Introduction to Markov chains

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Introduction

Markov chains are named after Andrei Markov (Russian mathematician). The idea is to deal with sequences of events based on the probabilities that dictate the motion of a population among different states.

1. A Markov chain is a stochastic model which models sequential data.
2. Composed of a transition scheme between states.
3. **Goals**: learn statistics, do prediction or estimation, recognize patterns.

Definition (Markov chain). A **Markov chain** is a series of discrete time intervals over which a population at a given time (say $t = n$) can be calculated based on the distribution at an earlier time $(n - 1)$ and the probabilities that model the population changes.

These probabilites are called **transition probabilities** and are assumed to be **constants**.

The next state depends only on the present state and not on the past history of the process.

Features of a Markov matrix

A matrix $T = [t_{ij}]_{,i,j=1\dots,n}$, called a **transition or Markov matrix**, describes the probabilistic motion of a population between various states. A Markov matrix has several **features**.

1. It is square matrix (all possible states must be used).
2. All entries lie in $[0,1]$ (they represent probabilities!).
3. The sum of all the elements of every column must be 1 (formally: $\sum_{i=1}^{n} t_{ij} = 1$ for each $j = 1, \dots, n$)

Example 1. The matrix

$$A = \begin{bmatrix} 1/2 & 1/3 \\ 1/2 & 2/3 \end{bmatrix}$$

is a Markov matrix.

Example 2. Is

$$B = \begin{bmatrix} 1/2 & 1/3 \\ 1/5 & 2/3 \end{bmatrix}$$

a Markov matrix?

The population proportion, in its various states, is given by a column vector

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

As you figured out in the opening activity, if we applied

$$T = \begin{bmatrix} .7 & .2 \\ .3 & .8 \end{bmatrix}$$

to a population vector such as

$$p = \begin{bmatrix} 120 \\ 80 \end{bmatrix}$$

we obtained the population distribution at the next month. More generally, we saw that

$T^n p$ gives the population distribution after nth months

More generally,

Suppose a Markov chain has initial probability vector

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

and transition matrix T. The probability vector after n repetitions of the experiment is $T^n p$.

Question. Is there any long-range prediction? More formally, given any probability vector p, does

$$\lim_{n \to \infty} P^n p$$

exist?

Yes. Such result is one of the main applications of Markov chains: find long-range predictions.

> **Theorem**. Let T be a Markov matrix, then for any probability vector p and for large enough n, $T^n p$ is constant. Formally, $\lim_{n\to\infty} T^n p$ exists.

The constant value of $\lim_{n\to\infty} T^n p$ is called the **equilibrium vector** or the **fixed vector** of the Markov chain.

In our example, the equilibrium vector associated to T was

$$\begin{bmatrix} 80 \\ 120 \end{bmatrix}$$

How can we find the fixed vector? It can be shown that if $\lim_{n \to \infty} T^n p$ exists, say V, then $TV = V$. Then

$$\begin{aligned} V &= TV \\ V - TV &= 0 \\ (I - T)V &= 0 \end{aligned}$$

Remark. The last equation means that we must take the matrix $I - T$ and solve the homogeneous system $(I - T)V = 0$. In other words, we apply **Gaussian-Jordan elimination** to the augmented matrix $[I - T|0]$.

An example

Example. Consider the example from the opening activity, in this case the Markov matrix is given by

$$T = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$$

Step 1 **Build the augmented matrix** $[I - T|0]$:

$$\left[\begin{array}{cc|c} 1-0.7 & 0-0.2 & 0 \\ 0-0.3 & 1-0.8 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 0.3 & -0.2 & 0 \\ -0.3 & 0.2 & 0 \end{array}\right]$$

Finally, row reduction yields

$$\left[\begin{array}{cc|c} 0.3 & -0.2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Step 2 **Solve the system.** The first equation gives $0.3x - 0.2y = 0$. In our case, $x + y = 200$, so $0.3x - 0.2(200 - x) = 0$; therefore $\boxed{(x, y) = (80, 120)}$