Homework 03 - Matrix Inversion

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HW3: Matrix inversion
Due Date: February 3-2019, 23:59 hrs.
Professor: Ph.D Daniel López Aguayo

Full names of team members: ______________________________

Instructions: Please write neatly on each page of your homework and send it in pdf format to dlopez.aguayo@tec.mx. Typed solutions in LaTeX (only) will be given extra credit; no late homework will be accepted. Each team should consist (of at most) 5 students.

I. Use Gaussian-Jordan elimination to compute (**by hand**) the inverse of each of the following matrices. In case the matrix is diagonal or triangular, you are allowed to use the results given in class. If the inverse does not exist state clearly why.

(a)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)
$$C = \begin{bmatrix} -\frac{4}{\sqrt{3}} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & e^2 \end{bmatrix}$$

(b)
$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(d)
$$D = \begin{bmatrix} \pi & \pi & \pi \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

II. Is the following matrix invertible?
$$E = \begin{bmatrix} 4 & 5 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 9 \end{bmatrix}$$
Justify your answer carefully.

III. Verify part **I** with Mathematica and please include your input and output.

IV. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary 2×2 matrix. It can be shown that A is invertible if and only if $ad - bc \neq 0$, and in this case
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
Use the above result to compute the inverse of each of the following matrices

(a)
$$A = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$$

(c)
$$C = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

(b)
$$B = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$$

(d)
$$D = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

V. Verify with Mathematica all the above answers (from part **IV**); please include your input and output.

VI. Give specific examples of square matrices A, B (not necessarily different), both of size 3, such that:

(a) $A+B$ is not invertible although A and B are invertible.

(b) $A+B$ is invertible although A and B are not invertible.

VII. Determine, **by inspection** (i.e no computations are allowed!), whether the given system has infinitely many solutions or only the trivial solution. *Hint*: use the theorems we saw in class.

(a)

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 0 \\ x_1 - x_2 + 2x_3 = 0 \end{cases}$$

(b)

$$\begin{cases} x_1 - x_3 + 2x_2 = 0 \\ x_2 + 2x_3 = 0 \\ \pi x_3 = 0 \end{cases}$$

VIII. Solve, **by hand**, the following system by matrix inversion (i.e first compute A^{-1} and then $A^{-1}B$.)

$$\begin{cases} x_1 - x_2 - 2x_3 = -5 \\ 2x_1 + 3x_2 + x_3 = 5 \\ 2x_2 + 3x_3 = 8 \end{cases}$$

IX. Consider the following system

$$\begin{cases} x_1 - x_2 - x_3 + 2x_4 = 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 = 3 \\ -x_1 + x_2 - x_3 = -3 \end{cases}$$

(a) Find the reduced row echelon form of $[A|B]$; then compute $\text{rank}(A)$ and $\text{rank}([A|B])$. Finally, use Mathematica to verify all your answers.

(b) Use the theorem given in class to prove that the system has infinitely many solutions.

(c) Compute the general form of the solution vector (the answer must be two-parametric).

(d) What are the free variables?

(e) Why we **cannot** use matrix inversion for this problem?

X. A Brazilian man, currently living in Canada, made phone calls within Canada, to the United States, and to Brazil. The rates per minute for these calls vary for the different countries. Use the information in the following table to determine the rates.

Month	**Time within Canada (min)**	**Time to the U.S. (min)**	**Time to Brazil (min)**	**Charges (U.S dollar)**
September	90	120	180	252
October	70	100	120	184
November	50	110	150	206

(a) Write down the corresponding linear system and state clearly what does each variable represent in this context.

(b) Compute the inverse of A using Mathematica (**not by hand!**).

(c) Use (b) to solve the system (recall you can use Mathematica to compute the product of matrices) and **interpret the answer** in terms of the context.

1 Answer to Problem I

2 Answer to Problem II

3 Answer to Problem III

III.a

$$\mathbf{varA} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix};$$

Inverse[varA];

MatrixForm[%]

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

III.b
Matrix is singular.

$$\mathbf{varB} = \begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} & \mathbf{3} \end{pmatrix};$$

Inverse[varB];

III.c

$$\mathbf{varC} = \begin{pmatrix} -\frac{\mathbf{4}}{\sqrt{\mathbf{3}}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\frac{\mathbf{2}}{\mathbf{3}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{e}^{\mathbf{2}} \end{pmatrix};$$

Inverse[varC];

MatrixForm[%]

$$\begin{pmatrix} -\frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & -\frac{3}{2} & 0 \\ 0 & 0 & \frac{1}{e^2} \end{pmatrix}$$

