LAGRANGIAN AND EULERIAN DESCRIPTIONS

$$\left.\begin{aligned} P &= P(x,y,z,t) \\ \vec{V} &= \vec{V}(x,y,z,t) \\ \vec{a} &= \vec{a}(x,y,z,t) \end{aligned}\right\} \text{Field variables}$$

$$\vec{V} = (u,v,w) = u(x,y,z,t)\vec{i} + v(x,y,z,t)\vec{j} + w(x,y,z,t)\vec{k}$$

$$\vec{a} = (a_x,a_y,a_z) = a_x(x,y,z,t)\vec{i} + a_y(x,y,z,t)\vec{j} + a_z(x,y,z,t)\vec{k}$$

- Material position vector

$$(x_{particle}(t), y_{particle}(t), z_{particle}(t))$$

* Acceleration field

$$\vec{F}_{particle} = m_{particle}\vec{a}_{particle}$$

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt}$$

$$\vec{V}_{particle}(t) \equiv \vec{V}(x_{particle}(t), y_{particle}(t), z_{particle}(t), t)$$

$$\vec{a}_{particle} = \frac{d\vec{V}_{particle}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}(x_{particle}, y_{particle}, z_{particle}, t)}{dt}$$

$$= \frac{\partial \vec{V}}{\partial t}\frac{dt}{dt} + \frac{\partial \vec{V}}{\partial x_{particle}}\frac{dx_{particle}}{dt} + \frac{\partial \vec{V}}{\partial y_{particle}}\frac{dy_{particle}}{dt} + \frac{\partial \vec{V}}{\partial z_{particle}}\frac{dz_{particle}}{dt}$$

$$\frac{dx_{particle}}{dt} = u \quad , \quad \frac{dy_{particle}}{dt} = v \quad , \quad \frac{dz_{particle}}{dt} = w$$

$$(x_{particle}, y_{particle}, z_{particle}) = (x,y,z)$$

$$\vec{a}_{particle}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{V}\cdot\vec{\nabla} = (u\vec{i} + v\vec{j} + w\vec{k})\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$(\vec{V}\cdot\vec{\nabla})\vec{V} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

$$\therefore \vec{a}(x,y,z,t) = \frac{d\vec{V}}{dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{Local acceleration}} + \underbrace{(\vec{V}\cdot\vec{\nabla})\vec{V}}_{\text{Advective acceleration}}$$

MATERIAL DERIVATIVE

$$\frac{D}{Dt} = \frac{d}{dt} = \underbrace{\frac{\partial}{\partial t}}_{\text{local}} + \overbrace{(\vec{V} \cdot \vec{\nabla})}^{\text{Advective}}$$

- Material acceleration

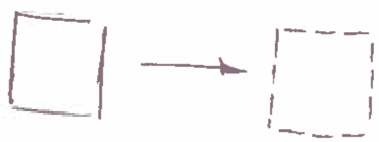
$$\vec{a}(x,y,z,t) = \frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \vec{\nabla})\vec{V}$$

- Material derivative of pressure

$$\frac{DP}{Dt} = \frac{dP}{dt} = \frac{\partial P}{\partial t} + (\vec{V} \cdot \vec{\nabla})P$$

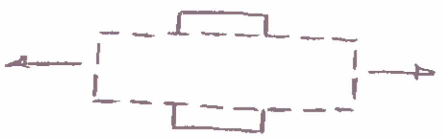
The material derivative D/Dt is defined by following a fluid particle as it moves throughout the flow field.

