Chapter 10 Review Questions & Problems pp 469

➢ Line Integrals (work integrals)

Evaluate $\int F(r) \cdot dr$ for given F and C by the method that seems most suitable. Remember that if F is a force, the integral gives the work done in the displacement along C. Show the details

12 $F = [y \cos(xy),\ x \cos(xy),\ e^z]$, C the straight-line segment from $(\pi, 1, 0)$ to $(0.5, \pi, 1)$.

- Let $F_1 = y\cos(xy)$, $F_2 = x\cos(xy)$, $F_3 = e^z$
- If $\nabla f = F$, then

:By Exactness and Independence of Path

$$\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] = [F_1, F_2, F_3]$$

$$\left.\begin{aligned}
\frac{\partial F_3}{\partial y} &= \frac{\partial F_2}{\partial z} = 0 \\
\frac{\partial F_1}{\partial z} &= \frac{\partial F_3}{\partial x} = 0 \\
\frac{\partial F_2}{\partial x} &= \frac{\partial F_1}{\partial y} = -xy\sin(xy) + \cos(xy)
\end{aligned}\right\} \text{it's exact.}$$

- If $\frac{\partial f}{\partial x} = F_1$

$$\frac{\partial f}{\partial x} = y\cos(xy) \Rightarrow f = y\frac{\sin(xy)}{y} + g(y)$$

//y is constant when integrating with respect to x

$$f = \sin(xy) + g(y)$$

- Let's do $\frac{\partial f}{\partial y} = x\cos(xy) + \frac{\partial g}{\partial y}$
- Since $\frac{\partial f}{\partial y} = F_2$,

$$x\cos(xy) + \frac{\partial g}{\partial y} = x\cos(xy)$$

$$\frac{\partial g}{\partial y} = 0 \Rightarrow g = h(z)$$

- Let's substitute g in $f = \sin(xy) + g(y)$

$f = \sin(xy) + h(z)$

$\hookrightarrow \frac{\partial f}{\partial z} = \frac{dh}{dz} = e^z = h$

$\hookrightarrow$ therefore $f = \sin(xy) + e^z$

- Let's compute.

$$\int_C F(r) \cdot dr = \left(\sin(xy) + e^z\right)\Big|_{(\pi, 1, 0)}^{(0.5, \pi, 1)}$$

$$= \left(\sin(0.5\pi) + e^1\right) - \left(\sin(\pi(1)) + e^0\right)$$

$$= \left(1 + e\right) - \left(1\right)$$

$$\underline{= e}\ //$$

$\boxed{14}$ $F = [-y^3,\ x^3 + e^{-y},\ 0]$, C the circle $x^2 + y^2 = 25$, $z = 2$

$$\int_C F(r) \cdot dr = \int_S \text{curl } F \cdot n \, dA$$: By Stokes's theorem

- $$\text{curl } F = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 + e^{-y} & 0 \end{bmatrix}$$

$$= 0\hat{i} + 0\hat{j} + (3x^2 + 3y^3)\hat{k}$$

- n shall point to the z direction, so

$$\int_C \text{curl } F \cdot n \, dA = \int_C \begin{bmatrix} 0 \\ 0 \\ 3x^2 + 3y^2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} dA$$

$$= \int 3x^2 + 3y^2 \, dA$$

$$\int_C 3x^2+3y^2\,dA = 3\int x^2+y^2\,dA$$ $\quad x^2+y^2=25 \Rightarrow r=5$

$$= 3\int_0^{2\pi}\int_0^{r} 25\,dr\,d\theta$$

$$= 3\int_0^{2\pi} 125\,d\theta$$

$$= 3(250\pi)$$

$$= 750\pi$$

$\boxed{16}$ $F=[x^2, y^2, y^2x]$, C the helix $r=[2\cos(t), 2\sin(t), 3t]$ from $(2,0,0)$ to $(-2,0,3\pi)$

