Mathematical Physical Modelling - Homework 03
Eigenvalues Problems

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Paper trail and process evidence are at the end of this document.

- **PROBLEMAS DE EIGENVALORES**

1. Encuentra los eigenvalores y los eigenvectores correspondientes.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

- **TRANSFORMACIONES LINEALES Y EIGENVALORES**

2. Encuentra la matriz $\mathbf{A}$ en la transformación lineal $\mathbf{y} = \mathbf{Ax}$, donde $\mathbf{x}$ son las coordenadas Cartesianas. Encuentra los eigenvalores y eigenvectores y explica su significado geométrico.

(a) Reflexión alrededor del eje x_1 en R^2.

(b) Proyección ortogonal de R^3 en el plano $x_2 = x_1$.

- **APLICACIONES: DEFORMACIONES ELÁSTICAS Y MODELOS DE POBLACIÓN**

3. Dada $\mathbf{A}$ en una deformación $\mathbf{y} = \mathbf{Ax}$, encuentra las direcciones principales y los correspondientes factores de extensión o contracción. Muestra los detalles de tu cálculo.

(a) $\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$ (c) $\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$

4. Encuentra el ritmo de crecimiento en el modelo de Leslie con la matriz dada. Muestra los detalles, manito.

(a) $\begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$

• Eigenvalues problems

Find the corresponding eigen values and eigenvectors

a) $\begin{bmatrix} 1 & 2 \\ \emptyset & 3 \end{bmatrix}$ $det(\lambda I - A)$ $\begin{bmatrix} \lambda & \emptyset \\ \emptyset & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ \emptyset & 3 \end{bmatrix} = \begin{bmatrix} \lambda-1 & -2 \\ \emptyset & \lambda-3 \end{bmatrix}$

$det(\lambda I - A) = (\lambda-1)(\lambda-3) - (-2 \cdot \emptyset) = \lambda^2 - 3\lambda - \lambda + 3 = \emptyset$

$\lambda^2 - 4\lambda + 3 = \emptyset$

$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\lambda = \frac{-(-4) \pm \sqrt{-4^2 - 4(3)(1)}}{2(1)} \longrightarrow \lambda = \frac{4 \pm \sqrt{16 - 12}}{2}$

$\lambda = \frac{4 \pm \sqrt{4}}{2} \longrightarrow \lambda = \frac{4 \pm 2}{2} \longrightarrow \lambda_1 = \frac{4+2}{2} = \frac{6}{2} = 3$

$\lambda_2 = \frac{4-2}{2} = \frac{2}{2} = 1$

$\lambda_1 = 3$

$\begin{bmatrix} 3-1 & -2 \\ \emptyset & 3-3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = (3-1)x_1 - 2x_2 = \emptyset = 2x_1 - 2x_2 = \emptyset$

$\emptyset + (3-3)x_2 = \emptyset = \emptyset + \emptyset = 0$

$X_1 = X_2$

$\rightarrow t\begin{pmatrix} x_1 \\ X_2 \end{pmatrix} \rightarrow t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$

$\begin{bmatrix} 1-1 & -2 \\ \emptyset & 1-3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = (1-1)x_1 - 2x_2 = \emptyset = \emptyset - 2x_2 = \emptyset$

$\emptyset + (1-3)x_2 = \emptyset \quad \emptyset - 2 \times x_2 = \emptyset$

$X_1 = t \qquad X_2 = \emptyset$

$\rightarrow t\begin{pmatrix} t \\ \emptyset \end{pmatrix} \rightarrow t\begin{pmatrix} 1 \\ \emptyset \end{pmatrix}$

b $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

$\Rightarrow$ Let's solve the eigen problem...