- RATE OF TRANSLATION VECTOR

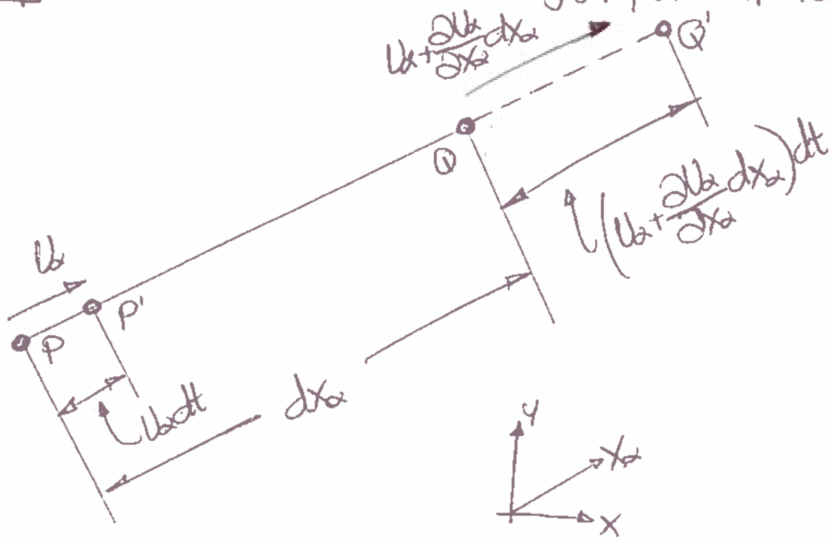


$$\vec{V} = u\vec{i} + v\vec{j} + w\vec{k}$$

- LINEAR STRAIN RATE



Rate of increase in length per unit length



$$\varepsilon_{\alpha\alpha} = \frac{1}{dt}\left(\frac{P'Q' - PQ}{PQ}\right)$$

$$PQ = dx_\alpha$$

$$P'Q' = \left(u_\alpha + \frac{\partial u_\alpha}{\partial x_\alpha}dx_\alpha\right)dt + dx_\alpha - u_\alpha dt$$

$$P'Q' - PQ = \left(u_\alpha + \frac{\partial u_\alpha}{\partial x_\alpha}dx_\alpha\right)dt + \cancel{dx_\alpha} - u_\alpha dt - \cancel{dx_\alpha}$$

$$= \cancel{u_\alpha dt} + \frac{\partial u_\alpha}{\partial x_\alpha}dx_\alpha dt - \cancel{u_\alpha dt}$$

$$= \frac{\partial u_\alpha}{\partial x_\alpha}dx_\alpha dt$$

$$\frac{P'Q' - PQ}{PQ} = \frac{\frac{\partial u_\alpha}{\partial x_\alpha}dx_\alpha dt}{dx_\alpha} = \frac{\partial u_\alpha}{\partial x_\alpha}dt$$

$$\varepsilon_{\alpha\alpha} = \frac{1}{dt}\left(\frac{\partial u_\alpha}{\partial x_\alpha}dt\right)$$

$$= \frac{\partial u_\alpha}{\partial x_\alpha}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}$$

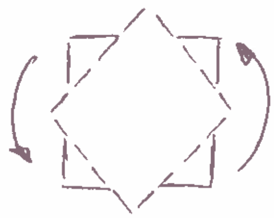
- VOLUMETRIC STRAIN RATE
(BULK STRAIN RATE)
(RATE OF VOLUMETRIC DILATATION)

Rate of increase of volume of a fluid element per unit volume

$$\frac{1}{V}\frac{DV}{Dt} = \frac{1}{V}\frac{dV}{dt} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

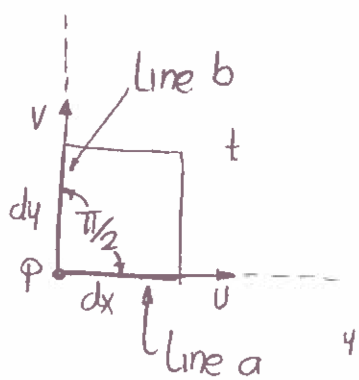
$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

• RATE OF ROTATION (ANGULAR VELOCITY)

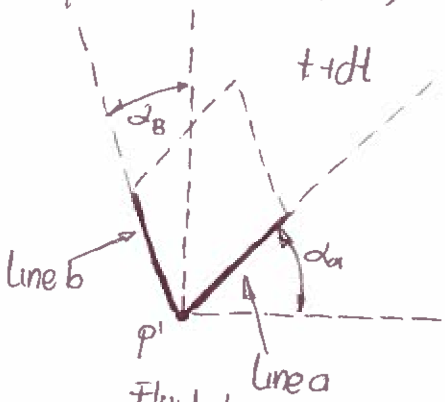


Average rotation rate of two initially perpendicular lines that intersect at that point

(counterclockwise is the mathematically positive direction)



