• Energy and Potential

* Energy expended in moving a point charge in an electric field

$$\vec{F}_E = Q\vec{E}$$

$$F_{E_L} = \vec{F}_E \cdot \vec{a}_L = Q\vec{E} \cdot \vec{a}_L$$

$$F_{appl} = -Q\vec{E} \cdot \vec{a}_L$$

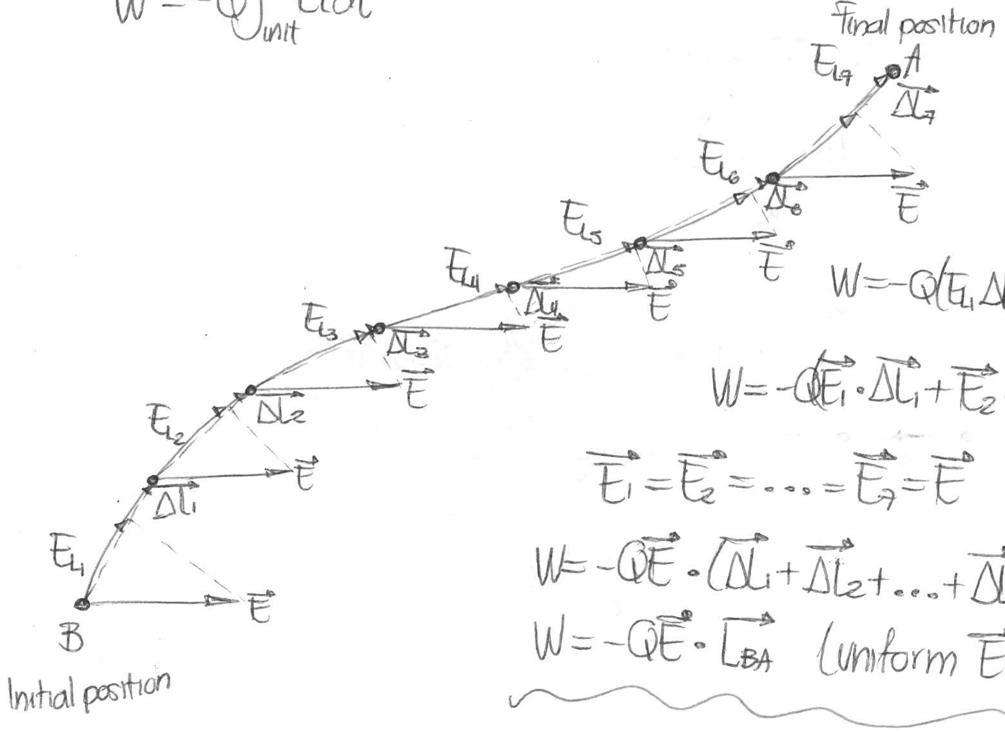
Differential work done by external source moving $Q = -Q\vec{E} \cdot \vec{a}_L dL$

$$dW = -Q\vec{E} \cdot \vec{dL}$$

$$W = -Q\int_{init}^{final} \vec{E} \cdot \vec{dL}$$

* The line integral

$$W = -Q\int_{init}^{final} E_L dL$$



$$W = -Q(E_{L_1}\Delta L_1 + E_{L_2}\Delta L_2 + \dots + E_{L_7}\Delta L_7)$$

$$W = -Q(\vec{E_1} \cdot \vec{\Delta L_1} + \vec{E_2} \cdot \vec{\Delta L_2} + \dots + \vec{E_7} \cdot \vec{\Delta L_7})$$

$$\vec{E_1} = \vec{E_2} = \dots = \vec{E_7} = \vec{E}$$

$$W = -Q\vec{E} \cdot (\vec{\Delta L_1} + \vec{\Delta L_2} + \dots + \vec{\Delta L_7})$$