$Ax = \lambda x$ $\qquad$ $Ax - \lambda x = 0$

$$\begin{cases} ax_1 + bx_2 = \lambda x_1 \\ -bx_1 + ax_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (a-\lambda)x_1 + bx_2 = 0 \\ -bx_1 + (a-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det \begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix} = 0$$

$(a-\lambda)^2 + b^2 = 0$

$\lambda^2 - 2a\lambda + (a^2+b^2) = 0$

$$\Rightarrow \lambda = \frac{-(-2a) \pm \sqrt{(-2a)^2 - 4(1)(a^2+b^2)}}{2(1)}$$

$$\lambda = \frac{2a \pm 2\sqrt{-b^2}}{2}$$

$\lambda = a \pm ib$

the eigen values are:

$\lambda_1 = a - ib$

for $\lambda = \lambda_1$, then the eigenvector is

$$\begin{cases} (a-(a-ib))x_1 + bx_2 = 0 \\ -bx_1 + (a-(a-ib))x_2 = 0 \end{cases}$$

$ix_1 + x_2 = 0$

$x_1 = ix_2$

$i(ix_2) + x_2 = 0$

$x_2 = x_2$

$x = t\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\lambda_2 = a + ib$

for $\lambda = \lambda_2$, then the eigen vector is

$$\begin{cases} (a-(a+ib))x_1 + bx_2 = 0 \\ -bx_1 + (a-(a+ib))x_2 = 0 \end{cases}$$

$-ix_1 + x_2 = 0$

$x_1 = -ix_2$

$-i(-ix_2) + x_2 = 0$

$x_2 = x_2$

$x = t\begin{pmatrix} -i \\ 1 \end{pmatrix}$

c) $\begin{bmatrix} 3 & 5 & 3 \\ \varnothing & 4 & 6 \\ \varnothing & \varnothing & 1 \end{bmatrix}$ When the matrix is upper, lower or diagonal, the eigenvalues are the entries of the main diagonal.

$\lambda_1 = 3 \quad \lambda_2 = 4 \quad \lambda_3 = 1$

$(\lambda I - A)(x)$ $\begin{bmatrix} \lambda & \varnothing & \varnothing \\ \varnothing & \lambda & \varnothing \\ \varnothing & \varnothing & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 5 & 3 \\ \varnothing & 4 & 6 \\ \varnothing & \varnothing & 1 \end{bmatrix} = \begin{bmatrix} \lambda-3 & -5 & -3 \\ \varnothing & \lambda-4 & -6 \\ \varnothing & \varnothing & \lambda-1 \end{bmatrix}$

$\lambda_1 = 3$

$\begin{bmatrix} \cancel{3-3} \to \varnothing & -5 & -3 \\ \varnothing & 3-4 & -6 \\ \varnothing & \varnothing & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$ $-5x_2 - 3x_3 = \varnothing$ $\quad x_3 = \varnothing$

$-x_2 - 6x_3 = \varnothing$

$2x_3 = \varnothing$ $\quad -x_2 - 6(\varnothing) = \varnothing$

$-x_2 = \varnothing$

$x_1 = t$, doesn't has any asigned value, so it can be any number $\quad x_2 = \varnothing$

$\to t\begin{pmatrix} t \\ \varnothing \\ \varnothing \end{pmatrix} = t\begin{pmatrix} 1 \\ \varnothing \\ \varnothing \end{pmatrix}$

$\lambda_2 = 4$

$\begin{bmatrix} 4-3 & -5 & -3 \\ \varnothing & \cancel{4-4} \to \varnothing & -6 \\ \varnothing & \varnothing & 4-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$ $x_1 - 5x_2 - 3x_3 = \varnothing$ $\quad x_3 = \varnothing$

$-6x_3 = \varnothing$ $\quad x_2 = t$

$3x_3 = \varnothing$

$x_2 = t$, So it doesn't has any asigned value, x_2 can be any number.

$x_1 - 5t - \varnothing = \varnothing$

$x_1 - 5t = \varnothing$

$x_1 = 5t$

$\to t\begin{pmatrix} 5t \\ t \\ \varnothing \end{pmatrix} \to t\begin{pmatrix} 5 \\ 1 \\ \varnothing \end{pmatrix}$

$\lambda_3 = 1$

$$\begin{pmatrix} 1-3 & -5 & -3 \\ 0 & 1-4 & -6 \\ 0 & 0 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{cases} -2x_1 - 5x_2 - 3x_3 = 0 \\ -3x_2 - 6x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow x_3 = -\frac{1}{2}x_2$$

$$-2x_1 - 5x_2 - 3\left(-\frac{1}{2}x_2\right) = 0$$

$$x_2 = -\frac{4}{7}x_1 \; ; \quad x_3 = -\frac{1}{2}\left(-\frac{4}{7}x_1\right)$$

$$= \frac{2}{7}x_1$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t\begin{pmatrix} 1 \\ -\frac{4}{7} \\ \frac{2}{7} \end{pmatrix}$$

Modelación Física Matemática

Ejercicio 2:

a) Encuentra la matriz A en la transformación lineal $y = Ax$, donde x son las coordenadas cartesianas. Encuentra los eigenvalores y eigenvectores y explica su significado geométrico.

