Importance of Dimensional analysis

- Revisit units and dimensions
- Dimensional analysis
- Dimensional homogeneity
- Theory of Physical Models, and scale-up (Model and Prototype)
- Modeling Law
- Methodologies to obtain dimensionless numbers
    - Non-dimensionalization of Differential equations
    - Similarity transformation in the PDE
    - Non-dimensionalization by collecting terms of equations
    - Exploration of forces, velocities, energies, geometries.
    - Buckingham's type approach
- Examples

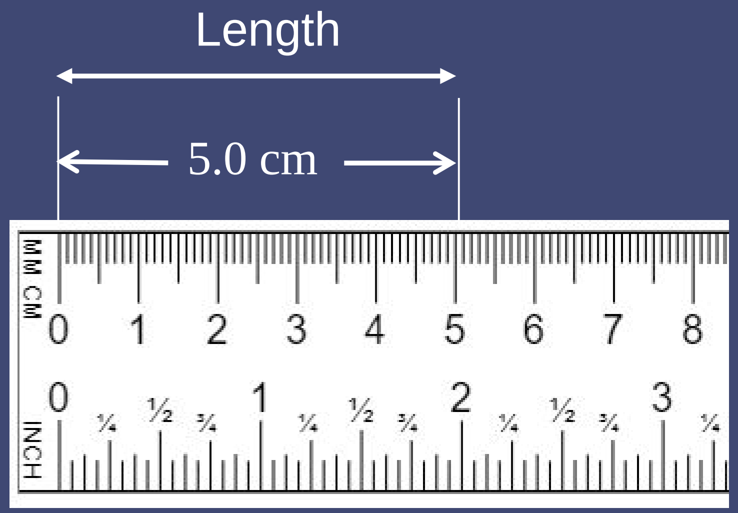
Dimensions and Units

Dimension: A measure of a physical quantity (without numerical values).

Unit: A way to assign a **number** to that dimension.

There are seven **primary dimensions** (also called **fundamental** or **basic dimensions**): mass, length, time, temperature, electric current(*), amount of light, and amount of matter.

All non-primary dimensions can be formed by some combination of the seven primary dimensions.



Dimensions of force {Force} [=] N[=] kg-m/s^2 [=] $M\text{-}L\text{-}\Theta^{-2}$

A **dimension** is a measure of a physical quantity without numerical values, while a **unit** is a way to assign a number to the dimension. For example, length is a dimension, but centimeter is a unit.

$$T = 100°\text{C} = 212°\text{F} = 373.25\ \text{K}\ [=]\ \text{T}$$

(*) Some people prefer charge unit as primary dimension but IUPAP suggest to use current instead, and then coulomb is non primary dimension because can be expressed as the product of current times time, i.e. (ampere) (second)

Primary dimensions and their associated primary SI units

Dimension		Symbol[*]	SI Unit
Mass	M	m	kg (kilogram)
Length	L	L	m (meter)
Time[†]	Θ	t	s (second)
Temperature	T	T	K (kelvin)
Electric current	I	I	A (ampere)
Amount of light	C	C	cd (candela)
Amount of matt	N	N	mol (mole)

Dımensıonal homogeneıty

The law of dimensional homogeneity: Every additive term in an equation must have the same dimensions.

$$\overbrace{\frac{d(m\,\hat{E})}{dt}}^{W} = \sum_{in} \dot{m}_i \left(\hat{U}_i + \alpha_i \frac{v_i^2}{2} + g\, z_i + \frac{p_i}{\rho_i}\right) - \underbrace{\sum_{out} \dot{m}_j \left(\hat{U}_j + \alpha_j \frac{v_j^2}{2} + g\, z_j + \frac{p_j}{\rho_j}\right)}_{W} + \underbrace{\dot{Q}}_{W} - \underbrace{\dot{W}}_{W}$$

kg/s	J/kg	m^2/s^2	$m\text{-}m/s^2$	$Pa\text{-}m^3/kg$
	N-m/kg	m^2/s^2	m^2/s^2	$N\text{-}m^3/m^2kg$
	$kg\text{-}m^2/kg\text{-}s^2$	m^2/s^2	m^2/s^2	$kg\text{-}m\text{-}m/kg\text{-}s^2$
	m^2/s^2	m^2/s^2	m^2/s^2	m^2/s^2

$kg\text{-}m\ /s^2\text{-}m/s$

N-m/s

J/s

W

Problem No.1 (Refresher of units, dimensional homogeneity law)

Review the differential operators (Gradient and Laplacian) in the lecture of "Units", and give the units and dimensions of:

a) The Darcy equation parameter (κ)

$$\underline{v} = -\frac{\kappa}{\mu}\left[\underline{\nabla} p - \rho\, \underline{g}\right]$$

b) The Brinkman equation parameters (β)

$$\underline{v} - \beta \nabla^2 \underline{v} = -\frac{\kappa}{\mu}\left[\underline{\nabla} p - \rho\, \underline{g}\right]$$

c) The Forchheimer's equation parameter (κ_I)

$$\underline{\nabla} p = \rho\, \underline{g} - \frac{\mu}{\kappa}\left[\underline{v}\right] - \frac{\rho}{\kappa_I}\left[\underline{v}\right]\left[\sqrt{\underline{v}\cdot\underline{v}}\right]$$

d) Challenge: the custom units of permeability is darcy. A porous medium with a permeability of 1 darcy permits a flow of 1 cm^3/s of a fluid with viscosity 1 cP (1 mPa·s) under a pressure gradient of 1 atm/cm acting across an area of 1 cm^2. Verify that these dimensions are consistent with your finding of part a)

ρ [=] kg/m^3

g [=] m/s^2

μ [=] Pa-s

$\underline{v}$ [=] m/s

p [=] Pa

$\underline{g} = -g\,\hat{k}$

1 cP =1 centipoise =1×10^{-3} Pa-s

Nondimensionalization of Equations

Nondimensional equation: If we divide each term in the equation by a collection of variables and constants whose product has those same dimensions, the equation is rendered nondimensional.

Normalized equation: If the nondimensional terms in the equation are of order unity, the equation is called normalized.

Each term in a nondimensional equation is dimensionless.

Nondimensional parameters: In the process of nondimensionalizing an equation of motion, nondimensional parameters often appear—most of which are named after a notable scientist or engineer (e.g., the Reynolds number and the Froude number, and so on).

This process is referred to by some authors as inspectional analysis.

The nondimensionalized Bernoulli equation

$$\underbrace{\frac{\Psi}{p_\infty}}_{\{\,1\,\}} = \underbrace{\frac{p_i}{p_\infty}}_{\{\,1\,\}} + \underbrace{\frac{\rho\, v_i^2}{2\, p_\infty}}_{\{\,1\,\}} + \underbrace{\frac{\rho\, g\, z_i}{p_\infty}}_{\{\,1\,\}} - \underbrace{\rho \frac{\Omega^2}{2\, p_\infty} r_i^2}_{\{\,1\,\}}$$

A nondimensionalized form of the Bernoulli equation is formed by dividing each additive term by a pressure (here we use P_∞). Each resulting term is **dimensionless** (dimensions of {1}).

p_∞ is any trivial variable, but if physical meaning is used will be better, e.g. stagnation pressure, saturation pressure, atmospheric pressure, etc.

Force balance in z-axis $$\frac{d^2z}{dt^2} = -g$$

Dimensional result $$z = z_0 + w_0 t - \frac{g\,t^2}{2}$$

Dimensional variables: Dimensional quantities that change or vary in the problem. Examples: z (dimension of length) and t (dimension of time).

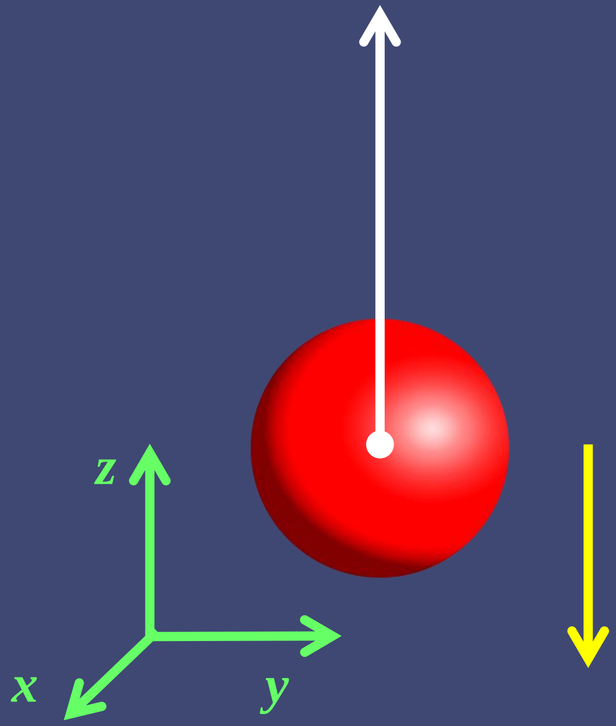
Nondimensional (or dimensionless) variables: Quantities that change or vary in the problem, but have no dimensions. Example: Angle of rotation, measured in degrees or radians, dimensionless units, specific gravity, specific heat capacity ratio $k=c_p/c_v$, or any normalized variable.

Dimensional constant: Gravitational constant g, while dimensional, remains constant.

Parameters: Refer to the combined set of dimensional variables, nondimensional variables, and dimensional constants in the problem.

Pure constants: The constant 1/2 and the exponent 2 in equation. Other common examples of pure constants are π and e.

w = component of velocity in the z-direction



g = gravitational force field, directed to negative z

Object falling in a vacuum. Vertical velocity is drawn positively, so $w < 0$ for a falling object.

Dımensıonal Analysıs And Sımılarıty

In most experiments, to save time and money, tests are performed on a geometrically scaled model, rather than on the full-scale prototype.

In such cases, care must be taken to properly scale the results. We introduce here a powerful technique called dimensional analysis.

The three primary purposes of dimensional analysis are

- To generate nondimensional parameters that help in the design of experiments (physical and/or numerical) and in the reporting of experimental results
- To obtain scaling laws so that prototype performance can be forecasted from model performance
- To (sometimes) forecast trends in the relationship between parameters

The principle of similarity

Three necessary conditions for complete similarity between a model and a prototype.

(1) Geometric similarity—the model must be the same shape as the prototype, but may be scaled by some constant scale factor.

(2) Kinematic similarity—the velocity at any point in the model flow must be proportional (by a constant scale factor) to the velocity at the corresponding point in the prototype flow.

(3) Dynamic similarity—When all forces in the model flow scale by a constant factor to corresponding forces in the prototype flow (force-scale equivalence).

Model and Prototype

A **Physical Model** is a projection of nature (nature can also be an actual or future technological process, technology, engine or device) or at least a sub-process (sub-system, element or component) that occurs in nature or in our world of experience to small scales. Then this small scale system is called physical model.

A **Prototype** is nature (e.g. natural phenomena, technological process, technology, engine or device) or sub-process occurring in nature.

Model is only a projection of e real process, some information of real process is lost during the projection , although model is used for executing experiments and transporting answers (extrapolating or forecasting) to the corresponding prototype.

There exists a point-to-point correlation between the model and prototype, and the corresponding points between the model and the prototype are called homologous points.

Model is short for **Physical Model**

This portion was adapted from Chung Fang, *An Introduction to Fluid Mechanics*, Sprineger, 2019

Consider an Euclidian space with the Cartesian coordinates and time (x_1,x_2,x_3,t), and let the coordinates and time used in the model and prototype be denoted respectively by (x_{1m},x_{2m},x_{3m},t_m), and (x_{1p},x_{2p},x_{3p},t_p), Since model is either an enlargement or a reduction in size of the prototype, it is plausible to define:

$$x_{1m} = k_{x1}x_{1p} \qquad x_{2m} = k_{x2}x_{2p} \qquad x_{3m} = k_{x3}x_{3p} \qquad t_m = k_t t_p$$

where (k_{x1},k_{x2},k_{x3}) are geometric scale factors in the spatial coordinates (x_1,x_2,x_3) and k_t is the time scale. A model is said to be **geometrically similar** to the prototype if $k_{x1}= k_{x2}= k_{x3}$; otherwise, the model is said to be **distorted**.

The model is said to be **kinematically similar** to the prototype, if the motions are similar, namely if homogeneous particles are to be found at homologous times in homologous points

$$v_{1m} = (k_{x1}/k_t)v_{1p} \qquad v_{2m} = (k_{x2}/k_t)v_{2p} \qquad v_{3m} = (k_{x2}/k_t)\, v_{3p}$$

$(x,y,z,t)=(x_1,x_2,x_3,t)$

Velocity and acceleration scale factors:

$$v_{1m} = k_{v1}\, v_{1p} \qquad v_{2m} = k_{v2}\, v_{2p} \qquad v_{3m} = k_{v3}\, v_{3p}$$

$$k_{vi} = (k_{xi}/k_t) \qquad k_{ai} = (k_{xi}/k_t^2)$$

The model is said to be **dynamically similar** to the prototype, if homologous points of the system are subject to similar forces, i.e. the force factors are invariant. If the masses of the model and the prototype be denoted respectively by m_m and m_p, and $k_m = m_m/m_p$ is defined as the scale factor for mass.

$$k_{Fi} = k_m(k_{xi}/k_t^2)$$

Scale factors for velocity and acceleration are not freely assignable, but must be computed from scale factors of geometry and time. Analogously, for dynamic similarity the force factors are obtained automatically from scale factors for geometry, mass, and time.

The complete similarity between a model and its prototype requires that the **geometric**, **kinematic**, and **dynamic** similarities all hold simultaneously.

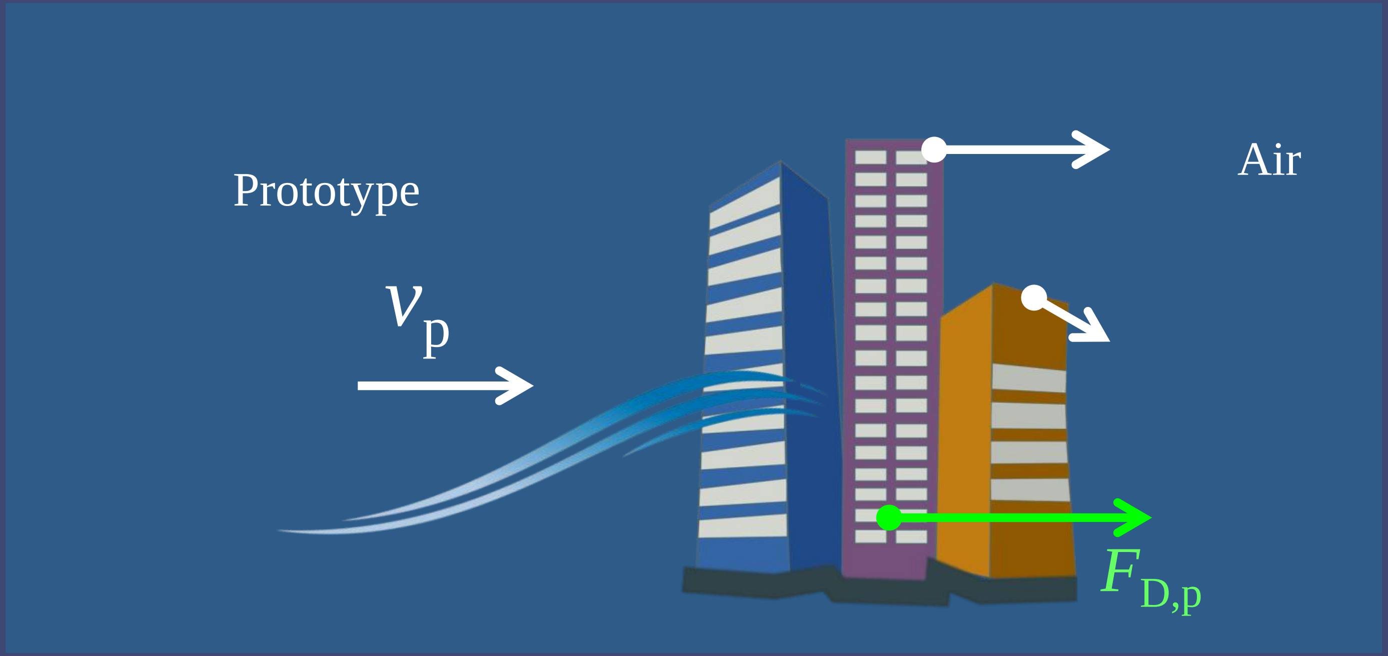
Similarity between a model and its prototype requires that the geometric, kinematic, and dynamic similarities all hold simultaneously. However, in practice, it is hardly possible to require all the dimensionless products to be the same in both the model and prototype, and one is regularly forced to hold only a reduced number of Π - products constant while the others are allowed to vary a dictated by the modeling law. Is such a case, an incomplete similarity between the model and the prototype is established and it is hopes that the Π-products which do not remain invariant in the projection will not, at least not much, influence the physical process that is studied. For such circumstance, the model is said to have **scale effects**. On the contrary, if a process depends only on the Π-products which remain invariant in a model projection, this process is called **scale invariant**.

