Homework 02 - Introduction to Systems of Linear Equations

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ITESM Campus Monterrey
Mathematical Physical Modelling F4005
HW2: Row echelon form and linear systems
Due Date: Tuesday 29-2019, 23:59 hrs.
Professor: Ph.D Daniel López Aguayo

Full names of team members: ______________________________

Instructions: Please write neatly on each page of your homework and send it in pdf format to dlopez.aguayo@tec.mx. Typed solutions in LaTeX (only) will be given extra credit; no late homework will be accepted. Each team should consist (of at most) 5 students.

I. In Exercises $1-6$, determine which equations are linear equations in the variables x, y and z. If any equation is not linear, please explain why not.

1. $x - \pi y + \sqrt[3]{5} \cdot z = 0$
2. $x^2 + y^2 + z^2 = 1$
3. $x^{-1} + 7y + z = \sin^2\left(\frac{\pi}{9}\right)$
4. $x + 7y + z = sin\left(\frac{\pi}{9}\right)$
5. $3 \cdot cos(x) - 4y + z = \sqrt{3}$
6. $cos(3) \cdot x - 4y + z = \sqrt{3}$

II. In Exercises $7-9$, draw graphs (you are allowed to use Mathematica) corresponding to the given linear systems. Determine geometrically whether each system has a unique solution, infinitely many solutions or no solutions. Then solve each system algebraically to confirm your answer.

7.
$$\begin{cases} x + y = 0 \\ 2x + y = 3 \end{cases}$$

8.
$$\begin{cases} x - 2y = 7 \\ 3x + y = 7 \end{cases}$$

9.
$$\begin{cases} 3x - 6y = 3 \\ -x + 2y = 1 \end{cases}$$

III. In Exercises $10-12$, solve the given system by back substitution.

10.
$$\begin{cases} x - y + z = 0 \\ 2y - z = 1 \\ 3z = -1 \end{cases}$$

11.
$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ -5x_2 + 2x_3 = 0 \\ 4x_3 = 0 \end{cases}$$

12.
$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 1 \\ x_2 + x_3 + x_4 = 0 \\ x_3 - x_4 = 0 \\ x_4 = 1 \end{cases}$$

IV. In Exercises $13-14$, the systems of equations are nonlinear. Find substitutions (changes of variables) that convert each system into a linear system and use this linear system to solve the original system. Also, verify your answer with Mathematica.

13.
$$\begin{cases} \frac{2}{x} + \frac{3}{y} = 0 \\ \frac{3}{x} + \frac{4}{y} = 1 \end{cases}$$

14.
$$\begin{cases} -2^a + 2 \cdot 3^b = 1 \\ 3 \cdot 2^a - 4 \cdot 3^b = 1 \end{cases}$$

V. In Exercises 15 – 18, determine whether the given matrix is in row echelon form (and justify your answer). If it is, state whether it is also in reduced row echelon form.

15. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

18. $\begin{bmatrix} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

VI. Use Gaussian elimination to find the rank of the following matrices and verify your answer with Mathematica.

19. $\begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

20. $\begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{bmatrix}$

VII. In Exercises 21 – 23, solve the given system of equations using Gaussian elimination; then verify your answer with Mathematica.

21.
$$\begin{cases} x_1 + 2x_2 - 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 0 \\ 4x_1 - x_2 + x_3 = 4 \end{cases}$$

22.
$$\begin{cases} 2r + s = 3 \\ 4r + s = 7 \\ 2r + 5s = -1 \end{cases}$$

23.
$$\begin{cases} w + x + 2y + z = 1 \\ w - x - y + z = 0 \\ x + y = -1 \\ w + x + z = 2 \end{cases}$$

VIII. For what value(s) of k, if any, will the following system have (a) no solution and (b) infinitely many solutions? No guessing is allowed! (i.e use an algebraic method)

24.
$$\begin{cases} kx + y = -2 \\ 2x - 2y = 4 \end{cases}$$

IX. Give an example of three planes that intersect in a single point. *Hint*: Keep it simple! think about familiar planes you learned in Mathematics III.

X. Let n be an arbitrary positive integer. Find the rank of the identity matrix I_n and justify your answer.

XI. For each of the following problems, write the corresponding linear system and solve it using Mathematica (**not by hand!**).

25. A coffee merchant sells three blends of coffee. A bag of the house blend contains 300 grams of Colombian beans and 200 grams of French roast beans. A bag of the special blend contains 200 grams of Colombian beans, 200 grams of Kenyan beans, and 100 grams of French roast beans. A bag of the gourmet blend contains 100 grams of Colombian beans, 200 grams of Kenyan beans, and 200 grams of French roast beans. The merchant has on hand 30 kilograms of Colombian beans, 15 kilograms of Kenyan beans, and 25 kilograms of French roast beans. If he wishes to use up all of the beans, how many bags of each type of blend can be made?

26. There are two fields whose total area is 1800 square yards. One field produces grain at the rate of $\frac{2}{3}$ bushel per square yard; the other field produces grain at the rate of $\frac{1}{2}$ bushel per square yard. If the total yield is 1100 bushels, what is the area of each field?

27. Find a parabola with an equation of the form $y = ax^2 + bx + c$ that passes through $(0, 1)$, $(-1, 4)$ and $(2, 1)$.

