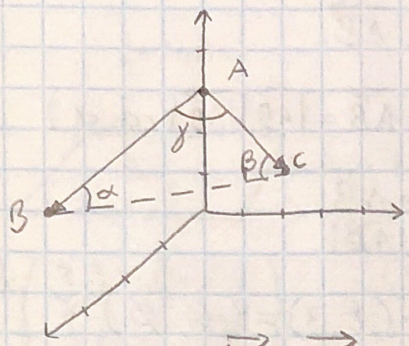
Vectores: Producto escalar

1- Encuentra los ángulos del triángulo con vértices $A(0,0,2)$ $B(3,0,2)$ y $C(1,1,1)$. Haz un dibujo del triángulo.



γ entre $\vec{AB}$, $\vec{AC}$

$\arccos(\vec{AB}\cdot\vec{AC} = |\vec{AB}||\vec{AC}|\cos\gamma)$

$\vec{AB} = (3,0,2)-(0,0,2) = (3,0,0)$

$\vec{AC} = (1,1,1)-(0,0,2) = (1,1,-1)$

$$\frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}||\vec{AC}|} = \frac{(3,0,0)\cdot(1,1,-1)}{|3,0,0|\ |1,1,1|} = \frac{(3\cdot1)+(0\cdot1)+(0\cdot1)}{\sqrt{3^2+0^2+0^2}\ \sqrt{1^2+1^2+1^2}} = \frac{\cancel{3}^{\,1}}{\cancel{\sqrt{9}}\ \sqrt{3}}$$

$$\frac{1}{\sqrt{3}} \qquad \arccos\left(\frac{1}{\sqrt{3}}\right) = 54.73^\circ$$

α entre $\vec{BA}$, $\vec{BC}$

$\vec{BA} = (0,0,2)-(3,0,2) = (-3,0,0)$

$\arccos(\vec{BA}\cdot\vec{BC} = |\vec{BA}||\vec{BC}|\cos\alpha)$

$\vec{BC} = (1,1,1)-(3,0,2) = (-2,1,-1)$

$$\frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}||\vec{BC}|} = \frac{(-3,0,0)\cdot(-2,1,-1)}{|3,0,0|\ |2,1,1|} = \frac{(-3\cdot-2)+(0\cdot1)+(0\cdot-1)}{\sqrt{3^2+0^2+0^2}\ \sqrt{2^2+1^2+1^2}} = \frac{\cancel{6}^{\,2}}{\cancel{\sqrt{9}}\ \sqrt{6}} = \frac{2}{\sqrt{6}}$$

$$\arccos\left(\frac{2}{\sqrt{6}}\right) = 35.26^\circ$$

La suma de los ángulos internos de un triángulo es igual a 180°

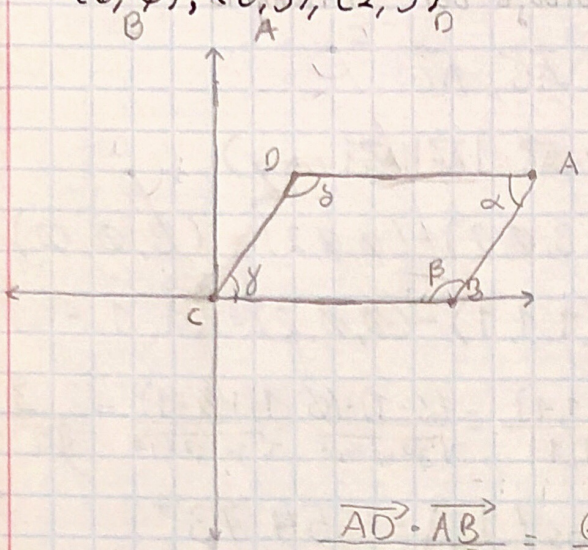
$$\begin{array}{r} +\ 54.73^\circ \\ 35.26^\circ \\ \hline 89.99^\circ \end{array} \qquad \begin{array}{r} -\ 180^\circ \\ 89.99^\circ \\ \hline 91.01^\circ \end{array}$$