- Let's concider $x(t)=2\cos(t)$, $y(t)=2\sin(t)$ & $z(t)=3t$

  @ $(2,0,0)$, $t=0$
  @ $(-2,0,3\pi)$, $t=\pi$ $\quad$ $=$ t varies from 0 to π

- So,

$$F=\begin{bmatrix} x^2 \\ y^2 \\ y^2x \end{bmatrix} = \begin{bmatrix} (2\cos(t))^2 \\ (2\sin(t))^2 \\ (2\sin(t))^2(2\cos(t)) \end{bmatrix} = \begin{bmatrix} 4\cos^2 t \\ 4\sin^2 t \\ 8\sin^2 t\cos t \end{bmatrix}$$

$r=[2\cos t, 2\sin t, 3t]$

$\hookrightarrow dr = [-2\sin t\,dt, 2\cos t\,dt, 3dt]$

- Therefore: $\quad$:By dot product

$$F\cdot dr = \begin{array}{l}(4\cos^2 t)(-2\sin t\,dt) + \\ (4\sin^2 t)(2\cos t\,dt) + \\ (8\sin^2 t\cos t)(3dt)\end{array}$$

$$= -8\cos^2 t\sin t\,dt + 8\sin^2 t\cos t\,dt + 24\sin^2 t\cos t\,dt$$

$$= (-8\cos^2 t\sin t + 32\sin^2 t\cos t)dt$$

$$\int_C F\cdot dr = \int_{t=0}^{\pi}(-8\cos^2 t\sin t + 32\sin^2 t\cos t)dt$$

$$\int_C F \cdot dr = -8\int_0^{\pi} \cos^2 t \sin t \, dt + 32\int_0^{\pi} \sin^2 t \cos t \, dt$$

$$= -8\left(\frac{2}{3}\right) + 32(0)$$

$$= -\frac{16}{3}$$

[18] $F = [\sin \pi y, \cos \pi x, \sin \pi x]$, C the boundary curve of $0 \leq x \leq 1$, $0 \leq y \leq 2$, $z = x$

: By Stokes's theorem

$$\int_C F \cdot dr = \int_S \text{curl } F \cdot n \, ds$$

- The surface g is given by $g = x - z$, so

$$n = \frac{1}{|\text{grad } g|} \text{grad } g \qquad \text{grad } g = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right]$$

$$= \left[\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right] \qquad = [1, 0, -1]$$

- curl of F is:

$$\text{curl } F = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin \pi y & \cos \pi x & \sin \pi x \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -(\pi \cos \pi x - 0) \\ -\pi \sin \pi x - \pi \cos \pi y \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -\pi \cos \pi x \\ -\pi(\sin \pi x + \cos \pi y) \end{bmatrix}$$

$$\frac{\partial}{\partial x} \cos \pi x - \frac{\partial}{\partial y} \sin \pi y$$

• So curl $F \cdot n$ is:

$$\begin{bmatrix} 0 \\ -\pi \cos \pi x \\ -\pi(\sin \pi x + \cos \pi y) \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \left(-\pi(\sin \pi x + \cos \pi y)\right)\left(-\frac{1}{\sqrt{2}}\right)$$

• From there:

$$\int_C F \cdot dr = \int_S \text{curl}\, F \cdot n\, ds$$

$$= \int_{y=0}^{2} \int_{x=0}^{1} \frac{\pi}{\sqrt{2}} \sin(\pi x) + \cos(\pi y)\, dx\, dy$$

$$= \frac{\pi}{\sqrt{2}} \int_{y=0}^{2} \frac{2}{\pi} + \cos(\pi y)\, dy$$

$$= \frac{\pi}{\sqrt{2}}\left(\frac{4}{\pi}\right)$$

$$= 2\sqrt{2} \,//$$

20 $F = [ze^{xz}, 2\sinh(2y), xe^{xz}]$, C the parabola $y = x$, $z = x^2$, $-1 \leq x \leq 1$