1 Answer to Problem I

I.1
$x - \pi\mathrm{y} + \sqrt[3]{5}z = 0$ is a linear equation

I.2
$x^2 + y^2 + z^2 = 1$ is NOT a linear equation
The variables shall occur only to the first power.

I.3
$x^{-1} + 7y + z = \left(\mathrm{Sin}\left[\frac{\pi}{9}\right]\right)^2$ is NOT a linear equation
The variables shall occur only to the first power.

I.4
$x + 7y + z = \mathrm{Sin}\left[\frac{\pi}{9}\right]$ is a linear equation

I.5
$3\mathrm{Cos}[x] - 4y + z = \sqrt{3}$ is NOT a linear equation
Linear equations shall not contain products, reciprocals or other functions of the variables.

I.6
$\mathrm{Cos}[3]x - 4y + z = \sqrt{3}$ is a linear equation

2 Answer to Problem II

II.7

ContourPlot$[\{x+y{=}{=}0, 2*x+y{=}{=}3\}, \{x,2,4\}, \{y,-4,-2\},$

ContourStyle $\rightarrow$ **{Blue, Orange}]**

(*

$$\left\{\begin{array}{ll} x+y & =0 \\ 2x+y & =3 \end{array}\right. \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \Rightarrow R_2 - 2R_1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \end{pmatrix} \Rightarrow \left\{\begin{array}{ll} x+y & =0 \\ -y & =3 \end{array}\right.$$

$\boxed{y=-3}$

$x+y=0$

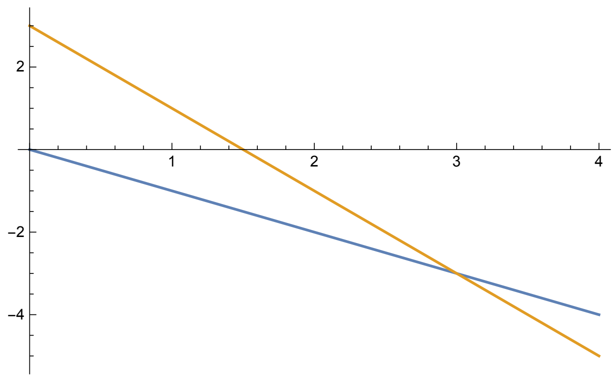
$x=-(-3)$

$\boxed{x=3}$

***)**

Solve$[\{x+y{=}{=}0, 2*x+y{=}{=}3\}, \{x,y\}]$

Out[ ]=



As shown in the plot, the system has one solution.

$\{\{x \rightarrow 3, y \rightarrow -3\}\}$

II.8

ContourPlot[{$x - 2*y$==7, $3*x+y$==7}, {x, 2, 4}, {y, −3, −1},

ContourStyle → {Blue, Orange}]

(*

$$\begin{cases} x-2y &= 7 \\ 3x+y &= 7 \end{cases} \Rightarrow \begin{pmatrix} 1 & -2 & 7 \\ 3 & 1 & 7 \end{pmatrix} \Rightarrow R_2 - 3R_1 \Rightarrow \begin{pmatrix} 1 & -2 & 7 \\ 0 & 7 & -14 \end{pmatrix} \Rightarrow \begin{cases} x-2y &= 7 \\ 7y &= -14 \end{cases}$$

$\boxed{y = -2}$

$x - 2y = 7$

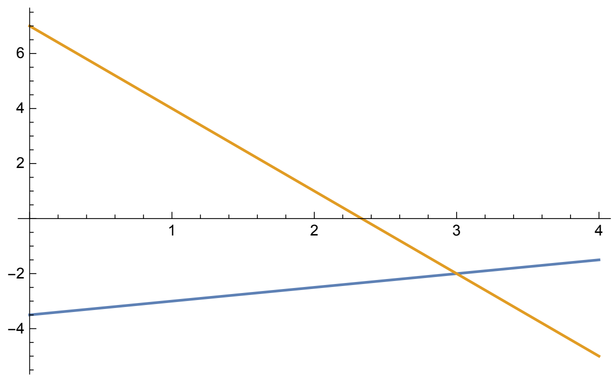
$x = 7 + 2(-2)$

$\boxed{x = 3}$

*)

Solve[{$x - 2*y$==7, $3*x+y$==7}, {x, y}]

Out[ ]=



As shown in the plot, the system has one solution.

$\{\{x \to 3, y \to -2\}\}$

II.9

ContourPlot[{3 * x − 6 * y==3, −x + 2 * y==1}, {x, −3.125, 4.5}, {y, −1.75, 1.75},

ContourStyle → {Blue, Orange}]

(*

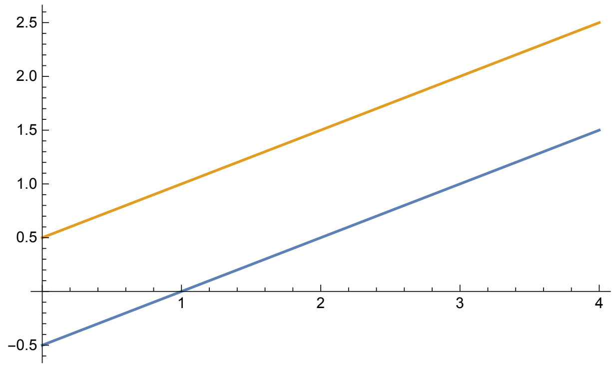
$$\left\{\begin{array}{ll} 3x-6y & =3 \\ -x+2y & =1 \end{array}\right. \Rightarrow \begin{pmatrix} 3 & -6 & 3 \\ -1 & 2 & 1 \end{pmatrix} \Rightarrow R_2+\tfrac{1}{3}R_1 \Rightarrow \begin{pmatrix} 3 & -6 & 3 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \left\{\begin{array}{ll} 3x-6y & =3 \\ 0 & =2 \end{array}\right.$$

$\boxed{\text{no solution}}$

*)

Solve[{3 * x − 6 * y==3, − x + 2 * y==1}, {x, y}]

Out[∘]=



As shown in the plot, the system has no solution.

