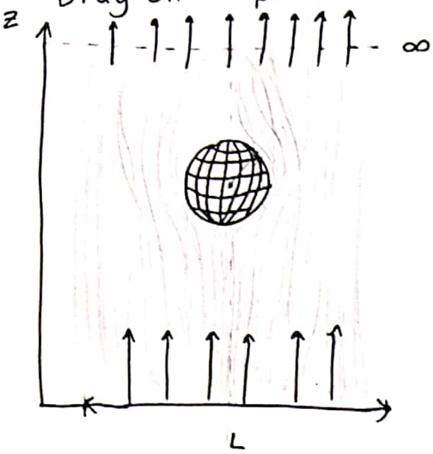
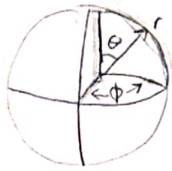
Drag on a sphere



Axisymmetric flow

Velocity $\vec{v} = (0, 0, W)$ where $W = V_\infty$ far away from the sphere.



radius $= a$

No slip boundary condition $\vec{v}(r=a) = 0$

Since it has symmetry for ϕ, $\vec{v}(r, \theta)$

Microchannels $\rightarrow$ Low Re

Dimensional Analysis

Drag(D)	MLT^{-2} (force)
μ	$ML^{-1}T^{-1}$
L	L
ρ	ML^{-3}
V	LT^{-1}

$n = 4$

$k = 3$

$n - k = 1$

Then $\Pi = f(\pi_1)$

$$\frac{D}{\rho V^2 L^2} = f(Re) \qquad \Pi = \frac{D}{\rho V^2 L^2}. \qquad \pi_1 = \frac{\rho L V}{\mu} = Re$$

Starting from the Navier stokes equation for incompressible, Newtonian fluid:

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla P + \mu \nabla^2 \vec{v} + f$$

Non dimensionalizing: $\tilde{v} = \frac{\vec{v}}{V_0}$ $\quad \tilde{P} = \frac{PL}{\mu V_0}$ $\quad \tilde{x} = \frac{x}{L}$ $\quad \tilde{t} = \frac{V_0 t}{L}$

$$\rho \frac{V_0^2}{L}\left((\tilde{v}\cdot\tilde{\nabla})\tilde{v}\right) = \mu \frac{V_0}{L^2} - \tilde{\nabla}\tilde{P} + \mu \frac{V_0}{L^2}\tilde{\nabla}^2 \tilde{v}$$

$$\rho \frac{V_0^2}{L}\left((\tilde{v}\cdot\tilde{\nabla})\tilde{v}\right) = \frac{\mu V_0}{L^2}\left(-\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2 \tilde{v}\right)$$

$$\frac{\rho V_0 L}{\mu}\left(\tilde{v}\cdot\tilde{\nabla}\right)\tilde{v} = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2 \tilde{v}$$

$$Re\left(\tilde{v}\cdot\tilde{\nabla}\right)\tilde{v} = -\tilde{\nabla}P + \tilde{\nabla}^2 \tilde{v} \qquad \text{Since } Re \rightarrow 0$$

$$\tilde{\nabla}P = \tilde{\nabla}^2 \tilde{v}$$

or

$$\boxed{\begin{aligned} \nabla P &= \mu \nabla^2 \vec{v} \\ \nabla \cdot V &= 0 \end{aligned}}$$

Stokes flow

Continuity equation

- Now we have a simplified expression for pressure gradient, thanks to the low Re number assumption

$$\mu \nabla^2 \vec{v} = \nabla P$$

- The continuity equation $\nabla \cdot \vec{v} = 0$ must also be satisfied. For an axisymmetric velocity field in spherical coordinates:

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (V_\theta \sin\theta)}{\partial \theta} + \cancel{\frac{1}{r\sin\theta}\frac{\partial (V_\varphi)}{\partial \varphi}}^{\,0}$$

Then the stream function is defined as

$$V_r = \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta} \qquad V_\theta = -\frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}$$

Such that

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(-\frac{1}{r\sin\theta}\sin\theta\frac{\partial \Psi}{\partial r}\right)$$

$$\nabla \cdot \vec{v} = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial r}\left(\frac{\partial \Psi}{\partial \theta}\right) - \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\partial \Psi}{\partial r}\right)$$

$$\boxed{\nabla \cdot \vec{v} = \frac{1}{r^2 \sin\theta}\frac{\partial^2 \Psi}{\partial r \partial \theta} - \frac{1}{r^2 \sin\theta}\frac{\partial^2 \Psi}{\partial r \partial \theta} = 0}$$

So indeed, the streamfunction satisfies that the divergence of velocity (continuity equation) is equal to zero

- Rewriting the simplified Navier Stokes equation in terms of the stream function:

$$\boxed{\begin{aligned} &\mu \nabla^2\left(\frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta}\hat{r} - \frac{1}{r\sin\theta}\frac{\partial \Psi}{\partial r}\hat{\theta}\right) = \nabla P \\ &\mu \nabla^2 \vec{v} = \nabla P \end{aligned}}$$

