Introduction to linear transformations

The idea of viewing matrices as functions

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Introduction

Suppose we have the following matrix C that represents costs of certain products from certain stores:

$$C = \begin{bmatrix} & Cost\ of\ product\ 1 & Cost\ of\ product\ 2 \\ Store1 & a & b \\ Store2 & c & d \end{bmatrix}$$

and N is the matrix that represents the number of products that we are going to buy

$$N = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Then:

$$CN = \begin{bmatrix} 5a + 6b \\ 5c + 6d \end{bmatrix} = \begin{bmatrix} \text{Cost of buying both products at store 1} \\ \text{Cost of buying both products at store 2} \end{bmatrix} = D$$

Remark: We can think of C as a **function (transformation)** that is being applied to the column matrix N. Mathematically:

$$\mathrm{C(N){=}D.}$$

.

Throughout this lecture, $\mathbb{R}$ will denote the set of all real numbers. We now consider some terminology.

A **transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ is a rule that assigns to each vector $v \in \mathbb{R}^n$ a unique vector $T(v) \in \mathbb{R}^m$.

1. The domain of T is the **maximal** subset of $\mathbb{R}^n$ for which T is defined.
2. For a vector v in the domain of T, the vector $T(v)$ is called the image of v.
3. The set of **all** possible images $T(v)$ (as v varies throughout the domain of T) is called the range of T.

Example 1. Consider the following function $T : \mathbb{R} \to \mathbb{R}^2$ defined as

$$T(w) = (w, 2w)$$

Some questions:

1. Is T ever undefined?
2. What is the domain of T?
3. What is the range of T?

Answers:

1. No, we can always compute $T(b)$ no matter what the value of b is.
2. $\mathbb{R}$.
3. The line $y = 2x$. Recall that a line in $\mathbb{R}^2$ is described by the set $\{(x, mx + b) : m, b \in \mathbb{R}\}$. In this case, $m = 2$ and $b = 0$. In other words, the range of T is the line that goes through the origin and that has slope 2.

A nice visual aid ☺ generated in Mathematica:

Example 2. Consider the following function $T : \mathbb{R} \to \mathbb{R}^2$ defined as

$$T(b) = (sinb, cosb)$$

Same questions:

1. Is T ever undefined?
2. What is the domain of T?
3. What is the range of T?

Answers:

1. No, we can always compute $T(b)$ no matter what the value of b is. Recall that sine and cosine are defined everywhere.
2. $\mathbb{R}$.
3. The unit circle! recall that the unit circle can be described as the set of points (x, y) that satisfy $x^2 + y^2 = 1$; hence, if $x = sinb$ and $y = cosb$ then $x^2 + y^2 = sin^2 b + cos^2 b = 1$.

A nice visual aid ☺ generated in Mathematica:

Example 3. Consider the following function $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as

$$T(a, b) = (a, b - a, -b)$$

Same questions:

1. Is T ever undefined?
2. What is the domain of T?
3. What is the range of T?

Answers:

1. No, linear terms are always defined!
2. $\mathbb{R}^2$.
3. The plane $x+y+z=0$. A plane consists of all the points (x,y,z) that satisfy $\alpha x+\beta y+\gamma z=\delta$ where $\alpha,\beta,\gamma,\delta$ are constants. Note that

   $T(a,b)=(\underbrace{a}_{x},\underbrace{b-a}_{y},\underbrace{-b}_{z})$. then:

   $$x+y+z=a+b-a+(-b)=0$$

   so the image corresponds to all the points lying on the plane $x+y+z=0$.

A nice visual aid ☺ generated in Mathematica:

Regarding the following example, we note that the function $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T(x, y) = (x, y - x, -y)$$

can be described via the matrix

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Indeed,

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y - x \\ -y \end{bmatrix} = T$$

Remark. This means that the function T is completely described by the matrix A.

A natural question arises

Can we describe every function $T : \mathbb{R}^n \to \mathbb{R}^m$ by a matrix?

No, only linear transformations.

This is the main motivation for the following.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a **linear transformation** if

1. $T(u+v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$ and
2. $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and all scalars c.

Remark. The concept of linear transformation is actually **more general** but it requires the notion of a vector space and field; however, in this course we will only work over $\mathbb{R}^n$ or $\mathbb{C}^n$.

Question. Do you know examples of linear transformations $T : U \to V$ where U and V are not $\mathbb{R}^n$?

Example 1. Define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x, y) = (x, x - y, x + y)$. Verify that T is a linear transformation.