Fluid element at time t_1



Fluid element at time t_2

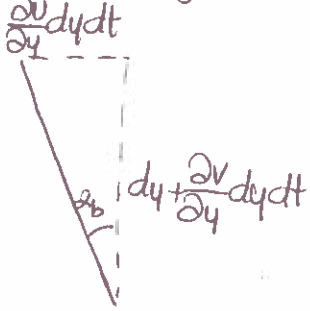
- $\alpha_a > \alpha_b$
- α_a (counterclockwise)
- α_b (counterclockwise)

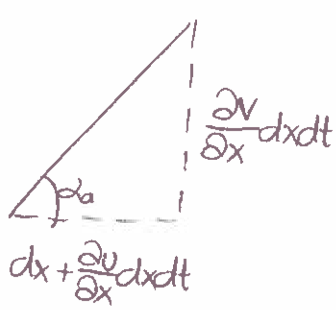
Average rotation angle

$$\frac{\alpha_a + \alpha_b}{2}$$

$$\therefore \omega = \frac{d}{dt}\left(\frac{\alpha_a + \alpha_b}{2}\right) = \frac{1}{2}\left(\frac{d\alpha_a}{dt} + \frac{d\alpha_b}{dt}\right)$$

* Considering the linear strain rate





$$\alpha_a \approx \tan^{-1}\frac{(\partial v/\partial x)dxdt}{dx + (\partial u/\partial x)dxdt}$$

$$\alpha_a = \lim_{dt\to 0}\left[\tan^{-1}\frac{(\partial v/\partial x)dxdt}{dx + (\partial u/\partial x)dxdt}\right] = \frac{\partial v}{\partial x}dt$$

$$\alpha_b \approx \tan^{-1}\frac{-(\partial u/\partial y)dydt}{dy + (\partial v/\partial y)dydt}$$

$$\alpha_b = \lim_{dt\to 0}\left[\tan^{-1}\frac{-(\partial u/\partial y)dydt}{dy + (\partial v/\partial y)dydt}\right] = -\frac{\partial u}{\partial y}dt$$

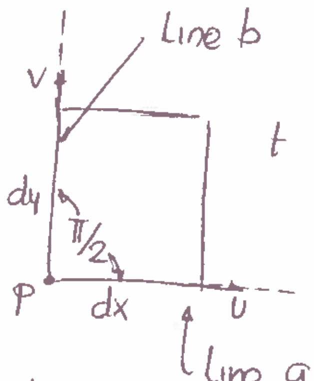
$$\therefore \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

in three dimensions

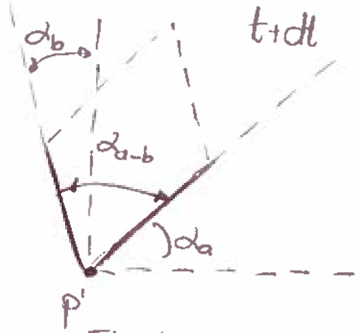
$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

- SHEAR STRAIN RATE

Half of the rate of decrease of the angle between two initially perpendicular lines that intersect at the point



Fluid element at time t_1



Fluid element at time t_2

- $\alpha_a > \alpha_b$
  $(\alpha_{a-b} < \pi/2)$

$$\varepsilon_{xy} = -\frac{1}{2}\frac{d}{dt}\alpha_{a-b}$$

$$-\frac{d}{dt}\alpha_{a-b} = \frac{d\alpha_a}{dt} - \frac{d\alpha_b}{dt}$$

$$\therefore \varepsilon_{xy} = \frac{1}{2}\left(\frac{d\alpha_a}{dt} - \frac{d\alpha_b}{dt}\right)$$

We know that

$$\alpha_a = \frac{\partial v}{\partial x}dt$$

and

$$\alpha_b = -\frac{\partial u}{\partial y}dt$$

$$\therefore \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

- STRAIN RATE TENSOR

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

• VORTICITY AND ROTATIONALITY

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = \text{curl}\,(\vec{V})$$

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\vec{V} = \vec{i}u + \vec{j}v + \vec{k}w$$

$$\therefore\ \vec{\zeta} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

And the RATE OF ROTATION VECTOR $\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\text{curl}(\vec{V}) = \frac{\vec{\zeta}}{2}$