{}

3 Answer to Problem III

10.

$$\begin{cases} x - y + z = 0 \\ \quad 2y - z = 1 \\ \qquad 3z = -1 \end{cases}$$

From last equation **z=-1/3**, therefore 2y+1/3=1 then **y=1/3**. For first equation It obtains **x**=1/3+1/3= **2/3**

11.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ \quad - 5x_2 + 2x_3 = 0 \\ \qquad 4x_3 = 0 \end{cases}$$

From last equation **x3=0**. Using second equation we obtain **x2**=(2(0))/5 **=0** then for first equation **x1**=-3(0)-2(0)= **0**

12.

$$\begin{cases} x_1 + x_2 - x_3 - x_4 = 1 \\ \quad x_2 + x_3 + x_4 = 0 \\ \qquad x_3 - x_4 = 0 \\ \qquad\quad x_4 = 1 \end{cases}$$

From last equation **x4=1** then **x3=1** therefore from second equation **x2**=-1-1 **=-2**. Finally from first equation **x1**=1+2+1+1 **=5**

4 Answer to Problem IV

IV . In Exercises 13 -14 , the systems of equations are non linear. Find substitutions (changes of variables) that convert each system into a linear system and use this linear system to solve the original system. Also, verify your answer with Mathematica.

In order to solve the first non linear system of equations we proceed to use change of variables, in this case :

u = 1/x , v = 1/y , now we have a new system of equations :
2u + 3v = 0
3u + 4v = 1 , solving using addition and substraction we get a new system:

-6u-9v = 0
6u+ 8v = 2 , where finally u = 3 and v = -2

Substituting in the change of variable:

3 = 1/x , therefore x = 1/3
-2 = 1/y, therefore y = -1/2

Solve[{2/x + 3/y == 0, 3/x + 4/y == 1}, {x, y}]

$\left\{\left\{x \to \frac{1}{3}, y \to -\frac{1}{2}\right\}\right\}$

In order to solve the second non linear system of equations we proceed to use change of variables, in this case:

2^a = t , 3^b = r

Substituting the new variables in the system of equations, we have the follow system :

-t + 2r = 1
3t - 4r = 1 , solving using addition and subtraction we get a new system:

-3t + 6r = 3
3t - 4r = 1 , where finally r = 2 and t = 3

Substituting in the change of variable :

3^b = 2	2^a = 3
Ln 3^b = Ln 2	Ln 2^a = Ln 3
b Ln(3) = Ln(2)	a Ln(2) = Ln 3
b = Ln(2)/Ln (3)	a = Ln (3)/Ln(2)

Solve$[\{-2^{\wedge}a + 2 * 3^{\wedge}b == 1, 3 * 2^{\wedge}a - 4 * 3^{\wedge}b == 1\}, \{a, b\}]$

$\left\{\left\{a \to \text{ConditionalExpression}\left[\frac{2i\pi C[1]+\text{Log}[3]}{\text{Log}[2]}, C[1] \in \mathbb{Z}\right], b \to \text{ConditionalExpression}\left[\frac{2i\pi C[2]+\text{Log}[2]}{\text{Log}[3]}, C[2] \in \mathbb{Z}\right.\right.\right.$

5 Answer to Problem V

V. In exercises 15 - 18, determine whether the given matrix is in row echelon form (and justify your answer). If it is, state whether it is also in reduced row echelon form.

15.

$a = \{\{1,0,1\},\{0,0,3\},\{0,1,0\}\}$

$\{\{1,0,1\},\{0,0,3\},\{0,1,0\}\}$

a//MatrixForm

$$\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{pmatrix}$$

$\{\{1,0,1\},\{0,0,3\},\{0,1,0\}\}$

$\{\{1,0,1\},\{0,0,3\},\{0,1,0\}\}$

RowReduce[a]

$\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$

MatrixForm[%]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The first matrix is not in the echelon form due to does not satisfy the condition of any row consisting entirely of zeros is at the bottom and there is a number one in the last row in the middle of the tow zeros.

16.

$b = \{\{1,2,3\},\{1,0,0\},\{0,1,1\},\{0,0,1\}\}$

$\{\{1,2,3\},\{1,0,0\},\{0,1,1\},\{0,0,1\}\}$

b//MatrixForm

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\{\{1, 2, 3\}, \{1, 0, 0\}, \{0, 1, 1\}, \{0, 0, 1\}\}$

RowReduce[*b*]

$\{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}, \{0, 0, 0\}\}$

MatrixForm[%]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The second matrix is not in the echelon form due to does not satisfy the condition of leading entry is not in a column to the right of the leading entry in the previous row because there is a number "1" in the second row immediately under the first leading entry.