We can simplify the problem if we take the curl of the stokes flow equation

$$\nabla \times (\mu \nabla^2 \vec{V}) - \nabla \times (\nabla P) = 0$$

By definition $\nabla \times (\nabla f) = 0$ $\therefore$ $\nabla \times (\mu \nabla^2 \vec{V}) = 0$

Using the vector identity

By continuity eq. $\nabla \cdot V = 0$

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} \quad \therefore \quad \nabla \times (\nabla \times \vec{V}) = \cancel{\nabla (\nabla \cdot \vec{V})} - \nabla^2 \vec{V}$$

$$\nabla \times (\nabla \times \vec{V}) = -\nabla^2 \vec{V}$$

Then

$$\nabla \times (-\mu \, \nabla \times (\nabla \times \vec{V})) = 0$$

Since μ = cte and we know the vorticity is defined as $\nabla \times \vec{V} = \text{curl}(\vec{V}) = \vec{\zeta}$

$$-\mu \nabla \times (\nabla \times (\vec{\zeta})) = 0 \quad \therefore \quad \nabla \times \nabla \times \vec{\zeta} = 0$$

In spherical coordinates

$$\nabla \times \vec{V} = \vec{\zeta} = \frac{1}{r \sin\theta} \left(\frac{\partial}{\partial \theta} (V_\phi \sin\theta) - \frac{\partial V_\theta}{\partial \phi} \right) \hat{r}$$

$$+ \frac{1}{r} \left(\frac{1}{\sin\theta} \frac{\partial V_r}{\partial \phi} - \frac{\partial}{\partial r} (r V_\phi) \right) \hat{\theta}$$

$$+ \frac{1}{r} \left(\frac{\partial (r V_\theta}{\partial r} - \frac{\partial V_r}{\partial \theta} \right) \hat{\phi}$$

Dropping the V_ϕ and $\partial \phi$ components because of the axisymmetric condition.

$$\nabla \times \vec{V} = \frac{1}{r} \left(\frac{\partial (r V_\theta)}{\partial r} - \frac{\partial V_r}{\partial \theta} \right) \hat{\phi}$$

Substituting the stream functions

$$\nabla \times \vec{V} = \vec{\zeta} = \frac{1}{r} \left[\frac{\partial}{\partial r} \left(r \left(\frac{-1}{r \sin\theta} \frac{\partial \Psi}{\partial r} \right) \right) - \frac{\partial}{\partial \theta} \left(\frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} \right) \right] \hat{\phi}$$

$$\nabla \times \vec{V} = \frac{1}{r} \left[\frac{-1}{\sin\theta} \frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r^2} \left(\frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \Psi}{\partial \theta} \right) \right) \right] \hat{\phi}$$

$$\nabla \times \vec{V} = \frac{-1}{r \sin\theta} \left[\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \Psi}{\partial \theta} \right) \right] \hat{\phi} \quad \therefore \quad \nabla \times \vec{V} = \frac{-1}{r \sin\theta} E^2 \Psi \, \hat{\phi} = \vec{\zeta}$$

Where E^2 is defined as:

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

It is useful to note that

$$\vec{v} = \nabla\times\left(\frac{\psi\,\hat{\varphi}}{r\sin\theta}\right) = \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\,\hat{r} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\,\hat{\theta}\right)$$

Which can be verified by

$$\nabla\times\left(\frac{\psi\,\hat{\varphi}}{r\sin\theta}\right) = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\psi\,\cancel{\sin\theta}}{r\,\cancel{\sin\theta}}\right) - \cancel{\frac{\partial}{\partial\varphi}(0)}^{\,0}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\cancel{\frac{1}{\sin\theta}\frac{\partial 0}{\partial\varphi}}^{\,0} - \frac{\partial}{\partial r}\left(\frac{\cancel{r}\,\psi}{\cancel{r}\sin\theta}\right)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\cancel{\frac{\partial(r\,0)}{\partial r}}^{\,0} - \cancel{\frac{\partial 0}{\partial\theta}}^{\,0}\right]\hat{\varphi}$$

$$\nabla\times\left(\frac{\psi}{r\sin\theta}\,\hat{\varphi}\right) = \frac{1}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial\theta}\right)\hat{r} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\,\hat{\theta}$$

then

In spherical coordinates $\nabla\times\nabla\times\vec{v} = \nabla^2\vec{v}$

$\nabla\times\nabla\times\left(\frac{\psi\,\hat{\varphi}}{r\sin\theta}\right) = \frac{-1}{r\sin\theta}E^2\psi\,\hat{\varphi}$

$$\vec{\xi} = \nabla\times\vec{v} = \nabla\times\nabla\times\left(\frac{\psi\,\hat{\varphi}}{r\sin\theta}\right) = \frac{-1}{r\sin\theta}E^2\psi\,\hat{\varphi} \;\therefore$$

So going back to

$$\nabla\times\nabla\times\vec{\xi} = \nabla\times\nabla\times\left[\nabla\times\nabla\times\left(\frac{\psi\,\hat{\varphi}}{r\sin\theta}\right)\right] = 0$$

we can identify the recursion pattern, where $\nabla\times\nabla\times$ leads to E^2

$$\frac{-1}{r\sin\theta}E^2(E^2(\psi)) = 0 \;\therefore\; E^2(E^2(\psi)) = 0 \;\therefore\; E^4\psi = 0 \;\therefore\; (E^2)^2\psi = 0$$

$$\boxed{\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2\psi = 0}$$

Now that we have the differential equation for the Stokes stream function:

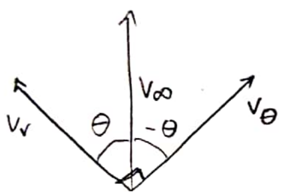
$$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0$$

We need to solve it with the correct boundary conditions.

