Chapter 8 Sols

(1) Go to the Jupyter notebook for Lecture 17, On cell 10, substitute np.linspace(1, 11, 200) with np.linspace(1, 11, 11).

Results for $\hat{P}$Die | Group = [0.03, 0.06, 0.12, 0.22, 0.38, 0.57, 0.74, 0.86, 0.93, 0.97, 0.98].

If we want E(Die | Group), just multiply times 10, Thr. (observed counts in blue)

If we want $E(\text{Die} \mid \text{Group})$, just multiply ... the # of mice in each group. Thus, (observed counts in blue)

$$E(\text{Die} \mid \text{Group}) = \begin{array}{ccccccccccc} 0 & 0 & 0 & 3 & 6 & 6 & 5 & 9 & 9 & 10 & 10 \\ \hline 0.3, & 0.6, & 1.2, & 2.2, & 3.8, & 5.7, & 7.4, & 8.6, & 9.3, & 9.7, & 9.8 \end{array}$$

② $f(y \mid \pi) = \binom{n}{y} \pi^{y} (1-\pi)^{n-y}$

Let's get $\ell_y(\pi)$ in order to gain intuition about what λ should be.

$$\ell_y(\pi) = y \cdot \log(\pi) + (n-y)\log(1-\pi)$$

λ Show'o

$$\ell_y(\pi) = y\cdot\log(\pi) + (n-y)\log(1-\pi)$$

$$= y\cdot\left(\log(\pi) - \log(1-\pi)\right) + n\log(1-\pi)$$

$$= y\cdot\log\left(\frac{\pi}{1-\pi}\right) + \underbrace{n\log(1-\pi)}$$

Let $\underline{\lambda = \log\left(\frac{\pi}{1-\pi}\right)}$ and

$$e^{\lambda} = \frac{\pi}{1-\pi} = e^{\lambda} - e^{\lambda}\cdot\pi = \pi \Rightarrow \pi(1+e^{\lambda}) = e^{\lambda} \Rightarrow \pi = \frac{e^{\lambda}}{1+e^{\lambda}}$$

$$\Rightarrow n\cdot\log(1-\pi) = n\cdot\log\left(1 - \frac{e^{\lambda}}{1+e^{\lambda}}\right) = n\cdot\log\left(\frac{1}{1+e^{\lambda}}\right)$$

$$= n\cdot\left[\log(1) - \log(1+e^{\lambda})\right]$$

$1-\pi$ $\qquad\qquad\qquad\qquad$ $1+e$

$$\Rightarrow n\cdot\log(1-\pi) = n\cdot\log\left(1-\frac{e^{\lambda}}{1+e^{\lambda}}\right) = n\cdot\log\left(\frac{1}{1+e^{\lambda}}\right)$$

$$= n\cdot\left[\log(1)-\log(1+e^{\lambda})\right]$$

$$= -n\cdot\log(1+e^{\lambda})$$

Thus, $\ell_y(\lambda) = y\cdot\lambda - n\cdot\log(1+e^{\lambda})$. Finally, taking the exponential

$$f(y\mid\lambda) = e^{\lambda y - n\psi(\lambda)}\binom{n}{y},$$ where $\psi(\lambda) = \log(1+e^{\lambda})$

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3 See Jupyter Notebook.

4 Departing from the general definition of Deviance (formula 8.31) ...

$$D(f_1, f_2) = 2 \int_Y f_1(y) \cdot \log \frac{f_1(y)}{f_2(y)} \cdot dy$$

and letting $y \sim Poi(\mu)$

$$\Rightarrow D(f_1, f_2) = 2 \cdot \left[\sum^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{} \cdot \log \left(\frac{\mu_1^y \cdot e^{-\mu_1}}{\mu_2^y \cdot e^{-\mu_2}} \right) \right]$$

$$D(f_1, f_2) = 2 \int_y f_1(y) \cdot \log \frac{f_1(y)}{f_2(y)} \cdot dy$$

and letting $y \sim \text{Poi}(\mu)$

$$\Rightarrow D(f_1, f_2) = 2 \cdot \left[\sum_{y=0}^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{y!} \cdot \log\left(\frac{\mu_1^y \cdot e^{-\mu_1}}{\mu_2^y \cdot e^{-\mu_2}} \right) \right]$$

$$= 2 \cdot \left[\sum_{y=0}^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{y!} \cdot \log\left(\frac{\mu_1}{\mu_2}\right)^y + \log\left(\frac{e^{-\mu_1}}{e^{-\mu_2}}\right) \right]$$

$$= 2 \cdot \left[\sum_{y=0}^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{y!} \cdot \left[y \cdot \log\left(\frac{\mu_1}{\mu_2}\right) - (\mu_1 - \mu_2) \right] \right]$$

$$\left[\sum^{\infty} \mu_1^y \, e^{-\mu_1} \quad \log\left(\mu_1\right) \quad \sum^{\infty} \mu_1^y \cdot e^{-\mu_1} (\mu_1 - \mu_2) \right]$$

