The Navier Stokes Equation

The Navier-Stokes Equation

https://www.britannica.com/science/Navier-Stokes-equation

Navier-Stokes equation, in fluid mechanics, a partial differential equation that describes the flow of incompressible fluids.

The equation is a generalization of the equation devised by Swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and frictionless fluids.

In 1821 French engineer Claude-Louis Navier introduced the element of viscosity (friction) for the more realistic and vastly more difficult problem of viscous fluids.

Throughout the middle of the 19th century, British physicist and mathematician Sir George Gabriel Stokes improved on this work, though complete solutions were obtained only for the case of simple two-dimensional flows.

The complex vortices and turbulence, or chaos, that occur in three-dimensional fluid (including gas) flows as velocities increase have proven intractable to any but approximate numerical analysis methods.

Part 1: Continuity Equation

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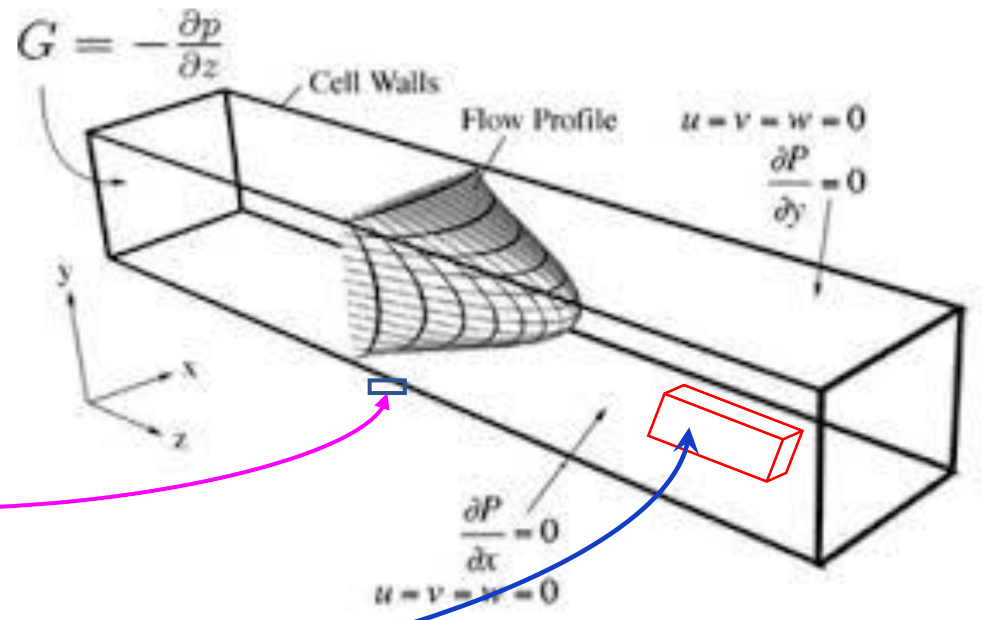
Reflections

The **Navier-Stokes equation**, in fluid mechanics, a partial differential equation that describes the flow of incompressible fluids.



How can we get such differential equation?

- In order to do so, we need to imagne a flow of a liquid in a channel…
- Then "take" a differential volume element and…
- Make a mass balance on that differential element (CONTINUITY EQUATION) and
- Make a force balance on that differential element (MOMENTUM EQUATION)
- Apply the elements of the moentum equation appropriate for the system you have a channel, die, etc.
- Afterwards you use a Constitutive Equation to relate the Stresses to the Memory function data

Image channel from: Fully developed flow of non-Newtonian fluids in a straight uniform square duct through porous medium. Journal of the Association of Arab Universities for Basic and Applied Sciences_ Volume 23, June 2017, Pages 66-74

Mass Balance
over differential element:

