Computer Age Statistical Inference: Exercises

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Many of these exercises use data used in the book. These datasets can be found on the book webpage `https://web.stanford.edu/~hastie/CASI`.

Chapter 1 Exercises

1. (a) Fit a cubic regression, as a function of age, to the `kidney` data of Figures 1.1 and 1.2, calculating estimates and standard errors at ages 20, 30, 40, 50, 60, 70, 80.

   (b) How do the results compare with those in Table 1.1?

2. The `lowess` curve in Figure 1.2 has a flat spot between ages 25 and 35. Discuss how one might use bootstrap replications like those in Figure 1.3 to suggest whether the flat spot is genuine or just a statistical artifact.

3. Suppose that there were no differences between AML and ALL patients for any gene, so that t in (1.6) exactly followed a student-t distribution with 70 degrees of freedom in all 7128 cases. *About* how big might you expect the largest observed t value to be? Hint: $1/7128 = 0.00014$.

4. (a) Perform 1000 nonparametric bootstrap replications of $\overline{\text{ALL}}$ (1.5). You can use program `bcanon` from the CRAN library "bootstrap" or type in the little program *Algorithm 10.1* on page 178.

   (b) Do the same for $\overline{\text{AML}}$.

   (c) Plot histograms of the results, and suggest an inference.

Chapter 2 Exercises

1. A coin with probability of heads θ is independently flipped n times, after which θ is estimated by
$$\hat{\theta} = \frac{s+1}{n+2},$$
with s equal the number of heads observed.

   (a) What are the bias and variance of $\hat{\theta}$?

(b) How would you apply the plug-in principle to get a practical estimate of $\text{se}(\hat{\theta})$?

2. Supplement Table 2.1 with entries for trimmed means, trim proportions 0.1, 0.2, 0.3, 0.4.

3. Page 14 presents two definitions of frequentism, one in terms of probabilistic accuracy and one in terms of an infinite sequence of future trials. Give a heuristic argument relating the two.

4. Suppose that in (2.15) we plugged in $\hat{\sigma}$ to get an approximate 95% normal theory hypothesis test for $H_0 : \theta = 0$. How would it compare with the student-t hypothesis test?

5. Recompute the Neyman–Pearson alpha-beta curve in Figure 2.2, now with $n = 20$. In qualitative terms, how does it compare with the $n = 10$ curve?

Chapter 3 Exercises

1. Suppose the parameter μ in the Poisson density (3.3) is known to have prior density $e^{-\mu}$. What is the posterior density of μ given x?

2. In Figure 3.1, suppose the doctor had said "1/2, 1/2" instead of "1/3, 2/3". What would be the answer to the physicist's question?

3. Let X be binomial,

$$\Pr_\pi\{X = x\} = \binom{n}{x}\pi^x(1-\pi)^{n-x} \qquad \text{for } x = 0, 1, \ldots, n.$$

What is the Fisher information $\mathcal{I}_\pi$ (3.16)? How does $\mathcal{I}_\pi$ relate to the estimate $\hat{\pi} = x/n$?

4. (a) Run the following simulation 200 times:
    - $x_i \overset{\text{ind}}{\sim} \mathcal{N}(\mu_i, 1)$ for $i = 1, 2, \ldots, 500$
    - $\mu_i = 3i/500$
    - $i_{\max} =$ index of largest x_i
    - $d = x_{i_{\max}} - \mu_{i_{\max}}$

   (b) Plot the histogram of the 200 d values.

   (c) What is the relation to Figure 3.4?

5. Give a brief nontechnical explanation of why $x_{610} = 5.29$ was likely to be an overestimate of θ_{610} in Figure 3.4.

6. Given prior density $g(\mu)$ and observation $x \sim \text{Poi}(\mu)$, you compute $g(\mu \mid x)$, the posterior density of μ given x. Later you are told that x could only be observed if it were greater than 0. (Table 6.2 presents an example of this situation.) Does this change the posterior density of μ given x?

Chapter 4 Exercises

1. (a) Verify formula (4.10).
   (b) Why isn't the formula for $\hat{\sigma}$ the one generally used in practice?