Modeling Law (Model design conditions or similarity requirement): A model is capable to reproduce a process in a prototype with complete similarity, if all the dimensionless products describing the process have the same values in model and prototype (Fang, 2019).

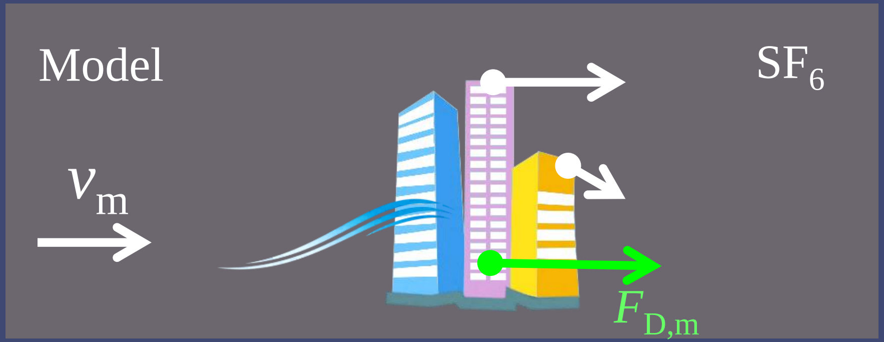
This portion was adapted from Chung Fang, *An Introduction to Fluid Mechanics*, Sprineger, 2019

Dimensional Analysis
(Buckingham Pi Theorem)

Suppose that $\boldsymbol{n}$ quantities describe some physical occurrence, and suppose further that each of these quantities are expressible in terms of certain fundamental dimensional quantities (e.g. mass, length, time) which are $\boldsymbol{m}$ in number. Then the occurrence can be described in terms of $\boldsymbol{n}$-$\boldsymbol{r}$ non dimensional quantities (Called Π - variables) where $\boldsymbol{r}$ is the rank of the $\boldsymbol{m} \times \boldsymbol{n}$ matrix formed from the dimensions of the n-quantities.



Kinematic similarity is achieved when, at all locations, the speed in the model flow is proportional to that at corresponding locations in the prototype flow, and points in the same direction.



In a general flow field, complete similarity between a model and prototype is achieved only when there is geometric, kinematic, and dynamic similarity.

True models have complete similarity, and distorted models only have partial similarity.

We let uppercase Greek letter Pi (Π) denote a nondimensional parameter.

In a general dimensional analysis problem, there is one Π that we call the dependent Π, giving it the notation Π_1.

The parameter Π_1 is in general a function of several other Π's, which we call independent Π's.

Functional relationship among Π's

$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_k)$$

To ensure complete similarity, the model and prototype must be geometrically similar, and all independent groups must match between model and prototype.

To achieve similarity

If $\Pi_{2,p} = \Pi_{2,m}$ and $\Pi_{3,p} = \Pi_{3,m}$ … and $\Pi_{k,p} = \Pi_{k,m}$

Then $\Pi_{1,p} = \Pi_{1,m}$

How to use the Π-numbers , to relate, properties, variables or phenomena

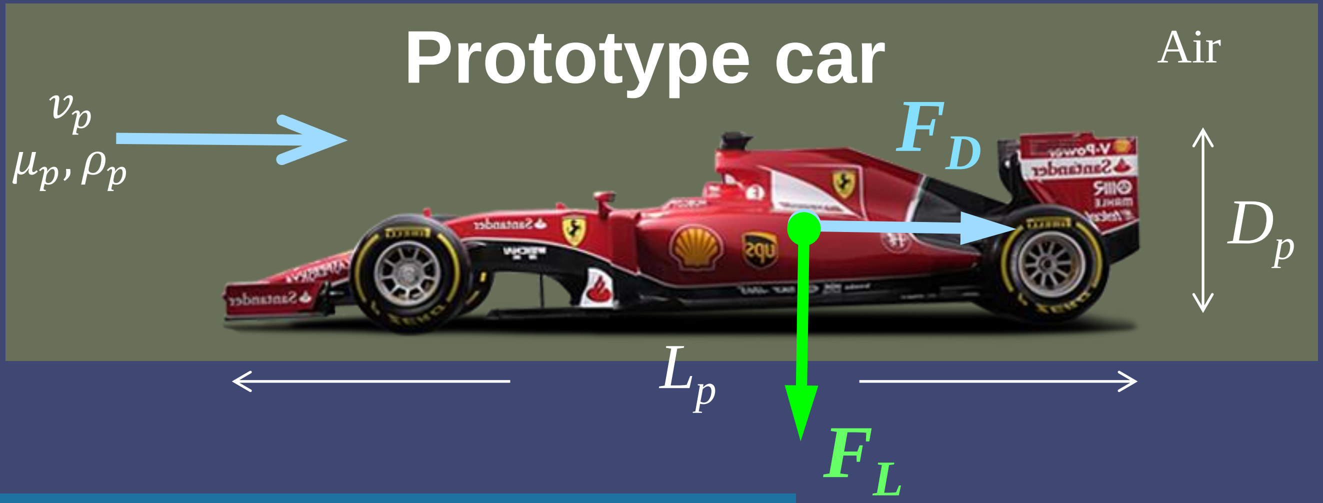
$$\Pi_1 = f(\Pi_2, \Pi_3, \cdots, \Pi_k)$$

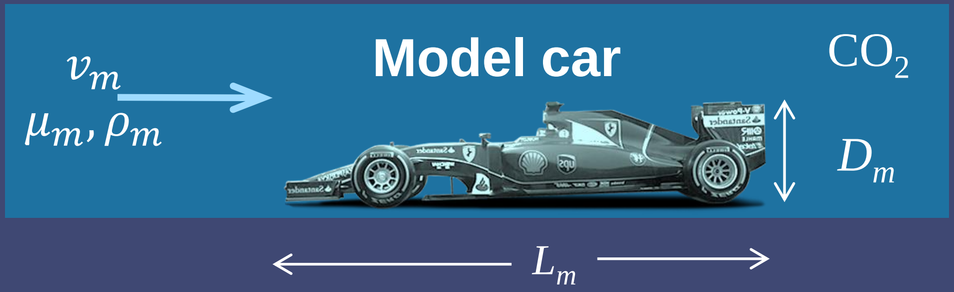
$$\Pi_1 = f_1(\Pi_3, \Pi_4, \cdots, \Pi_k)$$
$$\Pi_2 = f_2(\Pi_3, \Pi_4, \cdots, \Pi_k)$$

$$\Pi_1 = f_1(\Pi_4, \Pi_5, \cdots, \Pi_k)$$
$$\Pi_2 = f_2(\Pi_4, \Pi_5, \cdots, \Pi_k)$$
$$\Pi_3 = f_2(\Pi_4, \Pi_5, \cdots, \Pi_k)$$

In some cases are independent, and in some other cases are dependent

Example:
Dimensional analysis for race cars





Geometric similarity between a prototype car of length L_p and a model car of length L_m.

$D_p/L_p = D_m/L_m$

$$Re = \frac{\rho\, v\, L}{\mu} = \frac{v\, L}{\nu}$$

The **Reynolds number** Re is formed by the ratio of density, characteristic speed, and characteristic length to viscosity. Alternatively, it is the ratio of characteristic speed and length to kinematic viscosity, defined as $\nu = \mu/\rho$.

Dynamic similarity numbers

$$\Pi_2 = \frac{F_L}{\rho\, v^2 L^2} \qquad \Pi_1 = \frac{F_D}{\rho\, v^2 L^2} \qquad \Pi_3 = \frac{\rho\, v\, L}{\mu}$$

Kinematic similarity number $\Pi_4 = \dfrac{v}{c}$

Geometric similarity number $\Pi_5 = \dfrac{D}{L}$

Typical relationship For drag and lift

$$\begin{cases} \Pi_1 = f(\Pi_3, \Pi_4, \Pi_5) \\ \Pi_2 = g(\Pi_3, \Pi_4, \Pi_5) \end{cases}$$

The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics, and Mach number as well in aerodynamics.

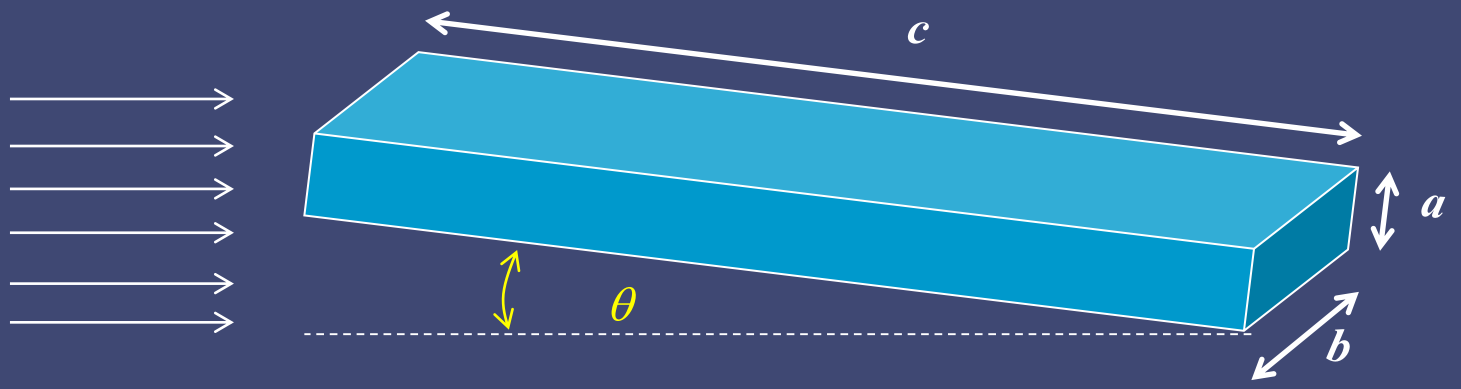
The characteristic length may be a matter of practical issues, and the same for area …

For drag coefficient, you may use the frontal area (or minimal frontal), but in some cases they can be referred to the platform area, so you must be carful regarding which area to use.

For lift force it is customary to use the platform area.

For surface ships, where the actual area in contact is not predictable, the square of the hull length is used instead, this for drag coefficient. We will call it trivial area

For fish, birds, and animals in general is complicated to measure either platform area or frontal area, then the volume to the power 2/3 is used for area



For a flat plate tilted in a stream, the areas discussed previously will be.

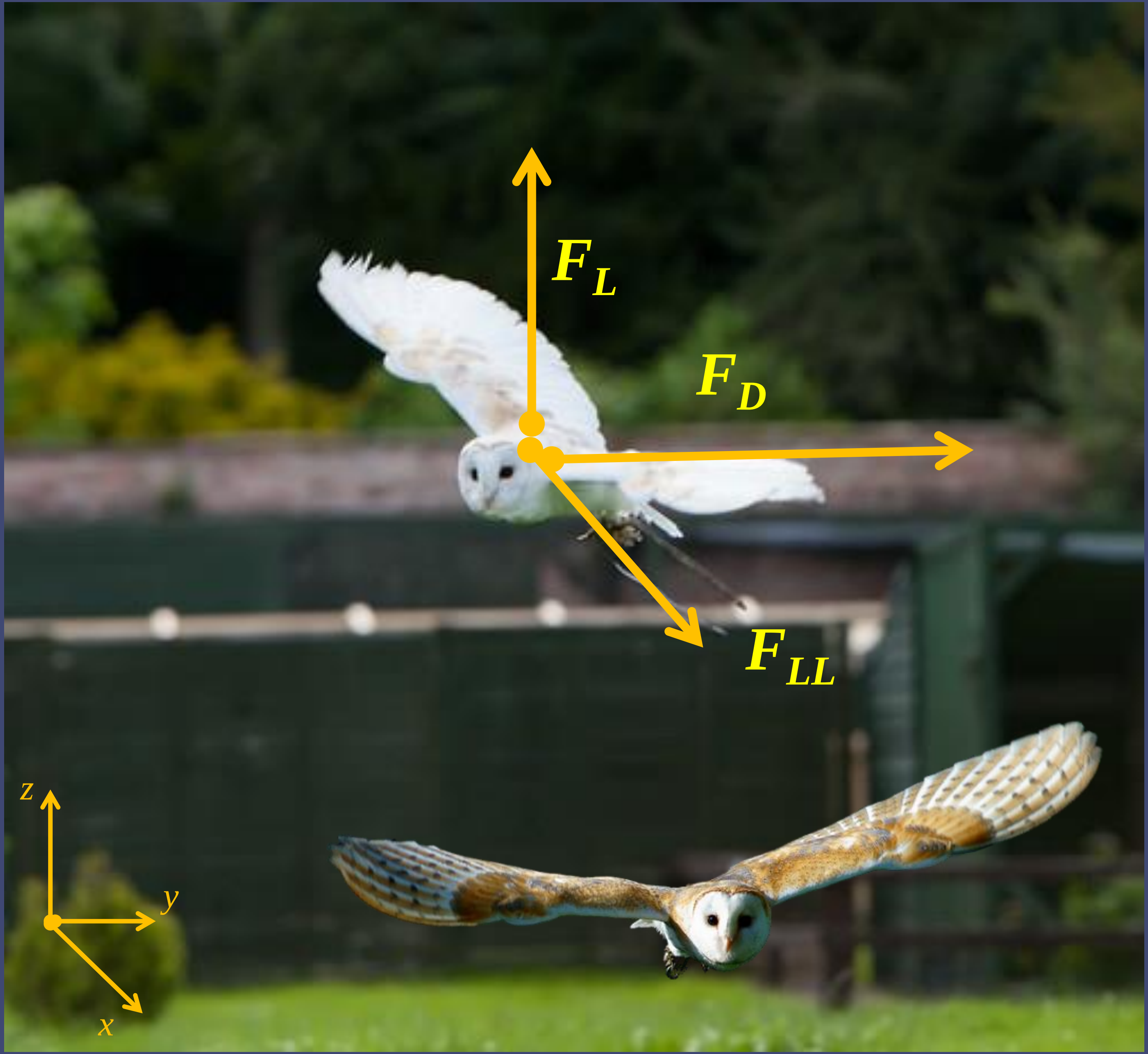
$$A_F = a\ b$$ Frontal (this is the way is calculated regardless the inclination).

$$A_P = c\ b$$ Platform area

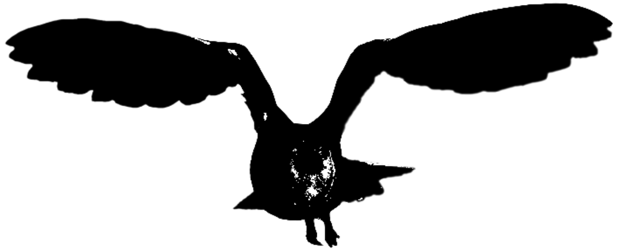
$$A_T = c^2$$ Trivial area

$$A_R = (a\ b\ c)^{2/3}$$ Reference area

For rocks, pebbles, grains, you use the frontal area of a sphere with the same volume of the rock



$$C_D = \frac{F_D}{\rho\, A_F v^2/2}$$



Frontal area (A_F)

$$C_L = \frac{F_L}{\rho\, A_B v^2/2}$$

Platform area(A_B)

The method of repeatıng varıables and the Buckıngham pı theorem

How to generate the nondimensional parameters, i.e., the Π's?

There are several methods that have been developed for this purpose, but the most popular (and simplest) method is the method of repeating variables.

The Method of Repeating Variables

Step 1: List the parameters in the problem and count their total number n.

Step 2: List the primary dimensions of each of the n parameters.

Step 3: Set the *reduction* j as the number of primary dimensions. Calculate k, the expected number of Π's,

$$k = n - j$$

Step 4: Choose j *repeating parameters*.

Step 5: Construct the k Π's, and manipulate as necessary.

Step 6: Write the final functional relationship and check your algebra.

A concise summary of the six steps that comprise the method of repeating variables.

Jolulo´s Method

$$h \qquad u \qquad c_p \qquad D \qquad k \qquad \rho \qquad \mu$$

Problem No.2 If the heat transfer coefficient (h [=] W/m^2-K), which can be seen as the ratio between thermal conductivity and the thickness of the boundary layer, is believed to be function of the stream velocity (u [=] m/s), the specific heat capacity (c_p [=] J/kg-K), diameter of the pipe (D [=] m), thermal conductivity of the fluid (k [=] W/m-K), and the fluid density (ρ [=] kg/m^3), state the relation between all these variables in dimensionless form.

Dimensionless analysis Π-Lopez method

- List all the variables starting with dependent, then independent and the end the properties
- Express the units, and write all of them in base units (m, kg, s, A, K, mol, cd) to understand the dimensions.

h	V	Cp	D	k	ρ	μ
W/(m^2-K)	m/s	J/(kg-K)	m	W/m-K	kg/m^3	Pa-s
kg/(s^3-K)	m/s	m^2/(s^2-K)	m	kg-m/s^3-K	kg/m^3	kg/(m-s)
$M^1T^{-3}\Theta^{-1}$	L^1T^{-1}	$L^2T^{-1}\Theta^{-1}$	L^1	$M^1L^1T^{-3}\Theta^{-1}$	M^1L^{-3}	$M^1L^{-1}T^{-1}$
$M^1L^0T^{-3}\Theta^{-1}$	$M^0\,L^1T^{-1}\Theta^0$	$M^0L^2T^{-1}\Theta^{-1}$	$M^0L^0T^0\Theta^0$	$M^1L^1T^{-3}\Theta^{-1}$	$M^1L^{-3}T^0\Theta^0$	$M^1L^{-1}T^{-1}\Theta^0$

N=kg-m/s^2
J=N-m = kg-m^2/s^2
W=J/s = N-m/s = kg-m^2/s^3
Pa=N/m^2 = kg / (m-s^2)
Pa-s= kg / (m-s)

M=mass [kg]
L = length [m]
T = time [s]
Θ = Temperature [K]

Select these variables, you want to correlate, estimate, or these that require more complex equipment to determine, or need to be measured

Dependent variable

Independent variables

$\underline{\underline{A}}$

Size, physical and transport properties

Select these variables, easy to obtain, by specifications, database, or any other reliable source.