$$\int_C F \cdot dr = \int_S \text{curl}\, F \cdot n\, ds$$: By Stokes's theorem

• Let's calculate curl of F

$$\text{curl}\, F = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{xz} & 2\sinh(2y) & xe^{xz} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial y} xe^{xz} - \frac{\partial}{\partial z} 2\sinh(2y) \\ \frac{\partial}{\partial z} ze^{xz} - \frac{\partial}{\partial x} xe^{xz} \\ \frac{\partial}{\partial x} 2\sinh(2y) - \frac{\partial}{\partial y} ze^{xz} \end{bmatrix}$$

$$\text{curl } F = \begin{bmatrix} 0 \\ e^{xz} - e^{xz} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

↳ therefore $\text{curl } F \cdot n = 0$

↳ $\int_C F \cdot dr = \int_S \text{curl } F \cdot n \, ds = 0$

➤ Double Integrals, Center of Gravity

Find the coordinates X, Y of the center of gravity of a mass of density $f(x,y)$ in the region R. Show the details.

$\boxed{22}$ $f = x^2 + y^2$, $R: x^2 + y^2 \leq a^2$, $y \geq 0$

- The center of gravity is given by:

$$X = \frac{1}{M} \int_R x f(x,y) \, dx dy \qquad \& \qquad M = \int_R f(x,y) \, dx dy$$

$$Y = \frac{1}{M} \int_R y f(x,y) \, dx dy$$

- Let's concider

$f = x^2 + y^2$, $y \geq 0$ and the surface $R: x^2 + y^2 \leq a^2$

with $x = r\cos\theta$ & $y = r\sin\theta$

↳ $y \geq 0$, $0 \leq \theta \leq \pi$

↳ $0 \leq r \leq a$

↳ $dx dy = r dr d\theta$

- Let's calculate the mass

$$M = \int_R (x^2 + y^2) \, dx dy = \int_{\theta=0}^{\pi} \int_{r=0}^{a} r r^2 \, dr d\theta = \int_0^{\pi} \frac{a^4}{4} \, d\theta = \frac{a^4 \pi}{4}$$

- Let's calculate the X coordinate

$$X = \frac{4}{\pi a^4} \int_R x(x^2+y^2)\,dx\,dy$$

$$= \frac{4}{\pi a^4} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \cos\theta \; r^2 \; r \, dr\, d\theta$$

$$= \frac{4}{\pi a^4} \int_0^{\pi} \left(\frac{1}{5} a^6 \cos\theta\right) d\theta$$

$$= \frac{4}{\pi a^4} \; \frac{1}{5} a^5 (0)$$

$$= 0$$

- Let's calculate the Y coordinate

$$Y = \frac{4}{\pi a^4} \int_R y(x^2+y^2)\,dx\,dy$$

$$= \frac{4}{\pi a^4} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r \sin\theta \; r^2 \; r \, dr\, d\theta$$

$$= \frac{4}{\pi a^4} \int_0^{\pi} \left(\frac{1}{5} a^5 \sin\theta\right) d\theta$$

$$= \frac{4}{\pi a^4} \cdot \frac{1}{5} a^5 (2)$$

$$= \frac{8a}{5\pi}$$

- therefore the center of gravity coordinates are:

$$\underline{\left(0, \frac{8a}{5\pi}\right)} \; //$$

Surface Integrals, Divergence theorem

Evaluate the integral directly or, if possible, by the divergence theorem. Show the details.

