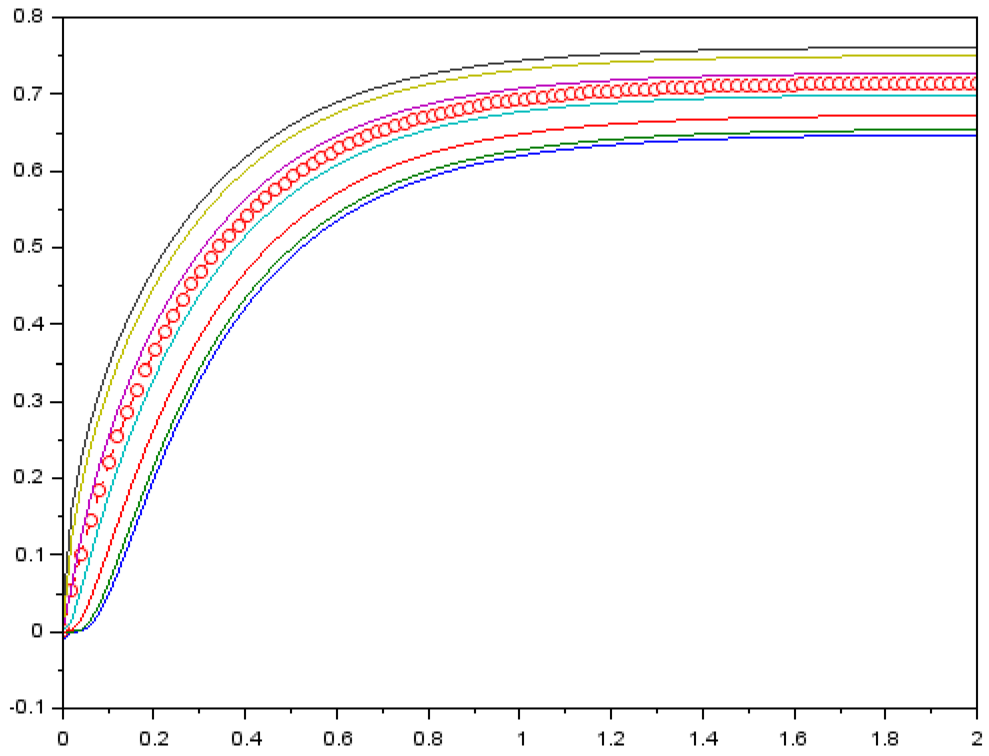


Solution(x) includes center and surface
C=y
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
τ


Solution(tau) different times
C=y
χ
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

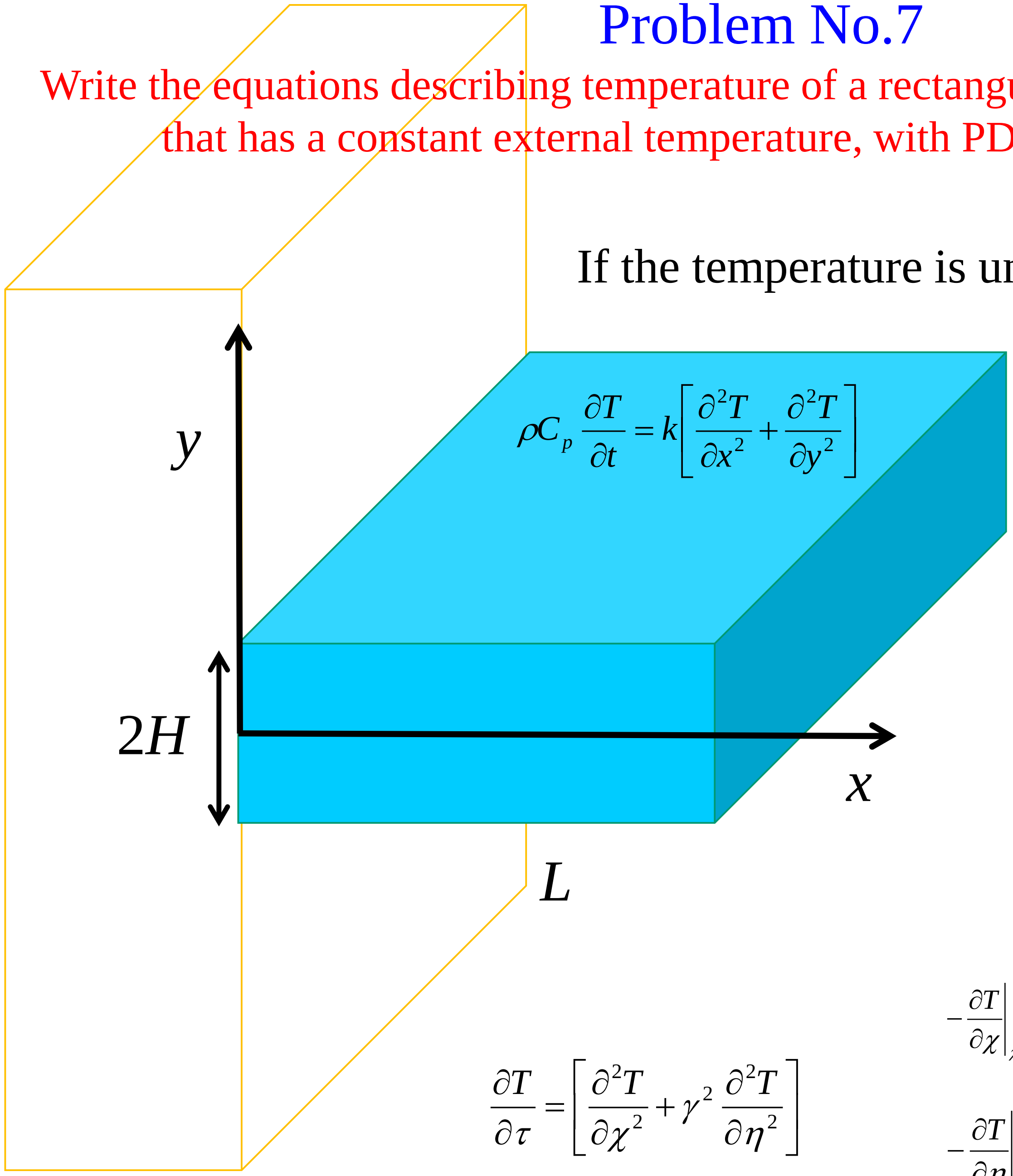


Solution
C=y
1
0.5
0
-0.5
-1
0
0.5
1
χ
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
τ


0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Problem No.7

Write the equations describing temperature of a rectangular fin attached to an oven that has a constant external temperature, with PDE, BC and IC given



$$\rho C_p \frac{\partial T}{\partial t} = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

If the temperature is uniform in z direction

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h(T - T_\infty)$$

$$-k\left.\frac{\partial T}{\partial y}\right|_{y=H} = h(T - T_\infty)$$

$T=T(t=0,x,y)$ =80°C

$$\tau = \frac{tk}{\rho C_p L^2} \qquad \gamma^2 = \left(\frac{L}{H}\right)^2$$

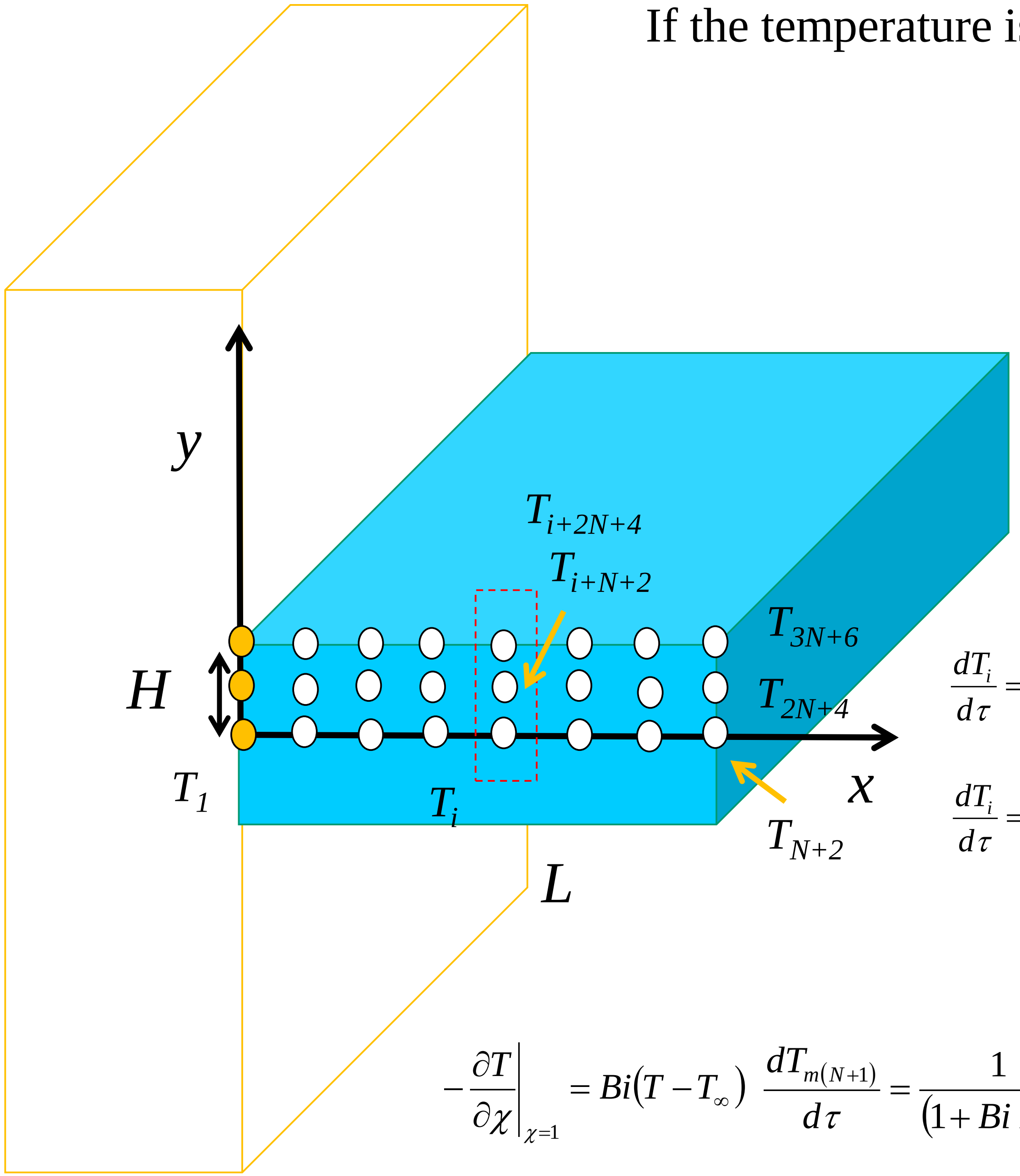
$$\chi = \frac{x}{L} \qquad \eta = \frac{y}{H}$$

$$\frac{\partial T}{\partial \tau} = \left[\frac{\partial^2 T}{\partial \chi^2} + \gamma^2 \frac{\partial^2 T}{\partial \eta^2}\right]$$

$$-\left.\frac{\partial T}{\partial \chi}\right|_{\chi=1} = Bi(T - T_\infty)$$

$$-\left.\frac{\partial T}{\partial \eta}\right|_{\eta=1} = \gamma\, Bi(T - T_\infty)$$

If the temperature is uniform in z direction



$$-\left.\frac{\partial T}{\partial \chi}\right|_{\chi=1} = Bi\left(T - T_\infty\right)$$

$$-\left.\frac{\partial T}{\partial \eta}\right|_{\eta=1} = \gamma\, Bi\left(T - T_\infty\right)$$

$$\frac{\partial T}{\partial \tau} = \left[\frac{\partial^2 T}{\partial \chi^2} + \gamma^2 \frac{\partial^2 T}{\partial \eta^2}\right]$$

$$\frac{dT_i}{d\tau} = \left[\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta\chi)^2} + \gamma^2 \frac{T_{i+N+2} - 2T_i + T_{i-(N+2)}}{(\Delta\eta)^2}\right]$$

$$\frac{dT_i}{d\tau} = \left[\frac{T_{i+1} - 2T_i + T_{i-1}}{(\Delta\chi)^2} + \gamma^2 \frac{T_{i+N+2} - 2T_i + T_{i+(N+2)}}{(\Delta\eta)^2}\right]$$

$$-\left.\frac{\partial T}{\partial \chi}\right|_{\chi=1} = Bi\left(T - T_\infty\right) \quad \frac{dT_{m(N+1)}}{d\tau} = \frac{1}{\left(1 + Bi\,\Delta\chi\right)} \frac{dT_{m(N+1)-1}}{d\tau}$$

Discretizate :

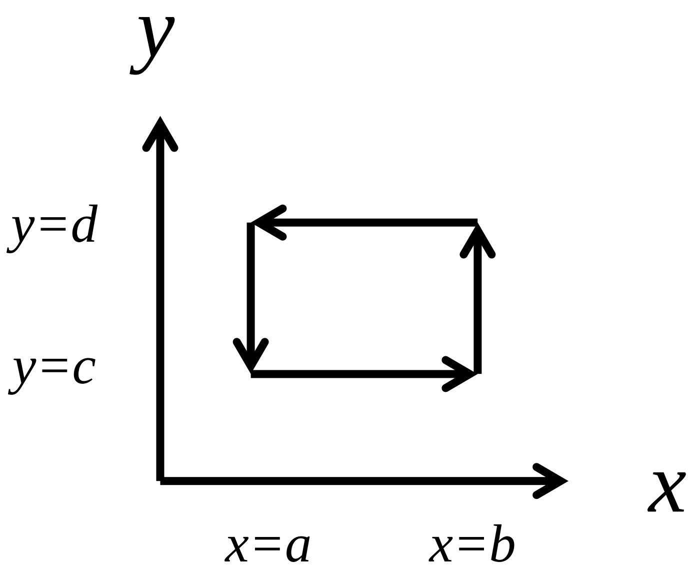
a) Using finite differences in x and y direction

b) Using finite differences in x and orthogonal collocation in y direction

c) Generalize the code, but define your variable (dependent, i.e. T) as a vector and not as a matrix

For a polynomial $P_n(x)$, write a program that gives as an output, y, the slope, the curvature, the radius of curvature, the unit tangent vector and the unit normal vector. You must use only codes you already have written or studied in class.

Some Math to digest



$$\underline{F} = P(x,y)\,\underline{i} + Q(x,y)\,\underline{j}$$

$$\underline{t}\,dl = \underline{i}dx + \underline{j}dy$$ Unit tangent vector

$$\underline{n}\,dl = \underline{i}dy - \underline{j}dx$$ Unit normal vector

$$\underline{F}\cdot\underline{n}dl = \left[P\,\underline{i} + Q\,\underline{j}\right]\cdot\left[\underline{i}dy - \underline{j}dx\right] = Pdy - Qdx$$

$$\oint \underline{F}\cdot\underline{n}\,dl = -\int_{x=a}^{x=b} Q(x,c)\,dx + \int_{y=c}^{y=d} P(b,y)\,dy - \int_{x=b}^{x=a} Q(x,d)\,dx + \int_{y=d}^{y=c} P(a,y)\,dy$$

$$\oint \underline{F}\cdot\underline{n}\,dl = \int_{y=c}^{y=d}\left[P(b,y) - P(a,y)\right]dy - \int_{x=b}^{x=a}\left[Q(x,d) - Q(x,c)\right]dx = \int_c^d\int_a^b \frac{\partial P}{\partial x}dxdy - \int_{x=b}^{x=a}\int_c^d \frac{\partial Q}{\partial y}\,dydx = \int_c^d\int_a^b \frac{\partial P}{\partial x}dxdy + \int_a^b\int_c^d \frac{\partial Q}{\partial y}\,dxdy$$

$$\oint \underline{F}\cdot\underline{n}\,dl = \iint \nabla\cdot\underline{F}dA$$

$$\oint \underline{F}\cdot\underline{n}\,dl = \int_c^d\int_a^b \frac{\partial P}{\partial x}dxdy + \int_a^b\int_c^d \frac{\partial Q}{\partial y}\,dxdy = \int_a^b\int_c^d \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right]dxdy = \iint \nabla\cdot\underline{F}dA$$

$$\oiint \underline{n}\cdot\underline{F}dA = \iiint \nabla\cdot\underline{F}dV$$

Divergence theorem
Gauss's theorem
Ostrogradsky's theorem

$$\oiint \underline{n} \cdot \underline{F} dA = \iiint \underline{\nabla} \cdot \underline{F} dV$$

$$\oint_{\partial U} \underline{n} \cdot \underline{F} dA_{n-1} = \int_U \underline{\nabla} \cdot \underline{F} dV_n$$

PDE’s solution

- FD, Finite differences
- FV, Finite volume
- FE, Finite element
- SE, Spectral Element
- BE, Boundary element

Finite differences

Discretize the domain, you do the discretization using Taylor Series approximations (for 1-D)

$$\frac{\partial\,(\rho\,\phi)}{\partial\,t} + \underline{\nabla}\cdot(\underline{v}\,\rho\,\phi) = \Gamma\,\underline{\nabla}\cdot\left[\underline{\nabla}\phi\right] + S_\phi(\phi)$$

$$\frac{d(\rho\phi)_i}{dt} + \frac{(\underline{v}\,\rho\,\phi)_{i+1} - (\underline{v}\,\rho\,\phi)_{i-1}}{2\Delta\chi} = \Gamma\,\frac{(\phi_{i-1} - 2\phi_i + \phi_{i+1})}{(\Delta\chi)^2} + S_\phi(\phi_i)$$

Finite Volume

Non overlapping small control volumes called cells

$$\frac{\partial\,(\rho\,\phi)}{\partial\,t} + \underline{\nabla}\cdot(\underline{v}\,\rho\,\phi) = \underline{\nabla}\cdot\left[\Gamma\underline{\nabla}\phi\right] + S_\phi$$

$$\int_\Omega \left[\underline{\nabla}\cdot(\underline{v}\,\rho\,\phi) - \underline{\nabla}\cdot\left[\Gamma\underline{\nabla}\phi\right]\right] d\Omega = \int_\Omega S_\phi\, d\Omega$$

$$\oint\left[\underline{v}\,\rho\,\phi - \Gamma\underline{\nabla}\phi\right]\cdot\underline{n}dA = \int_\Omega S_\phi\, d\Omega$$

$$\oint\left[\underline{v}\,\rho\,\phi\right]\cdot\underline{n}dA = \sum_f \left(\underline{v}\,\rho\,\phi\right)_f\cdot\underline{n}A_f$$

The method is conservative

Finite Element

Approximate numerical solutions, that predict the response of a physical system subjected to external influences

$$\rho \frac{\partial \phi}{\partial t} + \rho \underline{v} \cdot \underline{\nabla} \phi = \Gamma \underline{\nabla} \cdot \left[\underline{\nabla} \phi \right] + S_{\phi}$$

Differential equation

$$-\left.\frac{\partial \phi}{\partial \chi}\right|_{\chi=1} = \beta(\phi - \phi_{\infty})$$

Boundary condition

$$\Im_F(\phi) = \rho \frac{\partial \phi}{\partial t} + \rho \underline{v} \cdot \underline{\nabla} \phi - \Gamma \underline{\nabla} \cdot \left[\underline{\nabla} \phi \right] - S_{\phi} = \Re_a = 0$$

$$\Im_B(\phi) = -\left.\frac{\partial \phi}{\partial \chi}\right|_{\chi=1} - \beta(\phi - \phi_{\infty}) = \Re_b = 0$$

- Methods of weighted residuals (The collocation, The subdomain, the least-squares, and the Galerkin)
- Ritz variational method