Primary dimension

$$
\begin{array}{c|ccc|cccc}
 & h & u & c_p & D & k & \rho & \mu \\
\hline
M & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
L & 0 & 1 & 2 & 1 & 1 & -3 & -1 \\
\Theta & -3 & -1 & -2 & 0 & -3 & 0 & -1 \\
T & -1 & 0 & -1 & 0 & -1 & 0 & 0
\end{array}
$$

$\underline{\underline{B}}$

$$\Pi_1 = h \quad D^{\nu_{11}} \quad k^{\nu_{21}} \quad \rho^{\nu_{31}} \quad \mu^{\nu_{41}}$$

$$\Pi_2 = u \quad D^{\nu_{12}} \quad k^{\nu_{22}} \quad \rho^{\nu_{32}} \quad \mu^{\nu_{42}}$$

$$\Pi_3 = Cp \quad D^{\nu_{13}} \quad k^{\nu_{23}} \quad \rho^{\nu_{33}} \quad \mu^{\nu_{43}}$$

$$\underline{\underline{\nu}} = -\underline{\underline{B}}^{-1}\underline{\underline{A}}$$

k [=] W/m-K [=] kg-m /s^3-K [=] $\underbrace{M^{+1}L^{+1}\Theta^{-3}T^{-1}}$

- Organize in a matrix form, take the square matrix of physical properties, and obtain the coefficients of the dimensionless numbers.

We have used $\boldsymbol{u}$ for velocity, and the Greek letter $\boldsymbol{\nu}$ for exponents

$$\underline{\underline{\nu}} = -\underline{\underline{B}}^{-1}\,\underline{\underline{A}}$$

$$\underline{\underline{\nu}} = \begin{array}{ccc} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \\ \end{array}$$

Exponents of the common parameters in the dimensionless number 1	Exponents of the common parameters in the dimensionless number 2	Exponents of the common parameters in the dimensionless number 3

$$\Pi_1 = h \quad D^{1} \quad k^{-1} \quad \rho^{0} \quad \mu^{0}$$

$$\Pi_2 = u \quad D^{1} \quad k^{0} \quad \rho^{1} \quad \mu^{-1}$$

$$\Pi_3 = c_p \quad D^{0} \quad k^{-1} \quad \rho^{0} \quad \mu^{1}$$

$$\Pi_1 = h\,D/k \qquad Nu = h\,D/k$$

$$\Pi_2 = u\,D\,\rho/\mu \qquad Re = \rho\,u\,D/\mu$$

$$\Pi_3 = c_p\,\mu/k \qquad Pr = \mu\,c_p/k$$

$$Nu = f(Re, Pr)$$

Some empirical correlations have the form:

$$Nu = \alpha\,Re^{\beta}\,Pr^{\gamma}$$

Problem No.3: Try to work heat transfer coefficient, density difference, specific heat capacity, surface tension, heat of vaporization, diameter, temperature difference, gravity, conductivity, density and viscosity.

$$h \quad \Delta\rho \quad c_p \quad \sigma \quad \lambda \quad D \quad \Delta T \quad g \quad k \quad \rho \quad \mu$$

1. Organize variables, first dependent variable, then the independent variables, and at the end the physical and/or transport properties.
2. Write the units of each variable, and express all of them in base units.
3. If two variables have the same units, only include one of them, if you really need to include all of them, then include the variable as the product of the variable with any other variable, for instance in the present problem do not include $\Delta\rho$, include $\Delta\rho\ g$ instead. Or remove from the analysis one dimensionless number that is the result of dividing variables with the same units like $\Delta\rho/\rho$.
4. Obtain the matrix using each column for one variable and rows for the power of each base unit.
5. The square matrix of the end of the arrangement will be the matrix called ***B***, which contains the exponent of the base units for the terms that are common in the dimensionless numbers, and the remaining matrix will be the Matrix A, that contains all the variables that are the base of each dimensionless number

Group as a ratio or do not include variables with the same units as one variable taken previously

What about this, if you have previous knowledge and aware that you can group variables

$h \quad g\,\Delta\rho \quad c_p \quad \sigma_T \quad \lambda \quad \Delta T \quad D \quad k \quad \rho \quad \mu$

Prove that some of the dimensionless number can be:

$$Nu = h\,D / k$$

$$Gr = \rho\,\Delta\rho\,g\;D^3 / \mu^2$$

$$Pr = \mu\,c_p / k$$

$$Bo = \Delta\rho\,g\,D^2 / \sigma_T$$

$$Ja = c_p\,\Delta T / \lambda$$

Discuss about variations in surface tension with temperature….

$$Ma = (-d\sigma_T/dT)L\,\Delta T\,\rho\,c_p / (k\,\mu)$$

Similarity: Transformation may be employed to reduce a partial differential equation (PDE) from an equation in *n*-independent variable into an equation in (*n*-1) variables

$$k \frac{\partial^2 T}{\partial y^2} = \rho \, c_p \frac{\partial T}{\partial t}$$

$$\left[\frac{k}{\rho \, c_p}\right] \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial t}$$

$$D \frac{\partial^2 C}{\partial y^2} = \frac{\partial C}{\partial t}$$

$$[D] \frac{\partial^2 C}{\partial y^2} = \frac{\partial C}{\partial t}$$

$$\mu \frac{\partial^2 u}{\partial y^2} = \rho \frac{\partial u}{\partial t}$$

$$\left[\frac{\mu}{\rho}\right] \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

$$\eta = \frac{y}{2\sqrt{D \, t}}$$

$$\eta \frac{dF}{d\eta} + \frac{1}{2} \frac{d^2 F}{d\eta^2} = 0$$

$$F = \begin{cases} T \\ C \\ u \end{cases}$$

$$D = \begin{cases} \alpha \\ D \\ \nu \end{cases}$$

Similitude or analogy: problems from another field that have the same mathematical equation

Thermal diffusivity $\alpha = \dfrac{k}{\rho \, c_p}$

D
Molecular diffusivity

momentum diffusivity
Or kinematic viscosity $\nu = \dfrac{\mu}{\rho}$

In this case the dimensionless variable is $\boldsymbol{\eta}$

To estimate the mass flow rates at the beginning of the process, The Similarity solution approach can be used.

The diffusion equation coupled with the mass balance in a stagnant and inert fluid

$$D\frac{\partial^2 C}{\partial y^2} = \frac{\partial C}{\partial t}$$

Using separation of variables

$$C = C(t,z) = C(t,\eta)$$

Via dimension analysis, a dimensionless variable grouping the independent variables can be of the form:

$$\frac{C}{t}[=]D\frac{C}{z^2} \qquad \frac{z^2}{t\,D}[=]1$$

$$\frac{z}{\sqrt{t\,D}}[=]1 \qquad \eta = \frac{k\,z}{\sqrt{t\,D}}$$

This will be the new dimensionless variable

Lets assume the solution can be written as a product of two functions

$$C = C(t,\eta) = \Theta(t)F(\eta) \qquad dC = \Theta\,dF + Fd\Theta$$

$$\frac{\partial C}{\partial t} = \Theta\frac{\partial F}{\partial t} + F\frac{d\Theta}{dt} \qquad \frac{\partial C}{\partial t} = \Theta\frac{dF}{d\eta}\frac{\partial \eta}{\partial t} + F\frac{d\Theta}{dt}$$

$$\frac{\partial C}{\partial z} = \Theta\frac{\partial F}{\partial z} = \Theta\frac{dF}{d\eta}\left(\frac{\partial \eta}{\partial z}\right)$$

$$d\left(\frac{\partial C}{\partial z}\right) = d\left(\Theta\frac{\partial F}{\partial z}\right) = \Theta\,d\left[\frac{dF}{d\eta}\left(\frac{\partial \eta}{\partial z}\right)\right] + \frac{dF}{d\eta}\left(\frac{\partial \eta}{\partial z}\right)d\Theta$$

$$\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right) = \Theta\frac{dF}{d\eta}\frac{\partial}{\partial z}\left[\left(\frac{\partial \eta}{\partial z}\right)\right] + \Theta\left(\frac{\partial \eta}{\partial z}\right)^2\frac{d}{d\eta}\frac{dF}{d\eta}$$

Rewriting the new independent variables and the relationships with the original ones

$$D\frac{\partial^2 C}{\partial y^2} = \frac{\partial C}{\partial t}$$

$$C = C(t,z) = C(t,\eta)$$

$$C = C(t,z) = \Theta(t)F(\eta)$$

$$\eta = \frac{k\ z}{\sqrt{t\ D}}$$

$$\frac{\partial\eta}{\partial t} = \frac{k\ z}{\sqrt{D}}\frac{\partial}{\partial t}\left[t^{-1/2}\right]$$

$$\frac{\partial\eta}{\partial t} = -\left[t^{-\frac{3}{2}}\right]\frac{k\ z}{2\sqrt{D}} = -\frac{k\ z}{2\ t\sqrt{t\ \ D}} = -\frac{\eta}{2\ t}$$

$$\frac{\partial\eta}{\partial z} = \frac{k}{\sqrt{t\ \ D}}\frac{\partial}{\partial z}[z] = \frac{k}{\sqrt{t\ \ D}} = \frac{\eta}{z}$$

$$\frac{\partial^2\eta}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial\eta}{\partial z}\right] = \frac{\partial}{\partial z}\left[\frac{k}{\sqrt{t\ \ D}}\right] = 0$$

Rewriting the differential equation in terms of similarity transformation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial z^2}$$

$$\frac{\partial C}{\partial t} = \Theta \frac{dF}{d\eta}\frac{\partial \eta}{\partial t} + F\frac{d\Theta}{dt}$$

$$\frac{\partial C}{\partial t} = -\frac{\Theta\eta}{2t}\frac{dF}{d\eta} + F\frac{d\Theta}{dt}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right) = \Theta\frac{dF}{d\eta}\frac{\partial}{\partial z}\left[\left(\frac{\partial \eta}{\partial z}\right)\right] + \Theta\left(\frac{\partial \eta}{\partial z}\right)^2\frac{d}{d\eta}\left[\frac{dF}{d\eta}\right]$$

$$\frac{\partial}{\partial z}\left(\frac{\partial C}{\partial z}\right) = \Theta\left(\frac{\eta}{z}\right)^2\left[\frac{d^2F}{d\eta^2}\right]$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial z^2} \quad \Rightarrow \quad -\frac{\Theta\eta}{2t}\frac{dF}{d\eta} + F\frac{d\Theta}{dt} = D\Theta\left(\frac{\eta}{z}\right)^2\left[\frac{d^2F}{d\eta^2}\right]$$

$$-\frac{\eta}{2F}\frac{dF}{d\eta} + \frac{t}{\Theta}\frac{d\Theta}{dt} = \frac{1}{F}Dt\left(\frac{\eta}{z}\right)^2\left[\frac{d^2F}{d\eta^2}\right] \qquad \eta = \frac{k\,z}{\sqrt{tD}} \qquad \eta^2 = \frac{k^2\,z^2}{tD} \qquad tD\frac{\eta^2}{z^2} = k^2$$

$$-\frac{\eta}{2F}\frac{dF}{d\eta} + \frac{t}{\Theta}\frac{d\Theta}{dt} = \frac{1}{F}k^2\left[\frac{d^2F}{d\eta^2}\right] \quad \Rightarrow \quad \frac{t}{\Theta}\frac{d\Theta}{dt} = \frac{\eta}{2F}\frac{dF}{d\eta} + \frac{1}{F}k^2\left[\frac{d^2F}{d\eta^2}\right]$$

$$\frac{t}{\Theta}\frac{d\Theta}{dt} = \frac{\eta}{2F}\frac{dF}{d\eta} + \frac{1}{F}k^2\left[\frac{d^2F}{d\eta^2}\right]$$

The simplest trial function for Θ is 1, and then the differential equation is simplified to:

$$\frac{\eta}{2F}\frac{dF}{d\eta} + \frac{1}{F}k^2\left[\frac{d^2F}{d\eta^2}\right] = 0 \quad \Rightarrow \quad \frac{\eta}{2}\frac{dF}{d\eta} + k^2\left[\frac{d^2F}{d\eta^2}\right] = 0$$

Rewriting it and integrating

$$\frac{\eta}{2}\frac{dF}{d\eta} + k^2\frac{d}{d\eta}\left[\frac{dF}{d\eta}\right] = 0 \quad \Rightarrow \quad -\frac{\eta}{2k^2}d\eta = \frac{d\left[\frac{dF}{d\eta}\right]}{\left[\frac{dF}{d\eta}\right]}$$

$$-\frac{\eta^2}{4k^2} + k_2 = \ln\frac{dF}{d\eta} \quad \Rightarrow \quad k_3e^{-\eta^2} = \frac{dF}{d\eta}$$

If k^2=1/4, the equation takes a simpler form

$$\eta = \frac{z}{2\sqrt{tD}}$$

Integrating again

$$k_3 e^{-\eta^2} = \frac{dF}{d\eta} \qquad k_3 \int_{\eta_1}^{\eta} e^{-\eta^2} d\eta = F(\eta) - F(\eta_1)$$

This equation can be solved form $0<\eta<\infty$

The concentration takes the form:

$$\frac{\int_0^{\eta} e^{-\eta^2} d\eta}{\int_0^{\infty} e^{-\eta^2} d\eta} = \frac{F(\eta) - F(\eta_0)}{F(\eta_\infty) - F(\eta_0)}$$

The denominator is a known integral .

$$I = \int_0^\infty e^{-x^2} dx$$

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} dxdy = \int_0^\infty \int_0^{\pi/2} e^{-r^2} r d\theta dr = \frac{\pi}{2} \int_0^\infty \frac{e^{-r^2}}{(-2)} d\left(-r^2\right)$$

$$I^2 = -\frac{\pi}{4} \int_0^\infty e^{-r^2} d\left(-r^2\right) = -\frac{\pi}{4} e^{-r^2} \Big|_0^\infty = \frac{\pi}{4}$$

$$I = \frac{\sqrt{\pi}}{2}$$

"erf function"

$$\frac{F(\eta) - F(\eta_0)}{F(\eta_\infty) - F(\eta_0)} = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\eta^2} d\eta = erf(\eta)$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial z^2}$$

$$\eta = \frac{z}{2\sqrt{tD}}$$

The erf function can be used to calculate concentration in a semi-infinite medium

$$\frac{F(\eta)-F(\eta_0)}{F(\eta_\infty)-F(\eta_0)} = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\,d\eta = erf(\eta)$$

If we want to know the concentration of oxygen and/or nitrogen in the water

$$\frac{C-C_o}{C_\infty - C_o} = \frac{C_o - C}{C_o - C_\infty} = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\,d\eta = erf(\eta)$$

$$1-\frac{C-C_\infty}{C_o - C_\infty} = \frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\,d\eta = erf(\eta)$$

$$\frac{C-C_\infty}{C_o - C_\infty} = 1-\frac{2}{\sqrt{\pi}}\int_0^{\eta} e^{-\eta^2}\,d\eta = erfc(\eta)$$

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial z^2} \qquad\qquad \eta = \frac{z}{2\sqrt{tD}}$$

$$\frac{C - C_\infty}{C_o - C_\infty} = 1 - \frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\, d\eta = erfc(\eta)$$

To calculate the instantaneous flux, the first derivative of concentration respect to position is required , and evaluation at the origin in needed.

$$-\frac{1}{(C_0 - C_\infty)}\frac{\partial C}{\partial z} = \frac{2}{\sqrt{\pi}} e^{-\eta^2}\frac{\partial \eta}{\partial z} \qquad -D\frac{\partial C}{\partial z}\bigg|_{z=0} = D\frac{2(C_0 - C_\infty)}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{tD}}\right)e^{-\eta^2}\bigg|_{\eta=0}$$

$$J^* = -D\frac{\partial C}{\partial z}\bigg|_{z=0} = \left[\sqrt{\frac{D}{t\pi}}\right](C_0 - C_\infty)$$

Velocity of rising bubble

1. Write the equations of velocity of bubbles in dimensionless form.

Problem No.4

Verify if the equations given (The three equations are used to estimate the velocity of a rising bubble within a immiscible fluid, these correspond to three different flow regimes) obey the rule of dimensional homogeneity. One case will be analyzed and later all of them using a different approach

Spherical regime

$$v = \frac{g\,d_p^2 \Delta\rho}{6\,\mu_L}\left[\frac{1+\mu_G/\mu_L}{2+3\,\mu_G/\mu_L}\right]$$