1) Uniform flow speed V_∞ far away from the sphere

2) No slip boundary condition on the surface, such that $\vec{v}(r=a) = 0$

For boundary condition 1) we expect that the effect of the sphere is negligible far away from it. So the velocity field is uniform and constant as stated earlier $\vec{v} = (0, 0, V_\infty)$



We can define $V_r = \cos V_\infty$ and $V_\theta = -\sin\theta V_\infty$

when $r \to \infty$.

$$\lim_{r\to\infty} V_r = \cos V_\infty \qquad \lim_{r\to\infty} V_\theta = -\sin\theta V_\infty$$

Inserting the velocity in terms of the stream function.

$$V_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta} = \cos\theta V_\infty \quad \text{and} \quad \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\sin\theta V_\infty = V_\theta$$

Solving for ψ by separation of variables for V_r:

$$\partial\psi = r^2 V_\infty \sin\theta \cos\theta \, \partial\theta$$

function of r, since it was partial integration on $\partial\theta$

$$\psi = r^2 V_\infty \int u\,du \quad \text{where} \quad u = \sin\theta, \; du = \cos\theta \quad \therefore \quad \psi = r^2 V_\infty \frac{u^2}{2} + f(r)$$

$$\psi = r^2 V_\infty \frac{\sin^2\theta}{2} + f(r)$$

For V_θ we do the same

$$V_\theta = -\sin\theta V_\infty = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} \quad \therefore \quad \partial\psi = V_\infty \sin^2\theta \, r\,\partial r \quad \therefore \quad \psi = \sin^2\theta \frac{r^2}{2} V_\infty + f(\theta)$$

$$\psi = r^2 V_\infty \frac{\sin^2\theta}{2} + f(\theta)$$

So it is clear that $\dot{f}(\theta) = \dot{f}(r) = 0$. The Boundary condition then:

$$\lim_{r \to \infty} \Psi = \frac{1}{2} V_\infty r^2 \sin^2\theta$$

When approaching $r \to a$, we expect that only the r dependence component will change, as only r has been changed.

We formulate an ansatz based on this

$\Psi = F(r) \sin^2\theta$ where $F(r)$ is a function that depends solely on r.

We know that $E^2(E^2\Psi) = 0$

and that

$$E^2\Psi = \frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\left(\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right)\right)$$

We substitute the ansatz unto $E^2\Psi$

$$\frac{\partial^2\Psi}{\partial r^2} = \frac{\partial^2(F(r)\sin^2\theta)}{\partial r^2} = \sin^2\theta\, F''(r)$$

as constant since does not depend on θ

$$\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial(F(r)\sin^2\theta)}{\partial\theta}\right) = \frac{\sin\theta}{r^2}F(r)\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\sin^2\theta}{\partial\theta}\right)\right]$$

$$= \frac{\sin\theta}{r^2}F(r)\frac{\partial}{\partial\theta}\left(\frac{1}{\cancel{\sin\theta}}2\cancel{\sin\theta}\cos\theta\right)$$

$$= \frac{2\sin\theta}{r^2}F(r)\frac{\partial\cos\theta}{\partial\theta} = -\frac{2F(r)}{r^2}\sin^2\theta$$

Then

$$E^2\Psi = \sin^2\theta\left(F''(r) - \frac{2}{r^2}F(r)\right)$$

We can define a new function

$$g(r) = F''(r) - \frac{2}{r^2}F(r)$$

So that

$$E^2(E^2\Psi) = E^2\left(\sin^2\theta\left[F''(r) - \frac{2}{r^2}F(r)\right]\right) = E^2\left(\sin^2\theta\, g(r)\right) = \sin^2\theta\left(g''(r) - \frac{2}{r^2}g(r)\right) = 0$$

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Thus $g''(r) - \frac{2}{r^2}(g(r)) = 0$ because in general $\sin^2\theta \neq 0$

We can identify the form of the equation as a Cauchy Euler equation $\left[r^2 g'' - 2g = 0\right]$
a linear homogeneous ordinary differential equation with variable coefficients.