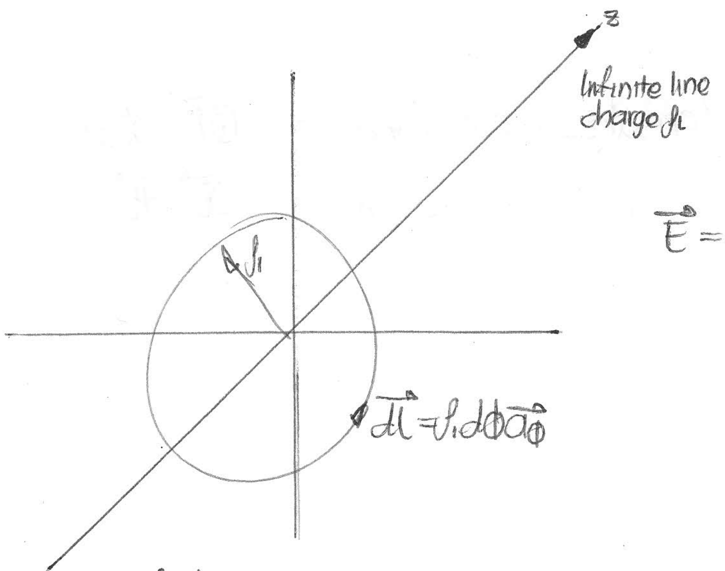
$$W = -Q\vec{E} \cdot \vec{L_{BA}} \quad (\text{uniform } \vec{E})$$

$$W = -Q\int_B^A \vec{E}\cdot\vec{dl} = -Q\vec{E}\cdot\int_B^A \vec{dl} = -Q\vec{E}\cdot\vec{L}_{BA}$$

$$\vec{dl} = dx\,\vec{a}_x + dy\,\vec{a}_y + dz\,\vec{a}_z$$

$$\vec{dl} = d\rho\,\vec{a}_\rho + \rho\, d\phi\,\vec{a}_\phi + dz\,\vec{a}_z$$

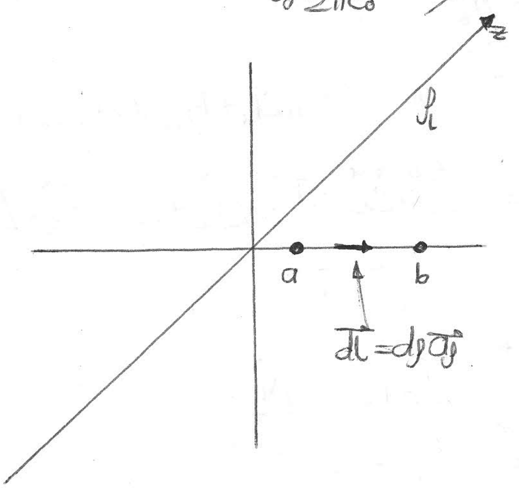
$$\vec{dl} = dr\,\vec{a}_r + r\,d\theta\,\vec{a}_\theta + r\sin\theta\, d\phi\,\vec{a}_\phi$$



$$\vec{E} = E_\rho \vec{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_0 \rho}\vec{a}_\rho$$

$$W = -Q\int_{init}^{final} \vec{E}\cdot\vec{dl} = -Q\int_{init}^{final} \frac{\rho_L}{2\pi\varepsilon_0 \cancel{\rho_1}}\vec{a}_\rho \cdot \cancel{\rho_1} d\phi\,\vec{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\varepsilon_0} d\phi\,\cancelto{0}{\vec{a}_\rho\cdot\vec{a}_\phi} = 0$$



$$W = -Q\int_{init}^{final} \frac{\rho_L}{2\pi\varepsilon_0 \rho}\vec{a}_\rho \cdot d\rho\,\vec{a}_\rho$$

$$= -Q\frac{\rho_L}{2\pi\varepsilon_0}\int_a^b \frac{d\rho}{\rho}$$

$$= \frac{-Q\rho_L}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

*Definition of Potential Difference and Potential

$$W=-Q\int_{init}^{final}\vec{E}\cdot\vec{dl}$$

$$\text{Potential difference}=V=-\int_{init}^{final}\vec{E}\cdot\vec{dl}$$

$$V_{AB}=-\int_{B}^{A}\vec{E}\cdot\vec{dl}$$

for point charges

$$\vec{E}=E_r\vec{a_r}=\frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}$$