Ellipsoidal regime 0.25<*Bo*<40

$$v^2 = \frac{2.14\,\sigma}{\rho_L\,d_p} + 0.505\,g\,d_p$$

Spherical cap regime *Bo*>40 and Re>1.2

$$v = \frac{2}{3}\sqrt{\frac{g d_p \Delta\rho}{2\rho_L}}$$

Analysis for the ellipsoidal regime

First step, check the units of each variable.

v = velocity [m/s]
σ = surface tension [N/m]
ρ_L = Density of liquid [kg/m^3]
g = gravitational field [m/s^2]
d_p = bubble diameter [m]

Second step, write the derived units in terms of base units:
Force units in base units is …[N] = [kg m / s^2]

Last step, make sure each term of the equation has the same units, in this case [m^2/s^2]

$$\left[\frac{\mathrm{m}}{\mathrm{s}}\right]^2 = \frac{\left[\frac{\mathrm{kg\ m}}{\mathrm{m\ s^2}}\right]}{\left[\frac{\mathrm{kg}}{\mathrm{m^3}}\right][\mathrm{m}]} + \left[\frac{\mathrm{m}}{\mathrm{s^2}}\right][\mathrm{m}]$$

Recast the previous equations in dimensionless form
And use any of the following dimensionless numbers

$Bo = \frac{g\ D^2 \Delta\rho}{\sigma}$	Bond or Eötvös
$Mo = \frac{g\ \mu_L^4 \Delta\rho}{\sigma^3 \rho_L^2}$	Morton
$Re = \frac{\rho_L\ v\ D}{\mu_L}$	Reynolds
$Fr = \frac{\rho_L\ v^2}{g\ D\ \Delta\rho}$	Froude
$We = \frac{D\ \rho_L\ v^2}{\sigma}$	Weber
$Ca = \frac{\mu_L\ v}{\sigma}$	Capillary
$Ar = \frac{g\ D^3\ (\Delta\rho/\rho_L)}{(\mu_L/\rho_L)^2}$	Archimedes

Relationship between
Dimensionless numbers

$$Bo = \frac{We}{Fr}$$

$$Ar^2 = \frac{Bo^3}{Mo}$$

$$Ar = \frac{Re^2}{Fr}$$

$$Re^4\ Mo = Bo\ We^2$$

$$Ri = \frac{Ar}{Re^2}$$

Richardson

Usually the properties of the continuous phase are used (in this case liquid or denser phase)

$Fo = \frac{t\ \mu_L}{\rho_L L^2}$	Fourier
$La = \frac{\rho_L \sigma\ L}{\mu_L^2}$	Laplace

All these dimensionless numbers will be studied later

Another easy approach to verify dimensional homogeneity is to recast in dimensionless form each equation (this is using dimensionless numbers). If only dimensionless terms are involved, then obeys the rule.

Spherical regime

$$v = \frac{g\, d_p^2 \Delta\rho}{6\, \mu_L}\left[\frac{1 + \mu_G/\mu_L}{2 + 3\, \mu_G/\mu_L}\right] \qquad Re = \frac{Ar}{6}\left[\frac{1 + \mu_G/\mu_L}{2 + 3\, \mu_G/\mu_L}\right]$$

Ellipsoidal regime 0.25<*Bo*<40

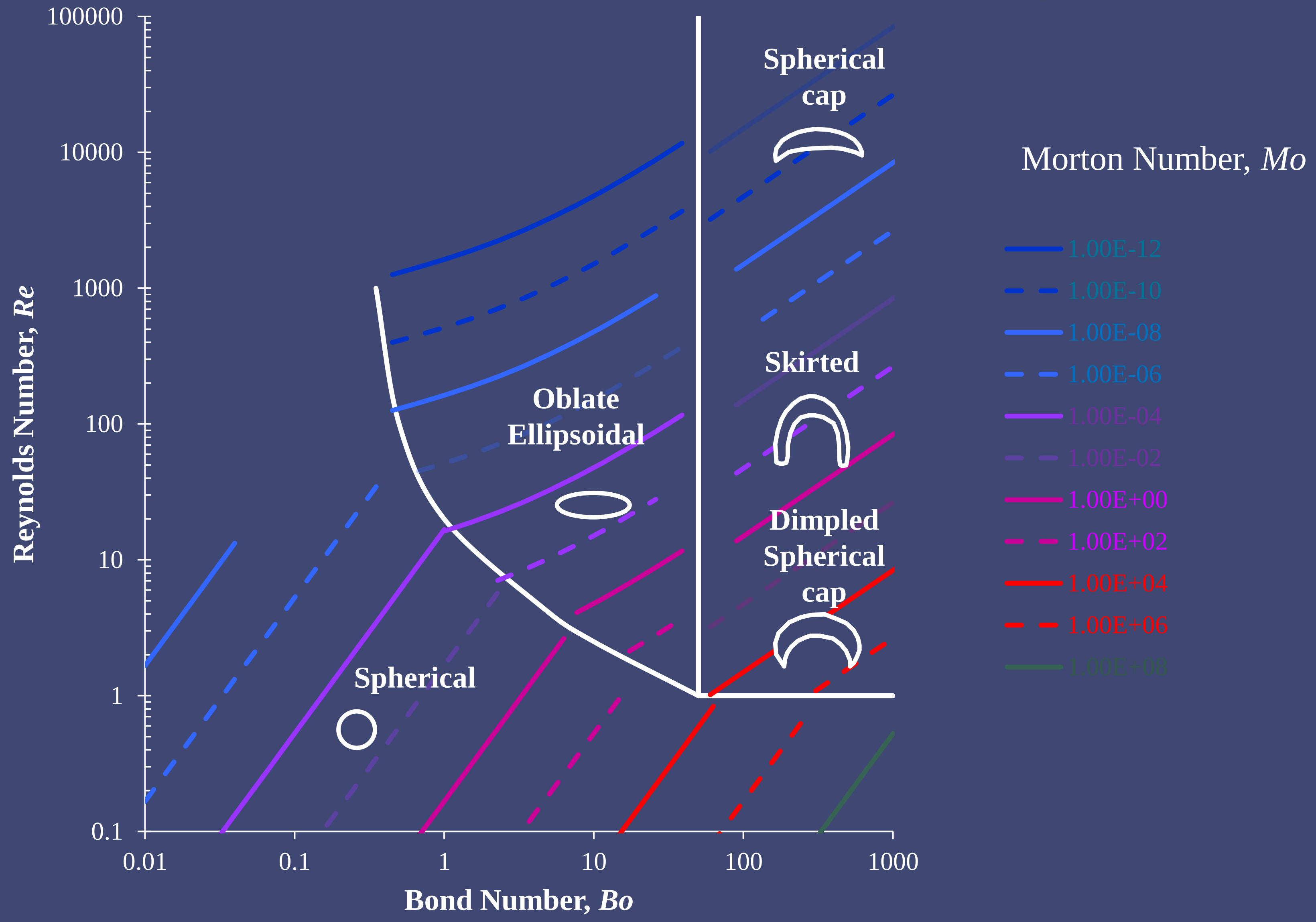
$$v^2 = \frac{2.14\, \sigma}{\rho_L\, d_p} + 0.505\, g\, d_p \qquad 1 = \frac{2.14}{We} + 0.505\, \frac{\rho_L}{Fr\, \Delta\rho}$$

Spherical cap regime *Bo*>40 and Re>1.2

$$v = \frac{2}{3}\sqrt{\frac{g\, d_p \Delta\rho}{2\rho_L}} \qquad Fr = \frac{2}{9}$$

Note: Any ratio of variables of the same dimensions is dimensionless, like ratio of viscosities, densities, and so on…

Background 3



Adapted from Luz Amaya-Bower and Taehum Lee, Computers & Fluids 39 (2010) 1191-1207

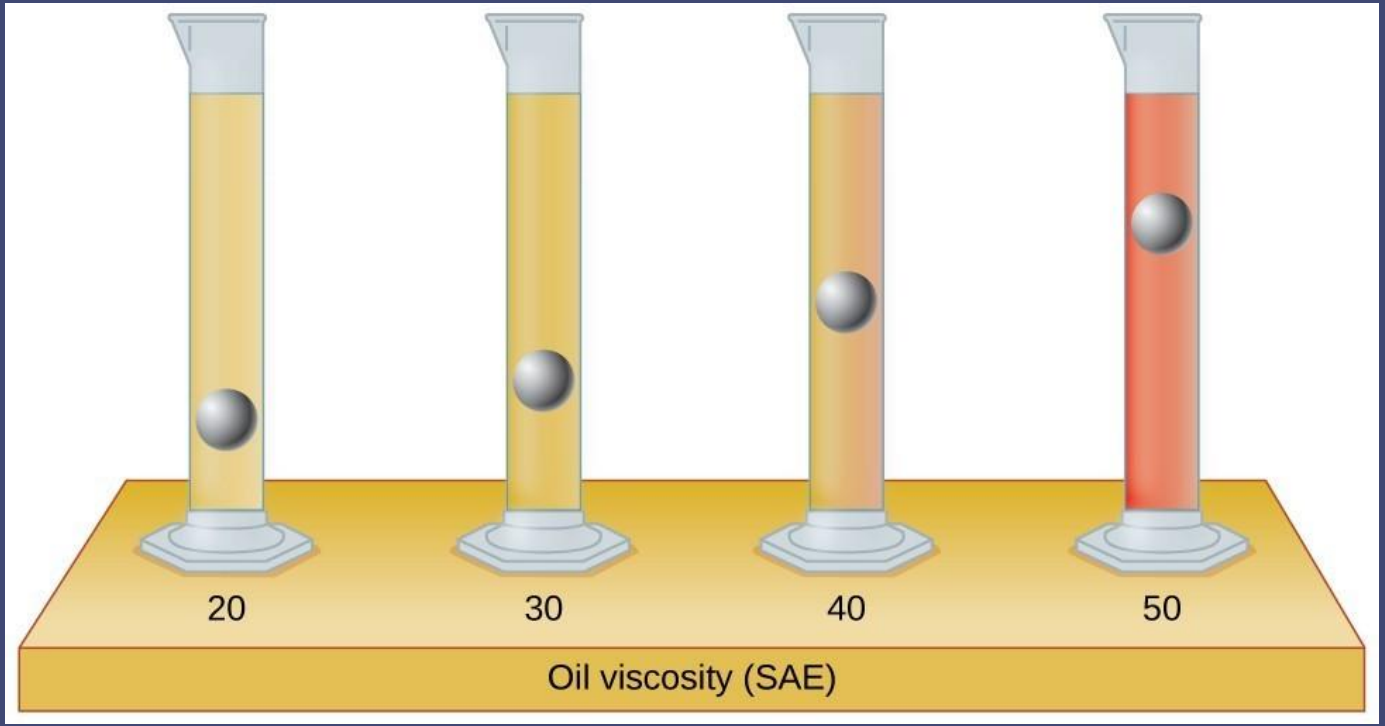
What are all the forces in a moving bubble or drop:

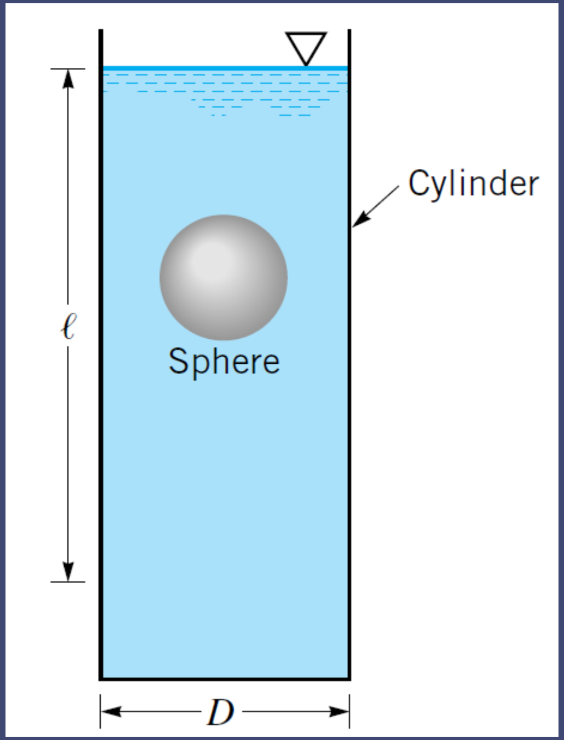
Drag
Lift
Pressure
Interactive
Gravity
Virtual Mass
Basset

Problem No.5 : The viscosity, μ, of a liquid can be measured by determining the time, t, it takes for a sphere of diameter, d, to settle slowly through a distance, ℓ, in a vertical cylinder of diameter, D, containing the liquid. Assume that

$$t = f\,(\ell, d, D, \mu, \Delta\gamma)$$

Where $\Delta\gamma$ is the difference in specific weights between the sphere and the liquid. Use dimensionless analysis to show how t, is related to μ, and describe how such an apparatus might be used to measure viscosity.





	t	d	μ	$\Delta\gamma$
M	0	0	1	1
L	0	1	-1	-2
Θ	1	0	-1	-2

$$\Pi_1 = \frac{d}{l} \qquad \Pi_2 = \frac{d}{D}$$

$$\Pi_3 = \Delta\gamma \quad t^{1} \quad d^{1} \quad \mu^{-1}$$

$$\Pi_3 = \frac{\Delta\gamma \, t \, d}{\mu}$$

Someone asked: it is feasible to spot dimensionless numbers discussed in class ?

Answer: Yes, for instance
This is the case, it collects the ratio between Archimedes and Reynolds, and a geometrical ratio

$$\Pi_3 = \frac{\Delta\gamma \, t \, d^2}{\mu \, l}\left(\frac{l}{d}\right) = \frac{Ar}{Re}\left(\frac{l}{d}\right)$$

$$\begin{array}{c|ccc} & \mu & D & L \\ \hline M & 1 & 0 & 0 \\ L & -1 & 1 & 1 \\ \Theta & -1 & 0 & 0 \end{array} \quad \underline{\underline{A}} \qquad \begin{array}{ccc} d & \Delta\gamma & t \\ \hline 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{array} \quad \underline{\underline{B}}$$

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{array} \qquad \begin{array}{ccc} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{array} \quad \underline{\underline{B}}^{-1}$$

$$\begin{array}{ccc} -1 & -1 & -1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{array} \quad \underline{\underline{\nu}} = -\underline{\underline{B}}^{-1}\,\underline{\underline{A}}$$

$$\Pi_1 = \mu\, d^{-1}\, \Delta\gamma^{-1}\, t^{-1}$$

$$\Pi_3 = L\, d^{-1}\, \Delta\gamma^{0}\, t^{0}$$

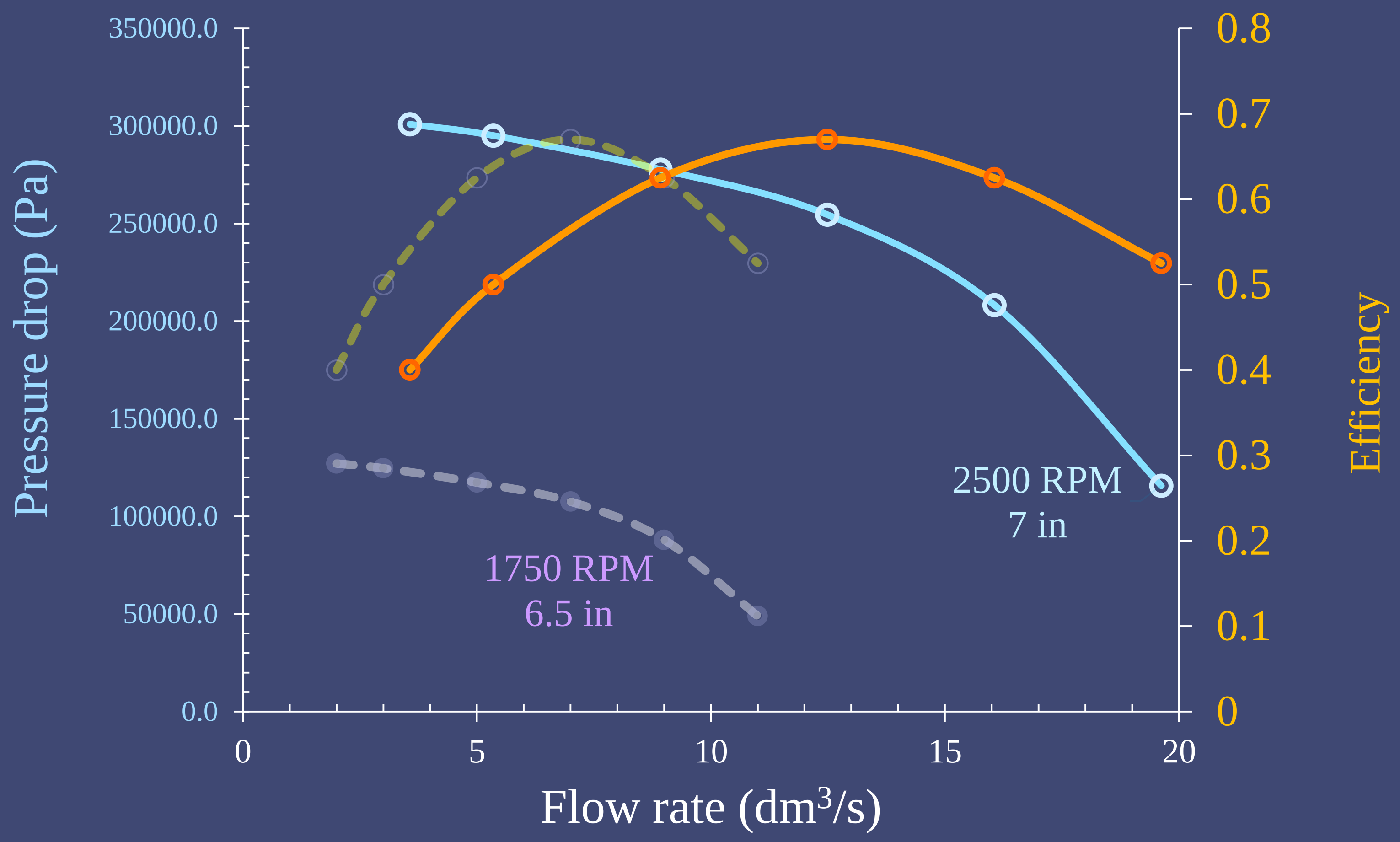
$$\Pi_2 = D\, d^{-1}\, \Delta\gamma^{0}\, t^{0}$$

ProblemNo.6: The curve of a pump, is used to estimate the pressure increase in that device as a function of volume flow rate. In Table 1, data for a pump operating a 1750 rpm, and a impeller diameter of 6.5-in is given. Determine the curve of the pump, when a similar 7-in impeller diameter pump runs at a speed of 2500 rpm.