$$= 2 \cdot \left[\sum_{y=0}^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{y!} \cdot \left[y \cdot \log\left(\frac{\mu_1}{\mu_2}\right) - (\mu_1 - \mu_2) \right] \right]$$

$$= 2 \cdot \left[\underbrace{\sum_{y=0}^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{y!} \cdot y}_{\text{Expected value of } Poi(\mu_1) = \mu_1} \cdot \log\left(\frac{\mu_1}{\mu_2}\right) - \underbrace{\sum_{y=0}^{\infty} \frac{\mu_1^y \cdot e^{-\mu_1}}{y!}}_{\text{Sum of } Poi(\mu_1) \text{ pmf.}} (\mu_1 - \mu_2) \right]$$

$$= 2 \cdot \left[\mu_1 \cdot \log\left(\frac{\mu_1}{\mu_2}\right) - (\mu_1 - \mu_2) \right]$$

$$= 2\mu_1 \cdot \left[\log\left(\frac{\mu_1}{\mu_2}\right) - \left(1 - \frac{\mu_2}{\mu_1}\right) \right]$$

$$= 2 \cdot \left[\mu_1 \cdot \log\left(\frac{\mu_1}{\mu_2}\right) - (\mu_1 - \mu_2) \right]$$

$$= 2\mu_1 \cdot \left[\log\left(\frac{\mu_1}{\mu_2}\right) - \left(1 - \frac{\mu_2}{\mu_1}\right) \right]$$

$$= 2\mu_1 \left[\left(\frac{\mu_2}{\mu_1} - 1\right) - \log\left(\frac{\mu_2}{\mu_1}\right) \right]$$

5 (8.28) states $X^T [y - \mu(\hat{\alpha})] = 0$

$X^T y - X^T \mu(\hat{\alpha}) = 0 \Rightarrow z - E(\hat{z}) = 0$

$\Rightarrow$

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5 (8.28) states $X^T[y - \mu(\hat{\alpha})] = 0$

$\Rightarrow X^T y - X^T \mu(\hat{\alpha}) = 0 \Rightarrow z - E(\hat{z}) = 0$

Thus, the optimality condition 8.28 states that the likelihood maximizing choice of $\hat{\alpha}$ is that which produces estimate $\hat{z} = X^T \mu(\hat{\alpha})$ s.t. $E(\hat{z}) = z$.

i.e. $\hat{z}$ is unbiased!

6 See Jupyter Notebook.

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(6) See Jupyter Notebook.

(7)

$$S^2_k = \sum_{y \in k} (y_i - m_k)^2$$

$$S^2_k = \sum_{y \in k,l} (y_i - m_k)^2 + \sum_{y \in k,r} (y_i - m_k)^2$$

$$= \sum_{y \in k,l} \left((y_i - m_{kl}) + (m_{kl} - m_k)\right)^2 + \sum_{y \in k,r} \left((y_i - m_{kr}) + (m_{kr} - m_k)\right)^2$$

$$= \underbrace{\sum_{y \in k,l} (y_i - m_{kl})^2}_{S^2_{k,l}} + \boxed{2 \cdot \sum_{y \in k,l} (y_i - m_{kl})(m_{kl} - m_k)} + \sum_{y \in k,l} (m_{kl} - m_k)^2$$

$$+ \sum_{y \in k,r} (y_i - m_{kr})^2 + \boxed{2 \sum_{y \in k,r} (y_i - m_{kr})(m_{kr} - m_k)} + \sum_{y \in k,r} (m_{kr} - m_k)^2$$

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$$= \sum_{y \in k,l} \left((y_i - m_{kl}) + (m_{kl} - m_k) \right) + \sum_{y \in k,r} \left(\cdots \right)$$

$$= \underbrace{\sum_{y \in k,l} (y_i - m_{kl})^2}_{S^2_{k,l}} + 2 \cdot \sum_{y \in k,l} (y_i - m_{kl})(m_{kl} - m_k) + \sum_{y \in k,l} (m_{kl} - m_k)^2$$

$$+ \underbrace{\sum_{y \in k,r} (y_i - m_{kr})^2}_{S^2_{k,r}} + 2 \sum_{y \in k,r} (y_i - m_{kr})(m_{kr} - m_k) + \sum_{y \in k,r} (m_{kr} - m_k)^2$$

Note that $(m_{kl} - m_k)$ are constants in these two sums.

Since we know $\sum (y_i - \bar{y}) = 0$

$$\Rightarrow \quad (m_{kl} - m_k) \cdot \sum_{y \in k,l} (y_i - m_{kl}) = 0$$

$$(m_{kr} - m_k) \cdot \sum_{y \in k,r} (y_i - m_{kr}) = 0$$

$$\Rightarrow S^2_k = S^2_{k,l} + S^2_{k,r} + \sum_{y \in k,l} (m_{kl} - m_k)^2 + \sum_{y \in k,r} (m_{kr} - m_k)^2$$

$$S^2_k = S^2_{k,l} + \cdots + N_{kl} (m_{kl} - m_k)^2 + N_{kr} (m_{kr} - m_k)^2$$

