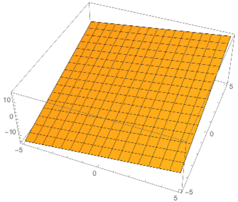
Introduction to systems of linear equations

Recall that the general equation of a line in $\mathbb{R}^2$ is of the form

$$ax + by = c$$

and that the general equation of a plane in $\mathbb{R}^3$ is of the form

$$ax + by + cz = d$$



Equations of this form are called **linear equations**.

This is the main motivation for the following

Definition (linear equation). A **linear equation** in the n variables $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where the **coefficients** $a_1, a_2, \ldots, a_n$ and the **constant term** b are constants.

The following equations **are linear**:

1. $3x - 4y = -1$
2. $r - \frac{s}{2} - \frac{15}{3}t = 9898$
3. $\sqrt{2} \cdot x + \frac{\pi}{4} \cdot y - sin\left(\frac{\pi}{5}\right) \cdot z = 1$
4. $x_1 + 5x_2 = 3 - x_3 + 2x_4$

Remark. Although in these examples the coefficients are elements of $\mathbb{R}$, in some applications (for example, in cryptography) they will be elements of $\mathbb{Z}_p$, where p is a prime number (modular arithmetic!).

Examples of nonlinear equations

The following equations are **not** linear:

1. $xy + 2z = 1$
2. $x_1^2 - x_2^3 = 3$
3. $\frac{x}{y} + z = 2$
4. $\sqrt{2} \cdot x + \frac{\pi}{4} \cdot y - sin\left(\frac{\pi z}{5}\right) = 1$

Remark. Linear equations **do not** contain products, reciprocals, or other functions of the variables; the variables occur only to the **first power** and are multiplied only by constants.

Questions

1. Is $sin\left(\frac{\pi}{7}\right) \cdot x_1 + 2^e \cdot x_2 + \sqrt{3} \cdot x_3 = 1$ linear?
2. Is $sin\left(\frac{\pi}{7}\right) \cdot x_1 + 2^e \cdot x_2^e + \sqrt{3} \cdot x_3 = 1$ linear?

Definition. A **solution** of a linear equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ is a vector $[s_1, s_2, \ldots, s_n]$ whose components satisfy the equation when we substitute $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$.

Example. $[5, 4]$ is a solution of $3x - 4y = -1$.

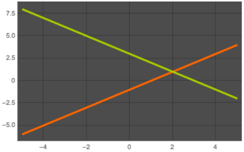
Definition. A **solution of a system of linear equations** is a vector that is **simultaneously** a solution of each equation in the system.

Question. Is $[1, -1]$ is a solution of the following system?

$$\begin{cases} 2x - y = 3 \\ x + 3y = 5 \end{cases}$$

As usual, there is a nice interplay between geometry and algebra. Suppose we have the following systems of equations:

1. $$\begin{cases} x - y = 1 \\ x + y = 3 \end{cases}$$
2. $$\begin{cases} x - y = 2 \\ 2x - 2y = 4 \end{cases}$$
3. $$\begin{cases} x - y = 1 \\ x - y = 3 \end{cases}$$

1. 

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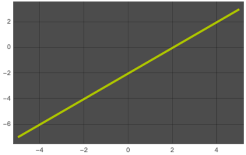
2. 

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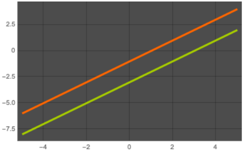
3. 

(no solution!)

Consider the system

$$\begin{cases} x - y - z = 2 \\ y + 3z = 5 \\ 5z = 10 \end{cases}$$

Question 1. How can we solve this system?

Answer. Work backwards! from the last equation we get $z = 2$. Using the second equation we obtain $y = 5 - 3(2) = -1$ and then the first equation gives $x = 3$.

The above procedure is known as **back substitution**.

Question 2. What is the geometric interpretation of this system in $\mathbb{R}^3$?

Definition (row echelon form). A matrix is in **row echelon form** if it satisfies the following properties:

1. Any rows consisting entirely of zeros are at the bottom.
2. In each nonzero row, the first **nonzero** entry (called the **leading entry**) is in a column to the right of the leading entry in the previous row.

Remark 1. The above properties guarantee that the leading entries form a staircase pattern.

Remark 2. In any column, all entries below the leading entry are zero.

The following matrices are in row echelon form (make sure you understand why!):

1. $$\begin{bmatrix} 2 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 11 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & -2 & 2 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

4. $$\begin{bmatrix} 0 & 2 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Which of the following matrices are in row echelon form? explain why!

1. $$\begin{bmatrix} 7 & 0 & 1 & 0 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
2. $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
3. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
4. $$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
5. $$\begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Answers ☺

1. Yes
2. Yes
3. Yes
4. No
5. No

We now describe the procedure by which any matrix can be reduced to a matrix in row echelon form.

Elementary row operations

1. Interchange two rows.
2. Multiply a row by a nonzero constant.
3. Add a multiple of a row to another row.

Notation for the previous operations

1. $R_i \longleftrightarrow R_j$ means interchange rows i and j.
2. kR_i means multiply row i by k.
3. $R_i + kR_j$ means add k times row j to i (and replace row i with the result).

Useful strategies to reducing to row echelon form

1. Work column by column (from left to right).
2. Work from the top to the bottom.
3. Create a leading entry in a column and use it to create zeros below it.

Example. Find the row echelon form of

$$\begin{bmatrix} 0 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

Solution. First, we consider the first nonzero column (which is column 1) and try to create a leading entry in this column. We can interchange rows 1 and 3: $R_1 \longleftrightarrow R_3$. We get:

$$\begin{bmatrix} 1 & -1 & -2 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

In place of 2 we must have 0, a simple way is to apply $R_2 - 2R_1$. The new elements of R_2 are:

$$2 - 2(1) = 0$$
$$3 - 2(-1) = 5$$
$$1 - 2(-2) = 5$$

so the new matrix is

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 5 & 5 \\ 0 & 2 & 3 \end{bmatrix}.$$

The leading term of the second row is 5, and below it we must have 0. It is easier to work when the leading term is 1; thus we can apply $\frac{1}{5}R_2$ to get

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Now in place of 2 we must have 0, so we can apply $R_3 - 2R_2$ to obtain

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

and this matrix is now in row echelon form. We are done!

Warning : The row echelon form of an arbitrary matrix is **NOT** unique! (point is: there is no a "correct" answer)

Question: Can you think of an application of row echelon form?

Definition (rank) The **rank** of a matrix is the number of nonzero rows in its row echelon form. The rank of a matrix A is usually denoted by $\operatorname{rank}(A)$.

Exercise. Prove that the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$

has rank 2 and verify your answer with Mathematica.