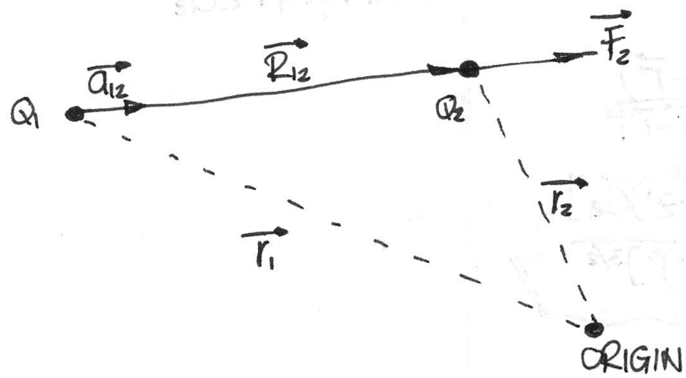
Coulomb's Law and Electric Field Intensity

$$F = k\frac{Q_1 Q_2}{R^2}$$

$$k = \frac{1}{4\pi\varepsilon_0}\ ,\ \varepsilon_0 = 8.854\times10^{-12} = \frac{1}{36\pi}10^{-9}\ F/m$$

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$$



$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$$

$$\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{12}$$

$$\vec{a}_{12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|} = \frac{\vec{R}_{12}}{R_{12}} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\vec{F}_1 = -\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{21} = -\frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{12}$$

* Electric Field Intensity

$$\vec{F}_t = \frac{Q_1 Q_t}{4\pi\varepsilon_0 R_{1t}^2}\vec{a}_{1t}$$

$$\frac{\vec{F}_t}{Q_t} = \frac{Q_1}{4\pi\varepsilon_0 R_{1t}^2}\vec{a}_{1t}$$

$$\vec{E} = \frac{\vec{F}_t}{Q_t} = \frac{Q_1}{4\pi\varepsilon_0 R_{1t}^2}\vec{a}_{1t}$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a_R}$$

For a charge Q_1 located at the origin of a spherical coordinate system

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 r^2}\vec{a_r} \quad \text{or} \quad E_r = \frac{Q_1}{4\pi\varepsilon_0 r^2}$$

For a charge Q located at the source point $\vec{r'} = x'\vec{a_x} + y'\vec{a_y} + z'\vec{a_z}$

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 |\vec{r}-\vec{r'}|^2}\frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|} = \frac{Q(\vec{r}-\vec{r'})}{4\pi\varepsilon_0 |\vec{r}-\vec{r'}|^3}$$

$$= \frac{Q[(x-x')\vec{a_x} + (y-y')\vec{a_y} + (z-z')\vec{a_z}]}{4\pi\varepsilon_0 [(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$$

For more than one point charge

$$\vec{E}(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r}-\vec{r_1}|^2}\vec{a_1} + \frac{Q_2}{4\pi\varepsilon_0 |\vec{r}-\vec{r_2}|^2}\vec{a_2} + \dots + \frac{Q_n}{4\pi\varepsilon_0 |\vec{r}-\vec{r_n}|^2}\vec{a_n}$$

$$\vec{E}(\vec{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0 |\vec{r}-\vec{r_m}|^2}\vec{a_m}$$

* Field due to a continuous volume charge distribution

$$\Delta Q = \rho_v \Delta v$$

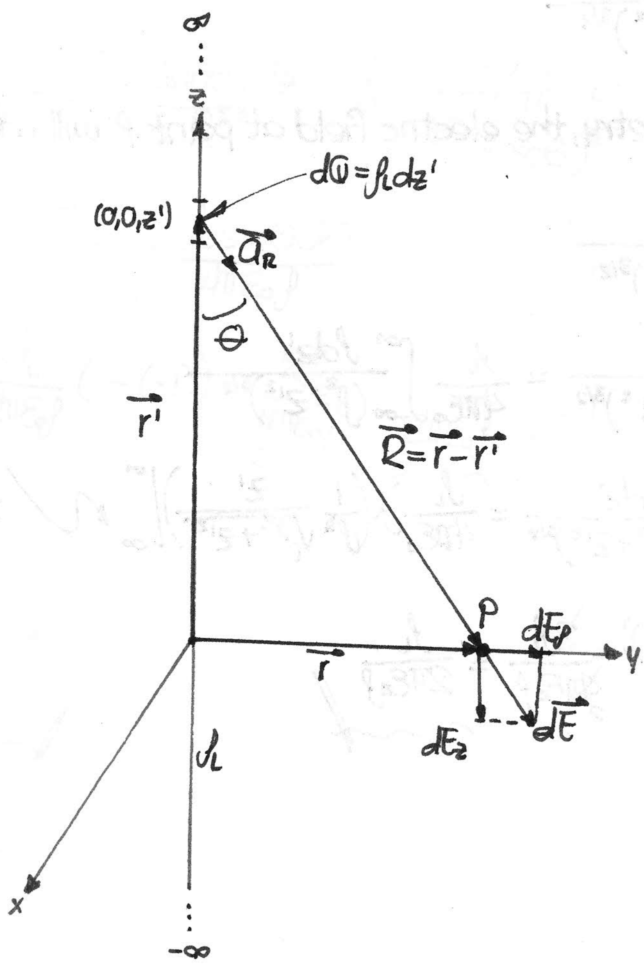
$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}$$