$c = \{\{1, 2, 3, 4\}, \{0, 0, 1, 3\}, \{0, 0, 0, 1\}, \{0, 0, 0, 0\}\}$

$\{\{1, 2, 3, 4\}, \{0, 0, 1, 3\}, \{0, 0, 0, 1\}, \{0, 0, 0, 0\}\}$

***c*//MatrixForm**

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\{\{1, 2, 3, 4\}, \{0, 0, 1, 3\}, \{0, 0, 0, 1\}, \{0, 0, 0, 0\}\}$

RowReduce[*c*]

$\{\{1, 2, 0, 0\}, \{0, 0, 1, 0\}, \{0, 0, 0, 1\}, \{0, 0, 0, 0\}\}$

MatrixForm[%]

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The third matrix is in the echelon form. However is not in the reduced form because in the third row, the number one is under number 3 and 4 respectively.

$\boldsymbol{d = \{\{1,2,0,0,-3,1,0\},\{0,0,1,0,4,-1,0\},\{0,0,0,1,3,-2,0\},\{0,0,0,0,0,0,1\},\{0,0,0,0,0,0,0\}\}}$

$\{\{1,2,0,0,-3,1,0\},\{0,0,1,0,4,-1,0\},\{0,0,0,1,3,-2,0\},\{0,0,0,0,0,0,1\},\{0,0,0,0,0,0,0\}\}$

***d*//MatrixForm**

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{4} & \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{3} & \mathbf{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$\{\{1,2,0,0,-3,1,0\},\{0,0,1,0,4,-1,0\},\{0,0,0,1,3,-2,0\},\{0,0,0,0,0,0,1\},\{0,0,0,0,0,0,0\}\}$

RowReduce[*d*]

$\{\{1,2,0,0,-3,1,0\},\{0,0,1,0,4,-1,0\},\{0,0,0,1,3,-2,0\},\{0,0,0,0,0,0,1\},\{0,0,0,0,0,0,0\}\}$

MatrixForm[%]

$$\begin{pmatrix} 1 & 2 & 0 & 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 1 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The fourth matrix is in the echelon form and is also in the reduced form due to every single number one has zeros at the top and bottom.

6 Answer to Problem VI

VI.Use Gaussian elimination to find the rank of the following matrices and verify your answer with Mathematica.

19. $\begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$ **20.** $\begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-4} & \mathbf{-4} & \mathbf{5} \\ \mathbf{2} & \mathbf{4} & \mathbf{0} & \mathbf{0} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{5} \\ \mathbf{-1} & \mathbf{1} & \mathbf{3} & \mathbf{6} & \mathbf{5} \end{pmatrix}$

In order to find the rank of the matrix 19 it is necessary to reducing to echelon form.

$$\begin{pmatrix} \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{1} & \mathbf{1} \end{pmatrix} \mathbf{R1(1/2)}$$

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix} \mathrm{R2} - \mathrm{R1}$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ -1 & 1 & 1 \end{pmatrix} \mathrm{R3} + \mathrm{R1}$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix} \mathrm{R3} - \mathrm{R2}$$
$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$
We have obtained two nonzero rows therefore the Matrix rank $= 2$

Applying the same method to matrix 20,

$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{pmatrix} \mathrm{R2} - \mathrm{R3}$$
$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 1 & -2 & -1 & -3 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{pmatrix} \mathrm{R3} - 2\mathrm{R1}$$
$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & -1 & 10 & 9 & -5 \\ -1 & 1 & 3 & 6 & 5 \end{pmatrix} \mathrm{R4} + \mathrm{R1}$$

$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \mathrm{R1} + 2\mathrm{R3}$$
$$\begin{pmatrix} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \mathrm{R3} + \mathrm{R2}$$
$$\begin{pmatrix} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \mathrm{R3}(1/8)$$
$$\begin{pmatrix} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 3 & -1 & 2 & 10 \end{pmatrix} \mathrm{R4} - 3\mathrm{R2}$$
$$\begin{pmatrix} 1 & 0 & 16 & 14 & -5 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 3 & 5 & 5 & 19 \end{pmatrix} \mathrm{R1} - 16\mathrm{R3}$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & -2 & -1 & -3 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 5 & 5 & 19 \end{pmatrix} \mathrm{R2} + 2\mathrm{R3}$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 5 & 5 & 19 \end{pmatrix} \mathrm{R4}(1/5)$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 19/5 \end{pmatrix} \mathrm{R4} - \mathrm{R3}$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 24/5 \end{pmatrix} \mathrm{R4}(5/24)$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 11 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{R1} - 11\mathrm{R4}$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{R2} + 5\mathrm{R4}$$
$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \mathrm{R3} + \mathrm{R4}$$

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore the matrix rank $= 4$

Now the above processes can be verified with Mathematica,

$a = \{\{2, 0, -1\}, \{1, 1, 0\}, \{-1, 1, 1\}\}//\text{MatrixForm}$

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{pmatrix}$$

RowReduce[%]

$\left\{\left\{1, 0, -\frac{1}{2}\right\}, \left\{0, 1, \frac{1}{2}\right\}, \{0, 0, 0\}\right\}$

MatrixForm[%]

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix};$$

$b = \{\{1, 2, -4, -4, 5\}, \{2, 4, 0, 0, 2\}, \{2, 3, 2, 1, 5\}, \{-1, 1, 3, 6, 5\}\}//\text{MatrixForm}$

$$\begin{pmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & 1 & 3 & 6 & 5 \end{pmatrix}$$

RowReduce[%]

$\{\{1, 0, 0, -2, 0\}, \{0, 1, 0, 1, 0\}, \{0, 0, 1, 1, 0\}, \{0, 0, 0, 0, 1\}\}$

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

7 Answer to Problem VII

VII. In Exercises $21-23,$ solve the given system of equations using Gaussian elimination; then verify your answer with Mathematica.