(a) Reflexión alrededor del eje X_1 en R^2

(b) Proyección ortogonal de R^3 en el plano $X_2 = X_1$

$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Matriz Identidad I multiplicada por λ $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

Eigen valores (det $\lambda I - A$

$\begin{bmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 1 \end{bmatrix} = (\lambda - 1)(\lambda + 1) + 0 = \lambda^2 - 1$

$\therefore$

· Valores propios: $\underline{\lambda_1 = 1}$, y $\underline{\lambda_2 = -1}$,

· Vectores propios: (Para $\lambda = 1$)

Resolviendo $(A - \lambda I)$:

$A - \lambda I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$

Reduciendo la matriz a su forma escalonada $R_2 \leftarrow -\frac{1}{2} R_2$

$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ Intercambiando filas $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

El sistema asociado con el valor propio $\lambda = 1$

$$(A - 1\lambda)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore x_2 = 0$ Entonces $\vec{V_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

x_1: Tomando valor de "1"

Para $\lambda = -1$

$$(A - \lambda I) : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

Reduciendo la matriz $R_1 \leftarrow \frac{1}{2} R_1$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

El sistema asociado para $\lambda = -1$

$$(A + 1I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Entonces:

$x_1 = 0$ $\vec{V_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Si $x_2 = 1$

Proyección ortogonal de R^3 en el plano $X_2 = X_1$

- Expresado como matriz, se tiene una proyección con la siguiente forma matricial:

$(1,0,0)$ en $(1/2, 1/2, 0)$

$(0,1,0)$ en $(1/2, 1/2, 0)$

$(0,0,1)$ en $(0,0,1)$ $\therefore$

Matriz:
$$\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Obteniendo eigenvalores y eigenvectores

$$\det \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \lambda(-\lambda^2 + 2\lambda - 1)$$

Resolviendo

$\lambda(-\lambda^2 + 2\lambda - 1) = 0$: $\lambda = 0$, $\lambda = 1$

$\therefore$ Valores propios son $= 0, 1$

Vectores propios para $\lambda = 0$

$$(A - \lambda I) : \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reduciendo la matriz a forma escalonada reducida

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \therefore (A - 0I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x + y = 0$

$z = 0$

$z = 0$

$\therefore \underline{x = -y}$

Sustituyendo

$$V = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} \quad y \neq 0 \quad \text{Sea } y = 1 \;\therefore$$

$$\vec{V} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Para $\lambda = 1$

$$(A - \lambda I) : \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Reduciendo a su forma escalonada tenemos:

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \therefore (A-1I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x - y = 0$, y z puede tomar cualquier valor

$\therefore \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Despejando

$x = y$

Sustituyendo en $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$V = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix}$ sea $y = 1$ $\therefore \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Entonces los vectores propios para $\lambda = 1$ son

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Para el eigenvalor $\underline{1}$ corresponden los vectores

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ y $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

lo cual indica que cada punto del plano $x_2 = x_1$ o $(y = x)$ es proyectado en si mismo o se asigna así mismo. Para el eigenvalor 0 un eigen vector es $(-1, 1, 0)$ y transpuesto es $(1, -1, 0)^T$. Esto muestra que cualquier punto en la linea $x_2 = -x_1, z = 0$ lo cual indica que es perpendicular al plano $x_2 = x_1$.