$$Q = \int_{vol} \rho_v dv$$

$$\Delta\vec{E}(\vec{r}) = \frac{\Delta Q}{4\pi\epsilon_0|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} = \frac{\rho_v\Delta v}{4\pi\epsilon_0|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$$\vec{E}(\vec{r}) = \int_{vol}\frac{\rho_v(\vec{r}')dv'}{4\pi\epsilon_0|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|}$$

* Field of a line charge



$$d\vec{E} = \frac{\rho_L dz'(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3}$$

$$\vec{r} = y\vec{a_y} = \rho\vec{a_\rho}$$
$$\vec{r}' = z'\vec{a_z}$$
$$\vec{R} = \vec{r}-\vec{r}' = \rho\vec{a_\rho} - z'\vec{a_z}$$

$$d\vec{E} = \frac{\rho_L dz'(\rho\vec{a_\rho} - z'\vec{a_z})}{4\pi\varepsilon_0(\rho^2+z'^2)^{3/2}}$$

because of symmetry, the electric field at point P will not have a z component

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\varepsilon_0(\rho^2+z'^2)^{3/2}}$$

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\varepsilon_0(\rho^2+z'^2)^{3/2}} = \frac{\rho_L}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{\rho dz'}{(\rho^2+z'^2)^{3/2}}$$

$$= \frac{\rho_L}{4\pi\varepsilon_0}\rho\int_{-\infty}^{\infty}\frac{dz'}{(\rho^2+z'^2)^{3/2}} = \frac{\rho_L}{4\pi\varepsilon_0}\rho\left(\frac{1}{\rho^2}\frac{z'}{\sqrt{\rho^2+z'^2}}\right)\Bigg|_{-\infty}^{\infty}$$

Using integral table

$$= \frac{\rho_L}{4\pi\varepsilon_0\rho}(1-(-1)) = \frac{\cancel{2}\rho_L}{\cancel{4}_2\pi\varepsilon_0\rho} = \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