21. $\mathrm{x1}+2\mathrm{x2}-3\mathrm{x3}=9$ 22. $2r+s=3$ 23. $w+x+2y+z=1$

$2\mathrm{x1}-\mathrm{x2}+\mathrm{x3}=0$ $4r+s=7$ $w-x-y+z=0$

$4\mathrm{x1}-\mathrm{x2}+\mathrm{x3}=4$ $2r+5s=-1$ $x+y=-1$

$w+x+z=2$

In order to solve the given system of equations we have to convert it into matrix form and only using the coeficients.

$$21. \begin{pmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{pmatrix} \; 22. \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{pmatrix} \; 23. \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix}$$

Now it is necessary to reducing to diagonal matrix each matrix by using Gaussian elimination.

21.

$$\begin{pmatrix} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{pmatrix} \mathrm{R2}-2\mathrm{R1}$$

$$\begin{pmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 4 & -1 & 1 & 4 \end{pmatrix} \mathrm{R3}-4\mathrm{R1}$$

$$\begin{pmatrix} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{pmatrix} \mathrm{R2}(-1/5)$$

$$\begin{pmatrix} 1 & 2 & -3 & 9 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{pmatrix} \mathrm{R1}-2\mathrm{R2}$$

$$\begin{pmatrix} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & -9 & 13 & -32 \end{pmatrix} \mathrm{R3}+9\mathrm{R2}$$

$$\begin{pmatrix} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 2/5 & 2/5 \end{pmatrix} \mathrm{R3}(5/2)$$

$$\begin{pmatrix} 1 & 0 & -1/5 & 9/5 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 1 & 1 \end{pmatrix} \mathrm{R1}+\mathrm{R3}(1/5)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & -7/5 & 18/5 \\ 0 & 0 & 1 & 1 \end{pmatrix} \mathrm{R2}+\mathrm{R3}(7/5)$$

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The solutions are, $\mathrm{x1}=2;\ \mathrm{x2}=5;\ \mathrm{x3}=1$

Now let$'s$ prove with Mathematica

Solve[{x1 + 2 * x2 − 3 * x3 == 9, 2 * x1 − x2 + x3 == 0, 4 * x1 − x2 + x3 == 4}, {x1, x2, x3}]
{{x1 → 2, x2 → 5, x3 → 1}}
22.

$$\begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{pmatrix} \text{R2} - 2\text{R1}$$
$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 2 & 5 & -1 \end{pmatrix} \text{R3} - \text{R1}$$
$$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{pmatrix} \text{R1} + \text{R2}$$
$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{pmatrix} \text{R2}(-1)$$
$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 4 & -4 \end{pmatrix} \text{R3} - 4\text{R2}$$
$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{R1}(1/2)$$
$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Solving with Mathematica,
Solve[{$2 * r + s == 3, 4 * r + s == 7, 2 * r + 5 * s == -1$}, {$r, s$}]
{{$r \to 2, s \to -1$}}

23.

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix} \text{R2} - \text{R1}$$
$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 2 \end{pmatrix} \text{R4} - \text{R1}$$
$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix} \text{R1} - \text{R3}$$
$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & -2 & -3 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{pmatrix} \text{R2}(-1/2)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right) \text{R3} - \text{R2}$$

$$\left(\begin{array}{ccccc} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 & -3/2 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right) \text{R3}(-2)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right) \text{R1} - \text{R3}$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right) \text{R2} - \text{R3}(3/2)$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -2 & 0 & 1 \end{array}\right)$$

We cannot proceed to solve anymore because is not possible to continue reducing, therefore there are not solutions.
Solving with Mathematica,

Solve$[\{w+x+2y+z == 1, w-x-y+z == 0, x+y == -1, w+x+z == 2\}, \{w, x, y, z\}]$
$\{\}$

8 Answer to Problem VIII

$$\left\{\begin{array}{ll} \text{kx}+y & =-2 \\ 2x-2y & =4 \end{array}\right. \Rightarrow \begin{pmatrix} k & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} \Rightarrow \frac{1}{k}R_1 \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & -\frac{2}{k} \\ 2 & -2 & 4 \end{pmatrix} \Rightarrow R_2-2R_1 \Rightarrow$$

$$\begin{pmatrix} 1 & \frac{1}{k} & -\frac{2}{k} \\ 0 & -\frac{2(1+k)}{k} & 4+\frac{4}{k} \end{pmatrix} \Rightarrow -\frac{k}{2(1+k)}R_2 \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & -\frac{2}{k} \\ 0 & 1 & -2 \end{pmatrix} \Rightarrow R_1-\frac{1}{k}R_2 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$$\boxed{\begin{array}{c} x=0 \\ y=-2;\text{for all } k \end{array}}$$

Verification using Mathematica:

$A=\begin{pmatrix} k & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix};$

RowReduce[A];

MatrixForm[%]

Solve[$\{k*x+y==-2, 2x-2y==4\}, \{x,y\}$]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$\{\{x\to 0, y\to -2\}\}$

9 Answer to Problem IX

First the Condition for a single point of intersection of three planes is: Rc=Rd=3
Rc is the Matrix Rank and Rd is the Extended Matrix Rank
To accomplish this, we must find a matrix that no row or column can be put as a linear combination of the rest.