2. Draw a schematic graph of $\dot{l}_x(\theta)$ versus θ. Use it to justify (4.25).

3. You observe $x_1 \sim \text{Bin}(20, \theta)$ and, independently, $x_2 \sim \text{Poi}(10 \cdot \theta)$. Numerically compute the Cramér–Rao lower bound (4.33). Hint: Fisher information adds for independent observations.

4. A coin with unknown probability of heads θ is flipped n_1 times, yielding x_1 heads; then it is flipped another x_1 times, yielding x_2 heads.
   (a) What is an intuitively plausible estimate of θ?
   (b) What Fisherian principle have you invoked?

5. Recreate a version of Figure 4.3 based on 1000 permutations.

6. A one-parameter family of densities $f_\theta(x)$ gives an observed value x. Statistician A computes the MLE $\hat{\theta}$. Statistician B uses a flat prior density $g(\theta) = 1$ to compute $\bar{\theta}$, the Bayes posterior expectation of θ given x. Describe the relationship between the two methods.

Chapter 5 Exercises

1. Suppose $X \sim \text{Poi}(\mu)$ where μ has a $\text{Gam}(\nu, 1)$ prior (as in Table 5.1).
   (a) What is the marginal density of X?
   (b) What is the conditional density of μ given $X = x$?

2. X is said to have an "F distribution with degrees of freedom ν_1 and ν_2", denoted $F_{\nu_1,\nu_2}(x)$, if
$$X \sim \frac{\nu_2}{\nu_1} \frac{\text{Gam}(\nu_1, \sigma)}{\text{Gam}(\nu_2, \sigma)},$$
the two gamma variates being independent. How does the F distribution relate to the beta distribution?

3. Draw a sample of 1000 bivariate normal vectors $x = (x_1, x_2)'$, with
$$x \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}\right).$$
   (a) Regress x_2 on x_1, and numerically check (5.18).
   (b) Do the same regressing x_1 on x_2.

Chapter 1 Exercises

1. (a) Fit a cubic regression, as a function of age, to the kidney data of Figures 1.1 and 1.2, calculating estimates and standard errors at ages 20, 30, 40, 50, 60, 70, 80.

See answer in attached Jupyter Notebook

(b) How do the results compare with those in Table 1.1?

See answer in attached Jupyter Notebook

2. The lowess curve in Figure 1.2 has a flat spot between ages 25 and 35. Discuss how one might use bootstrap replications like those in Figure 1.3 to suggest whether the flat spot is genuine or just a statistical artifact.

Ans: One way I can think of is to Bootstrap sample only from observations from subjects with ages 25 to 35. On each bootstrap dataset I would compute a linear regression using age and and intercept. Then I would collect the value of the coefficient for age on each bootstrap iteration. Finally, I would create a histogram with the values of the age coefficients across bootstrap samples and compute the confidence interval on the age coefficients using it. If the CI ranges from negative to positive values, then I would conclude the flat spot it legitimate.

3. Suppose that there were no differences between AML and ALL patients for any gene, so that t in (1.6) exactly followed a student-t distribution with 70 degrees of freedom in all 7128 cases. About how big might you expect the largest observed t value to be? Hint: $1/7128$ = 0.00014.

Ans: If the distribution of the difference in the means from AML and ALL patients was indeed t with 70 degrees of freedom, then by the hint, we´d expect the largest value to be larger than the other 7127 just by chance (the probability that a random value is the largest would be 1/7128). If we look for such a value in the t distribution with 70 degrees of freedom, we get 3.826.

4. (a) Perform 1000 nonparametric bootstrap replications of ALL (1.5). You can use program bcanon from the CRAN library "bootstrap" or type in the little program Algorithm 10.1 on page 178.

(b) Do the same for AML.

(c) Plot histograms of the results, and suggest an inference.