Finite Element

Approximate numerical solutions, that predict the response of a physical system subjected to external influences

$$\Im_F(\phi) = \rho\frac{\partial\phi}{\partial t} + \rho\underline{v}\cdot\underline{\nabla}\phi - \Gamma\underline{\nabla}\cdot\left[\underline{\nabla}\phi\right] - S_\phi = \Re_a = 0$$

$$\Im_B(\phi) = -\left.\frac{\partial\phi}{\partial\chi}\right|_{\chi=1} - \beta(\phi - \phi_\infty) = \Re_b = 0$$

$$\int \Re_a(\chi)\, w_k(\chi)\, d\chi = 0$$

w_{k} = Test function

$$\phi_a(\chi) = \phi_0(\chi) + \sum_{i=1}^{N} a_i\psi_i(\chi)$$

ψ_{i} = Trial function

Rabbits and wolves in an isolated forest

The dynamics of population of wolves and rabbits can be described by the equations

$$\frac{dB}{dt} = (k - k_D)B - aB(W + S) + k_{mr}BD$$

$$\frac{dD}{dt} = (k - k_D)D - aD(W + S) + k_{mr}BD$$

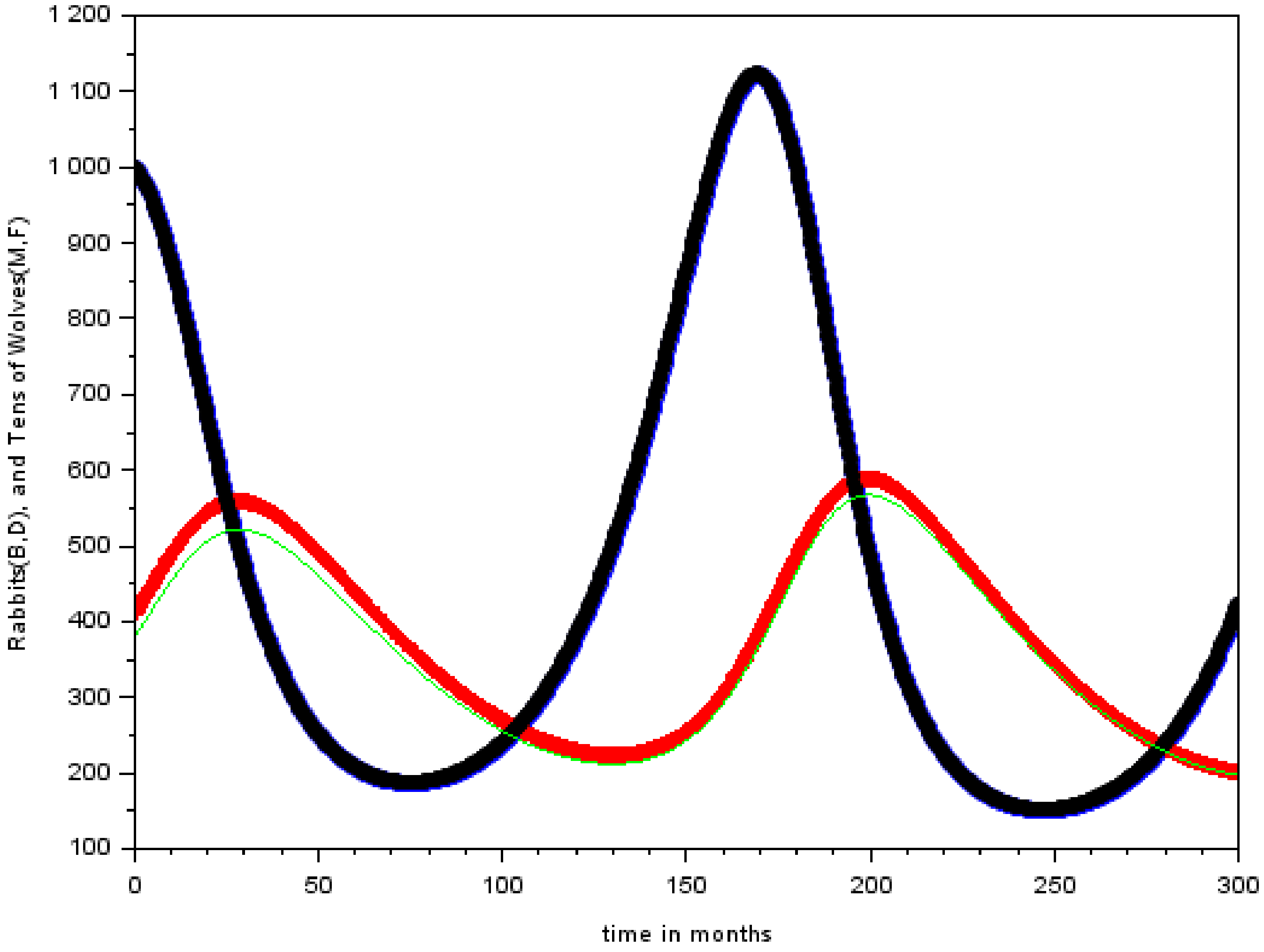
$$\frac{dW}{dt} = -rW + bW(B + D) + k_{mw}WS - k_f W^2$$

$$\frac{dS}{dt} = -rS + bS(B + D) + k_{mw}WS$$

B=Buck, D=Doe, W=Male Wolf (loup), S=Female Wolf (louve)

k=0.08 1/month	Shelter growth index
a=0.001 1/(wolf-month)	Predator-prey feeding rate index
r=0.02 1/month	Death rate index for wolves because of ageing.
b=0.00002 1/(rabbit-month)	Predator-prey growth rate index
k_{mr}=0.000004 1/(rabbit-month)	Rabbits reproduction rate index
k_{mw}=0.000001 1/(wolf-month)	Wolves reproduction rate index
k_f=0.000005 1/(wolf-month)	Death rate index for wolves because of alpha male fights.
k_D=0.01 1/month	Death rate index for rabbits because of ageing.

What is the size of the pack of wolves (males and females), needed in order to have a population of 20 wolves, 20 she-wolves after 300 months, if you start with 1000 bucks and 1000 does.


Rabbits(B,D), and Tens of Wolves(M,F)
1 200
1 100
1 000
900
800
700
600
500
400
300
200
100
0
50
100
150
200
250
300
time in months

Flow throw porous media

$$-\underline{\nabla} p + \rho \underline{g} - \mu \underline{\underline{\kappa}}^{-1}\cdot \underline{u} - \rho \underline{\beta}\cdot \left|\underline{u}\right|\underline{u} + \frac{\mu}{\phi}\underline{\nabla}\cdot\left(\underline{\nabla}\,\underline{u}\right) = \frac{1}{\phi}\frac{\partial \underline{u}}{\partial t}$$

For a 1-D flow through parallel plates, collecting terms and recasting in dimensionless form:

$$\kappa = \frac{\phi^3 D_p^2}{150(1-\phi)^2}$$

$$\beta = \frac{1.75(1-\phi)}{\phi^3 D_p}$$

$$\left[-\frac{\partial p}{\partial x} + \rho g_x - \frac{\mu}{\kappa}u - \rho\beta u^2 + \frac{\mu}{\phi}\frac{d^2 u}{dy^2} = \frac{1}{\phi}\frac{\partial u}{\partial t}\right]\left(\frac{\phi h^2}{\mu}\right)\left(\frac{\rho h}{\mu}\right)$$

$$\frac{\partial^2 w}{\partial \eta^2} + c - b^2 w - a\,w^2 = \frac{\partial w}{\partial \tau}$$

1-D, parallel plates

$$\frac{\partial^2 w}{\partial \eta^2} + \frac{\partial^2 w}{\partial \chi^2} + c - b^2 w - a\,w^2 = \frac{\partial w}{\partial \tau}$$

2-D, square duct

$$c = \frac{\rho h^3 \phi(-\nabla P)}{\mu^2}$$

$$b^2 = \frac{\phi\, h^2}{\kappa}$$

$$a = \phi\, h\, \beta$$

$$w = \frac{\rho\, u\, h}{\mu} \qquad \eta = \frac{y}{h}$$

Boundary conditions

$$u\Big|_{y=h} = 0 \qquad w\Big|_{\eta=1} = 0$$

$$\left.\frac{du}{dy}\right|_{y=0} = 0 \qquad \left.\frac{dw}{d\eta}\right|_{\eta=0} = 0$$

$$\frac{\partial^2 w}{\partial \eta^2}+\frac{\partial^2 w}{\partial \chi^2}+c-b^2 w-a\,w^2=\frac{\partial w}{\partial \tau}$$

2-D, square cross sectional porous media

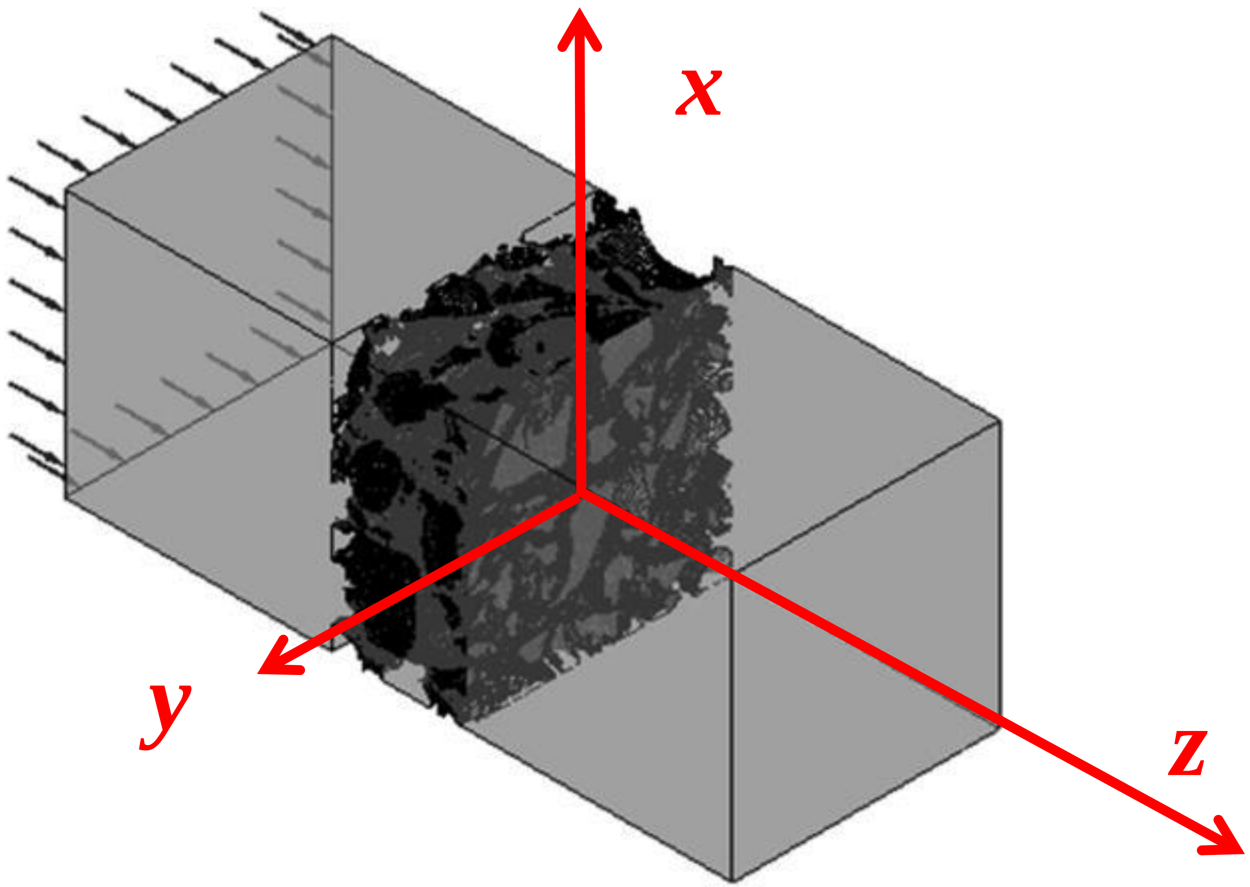
$$\frac{\partial^2 w}{\partial \eta^2}+\frac{\partial^2 w}{\partial \chi^2}+c-b^2 w-a\,w^2=\frac{\partial w}{\partial \tau}$$

$$\frac{w_{i,j-1}-2w_{i,j}+w_{i,j+1}}{(\Delta\eta)^2}+\frac{w_{i-1,j}-2w_{i,j}+w_{i+1,j}}{(\Delta\chi)^2}+c-b^2 w_{i,j}-a\,{w_{i,j}}^2=\frac{dw_{i,j}}{d\tau}$$

$$\left.\frac{\partial w}{\partial \eta}\right|_{\eta=0}=\left.\frac{\partial w}{\partial \chi}\right|_{\chi=0}=0$$

$$\left.w\right|_{\eta=1}=0$$

$$\left.w\right|_{\chi=1}=0$$


x
y
z

$$-\underline{\nabla} p + \rho \underline{g} - \mu \underline{\underline{\kappa}}^{-1} \cdot \underline{u} - \rho \underline{\beta} \cdot |\underline{u}| \underline{u} + \frac{\mu}{\phi} \underline{\nabla} \cdot \left(\underline{\nabla} \underline{u} \right) = 0$$

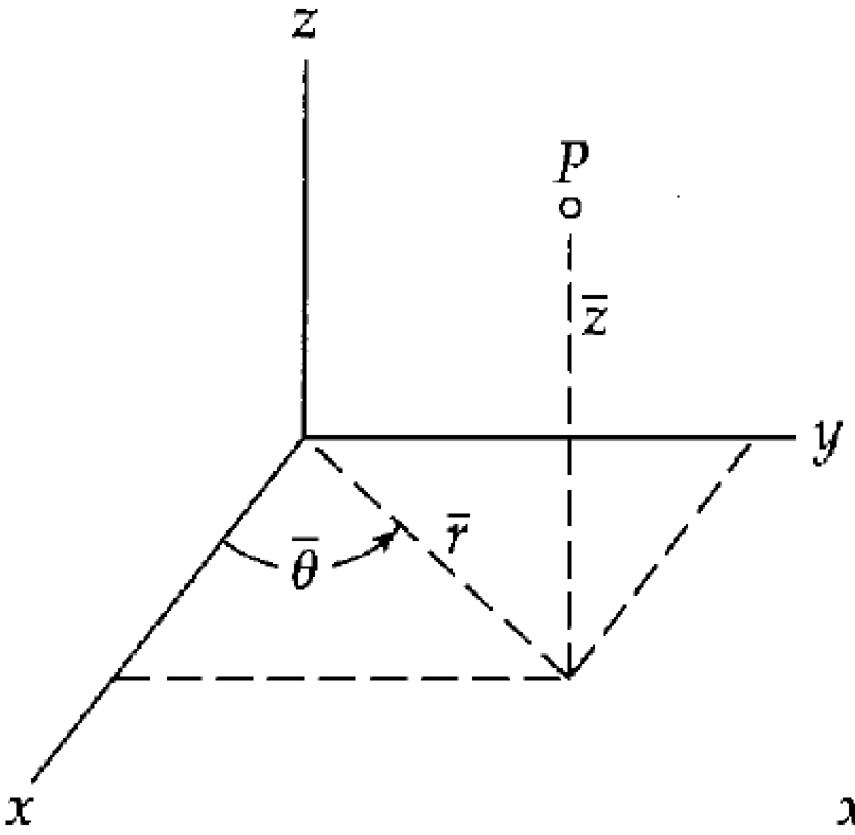
This is called Darcy-Brinkman-Forchheimer equation

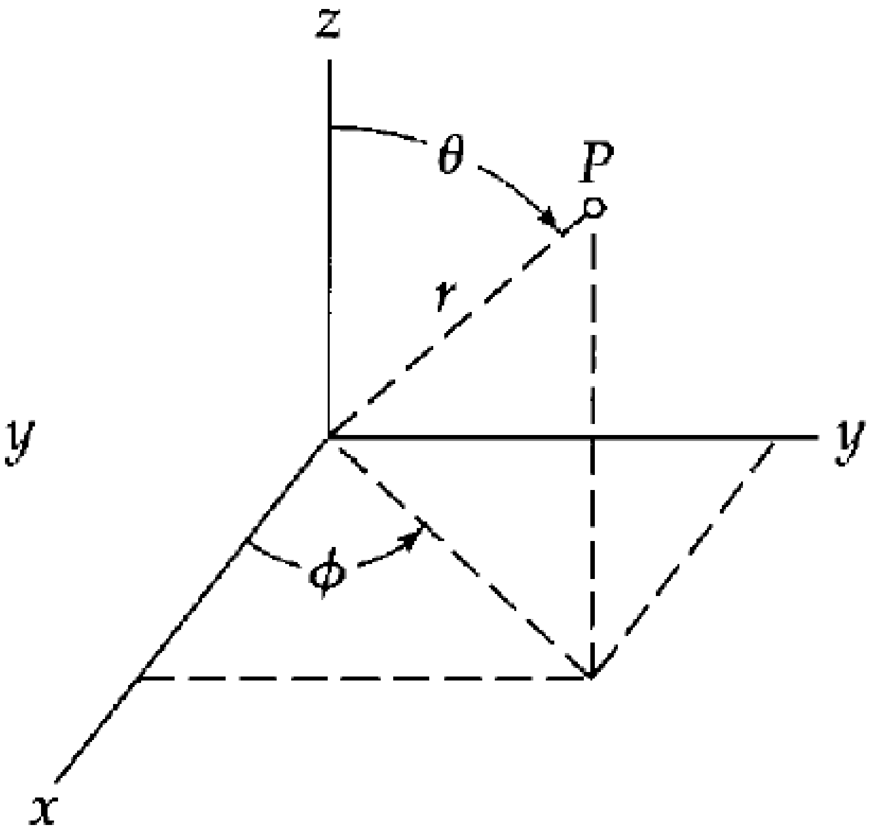
$$\frac{d[R]}{dt} = k[R] - a[R][W] + k_{mr}[R]^2 - d[R][E]$$

$$\frac{d[E]}{dt} = -r_E[E] + c[R][E] + k_{me}[E]^2$$

$$\frac{d[W]}{dt} = -r[W] + b[W][R] + k_{mw}[W]^2$$

Engineering Jargon for cylindrical and spherical coordinates





$$\begin{cases} x = r\cos\theta & \text{(A.6-1)} \\ y = r\sin\theta & \text{(A.6-2)} \\ z = z & \text{(A.6-3)} \end{cases} \qquad \begin{array}{ll} r = +\sqrt{x^2 + y^2} & \text{(A.6-4)} \\ \theta = \arctan(y/x) & \text{(A.6-5)} \\ z = z & \text{(A.6-6)} \end{array}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = (\cos\theta)\dfrac{\partial}{\partial r} + \left(-\dfrac{\sin\theta}{r}\right)\dfrac{\partial}{\partial\theta} + (0)\dfrac{\partial}{\partial z} \\[2ex] \dfrac{\partial}{\partial y} = (\sin\theta)\dfrac{\partial}{\partial r} + \left(\dfrac{\cos\theta}{r}\right)\dfrac{\partial}{\partial\theta} + (0)\dfrac{\partial}{\partial z} \\[2ex] \dfrac{\partial}{\partial z} = (0)\dfrac{\partial}{\partial r} + (0)\dfrac{\partial}{\partial\theta} + (1)\dfrac{\partial}{\partial z} \end{cases}$$

for a scalar function $\chi(x, y, z) = \psi(r, \theta, z)$:

$$\left(\frac{\partial \chi}{\partial x}\right)_{y,z} = \left(\frac{\partial r}{\partial x}\right)_{y,z}\left(\frac{\partial \psi}{\partial r}\right)_{\theta,z} + \left(\frac{\partial \theta}{\partial x}\right)_{y,z}\left(\frac{\partial \psi}{\partial \theta}\right)_{r,z} + \left(\frac{\partial z}{\partial x}\right)_{y,z}\left(\frac{\partial \psi}{\partial z}\right)_{r,\theta}$$

$$\begin{cases} x = r \sin\theta \cos\phi & \text{(A.6-19)} \\ y = r \sin\theta \sin\phi & \text{(A.6-20)} \\ z = r \cos\theta & \text{(A.6-21)} \end{cases} \qquad \begin{aligned} r &= +\sqrt{x^2 + y^2 + z^2} && \text{(A.6-22)} \\ \theta &= \arctan(\sqrt{x^2 + y^2}/z) && \text{(A.6-23)} \\ \phi &= \arctan(y/x) && \text{(A.6-24)} \end{aligned}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = (\sin\theta\cos\phi)\dfrac{\partial}{\partial r} + \left(\dfrac{\cos\theta\cos\phi}{r}\right)\dfrac{\partial}{\partial\theta} + \left(-\dfrac{\sin\phi}{r\sin\theta}\right)\dfrac{\partial}{\partial\phi} \\[2ex] \dfrac{\partial}{\partial y} = (\sin\theta\sin\phi)\dfrac{\partial}{\partial r} + \left(\dfrac{\cos\theta\sin\phi}{r}\right)\dfrac{\partial}{\partial\theta} + \left(\dfrac{\cos\phi}{r\sin\theta}\right)\dfrac{\partial}{\partial\phi} \\[2ex] \dfrac{\partial}{\partial z} = (\cos\theta)\dfrac{\partial}{\partial r} + \left(-\dfrac{\sin\theta}{r}\right)\dfrac{\partial}{\partial\theta} + (0)\dfrac{\partial}{\partial\phi} \end{cases}$$

$$\nabla = \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}$$

$$\nabla = \boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \boldsymbol{\delta}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_0}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}$$

$$
\begin{aligned}
\nabla \mathbf{v} &= \left\{\boldsymbol{\delta}_r \frac{\partial}{\partial r} + \boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{\delta}_z \frac{\partial}{\partial z}\right\}\{\boldsymbol{\delta}_r v_r + \boldsymbol{\delta}_\theta v_\theta + \boldsymbol{\delta}_z v_z\} \\
&= \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z \frac{\partial v_z}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r \frac{1}{r}\frac{\partial v_r}{\partial \theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z \frac{1}{r}\frac{\partial v_z}{\partial \theta} \\
&\quad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta \frac{v_r}{r} - \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r \frac{v_\theta}{r} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial v_z}{\partial z} \\
&= \boldsymbol{\delta}_r\boldsymbol{\delta}_r \frac{\partial v_r}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial r} + \boldsymbol{\delta}_r\boldsymbol{\delta}_z \frac{\partial v_z}{\partial r} + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_r\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}\right) + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_\theta\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \\
&\quad + \boldsymbol{\delta}_\theta\boldsymbol{\delta}_z \frac{1}{r}\frac{\partial v_z}{\partial \theta} + \boldsymbol{\delta}_z\boldsymbol{\delta}_r \frac{\partial v_r}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_\theta \frac{\partial v_\theta}{\partial z} + \boldsymbol{\delta}_z\boldsymbol{\delta}_z \frac{\partial v_z}{\partial z}
\end{aligned}
$$

$$[\nabla \cdot \boldsymbol{\tau}]_r = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{rr}) + \frac{\tau_{\theta r}}{r}\cot\theta + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi r}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}$$