$\gamma = 54.73^\circ$

$\alpha = 35.26^\circ$

$\beta = 91.01^\circ$

2- Encuentra los ángulos de un paralelogramo si los vértices son $(\emptyset,\emptyset)$ $(6,\emptyset)$, $(8,3)$, $(2,3)$



α entre $\overrightarrow{AD}, \overrightarrow{AB}$

$\arccos(\overrightarrow{AD}\cdot\overrightarrow{AB} = |\overrightarrow{AD}||\overrightarrow{AB}|\cos\alpha)$

$$\alpha = \frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{|\overrightarrow{AD}|\,|\overrightarrow{AB}|}$$

$\overrightarrow{AD} = (2,3)-(8,3) = (-6,\emptyset)$

$\overrightarrow{AB} = (6,\emptyset)-(8,3) = (-2,-3)$

$$\frac{\overrightarrow{AD}\cdot\overrightarrow{AB}}{|\overrightarrow{AD}||\overrightarrow{AB}|} = \frac{(-6,\emptyset)\cdot(-2,-3)}{|6,\emptyset|\,|2,3|} = \frac{-6\cdot-2+\emptyset\cdot-3}{\sqrt{6^2}\,\sqrt{2^2+3^2}} = \frac{\cancel{12}^{\,2}}{\cancel{6}\,\sqrt{13}}$$

$$= \frac{2}{\sqrt{13}} \qquad \arccos\left(\frac{2}{\sqrt{13}}\right) = 56.30^\circ$$

β entre $\overrightarrow{BA}, \overrightarrow{BC}$ $\qquad \overrightarrow{BA} = (8,3)-(6,\emptyset) = (2,3)$

$\arccos(\overrightarrow{BA}\cdot\overrightarrow{BC} = |BA||BC|\cos\beta)$ $\qquad \overrightarrow{BC} = (\emptyset\emptyset)-(6,\emptyset) = (-6,\emptyset)$

$$\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} = \frac{(2,3)\cdot(-6,\emptyset)}{|2,3|\,|6,\emptyset|} = \frac{2\cdot-6+3\cdot\emptyset}{\sqrt{2^2+3^2}\,\sqrt{6^2}} = \frac{\cancel{-12}^{\,-2}}{\sqrt{13}\,\cancel{6}} = \frac{-2}{\sqrt{13}} \qquad \arccos\left(\frac{-2}{\sqrt{13}}\right) = 123.69^\circ$$

$\alpha = 56.30^\circ$

$\beta = 123.69^\circ$

$\gamma = 56.30^\circ$

$\delta = 123.69^\circ$

La suma de dos vértices contiguos de un paralelogramo es igual a 180° y la suma total de los ángulos internos es 360°. Dos ángulos opuestos por el vértice son iguales

$\alpha = \gamma \qquad \beta = \delta$, para probar esto buscaremos γ y δ con el arcos

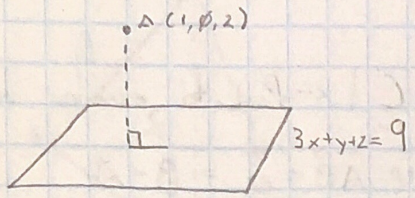
γ $\overrightarrow{CD} = (2,3)-(\emptyset,\emptyset) = (2,3)$ $\qquad \overrightarrow{DA} = (8,3)-(2,3) = (6,\emptyset)$