$$\vec{dl}=dr\vec{a_r}$$

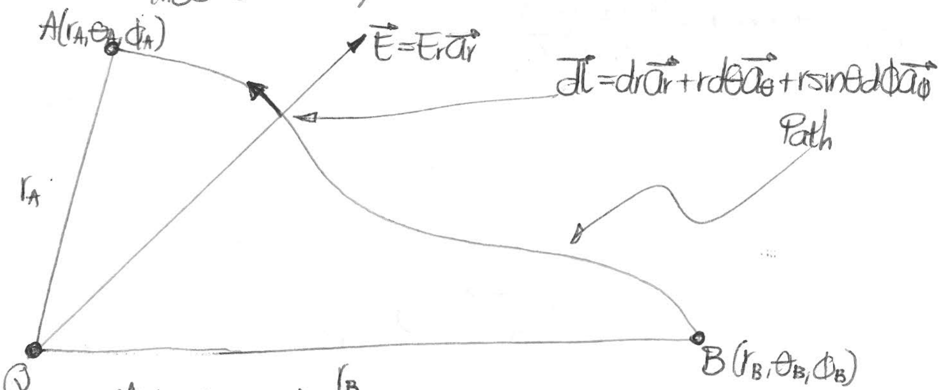
$$V_{AB}=-\int_{B}^{A}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}\right)\cdot\left(dr\vec{a_r}\right)=-\int_{B}^{A}\frac{Q}{4\pi\varepsilon_0 r^2}dr=-\frac{Q}{4\pi\varepsilon_0}\int_{B}^{A}\frac{dr}{r^2}$$

$$=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r}\Big|_{B}^{A}\right)=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right)$$

$$V_{AB}=V_A-V_B$$

*The potential field of a point charge

$$V_{AB}=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right)=V_A-V_B$$



$$V_{AB}=-\int_{B}^{A}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}\right)\cdot\left(dr\vec{a_r}+rd\theta\vec{a_\theta}+r\sin\theta d\phi\vec{a_\phi}\right)=-\int_{r_B}^{r_A}\frac{Q}{4\pi\varepsilon_0 r^2}dr=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A}-\frac{1}{r_B}\right)$$

if the point $r=r_B$ recedes to infinity

$$V_A = \frac{Q}{4\pi\varepsilon_0 r_A}$$

$$\therefore V = \frac{Q}{4\pi\varepsilon_0 r}$$

* The potential field of a system of charges: conservative property

Q_1 at $\vec{r_1}$

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r_1}|}$$

two charges

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r_1}|} + \frac{Q_2}{4\pi\varepsilon_0 |\vec{r}-\vec{r_2}|}$$

n charges

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r_1}|} + \frac{Q_2}{4\pi\varepsilon_0 |\vec{r}-\vec{r_2}|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0 |\vec{r}-\vec{r_n}|}$$

$$V(\vec{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 |\vec{r}-\vec{r_m}|}$$

$$V(\vec{r}) = \frac{\rho_v(\vec{r_1})\Delta v_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r_1}|} + \frac{\rho_v(\vec{r_2})\Delta v_2}{4\pi\varepsilon_0 |\vec{r}-\vec{r_2}|} + \cdots + \frac{\rho_v(\vec{r_n})\Delta v_n}{4\pi\varepsilon_0 |\vec{r}-\vec{r_n}|}$$

$$V(\vec{r}) = \int_{vol} \frac{\rho_v(\vec{r'})dv'}{4\pi\varepsilon_0 |\vec{r}-\vec{r'}|}$$