Homework 3 Eigenvalue probleme

3 Deformaciones elásticas

Dada A en una deformación $y = Ax$, encuentra las direcciones principales y los correspondientes factores de extensión o contracción. Muestra los detalles de tu cálculo.

a

$$A = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix}$$

Note: The principal directions are the directions of the position vector x for which the direction of the position vector y is the same or exact opposite.

$$y = Ax$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} y_1 &= 2.0x_1 + 0.4x_2 \\ y_2 &= 0.4x_1 + 2.0x_2 \end{aligned}$$

▷ Let's solve the eigenvalue problem...

$$Ax = \lambda x \qquad\qquad Ax - \lambda x = 0$$

$$\begin{cases} 2.0x_1 + 0.4x_2 = \lambda x_1 \\ 0.4x_1 + 2.0x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (2.0-\lambda)x_1 + 0.4x_2 = 0 \\ 0.4x_1 + (2.0-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det\begin{pmatrix} 2.0-\lambda & 0.4 \\ 0.4 & 2.0-\lambda \end{pmatrix} = 0$$ is the characteristic equation

$$(2.0-\lambda)^2 - 0.4^2 = 0$$
$$\lambda^2 - 4.0\lambda + 3.84 = 0$$

the eigenvalues are:

$$\lambda_1 = -\frac{8}{5}$$

for $\lambda = \lambda_1$, then

$$\begin{cases} 3.6x_1 + 0.4x_2 = 0 \\ 0.4x_1 + 3.6x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda_2 = -\frac{12}{5}$$; where λ is the strech factor

for $\lambda = \lambda_2$, then

$$\begin{cases} 4.4x_1 + 0.4x_2 = 0 \\ 0.4x_1 + 4.4x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$; where x is the eigen vector

▷ The eigenvalues show a contraction, with no direction b $A = \begin{pmatrix} 5 & 2 \\ 2 & 13 \end{pmatrix}$ The eigenvalues show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2, respectively.

▷ Let's solve the eigen problem...

$$Ax = \lambda x \qquad Ax - \lambda x = 0$$

$$\begin{cases} 5x_1 + 2x_2 = \lambda x_1 \\ 2x_1 + 13x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (5-\lambda)x_1 + 2x_2 = 0 \\ 2x_1 + (13-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{pmatrix} = 0$$
$$(5-\lambda)(13-\lambda) - 4 = 0$$
$$\lambda^2 - 18\lambda + 61 = 0$$

the eigenvalues are:

$\lambda_1 = 9 + 2\sqrt{5}$

for $\lambda = \lambda_1$, then

$$\begin{cases} (-4-2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4-2\sqrt{5})x_2 = 0 \end{cases}$$

$$x_1 = \frac{-(4-2\sqrt{5})x_2}{2}$$
$$x_1 = (-2+\sqrt{5})x_2$$

$$-2x_2 + 2x_2 = 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-2+\sqrt{5})x_2 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -2+\sqrt{5} \\ 1 \end{pmatrix}$$

$\lambda_2 = 9 - 2\sqrt{5}$

for $\lambda = \lambda_2$

$$\begin{cases} (-4+2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4+2\sqrt{5})x_2 = 0 \end{cases}$$

$$x_1 = \frac{-(4+2\sqrt{5})x_2}{2} = (-2-\sqrt{5})x_2$$

$$-2x_2 + 2x_2 = 0$$

$$x = \begin{pmatrix} -(4+2\sqrt{5})x_2 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -4-2\sqrt{5} \\ 1 \end{pmatrix}$$

▷ Let's compute the principal directions

$$\theta_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(\frac{1}{-2+\sqrt{5}}\right)$$
$= 76°43'2.91''$ with the positive x_1 direction

$$\theta_2 = \tan^{-1}\left(\frac{1}{-4-2\sqrt{5}}\right)$$
$= 173°16'5.87''$ with the positive x_1 direction

c $A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$

▷ Let's solve the eigen problem...

$Ax = \lambda x$ $\qquad\qquad$ $Ax - \lambda x = 0$

$$\begin{cases} 1.25x_1 + 0.75x_2 = \lambda x_1 \\ 0.75x_1 + 1.25x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (1.25-\lambda)x_1 + 0.75x_2 = 0 \\ 0.75x_1 + (1.25-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det\begin{pmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 2.5\lambda + 1 = 0$$

the eigen values are:

$\lambda_1 = 2$

for $\lambda = \lambda_1$, then

$$\begin{cases} -0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 - 0.75x_2 = 0 \end{cases}$$

$x_1 = x_2$

$x = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 0.5$

for $\lambda = \lambda_2$, then

$$\begin{cases} 0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 + 0.75x_2 = 0 \end{cases}$$

$x_1 = -x_2$

$x = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

▷ Let's compute the principal directions.