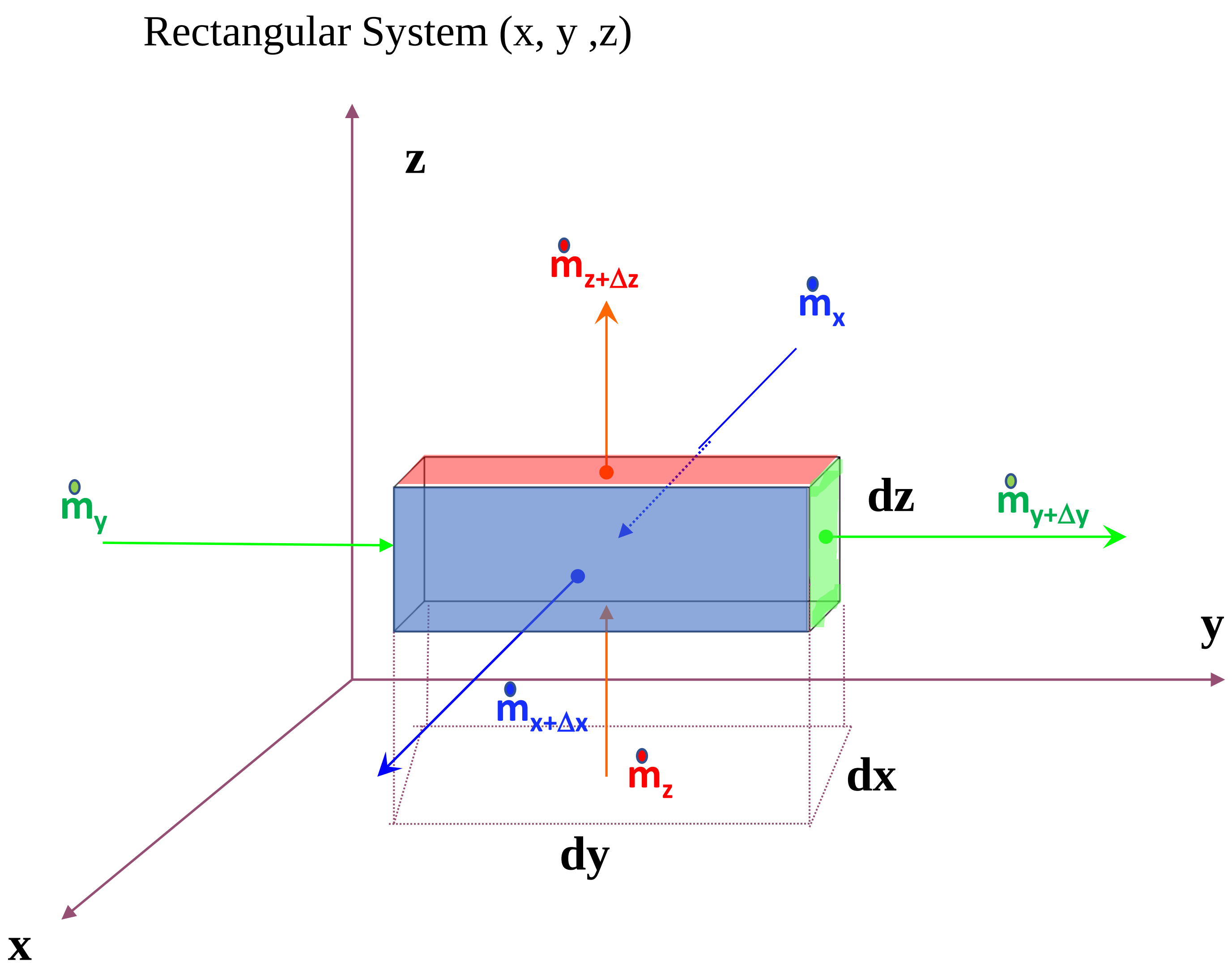
Input Rate to the System	-	Output Rate from the System	=	System's Accumulation

The differential element on cartesian coordinates

Imagine now that the material (mass) can enter and exit the volume through the faces of the element:

- The mass flowing (m upper dot) in the x direction enters and leaves the element through a surface equal to $\Delta y \Delta z$.
- The mass flowing (m upper dot) in the y direction enters and leaves the element through a surface equal to $\Delta x \Delta z$.
- The mass flowing (m upper dot) in the z direction enters and leaves the element through a surface equal to $\Delta x \Delta y$.

- Now the mass flow rate (m upper dot) can be expressed in terms of density and velocity flowing through a surface area:



Mass flux concept

The mass flow rate (kilograms/second) can be converted to something that is called Mass Flux, that is kilograms/(second meter^2).

Such flux can be calculated as follows:

ρ is density $\quad \dfrac{\text{Mass}}{\text{Vol}}$

v is velocity $\quad \dfrac{\text{meters}}{\text{second}}$

$$\rho\, v = \frac{\text{Mass}}{\text{Volume}} \; \frac{\text{meters}}{\text{second}} = \frac{\text{Kilograms}}{\text{m}^2\ \text{s}}$$