See answer in attached Jupyter Notebook

Homework 2 Solutions:

① a) · Bias calculation: First, compute $E(\hat{\theta})$

Since $\hat{\theta} = \frac{s+1}{n+2} \Rightarrow E(\hat{\theta}) = \frac{1}{n+2} \cdot E(S+1)$

$$= \frac{1}{n+2} \cdot [E(S)+1]$$

Expected # of successes in Binomial setting is np

$$= \frac{1}{n+2} \cdot (n\theta + 1)$$

$$= \frac{n\theta+1}{n+2}$$

Then, compute $\theta - E(\hat{\theta})$

$$\text{Bias} = \theta - \left[\frac{n\theta}{n+2} + \frac{1}{n+2}\right] = \theta\Big(\underbrace{\frac{n+2}{n+2}}_{=1} - \frac{n}{n+2}\Big) - \frac{1}{n+2}$$

$$= \frac{2\theta - 1}{n+2}$$

· Variance Calculation

$$V(\hat{\theta}) = V\left(\frac{s+1}{n+2}\right) = \frac{1}{(n+2)^2} \cdot V(S+1)$$

$$= \frac{V(S)}{(n+2)^2} = \frac{n \cdot \theta \cdot (1-\theta)}{(n+2)^2}$$

b) $se(\hat{\theta})$ is just $\sqrt{V(\hat{\theta})}$

Thus, $se(\hat{\theta}) = \frac{[n \cdot \theta \cdot (1-\theta)]^{1/2}}{n+2}$

- Since we don't know θ, we use our estimator $\theta = \frac{}{n+2}$ in its place (this is the plug-in principle) and get

$$se(\hat{\theta}) = \frac{\left[n \cdot \left(\frac{s+1}{n+2}\right) \cdot \left(1 - \frac{s+1}{n+2}\right)\right]^{1/2}}{n+2}$$

and then, of course, you can further simplify it.

② See Jupyter Notebook w/ sol.

③ Frequentism assigns the properties of the estimator (like its bias & variance) to the value of it we got from our very specific sample. The reason for doing this (and the link to the repeated sampling assumption), is that your very specific sample is a random one from the infinite possible samples.

④ The test would asymptotically converge to the t-student's test for larger $n_1 + n_2$, as sample increase would be accompanied by asymptotic convergence of $\hat{\sigma}$ to σ.

⑤ See Jupyter Notebook w/ sol.

Chapter 3 sols

(1) $f_\mu(x) = \dfrac{e^{-\mu} \cdot \mu^x}{x!}$

Will assume x is one observation, as vector notation not used explicitly in the problem.

$\pi(\mu) = e^{-\mu}$

$$P(\mu|x) \propto \pi(\mu) \cdot f_\mu(x)$$
$$\propto e^{-\mu} \cdot e^{-\mu} \cdot \mu^x$$
$$\propto e^{-2\mu} \cdot \mu^x$$

Compare this with the Gamma distribution

$y \sim Ga(\alpha, \beta)$

then $f(y) = \dfrac{\beta^\alpha}{\Gamma(\alpha)} y^{\alpha-1} \cdot e^{-\beta y}$

$$\propto y^{\alpha-1} \cdot e^{-\beta y}$$

Dropping the constants & just keeping the kernel

Thus, $y = \mu$, $\beta = 2$ and $\alpha - 1 = x$
$\Rightarrow \underline{\alpha = x+1}$

and $P(\mu|x) \propto e^{-2\mu} \cdot \mu^x \propto \underline{Ga(x+1, 2)}$

(3) Going step-by-step, let's first compute the score function.

(2) is

$$\dot{\ell}_{\underline{x}}(\pi) = \frac{\partial}{\partial \pi}\left\{ \log\left[\binom{n}{x}\right] + x \cdot \log(\pi) + (n-x) \cdot \log(1-\pi) \right\}$$

below.

$$= \frac{\partial}{\partial \pi} \cancel{\log \binom{n}{x}}^{\,0} + \frac{x}{\pi} - \frac{(n-x)}{1-\pi}$$

$$= \frac{x \cdot (1-\pi) - (n-x) \cdot \pi}{\pi(1-\pi)} = \frac{x - \cancel{x\pi} - n\pi + \cancel{x\pi}}{\pi(1-\pi)}$$