Flow (dm^3/s)	**ΔP(Pa)**	**Efficiency**	**NPSHR (m)**
2	127000	0.4	1.67
3	124700	0.5	1.67
5	117300	0.625	1.67
7	107500	0.67	1.67
9	8800	0.625	2.5
11	48900	0.525	4.29

It is suggested from dimensional analysis that pumps performance is generalized with the following dimensionless numbers.

$$\Pi_P = \frac{\Delta p}{\rho(\Omega D)^2} \quad \Pi_F = \frac{\dot{V}}{\Omega\, D^3} \quad \Pi_{NPSH} = \frac{g\ NPSH}{(\Omega D)^2} \quad \Pi_E = \eta$$



By increasing RPM’s one may have higher pressure difference, and increase the pump efficiency for flow rates higher than 12 liters/s

flow m^3/s	Flow (dm^3/s)	$\Delta p(Pa)$	η	$NPSH_R$ (m)
0.002	2	127000	0.4	1.67
0.003	3	124700	0.5	1.67
0.005	5	117300	0.625	1.67
0.007	7	107500	0.67	1.67
0.009	9	8800	0.625	2.5
0.011	11	48900	0.525	4.29

Π_P	Π_F	Π_{NPSH}
0.139149	0.002425	0.017896
0.136629	0.003638	0.017896
0.128521	0.006063	0.017896
0.117784	0.008488	0.017896
0.009642	0.010913	0.026791
0.053578	0.013338	0.045973

flow m^3/s	Flow (dm^3/s)	$\Delta p(Pa)$	η	$NPSH_R$ (m)
0.003569	3.568503	300591.716	0.408827	3.9526627
0.005353	5.352754	295147.929	0.507356	3.9526627
0.008921	8.921256	277633.136	0.630517	3.9526627
0.01249	12.48976	254437.87	0.674855	3.9526627
0.016058	16.05826	20828.4024	0.630517	5.9171598
0.019627	19.62676	115739.645	0.531988	10.153846

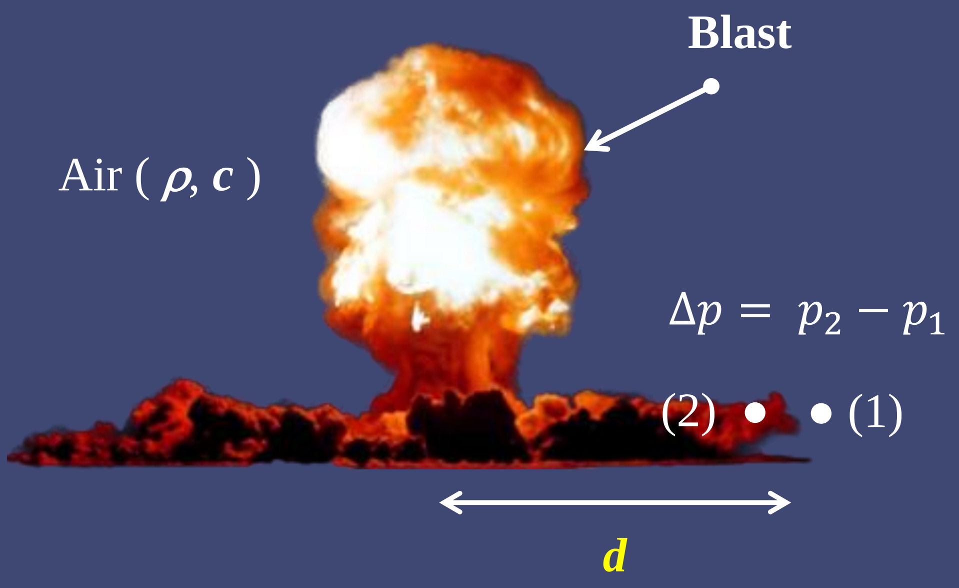
ρ	997	kg/m^3	D	0.1651	m
Ω	1750	rpm	D	6.5	in
Ω	183.2596	1/s	g	9.81	m/s^2

ρ	997	kg/m^3	D	0.1778	m
Ω	2500	rpm	D	7	in
Ω	261.799388	1/s	g	9.81	m/s^2

$$\Pi_P = \frac{\Delta p}{\rho(\Omega D)^2} \qquad \Pi_F = \frac{\dot{V}}{\Omega\, D^3} \qquad \Pi_{NPSH} = \frac{g\ NPSH}{(\Omega D)^2}$$

$$\frac{1-\eta_2}{1-\eta_1} = \left(\frac{D_1}{D_2}\right)^{1/5}$$

Problem No.7: The pressure rise, ***Δp***, across a blast wave, as shown in the figure, is assumed to be a function of the amount of energy released in the explosion, ***E***, the air density, ***ρ***, the speed of sound, ***c***, and the distance from the blast, ***d*** (a) Put this relationship in dimensionless form. (b) Consider two blasts: the prototype blast with energy release ***E*** and a model blast with 1/1000th the energy release (E_m = ***E*** / 1000), At what distance from the model blast will the pressure rise be the same as that a distance of 1 mile from the prototype blast ?



	Δp	E	ρ	c	d
M	1	1	1	0	0
L	-1	2	-3	1	1
Θ	-2	-2	0	-1	0

$$\underbrace{\Delta p \quad E}_{\underline{\underline{A}}} \qquad \underbrace{\rho \quad c \quad d}_{\underline{\underline{B}}}$$

$$\underline{\underline{\nu}} = -\underline{\underline{B}}^{-1}\underline{\underline{A}} =$$

-1	-1
-2	-2
0	-3

$$\Pi_P = \Delta p\, \rho^{-1} c^{-2} d^{0} = \frac{\Delta p}{\rho\, c^2}$$

$$\Pi_E = E\, \rho^{-1} c^{-2} d^{-3} = \frac{E}{\rho\, c^2 d^3}$$

$$\frac{\Delta p}{\rho\, c^2} = f\left(\frac{E}{\rho\, c^2 d^3}\right)$$

$$\frac{\Delta p_p}{\rho_p\, c_p^2} = \frac{\Delta p_m}{\rho_m\, c_m^2}$$

If both prototype and model are to operate under atmospheric conditions, then density and speed of sound remain the same under both conditions, then the pressure number keeps the same, and the energy number must be the same

$$\frac{E_p}{\rho_p\, c_p^2\, d_p^3} = \frac{E_m}{\rho_m\, c_m^2\, d_m^3}$$

$$\frac{d_m}{d_p} = \left[\frac{E_m \rho_p\, c_p^2}{E_p\, \rho_m\, c_m^2}\right]^{1/3} = \frac{1}{10}$$

Answer: 0.1 miles

Exploring variables technique

So far we have identified four different types of forces, if we group them in ratios, six dimensionless groups can be constructed

$$\frac{N\,!}{m\,!\,(N-m)!} = \frac{4\,!}{2!\,2!} = 6$$

Inertial Force: $\underline{F} = \underline{v}\ \dot{m}$

$$\underline{F} = \frac{d(m\underline{v})}{dt} \sim \underline{v}\frac{dm}{dt} \sim \underline{v}\frac{\rho\, dV}{dt} \sim \underline{v}\frac{\rho\, A\, d\underline{x}\cdot\underline{\hat{n}}}{dt} \sim \underline{v}\,\rho\, A\, \underline{v}\cdot\underline{\hat{n}}$$

$$\underline{F} \sim \underline{v}\,\rho\, L^2\, \underline{v}\cdot\underline{\hat{n}} \sim \rho v^2 L^2$$

Gravitational Force: $\underline{F} = m\,\underline{g}$

$$\underline{F} = m\,\underline{g} \sim \rho\, V\, \underline{g} \sim \rho\, L^3\, g$$

Surface tension force: $\underline{F} = \sigma\, L\, \underline{\hat{t}}$

$$\underline{F} = \sigma\, L\, \underline{\hat{t}} \sim \sigma\, L$$

Viscous Force: $\underline{F} = \underline{\hat{n}}\cdot\underline{\underline{\tau}}\, A$

$$\underline{F} \sim \underline{\hat{n}}\cdot\mu\left(\left[\underline{\nabla}\,\underline{v}\right] + \left[\underline{\nabla}\,\underline{v}\right]^T\right) A/2 \sim \mu\, v\, L$$

$\underline{\hat{n}}$	Unit normal vector	$\underline{\underline{\tau}}$	Viscous stress tensor	σ	Surface tension	μ	Viscosity
$\underline{\hat{t}}$	Unit tangent vector	ρ	Density	v	Velocity	L	Contact length or characteristic length

Inertial Force: $\underline{F} = \underline{v}\ \dot{m}$ $\quad \underline{F} \sim \rho\, v^2\, L^2$

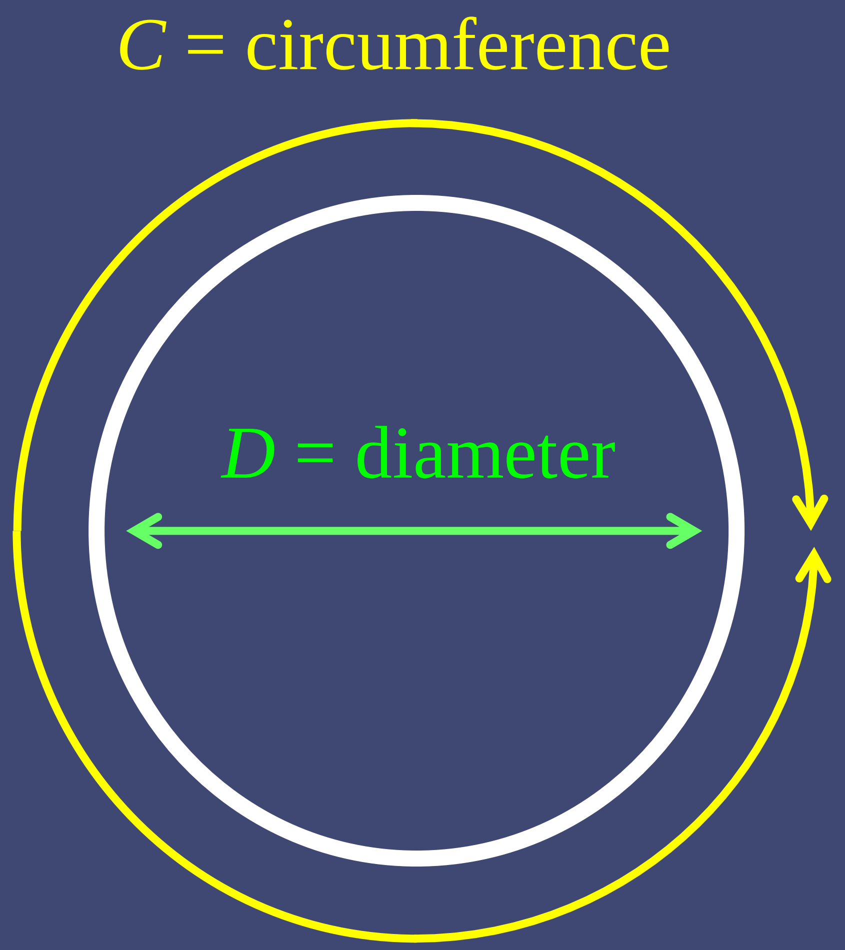
Gravitational Force: $\underline{F} = m\, \underline{g}$ $\quad \underline{F} \sim \rho\, L^3\, g$

Surface tension force: $\underline{F} = \sigma\, L\, \underline{\hat{\tau}}$ $\quad \underline{F} \sim \sigma\, L$

Viscous Force: $\underline{F} = \underline{\hat{n}} \cdot \underline{\underline{\tau}}\, A$ $\quad \underline{F} \sim \mu\, v\, L$

$\underline{\hat{n}}$	Unit normal vector	$\underline{\underline{\tau}}$	Viscous stress tensor	σ	Surface tension	μ	Viscosity
$\underline{\hat{\tau}}$	Unit tangent vector	ρ	Density	v	Velocity	L	Contact length or characteristic length

Dimensionless numbers, or dimensionless ratios



$$\pi = C/D$$

Inertial Force: $\underline{F} \sim \rho\, v^2\, L^2$

Gravitational Force: $\underline{F} \sim \rho\, L^3\, g$

Surface tension force: $\underline{F} \sim \sigma\, L$

Viscous Force: $\underline{F} \sim \mu\, v\, L$

$Re =$ Inertial/Viscous	Reynolds	$Re = \rho\, v^2\, L^2 / [\mu\, v\, L]$
$Fr =$ Inertial/Gravitational	Froude	$Fr = \rho\, v^2\, L^2 / [\rho\, L^3\, g]$
$We =$ Inertial/Surface tension	Weber	$We = \rho\, v^2\, L^2 / [\sigma\, L]$
$\chi = Fr/Re =$ Viscous/Gravitational	Unnamed	$\chi = Fr/Re = \mu\, v\, L / [\rho\, L^3\, g]$
$Ca =$ Viscous/Surface tension	Capillary	$Ca = \mu\, v\, L / [\sigma\, L]$
$Bo =$ Gravitational/Surface tension	Bound	$Bo = \rho\, L^3\, g / [\sigma\, L]$

$Re =$ Inertial/Viscous

$Fr =$ Inertial/Gravitational

$We =$ Inertial/Surface tension

$\chi = Fr/Re =$ Viscous/Gravitational

$Ca =$ Viscous/Surface tension

$Bo =$ Gravitational/Surface tension

$$Re = \frac{\rho\, v\, L}{\mu}$$

$$Fr = \frac{v^2}{L\, g}$$

$$We = \rho\, v^2 L\, /\sigma$$

$$\chi = Fr/Re = \frac{\mu\, v}{\rho\, L^2\, g}$$

$$Ca = \mu\, v\, /\sigma$$

$$Bo = \rho\, L^2\, g\, /\sigma$$

For two phase flow

Re = Inertial/Viscous

Fr = Inertial/Gravitational

We = Inertial/Surface tension

$\chi = Fr/Re$ = Viscous/Gravitational

Ca = Viscous/Surface tension

Bo = Gravitational/Surface tension

$$Re = \frac{\rho\, v\, L}{\mu}$$

$$Fr = \frac{\rho\, v^2}{\Delta\rho\, L\, g}$$

$$We = \rho\, v^2 L\, / \sigma$$

$$Fr/Re = \frac{\mu\, v}{\Delta\rho\, L^2\, g}$$

$$Ca = \mu\, v\, / \sigma$$

$$Bo = \Delta\rho\, L^2\, g\, / \sigma$$

$\Delta\rho$ Density difference

ρ Density

σ Interfacial tension

v Velocity

μ Viscosity

L Contact length or characteristic length

Viscosity and density are properties of the external phase

You can combine two of tem in any desired fashion and can obtain useful dimensionless numbers

$$Re = \frac{\rho\, v\, L}{\mu} \qquad Fr/Re = \frac{\mu\, v}{\Delta\rho\, L^2\, g}$$

$$Ar = \frac{Re}{(Fr/Re)} = \frac{Re^2}{Fr} = \frac{(\Delta\rho/\rho)\, L^3\, g}{(\mu/\rho)^2}$$

Archimedes number

We had identified different types of forces, but how to know which ones should be taken into consideration…

Problem No.8. Deep-water waves. Which forces most affect waves in large bodies of water ? Consider a series of waves that is characterized by wavelength (λ), a wave height (h), and wave speed (c). Use data observed in nature, where waves travel at a speed of 1 m/s, and a height of 10 cm.

$$Bo = \Delta\rho\, L^2\, g\, /\sigma$$

$$Bo = (996\ \mathrm{kg/m^3})(0.1\ \mathrm{m})^2(9.81\ \mathrm{m/s^2})/(72.8\times 10^{-3}\mathrm{N/m})$$

$$Bo = 1{,}342.14$$

$$Re = \frac{\rho\, v\, L}{\mu}$$

$$Re = \frac{(997\,\mathrm{kg/m^3})(1\ \mathrm{m/s})(0.1\ \mathrm{m})}{(0.89\times 10^{-3}\mathrm{Pa\ s})}$$

$$Re = 112{,}022.5$$

$$Fr = \frac{\rho\, v^2}{\Delta\rho\, L\, g}$$

$$Fr = \frac{(997\,\mathrm{kg/m^3})(1\ \mathrm{m/s})^2}{(996\,\mathrm{kg/m^3})(0.1\ \mathrm{m})(9.81\ \mathrm{m/s^2})}$$

$$Fr = 1.02$$

Conclusion: Only gravity and inertia affect the motion of Deep-water waves, Froude number indicates that gravitational forces, are as important as inertial. Reynolds number indicates that viscous forces are smaller compared with inertial, and Bond number indicates that gravitational forces are more important than Surface tension.