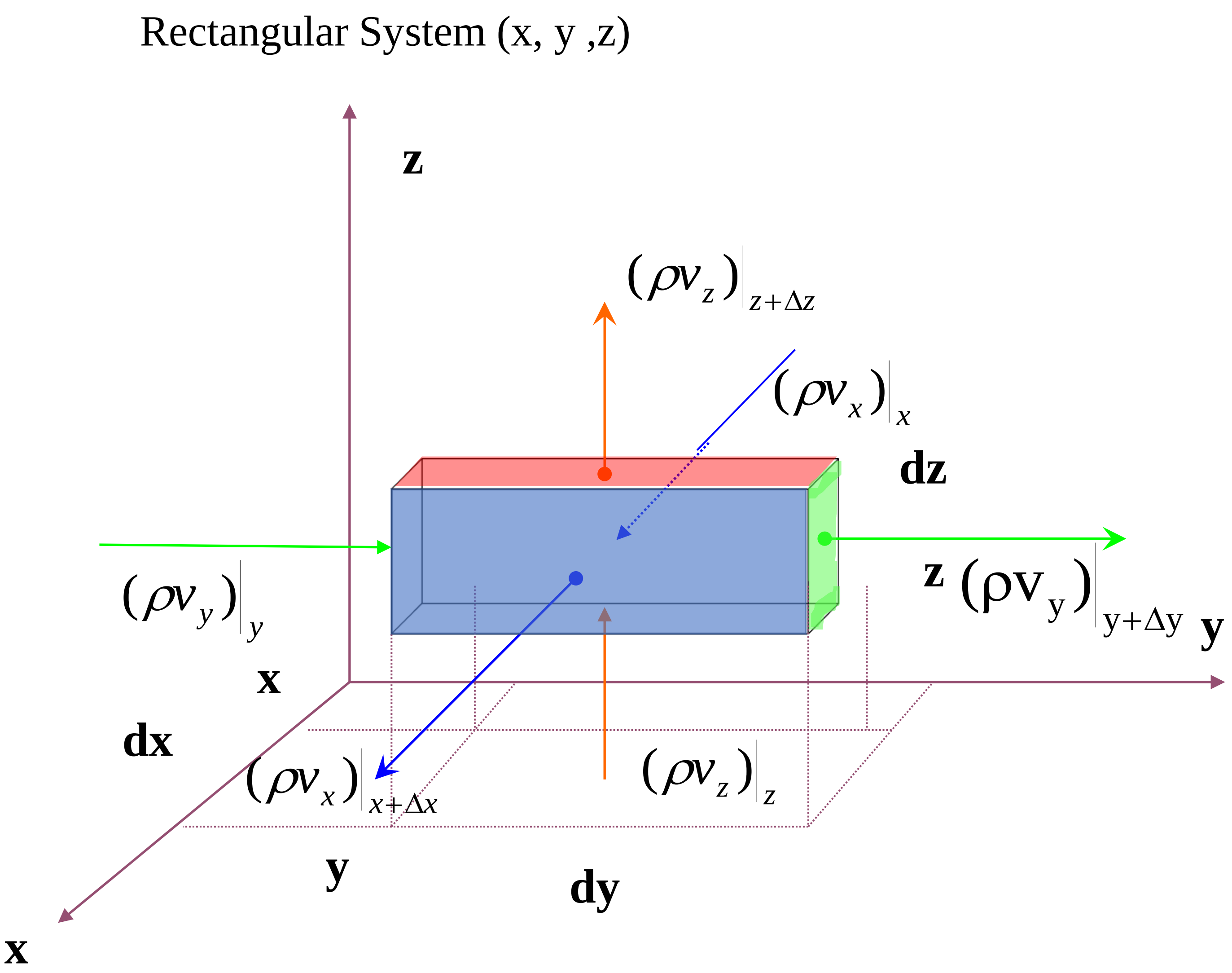
The differential element on cartesian coordinates

Then the mass flux entering and leaving, in every drection can be expresed as:

$\rho\ v_x$ $\rho\ v_y$ $\rho\ v_z$

And that (see the figure)

- the mass flux at x enters and leaves at x+ Δx, through an area ΔyΔz
- the mass flux at y enters and leaves at y+ Δy, through an area ΔxΔz
- the mass flux at z enters and leaves at z+ Δz, through an area ΔyΔx



If we perform a balance in x direction:

Mass flux entering at distance x

Through an area

Mass flux leaving at distance x +ΔX

Through an area

$$\boxed{(\rho v_x)\Big|_{X}}\ \Delta y \Delta z \ - \ (\rho v_x)\Big|_{X+\Delta X}\ \Delta y \Delta z$$

Since the area is the same:

$$\left[(\rho v_x)\Big|_{x} - (\rho v_x)\Big|_{x+\Delta x} \right] \Delta y \Delta z$$

Then we can do a mass balance on the differential volume elementy : $\Delta x \Delta y \Delta z$