Step 1 Check that T preserves sums: let $u = (a, b)$, $v = (c, d)$ be elements of $\mathbb{R}^2$. On one hand

$$\begin{aligned} T(u+v) &= T((a,b)+(c,d)) \\ &= T((\underbrace{a+c}_{x}, \underbrace{b+d}_{y})) \\ &= (a+c, a+c-b-d, a+c+b+d) \end{aligned}$$

On the other hand

$$\begin{aligned} T(u)+T(v) &= T((a,b)) + T((c,d)) \\ &= (a, a-b, a+b) + (c, c-d, c+d) \\ &= (a+c, a-b+c-d, a+b+c+d) \\ &= (a+c, a+c-b-d, a+c+b+d) \end{aligned}$$

It follows that $T(u+v) = T(u) + T(v)$, so T preserves sums.

Step 2 Check that T preserves scalars. Recall that $T(x,y) = (x, x-y, x+y)$. Let $k \in \mathbb{R}$ and $v = (a,b) \in \mathbb{R}^2$. Then

$$
\begin{aligned}
T(kv) &= T(k(a,b)) \\
&= T((\underbrace{ka}_{x}, \underbrace{kb}_{y})) \\
&= (ka, ka - kb, ka + kb) \\
&= (ka, k(a-b), k(a+b)) \\
&= k\underbrace{(a, a-b, a+b)}_{T(a,b)} \\
&= kT(a,b) \\
&= kT(v)
\end{aligned}
$$

Hence $T(kv) = kT(v)$. Therefore scalars can be pulled out of T.

By Step 1 and Step 2 we conclude that T is a linear transformation since **all** conditions are satisfied.

Remark 1. It can be shown (**try!**) that if T is linear then $T(\vec{0}) = \vec{0}$, where $\vec{0}$ denotes the zero vector that corresponds to the domain and codomain of T. It follows that if $T(\vec{0}) \neq \vec{0}$ then T **is not** a linear transformation.

Remark 2. Think of $\vec{0}$ as the origin corresponding to the working spaces.

An example to illustrate this.

Example 2. Consider the map $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, x+1)$.

Question. Is T a linear transformation?

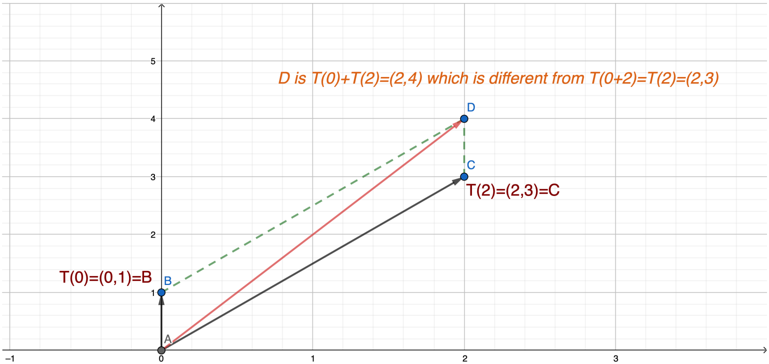
Answer. No. First, the origin that corresponds to $\mathbb{R}$ is 0, while the origin corresponding to $\mathbb{R}^2$ is $(0,0)$. If T is linear, then by Remark 1, T must satisfy $T(0) = (0,0)$. However, $T(0) = (0, 0+1) = (0,1) \neq (0,0)$.

A geometric interpretation

Remark. Consider again the example $T : \mathbb{R} \to \mathbb{R}^2$ given by $T(x) = (x, x+1)$. We just saw that T is **not** linear.

Is there any geometric way to see this?

Yes! To be linear it must satisfy $T(u) + T(v) = T(u+v)$ for any points u, v (plus the scalar condition). Now, consider the following picture:



Exercise.

Make an educated guess ☺ of a linear transformation $T : \mathbb{R} \to \mathbb{R}^2$.

Hint: Think about how can we avoid the situation from the previous slides! What about modifying the direction of B?

Theorem. It can be shown that the image of a **linear** transformation whose codomain is $\mathbb{R}^2$, is one of the following:

1. A line through the origin (think why it must go through the origin!).
2. The whole space $\mathbb{R}^2$.
3. The zero space (i.e the set containing the origin (0,0)).

Problems ☺

Problem 1.

According to the theorem given on the previous slide, if we have a linear transformation $T : \mathbb{R} \to \mathbb{R}^2$, then the image of T should be a line, the origin or the whole plane.

However, according to the slide on page 8, the image is the unit circle.

Is this a contradiction to the theorem? What is going on here?

Problem 2. Consider the following transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (y, x)$.

1. Prove that T is linear (you have to check two properties).
2. What is the domain of T?
3. What is the geometric interpretation of the function T? compute some values to figure it out! Using this interpretation, compute the range of T.