$\theta_1 = \operatorname{Tan}^{-1}\left(\frac{1}{1}\right)$

$= 45°0'0''$ with the positive x_1 direction

$\theta_2 = \operatorname{Tan}^{-1}\left(\frac{1}{-1}\right)$

$= 135°0'0''$ with the positive x_1 direction

The eigen values show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2, respectively.

Homework 3 Eigenvalue problems.

4 Modelos de población.

a

$$L = \begin{pmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}$$

Note: The Leslie model describes age-specified population growth.

Assumptions:
- i) Let the oldest age attained be 99 years.
- ii) Since the provided matrix L has 3 columns, divide the population into three age classes of 33 years each.

Note: Given a Leslie matrix $L = [l_{jk}]$,
1) l_{1k} is the average number of new borns to a single member durig the time he/she is in age class k.
2) $l_{j,j-1}$ is the fraction of members in age class $j-1$ that will survive and pass into class j.

Note: Proportional change means that we are looking for a distribution vector x such that $Lx = \lambda x$, where λ is the rate of change (growth if $\lambda > 1$, decrease if $\lambda < 1$).

Encuentra el ritmo de crecimiento en el modelo de Leslie con la matrix dada.

▷ Let's find the λ's

$$\det(L - \lambda I)x = 0$$

$$\det\begin{pmatrix} -\lambda & 9.0 & 5.0 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{pmatrix} = 0$$

$$-0.4 \det\begin{pmatrix} -\lambda & 5.0 \\ 0.4 & -\lambda \end{pmatrix} + (-\lambda)\det\begin{pmatrix} -\lambda & 9.0 \\ 0.4 & -\lambda \end{pmatrix} = 0$$

$$(-0.4)(\lambda^2 - 2) + (-\lambda)(\lambda^2 - 3.6) = 0$$

$$-0.4\lambda^2 + 0.8 - \lambda^3 + 3.6\lambda = 0$$

$$-\lambda^3 - 0.4\lambda^2 + 3.6\lambda + 0.8 = 0$$

$$\lambda_1 = 1.82, \quad \lambda_2 = -0.23, \quad \lambda_3 = -2$$

▷ A positive root is found to be $\lambda = 1.82$

The growth rate will be 1.82 per 33 years.

$\boxed{b}$

$$L = \begin{pmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}$$

Assumptions:

i) Let the oldest age attained be 100 years

ii) Since the provided matrix L has four columns, let's divide the population into four age classes of 25 years each.

$\triangleright$ let's find the λ's

$$\det(L - \lambda I)x = 0$$

$$\det\begin{pmatrix} -\lambda & 3.0 & 2.0 & 2.0 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{pmatrix} = +(-\lambda)\begin{pmatrix} -\lambda & 0 & 0 \\ 0.5 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix}$$

$$-(3.0)\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix}$$

$$+(2.0)\begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$-(2.0)\begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \\ 0 & 0 & 0.1 \end{pmatrix}$$

$$= -\lambda\left[+(-\lambda)\begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0\right]$$

$$-3.0\left[+(0.5)\begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0\right]$$

$$+2.0\left[+(0.5)\begin{pmatrix} 0.5 & 0 \\ 0 & -\lambda \end{pmatrix} \cancel{-(-\lambda)\begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix}} + 0\right]$$

$$-2.0\left[+(0.5)\begin{pmatrix} 0.5 & -\lambda \\ 0 & 0.1 \end{pmatrix} \cancel{-(-\lambda)\begin{pmatrix} 0 & -\lambda \\ 0 & 0.1 \end{pmatrix}} + 0\right]$$

$$= -\lambda(-\lambda(\lambda^2)) - 3.0(0.5(\lambda^2)) + 2.0(0.5(-0.5\lambda))$$
$$-2.0(0.5(0.5(0.1)))$$
$$= \lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05$$

$\lambda_1 = 1.37 \quad \lambda_2 = -1.03 \quad \lambda_3 = -0.17 + 0.07i \quad \lambda_4 = -0.17 - 0.07i$

$\triangleright$ A positive root is found to be $\lambda = 1.37$

The growth rate will be 1.37 per 25 years.