Mass balance

i means entering (inn)

O means leaving (out)

$$(\dot{m}_{ix} + \dot{m}_{iy} + \dot{m}_{iz}) -$$
$$(\dot{m}_{ox} + \dot{m}_{oy} + \dot{m}_{oz}) =$$
$$= dM_{syst} / dt$$

Where the accumulation of mass in the element volume is dM/dt

$$\Delta y \Delta z \left[(\rho v_x)\big|_x - (\rho v_x)\big|_{x+\Delta x} \right] + \Delta x \Delta z \left[(\rho v_y)\big|_y - (\rho v_y)\big|_{y+\Delta y} \right] +$$
$$\Delta x \Delta y \left[(\rho v_z)\big|_z - (\rho v_z)\big|_{z+\Delta z} \right] = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z)$$

The Continuity Equation in Rectangular System

By dividing this entire equation by (ΔxΔyΔz), and taking the limits these dimensions approach zero, we get:

Assuming that dV = dxdydz ≠ f(x,y,z,t).

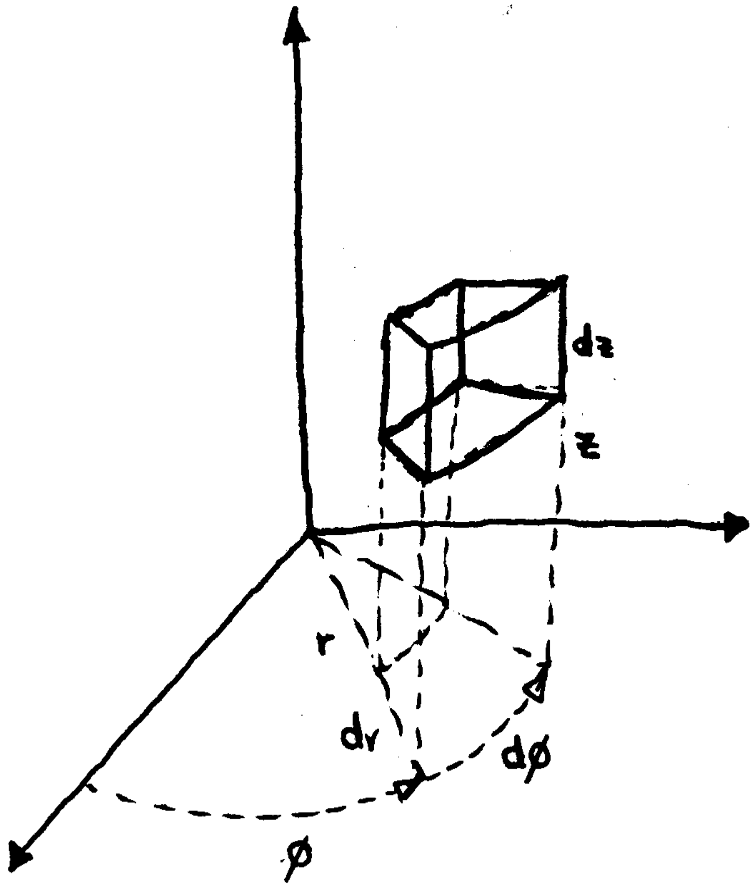
$$-\left[\frac{\partial}{\partial x}(\rho_x v_x)+\frac{\partial}{\partial y}(\rho_y v_y)+\frac{\partial}{\partial z}(\rho_z v_z)\right]=\frac{\partial \rho}{\partial t}$$

$$-(\overline{\nabla}\bullet\rho\overline{v})=\frac{\partial \rho}{\partial t}$$

$$\text{where} \qquad \overline{\nabla}\equiv Nabla \quad Operator \equiv \frac{\partial \hat{i}}{\partial x}+\frac{\partial \hat{j}}{\partial y}+\frac{\partial \hat{k}}{\partial z}$$

Continuity Equation (Mass Conservation)

Cilindric System (r, ϕ, z)



Mass Balance
over diferencial element:

Input Rate to the System	-	Output Rate from the System	=	System's Accumulation

$$A_{in\ r} = \iint r\,d\phi\,dz \qquad A_{out\ r} = \iint r\,d\phi\,dz$$

$$A_{in\ \phi} = \iint dr\,dz \qquad A_{out\ \phi} = \iint dr\,dz$$

$$A_{in\ z} = \iint r\,d\phi\,dr \qquad A_{out\ z} = \iint r\,d\phi\,dr$$

$$V = \iiint r\,d\phi\,dr\,dz$$

$$(\dot{m}_{ir} + \dot{m}_{i\phi} + \dot{m}_{iz}) - (\dot{m}_{or} + \dot{m}_{o\phi} + \dot{m}_{oz}) = dM_{sys}/dt$$