We assume a trial solution of the form $y = cx^m$ in this case $F = cr^m$

So $F' = mcr^{m-1}$

$F'' = m(m-1)cr^{m-2}$

Then $F'' - \frac{2}{r^2}F(r) = m(m-1)cr^{m-2} - \frac{2}{r^2}r^m = m(m-1)cr^{m-2} - 2r^{m-2}$

$cr^{m-2}(m(m-1)-2) = cr^{m-2}(m^2-m-2) = cr^{m-2}((m+1)(m-2))$

We know then that $g(r) = cr^{m-2}((m+1)(m-2))$

Since it has the same form as $F(r)$, we expect a similar solution. $g = cr^n$

$g(r) = (m+1)(m-2)cr^{m-2}$

$g'(r) = (m+1)(m-2)(m-2)cr^{m-3}$

$g''(r) = (m+1)(m-2)(m-2)(m-3)cr^{m-4}$

$(m+1)(m-2)(m-2)(m-3)cr^{m-4} - 2(m+1)(m-2)cr^{m-4}$

$g''(r) - \frac{2}{r^2}g(r) = 0$

$cr^{m-4}\left((m+1)(m-2)^2(m-3) - 2(m+1)(m-2)\right) = 0$

So

$(m+1)(m-2)^2(m-3) - 2(m+1)(m-2) = 0$

$(m+1)(m-2)\left[(m-2)(m-3) - 2\right] = 0$

$(m-2)(m-3) - 2 = 0 \therefore m^2 - 5m + 6 - 2 = 0 \therefore m^2 - 5m + 4 = 0 \therefore (m-4)(m-1) = 0$

then the roots are:

$m = 1, 4, 2, -1$ because of the fact $(m+1)(m-2)(m-4)(m-1) = 0$

Finally $F(r) = Ar + Br^4 + Dr^2 + \frac{C}{r}$

So $\Psi = \sin^2\theta \left[Ar + Br^4 + \frac{C}{r} + Dr^2 \right]$

we know that $\lim_{r \to \infty} \Psi = \frac{r^2}{2} V_\infty \sin^2\theta$

Then $\sin^2\theta \left[A(r) + \cancel{B(r^4)}^{\,0} + \frac{C}{r} + D(r^2) \right]$ Since there is no order 4 as $r \to \infty$

this implies $B = 0$ and D must be equal to $\frac{V_\infty}{2}$ $\therefore D = \frac{V_\infty}{2}$

Now for the no-slip boundary condition

$V_r(r=a) = 0$ and $V_\theta(r=a) = 0$

Then $V_r = \frac{1}{r^2 \sin\theta} \frac{\partial \Psi}{\partial \theta} = 0$ $\therefore V_r = \frac{1}{r^2 \cancel{\sin\theta}} 2 \cancel{\sin\theta} \cos\theta \left[Ar + \frac{C}{r} + \frac{V_\infty}{2} r^2 \right]$

when $r = a$, $V_r = 0$

$V_r = 2\cos\theta \left[\frac{A}{a} + \frac{C}{a^3} + \frac{V_\infty}{2} \right] = 0$ $\therefore$ $\frac{A}{a} + \frac{C}{a^3} + \frac{V_\infty}{2} = 0$

Now for V_θ

$V_\theta = \frac{-1}{r\sin\theta} \frac{\partial \Psi}{\partial r} = 0$ $\therefore$ $V_\theta = \frac{-1}{r \cancel{\sin\theta}} \sin^{\cancel{2}}\theta \left[A + \frac{\cancel{2} V_\infty r}{\cancel{2}} - \frac{C}{r^2} \right]$

$V_\theta = -\sin\theta \left[\frac{A}{r} - \frac{C}{r^3} + V_\infty \right]$ is equal to 0 when $r = a$

$\left[\frac{A}{a} - \frac{C}{a^3} + V_\infty \right] = 0$

Now we have two equations with two incognitas, we equal both equations

$\cancel{\frac{A}{a}} - \frac{C}{a^3} + V_\infty = \cancel{\frac{A}{a}} + \frac{C}{a^3} + \frac{V_\infty}{2}$ $\therefore$ $\frac{V_\infty}{2} = \frac{2C}{a^3}$ $\therefore$ $C = \frac{V_\infty a^3}{4}$

Then $\frac{A}{a} - \frac{V_\infty a^3}{4\, a^3} + V_\infty = 0$ $\therefore$ $A = \frac{-3}{4} V_\infty a$

Then, we can define Ψ as:

$$\Psi = \sin^2\theta\left[-\frac{3}{4}V_\infty a r + \frac{V_\infty a^3}{4r} + \frac{V_\infty r^2}{2}\right]$$

$$\boxed{\Psi = \frac{V_\infty}{2}\left[r^2 + \frac{a^3}{2r} - \frac{3ar}{2}\right]\sin^2\theta}$$

○ Equation for Stokes Stream function

Then we can substitute to get the velocity vectors V_θ and V_r

$$V_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} = \frac{V_\infty}{\cancel{2}}\frac{1}{r^2}\left[r^2 + \frac{a^3}{2r} - 3ar\right]\frac{1}{\cancel{\sin\theta}}\cancel{2}\cancel{\sin\theta}\cos\theta$$

$$V_r = V_\infty\cos\theta\left[1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right]$$

$$V_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} = -\frac{1}{r\cancel{\sin\theta}}\sin^{\cancel{2}}\theta\,\frac{V_\infty}{2}\left[2r - \frac{a^3}{2r^2} - \frac{3a}{2}\right]$$

$$V_\theta = -V_\infty\sin\theta\left[1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right]$$

○ The velocity field is then:

$$\boxed{\vec{V} = V_\infty\cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\hat{r} - V_\infty\sin\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\hat{\theta}}$$

○ The derivation of the pressure gradient is straight forward now that we know the velocity field.