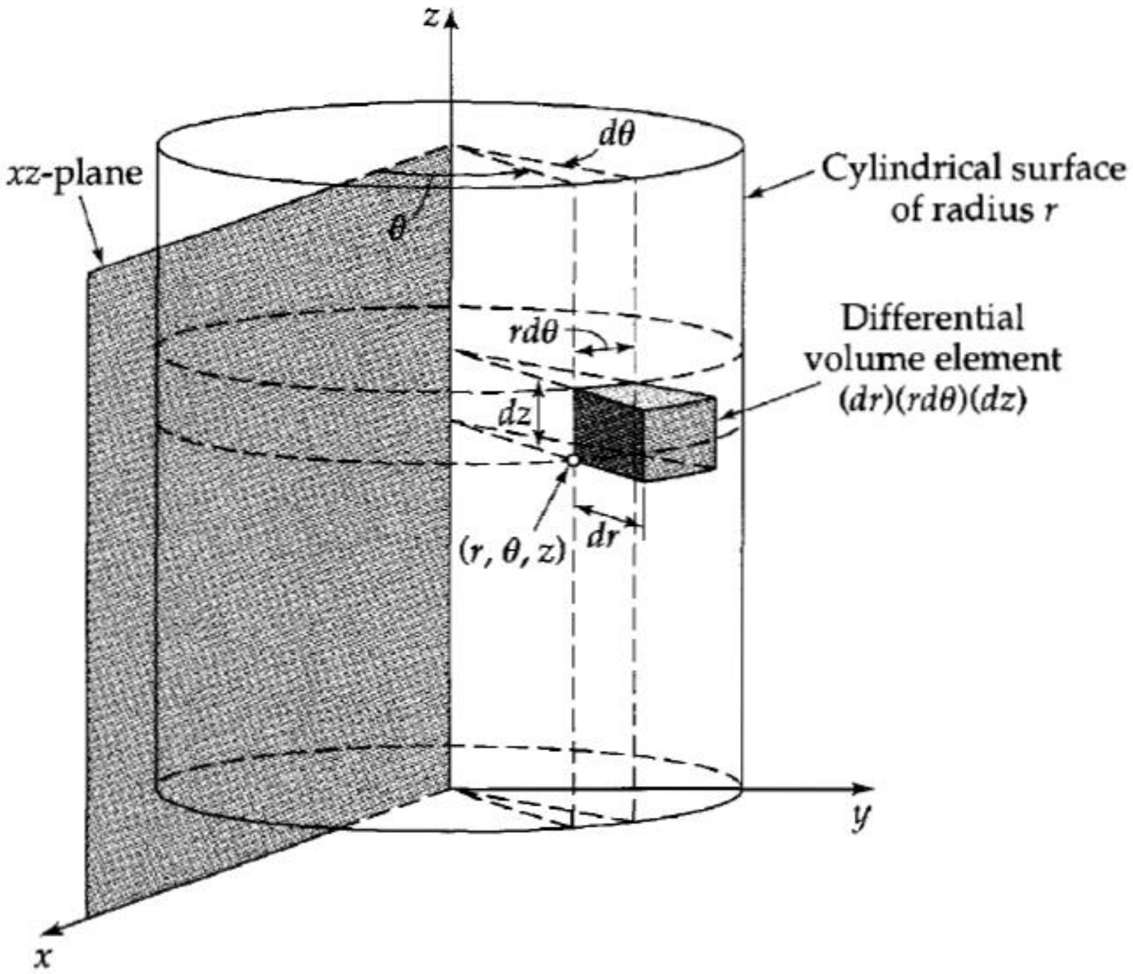
$$(\nabla \cdot \nabla) = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - [\nabla \times [\nabla \times \mathbf{v}]]$$

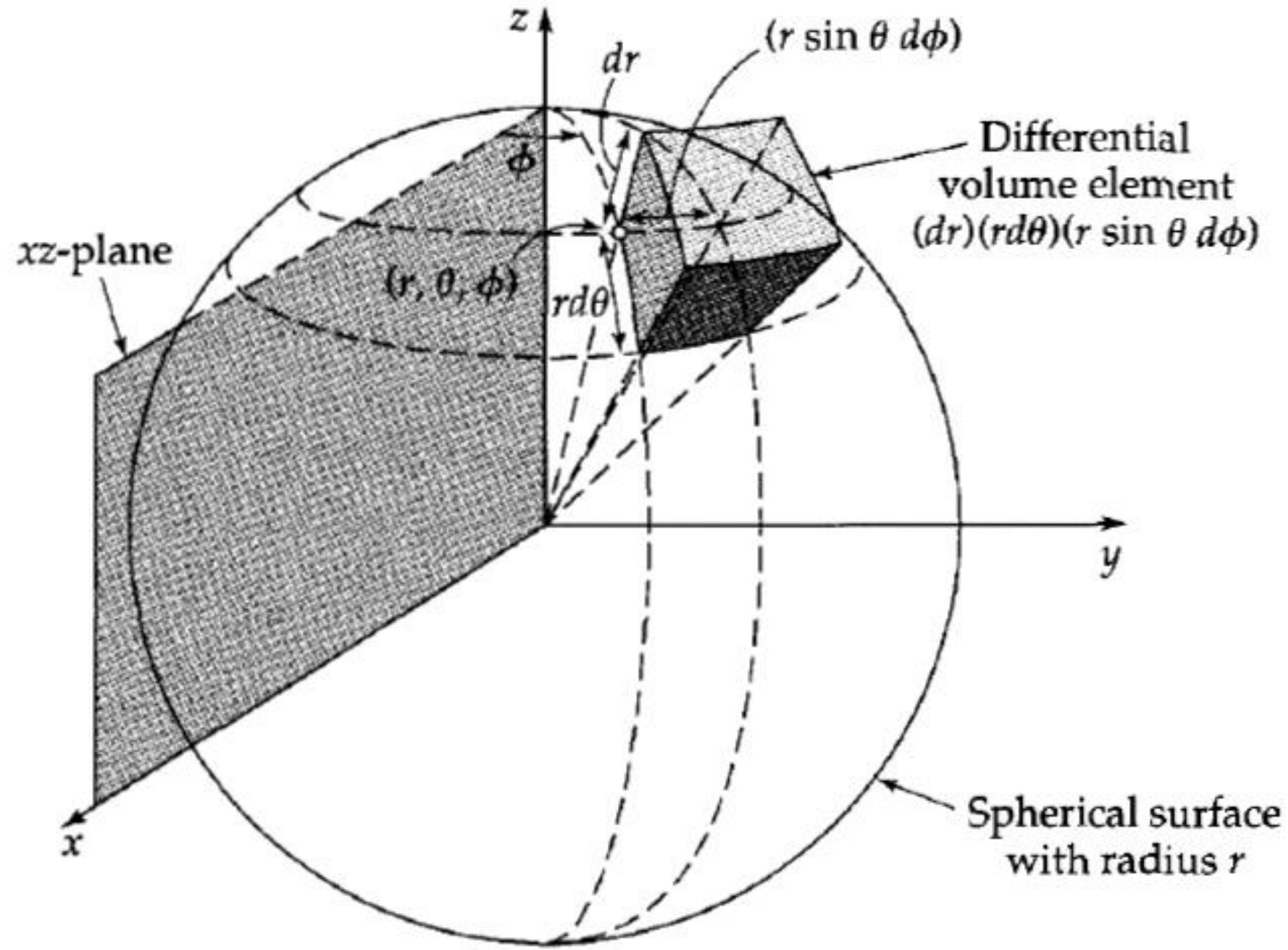
$$\int_{z_1}^{z_2}\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r, \theta, z) r \, dr \, d\theta \, dz$$

$$\int_{z_1}^{z_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r, \theta, z) r \, dr \, d\theta \, dz$$

$$\int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} g(r, \theta, \phi) r^2 \, dr \ \sin \ \theta \, d\theta \, d\phi$$



z
dθ
xz-plane
Cylindrical surface
of radius r
θ
Differential
volume element
(dr)(rdθ)(dz)
rdθ
dz
(r, θ, z)
dr
y
x



z
dr
$(r \sin\theta \, d\phi)$
Differential
volume element
$(dr)(rd\theta)(r \sin\theta \, d\phi)$
ϕ
xz-plane
(r, θ, ϕ)
$rd\theta$
y
x
Spherical surface
with radius r

Gibbs notation	Expanded notation in terms of unit vectors and unit dyads	Cartesian tensor notation
$(\mathbf{v} \cdot \mathbf{w})$	$\sum_i v_i w_i$	$v_i w_i$
$[\mathbf{v} \times \mathbf{w}]$	$\sum_i \sum_j \sum_k \varepsilon_{ijk} \boldsymbol{\delta}_i v_j w_k$	$\varepsilon_{ijk} v_j w_k$
$[\nabla \cdot \boldsymbol{\tau}]$	$\sum_i \sum_j \boldsymbol{\delta}_i \frac{\partial}{\partial x_j} \tau_{ji}$	$\partial_j \tau_{ji}$
$\nabla^2 s$	$\sum_i \frac{\partial^2}{\partial x_i^2} s$	$\partial_i \partial_i s$
$[\nabla \times [\nabla \times \mathbf{v}]]$	$\sum_i \sum_j \sum_k \sum_m \sum_n \boldsymbol{\delta}_i \varepsilon_{ijk} \varepsilon_{kmn} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_m} v_n$	$\varepsilon_{ijk} \varepsilon_{kmn} \partial_j \partial_m v_n$
$\{\boldsymbol{\tau} \times \mathbf{v}\}$	$\sum_i \sum_j \sum_k \sum_l \varepsilon_{jkl} \boldsymbol{\delta}_i \boldsymbol{\delta}_l \tau_{ij} v_k$	$\varepsilon_{jkl} \tau_{ij} v_k$

$$[\boldsymbol{\tau} = \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\dagger}) - (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})\boldsymbol{\delta}]$$

$$-\ \tau_{xx} = -\mu\left[2\frac{\partial v_x}{\partial x}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})$$

$$-\ \tau_{yy} = -\mu\left[2\frac{\partial v_y}{\partial y}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})$$

$$-\ \tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + (\tfrac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})$$

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$

$$\tau_{yz} = \tau_{zy} = \mu\left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z}\right]$$

$$\tau_{zx} = \tau_{xz} = \mu\left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Cylindrical coordinates (r, θ, z):

$$-\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})$$

$$-\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})$$

$$-\tau_{zz} = -\mu\left[2\frac{\partial v_z}{\partial z}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla\cdot\mathbf{v})$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right]$$

$$\tau_{zr} = \tau_{rz} = \mu\left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right]$$

$$(\nabla\cdot\mathbf{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

Spherical coordinates (r, θ, ϕ):

$$\tau_{rr} = -\mu\left[2\frac{\partial v_r}{\partial r}\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{\theta\theta} = -\mu\left[2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{\phi\phi} = -\mu\left[2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta\cot\theta}{r}\right)\right] + \left(\tfrac{2}{3}\mu - \kappa\right)(\nabla \cdot \mathbf{v})$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right]$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

$$\tau_{\phi r} = \tau_{r\phi} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

$$[\partial\rho/\partial t + (\nabla \cdot \rho\mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical coordinates (r, θ, ϕ):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial\theta}(\rho v_\theta\ \sin\ \theta) + \frac{1}{r\ \sin\ \theta}\frac{\partial}{\partial\phi}(\rho v_\phi) = 0$$

$$[\rho D\mathbf{v}/Dt = -\nabla p + [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \left[\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \left[\frac{\partial}{\partial x}\tau_{xy} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left[\frac{\partial}{\partial x}\tau_{xz} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z$$

$$[\rho D\mathbf{v}/Dt = -\nabla p + [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}\right] + \rho g_z$$

$$[\rho D\mathbf{v}/Dt = -\nabla p + [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$

$$+ \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{rr}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta r} \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi r} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+ \left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3 \tau_{r\theta}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\theta} \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi\theta} + \frac{(\tau_{\theta r} - \tau_{r\theta}) - \tau_{\phi\phi}\cot\theta}{r}\right) + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot\theta}{r}\right) = -\frac{1}{r \sin\theta}\frac{\partial p}{\partial \phi}$$

$$+ \left[\frac{1}{r^3}\frac{\partial}{\partial r}(r^3 \tau_{r\phi}) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\tau_{\theta\phi} \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\tau_{\phi\phi} + \frac{(\tau_{\phi r} - \tau_{r\phi}) + \tau_{\phi\theta}\cot\theta}{r}\right] + \rho g_\phi$$

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}]$$

Cylindrical coordinates $(\mathbf{r}, \theta, z)$:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z\frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z\frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}]$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2\cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial\phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi\cot\theta}{r}\right) = -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(v_\phi\sin\theta)\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial\phi^2} + \frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial\phi} + \frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right] + \rho g_\phi$$

$$I(t) = \int_{\alpha(t)}^{\beta(t)} f(x, t)\, dx$$

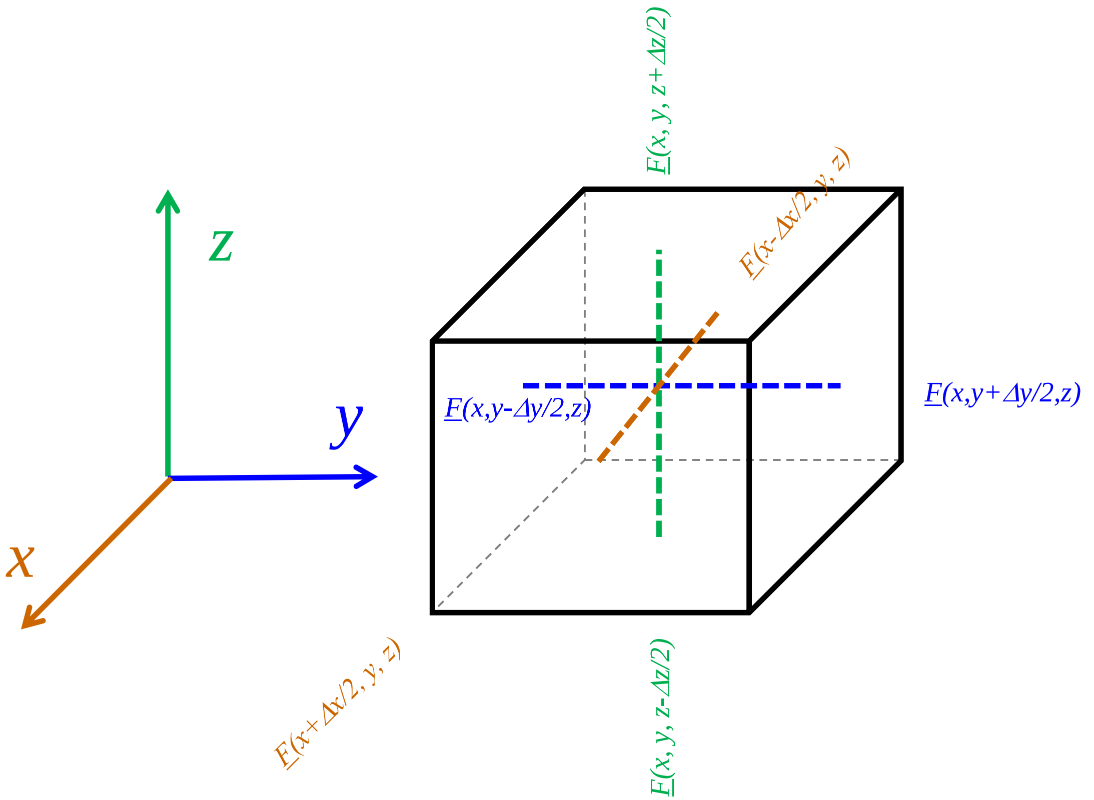
Leibniz formula

$$\frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x, t)\, dx = \int_{\alpha(t)}^{\beta(t)} \frac{\partial}{\partial t} f(x, t)\, dx + \left(f(\beta, t)\,\frac{d\beta}{dt} - f(\alpha, t)\,\frac{d\alpha}{dt}\right)$$

Gauss Theorem 3D

Vector field

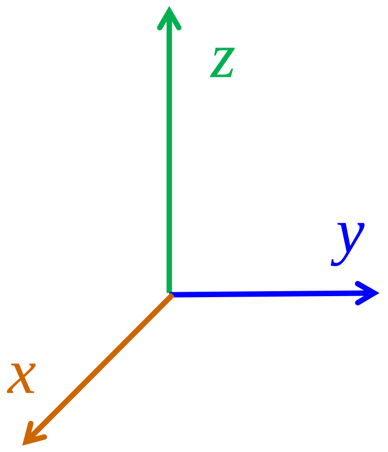
$$\underline{F}(x,y,z) = \hat{\imath}\,P(x,y,z) + \hat{\jmath}\,Q(x,y,z) + \hat{k}\,R(x,y,z)$$

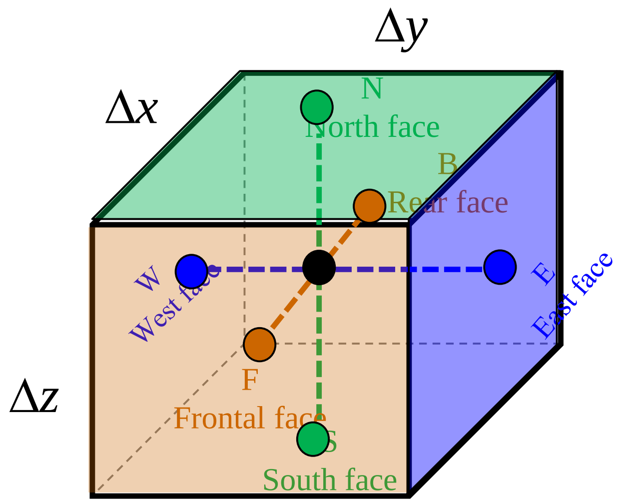


Vector field in this case is a flux of a property, and flux in the sense of flow per unit area. (Mathematicians use the concept of flux as flow, but will cause confusion in this context, the term will be avoided in the mathematics' jargon)