III.d
Matrix is singular.

$$\mathbf{varD} = \begin{pmatrix} \pi & \pi & \pi \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix};$$

Inverse[varD];

4 Answer to Problem IV

IV.a

$$A^{-1} = \frac{1}{1(8)-2(2)} \begin{pmatrix} 8 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 8 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

IV.b

$$B^{-1} = \frac{1}{0(0)-2(3)} \begin{pmatrix} 0 & -2 \\ -3 & 0 \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 0 & -2 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}$$

IV.c

$$C^{-1} = \frac{1}{2(0)-0(4)} \begin{pmatrix} 0 & 0 \\ -4 & 2 \end{pmatrix} \Rightarrow ad - bc\ isnotequalto\ 0,\ therefore\ C^{-1}\ does\ not\ exist.$$

IV.d

$$D^{-1} = \frac{1}{\mathrm{Cos}[\theta](\mathrm{Cos}[\theta])-(-\mathrm{Sin}[\theta])(\mathrm{Sin}[\theta])} \begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix} = \frac{1}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} \begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\mathrm{Cos}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} & \frac{\mathrm{Sin}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} \\ -\frac{\mathrm{Sin}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} & \frac{\mathrm{Cos}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} \end{pmatrix}$$

5 Answer to Problem V

V.a

```
Custom2by2Inverse[varM_]:=Module[{vM = varM, a, b, c, d, Res},
(* Let's check if the matrix is of size 2x2 *)
If[Dimensions[vM][[1]] == 2&&Dimensions[vM][[2]] == 2,
a = vM[[1, 1]];
b = vM[[1, 2]];
c = vM[[2, 1]];
d = vM[[2, 2]];
(* The matrix is invertible if and only if ad − bc ≠ 0 ... *)
If[a ∗ d − b ∗ c == 0,
Print["[ERROR] The matrix is not invertible."]
];
(* Use the provided equation to calculate the inverse of the 2x2 matrix *)
Res = 1/(a∗d−b∗c) ( d  −b
                    −c   a );
(* Use Mathematica's function Inverse[] to verify the calculation *)
If[Res == Inverse[varM],
Print["[PASS] The calculation is EQUAL to Mathematica's function Inverse[",
MatrixForm[varM], "]."],
Print["[FAIL] The calculation is NOT EQUAL to Mathematica's function Inverse[",
MatrixForm[varM], "]."]
];
(*Return the calculated inverse of vM*)
Res,
(*Display an error message if the matrix size is not 2x2*)
Print("[ERROR] Unfortunately, this function only works with 2x2 matrices.")
];
];
```

$$\mathbf{Custom2by2Inverse}\left[\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{8} \end{pmatrix}\right];$$

MatrixForm[%]

[PASS] The calculation is EQUAL to Mathematica's function Inverse[$\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix}$].

$$\begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

V.b

$$\mathbf{Custom2by2Inverse}\left[\begin{pmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{3} & \mathbf{0} \end{pmatrix}\right];$$

MatrixForm[%]

[PASS] The calculation is EQUAL to Mathematica's function Inverse[$\begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix}$].

$$\begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}$$

V.c
Infinite expression encountered.
Matrix is singular.

$$\mathbf{Custom2by2Inverse}\left[\begin{pmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{4} & \mathbf{0} \end{pmatrix}\right];$$

MatrixForm[%]

[ERROR] The matrix is not invertible.

V.d

$$\mathbf{Custom2by2Inverse}\left[\begin{pmatrix} \mathbf{Cos}[\boldsymbol{\theta}] & -\mathbf{Sin}[\boldsymbol{\theta}] \\ \mathbf{Sin}[\boldsymbol{\theta}] & \mathbf{Cos}[\boldsymbol{\theta}] \end{pmatrix}\right];$$

MatrixForm[%]

[PASS] The calculation is EQUAL to Mathematica's function Inverse[$\begin{pmatrix} \mathrm{Cos}[\theta] & -\mathrm{Sin}[\theta] \\ \mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] \end{pmatrix}$].

$$\begin{pmatrix} \frac{\mathrm{Cos}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} & \frac{\mathrm{Sin}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} \\ -\frac{\mathrm{Sin}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} & \frac{\mathrm{Cos}[\theta]}{\mathrm{Cos}[\theta]^2+\mathrm{Sin}[\theta]^2} \end{pmatrix}$$

6 Answer to Problem VI

7 Answer to Problem VII

8 Answer to Problem VIII

9 Answer to Problem IX

10 Answer to Problem X