one example of this is:

$$\begin{cases} x + 2y + 3z = 7 \\ 3x + 5y + 7z = 21 \\ 4x + 6y + 5z = 26 \end{cases}$$

The matrix is

$\boldsymbol{A = \{\{1,2,3\},\{3,5,7\},\{4,6,5\}\}}$

$\{\{1,2,3\},\{3,5,7\},\{4,6,5\}\}$

We can see that no row or column can be put as a linear equation of the rest, this means that it´s not possible to reduce the matrix in zeros and ones if you sum or rest the rows

The extended matrix is

$\boldsymbol{B = \{\{1,2,3,7\},\{3,5,7,21\},\{4,6,5,26\}\}}$

$\{\{1,2,3,7\},\{3,5,7,21\},\{4,6,5,26\}\}$

The ranks are:

MatrixRank[A]

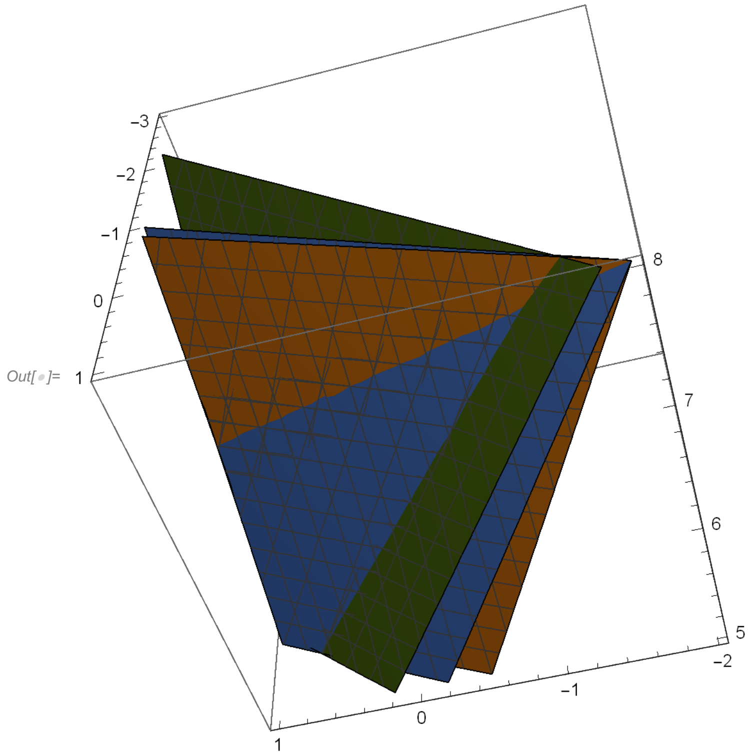
3

MatrixRank[B]

3

For this we can infer that the three planes intersect in a single point.
We can plot to prove that:

```
ContourPlot3D[{x+2 y+3 z==7,3 x+5 y+7 z==21,4 x+6 y+5 z==26},{x,5,8},{y,-2,1},{z,-3,1}]
```



Also we can prove it if we solve the System:

Solve$[\{\boldsymbol{x} + \mathbf{2}\boldsymbol{y} + \mathbf{3}\boldsymbol{z} == \mathbf{7}, \mathbf{3}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} + \mathbf{7}\boldsymbol{z} == \mathbf{21}, \mathbf{4}\boldsymbol{x} + \mathbf{6}\boldsymbol{y} + \mathbf{5}\boldsymbol{z} == \mathbf{26}\}]$

$\left\{\left\{x \to \frac{23}{3}, y \to -\frac{4}{3}, z \to \frac{2}{3}\right\}\right\}$

For this the example is a system of planes that intersect in one single point

10 Answer to Problem X

X. Let n be an arbitrary positive integer. Find the rank of the identity matrix I_n and justify your answer.

The identity matrix of size n is a square matrix with ones on the main diagonal and zeros elsewhere. Let n=4:

I4 = DiagonalMatrix[{1, 1, 1, 1}];

MatrixForm[I4]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

MatrixRank[I4]

4

This can be justified by finding the Reduced Row Echelon Form of the Identity Matrix which is equal to the Identity Matrix itself:

MatrixForm[RowReduce[I4]]

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where the number of non-zero rows will always be equal to n.

11 Answer to Problem XI

XI. For each of the following problems, write the corresponding linear system and solve it using Mathematica (not by hand!).

25. A coffee merchant sells three blends of coffee. A bag of the house blend contains 300 grams of Colombian beans and 200 grams of French roast beans. A bag of the special blend contains 200 grams of Colombian beans, 200 grams of Kenyan beans, and 100 grams of French roast beans. A bag of the gourmet blend contains 100 grams of Colombian beans, 200 grams of Kenyan beans, and 200 grams of French roast beans. The merchant has on hand 30 kilograms of Colombian beans, 15 kilograms of Kenyan beans, and 25 kilograms of French roast beans. If he wishes to use up all of the beans, how many bags of each type of blend can be made?