$\overrightarrow{CB} = (6,\emptyset)-(\emptyset,\emptyset) = (6,\emptyset)$ $\qquad \delta$ $\overrightarrow{DC} = (\emptyset,\emptyset)-(2,3) = (-2,-3)$

$$\frac{\overrightarrow{CD}\cdot\overrightarrow{CB}}{|\overrightarrow{CD}||\overrightarrow{CB}|} = \frac{(2,3)\cdot(6,\emptyset)}{|2,3|\,|6,\emptyset|} = \frac{2\cdot6+3\cdot\emptyset}{\sqrt{2^2+3^2}\,\sqrt{6^2}} = \frac{\cancel{12}^{\,2}}{\sqrt{13}\,\cancel{6}} = \frac{2}{\sqrt{13}} \qquad \arccos\left(\frac{2}{\sqrt{13}}\right) = 56.30^\circ$$

$$\frac{\overrightarrow{DA}\cdot\overrightarrow{DC}}{|\overrightarrow{DA}||\overrightarrow{DC}|} = \frac{(6,\emptyset)\cdot(-2,-3)}{|6,\emptyset|\,|2,3|} = \frac{6\cdot-2+\emptyset\cdot-3}{\sqrt{6^2}\,\sqrt{2^2+3^2}} = \frac{\cancel{-12}^{\,-2}}{\cancel{6}\,\sqrt{13}} = \frac{-2}{\sqrt{13}} \qquad \arccos\left(\frac{-2}{\sqrt{13}}\right) = 123.69^\circ$$

3- Encuentra la distancia del punto $A(1,\emptyset,2)$ al Plano $3x+y+z=9$ haz un dibujo del problema



$$\underset{\sim}{P}\cdot\underset{\sim}{A} = ax+by+cz=d$$

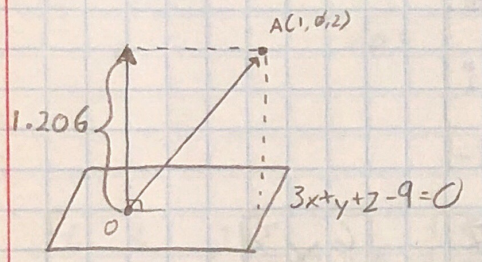
$$P=\begin{pmatrix}a\\b\\c\end{pmatrix}\neq\emptyset \qquad A=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

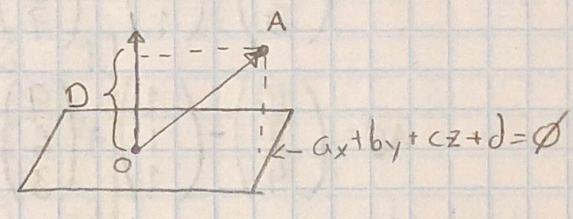
$$P=\begin{pmatrix}3\\1\\1\end{pmatrix} \quad A=\begin{pmatrix}1\\\emptyset\\2\end{pmatrix} \qquad D=P\cdot A=\frac{\underset{\sim}{P}\cdot\underset{\sim}{A}}{|\underset{\sim}{P}|} \qquad \hat{P}\cdot\hat{A}=\frac{d}{|P|}$$

$$P=3x+y+z-9=\emptyset \qquad D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}} = \text{Proyección } U=\frac{|\underset{\sim}{u}\cdot\underset{\sim}{v}|}{|\underset{\sim}{v}|}$$

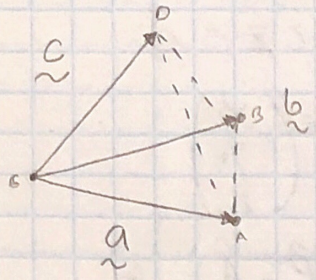
$$D=\frac{|(3\cdot1)+(1\cdot\emptyset)+(1\cdot2)-9)|}{\sqrt{3^2+1^2+1^2}}=\frac{|3+\emptyset+2-9|}{\sqrt{11}}=\frac{|4|}{\sqrt{11}}$$

$$D=\underline{1.206\ u}$$





7- Encuentra el volumen de un tetaedro con vértices (1, 1, 1) (5, -7, 3) (7, 4, 8) y (10, 7, 4)



$$V_T = \frac{1}{6} V_p$$

$$V_p = |(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}|$$ ← Producto escalar triple

$$V_p = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$\overrightarrow{AB} = \underset{\sim}{a} = B - A$