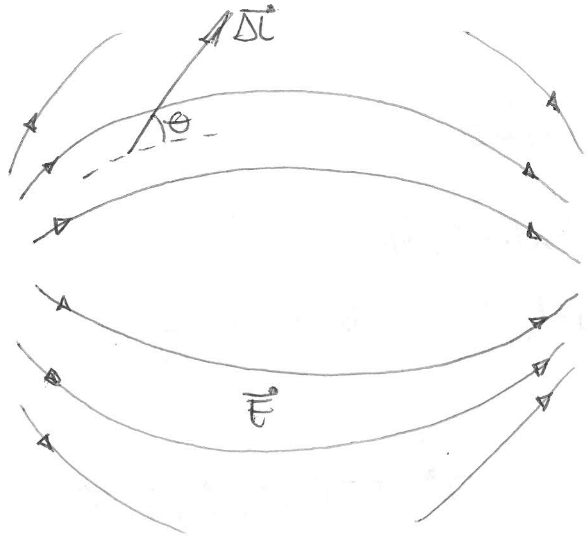
$$V(\vec{r}) = \int \frac{\rho_L(\vec{r'})dL'}{4\pi\varepsilon_0 |\vec{r}-\vec{r'}|}$$

$$\boxed{\oint \vec{E}\cdot d\vec{L} = 0}$$

$$V(\vec{r}) = \int_s \frac{\rho_s(\vec{r'})ds'}{4\pi\varepsilon_0 |\vec{r}-\vec{r'}|}$$

*Potential gradient

$$V = -\int \vec{E} \cdot d\vec{L}$$

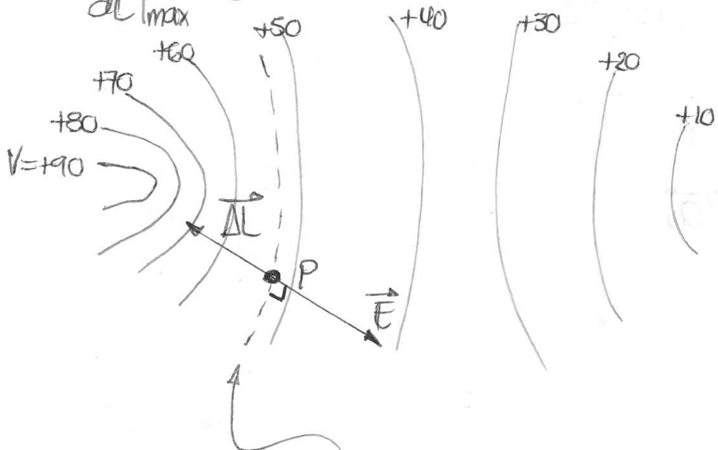


$$\Delta V \doteq -\vec{E} \cdot \vec{\Delta L}$$

$$\Delta V \doteq -E \Delta L \cos\theta$$

$$\frac{dV}{dL} = -E\cos\theta$$

$$\left.\frac{dV}{dL}\right|_{max} = E, \text{ when } \theta = \pi$$



Equipotential Surface

$$\Delta V = -\vec{E} \cdot \vec{\Delta L} = 0, \text{ when } \theta = \pi/2$$

$$\left.\frac{dV}{dL}\right|_{max} = E, \text{ when } \theta = \pi$$

$$\therefore \vec{E} = -\left.\frac{dV}{dL}\right|_{max} \vec{a}_N$$

$$\left.\frac{dV}{dL}\right|_{max} = \frac{dV}{dN}$$

and $\vec{E} = -\frac{dV}{dN}\vec{a}_N$

$$\text{Gradient of } T = \text{grad } T = \frac{dT}{dN}\vec{a}_N$$

$$\therefore \vec{E} = -\text{grad } V$$

$V(x,y,z)$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$dV = -\vec{E}\cdot d\vec{l} = -E_x dx - E_y dy - E_z dz$$

$$\therefore -E_x dx - E_y dy - E_z dz = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$E_x = -\frac{\partial V}{\partial x}$$

$$E_y = -\frac{\partial V}{\partial y} \qquad \vec{E} = E_x\vec{a_x} + E_y\vec{a_y} + E_z\vec{a_z}$$

$$E_z = -\frac{\partial V}{\partial z} \qquad \vec{E} = -\left(\frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}\right)$$

$$\vec{E} = -\operatorname{grad} V$$

$$\therefore \operatorname{grad} V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z} \,//$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}$$