28. $F = [x+y^2, y+z^2, z+x^2]$, S the ellipsoid with semi-axes of length a, b, c

$$\int_S F \cdot n \, dA = \int_T \operatorname{div} F \, dT$$: By divergence theorem

- let's compute div F

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x+y^2) + \frac{\partial}{\partial y}(y+z^2) + \frac{\partial}{\partial z}(z+x^2)$$

$$= 1 + 1 + 1$$

$$= 3$$

- The surface is the ellipsoid

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, where x varies from 0 to a, y varies from 0 to b & z varies from 0 to c

- So:

$$\int_S F \cdot n \, dA = \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} 3 \, dx \, dy \, dz$$

$$= \underline{3abc} \text{ //}$$

$\boxed{30}$ $F = [1, 1, 1]$, $S: x^2 + y^2 + 4z^2 = 4$, $z \geq 0$

$\int_S F \cdot n \, dA = \int_T \operatorname{div} F \, dT$:By divergence theorem

- Let's compute div F

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad / \; F_1 = F_2 = F_3 = 1$$

$$= 0$$

$$\hookrightarrow \int_S F \cdot n \, dA = \int_T 0 \, dT = \underline{0} \,/\!/$$

$\boxed{32}$ $F = [y^2, x^2, z^2]$, S the portion of the paraboloid $z = x^2 + y^2$, $z \leq 9$

$\int_S F \cdot n \, dA = \int_R F \cdot N \, du \, dv$:By direct evalvation

- Let's concider $x = u\cos v$, $y = u\sin v$

$\hookrightarrow z = u^2$

$\hookrightarrow$ u varies from 0 to 3
v varies from 0 to 2π

- So:

$r = [u\cos v, u\sin v, u^2]$

$\hookrightarrow r_u = [\cos v, \sin v, 2u]$
$r_v = [-u\sin v, u\cos v, 0]$

- From there

$$N = r_u \times r_v$$
$$= \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 2u \\ -u\sin v & u\cos v & 0 \end{bmatrix} = \begin{bmatrix} -2u^2\cos v \\ -2u^2\sin v \\ u \end{bmatrix}$$

- $F = [y^2, x^2, z^2]$

$= [u^2 \sin^2 v,\ u^2 \cos^2 v,\ u^4]$

- So:

$$F \cdot N = \begin{bmatrix} u^2 \sin^2 v \\ u^2 \cos^2 v \\ u^4 \end{bmatrix} \cdot \begin{bmatrix} -2u^2 \cos v \\ -2u^2 \sin v \\ u \end{bmatrix}$$

$$= \begin{array}{l} -2u^4 \sin^2 v \cos v \\ -2u^4 \cos^2 v \sin v \\ +u^5 \end{array}$$

- Let's evaluate

$$\int_S F \cdot n\, dA = \int_{v=0}^{2\pi} \int_{u=0}^{3} (-2u^4 \sin^2 v \cos v - 2u^2 \cos^2 v \sin v + u^5)\, du\, dv$$

$$= \int_0^{2\pi} \frac{243}{2} + \frac{243}{5}(-2\cos^2 v \sin v - 2\cos v \sin^2 v)\, dv$$

$$\underline{= 243\pi/4}$$

$\boxed{34}$ $F = [x, xy, z]$, S the boundary of $x^2 + y^2 \leq 1$, $0 \leq z \leq 5$

- Let's go polar : By divergence theorem

$$\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \Rightarrow \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta \leq 2\pi \end{array}$$

$\hookrightarrow dx\,dy\,dz = r\,dr\,d\theta\,dz$

- From the Theorem

$$\int_S F \cdot n\, dA = \int_T \operatorname{div} F\, dT$$

- Let's compute div F

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} xy + \frac{\partial}{\partial z} z$$

$$= 1 + x + 1$$

$$= 2 + x$$

$$= 2 + r\cos\theta$$

- So:

$$\int_S F \cdot n \, dA = \int_{z=0}^{5} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} (2 + r\cos\theta) r \, dr \, d\theta \, dz$$

$$= \int_0^5 \int_0^{2\pi} \left(1 + \frac{\cos\theta}{3}\right) d\theta \, dz$$

$$= \int_0^5 2\pi \, dz$$

$$= \underline{10\pi} //$$