$\overrightarrow{AC} = \underset{\sim}{b} = C - A$

$AD = \underset{\sim}{c} = D - A$

$A = (1, 1, 1)$

$B = (5, -7, 3)$

$C = (7, 4, 8)$

$D = (10, 7, 4)$

$$\overrightarrow{AB} = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AC} = \begin{pmatrix} 7 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix} 10 \\ 7 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix}$$

$$V_p = \begin{bmatrix} 4 & -8 & 2 \\ 6 & 3 & 7 \\ 9 & 6 & 3 \end{bmatrix}$$

Sacamos los determinantes

$$\begin{bmatrix} 4 & -8 & 2 \\ 6 & 3 & 7 \\ 9 & 6 & 3 \end{bmatrix} = \det\left(4\begin{bmatrix} 3 & 7 \\ 6 & 3 \end{bmatrix} + (-1 \cdot -8)\begin{bmatrix} 6 & 7 \\ 9 & 3 \end{bmatrix} + 2\begin{bmatrix} 6 & 3 \\ 9 & 6 \end{bmatrix}\right)$$

$$\det(4((3 \cdot 3) - (6 \cdot 7) + 8((6 \cdot 3) - (9 \cdot 7)) + 2((6 \cdot 6) - (9 \cdot 3)))$$

$$\det(4(9 - 42) + 8(18 - 63) + 2(36 - 27))$$

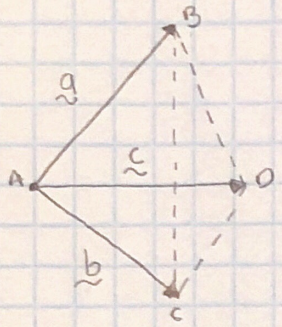
$$\det(4(-33) + 8(-45) + 2(9)$$

$$\det(-132 - 360 + 18) = -474$$

$$V_p = |474|$$

$$V_T = \frac{1}{6} 474 = \underline{79\ u^3}$$

8- Encuentra el volumen de un tetraedro con vértices $(1,3,6)$, $(3,7,12)$ $(8,8,9)$ y $(2,2,8)$



$$V_T = \frac{1}{6} V_P$$

$$V_P = |(a \times b) \cdot c|$$

$$\overrightarrow{AB} = \underset{\sim}{a} \qquad \overrightarrow{AC} = \underset{\sim}{b} \qquad \overrightarrow{AD} = \underset{\sim}{c}$$

$$V_P = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$A = (2,2,8)$

$B = (8,8,9)$ $\qquad \overrightarrow{AB} = (8,8,9) - (2,2,8) = (6,6,1)$

$C = (3,7,12)$ $\qquad \overrightarrow{AC} = (3,7,12) - (2,2,8) = (1,5,4)$

$D = (1,3,6)$ $\qquad \overrightarrow{AD} = (1,3,6) - (2,2,8) = (-1,1,-2)$

$$V_P = \begin{bmatrix} 6 & 6 & 1 \\ 1 & 5 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 & 1 \\ 1 & 5 & 4 \\ -1 & 1 & -2 \end{bmatrix} = \det\left(6\begin{bmatrix} 5 & 4 \\ 1 & -2 \end{bmatrix} + (-6)\begin{bmatrix} 1 & 4 \\ -1 & -2 \end{bmatrix} + 1\begin{bmatrix} 1 & 5 \\ -1 & 1 \end{bmatrix}\right)$$

$$6((5\cdot -2)-(4\cdot 1)) - 6((1\cdot -2)-(-1\cdot 4)) + 1((1\cdot 1)-(-1\cdot 5))$$

$$6(-10-4) - 6(-2+4) + 1(1+5)$$

$$6(-14) - 6(2) + 1(6) = -90$$

$$V_P = |90|$$

$$V_T = \frac{1}{6} 90 = \underline{15\ u^3}$$