Now, using the definition of Fisher Information:

$$I_\pi = \sum_{x=0}^{n} \binom{n}{x} \cdot \pi^x \cdot (1-\pi)^{n-x} \cdot \left(\frac{x - n\pi}{\pi(1-\pi)} \right)^2$$

$$= \frac{1}{(\pi(1-\pi))^2} \cdot \sum_{x=0}^{n} \binom{n}{x} \cdot \pi^x \cdot (1-\pi)^{n-x} \cdot (x - n\pi)^2$$

$$= \frac{1}{(\pi(1-\pi))^2} \cdot \underbrace{\sum_{x=0}^{n} \binom{n}{x} \cdot \pi^x \cdot (1-\pi)^{n-x} \cdot (x - E(x))^2}$$

This is the definition of the variance for a Binomial(n, π) distr, which is known & equal to $n \cdot \pi \cdot (1-\pi)$

$$= \frac{n \cancel{\pi} \cancel{(1-\pi)}}{(\pi \cdot (1-\pi))^{\cancel{2}}} = \frac{n}{\pi(1-\pi)}.$$:)

(2) Sonogram

Twins are:

	SS	Diff	
I	1/2	0	1/2

Then

$$\frac{g(I \mid s)}{g(F \mid s)} = \frac{\cancel{1/2}}{\cancel{1/2}} \cdot \frac{1}{1/2} =$$

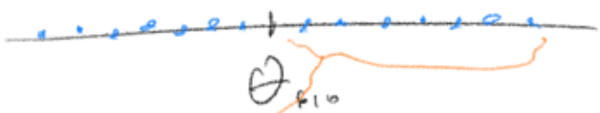
F	1/4	1/4	1/2

$= 2$

Identical twins would be twice as likely as fraternals after sonogram results.

(4) See Jupyter Notebook.

(5) The Effect of gene 610 measured on our sample, X_{610} would have been a different number in other samples from the same population. If we think about θ_{610}, the true effect size vs. $X^{(i)}_{610}$ for different samples, it would have looked like:



The fact that we chose X_{610} because it was large vs. the other gene effects, makes it more likely to be one of the larger measurements across samples, biasing the estimation of its effect.

(6) Posterior of μ given x doesn't change, as our belief of what the parameter μ is only depends on the data (x) through the likelihood.

Now, if we let y: "Information telling us only $x \geq 1$ can be observed", then posterior μ given x & y would change, as the new ... the likelihood func.

information on y would modify the [illegible]

Chapter 4 Sols + 5.1 Sols

① a) ~s

$$f_x(\theta,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\frac{(x-\theta)^2}{\sigma^2}}$$ (Density for 1 observation)

$$f_{\underline{x}}(\theta,\sigma) = (2\pi\sigma^2)^{-n/2} \cdot \exp\left\{\sum_{i=1}^{n} -\frac{1}{2}\frac{(x_i-\theta)^2}{\sigma^2}\right\}$$ (likelihood of sample)

$$l_{\underline{x}}(\theta,\sigma) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2}\cdot\sum_{i=1}^{n}\frac{(x_i-\theta)^2}{\sigma^2}$$ (log-likelihood of sample)

Optimality condition

$$\frac{\partial l_x}{\partial\theta} = \sum_{i=1}^{n}\frac{(x_i-\theta)}{\sigma^2} = 0 \Rightarrow \sum_{i=1}^{n} x_i - n\theta \overset{\checkmark}{=} 0$$

Derivative w.r.t. θ

$$\Rightarrow \boxed{\hat{\theta}^{mle} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X}}$$ (mean estimator)

Optimality condition

$$\frac{\partial l_x}{\partial\sigma} = -n\cdot\frac{\cancel{2\pi}\,\sigma}{\cancel{2\pi}\,\sigma^2} + \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{\sigma^3} \overset{\checkmark}{=} 0$$

Derivative w.r.t. σ

$$\Rightarrow \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{\sigma^3} = \frac{n}{\sigma}$$

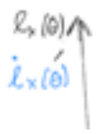
$$\sigma^2 = \frac{\sum_{i=1}^{n}(x_i-\theta)^2}{n}$$ (Estimator for variance.)