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\vec{a_x} + \frac{\partial T}{\partial y}\vec{a_y} + \frac{\partial T}{\partial z}\vec{a_z} = \operatorname{grad} T$$

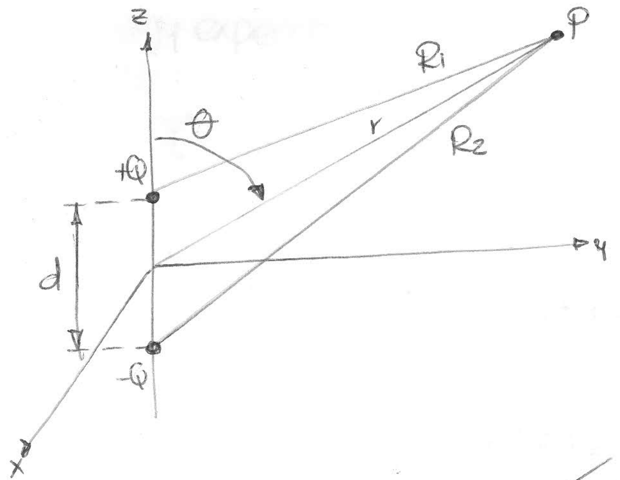
$$\vec{E} = -\vec{\nabla}V \,//$$

$$\vec{\nabla}V = \frac{\partial V}{\partial x}\vec{a_x} + \frac{\partial V}{\partial y}\vec{a_y} + \frac{\partial V}{\partial z}\vec{a_z}$$

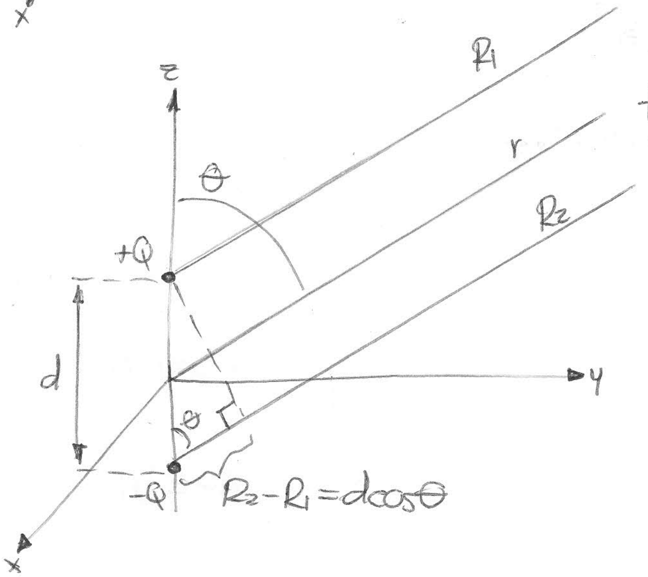
$$\vec{\nabla}V = \frac{\partial V}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\vec{a_\phi} + \frac{\partial V}{\partial z}\vec{a_z}$$

$$\vec{\nabla}V = \frac{\partial V}{\partial r}\vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a_\phi}$$

* The dipole



$$V=\frac{Q}{4\pi\varepsilon_0 R_1}-\frac{Q}{4\pi\varepsilon_0 R_2}$$
$$=\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$
$$=\frac{Q}{4\pi\varepsilon_0}\left(\frac{R_2-R_1}{R_1R_2}\right)$$
$$=\frac{Q}{4\pi\varepsilon_0}\left(\frac{d\cos\theta}{r^2}\right)$$
$$=\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}$$



for a distant point P
$R_1\approx R_2\approx r$

$$\vec{E}=-\vec{\nabla}V=-\left(\frac{\partial V}{\partial r}\vec{a_r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\vec{a_\theta}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\vec{a_\phi}\right)$$