$\tan\theta = \frac{\rho}{z'}$

$\longrightarrow z' = \rho\cot\theta$

$\longrightarrow dz' = -\rho\csc^2\theta\, d\theta$

$\sin\theta = \frac{\rho}{R}$

$\longrightarrow R = \rho\csc\theta$

$\longrightarrow \rho = R\sin\theta$

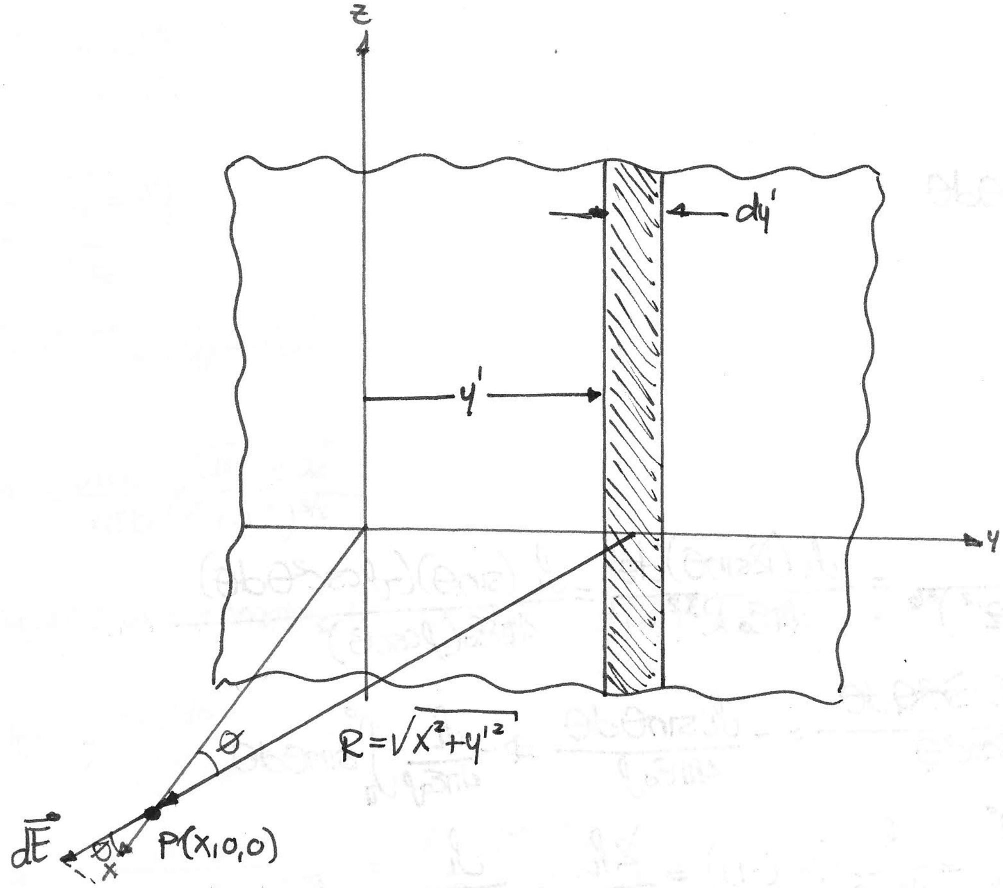
$$dE_\rho = \frac{\rho_L \rho\, dz'}{4\pi\varepsilon_0(\rho^2+z'^2)^{3/2}} = \frac{\rho_L (R\sin\theta)\, dz'}{4\pi\varepsilon_0 R^{3/2}} = \frac{\rho_L(\sin\theta)(-\rho\csc^2\theta\, d\theta)}{4\pi\varepsilon_0(\rho\csc\theta)^2}$$

$$= -\frac{\rho_L \cancel{\rho}\sin\theta\,\cancel{\csc^2\theta}\, d\theta}{4\pi\varepsilon_0\rho^{\cancel{2}}\cancel{\csc^2\theta}} = -\frac{\rho_L\sin\theta\, d\theta}{4\pi\varepsilon_0\rho} \Rightarrow -\frac{\rho_L}{4\pi\varepsilon_0\rho}\int_\pi^0 \sin\theta\, d\theta$$

$$= \frac{\rho_L\cos\theta}{4\pi\varepsilon_0\rho}\Big|_\pi^0 = \frac{\rho_L}{4\pi\varepsilon_0\rho}(1-(-1)) = \frac{\cancel{2}\rho_L}{\cancel{4}_2\pi\varepsilon_0\rho} = \frac{\rho_L}{2\pi\varepsilon_0\rho} = E_\rho$$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a_\rho}$$

* Field of a sheet of charge



For a line of charge

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho} \vec{a_\rho}$$

In our present case $\rho_L = \rho_s dy'$

$$dE_x = \frac{\rho_s dy'}{2\pi\epsilon_0 \sqrt{x^2+y'^2}} \cos\theta$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y'^2}}$$

$$dE_x = \frac{\rho_s}{2\pi\epsilon_0} \frac{x\,dy'}{x^2+y'^2}$$

$$E_x = \int dE_x = \frac{\rho_s}{2\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{x\,dy'}{x^2+y'^2} = \frac{\rho_s}{2\pi\epsilon_0} x \int_{-\infty}^{\infty} \frac{dy'}{x^2+y'^2} = \frac{\rho_s}{2\pi\epsilon_0} \cancel{x} \left(\frac{1}{\cancel{x}}\right) \tan^{-1}\frac{y'}{x}\Big|_{-\infty}^{\infty}$$

$$= \frac{\rho_s}{2\pi\epsilon_0} \tan^{-1}\frac{y'}{x}\Big|_{-\infty}^{\infty} = \frac{\rho_s}{2\pi\epsilon_0}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\rho_s}{2\cancel{\pi}\epsilon_0}(\cancel{\pi}) = \frac{\rho_s}{2\epsilon_0}$$

$$\vec{E} = \frac{\rho_s}{2\epsilon_0} \vec{a_N}$$