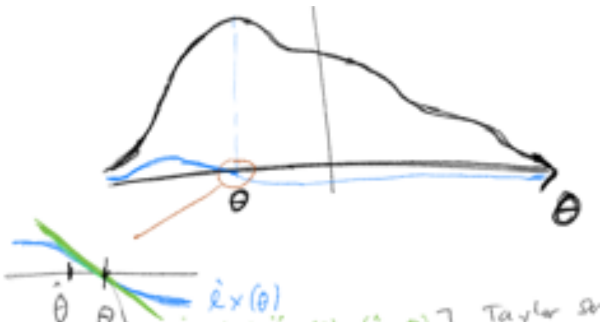
$$\boxed{\sigma = \left[\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}\right]^{1/2}}$$ (Using plug-in principle $\theta = \bar{x}$ and taking square root.)

b) ~s

Estimator for variance only takes into account current sample when maximizing likelihood. It ignores the fact that $\bar{x}$, our plug-in estimate for θ, results in loss of 1 degree of freedom for the residuals.

② Given a **log-likelihood** function $l_x(\theta)$, we observe that score function $\dot{l}_x(\theta)$ is zero at the value of θ **where** $l_x(\theta)$ is maximized.





$\dot{\ell}_X(\theta)$

$\dot{\ell}(\theta) + \ddot{\ell}_X(\theta)\cdot(\hat{\theta}-\theta)$] - Taylor series around true parameter θ

$$\Rightarrow 0 = \dot{\ell}(\theta) + \ddot{\ell}_X(\theta)\cdot(\hat{\theta}-\theta)$$

$$(\theta-\hat{\theta})\,\ddot{\ell}_X(\theta) = \dot{\ell}_x(\theta)$$

$$\hat{\theta} = \theta - \frac{\dot{\ell}_X(\theta)}{\ddot{\ell}_X(\theta)} = \theta - \frac{\dot{\ell}_X(\theta)/n}{\ddot{\ell}_X(\theta)/n}$$

③ Cramér-Rao Bound

$$V_\theta\{\tilde{\theta}\} \geq \frac{1}{I_\theta}$$

Total Fisher Information, if observations are iid then $= n\cdot I_\theta^{(1)}$ where $I_\theta^{(1)}$ is the Fisher Inf. of 1 observation.

$X_1 \sim Bin(20,\theta)$ sampled Indep.
$X_2 \sim Poi(10\cdot\theta)$

$$f(x_1) = \binom{n}{x_1}\cdot\theta^{x_1}\cdot(1-\theta)^{n-x_1}$$

$$\ell_{x_1}(\theta) = \log\left[\binom{n}{x_1}\right] + x_1\log(\theta) + (n-x_1)\log(1-\theta)$$

$$\dot{\ell}_{x_1}(\theta) = \frac{x_1}{\theta} + \frac{(n-x_1)}{1-\theta}$$

$$\ddot{\ell}_{x_1}(\theta) = -\frac{(n-x_1)}{(1-\theta)^2} - \frac{x_1}{\theta^2}$$

$$I_\theta = -E\left[\ddot{\ell}_{x_1}(\theta)\right] = \frac{(n-n\cdot\theta)}{(1-\theta)^2} + \frac{n\cancel{\theta}}{\theta^{\cancel{2}}}$$

$$= \frac{n\cancel{(1-\theta)}}{(1-\theta)^{\cancel{2}}} + \frac{n}{\theta} = \frac{\cancel{n\theta} + n - \cancel{n\theta}}{\theta(1-\theta)} = \frac{n}{\theta(1-\theta)}$$

$$= \boxed{\frac{20}{\theta(1-\theta)}}$$

Let $\lambda = 10 \cdot \theta$

$$f_{x_2}(\lambda) = \frac{e^{-\lambda} \cdot \lambda^{x_2}}{x_2!}$$

$$\ell_{x_2}(\lambda) = -\lambda + x_2 \log(\lambda) - \log(x_2!)$$

$$\dot{\ell}_{x_2}(\lambda) = -1 + \frac{x_2}{\lambda}$$

$$\ddot{\ell}_{x_2}(\lambda) = \frac{-x_2}{\lambda^2}$$

Then, taking the expectation & substituting θ back...

$$I_\theta = -E\left[\ddot{\ell}_{x_2}(\theta)\right] = -E\left[\frac{-x_2}{(10\theta)^2}\right] = \frac{\cancel{(10\cdot\theta)}}{(10\cdot\theta)^{\cancel{2}}} = \frac{1}{10\theta}$$

Finally, $$\boxed{\mathcal{I}_{\theta|x_1,x_2} = \frac{20}{\theta(1-\theta)} + \frac{1}{10\theta}}$$ Total Fisher Inf. of both observations.