$$\frac{\partial V}{\partial r}=\frac{\partial}{\partial r}\left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}\right)=\frac{Qd\cos\theta}{\cancel{4}_{2}\pi\varepsilon_0}(-\cancel{2})r^{-3}$$
$$=-\frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}$$

$$\frac{\partial V}{\partial\theta}=\frac{\partial}{\partial\theta}\left(\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}\right)=\frac{Qd}{4\pi\varepsilon_0 r^2}(-\sin\theta)$$
$$=-\frac{Qd\sin\theta}{4\pi\varepsilon_0 r^2}$$

$$\frac{\partial V}{\partial\phi}=0$$

$$\vec{E}=-\nabla V=-\left(-\frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\vec{a_r}-\frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\vec{a_\theta}\right)$$
$$\vec{E}=\frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\vec{a_r}+\sin\theta\vec{a_\theta}\right)$$

$$\vec{p}=Q\vec{d}$$
$$V=\frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}=\frac{\vec{p}\cdot\vec{a_r}}{4\pi\varepsilon_0 r^2}$$

* Energy density in the electrostatic field

Work to position $Q_2 = Q_2 V_{2,1}$
Work to position $Q_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$
Work to position $Q_4 = Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3}$

Total positioning work = potential energy of field

$$= W_E = Q_2 V_{2,1} + \underbrace{Q_3 V_{3,1}} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \ldots$$

$$\to Q_3 V_{3,1} = Q_3 \frac{Q_1}{4\pi\varepsilon_0 R_{13}} = Q_1 \frac{Q_3}{4\pi\varepsilon_0 R_{31}} = Q_1 V_{1,3}$$

$$\therefore W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \ldots$$

$$\begin{aligned} 2W_E = {} & Q_1 (V_{1,2} + V_{1,3} + V_{1,4} + \ldots) \\ & + Q_2 (V_{2,1} + V_{2,3} + V_{2,4} + \ldots) \\ & + Q_3 (V_{3,1} + V_{3,2} + V_{3,4} + \ldots) \\ & + \ldots \end{aligned}$$

$$V_{1,2} + V_{1,3} + V_{1,4} + \ldots = V_1$$

$$2W_E = (Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \ldots)$$

$$W_E = \frac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3 + \ldots) = \frac{1}{2}\sum_{m=1}^{N} Q_m V_m = \frac{1}{2}\int_{Vol} \rho_v V \, dv$$

$$\rho_v = \vec{\nabla} \cdot \vec{D}$$

$$W_E = \frac{1}{2}\int_{Vol} (\vec{\nabla} \cdot \vec{D}) V \, dv, \quad \vec{\nabla} \cdot (V\vec{D}) \equiv V(\vec{\nabla} \cdot \vec{D}) + \vec{D} \cdot (\vec{\nabla} V)$$

$$W_E = \frac{1}{2}\int_{Vol} [\vec{\nabla} \cdot (V\vec{D}) - \vec{D} \cdot (\vec{\nabla} V)] \, dv$$

$$W_E = \frac{1}{2}\int_{Vol} \vec{\nabla} \cdot (V\vec{D}) \, dv - \frac{1}{2}\int_{Vol} \vec{D} \cdot (\vec{\nabla} V) \, dv, \quad \oint_S \vec{A} \cdot d\vec{S} = \int_{Vol} \vec{\nabla} \cdot \vec{A} \, dv$$

$$W_E = \cancel{\frac{1}{2}\oint_S (V\vec{D}) \cdot d\vec{S}}^{\,\to 0} - \frac{1}{2}\int_{Vol} \vec{D} \cdot (\vec{\nabla} V) \, dv, \quad E = -\vec{\nabla} V$$

$$W_E = \frac{1}{2}\int_{Vol} \vec{D} \cdot \vec{E} \, dv = \frac{1}{2}\int_{Vol} \varepsilon_0 E^2 \, dv \quad //$$