Vector field

$$\underline{F}(x,y,z) = \hat{\imath}\, P(x,y,z) + \hat{\jmath}\, Q(x,y,z) + \hat{k}\, R(x,y,z)$$



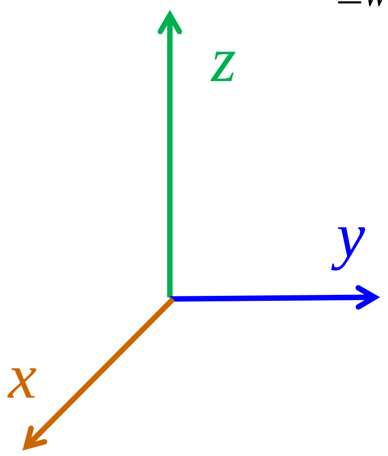


Coordinates of the points

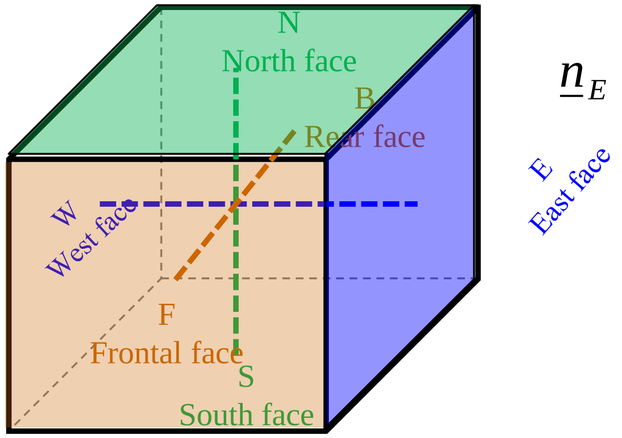
Point	*x*	*y*	*z*	Normal
C	x	y	z	
F	$x+\Delta x/2$	y	z	$\underline{n}_F = \hat{\imath}$
B	$x-\Delta x/2$	y	z	$\underline{n}_B = -\hat{\imath}$
E	x	$y+\Delta y/2$	z	$\underline{n}_E = \hat{\jmath}$
W	x	$y-\Delta y/2$	z	$\underline{n}_W = -\hat{\jmath}$
N	x	y	$z+\Delta z/2$	$\underline{n}_N = \hat{k}$
S	x	y	$z-\Delta z/2$	$\underline{n}_S = -\hat{k}$

Vector field

$$\underline{F}(x,y,z)=\hat{\imath}\,P(x,y,z)+\hat{\jmath}\,Q(x,y,z)+\hat{k}\,R(x,y,z)$$



$\underline{n}_W = -\hat{\jmath}$



$\underline{n}_E = \hat{\jmath}$

Integrate the flow of the vector field $\underline{F}$ $\quad \Phi = \oiint \underline{F}\cdot\underline{n}\,dA$

$$\Phi_W+\Phi_E=\underline{F}(x,y-\Delta y/2,z)\cdot\underline{n}_W\Delta x\Delta z+\underline{F}(x,y+\Delta y/2,z)\cdot\underline{n}_E\Delta x\Delta z$$

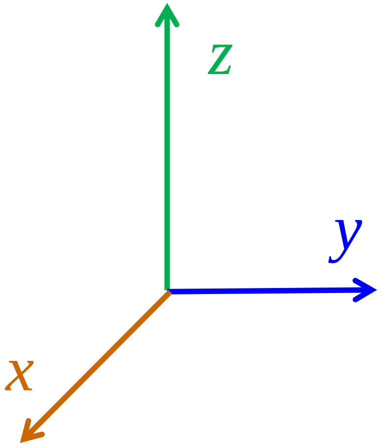
$$\Phi_E+\Phi_W=Q(x,y+\Delta y/2,z)\Delta x\Delta z-Q(x,y-\Delta y/2,z)\Delta x\Delta z$$

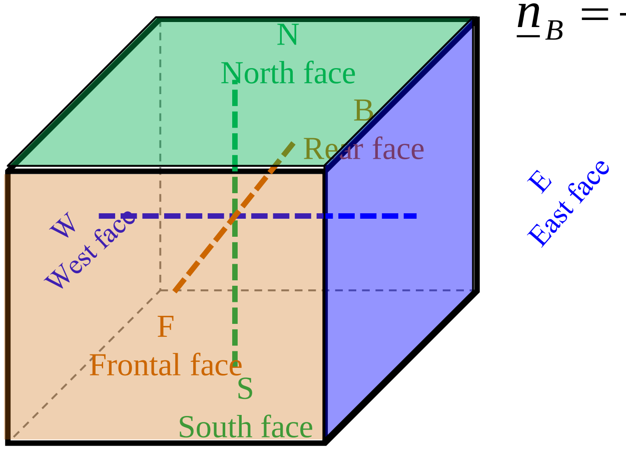
$$\Phi_E+\Phi_W=\left[\frac{Q(x,y+\Delta y/2,z)-Q(x,y-\Delta y/2,z)}{\Delta y}\right]\Delta x\Delta z\Delta y=\frac{\partial Q}{\partial y}\,dV$$

Vector field

$$\underline{F}(x,y,z)=\hat{\imath}\,P(x,y,z)+\hat{\jmath}\,Q(x,y,z)+\hat{k}\,R(x,y,z)$$

$$\underline{n}_B=-\hat{\imath}$$





$$\underline{n}_F=\hat{\imath}$$

Integrate the flow of the vector field $\underline{F}$

$$\Phi_F+\Phi_B=\underline{F}(x+\Delta x/2,y,z)\cdot\underline{n}_F\Delta y\Delta z+\underline{F}(x-\Delta x/2,y,z)\cdot\underline{n}_B\Delta y\Delta z$$

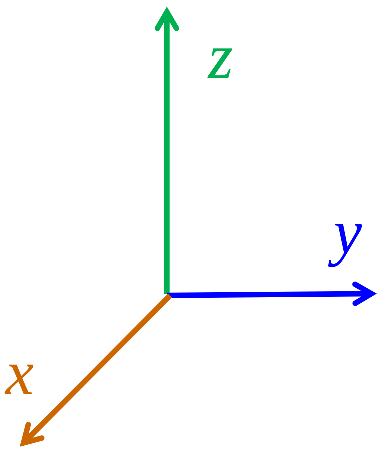
$$\Phi_F+\Phi_B=P(x+\Delta x/2,y,z)\Delta y\Delta z-P(x-\Delta x/2,y,z)\Delta y\Delta z$$

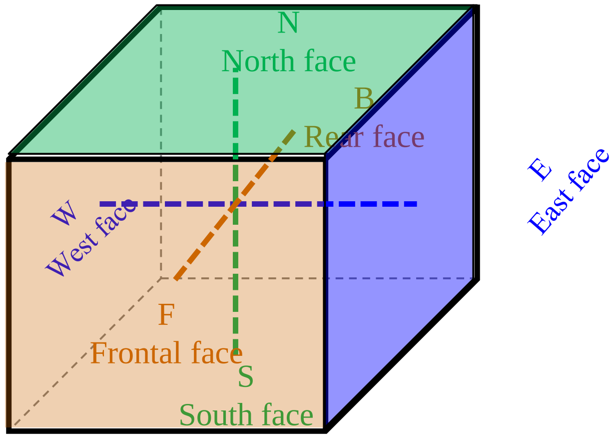
$$\Phi_F+\Phi_B=\left[\frac{P(x+\Delta x/2,y,z)-P(x-\Delta x/2,y,z)}{\Delta x}\right]\Delta x\Delta z\Delta y=\frac{\partial P}{\partial x}dV$$

Vector field

$$\underline{F}(x,y,z)=\hat{\mathrm{i}}\,P(x,y,z)+\hat{\mathrm{j}}\,Q(x,y,z)+\hat{\mathrm{k}}\,R(x,y,z)$$

$$\underline{n}_N=\hat{\mathrm{k}}$$





$$\underline{n}_S=-\hat{\mathrm{k}}$$

Integrate the flow of the vector field $\underline{F}$

$$\Phi_N+\Phi_S=\underline{F}(x,y,z+\Delta z/2)\cdot\underline{n}_N\Delta y\Delta x+\underline{F}(x,y,z-\Delta z/2)\cdot\underline{n}_S\Delta y\Delta x$$

$$\Phi_N+\Phi_S=R(x,y,z+\Delta z/2)\Delta y\Delta x-R(x,y,z-\Delta z/2)\Delta y\Delta x$$

$$\Phi_F+\Phi_B=\left[\frac{R(x,y,z+\Delta z/2)-R(x,y,z-\Delta z/2)}{\Delta z}\right]\Delta x\Delta z\Delta y=\frac{\partial R}{\partial z}dV$$

$$\Phi_W + \Phi_E + \Phi_F + \Phi_B + \Phi_N + \Phi_S = \sum \underline{F_i} \cdot \underline{n_i}\ \mathit{\Delta} A_i$$

$$\sum \underline{F_i} \cdot \underline{n_i}\ \mathit{\Delta} A_i = \left[\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right] \Delta V$$

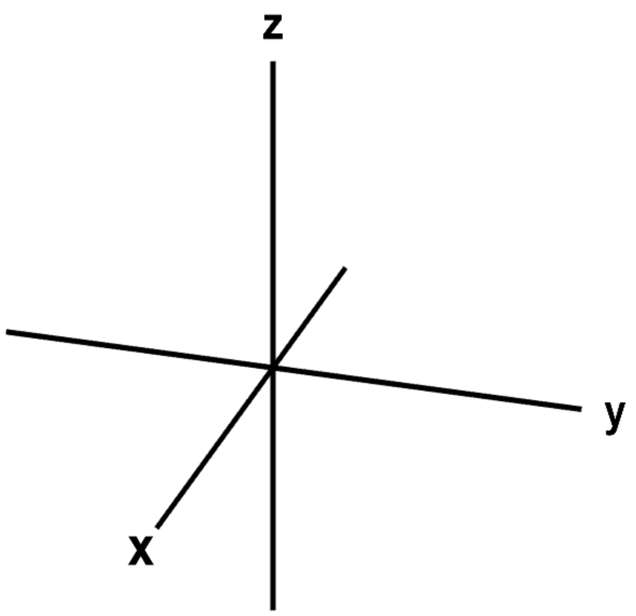
If we integrate in the entire control volume

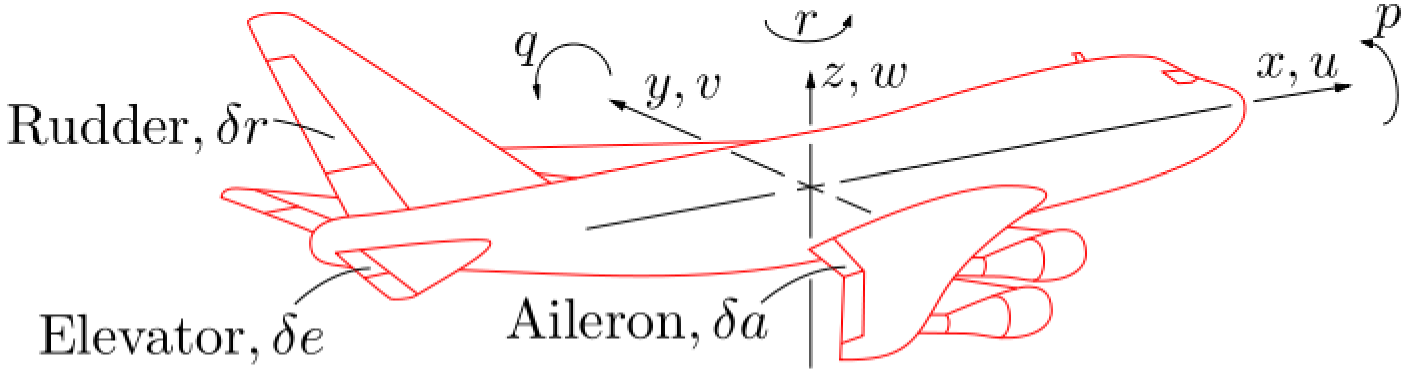
$$\mathit{\Phi} = \oiint \underline{F} \cdot \underline{n}\, dA = \iiint \underline{\nabla} \cdot \underline{F}\, dV$$

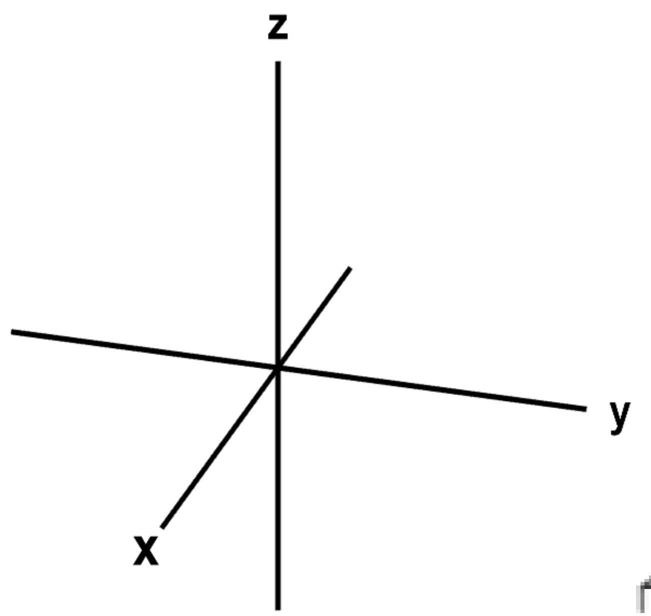
This is called the Divergence theorem

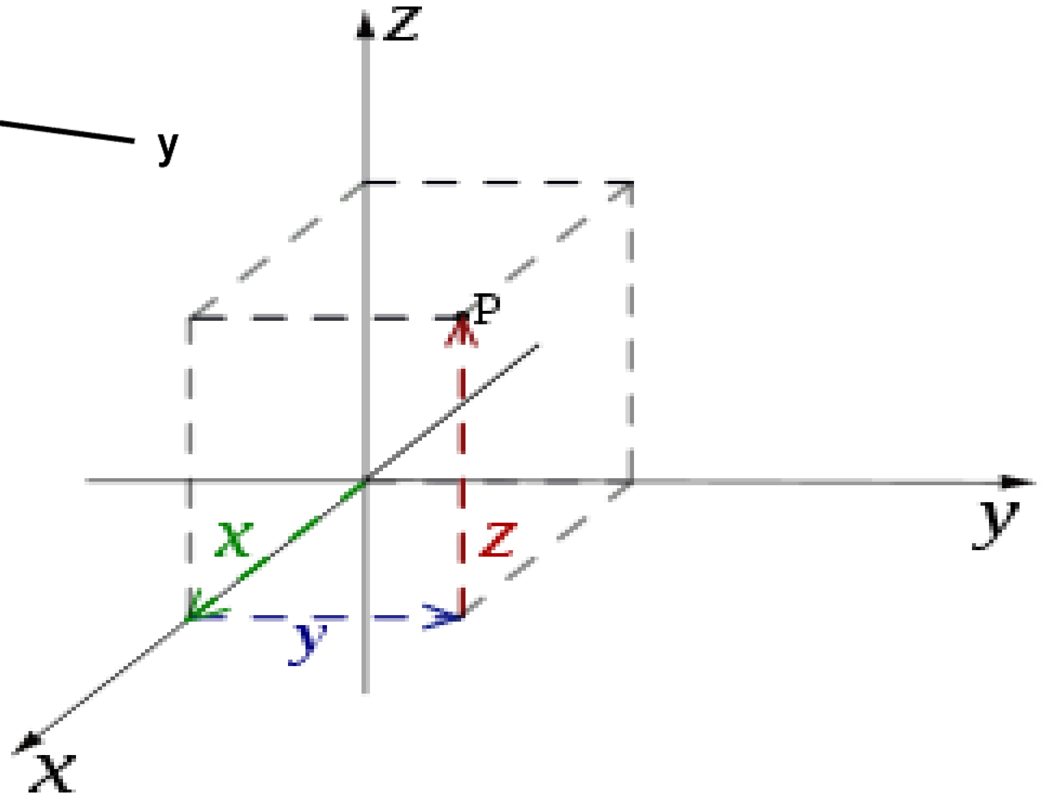
In this case $\mathit{\Phi}$ is the total outward flow of a vector field ($\underline{F}$) through a closed surface. $\underline{F}$ is the vector, i.e. the flux of property

Warning: Remember mathematicians use the term flux instead of flow, please avoid the use of this jargon.


z
y
x


Rudder, δr
q
y, v
r
z, w
x, u
p
Elevator, δe
Aileron, δa


z
y
x


z
P
x
z
y
y
x

$$\frac{\partial\left(\rho\,\underline{v}\right)}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\underline{v}\underline{v}\right)=\underline{\nabla}\cdot\left[\underline{\underline{\tau}}\right]-\underline{\nabla}p+\rho\underline{g}$$

Cauchy

$$\rho\left[\frac{\partial\,\underline{v}}{\partial t}+\underline{v}\cdot\underline{\nabla}\underline{v}\right]=\mu\nabla^{2}\underline{v}-\underline{\nabla}p+\rho\underline{g}$$

Navier-Stokes

$$\frac{\partial\left(\rho\,\underline{v}\right)}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\underline{v}\underline{v}\right)=-\underline{\nabla}p+\rho\underline{g}$$

Euler

$$\mu\nabla^{2}\underline{v}-\underline{\nabla}p=0$$

Stokes

Watch out, azimuth angle is θ en Fig 1, and ϕ in Fig 2

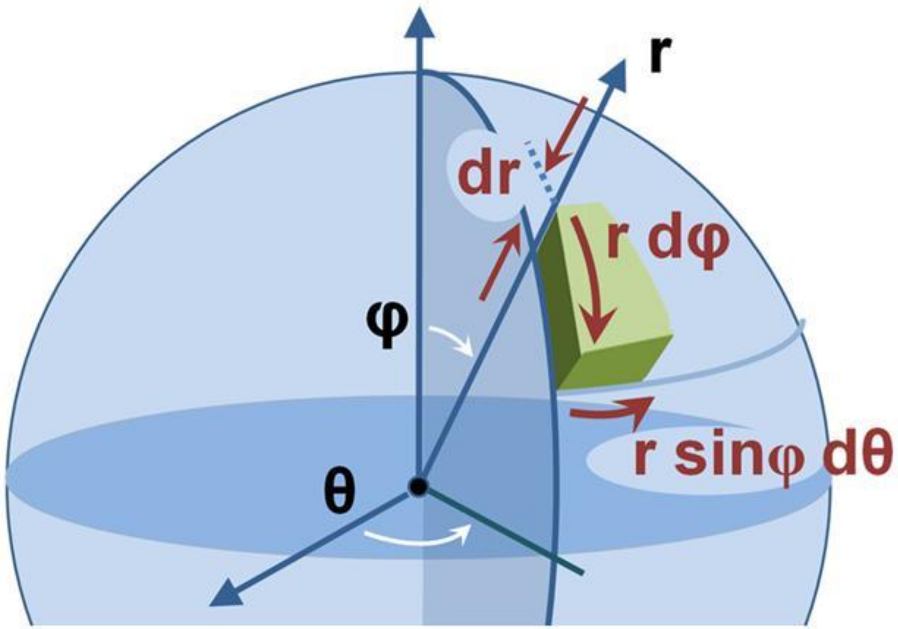


Fig. 1

Θ : azimuth angle

ϕ : zenith angle

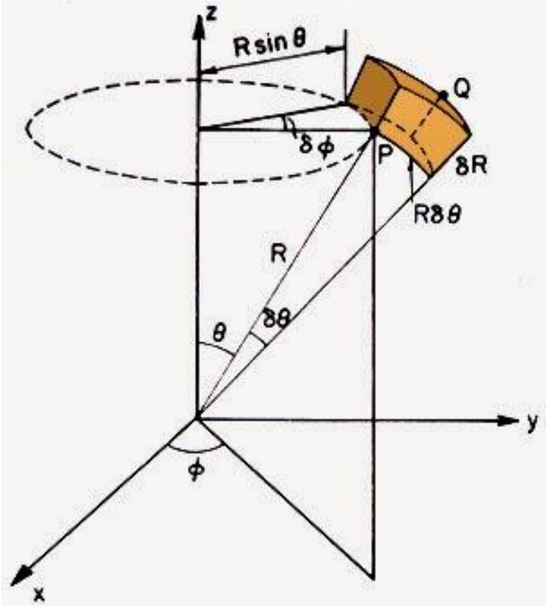


Fig. 2

ϕ : azimuth angle

Θ : zenith angle

Finite Element

1D-PDE

$u = u\,(x,t)$

PDE

$$c\left(x,t,u,\frac{\partial u}{\partial t}\right)\frac{\partial u}{\partial t} = \frac{1}{x^m}\,\frac{\partial}{\partial x}\left(x^m\,f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial t}\right)$$

$$c\,\frac{\partial u}{\partial t} = \frac{1}{x^m}\,\frac{\partial}{\partial x}\left(x^m\,f\right) + s$$