Establishing the equations for the linear system:
x = # of House Blend bags
y = # of Special Blend bags
z = # of Gourmet Blend bags

$\mathbf{300 * \boldsymbol{x} + 200 * \boldsymbol{y} + 100 * \boldsymbol{z}{=}{=}30000;}$

$\mathbf{200 * \boldsymbol{y} + 200 * \boldsymbol{z}{=}{=}1500;}$

$\mathbf{200 * \boldsymbol{x} + 100 * \boldsymbol{y} + 200 * \boldsymbol{z}{=}{=}25000;}$

$\boldsymbol{M = \{\{300 * x, 200 * y, 100 * z, 30000\}, \{0, 200 * y, 200 * z, 1500\}, \{200 * x, 100 * y, 200 * z, 25000\}\};}$

MatrixForm[$\boldsymbol{M}$]

$$\begin{pmatrix} 300x & 200y & 100z & 30000 \\ 0 & 200y & 200z & 1500 \\ 200x & 100y & 200z & 25000 \end{pmatrix}$$

$\mathbf{Solve[300\boldsymbol{x} + 200\boldsymbol{y} + 100\boldsymbol{z} == 30000\&\&200\boldsymbol{x} + 100\boldsymbol{y} + 200\boldsymbol{z} == 25000\&\&200\boldsymbol{y} + 200\boldsymbol{z} == 15000, \{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}]}$

$\{\{x \to 65, y \to 30, z \to 45\}\}$

The total amount of bags that can be made of each blend is 65 bags of House Blend, 30 bags of Special Blend and 45 bags of Gourmet Blend.

26. There are two fields whose total area is 1800 square yards. One field produces grain at the rate of 2/3 bushel per square yard; the other field produces grain at the rate of 1/2 bushel per square yard. If the total yield is 1100 bushels, what is the area of each field?

Establishing the equations for the linear system:
x = Square Yards of Field 1

y = Square Yards of Field 2

$\boldsymbol{x + y == 1800;}$

$\mathbf{(2/3)}\boldsymbol{x} + \mathbf{(1/2)}\boldsymbol{y} == \mathbf{1100;}$

$\mathbf{MF} = \{\{\boldsymbol{x}, \boldsymbol{y}, \mathbf{1800}\}, \{\mathbf{(2/3)}\boldsymbol{x}, \mathbf{(1/2)}\boldsymbol{y}, \mathbf{1100}\}\};$

MatrixForm[MF]

$$\begin{pmatrix} x & y & 1800 \\ \frac{2x}{3} & \frac{y}{2} & 1100 \end{pmatrix}$$

$\mathbf{Solve}[\boldsymbol{x} + \boldsymbol{y} == \mathbf{1800\&\&(2/3)}\boldsymbol{x} + \mathbf{(1/2)}\boldsymbol{y} == \mathbf{1100}, \{\boldsymbol{x}, \boldsymbol{y}\}]$

$\{\{x \to 1200, y \to 600\}\}$

Field 1 has an area of 1200 square yards while field 2 has an area of 600 square yards.

27. Find a parabola with an equation of the form y = ax2 + bx + c that passes through (0, 1), (-1, 4) and (2, 1).

Establishing the equations for the linear system:

$\mathbf{1} == \boldsymbol{a} * \mathbf{0\verb|^|2} + \boldsymbol{b} * \mathbf{0} + \boldsymbol{c};$

$\mathbf{4} == \boldsymbol{a} * \mathbf{(-1)\verb|^|2} + \boldsymbol{b} * \mathbf{(-1)} + \boldsymbol{c};$

$\mathbf{1} == \boldsymbol{a} * \mathbf{2\verb|^|2} + \boldsymbol{b} * \mathbf{2} + \boldsymbol{c};$

$\mathbf{MP} = \{\{\mathbf{0}, \mathbf{0}, \boldsymbol{c}, \mathbf{1}\}, \{\boldsymbol{a}, -\boldsymbol{b}, \boldsymbol{c}, \mathbf{4}\}, \{\mathbf{4}\boldsymbol{a}, \mathbf{2}\boldsymbol{b}, \boldsymbol{c}, \mathbf{1}\}\};$

MatrixForm[MP]

$$\begin{pmatrix} 0 & 0 & c & 1 \\ a & -b & c & 4 \\ 4a & 2b & c & 1 \end{pmatrix}$$

$\mathbf{Solve}[\mathbf{1} == \boldsymbol{a} * \mathbf{0\verb|^|2} + \boldsymbol{b} * \mathbf{0} + \boldsymbol{c}\mathbf{\&\&4} == \boldsymbol{a} * \mathbf{(-1)\verb|^|2} + \boldsymbol{b} * \mathbf{(-1)} + \boldsymbol{c}\mathbf{\&\&1} == \boldsymbol{a} * \mathbf{2\verb|^|2} + \boldsymbol{b} * \mathbf{2} + \boldsymbol{c}, \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}]$

$\{\{a \to 1, b \to -2, c \to 1\}\}$

$\boxed{\text{The equation for the parabola that passes though the three points is: y = x}^2 - 2x + 1.}$

12 Answer to Optional Problem 1

Consider the following matrix

A= $\begin{pmatrix} \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi^2 \\ \pi^3 & \pi^3 & \pi^3 \end{pmatrix}$

1. Find the reduced row echelon from of A; then find the rank of A.
2. How can you enter in Mathenatica (in one line) the matrix A? (without typing every entry!). Hint: Consider the Table command.
3. Now generalize the result as follows: let X be the following arbitrary square matrix of size n, where c is any non-zero number. Compute the rank of X.