The pressure gradient (from stokes flow) is given by

$$\nabla P = \mu\nabla^2\vec{V}$$

$$\frac{\partial P}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial P}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial P}{\partial\varphi}\hat{\varphi} = \mu\nabla^2\vec{V}$$

where $\nabla^2\vec{V}$ is defined as:

$$\nabla^2\vec{V} = \left(\nabla^2 V_r - \frac{2V_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(V_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}\right)\hat{r}$$

$$+\left(\nabla^2 V_\theta - \frac{V_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\varphi}{\partial\varphi}\right)\hat{\theta}$$

$$+\left(\nabla^2 V_\varphi - \frac{V_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial V_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\theta}{\partial\varphi}\right)\hat{\varphi}$$

Because of Axisymmetric condition all $\hat{\varphi}$ terms are eliminated.

Starting with the $\hat{r}$ term:

$$\frac{\partial P}{\partial r} = \mu\left[\nabla^2 V_r - \frac{2V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} - \frac{2V_\theta \cot\theta}{r^2} - \cancelto{0}{\frac{2}{r^2\sin\theta}\frac{\partial V_\varphi}{\partial \varphi}}\right]$$

Note:

$$\frac{2}{r^2\sin\theta}\frac{\partial (V_\theta \sin\theta)}{\partial \theta} = \frac{2}{r^2\sin\theta}\left(\frac{\partial V_\theta}{\partial \theta}\sin\theta + V_\theta\cos\theta\right) = \frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta} + \frac{2V_\theta\cot\theta}{r^2}$$

The Laplace operator $\nabla^2 f$ is defined as: (in spherical coordinates)

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Then

$$\nabla^2 V_r = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V_r}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V_r}{\partial\theta}\right) + \cancelto{0 \text{ axisymmetric}}{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V_r}{\partial\varphi^2}}$$

$$\nabla^2 V_r = \frac{1}{r^2}\left(\frac{\partial V_r}{\partial r}(2r) + r^2\frac{\partial^2 V_r}{\partial r^2}\right) + \frac{1}{r^2\sin\theta}\left(\cos\theta\frac{\partial V_r}{\partial\theta} + \frac{\partial^2 V_r}{\partial\theta^2}\sin\theta\right)$$

$$\nabla^2 V_r = \frac{2}{r}\frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial V_r}{\partial\theta}\cot\theta + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2}$$

Then,

$$\frac{1}{\mu}\frac{\partial P}{\partial r} = \frac{2}{r}\frac{\partial V_r}{\partial r} + \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r^2}\left(\frac{\partial V_r}{\partial\theta}\right)\cot\theta + \frac{1}{r^2}\frac{\partial^2 V_r}{\partial\theta^2} - \frac{2V_r}{r^2} - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{2V_\theta\cot\theta}{r^2}$$

Now for $\hat{\theta}$,

$$\frac{1}{\mu r}\frac{\partial P}{\partial\theta} = \nabla^2 V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r^2\sin^2\theta} - \cancelto{0}{\frac{2\cot}{r^2\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}}$$

we already know the form of $\nabla^2 V_\theta$, then

$$\frac{1}{\mu r}\frac{\partial P}{\partial\theta} = \frac{2}{r}\frac{\partial V_\theta}{\partial r} + \frac{\partial^2 V_\theta}{\partial r^2} + \frac{1}{r^2}\cot\theta\frac{\partial V_\theta}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{V_\theta}{r^2\sin^2\theta}$$

we need to find the expressions of $\frac{\partial V_r}{\partial r}$, $\frac{\partial^2 V_r}{\partial r^2}$, $\frac{\partial V_\theta}{\partial\theta}$, $\frac{\partial^2 V_\theta}{\partial\theta^2}$, $\frac{\partial V_r}{\partial\theta}$, $\frac{\partial^2 V_r}{\partial\theta^2}$, $\frac{\partial V_\theta}{\partial r}$, $\frac{\partial^2 V_\theta}{\partial r^2}$.

$$V_r = V_\infty \cos\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right] \qquad V_\theta = -V_\infty \sin\theta\left[1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right]$$

$$\frac{\partial V_r}{\partial r} = V_\infty \cos\theta\left[0 + \frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right] = V_\infty \cos\theta\left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right] \qquad \frac{\partial V_\theta}{\partial r} = -V_\infty \sin\theta\left[0 + \frac{3a^3}{4r^4} + \frac{3a}{4r^2}\right] = -V_\infty \sin\theta\left[\frac{3a}{4r^2} + \frac{3a^3}{4r^4}\right]$$

$$\frac{\partial^2 V_r}{\partial r^2} = \frac{\partial}{\partial r}\left[V_\infty \cos\theta\left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right]\right] = V_\infty \cos\theta\left[-\frac{3a}{r^3} + \frac{6a^3}{r^5}\right] \qquad \frac{\partial^2 V_\theta}{\partial r^2} = -V_\infty \sin\theta\left[-\frac{6a}{4r^3} - \frac{12a^3}{4r^5}\right] = -V_\infty \sin\theta\left[-\frac{3a}{2r^3} - \frac{3a^3}{r^5}\right]$$

$$\frac{\partial V_r}{\partial \theta} = -V_\infty \sin\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right] \qquad \frac{\partial V_\theta}{\partial \theta} = -V_\infty \cos\theta\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]$$

$$\frac{\partial^2 V_r}{\partial \theta^2} = -V_\infty \cos\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right] \qquad \frac{\partial^2 V_\theta}{\partial \theta^2} = V_\infty \sin\theta\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right]$$