PDE- BC

Left boundary

$$p_L(x_L,t,u_L\,) + q_L(x_L,t)\,\left[f\left(x,t,u,\frac{\partial u}{\partial x}\right)\Bigg|_{x=x_L}\right] = 0$$

$$p_L + q_L\,f\, = 0$$

Right boundary

$$p_R(x_R,t,u_R\,) + q_R(x_R,t)\,\left[f\left(x,t,u,\frac{\partial u}{\partial x}\right)\Bigg|_{x=x_R}\right] = 0$$

$$p_R + q_R\,f\, = 0$$

PDE- IC

$$u(x,t=0) = u_o(x)$$

$$u_o = g$$

Matlab: pdepe (parabolic PDE)

m = 0 for slab, 1 cylinder, 2 sphere

PDE $$\frac{\partial u}{\partial t} = D\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

BC $$u(0,t) = 1 \quad 0 < t < \infty$$

$$\frac{\partial u(L,t)}{\partial x} = 0 \quad 0 < t < \infty$$

IC $$u(0,x) = 0 \quad 0 < x \leq 1$$

```matlab
function Diffusion
% This is the main function. Within this function the meshes are defined,
% PDEPE is called and the results are plotted
clear; close all;

%% Parameters
P(1) = 1; %Diffusion coefficient D
P(2) = 1; %c0
L = 1; %Length of domain
maxt = 1; %Max. simulation time
m = 0; %Parameter corresponding to the symmetry of the problem (see help)
t = linspace(0,maxt,100); %tspan
x = linspace(0,L,100); %xmesh

%%
% Call of PDEPE. It needs the following arguments
% m: see above
% DiffusionPDEfun: Function containg the PDEs
% DiffusionICfun: Function containing the ICs for t = 0 at all x
% DiffusionBCfun: Function containing the BCs for x = 0 and x = L
% x: xmesh and t: tspan
% PDEPE returns the solution as multidimensional array of size
% xmesh x tspan x (# of variables)
sol = pdepe(m,@DiffusionPDEfun,@DiffusionICfun,@DiffusionBCfun,x,t,[],P);
u = sol;

%% Plotting
% 3-D surface plot
```

```
figure(1)
surf(x,t,u,'edgecolor','none');
xlabel('Distance x','fontsize',20,'fontweight','b','fontname','arial')
ylabel('Time t','fontsize',20,'fontweight','b','fontname','arial')
zlabel('Species u','fontsize',20,'fontweight','b','fontname','arial')
axis([0 L 0 maxt 0 P(2)])
set(gcf(), 'Renderer', 'painters')
set(gca,'FontSize',18,'fontweight','b','fontname','arial')

% 2-D line plot
figure(2)
hold all
for n = linspace(1,length(t),10)
    plot(x,sol(n,:),'LineWidth',2)

end
xlabel('Distance x','fontsize',20,'fontweight','b','fontname','arial')
ylabel('Species u','fontsize',20,'fontweight','b','fontname','arial')
axis([0 L 0 P(2)])
set(gca,'FontSize',18,'fontweight','b','fontname','arial')
```

```matlab
function [c,f,s] = DiffusionPDEfun(x,t,u,dudx,P)
% Function defining the PDE

% Extract parameters
D = P(1);
% PDE
c = 1;
f = D.*dudx;
s = 0;
```

```matlab
function u0 = DiffusionICfun(x,P)
% Initial conditions for t = 0; can be a funciton of x
u0 = 0;
```

```matlab
function [pl,ql,pr,qr] = DiffusionBCfun(xl,ul,xr,ur,t,P)
% Boundary conditions for x = 0 and x = L;

% Extract parameters
c0 = P(2);

% BCs: No flux boundary at the right boundary and constant concentration
% the left boundary
pl = ul-c0;     ql = 0;     pr = 0;          qr = 1;
```

Name	Form on 1st part of boundary	Form on 2nd part of boundary
Dirichlet	$y = f$	
Neumann	$\dfrac{\partial y}{\partial n} = f$	
Robin	$c_0 y + c_1 \dfrac{\partial y}{\partial n} = f$	
Mixed	$y = f$	$c_0 y + c_1 \dfrac{\partial y}{\partial n} = f$
Cauchy	both $y = f$, and $c_0 \dfrac{\partial y}{\partial n} = g$	

u is the dependent variable

Five types of boundary conditions are defined at physical boundaries, and a "**zeroth**" type designates those cases with no physical boundaries.

In the equations below the coordinate at the boundary is denoted $\boldsymbol{x}_B$ and B indicates one of the boundaries. **Type 1**. Prescribed field variable (Dirichlet condition): $u = u(x_B, t)$

Type 2. Prescribed flux. Flux produced by potential gradient of field variable u (Neumann condition):

$$\left.\frac{\partial u}{\partial x}\right|_{x=x_B} = \mathrm{g}(x_B, t)$$

B index refers to any boundary where information related to the dependent variable is known (by information I mean the dependent variable, or any other derivative of order "*n*-1", where "*n*" is the order of the differential equation). In the particular case of 1-D, the boundary may be the left or the right boundary.

Here n_i is the outward-facing normal vector on the body surface.

Type 3. Convective boundary condition (sometimes called the Robin condition):

$$k\left.\frac{\partial u}{\partial x}\right|_{x=x_B}+h\,u(x,t)=\mathrm{g}(x_B,t)$$

Type 4. Thin, high-conductivity film at the body surface:

$$k\left.\frac{\partial u}{\partial x}\right|_{x=x_B}=g(x_B,t)-[\rho\,c_p\delta_i]\left.\frac{\partial u}{\partial t}\right|_{x=x_B}$$

Type 5. Thin, high-conductivity film at the body surface, with the addition of convection heat losses from the surface:

$$k\left.\frac{\partial u}{\partial x}\right|_{x=x_B}+h\,u(x,t)=\mathrm{g}(x_B,t)-[\rho\,c_p\delta_i]\left.\frac{\partial u}{\partial t}\right|_{x=x_B}$$

http://www.greensfunction.unl.edu/OrganizeGF/node1.html

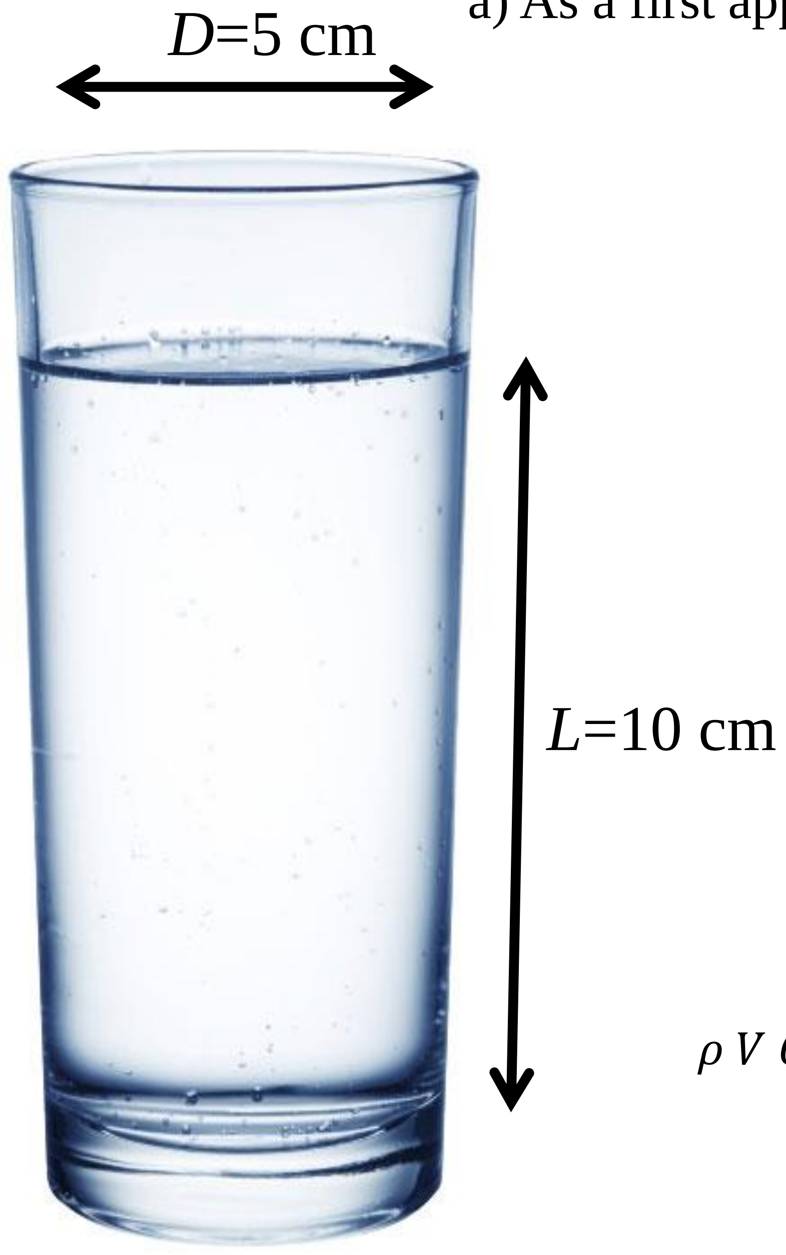
$$k \left. \frac{\partial u}{\partial x} \right|_{x=x_B} + h\, u(x,t) = \mathrm{g}(x_B, t) - \left[\rho\, c_p \delta_i\right] \left. \frac{\partial u}{\partial t} \right|_{x=x_B}$$

Boundary condition	Typical boundary condition
Zeroth kind	$u = u(x \to \infty, t) = f_\infty = finite\ value$
First kind	$u = u(x_B, t) = f(t)$
Second kind	$\left.\frac{\partial u}{\partial x}\right\|_{x=x_B} = g(x_B, t)$
Third kind	$k \left.\frac{\partial u}{\partial x}\right\|_{x=x_B} + h\, u(x,t) = g(x_B, t)$
Fourth kind	$k \left.\frac{\partial u}{\partial x}\right\|_{x=x_B} = g(x_B, t) - [\rho\, c_p \delta_i] \left.\frac{\partial u}{\partial t}\right\|_{x=x_B}$
Fifth kind	$k \left.\frac{\partial u}{\partial x}\right\|_{x=x_B} + h\, u(x,t) = g(x_B, t) - [\rho\, c_p \delta_i] \left.\frac{\partial u}{\partial t}\right\|_{x=x_B}$

Differential equations
And
System of ODEs

A glass of water, is standing over an insulated table in contact with the dry environment. Calculate the evolution of temperature and mass as a function of time. The initial temperature of water is 85°C, you can assume that the mass of the glass is negligible compared with the mass of water

a) As a first approach you can assume there is no evaporation.



T_{air}=298.15 K

ρ = 997 kg/m3

C_p=4187 J/kg-K

h_o=10 W/m^2-K

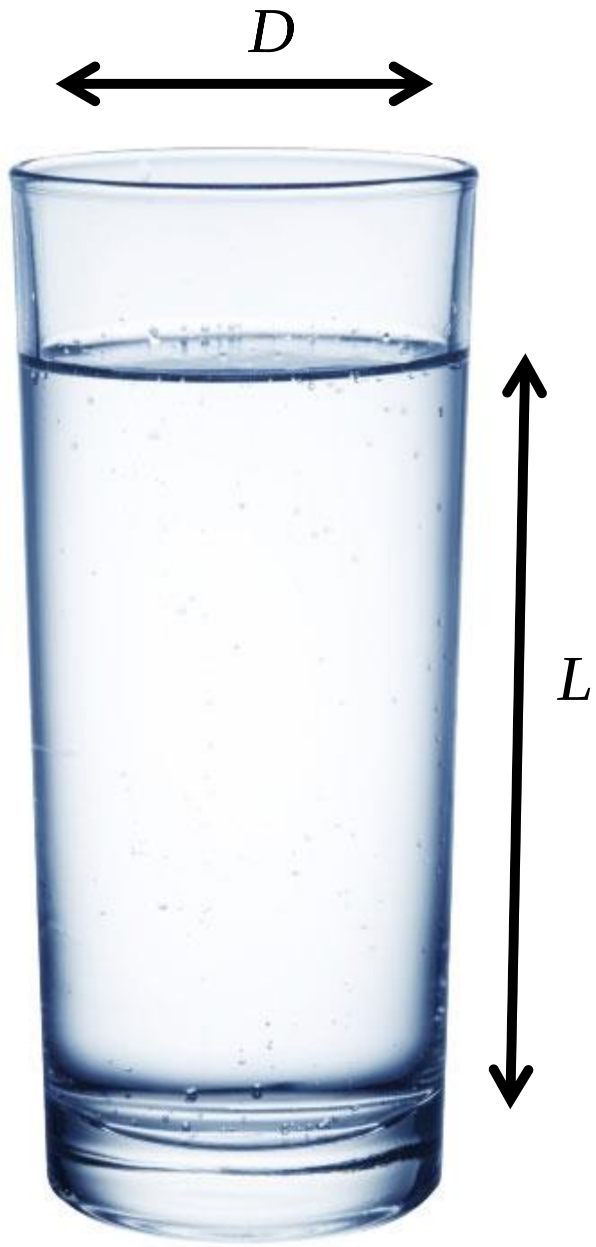
ε = 0.92

σ = 5.67×10^{-8} W/m^2-K^4

T_{walls}=298.15 K

The mathematical model has the form:

$$\rho\, V\, C_p \frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right]$$



b) As a second approach approach you can the air temperature in the surroundings has a periodic behavior of the form:

$$T_{air} = T_a + \Delta T_a \sin\left(\frac{2\pi t}{t_p}\right)$$

T_a=293.15 K ΔT_a=5 K t_p=86 400 s

The model keeps the same form

$$\rho\, V\, C_p \frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right]$$

c) If evaporation is important, the model has the form:

$$m\,C_p\frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right] - h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}\lambda$$

$$\frac{dm}{dt} = -h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}$$

Repeat calculations using the data and parameters given below

$$\ln(P^{sat}) = A_A - \frac{B_A}{T + C_A}$$

A_A=23.4843; B_A=3991.44; C_A= -39.3353

M_w=18 kg/kmol $\quad$ R=8314.34 Pa-m^3/kmol-K

h_m=2.84×10^{-3} m/s $\quad$ λ =2 230 000 J/kg

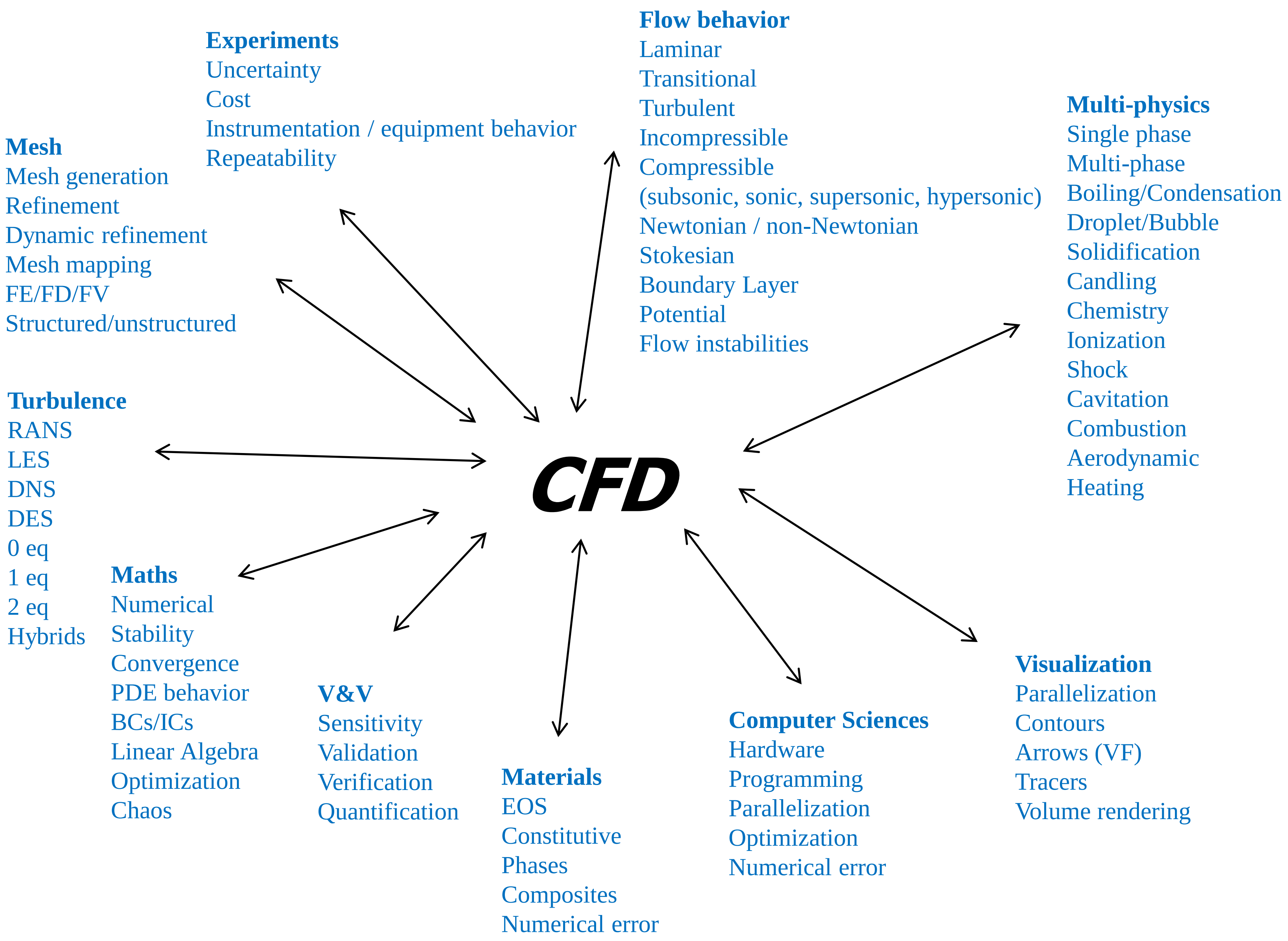
y_w=0

$$\rho = \frac{m}{V}$$

$$A = \frac{\pi D^2}{4} + \pi D L$$

$$A_M = \frac{\pi D^2}{4}$$

$$V = \frac{\pi D^2}{4} L$$



Sal Rodriguez *Applied Computational Fluid Dynamics and Turbulence Modeling*. *Springer*

https://doi.org/10.1007/978-3-030-28691-0

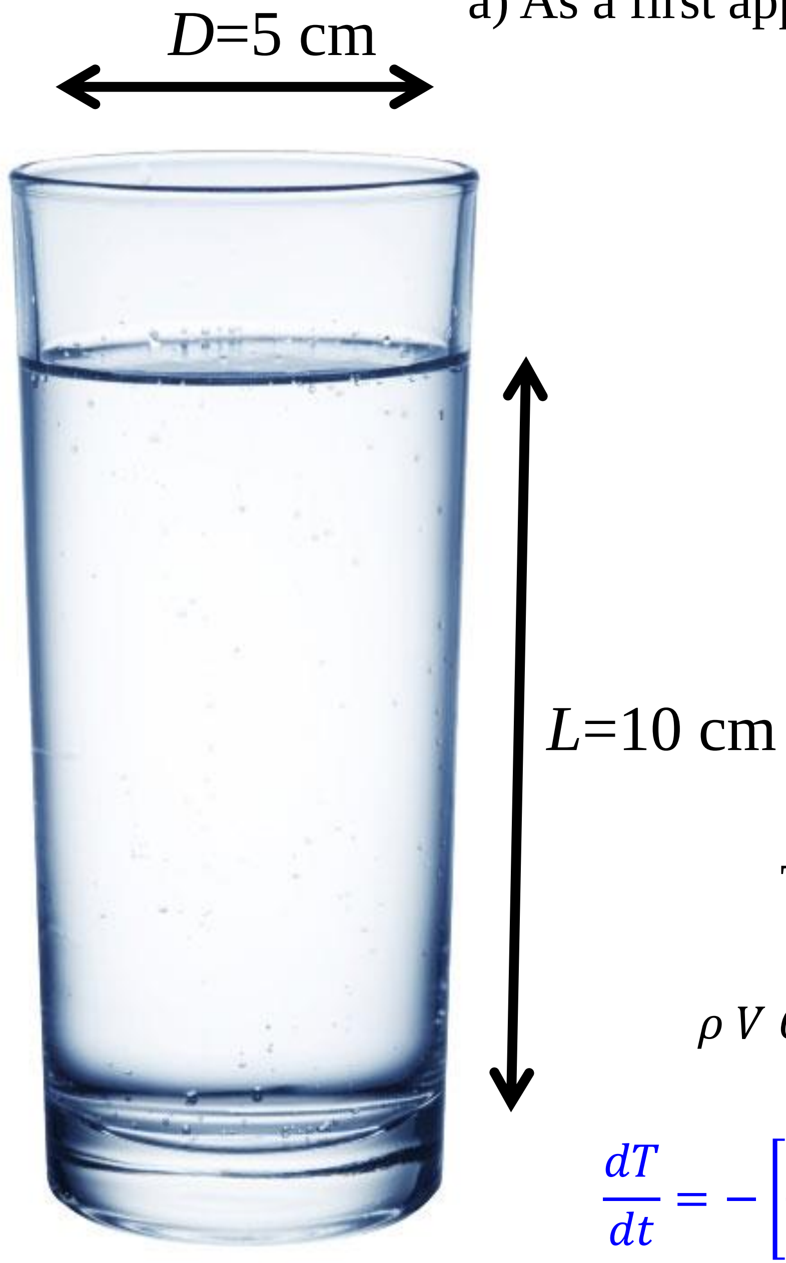
Direct Numerical Solution (DNS)