X= $\begin{pmatrix} c & c & \cdots & c \\ c^2 & c^2 & \cdots & c^2 \\ \vdots & \vdots & \vdots & \vdots \\ c^n & c^n & \cdots & c^n \end{pmatrix}$

Optional 1.1

Print["RowReduce[A]="MatrixForm[RowReduce[*A*]]]

Print["MatrixRank[A]="]

MatrixRank[*A*]

RowReduce[A]= $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

MatrixRank[A]= 1

Optional 1.2

A = Table[Pi^i, {i, 1, 3}, {j, 1, 3}];

MatrixForm[*A*]

$$\begin{pmatrix} \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi^2 \\ \pi^3 & \pi^3 & \pi^3 \end{pmatrix}$$

Optional 1.3

$$\text{X=}\begin{pmatrix} c & c & \cdots & c \\ c^2 & c^2 & \cdots & c^2 \\ \vdots & \vdots & \vdots & \vdots \\ c^n & c^n & \cdots & c^n \end{pmatrix} \Rightarrow \begin{array}{c} \frac{1}{X_{11}}R_1 \to R_1 \\ \frac{1}{X_{21}}R_2 \to R_2 \\ \vdots \\ \frac{1}{X_{n1}}R_n \to R_n \end{array} \Rightarrow \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \Rightarrow$$

$$\begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ \vdots \\ R_n - R_1 \to R_n \end{array} \Rightarrow \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

13 Answer to Optional Problem 2

For what value(s) of k, if any, will the following system:

$$\left\{\begin{array}{ll} x+y+\mathrm{kz} & =1 \\ x+\mathrm{ky}+z & =1 \\ \mathrm{kx}+y+z & =-2 \end{array}\right.$$

have

1. No solution
2. A unique solution
3. Infinitely many solutions

Hint. Find the reduced echelon form of the augmented matrix, then analyze different cases (beware of division by zero!).

$$\left\{\begin{array}{ll} x+y+\mathrm{kz} & =1 \\ x+\mathrm{ky}+z & =1 \\ \mathrm{kx}+y+z & =-2 \end{array}\right. \Rightarrow \begin{pmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{pmatrix} \Rightarrow R_1 \leftrightarrow R_3 \Rightarrow \begin{pmatrix} k & 1 & 1 & -2 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \Rightarrow$$

$$\frac{1}{k}R_1 \to R_1 \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & \frac{1}{k} & -\frac{2}{k} \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \Rightarrow \begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & \frac{1}{k} & -\frac{2}{k} \\ 0 & k-\frac{1}{k} & 1-\frac{1}{k} & 1+\frac{2}{k} \\ 0 & 1-\frac{1}{k} & k-\frac{1}{k} & 1+\frac{2}{k} \end{pmatrix} \Rightarrow$$

$$R_1 - \frac{1}{-1+k^2}R_2 \to R_1 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & k-\frac{1}{k} & 1-\frac{1}{k} & 1+\frac{2}{k} \\ 0 & 1-\frac{1}{k} & k-\frac{1}{k} & 1+\frac{2}{k} \end{pmatrix} \Rightarrow \frac{k}{-1+k^2}R_2 \to R_2 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & 1 & \frac{1}{1+k} & \frac{2+k}{-1+k^2} \\ 0 & 1-\frac{1}{k} & k-\frac{1}{k} & 1+\frac{2}{k} \end{pmatrix}$$

$$R_3 - \left(1-\frac{1}{k}\right)R_2 \to R_3 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & 1 & \frac{1}{1+k} & \frac{2+k}{-1+k^2} \\ 0 & 0 & k-\frac{2}{1+k} & 1+\frac{1}{1+k} \end{pmatrix} \Rightarrow \frac{1+k}{-2+k+k^2}R_3 \to R_3 \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & 1 & \frac{1}{1+k} & \frac{2+k}{-1+k^2} \\ 0 & 0 & 1 & \frac{1}{-1+k} \end{pmatrix} \Rightarrow \begin{array}{c} R_2 - \frac{1}{1+k}R_3 \to R_2 \\ R_1 - \frac{1}{1+k}R_3 \to R_1 \end{array} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{-1+k} \\ 0 & 1 & 0 & \frac{1}{-1+k} \\ 0 & 0 & 1 & \frac{1}{-1+k} \end{pmatrix}$$

$$\begin{aligned} z &= \frac{1}{-1+k} \\ y &= \frac{1}{-1+k} \\ x &= -\frac{2}{-1+k} \end{aligned}$$

Optional 2.1

The system does not have a solution for k = 1.

Optional 2.2 & 2.3

Since y = z, the system does not have a unique solution. Therefore the system has infinitely many solutions for k $\neq$ 1.

Verification using Mathematica:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \boldsymbol{k} & \mathbf{1} \\ \mathbf{1} & \boldsymbol{k} & \mathbf{1} & \mathbf{1} \\ \boldsymbol{k} & \mathbf{1} & \mathbf{1} & \mathbf{-2} \end{pmatrix};$$

RowReduce[*A*];

MatrixForm[%]

Solve[$\{x+y+k*z==1, x+k*y+z==1, k*x+y+z==-2\}, \{x,y,z\}$**]**

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{-1+k} \\ 0 & 1 & 0 & \frac{1}{-1+k} \\ 0 & 0 & 1 & \frac{1}{-1+k} \end{pmatrix}$$

$$\left\{\left\{x\to -\frac{2}{-1+k}, y\to \frac{1}{-1+k}, z\to \frac{1}{-1+k}\right\}\right\}$$