Substituting,

$$\frac{1}{\mu}\frac{\partial P}{\partial r} = \frac{2}{r}\left[V_\infty \cos\theta\left(\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right)\right] + V_\infty \cos\theta\left(-\frac{3a}{r^3} + \frac{6a^3}{r^5}\right) + \frac{1}{r^2}\left[-V_\infty \sin\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\frac{\cos\theta}{\sin\theta}\right]$$

$$+ \frac{1}{r^2}\left[-V_\infty \cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\right] - \frac{2}{r^2}\left[V_\infty \cos\theta\left(1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right)\right]$$

$$- \frac{2}{r^2}\left[-V_\infty \cos\theta\left(1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right)\right] - \frac{2}{r^2}\left[-V_\infty \cancel{\sin\theta}\,\frac{\cos\theta}{\cancel{\sin\theta}}\left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right)\right]$$

Expanding, and introducing $\alpha = V_\infty \cos\theta$

$$\alpha\left(\frac{3a}{r^3}\right) - \alpha\left(\frac{3a^3}{r^5}\right) - \alpha\left(\frac{3a}{r^3}\right) + \alpha\left(\frac{6a^3}{r^5}\right)$$

$$-\alpha\left(\frac{1}{r^2}\right) + \alpha\left(\frac{3a}{2r^3}\right) - \alpha\left(\frac{a^3}{2r^5}\right) - \alpha\left(\frac{1}{r^2}\right) + \alpha\left(\frac{3a}{2r^3}\right) - \alpha\left(\frac{a^3}{2r^5}\right)$$

$$-\alpha\left(\frac{2}{r^2}\right) + \alpha\left(\frac{6a}{2r^3}\right) - \alpha\left(\frac{a^3}{r^5}\right) + \alpha\left(\frac{2}{r^2}\right) - \alpha\left(\frac{6a}{4r^3}\right) - \alpha\left(\frac{a^3}{2r^5}\right)$$

$$+\alpha\left(\frac{2}{r^2}\right) - \alpha\left(\frac{6a}{4r^3}\right) - \alpha\left(\frac{a^3}{2r^5}\right)$$

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Grouping all terms respectively

$$\frac{\alpha}{r^2}\left(-2-2+2+2\right) = \frac{\alpha}{r^2}(0)$$

$$\frac{\alpha a}{r^3}\left(3-3+\frac{3(2)}{2}+3-\frac{6}{4}\times 2\right) = \frac{3\alpha a}{r^3}$$

$$\frac{\alpha a^3}{r^5}\left(-3+6-\frac{1}{2}-\frac{1}{2}-1-\frac{1}{2}-\frac{1}{2}\right) = \frac{\alpha a^3}{r^5}(0)$$

Then

$$\frac{1}{\mu}\frac{\partial P}{\partial r} = \frac{\alpha\, 3a}{r^3} = V_\infty \cos\theta\left(\frac{3a}{r^3}\right)$$

By separation of variables

$$\frac{1}{\mu}\int \partial P = V_\infty \cos\theta\, 3a \int \frac{\partial r}{r^3}$$

$$\frac{P}{\mu} = -V_\infty \cos\theta\left(\frac{3a}{2r^2}\right) + P_1(\theta)$$

$$P = -V_\infty \cos\theta\left(\frac{3a\mu}{2r^2}\right) + P_1(\theta)$$

Now we do the same for $\frac{\partial P}{\partial \theta}$, we introduce the variable $\beta = V_\infty \sin\theta$.

$$\frac{1}{\mu r}\frac{\partial P}{\partial \theta} = \frac{2}{r}\left(-\beta\left(\frac{3a}{4r^2}+\frac{3a^3}{4r^4}\right)\right) - \beta\left(\frac{-3a}{2r^3}-\frac{3a^3}{r^5}\right) + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\left(-V_\infty \cos\theta\right)\left(\frac{1}{1}-\frac{3a}{4r}-\frac{a^3}{4r^3}\right)$$

$$+\frac{1}{r^2}\beta\left(1-\frac{3}{4}\frac{a}{r}-\frac{a^3}{4r^3}\right) - \frac{2}{r^2}\beta\left(1-\frac{3}{2}\frac{a}{r}+\frac{a^3}{2r^3}\right) + \frac{V_\infty}{r^2}\frac{\sin\theta}{\sin^2\theta}\left(1-\frac{a^3}{4r^3}-\frac{3}{4}\frac{a}{r}\right)$$