Reynolds-averaged Navier-Stokes equations (RANS)
Large eddy simulation equations (LES)

Unsteady Reynolds-averaged Navier-Stokes
(URANS)

Differential equations
And
System of ODEs

Problem 1. A glass of water, is standing over an insulated table in contact with the dry environment. Calculate the evolution of temperature and mass as a function of time. The initial temperature of water is 85°C, you can assume that the mass of the glass is negligible compared with the mass of water



a) As a first approach you can assume there is no evaporation.

T_{air}=298.15 K

ρ = 997 kg/m3

C_p=4187 J/kg-K

h_o=10 W/m^2-K

ε = 0.92

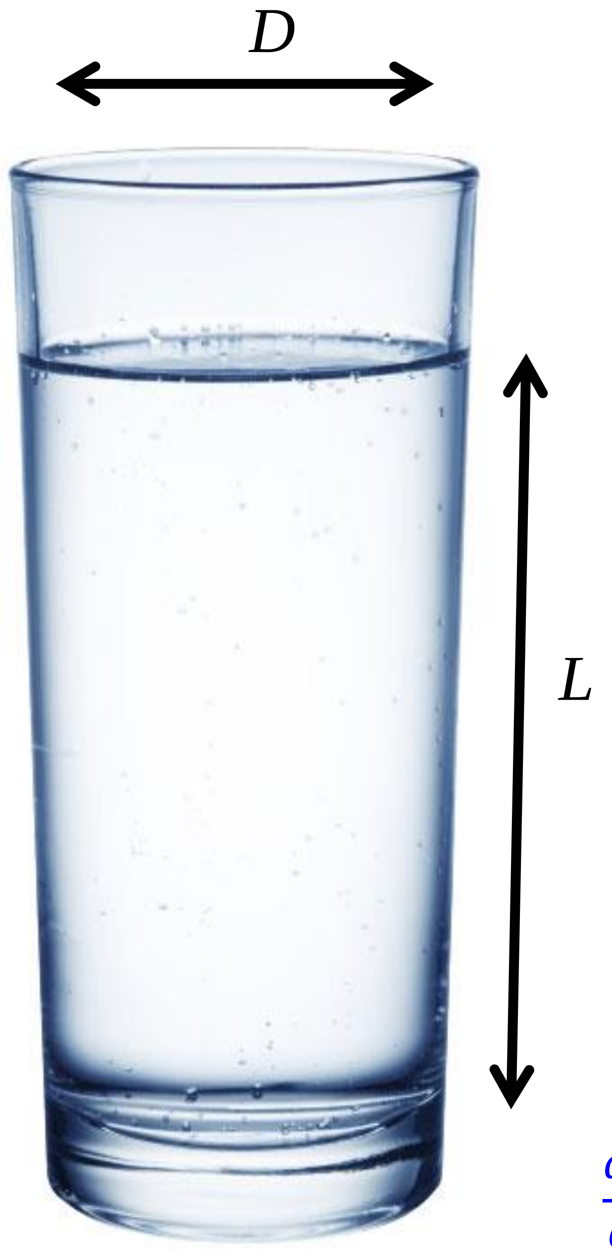
σ = 5.67×10^{-8} W/m^2-K^4

T_{walls}=298.15 K

The mathematical model has the form:

$$\rho\, V\, C_p \frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right]$$

$$\frac{dT}{dt} = -\left[\frac{h_o A[T - T_{air}] + \sigma\varepsilon A\left[T^4 - T_{walls}^4\right]}{\rho\, V\, C_p}\right] = F_1(T)$$



b) As a second approach approach you can the air temperature in the surroundings has a periodic behavior of the form:

$$T_{air} = T_a + \Delta T_a \sin\left(\frac{2\pi t}{t_p}\right)$$

T_a=293.15 K ΔT_a=5 K t_p=86 400 s

The model keeps the same form

$$\rho\, V\, C_p \frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right]$$

$$T_{air} = T_a + \Delta T_a \sin\left(\frac{2\pi t}{t_p}\right)$$

$$\frac{dT}{dt} = -\left[\frac{h_o A[T - T_{air}] + \sigma\varepsilon A\left[T^4 - T_{walls}^4\right]}{\rho\, V\, C_p}\right] = F_1(T,t)$$

c) If evaporation is important, the model has the form:

$$m\,C_p\frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right] - h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}\lambda$$

$$\frac{dm}{dt} = -h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}$$

Repeat calculations using the data and parameters given below

$$\ln(P^{sat}) = A_A - \frac{B_A}{T + C_A}$$

A_A=23.4843; B_A=3991.44; C_A= -39.3353

M_w=18 kg/kmol R=8314.34 Pa-m^3/kmol-K

h_m=0.01 m/s λ =2 230 000 J/kg

y_w=0

$$\rho = \frac{m}{V}$$

$$A = \frac{\pi D^2}{4} + \pi D L$$

$$A_M = \frac{\pi D^2}{4}$$

$$V = \frac{\pi D^2}{4} L$$

$$m\ C_p\frac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A\left[T^4 - T_{walls}^4\right] - h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}\lambda$$

$$\frac{dm}{dt} = -h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}$$

$$\frac{dT}{dt} = -\left[\frac{h_o A[T - T_{air}] + \sigma\varepsilon A\left[T^4 - T_{walls}^4\right] + h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}}\lambda}{m\ \ C_p}\right] = F_1(T, m, t)$$

$$\frac{dm}{dt} = -h_m A_M M_w \frac{[p^{sat} - y_w p_{air}]}{RT_{air}} = F_2(T, m, t)$$

$$\frac{dy_1}{dt} = F_1(y_1, y_2, t)$$

$$\frac{dy_2}{dt} = F_2(y_1, y_2, t)$$

Matlab code

Time span

Initial conditions

Call or retrieve of the parameters

Integration of the differential equations

```
function[Yo,time]=GlassofWater()
    tspan=linspace(0,36600,3600);
    To=85+273.15;
    mo=0.1958;
    yo=[To,mo]';
    time0=0;
    [par]=propsf();
% par=[D,rho,cp,tp,DT,Ta,R,Mm,yw,ho,p,hm,lambda,Aa,Ba,Ca,sigma,epsilon,....]
    [t,Y]=ode45(@fun,tspan,yo,[],par);
    time=t;
    subplot(2,1,1)
    plot(t/3600,Y(:,1));
    xlabel('time [h]');ylabel('Temperature[K]');
    subplot(2,1,2);
    plot(t/3600,Y(:,2));
    xlabel('time [h]');ylabel('mass [kg]');
    Yo=Y;
end
```

Main program

```
% ====Differential equations
% ydot=vector of the set of diffrential equations
% ydot(1)=dy(1)/dt, ydot(2)=dy(2)/dt, ydot(3)=dy(3)/dt, ...ydot(N)=dy(N)/dt
% t = independent vriable , in this case time
% par= vector with all the parameters needed (fixed values)
function[ydot]=fun(x,y,par)
   time=x;
   T=y(1);
   m=y(2);
   %===========================================
   % Parmaeters
    D=par(1);
   rho=par(2);    cp=par(3);    tp=par(4);
   DT=par(5);     Ta=par(6);    R=par(7);
   Mw=par(8);     yw=par(9);      ho=par(10);
   p=par(11);      hm=par(12);  lambda=par(13);
   Aa=par(14);    Ba=par(15);   Ca=par(16);
   sigma=par(17);     epsilon=par(18);
   %===========================================
   % Volume of liquid within the glass (V), area for mass transfer (Am) and
   % water depth (L)
   V=m/rho;
   Am=pi*D^2/4;
   L=V/Am;
   % Saturation pressure of the liquid
   psat=exp(Aa-Ba/(T+Ca));
   % Air temperature during the day
   Tair=Ta+DT*sin(2*pi*time/tp);
   % Area for heat transfer
   A=pi*D^2/4+pi*D*L;
   Twall=Tair;
   %=========== Actual Differential Equations================
   ydot(2)=-hm*Am*Mw*(psat-yw*p)/(R*Tair);
   ydot(1)=-ho*A*(T-Tair)-sigma*epsilon*A*(T^4-Twall^4)+lambda*ydot(2);
   ydot(1)=ydot(1)/(m*cp);
   ydot=ydot';
end
```

$$\ln(P^{sat}) = A_A - \frac{B_A}{T + C_A}$$

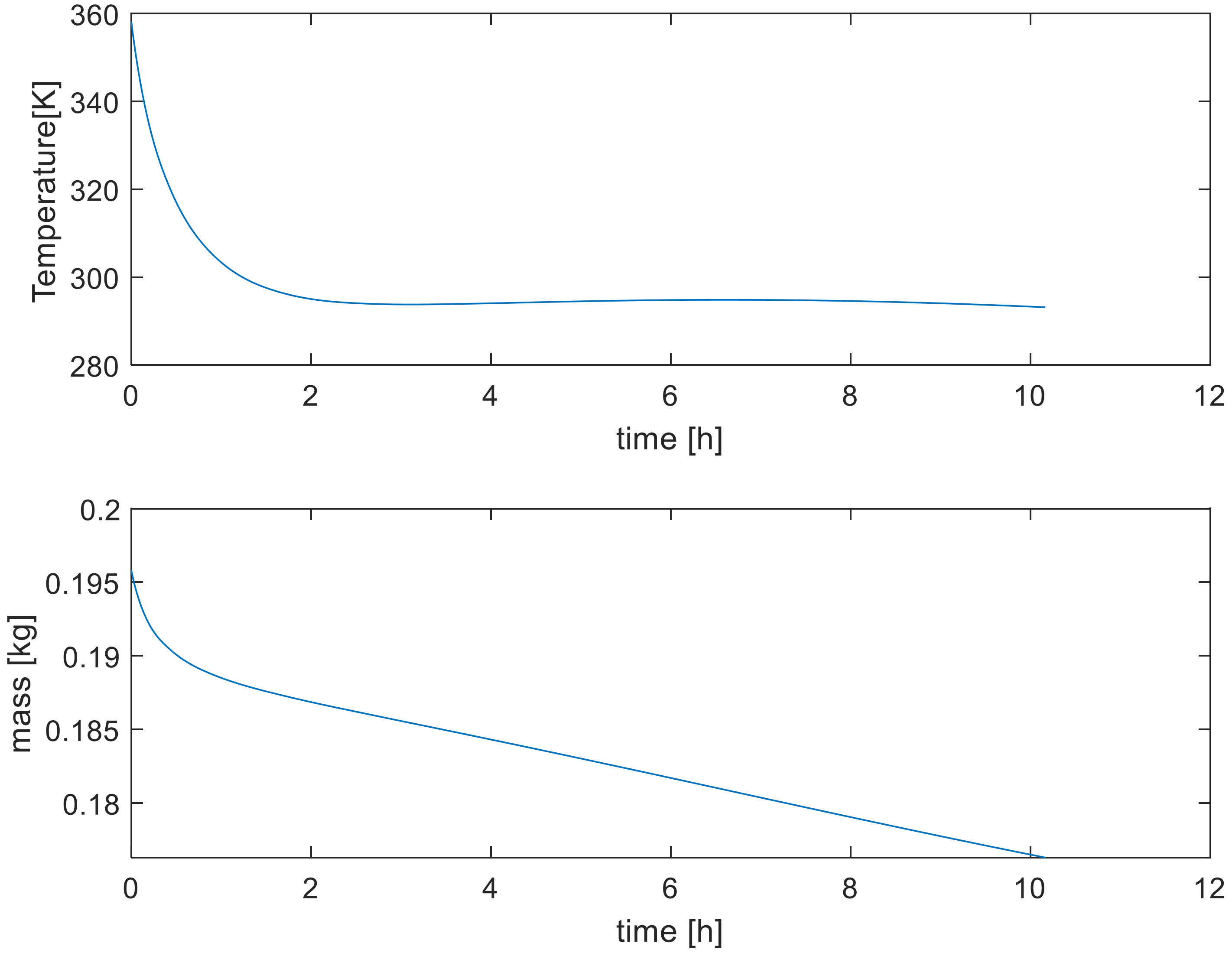
$$T_{air} = T_a + \Delta T_a \sin\left(\frac{2\pi t}{t_p}\right)$$

Function to specify the Differential equations

```
function[par]=propsf()
   % Geometrical parameters
   D=5e-2;         L=10e-2;
   Gp=[D; L];
   % D=Gp(1);L=Gp(2);
   % Physical Properties
   rho=997;        cp=4187.0;
   lambda=2.23e6;  Mw=18;
   Aa=23.4843;     Ba=3991.44;   Ca=-39.3356;
   epsilonw=0.92;
   Pp=[rho; cp; lambda; Mw; Aa; Ba; Ca; epsilonw];
   DT=5;           Ta=293.15;
   yw=0.0;         ho=10;
   p=101325;       hm=0.01;
   epsilon=0.92;   tp=86400;
   Mp=[DT; Ta; yw; ho; p; hm; epsilon; tp];
   sigma=5.67e-8;  R=8314.34;
   Pc=[R ; sigma];
   %===========================
   % Parameters
   par=[D rho cp tp];
   par=[par DT Ta R Mw];
   par=[par yw ho p hm];
   par=[par lambda Aa Ba Ca];
   par=[par sigma epsilon];
end
```

Function to retrieve of the parameters

$$\begin{cases} \dfrac{dm}{dt} = -h_m A_M M_w \dfrac{[p^{sat} - y_w p_{air}]}{R T_{air}} \\ m\, C_p \dfrac{dT}{dt} = -h_o A[T - T_{air}] - \sigma\varepsilon A[T^4 - T_{walls}^4] - h_m A_M M_w \dfrac{[p^{sat} - y_w p_{air}]}{R T_{air}} \lambda \end{cases}$$


360
340
320
300
280
Temperature[K]
0
2
4
6
8
10
12
time [h]
0.2
0.195
0.19
0.185
0.18
mass [kg]
0
2
4
6
8
10
12
time [h]

Problem 2. The growth and collapse of bubbles is given by the equation given below

$$\ddot{y}y + \frac{3}{2}(\dot{y})^2 = -B(\tau) + \frac{1}{y^{3\,k}} \qquad \ddot{y} = \frac{d^2y}{d\tau^2} \qquad \dot{y} = \frac{dy}{d\tau}$$

Here y is the ratio of the actual radius of the bubble and the initial radius of the bubble, and τ is the dimensionless time

$$y = \frac{R}{R_0} \qquad \tau = \frac{t}{R_0\sqrt{\rho/p_0}}$$

The initial conditions are: $\left. y\right|_{\tau=0} = 1$ and $\left. \dot{y}\right|_{\tau=0} = 0$

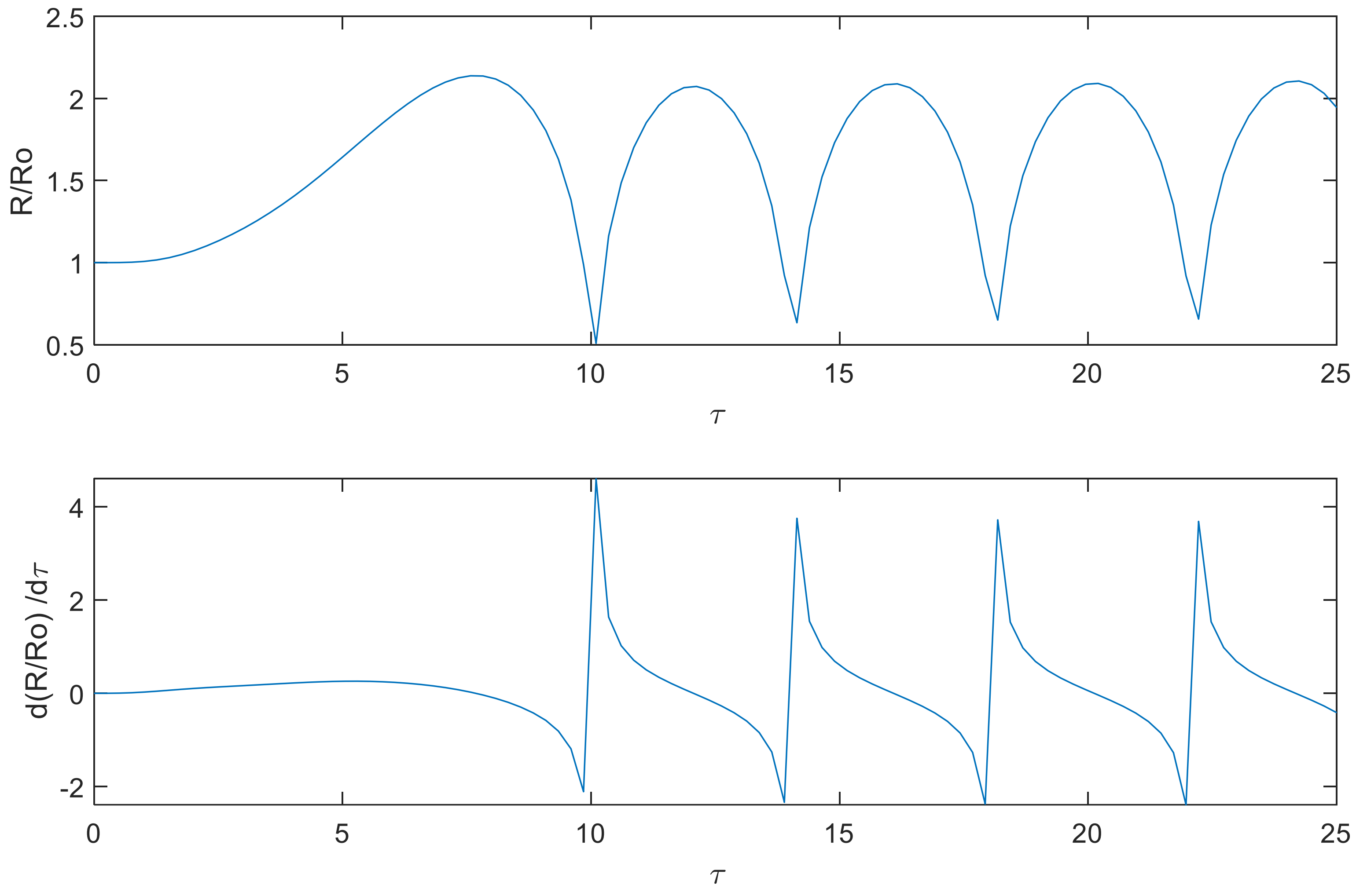
For a pressure perturbation given by:

$$B(\tau) = \begin{cases} (1 + \cos(\pi\,\tau/5))/2 & 0 \le \tau \le 10 \\ 1 & \text{otherwise} \end{cases}$$