Deen, William M. Chem.Eng. Fluid Mechanics, Cambridge University Press, 2016

Problem No.9: Inkjet printing The breakup of liquid jet into drops is crucial for the performance on an inkjet printer. Which forces most affect the flow, once the jet has left the print nozzle. Reasonable length and velocity scales are the nozzle diameter and jet speed, respectively. Based on representative data in Wijsoff (2010), assume L=32 μm, v =7 m/s, μ = 10 cP, σ = 30 dyne/cm, ρ = 1 g/cm^3.

$$Re = \frac{\rho\, v\, L}{\mu}$$

$$Re = \frac{(1000\,\mathrm{kg/m^3})(7\ \mathrm{m/s})(32 \times 10^{-6}\mathrm{m})}{(10.0 \times 10^{-3}\mathrm{Pa\ s})}$$

$$Re = 22.4$$

$$Bo = \Delta\rho\, L^2\, g\, /\sigma$$

$$Bo = (999\ \ \mathrm{kg/m^3})(32 \times 10^{-6}\mathrm{m})^2(9.81\ \ \mathrm{m/s^2})/(30 \times 10^{-3}\mathrm{N/m})$$

$$Bo = 3.345 \times 10^{-4}$$

$$Fr = \frac{\rho\, v^2}{\Delta\rho\, L\, g}$$

$$Fr = \frac{(1000\,\mathrm{kg/m^3})(7\ \mathrm{m/s})^2}{(999\,\mathrm{kg/m^3})(32 \times 10^{-6}\mathrm{m})(9.81\ \ \mathrm{m/s^2})}$$

$$Fr = 156{,}246.97$$

$$We = \rho\, v^2 L\, /\sigma$$

$$We = \frac{(1000\,\mathrm{kg/m^3})(7\ \mathrm{m/s})^2(32 \times 10^{-6}\mathrm{m})}{(30 \times 10^{-3}\mathrm{N/m})}$$

$$We = 52.26$$

Conclusion: In this scenario gravitational forces are negligible, Bond number indicates that gravitational forces are negligible, compared with surface tension, Froude indicates as well that gravitational forces are negligible compared with inertial, .

H. Wijshoff / Physics Reports 491 (2010) 77177

Deen, William M. Chem.Eng. Fluid Mechanics, Cambridge University Press, 2016

In the original paper “H. Wijshoff / Physics Reports 491 (2010) 77177” they used the Ohnesorge number as a parameter to analyze the experimental, data. Calculate this dimensionless number and explain its physical meaning, and under what circumstances can be used and the purpose of using it.

The Ohnesorge Number

$$Re = \frac{\rho\, v\, L}{\mu}$$

$$We = \rho\, v^2 L\, /\sigma$$

$$Re = \text{Inertial}/\text{Viscous}$$

$$We = \text{Inertial}/\text{Surface tension}$$

$$Oh = \frac{We}{Re^2}$$

$$Oh = \frac{(\text{Viscous})^2}{(\text{Inertial})(\text{Surface tension})}$$

$$Oh = 0.104$$

Problem No.10 (Advanced): Recast the equation of flow through porous media in dimensionless form (Hint: Use J Leverett J-Function to replace dimensionless capillary pressure)

$$\frac{\partial[\varphi\, S]}{\partial t}+\underline{\nabla}\cdot\left[F_w\left(\underline{u}+\frac{k_{ro}}{\mu_o}\frac{dp_c}{dS}\underline{\underline{k}}\cdot\underline{\nabla}S-\frac{k_{ro}}{\mu_o}\Delta\rho g\underline{\underline{k}}\cdot\underline{\nabla}z\right)\right]=0$$

$$\frac{\partial[\varphi\, S]}{\partial t}+\underline{\nabla}\cdot\left[F_w\left(\underline{u}+\frac{k_{ro}}{\mu_o}\frac{dp_c}{dS}k\,\underline{\nabla}S-\frac{k_{ro}}{\mu_o}\Delta\rho\, g\, k\ \sin\alpha\right)\right]=0$$

$$\frac{\partial[\varphi\, S]}{\partial t}+\underline{\nabla}\cdot\left[F_w\left(\underline{u}+\frac{k_{ro}}{\mu_o}\frac{dp_c}{dS}k\,\underline{\nabla}S+\frac{k_{ro}}{\mu_o}k\,\Delta\rho\,\underline{g}\right)\right]=0$$

$$\underline{g}=-g[\ \hat{\imath}\ \sin\alpha+\hat{\jmath}\ \cos\alpha]$$

$$J_S=\frac{p_C}{\sigma\ \cos\theta}\sqrt{\frac{k}{\varphi}}$$

$$\frac{d\,p_C}{dS}=\frac{\sigma\ \cos\theta}{\sqrt{k/\varphi}}\frac{d\,J_S}{d\,S}$$

$$\frac{\partial[\varphi\, S]}{\partial t}+\underline{\nabla}\cdot\left[F_w\left(\underline{u}+\frac{k\,k_{ro}}{\mu_o}\frac{\sigma\ \cos\theta}{\sqrt{k/\varphi}}\frac{d\,J_S}{d\,S}\,\underline{\nabla}S-\frac{k_{ro}}{\mu_o}\Delta\rho\, g\, k\ \sin\alpha\right)\right]=0$$

$$\frac{\partial[\varphi\, S]}{\partial t} + \underline{\nabla} \cdot \left[F_w \left(\underline{u} + \frac{k\, k_{ro}}{\mu_o} \frac{\sigma\ \cos\theta}{\sqrt{k/\varphi}} \frac{d\, J_S}{d\, S}\, \underline{\nabla} S - \frac{k_{ro}}{\mu_o} \Delta\rho\ g\ k\ \sin\alpha \right) \right] = 0$$

$$\frac{\partial[\varphi\, S]}{\partial t} + \frac{1}{L}\underline{\check{\nabla}} \cdot \left[F_w \left(\underline{u} + \frac{k\, k_{ro}}{\mu_o} \frac{\sigma\ \cos\theta}{L\sqrt{k/\varphi}} \frac{d\, J_S}{d\, S}\, \underline{\check{\nabla}} S - \frac{k_{ro}}{\mu_o} \Delta\rho\ g\ k\ \sin\alpha \right) \right] = 0 \qquad \underline{\nabla} = \frac{1}{L}\underline{\check{\nabla}}$$

$$\frac{\partial[\varphi\, S]}{\partial t} + \frac{1}{L}\left[\frac{k\ \sigma\ \cos\theta}{\mu_o\, L\sqrt{k/\varphi}}\right] \underline{\check{\nabla}} \cdot \left[F_w \left(\underline{u}\ \frac{\mu_o}{\varphi\ (k/\varphi)} \frac{L\sqrt{k/\varphi}}{\sigma\ \cos\theta} + k_{ro} \frac{d\, J_S}{d\, S}\, \underline{\check{\nabla}} S - \frac{\Delta\rho\ [g \sin\alpha\,]\ [k/\varphi]}{\sigma\ \cos\theta} k_{ro} \left[L\sqrt{\varphi/k}\right] \right) \right] = 0$$

$$\left[\frac{\varphi\ \mu_o}{k} \frac{L^2\sqrt{k/\varphi}}{\sigma\ \cos\theta}\right] \frac{\partial[\,S]}{\partial t} + \underline{\check{\nabla}} \cdot \left[F_w \left(\underline{Ca}\ \lambda + k_{ro} \frac{d\, J_S}{d\, S}\, \underline{\check{\nabla}} S - Bo\ \lambda\ k_{ro} \right) \right] = 0$$

$$\frac{\partial[\,S]}{\partial \tau} + \underline{\check{\nabla}} \cdot \left[F_w \left(\underline{Ca}\ \lambda + k_{ro} \frac{d\, J_S}{d\, S}\, \underline{\check{\nabla}} S - Bo\ \lambda\ k_{ro} \right) \right] = 0$$

$$Bo = \frac{\Delta\rho\ [g \sin\alpha\,]\ [k/\varphi]}{\sigma\ \cos\theta}$$

$$\underline{Ca} = \frac{\underline{u}\ \mu_o}{\varphi\ \sigma\ \cos\theta}$$

$$J_S = \frac{p_C}{\sigma\ \cos\theta}\sqrt{\frac{k}{\varphi}} \qquad \lambda = \frac{L}{\sqrt{k/\varphi}} \qquad \tau = \left[\frac{\sqrt{k/\varphi}}{\mu_o} \frac{\sigma\ \cos\theta}{L^2}\right] t$$

Laplace and Fourier Number

$$\tau = \left[\frac{\sqrt{k/\varphi}}{\mu_o}\frac{\sigma\ \cos\theta}{L^2}\right]t \qquad \tau = \left[\frac{\rho\ \sigma\ \cos\theta\sqrt{k/\varphi}}{\mu_o^2}\right]\frac{t\ \mu_o}{\rho L^2} \qquad \tau = La\ Fo$$

Orlando Castilleja-Escobedo, Rubén E.Sánchez-García, Krishna D.P.Nigam, José L.López-Salinas Directional displacement of non-aqueous fluids through spontaneous aqueous imbibition in porous structures, ***Chemical Engineering Science*** 228 (2020) 1-14, Volume 228, 30 December 2020. https://doi.org/10.1016/j.ces.2020.115959

Experimental Study of Forced Convective Heat Transfer in a Coiled Flow Inverter using TiO_2–Water Nanofluids, Bárbara Arévalo-Torres, Jose Luis Lopez-Salinas, Alejandro Javier García-Cuéllar, ***Applied Sciences (2020)***

Problem No.11 (Advanced) : Read the paper from Venerus (2010) and give a dimensionless equation for pressure drop of compressible fluid within micro-channels. If you don't find the paper see chapter 8 of Venerus et.al 2018

$$\frac{Re}{16\,\psi\,(L/D)\,Eu} = \frac{2}{1+\sqrt{1+2\,\alpha}}$$

Solution

$$\alpha = \frac{16\,\psi\,(L/D)\,Cu}{Re} \qquad \alpha = \frac{16\,\psi\,(L/D)\,[\kappa_T\,\rho\,v^2/2]}{Re}$$

$$Re = \frac{\rho\,v\,D}{\mu} \qquad \frac{1}{Eu} = 2\,\Delta p\,/(\rho\,v^2)$$

$$Cu = [\kappa_T\,\rho\,v^2/2]$$ Cauchy Number

$$Eu = (\rho\,v^2)/(2\,\Delta p)$$ Euler Number

$$Re = \frac{\rho\,v\,D}{\mu}$$ Reynolds Number

$$\kappa_T = [1/B]$$ B = Bulk modulus

$\psi = 1$, for cylinder, $\frac{3}{2}$ for parallel plates

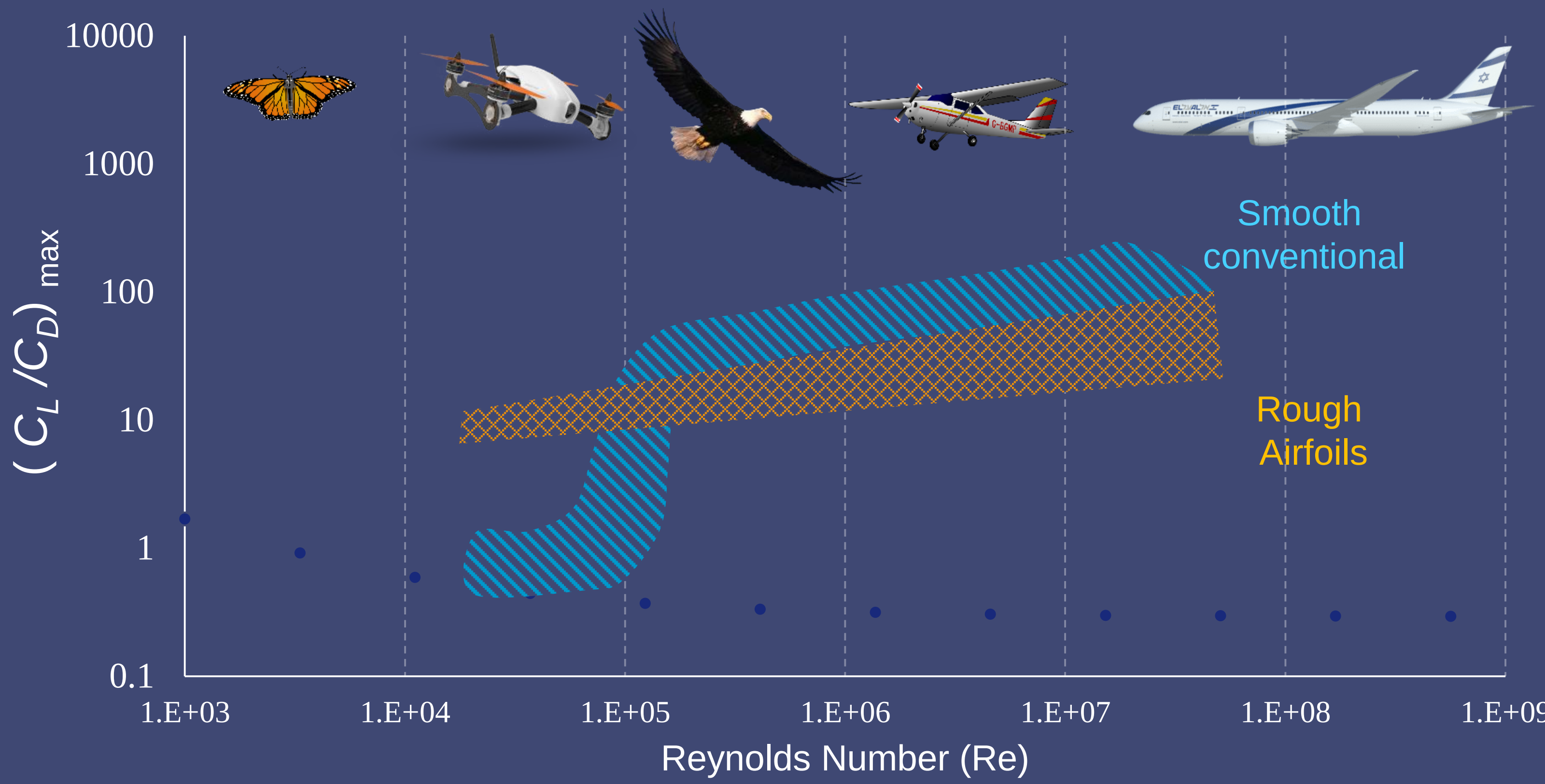
$$B = -V\left(\frac{\partial p}{\partial V}\right)\bigg|_T \qquad B = \rho\left(\frac{\partial p}{\partial \rho}\right)\bigg|_T \qquad \kappa_T = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)\bigg|_T$$

Isothermal compressibility coefficient

D. C. Venerus *and* D. J. Bugajsky. Compressible laminar flow in a channel, *Physics of Fluids* **22**, 046101 (2010); https://doi.org/10.1063/1.3371719

Venerus D.C., Ottinger H.C. A Modern course in Transport Phenomena, Cambridge 2018

https://edpuzzle.com/join/ruzmadm



Effect of Reynolds number of airfoil maximum sectional lift-to-drag ratio (Adapted from McMasters and Henderson. And Muller)