$$\Rightarrow \quad (m_{k} \ldots \sum_{y \in k,l} \ldots$$

$$(m_{kr} - m_k) \cdot \sum_{x \in k,r} (y_i - m_{kr}) = 0$$

$$\Rightarrow S^2_{k} = S^2_{k,l} + S^2_{k,r} + \sum_{y \in k,l} (m_{kl} - m_k)^2 + \sum_{y \in k,r} (m_{kr} - m_k)^2$$

$$= S^2_{k,l} + S^2_{k,r} + N_{kl}(m_{kl} - m_k)^2 + N_{kr}(m_{kr} - m_k)^2$$

$$= S^2_{k,l} + S^2_{k,r} + N_{kl}(m_{kl} - m_k)^2 + N_{kr}(m_k - m_{kr})^2$$

$$= S^2_{k,l} + S^2_{k,r} + N_{kl} \cdot m_{kl}^2 - 2 \cdot N_{kl} \cdot m_k \cdot m_{kl} + N_{kl} \cdot m_k^2 + N_{kr} \cdot m_k^2 - 2 N_{kr}\, m_k \cdot m_{kr} + N_{kr} \cdot m_{kr}^2$$

$$= S^2_{k,l} + S^2_{k,r} + N_{kl} \cdot m^2_{kl} + N_{kr} \cdot m^2_{kr} - 2\, m_k (N_{kl} \cdot m_{kl} + N_{kr}\, m_{kr})$$

$$(\ldots) \cdot m_k^2$$

$$= s^2_{k,l} + s^2_{k,r} + N_{kl} \cdot m^2_{kl} + N_{kr} \cdot m^2_{kr} - 2\, m_k \left(N_{kl} \cdot m_{kl} + N_{kr}\, m_{kr}\right) + (N_{kl} + N_{kr}) \cdot m_k^2$$

$$= s^2_{k,l} + s^2_{k,r} + N_{kl} \cdot m^2_{k,l} + N_{kr} \cdot m^2_{kr} - 2\, \frac{(N_{kl} \cdot m_{kl} + N_{kr} \cdot m_{kr})^2}{N_k} + \overbrace{\cancel{(N_{kl} + N_{kr})}}^{= N_k} \cdot \frac{(N_{kl} \cdot m_{kl} + N_{kr} \cdot m_{kr})^2}{N_k^{\cancel{2}}}$$

$$= s^2_{k,l} + s^2_{k,r} + N_{k,l} \cdot m^2_{k,l} + N_{k,r} \cdot m^2_{k,r} - \frac{(N_{kl} \cdot m_{kl} + N_{kr} \cdot m_{kr})^2}{N_k}$$

$$= S^2_{k,l} + S^2_{k,r}$$

$$+ \frac{N_k \left(N_{k,l} \cdot m^2_{k,l} + N_{k,r} \cdot m^2_{k,r}\right) - \left(N_{kl} \cdot m_{kl} + N_{kr} \cdot m_{kr}\right)^2}{N_k}$$

$$= S^2_{k,l} + S^2_{k,r}$$

$$+ \frac{\left(N_{k,l} + N_{k,r}\right)\left(N_{k,l} \cdot m^2_{k,l} + N_{k,r} \cdot m^2_{k,r}\right) - \left(N_{kl} \cdot m_{kl} + N_{kr} \cdot m_{kr}\right)^2}{N_k}$$

$$= S^2_{k,l} + S^2_{k,r} +$$

$$\frac{1}{N_k} \cdot \Big[\cancel{N^2_{k,l}\, m^2_{k,l}} + N_{k,l} \cdot N_{k,r}\, m^2_{k,r} + N_{k,r} \cdot N_{k,l} \cdot m^2_{k,l} + \cancel{N^2_{kr}\, m^2_{k,r}}$$

$$- \cancel{N^2_{kl}\, m^2_{kl}} - 2 N_{kr} \cdot N_{kl} \cdot m_{kl} \cdot m_{kr} - \cancel{N^2_{kr}\, m^2_{kr}} \Big]$$

$$N_k$$

$$= s^2_{k,l} + s^2_{k,r} +$$

$$\frac{1}{N_k} \cdot \Big[\cancel{N^2_{k,l}\, m^2_{k,l}} + N_{k,l} \cdot N_{k,r}\, m^2_{k,r} + N_{k,r} \cdot N_{k,l} \cdot m^2_{k,l} + \cancel{N^2_{k,r}\, m^2_{k,r}}$$

$$- \cancel{N^2_{k,l}\, m^2_{k,l}} - 2 N_{k,r} \cdot N_{k,l} \cdot m_{k,l} \cdot m_{k,r} - \cancel{N^2_{k,r}\, m^2_{k,r}} \Big]$$

$$= s^2_{k,l} + s^2_{k,r} + \frac{N_{k,l} \cdot N_{k,r}}{N_k} \cdot \left(m^2_{k,l} - 2 m_{k,l} m_{k,r} + m^2_{k,r} \right)$$

$$\Rightarrow \quad s^2_k = s^2_{k,l} + s^2_{k,r} + \frac{N_{k,l} \cdot N_{k,r}}{N_k} \left(m_{k,l} - m_{k,r} \right)^2$$