Find how the air bubbles (k=1.4) radius evolves with time:

```matlab
function[Yout,time]=Bubble()
tspan=linspace(0,25,100);
    y1=1;
    y2=0;
    yo=[y1,y2]';
    par=1.4;
    [t,Y]=ode45(@fun,tspan,yo,[],par);
    time=t;
    Yout=Y;
    subplot(2,1,1);
    plot(t,Y(:,1));
    xlabel('\tau');ylabel('R/Ro');
    subplot(2,1,2);
     plot(t,Y(:,2));
    xlabel('\tau');ylabel('d(R/Ro) /d\tau');
end
```

```matlab
function[ydot]=fun(x,y,par)
    k=par;
    tau=x;
    B=(1+cos(pi*tau/5))/2;
    if tau>10
        B=1;
    end
    ydot(1)=y(2);
    ydot(2)=(-B+1/(y(1))^(3*k)-(3/2)*y(2)^2)/y(1);
    ydot=ydot';
end
```



2.5
2
1.5
1
0.5
R/Ro
0
5
10
15
20
25
τ
4
2
0
-2
d(R/Ro) /dτ
0
5
10
15
20
25
τ

The previous equation was a simplification of a more complex equation known as Rayleigh-Plesset

$$\ddot{R}R + \frac{3}{2}\left(\dot{R}\right)^2 + \frac{4\,\mu\,\dot{R}}{\rho\,R} + \frac{2\,\sigma}{\rho\,R} = \frac{\Delta p}{\rho} + \frac{p_0^g}{\rho}\left(\frac{R_0}{R}\right)^{3k}$$

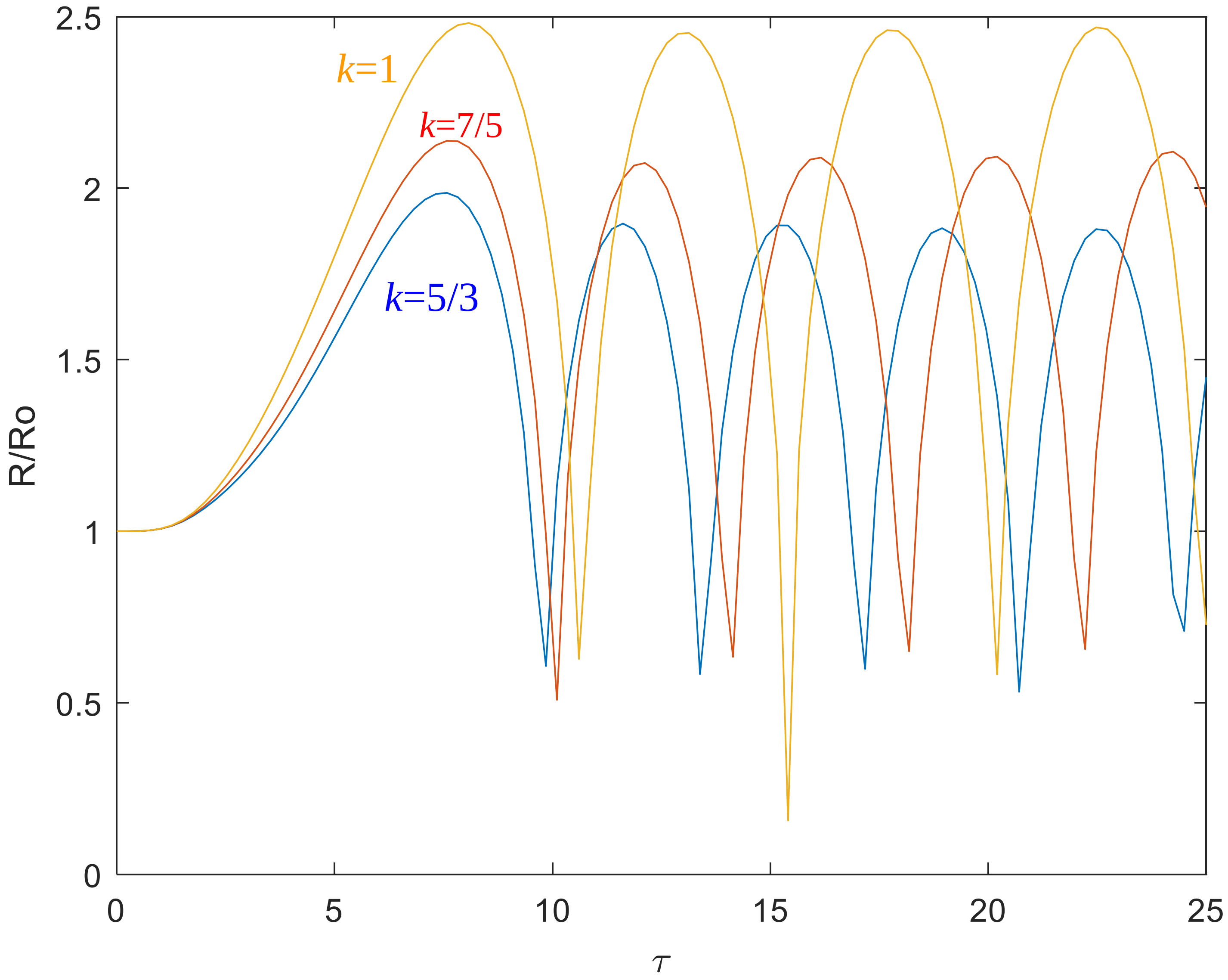
Explain the effect that viscosity and surface tension have over the evolution of radius

$$\ddot{y}y + \frac{3}{2}(\dot{y})^2 + \alpha\frac{\dot{y}}{y} + \beta\left(\frac{1}{y}\right) = -B(\tau) + \frac{1}{y^{3\,k}}$$

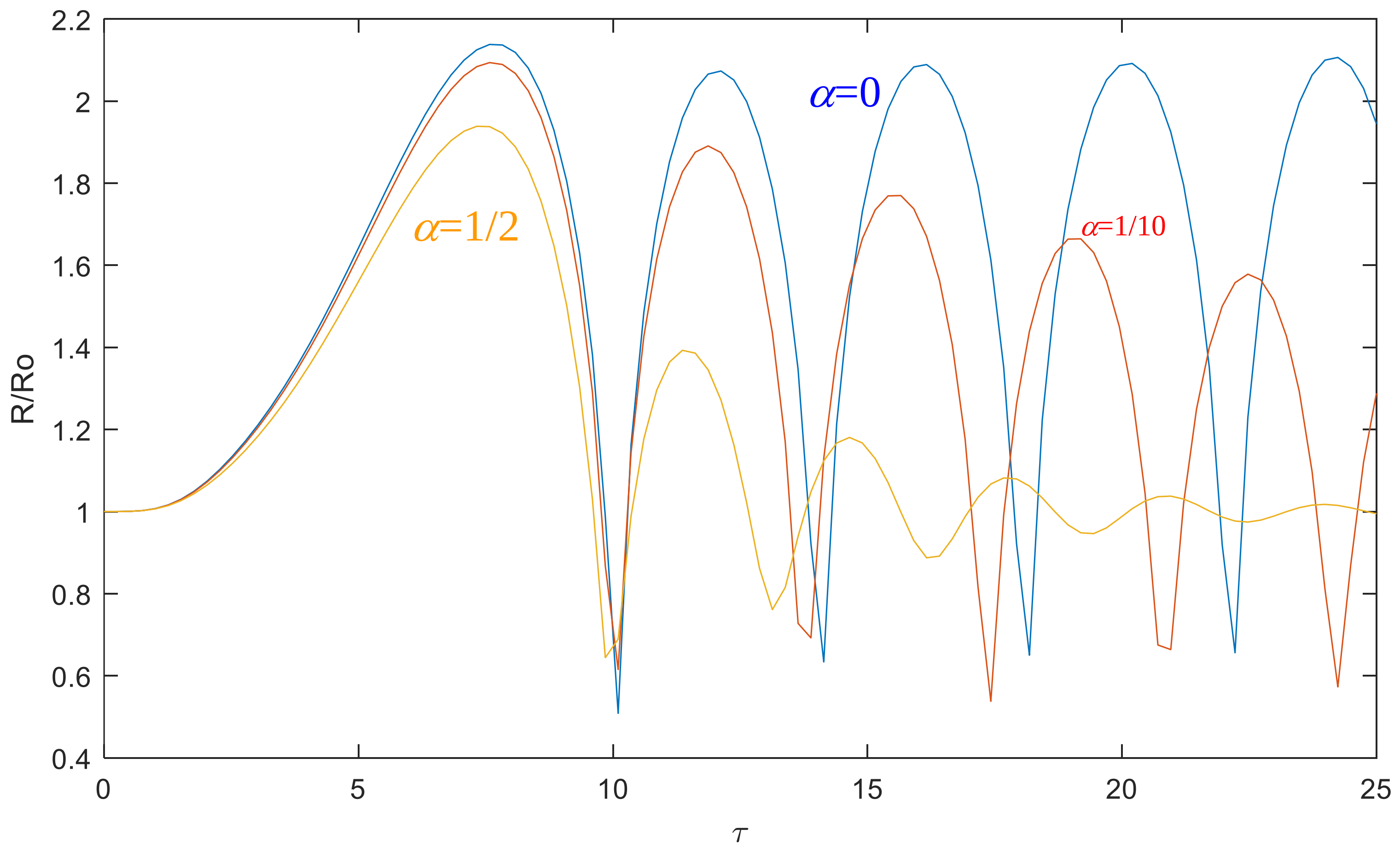
```
function[Yout,time]=Bubble()
tspan=linspace(0,35,500);
    y1=1;
    y2=0;
    yo=[y1,y2]';
    % k=1.4 adiabatic process, k=1 isothermic
    % alphaM=0 inviscid
    % betaM =0 negligible surface tension
    k=1.4;alphaM=0.1;betaM=0.1;
    par(1)=k;
    par(2)=alphaM;
    par(3)=betaM;
    [t,Y]=ode45(@fun,tspan,yo,[],par);
    time=t;
    Yout=Y;
    subplot(2,1,1);
    plot(t,Y(:,1));
    xlabel('\tau');ylabel('R/Ro');
    subplot(2,1,2);
     plot(t,Y(:,2));
    xlabel('\tau');ylabel('d(R/Ro) /d\tau');
end
```

```
function[ydot]=fun(x,y,par)
    k=par(1);
    a1=par(2);
    b1=par(3);
    tau=x;
    B=(1+cos(pi*tau/5))/2;
    if tau>10
        B=1;
    end
    ydot(1)=y(2);
    ydot(2)=(-B+1/(y(1))^(3*k)-(3/2)*y(2)^2-a1*y(2)/y(1)-b1/y(1))/y(1);
    ydot=ydot';
    end
```

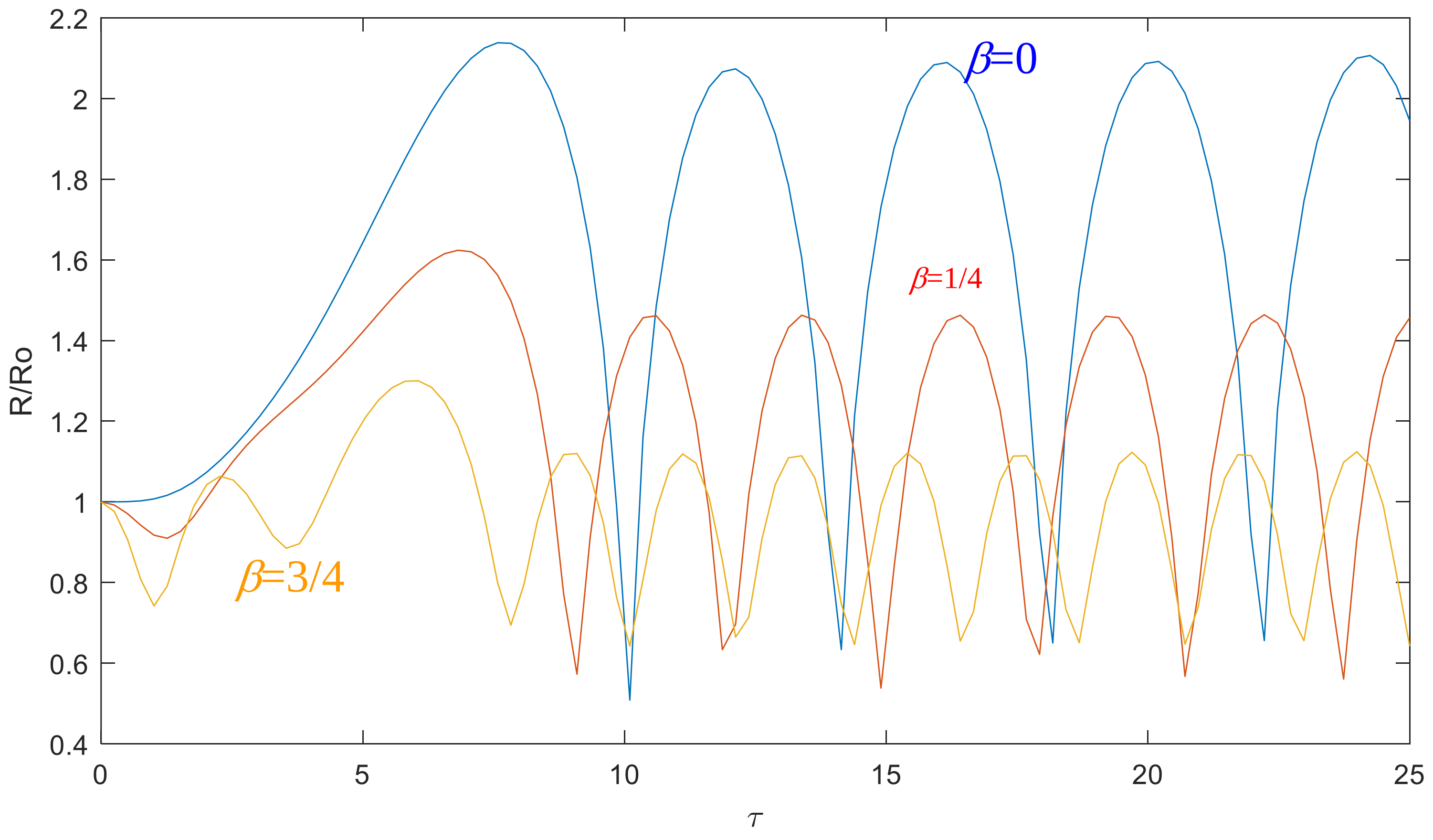
Gas molecule shape and size effect (α=0,β=0)



Effect of the viscosity (k=7/5, β=0)



Effect of the surface tension (k=7/5, α=0)



Problem 3. Some complex system of partial differential equations can be converted into a set of strongly non-linear system of ODEs with boundary conditions, and under these simplification the shooting Newton Raphson method can be used to solve the problem, and in some instances a continuation approach is suggested to get the desired convergence. Solve the following PDEs using the description given in the problem statement. (Give the solution for *Pr* =0.1, 1.0 and 10.0)

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

Mass conservation

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2} + g\beta(T - T_\infty)$$

Linear momentum conservation

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2}$$

Energy conservation

This set of differential equations represents the natural convection of heat transfer In a vertical wall

$$\frac{\partial(\rho)}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$

$$\frac{\partial(\rho \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}\ \underline{v}) = \rho \underline{g} - \underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}}$$

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \frac{\mu}{\rho} \frac{\partial^2 v_x}{\partial y^2} + g\beta(T - T_\infty)$$

$$\frac{\partial[\rho \hat{E}]}{\partial t} + \underline{\nabla} \cdot [\rho\ \underline{v}\ \hat{E}] = \underline{\nabla} \cdot \left[\underline{\underline{\tau}} \cdot \underline{v}\right] - \underline{\nabla} \cdot \left[p\ \underline{\underline{I}} \cdot \underline{v}\right] + k\nabla^2 T + \dot{Q}_v$$

$$v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2}$$

The boundary conditions are:

$$v_x\Big|_{y=0} = 0 \qquad f'(0) = 0$$

$$v_y\Big|_{y=0} = 0 \qquad f(0) = 0$$

$$T\Big|_{y=0} = T_0 \qquad \theta(0) = 0$$

$$v_x\Big|_{y\to\infty} = 0 \qquad f'(\infty) = 0$$

$$T\Big|_{y\to\infty} = T_\infty \qquad \theta(\infty) = 1$$

$$v_x\Big|_{x=0} = 0 \qquad f'(\infty) = 0$$

$$T\Big|_{x=0} = T_\infty \qquad \theta(\infty) = 1$$

Using the following transformation rules you end up with:

$$C = \left[\frac{g\beta\rho^2(T_s - T_\infty)}{4\,\mu^2}\right]^{1/4} \qquad \eta = C\,\frac{y}{x^{1/4}}$$

$$v_x = 4\frac{\mu}{\rho}C^2x^{1/2}f' \qquad v_y = \frac{\mu}{\rho}Cx^{-1/2}[\eta f' - 3f]$$

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$f''' + 3ff'' - 2(f')^2 + 1 - \theta = 0$$

$$\theta'' + 3\,\mathrm{Pr}\,f\,\theta' = 0$$

The boundary conditions are:

$$\eta = C\frac{y}{x^{1/4}}$$

$$f = f(\eta)$$

$$f''' + 3ff'' - 2(f')^2 + 1 - \theta = 0$$

$$\theta = \theta(\eta)$$

$$\theta'' + 3\,\mathrm{Pr}\,f\,\theta' = 0$$

You can solve the previous system, but we will scale up to make easier the solution technique

$$\theta = s\,\Theta$$

$$f = s\,F \qquad f' = s^2\,F' \qquad f'' = s^3\,F'' \qquad f''' = s^4\,F'''$$

$$\zeta = s\,\eta$$

$$s = 1/\Theta(\infty) = 1/\Theta_\infty$$

The final equations are: $F = F(\zeta) \quad \Theta = \Theta(\zeta)$

$$F''' + 3FF'' - 2(F')^2 + \Theta_\infty^3(\Theta_\infty - \Theta) = 0$$

$F(0) = 0 \quad F'(0) = 0 \quad F'(\infty) = 0$

$$\Theta'' + 3\,\mathrm{Pr}\,F\,\Theta' = 0$$

$\Theta(0) = 0 \quad \Theta'(0) = 1$

In canonical form the equations will look in the form:

$$F''' + 3FF'' - 2(F')^2 + \Theta_\infty^3(\Theta_\infty - \Theta) = 0 \qquad F(0) = 0 \quad F'(0) = 0 \quad F'(\infty) = 0$$

$$\Theta'' + 3\,\mathrm{Pr}\,F\,\Theta' = 0 \qquad \Theta(0) = 0 \quad \Theta'(0) = 1$$

Canonical:

$$y_1' = y_2 \qquad y_1(0) = 0$$

$$y_2' = y_3 \qquad y_2(0) = 0 \qquad y_2(\infty) = 0$$

$$y_3' = 2y_2^2 - 3y_1y_3 + \beta^3(y_4 - \beta) \qquad y_3(0) = \alpha$$

$$y_4' = y_5 \qquad y_4(0) = 0 \qquad y_4(\infty) = \beta$$

$$y_5' = -3\,Pr\,y_1y_5 \qquad y_5(0) = 1$$

We have 5 known boundary conditions, one unknown left condition and one right boundary condition, this problem can be solved by iteration using initial conditions instead, and generate two residuals and use Newton-Raphson shooting technique to make this residual set to zero :

Canonical:

$$y_1' = y_2 \qquad y_1(0) = 0$$

$$y_2' = y_3 \qquad y_2(0) = 0 \qquad R_2(\alpha, \beta) = y_2(\infty) - 0$$

$$y_3' = 2y_2^2 - 3y_1y_3 + \beta^3(y_4 - \beta) \qquad y_3(0) = \alpha$$

$$y_4' = y_5 \qquad y_4(0) = 0 \qquad R_4(\alpha, \beta) = y_4(\infty) - \beta$$

$$y_5' = -3\,Pr\,y_1y_5 \qquad y_5(0) = 1$$

You can guess for α and β, and use the residuals to guide the iteration process, and update the values for the unknowns:

$$R_2 = R_{2,0} + \left(\frac{dR_2}{d\alpha}\right)(\alpha - \alpha_0) + \left(\frac{dR_2}{d\beta}\right)(\beta - \beta_0) = 0$$

$$R_4 = R_{4,0} + \left(\frac{dR_4}{d\alpha}\right)(\alpha - \alpha_0) + \left(\frac{dR_4}{d\beta}\right)(\beta - \beta_0) = 0$$

$$\left(\frac{dR_2}{d\alpha}\right)(\alpha-\alpha_0)+\left(\frac{dR_2}{d\beta}\right)(\beta-\beta_0)=-R_{2,0}$$

$$\left(\frac{dR_4}{d\alpha}\right)(\alpha-\alpha_0)+\left(\frac{dR_4}{d\beta}\right)(\beta-\beta_0)=-R_{4,0}$$

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}-\begin{bmatrix}\frac{dR_2}{d\alpha} & \frac{dR_2}{d\beta}\\ \frac{dR_4}{d\alpha} & \frac{dR_4}{d\beta}\end{bmatrix}^{-1}\begin{bmatrix}R_{2,0}\\ R_{4,0}\end{bmatrix} \qquad \begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix}-\begin{bmatrix}y_7 & y_{12}\\ y_9 & (y_{14}-1)\end{bmatrix}^{-1}\begin{bmatrix}R_{2,0}\\ R_{4,0}\end{bmatrix}$$

1. You can guess for α and β.
2. Integrate the system of equations using a RK type algorithm
3. Calculate the residuals, the residual derivatives and update the guess for α and β, and repeat the algorithm until residuals are equal to zero.