Let's now compute numerically, as requested by problem. Since θ is the success probability of a Binomial distribution, we know $0 \le \theta \le 1$. Making a grid in increments of 0.1, we get

θ	$\mathcal{I}_{\theta\|x_1,x_2}$	$1/\mathcal{I}_\theta$
0.1	223	0.0044
0.2	125	0.0080
0.3	95	0.0104
0.4	84	0.0119
0.5	96	0.0125
0.6	84	0.0119
0.7	95	0.105
0.8	125	0.0080
0.9	222	0.0045

Information given is greater at the edges !!

Variance of any unbiased estimator $\ge 1/I_\theta$ for this sample.

(4)

a) $\frac{x_1 + x_2}{n_1 + n_2}$, the total amount of successes over the total amount of trials.

b) Conditional Inference, because we considered the outcomes of the individual samples, regardless of how the second sample was generated (i.e., ignoring the distribution of x_i).

(5) Just Change B from 10000 to 1000 in our Jupyter from Class.

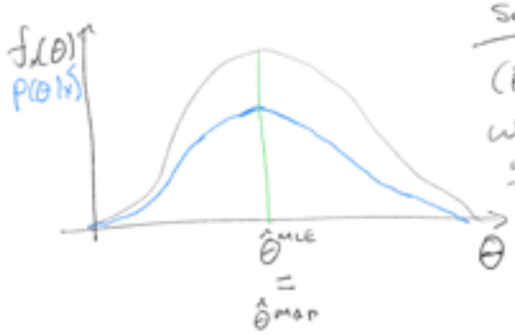
(6) Since the posterior computed by statistician B is proportional to the likelihood

$$P(\theta \mid x) \propto f_x(\theta) \cdot g(\theta)$$
$$\propto f_x(\theta)$$

which is the function statistician A is trying to maximize. Thus, under a flat prior, they have the same maximizer.

$(\hat{\theta}^{MLE} = \hat{\theta}^{map})$

where $\hat{\theta}$ is the maximum a posteriori estimate.



Chapter 5, (1).

$X \sim Poi(\mu)$, $\pi(\mu) \sim Ga(V, 1)$

a) Before getting the marginal density of X, let's get the joint density for x, μ.

$$f(x, \mu \mid V) = f_x(\mu) \cdot \pi(\mu) \quad \text{(using conditional probability)}$$

$$= \frac{\mu^x \cdot e^{-\mu}}{x!} \cdot \frac{\mu^{v-1} \cdot e^{-\mu/1}}{1^v \, \Gamma(v)}$$

$$\frac{\mu^{(v+x)-1} \cdot e^{-2\mu}}{\Gamma(x+1) \cdot \Gamma(v)}$$

Recall $\Gamma(y+1) = y!$

Now, get marginal density $f(x)$ by integrating over μ.

$$f(x|V) = \frac{1}{\;} \int^{\infty} \mu^{(v+x)-1} \, e^{-\mu/k(n)} \, d\mu$$

$$f(x|v) = \frac{(1/2)^{v+x}}{\Gamma(x-1)} \cdot \underbrace{\int_0^\infty \frac{\mu^{(v+x)-1} \cdot e^{-\mu(1/2)}}{(1/2)^{v+x} \cdot \Gamma(v)} d\mu}_{1}$$

Gamma density $Ga(v+x, 1/2)$

$$\boxed{\Rightarrow f(x|v) = \frac{(1/2)^{v+x}}{\Gamma(x-1)}}$$

b) Using Bayes Rule

$$f(\mu|x,v) = \frac{f(\mu, x|v)}{f(x|v)}$$

So just substitute them & simplify ☺.