Similarly, and if $dV = rdrd\phi dz \neq f(r,\phi,z,t)$

$$-\left[\frac{\partial}{r\partial r}(\rho_r v_r r) + \frac{\partial}{r\partial\phi}(\rho_\phi v_\phi) + \frac{\partial}{\partial z}(\rho_z v_z)\right] = \frac{\partial\rho}{\partial t}$$

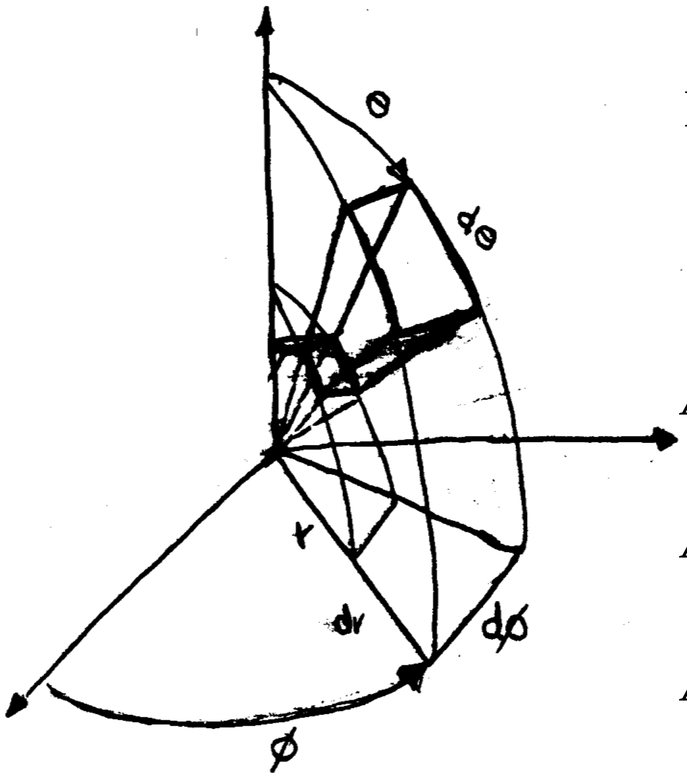
$$-(\overline{\nabla} \bullet \rho\overline{v}) = \frac{\partial\rho}{\partial t}$$

The Continuity Equation in Cylindrical System

where $\overline{\nabla} \equiv Nabla \quad Operator \equiv \frac{\partial\hat{i}}{\partial r} + \frac{\partial\hat{j}}{r\partial\phi} + \frac{\partial\hat{k}}{\partial z}$

Continuity Equation (Mass Conservation)

Spheric System (r, ϕ, θ)



Mass Balance
over diferencial element:

Input Rate to the System	-	Output Rate from the System	=	System's Accumulation

$$A_{in\ r} = \iint rd\theta rSen\theta d\phi \qquad A_{out\ r} = \iint rd\theta rSen\theta d\phi$$

$$A_{in\ \phi} = \iint rd\theta dr \qquad A_{out\ \phi} = \iint rd\theta dr$$

$$A_{in\ \theta} = \iint rSen\theta d\phi dr \qquad A_{out\ \theta} = \iint rSen\theta d\phi dr$$

$$V = \iiint r^2 Sen\theta d\phi dr d\theta$$

$$(\dot{m}_{ir} + \dot{m}_{i\phi} + \dot{m}_{i\theta}) - (\dot{m}_{or} + \dot{m}_{o\phi} + \dot{m}_{o\theta}) = dM_{sys}/dt$$

Similarly and if $dV = r^2 Sen\Theta\, dr d\phi d\Theta \neq f(r,\phi,\Theta,t)$

$$-\left[\left(\frac{\partial}{r^2\partial r}(r^2\rho_r v_r)\right)+\left(\frac{\partial}{rSen\theta\partial\phi}(\rho_\phi v_\phi)\right)+\left(\frac{\partial}{rSen\theta\partial\theta}(\rho_\theta v_\theta Sen\theta)\right)\right]=\frac{\partial\rho}{\partial t}$$

$$-(\bar{\nabla}\bullet\rho\bar{v})=\frac{\partial\rho}{\partial t}$$

The Continuity Equation in Spherical System

where $\nabla \equiv Nabla \;\; Operator \equiv \frac{\partial\hat{i}}{\partial r}+\frac{\partial\hat{j}}{rsen\theta\,\partial\phi}+\frac{1\partial\hat{k}}{r\partial\theta}$