To minimize evaluations, you can use parametric equations to estimate the derivatives of the residuals, and the algorithm will be generalized in the form

$$y_1' = f_1$$
$$y_2' = f_2$$
$$y_3' = f_3$$
$$y_4' = f_4$$
$$y_5' = f_5$$

$$y_i' = f_i(y_1, y_2, y_3, y_4, y_5, \beta)$$

Using the chain rule of derivations, and analyzing the variation of the functions respect to the unknowns we have:

$$\frac{df_i}{d\alpha} = \frac{\partial f_i}{\partial y_1}\frac{dy_1}{d\alpha} + \frac{\partial f_i}{\partial y_2}\frac{dy_2}{d\alpha} + \frac{\partial f_i}{\partial y_3}\frac{dy_3}{d\alpha} + \frac{\partial f_i}{\partial y_4}\frac{dy_4}{d\alpha} + \frac{\partial f_i}{\partial y_5}\frac{dy_5}{d\alpha} + \frac{\partial f_i}{\partial \beta}\frac{d\beta}{d\alpha}$$

$$\frac{df_i}{d\alpha} = \frac{d}{d\alpha}\left(\frac{dy_i}{d\zeta}\right) = \frac{d}{d\zeta}\underbrace{\left(\frac{dy_i}{d\alpha}\right)} = \frac{\partial f_i}{\partial y_1}\frac{dy_1}{d\alpha} + \frac{\partial f_i}{\partial y_2}\frac{dy_2}{d\alpha} + \frac{\partial f_i}{\partial y_3}\frac{dy_3}{d\alpha} + \frac{\partial f_i}{\partial y_4}\frac{dy_4}{d\alpha} + \frac{\partial f_i}{\partial y_5}\frac{dy_5}{d\alpha}$$

The variation of the dependent variables respect to the parameters to estimate will be the parametric variables, that will be treated as new dependent variables:

This is for the unknown α

$$\frac{d}{d\zeta}\left(\frac{dy_i}{d\alpha}\right)=\frac{\partial f_i}{\partial y_1}\frac{dy_1}{d\alpha}+\frac{\partial f_i}{\partial y_2}\frac{dy_2}{d\alpha}+\frac{\partial f_i}{\partial y_3}\frac{dy_3}{d\alpha}+\frac{\partial f_i}{\partial y_4}\frac{dy_4}{d\alpha}+\frac{\partial f_i}{\partial y_5}\frac{dy_5}{d\alpha}$$

$$\frac{d}{d\zeta}(y_{i+5})=\frac{\partial f_i}{\partial y_1}y_6+\frac{\partial f_i}{\partial y_2}y_7+\frac{\partial f_i}{\partial y_3}y_8+\frac{\partial f_i}{\partial y_4}y_9+\frac{\partial f_i}{\partial y_5}y_{10}$$

Similar is for β, and if you scale the functions using this unknown in the set of equations, this scaling variable may appear explicit in the equations, so an extra term should be included

$$\frac{df_i}{d\beta}=\frac{d}{d\zeta}\left(\frac{dy_i}{d\beta}\right)=\frac{\partial f_i}{\partial y_1}\frac{dy_1}{d\beta}+\frac{\partial f_i}{\partial y_2}\frac{dy_2}{d\beta}+\frac{\partial f_i}{\partial y_3}\frac{dy_3}{d\beta}+\frac{\partial f_i}{\partial y_4}\frac{dy_4}{d\beta}+\frac{\partial f_i}{\partial y_5}\frac{dy_5}{d\beta}+\frac{\partial f_i}{\partial \beta}$$

$$\frac{d}{d\zeta}(y_{i+10})=\frac{\partial f_i}{\partial y_1}y_{11}+\frac{\partial f_i}{\partial y_2}y_{12}+\frac{\partial f_i}{\partial y_3}y_{13}+\frac{\partial f_i}{\partial y_4}y_{14}+\frac{\partial f_i}{\partial y_5}y_{15}+\frac{\partial f_i}{\partial \beta}$$

$$y_6' = \frac{d}{d\zeta}(y_6) = \frac{\partial f_1}{\partial y_1} y_6 + \frac{\partial f_1}{\partial y_2} y_7 + \frac{\partial f_1}{\partial y_3} y_8 + \frac{\partial f_1}{\partial y_4} y_9 + \frac{\partial f_1}{\partial y_5} y_{10}$$

$$y_7' = \frac{d}{d\zeta}(y_7) = \frac{\partial f_2}{\partial y_1} y_6 + \frac{\partial f_2}{\partial y_2} y_7 + \frac{\partial f_2}{\partial y_3} y_8 + \frac{\partial f_2}{\partial y_4} y_9 + \frac{\partial f_2}{\partial y_5} y_{10}$$

$$y_8' = \frac{d}{d\zeta}(y_8) = \frac{\partial f_3}{\partial y_1} y_6 + \frac{\partial f_3}{\partial y_2} y_7 + \frac{\partial f_3}{\partial y_3} y_8 + \frac{\partial f_3}{\partial y_4} y_9 + \frac{\partial f_3}{\partial y_5} y_{10}$$

$$y_9' = \frac{d}{d\zeta}(y_9) = \frac{\partial f_4}{\partial y_1} y_6 + \frac{\partial f_4}{\partial y_2} y_7 + \frac{\partial f_4}{\partial y_3} y_8 + \frac{\partial f_4}{\partial y_4} y_9 + \frac{\partial f_4}{\partial y_5} y_{10}$$

$$y_{10}' = \frac{d}{d\zeta}(y_{10}) = \frac{\partial f_5}{\partial y_1} y_6 + \frac{\partial f_5}{\partial y_2} y_7 + \frac{\partial f_5}{\partial y_3} y_8 + \frac{\partial f_5}{\partial y_4} y_9 + \frac{\partial f_5}{\partial y_5} y_{10}$$

$$y_{11}' = \frac{d}{d\zeta}(y_{11}) = \frac{\partial f_1}{\partial y_1} y_{11} + \frac{\partial f_1}{\partial y_2} y_{12} + \frac{\partial f_1}{\partial y_3} y_{13} + \frac{\partial f_1}{\partial y_4} y_{14} + \frac{\partial f_1}{\partial y_5} y_{15} + \frac{\partial f_1}{\partial \beta}$$

$$y_{12}' = \frac{d}{d\zeta}(y_{12}) = \frac{\partial f_2}{\partial y_1} y_{11} + \frac{\partial f_2}{\partial y_2} y_{12} + \frac{\partial f_2}{\partial y_3} y_{13} + \frac{\partial f_2}{\partial y_4} y_{14} + \frac{\partial f_2}{\partial y_5} y_{15} + \frac{\partial f_2}{\partial \beta}$$

$$y_{13}' = \frac{d}{d\zeta}(y_{13}) = \frac{\partial f_3}{\partial y_1} y_{11} + \frac{\partial f_3}{\partial y_2} y_{12} + \frac{\partial f_3}{\partial y_3} y_{13} + \frac{\partial f_3}{\partial y_4} y_{14} + \frac{\partial f_3}{\partial y_5} y_{15} + \frac{\partial f_3}{\partial \beta}$$

$$y_{14}' = \frac{d}{d\zeta}(y_{14}) = \frac{\partial f_4}{\partial y_1} y_{11} + \frac{\partial f_4}{\partial y_2} y_{12} + \frac{\partial f_4}{\partial y_3} y_{13} + \frac{\partial f_4}{\partial y_4} y_{14} + \frac{\partial f_4}{\partial y_5} y_{15} + \frac{\partial f_4}{\partial \beta}$$

$$y_{15}' = \frac{d}{d\zeta}(y_{15}) = \frac{\partial f_5}{\partial y_1} y_{11} + \frac{\partial f_5}{\partial y_2} y_{12} + \frac{\partial f_5}{\partial y_3} y_{13} + \frac{\partial f_5}{\partial y_4} y_{14} + \frac{\partial f_5}{\partial y_5} y_{15} + \frac{\partial f_5}{\partial \beta}$$

$$f_{ij} = \frac{\partial f_i}{\partial y_j}$$

$f_1 = y_2$					
$f_{11} = 0$	$f_{12} = 1$	$f_{13} = 0$	$f_{14} = 0$	$f_{15} = 0$	$f_{1\beta} = 0$
$f_2 = y_3$					
$f_{21} = 0$	$f_{22} = 0$	$f_{23} = 1$	$f_{24} = 0$	$f_{25} = 0$	$f_{2\beta} = 0$
$f_3 = 2y_2^2 - 3y_1y_3 + \beta^3(y_4 - \beta)$					
$f_{31} = -3y_3$	$f_{32} = 4y_2$	$f_{33} = -3y_1$	$f_{34} = \beta^3$	$f_{35} = 0$	$f_{3\beta} = \beta^2(3y_4 - 4\beta)$
$f_4 = y_5$					
$f_{41} = 0$	$f_{42} = 0$	$f_{43} = 0$	$f_{44} = 0$	$f_{45} = 1$	$f_{4\beta} = 0$
$f_5 = -3\,Pr\,y_1y_5$					
$f_{51} = -3\,Pr\,y_5$	$f_{52} = 0$	$f_{53} = 0$	$f_{54} = 0$	$f_{55} = -3\,Pr\,y_1$	$f_{5\beta} = 0$

$$y_1' = y_2 \qquad y_1(0) = 0$$

$$y_2' = y_3 \qquad y_2(0) = 0$$

$$y_3' = 2y_2^2 - 3y_1y_3 + \beta^3(y_4 - \beta) \qquad y_3(0) = \alpha$$

$$y_4' = y_5 \qquad y_4(0) = 0$$

$$y_5' = -3\ Pr\ y_1y_5 \qquad y_5(0) = 1$$

Original Problem

$$y_6' = y_7 \qquad y_6(0) = 0$$

$$y_7' = y_8 \qquad y_7(0) = 0$$

$$y_8' = -3y_3y_6 + 4y_2y_7 + -3y_1y_8 + \beta^3 y_9 \qquad y_8(0) = 1$$

$$y_9' = y_{10} \qquad y_9(0) = 0$$

$$y_{10}' = -3\ Pr\ y_5y_6 - 3\ Pr\ y_1y_{10} \qquad y_{10}(0) = 0$$

Parametrical equations to track the evolution of guess α

$$y_{11}' = y_{12} \qquad y_{11}(0) = 0$$

$$y_{12}' = y_{13} \qquad y_{12}(0) = 0$$

$$y_{13}' = -3y_3y_{11} + 4y_2y_{12} + -3y_1y_{13} + \beta^3 y_{14} + \beta^2(3y_4 - 4\beta) \qquad y_{13}(0) = 0$$

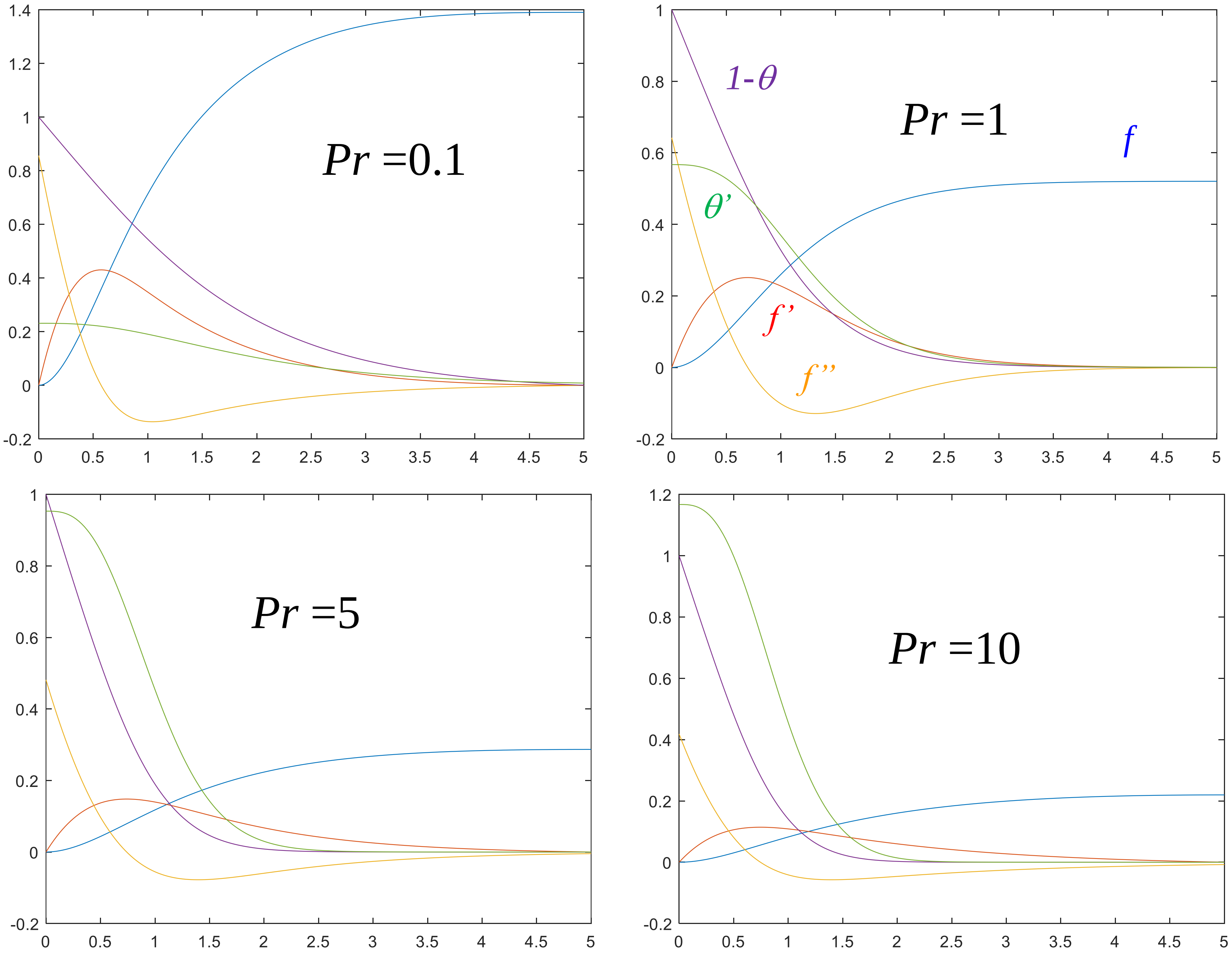
$$y_{14}' = y_{15} \qquad y_{14}(0) = 0$$

$$y_{15}' = -3\ Pr\ y_5y_{11} - 3\ Pr\ y_1y_{15} \qquad y_{15}(0) = 0$$

Parametrical equations to track the evolution of guess β

```
function[Y,t,aM,bM,R20,R40]=TSVM()
% number of point within the domain
M=200;
xspan=linspace(0,5,M);
aM=0.33;
bM=0.9201;
Pr=10;
NM=20;
% This problem is strongly no linear
% then a loop for will be used, and convergence will
% be assessed by the residuals, it is recommended to use contination
for i=1:NM
y0=[0, 0, aM, 0, 1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0]';
eta=bM;
par=[eta,Pr];
[t,Y]=ode45(@funTSVM,xspan,y0,[],par);
Q=[Y(M,7), Y(M,12); Y(M,9), Y(M,14)-1];
R20=Y(M,2)-0;
R40=Y(M,4)-eta;
dv=-inv(Q)*[R20;R40];
aM=aM+dv(1);
bM=bM+dv(2);
% Iteration ends here
plot(t,Y(:,1)/eta,t,Y(:,2)/eta^2,t,Y(:,3)/eta^3,t,1-Y(:,4)/eta,t,Y(:,5)/eta^2);
end
end
```

```
function[ydot]=funTSVM(x,y,par)
eta=par(1);Pr=par(2);
ydot(1)=y(2);
f11=0;f12=1;f13=0;f14=0;f15=0;
ydot(2)=y(3);
f21=0;f22=0;f23=1;f24=0;f25=0;
ydot(3)=-3*y(1)*y(3)+2*y(2)^2-eta^3*(eta-y(4));
f31=-3*y(3);f32=4*y(2);f33=-3*y(1);f34=eta^3;f35=0;
ydot(4)=y(5);
f41=0;f42=0;f43=0;f44=0;f45=1;
ydot(5)=-3*Pr*y(1)*y(5);
f51=-3*Pr*y(5);f52=0;f53=0;f54=0;f55=-3*Pr*y(1);
ydot(6)=f11*y(6)+f12*y(7)+f13*y(8)+f14*y(9)+f15*y(10);
ydot(7)=f21*y(6)+f22*y(7)+f23*y(8)+f24*y(9)+f25*y(10);
ydot(8)=f31*y(6)+f32*y(7)+f33*y(8)+f34*y(9)+f35*y(10);
ydot(9)=f41*y(6)+f42*y(7)+f43*y(8)+f44*y(9)+f45*y(10);
ydot(10)=f51*y(6)+f52*y(7)+f53*y(8)+f54*y(9)+f55*y(10);

ydot(11)=f11*y(11)+f12*y(12)+f13*y(13)+f14*y(14)+f15*y(15);
ydot(12)=f21*y(11)+f22*y(12)+f23*y(13)+f24*y(14)+f25*y(15);
ydot(13)=f31*y(11)+f32*y(12)+f33*y(13)+f34*y(14)+f35*y(15);
ydot(14)=f41*y(11)+f42*y(12)+f43*y(13)+f44*y(14)+f45*y(15);
ydot(15)=f51*y(11)+f52*y(12)+f53*y(13)+f54*y(14)+f55*y(15);

ydot(13)=ydot(13)+3*eta^2*(y(4)-eta)-eta^3;

ydot=ydot';
end
```



Pr =0.1
Pr =1
1-θ
f
θ'
f'
f''
Pr =5
Pr =10