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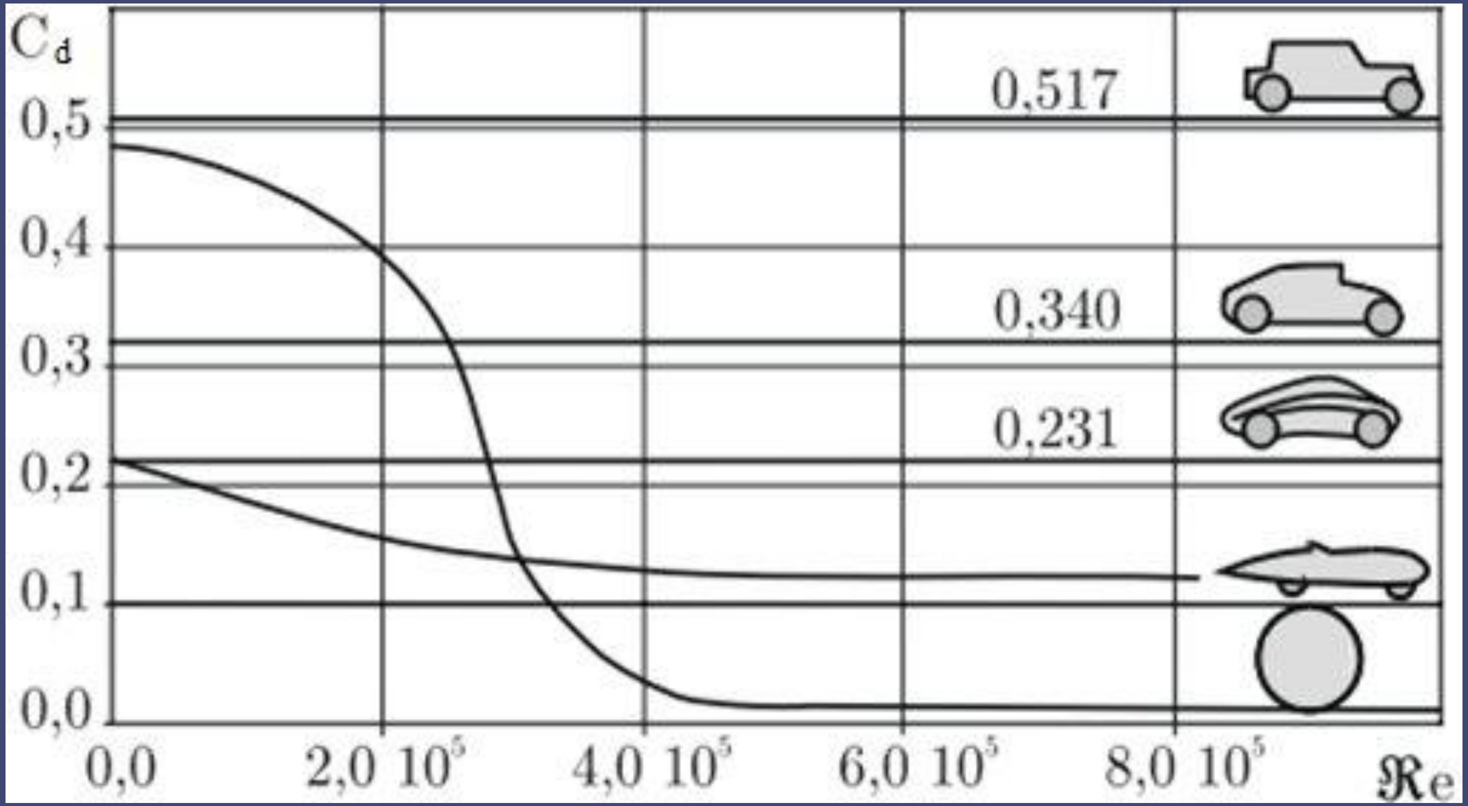
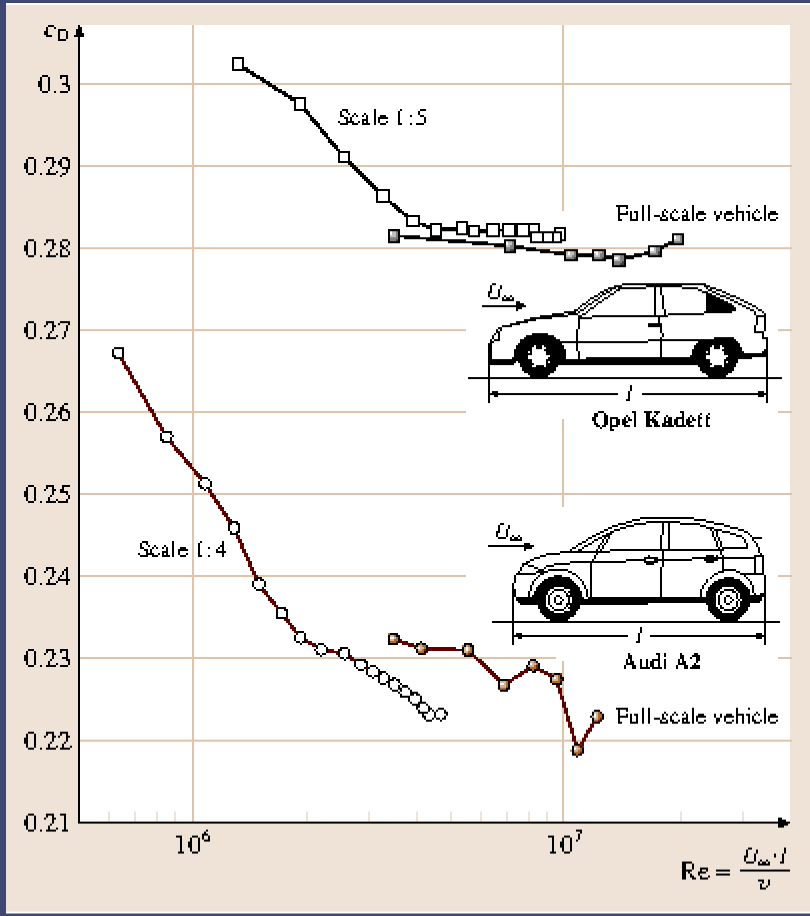


Cd
0,5
0,4
0,3
0,2
0,1
0,0
0,517
0,340
0,231
0,0
2,0 10^5
4,0 10^5
6,0 10^5
8,0 10^5
ℜe



cD
0.3
0.29
0.28
0.27
0.26
0.25
0.24
0.23
0.22
0.21
Scale 1:5
Full-scale vehicle
Opel Kadett
Scale 1:4
Audi A2
Full-scale vehicle
10^6
10^7
Re = U∞·l / ν

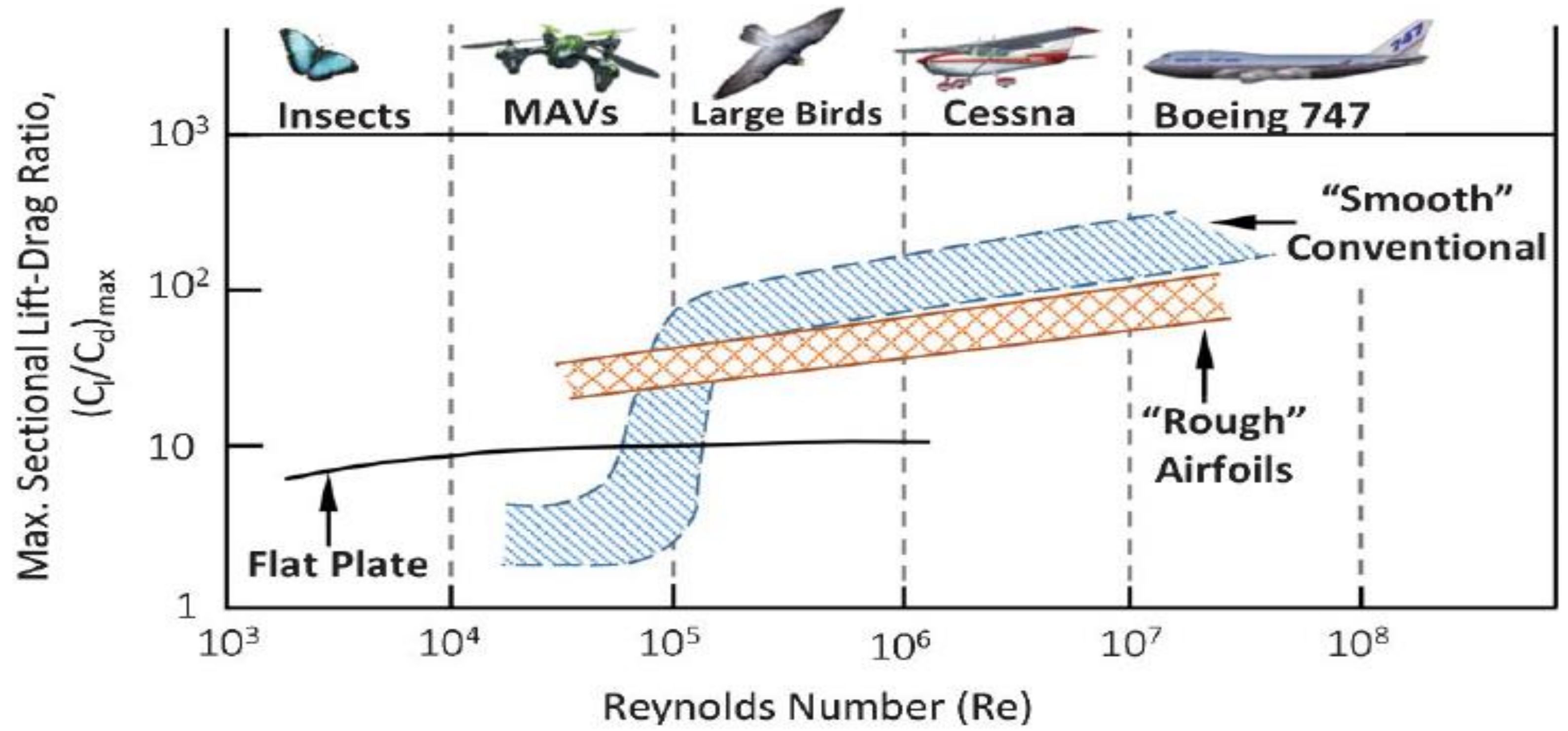
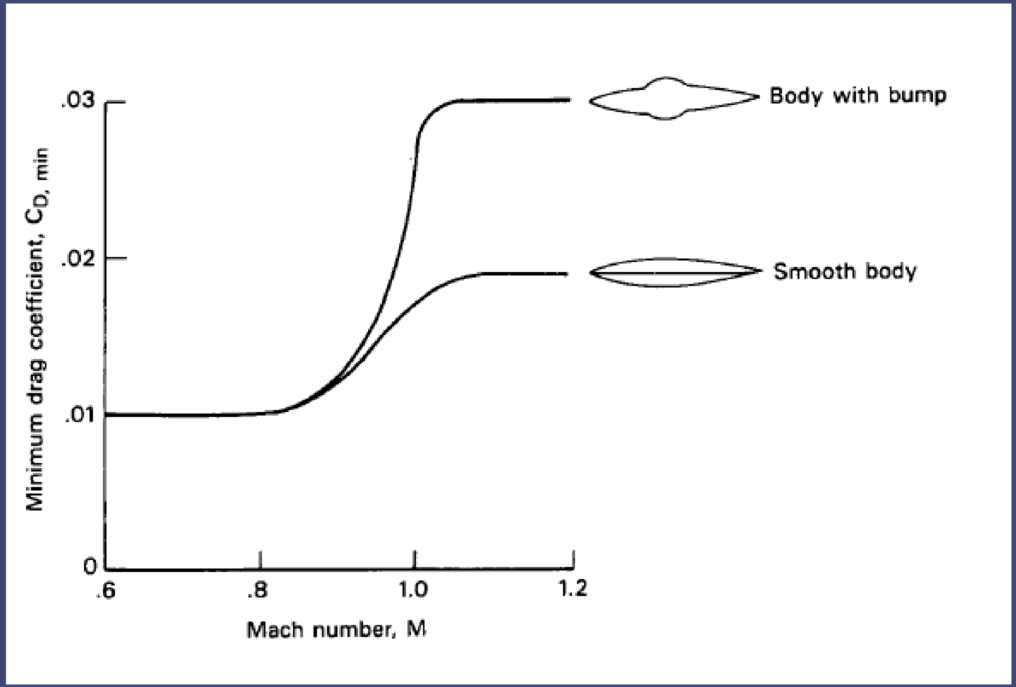
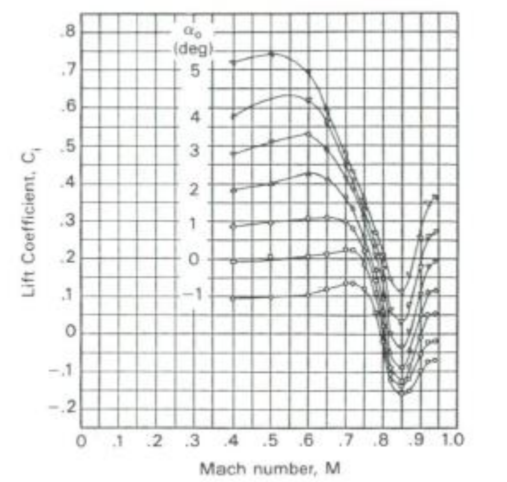


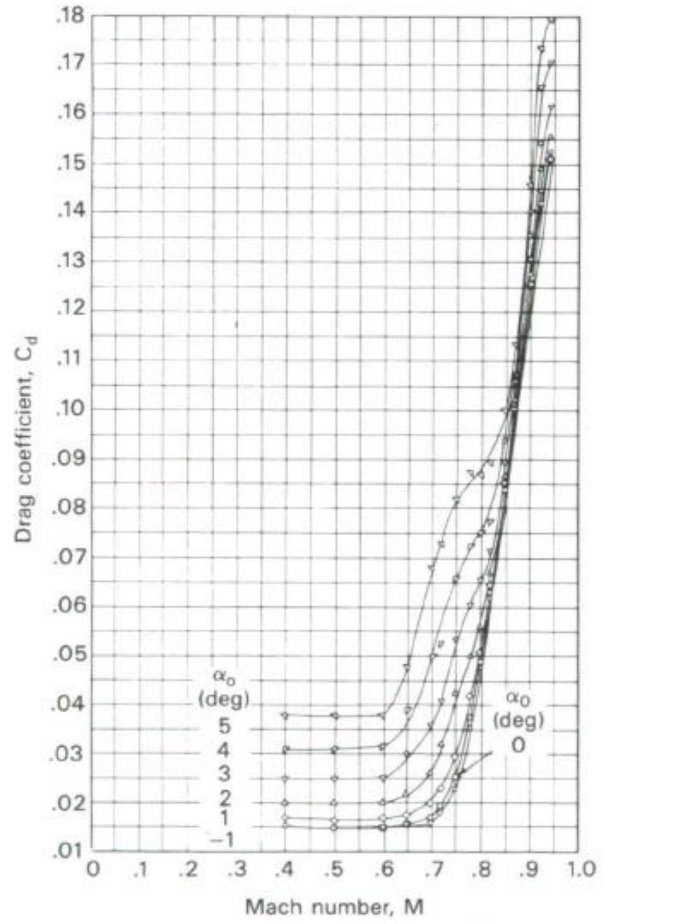
Fig. 1 Effect of Reynolds number on airfoil maximum sectional lift-to-drag ratio. (Adapted from McMasters and Henderson [2] and Mueller [3].)

Effect of Reynolds number of airfoil maximum sectional lift-to-drag ratio (Adapted from McMasters and Henderson. And Muller)





(a) Section lift coefficient.



(b) Section drag coefficient.

(b) Section drag coefficient.