So we work with the terms that do not include β

$$-\frac{V_\infty}{r^2}\left(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\right)\frac{\cos\theta\cos\theta}{\sin\theta} + \frac{V_\infty}{r^2}\frac{\sin\theta}{\sin^2\theta}\left(1-\frac{a^3}{4r^3}-\frac{3}{4}\frac{a}{r}\right)$$

we know $\cos^2\theta = 1-\sin^2\theta$

$$\frac{-V_\infty}{r^2}\left(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\right)\frac{1-\sin^2\theta}{\sin\theta}+\frac{V_\infty}{r^2}\left(1-\frac{a^3}{4r^3}-\frac{3}{4}\frac{a}{r}\right)\frac{1}{\sin\theta}$$

$$\cancel{\frac{-V_\infty}{r^2}\left(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\right)\frac{1}{\sin\theta}}+\frac{V_\infty}{r^2}\frac{\sin^{\cancel{2}}\theta}{\cancel{\sin\theta}}\left(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\right)+\cancel{\frac{V_\infty}{r^2}\left(1-\frac{a^3}{4r^3}-\frac{3a}{4r}\right)\frac{1}{\sin\theta}}$$

$$\frac{V_\infty\sin\theta}{r^2}\left(1-\frac{3a}{4r}-\frac{a^3}{4r^3}\right)=\beta\left(\frac{1}{r^2}-\frac{3a}{4r^3}-\frac{a^3}{4r^5}\right)$$

Now expanding

$$-\beta\left(\frac{3a}{2r^3}\right)-\beta\left(\frac{3}{2}\frac{a^3}{r^5}\right)+\beta\left(\frac{3a}{2r^3}\right)+\beta\left(\frac{3a^3}{r^5}\right)+\beta\left(\frac{1}{r^2}\right)-\beta\left(\frac{3a}{4r^3}\right)-\beta\left(\frac{a^3}{4r^5}\right)$$

$$+\beta\left(\frac{1}{r^2}\right)+\beta\left(\frac{-3}{4}\frac{a}{r^3}\right)-\beta\left(\frac{a^3}{4r^5}\right)-\beta\left(\frac{2}{r^2}\right)+\beta\left(\frac{3a}{r^3}\right)-\beta\left(\frac{2a^3}{2r^5}\right)$$

Again, grouping by power

$$\frac{\beta a}{r^3}\left(-\frac{3}{2}+\frac{3}{2}-\frac{3}{4}-\frac{3}{4}+3\right)=\frac{3}{2}\beta\frac{a}{r^3}$$

$$\frac{\beta a^3}{r^5}\left(-\frac{3}{2}+3-\frac{1}{4}-\frac{1}{4}-1\right)=\cancel{\frac{\beta a^3}{r^5}(0)}$$

$$\frac{\beta}{r^2}\left(\frac{1}{1}+1-2\right)=\cancel{\frac{\beta}{r^2}(0)}$$

Then

$$\frac{1}{\mu r}\frac{\partial P}{\partial\theta}=\frac{3}{2}\beta\frac{a}{r^3}\quad\therefore\quad\frac{1}{\mu}\partial P=\frac{3}{2}\beta\frac{a}{r^2}\partial\theta\quad\therefore\quad\frac{1}{\mu}\int\partial P=\frac{3}{2}V_\infty\frac{a}{r^2}\int\sin\theta\,\partial\theta$$

$$P=-\frac{3}{2}V_\infty\frac{a\mu}{r^2}\cos\theta+P_1(r)$$

Since both integrals present the same form

$$P = -V_\infty \cos\theta \left(\frac{3a\mu}{2r^2}\right) + P_1(\theta)$$

$$P = -V_\infty \cos\theta \left(\frac{3a\mu}{2r^2}\right) + P_2(r)$$

Then we can infer

$$P = -V_\infty \cos\theta \left(\frac{3a\mu}{2r^2}\right) + C$$

as $r \to \infty$, $P \to P_\infty$ then $C = P_\infty$

$$\boxed{P = P_\infty - V_\infty \cos\theta \left(\frac{3a\mu}{2r^2}\right)}$$

We know that for spherical coordinates the stress tensors are given by:

$$\sigma_r = \tau_{rr} = -p + 2\mu \frac{\partial V_r}{\partial r}$$

$$\sigma_\theta = \tau_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{\mu}{r}\frac{\partial V_r}{\partial \theta}$$

$$\sigma_\varphi = \tau_{r\varphi} = 0$$

So for σ_r

$$\tau_{rr} = -\left[P_\infty - V_\infty \cos\theta \left(\frac{3a\mu}{2r^2}\right)\right] + 2\mu V_\infty \cos\theta \left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right]$$

$$\tau_{rr} = V_\infty \cos\theta \left[\frac{3a\mu}{2r^2} + \frac{2\mu 3a}{2r^2} - \frac{2\,3a^3}{2r^4}\right] + P_\infty$$

This must be evaluated at $r = a$

$$\tau_{rr} = V_\infty \cos\theta \left[\frac{3\mu}{2a} + \frac{3\mu}{a} - \frac{3\cdot}{a}\right] + P_\infty$$

$$\boxed{\sigma_r = \tau_{rr} = V_\infty \cos\theta \left[\frac{3\mu}{2a}\right] + P_\infty}$$

for σ_θ

$$\sigma_\theta = \tau_{r\theta} = \mu r \frac{\partial}{\partial r}\left(\frac{V_\theta}{r}\right) + \frac{\mu}{r}\frac{\partial V_r}{\partial \theta}$$

$$= \mu r \frac{\partial}{\partial r}\left(-V_\infty \sin\theta\left(\frac{1}{r} - \frac{a^3}{4r^4} - \frac{3a}{4r^2}\right)\right) + \frac{\mu}{r}\left(-V_\infty \sin\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right]\right)$$

