The inverse of a matrix

Suppose we have the system

$$AX = B \tag{1}$$

where A is a square matrix. Now **suppose** there exists a matrix C such that $AC = CA = I$, where I is the identity matrix.

Multiplying eq. (1) by C on the left we obtain

$$\begin{aligned} CAX &= CB \\ IX &= CB \\ X &= CB \end{aligned}$$

Conclusion. If C exists, then we can solve for X and find the vector of unknowns!

Definition (the inverse of a matrix). Let A be a **square** matrix and suppose there exists a matrix C such that $AC = CA = I$. We say C is the inverse matrix of A and denote it by A^{-1}.

Remarks

1. The notation A^{-1} comes from the fact that $AA^{-1} = A^{-1}A = I$ and the identity matrix represents the unit, just like in $\mathbb{R} : a \cdot \frac{1}{a} = 1$ if $a \neq 0$.
2. The inverse of A might not exist.
3. If A^{-1} exists then it is unique.
4. There are many methods to compute the inverse of a matrix.

Theorem. Let A be a square matrix of size n. A is invertible if and only if the reduced row echelon form of A is I_n.

To determine whether a matrix A is invertible or not, we will apply the **Gaussian-Jordan** algorithm as follows.

Step 1 Create the augmented matrix $[A \mid I]$.

Step 2 Reduce A to the identity matrix (via reduced row echelon form). Thus the idea is to transform $[A \mid I] \to [I \mid A^{-1}]$.

Step 3 If A cannot be reduced to I then A is not invertible. This happens when the reduced row echelon form of A contains a zero row.

Example 1. Find the inverse of the following matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Step 1 Create the augmented matrix $[A \mid I]$.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

Step 2 Try to convert A to reduced row echelon form. We first apply $\frac{R_1}{2} \to R_1$, to obtain

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{array}\right]$$

Then we can apply $R_2 - 4R_1 \to R_2$; $R_3 + 2R_1 \to R_3$ and get

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

Then we apply $-\frac{R_2}{8} \to R_2$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{array}\right]$$

Then $R_3 - 8R_2 \to R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

$R_1 - \frac{R_2}{2} \to R_1$ to obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{8} & \frac{3}{8} & \frac{1}{16} & 0 \\ 0 & 1 & \frac{1}{4} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Now are done if we apply $R_2 - \frac{1}{4}R_3 \to R_2$; $R_1 - \frac{3}{8}R_3 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Step 3 Analyze the reduced row echelon form:

1. Notice that the matrix on the left-hand side is precisely the **identity matrix**.
2. Therefore the **inverse exists** and it is equal to the matrix on the right hand side.

$$\boxed{\textit{Conclusion}: A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix}}$$

Homework: Check $AA^{-1} = A^{-1}A = I$.

Remark. Given a system $AX = B$, if A^{-1} exists then $AX = B$ has a **unique** solution and it is given by $X = A^{-1}B$.

Example. Suppose we have the system

$$\begin{cases} 2x + y + z = 12 \\ 4x - 6y = -2 \\ -2x + 7y + 2z = 21 \end{cases}$$

We already know A^{-1}, so we simply compute

$$X = A^{-1}B = \begin{bmatrix} \frac{3}{4} & -\frac{5}{16} & -\frac{3}{8} \\ \frac{1}{2} & -\frac{3}{8} & -\frac{1}{4} \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 12 \\ -2 \\ 21 \end{bmatrix} = \begin{bmatrix} \frac{7}{4} \\ \frac{3}{2} \\ 7 \end{bmatrix}$$

Therefore the solution is given by $\boxed{(x, y, z) = \left(\frac{7}{4}, \frac{3}{2}, 7\right)}$

Question. Given a linear system $AX = B$, what if A^{-1} does not exist?

Answer. In this case there are two possibilities:

1. There are infinitely many solutions.
2. No solution.

Then one can figure out the answer by using the following

Theorem. Let $AX = B$ be a linear system, where A is a square matrix of size $m \times n$.

1. If $\text{rank}(A) < \text{rank}([A|B])$, then the system has **no solution**.
2. If $\text{rank}(A) = \text{rank}([A|B])$ and **both** are less than n, then the system has **infinitely many solutions**.
3. If $\text{rank}(A) = \text{rank}([A|B]) = n$, then the system has a **unique solution**.
4. If the system is homogeneous (B is the zero vector) and $n > m$ (more variables than equations) then the system has **infinitely many solutions**.

Example. Solve the system

$$\begin{cases} x - y + 2z = 3 \\ x + 2y - z = -3 \\ 2y - 2z = 1 \end{cases}$$

Check (exercise!) that the row reduced echelon form of the **augmented matrix** is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Since there exists a row consisting of zeros then A^{-1} does not exist. Moreover,

1. $\text{rank}(A) = 2$
2. $\text{rank}(A|B) = 3$

Because $2 < 3$ then the theorem says that no solution exist (which is clear if we look at the last equation!).

Homework ☺: Plot in Mathematica the previous system and explain what is going on!

To compute the **inverse** matrix in Mathematica we simply type:

Inverse[name of the matrix]

Definition (diagonal matrix). A square matrix is **diagonal** if the entries outside the **main diagonal** are all 0. For example:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{\pi}{2} \end{bmatrix}, B = \begin{bmatrix} 11 & 0 & 0 \\ 0 & \frac{\pi^2}{99} & 0 \\ 0 & 0 & 6 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

are all diagonal matrices. When A is diagonal we write $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$. In this case, A^{-1} is given by

$$\boxed{A^{-1} = \operatorname{diag}\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right)}$$

provided each division is well-defined. Therefore

$$A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{\pi} \end{bmatrix}, B^{-1} = \begin{bmatrix} \frac{1}{11} & 0 & 0 \\ 0 & \frac{99}{\pi^2} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix}, C^{-1} = ?$$

Definition (lower triangular matrix). A square matrix is called **lower triangular** if all the entries above the main diagonal are 0.

Example:

$$\begin{bmatrix} -11 & 0 & 0 \\ 55 & 3 & 0 \\ 34 & 0 & -e \end{bmatrix}$$

Definition (upper triangular matrix). A square matrix is called **upper triangular** if all the entries below the main diagonal are 0.

Example:

$$\begin{bmatrix} 1 & 44 & -12 \\ 0 & 3 & 22 \\ 0 & 0 & -\frac{2}{\sqrt{3}} \end{bmatrix}$$

Theorem. A triangular matrix (upper, lower or diagonal) is invertible if and only if **no** element on its main diagonal is 0.

Problem 1. What happens if a matrix is **both** upper triangular and lower triangular? What kind of matrix do we get?

Problem 2. Justify which of the following matrices are invertible. *Hint*: no long calculations are needed!

1. $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
2. The identity matrix of any size.
3. $$\begin{bmatrix} 1 & 2 & \sqrt{\pi} \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$
4. $$\begin{bmatrix} 1 & 0 & 0 \\ -35 & 113 & 0 \\ 234 & 34 & cos(0) \end{bmatrix}$$
5. $$\begin{bmatrix} 1 & 0 & 0 \\ -35 & 113 & 0 \\ 234 & 34 & cos(\frac{\pi}{2}) \end{bmatrix}$$

Problem 3. Consider the system:

$$\begin{cases} x + 2y - z = 7 \\ 2x - 3y - 4z = -3 \\ x + y + z = 0 \end{cases}$$

1. Write down the augmented matrix $[A \mid I]$.
2. Compute the reduced row echelon form of A.
3. Is A invertible? why?
4. Use Mathematica to compute A^{-1}.
5. Solve the system (by hand).
6. Verify your answer with Mathematica.