— *Lift and drag characteristics of NACA 2315 airfoil section as function of Mach number for several angles of attack.* [data from ref. 48]

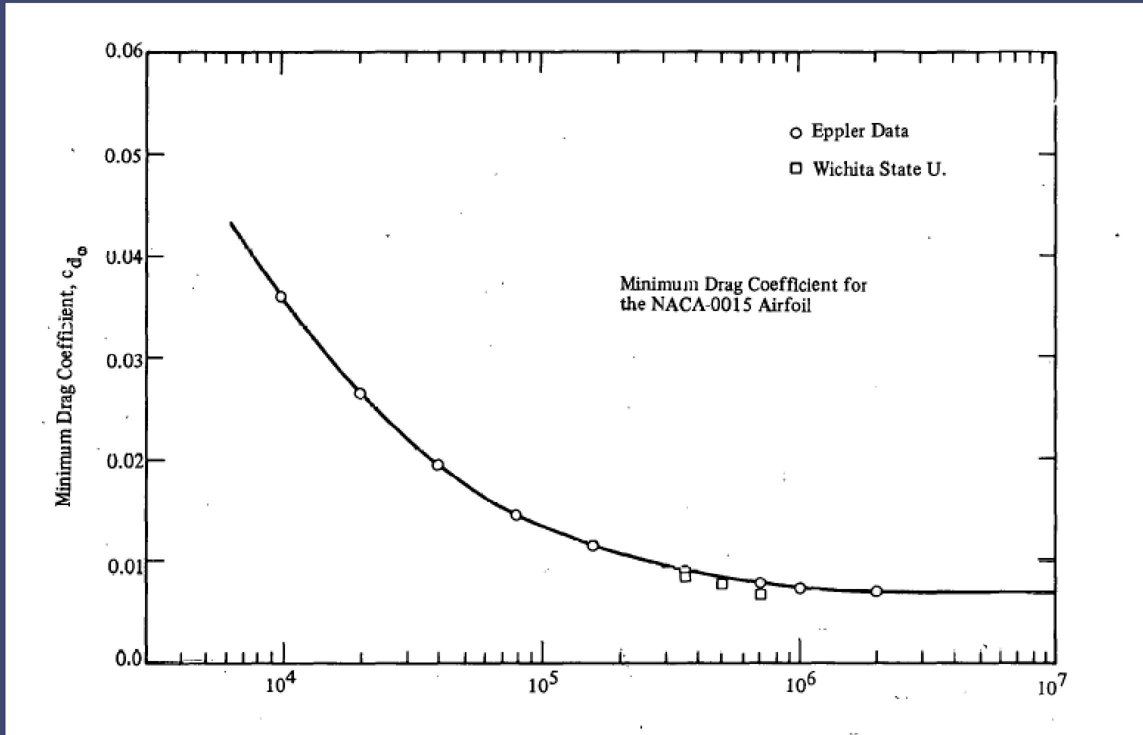


Figure 34. Predicted and Measured Values of Minimum Section Drag Coefficients, c_{d_o}, as a Function of Reynolds Number, Re

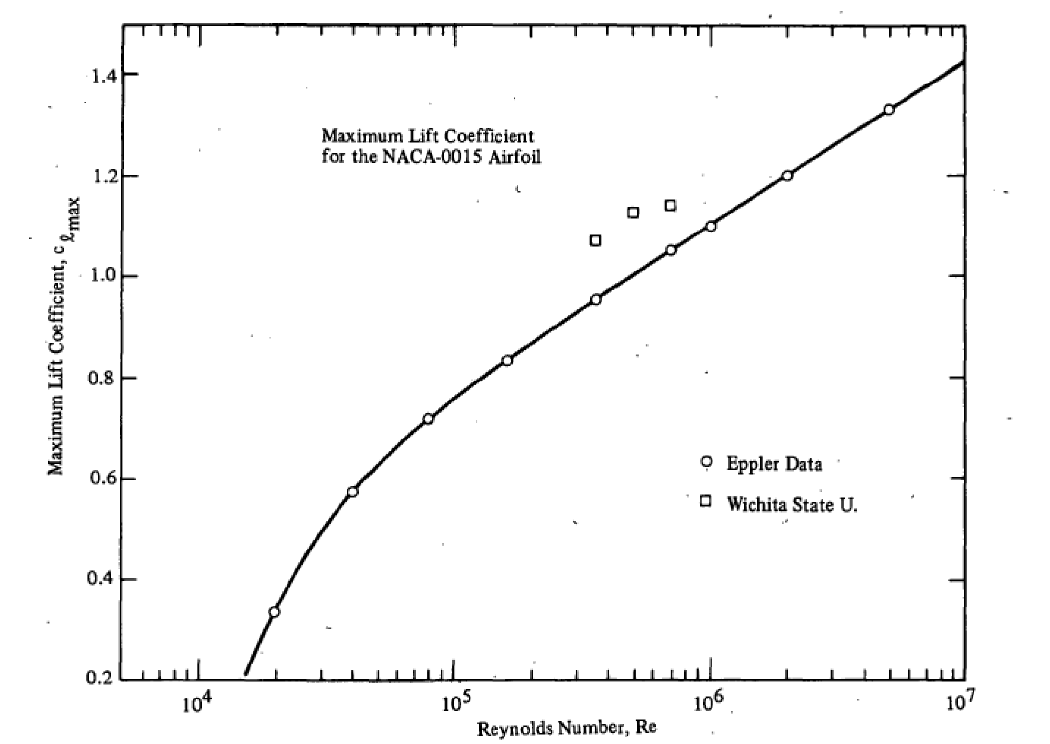


Figure 35. Predicted and Measured Values of Section Maximum Lift Coefficients, $c_{\ell_{max}}$, as a Function of Reynolds Number, Re

$$C_D = C_{D0} + C_{Di}$$

$$C_{Di} = \frac{C_L^2}{\pi A \epsilon} \qquad A = \frac{K^2 b^2}{S}$$

S = Wing Area
b= Wing span
K = Munk's span factor ≈1.1
$\epsilon \approx 1$

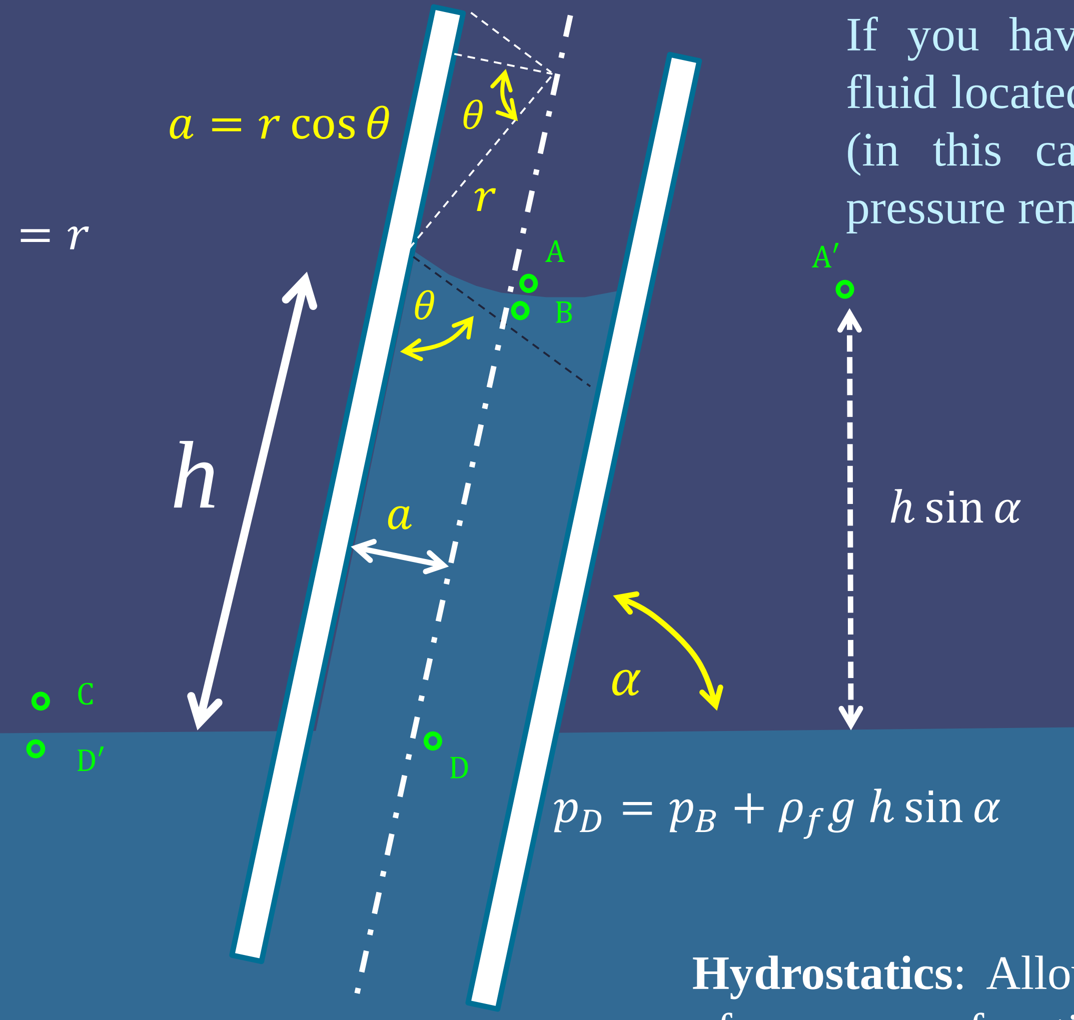
Collecting terms

Dimensional analysis of a simple system, a capillary rise. The pressure difference across a curved interfaces is calculated using Young-Laplace, this can be used to predict the capillary rise, with aid of hydrostatic pressure we have learn so far.

Thermodynamics: Young-Laplace will help to calculate pressure difference across the curved interface

$$\Delta p = p_A - p_B = \sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right) \quad r_1 = r_2 = r$$

$$a = r\cos\theta$$



If you have a continuous stagnate fluid located at the same energy level (in this case potential energy) its pressure remains.

" *air* " = air over the liquid

$$p_C = p_{A'} + \rho_{air} g h \sin\alpha$$

$$p_D = p_C$$

$$p_D = p_B + \rho_f g\, h \sin\alpha$$

Flat interfaces have equal pressure

" *f* " = liquid fluid

Hydrostatics: Allows calculation of pressure as function of depth

$$\Delta p = p_A - p_B = \sigma\left(\frac{1}{r_1}+\frac{1}{r_2}\right) = \frac{2\sigma}{r} = \frac{2\sigma\cos\theta}{a}$$

$$p_D = p_B + \rho_f g\, h \sin\alpha$$

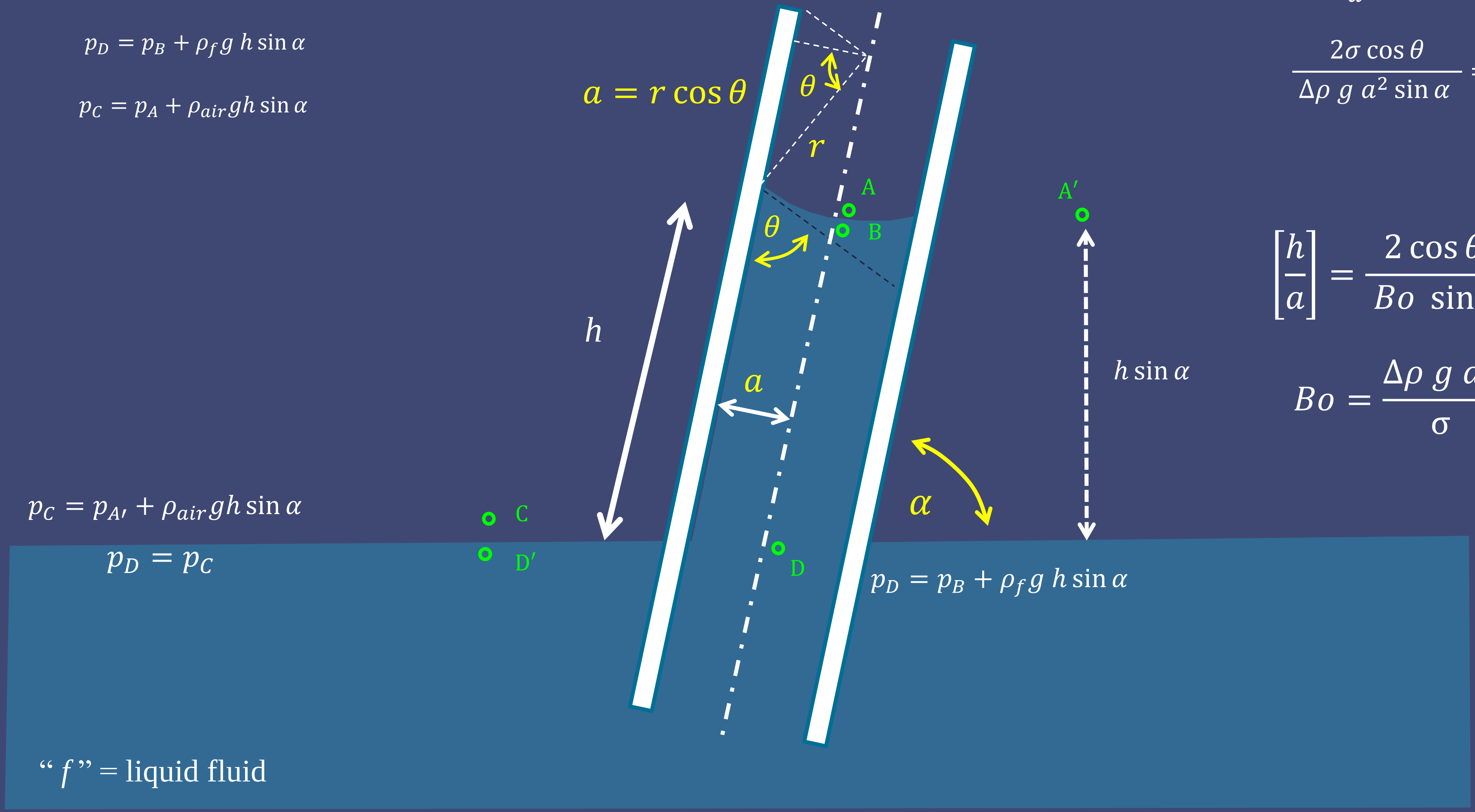
$$p_C = p_A + \rho_{air} g h \sin\alpha$$

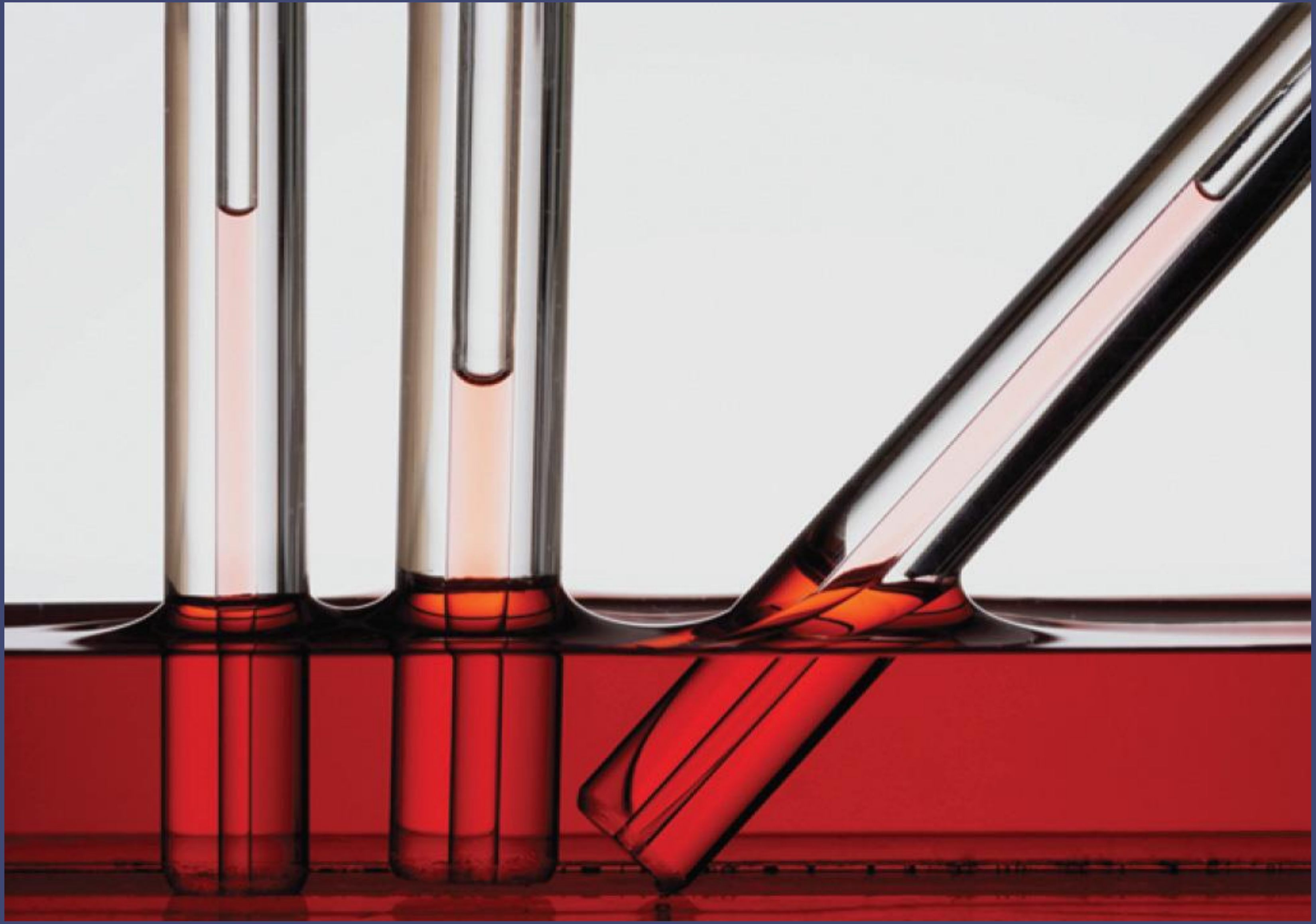
$$p_A - p_B = \frac{2\sigma\cos\theta}{a} = (\rho_f - \rho_{air}) g\, h \sin\alpha$$

$$\frac{2\sigma\cos\theta}{\Delta\rho\, g\, a^2 \sin\alpha} = \frac{h}{a}$$

$$\left[\frac{h}{a}\right] = \frac{2\cos\theta}{Bo\ \sin\alpha}$$

$$Bo = \frac{\Delta\rho\ g\ a^2}{\sigma}$$





$$\left[\frac{h}{a}\right] = \frac{2\cos\theta}{Bo\ \sin\alpha}$$

$$Bo = \frac{\Delta\rho\ g\ a^2}{\sigma}$$

Bond number

Contact angle

$$\left[\frac{h}{a}\right] = f(Bo, \alpha, \theta)$$

Inclination of capillary respect to the horizon

Dimensionless capillary rise

The dimensionless capillary rise, is function of Bond number, contact angle and orientation. For the specific scenario of small circular capillary tube indicates that for the case of high values of Bond (Bo>>1) number the dimensionless capillary rise is negligible, and on the contrary, if the Bond number is low (Bo<<1) the dimensionless capillary rise with be important.

By: Dr. José Luis López Salinas
This material can be used only by the instructor to plan his lecture.
It is by no means, a template nor lecture notes