$$= \mu r\left[-V_\infty \sin\theta\left(-\frac{1}{r^2} + \frac{4a^3}{4r^5} + \frac{2\cdot 3a}{4r^3}\right)\right] + \left(-V_\infty \mu \sin\theta\left[\frac{1}{r} - \frac{3a}{2r^2} + \frac{a^3}{2r^4}\right]\right)$$

$$= -V_\infty \mu \sin\theta\left[-\frac{1}{r} + \frac{a^3}{r^4} + \frac{3a}{2r^2}\right] - V_\infty \mu \sin\theta\left[\frac{1}{r} - \frac{3a}{2r^2} + \frac{a^3}{2r^4}\right]$$

Evaluated at $r = a$

$$= -V_\infty \mu \sin\theta\left[\cancel{-\frac{1}{a}} + \frac{1}{a} + \cancel{\frac{3}{2a}} + \cancel{\frac{1}{a}} - \cancel{\frac{3}{2a}} + \frac{1}{2a}\right]$$

$$\boxed{\sigma_\theta = -V_\infty \sin\theta\left[\frac{3\mu}{2a}\right]}$$

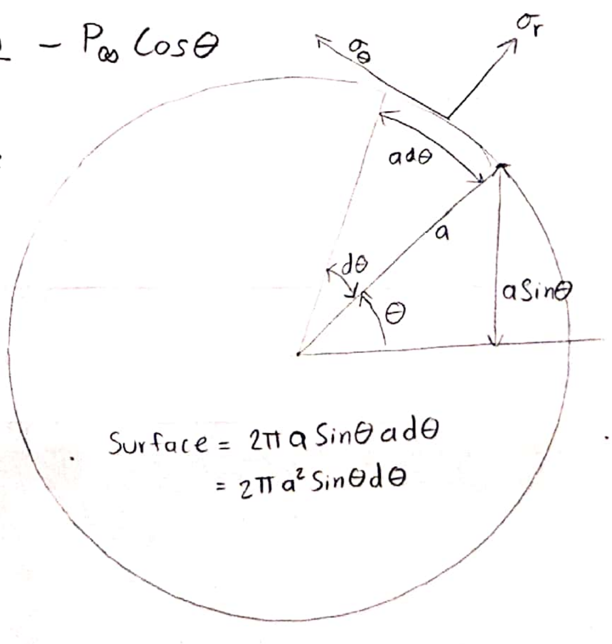
Because of the axisymmetric symmetry of the problem, it is expected that the net drag force is in the direction of the uniform flow far from the sphere w, V_∞

The component of the stress tensor in this direction is:

$$\sigma = \sigma_r \cos\theta - \sigma_\theta \sin\theta = V_\infty \cos^2\theta\left[\frac{3\mu}{2a}\right] - P_\infty \cos\theta + V_\infty \sin^2\theta\left[\frac{3\mu}{2a}\right]$$

$$\sigma = \frac{3\mu}{2a}V_\infty\left[\underbrace{\cos^2\theta + \sin^2\theta}_{1}\right] - P_\infty \cos\theta = \frac{3\mu V_\infty}{2a} - P_\infty \cos\theta$$

So drag force is expressed as the sum of all forces of the fluid exerted on the sphere. This is done by integrating over the surface of the sphere.



$$\text{Drag} = \int_0^{\pi}\int_0^{2\pi} \sigma a^2 \sin\theta\, d\theta\, d\varphi = \int_0^{\pi}\int_0^{2\pi}\left(\frac{3\mu V_\infty}{2a} - P_\infty \cos\theta\right) a^2 \sin\theta\, d\theta\, d\varphi$$

$$\text{Drag} = \int_0^{2\pi}\int_0^{\pi} \frac{3\mu V_\infty a \sin\theta}{2} - P_\infty a^2 \cos\theta \sin\theta \, d\theta \, d\varphi$$

$$= \int_0^{2\pi}\left[V_\infty\left(\frac{3\mu a}{2}\right)\int_0^{\pi} \sin\theta \, d\theta - P_\infty a^2 \int_0^{\pi} \cos\theta \sin\theta \, d\theta \right] d\varphi$$

$$\int_0^{\pi} \sin\theta \, d\theta = -\cos(\theta)\Big|_0^{\pi} = -\cos(\pi) + \cos(0) = -(-1) + 1 = 2$$

$$\int_0^{\pi} \cos\theta \sin\theta \, d\theta = -\int_0^{\pi} u \, du = -\frac{u^2}{2}\Big|_0 = -\frac{\cos^2(\theta)}{2}\Big|_0^{\pi} = -\frac{\cos^2(\pi)}{2} + \frac{\cos^2(0)}{2} = -\frac{1}{2} + \frac{1}{2}$$

if $u = \cos(\theta)$
$du = -\sin(\theta) d\theta$

$$\text{Drag} = \left[V_\infty\left(\frac{3\mu a}{2}\right)(2) - \cancel{P_\infty a^2 (0)} \right] 2\pi$$

$$\boxed{\text{Drag} = V_\infty 3\mu a (2\pi) = 6\pi\mu a V_\infty}$$