Chapter 5 Solutions

① $X \sim Poi(\mu)$, $\pi(\mu) \sim Ga(\nu, 1)$

a) Before getting marginal of X, let's get the joint density for X, μ.

$$f(x, \mu \mid \nu) = f_X(\mu) \cdot \pi(\mu)$$

(using conditional probability)

$$= \mu^x \cdot e^{-\mu} \quad \mu^{\nu-1} \cdot e^{-\mu/1}$$

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$$f(x, \mu \mid \nu) = f_x(\mu) \cdot \pi(\mu)$$

(using conditional probability)

$$= \frac{\mu^x \cdot e^{-\mu}}{x!} \cdot \frac{\mu^{\nu-1} \cdot e^{-\mu/1}}{1^\nu \cdot \Gamma(\nu)}$$

(Recall $\Gamma(y+1) = y!$)

$$= \frac{\mu^{(\nu+x)-1} \cdot e^{-2\mu}}{\Gamma(x-1) \cdot \Gamma(\nu)}.$$

Now, get marginal density $f(x)$ by integrating

over μ. ∞

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Now, get <u>marginal density</u> $f(x)$ by integrating over μ.

$$f(x \mid \nu) = \frac{1}{\Gamma(x-1)} \cdot \int_0^\infty \frac{\mu^{(\nu+x)-1} \cdot e^{-\mu/(1/2)}}{\Gamma(\nu)} \, d\mu$$

$$f(x \mid \nu) = \frac{(1/2)^{\nu+x}}{\Gamma(x-1)} \cdot \underbrace{\int_0^\infty \frac{\mu^{(\nu+x)-1} \cdot e^{-\mu/(1/2)}}{(1/2)^{\nu+x} \cdot \Gamma(\nu)} \, d\mu}_{1}$$

Gamma Density

$Ga(\nu + x, 1/2)$

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Gamma Density

$Ga(v+x, 1/2)$

$$\Rightarrow \boxed{f(x|v) = \frac{(1/2)^{v+x}}{\Gamma(x-1)}}$$

b) Using Bayes Rule

$$f(\mu|x,v) = \frac{f(\mu,x|v)}{f(x|v)}$$

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(2) $X \sim F_{\nu_1,\nu_2}$ iff

$$X \sim \frac{\nu_2}{\nu_1} \cdot \frac{Ga(\nu_1, \sigma)}{Ga(\nu_2, \sigma)}$$

while $X \sim Be_{\nu_1,\nu_2}$ iff

$$X \sim \frac{Ga(\nu_1, \sigma)}{Ga(\nu_1, \sigma) + Ga(\nu_2, \sigma)}$$

Then, working with the densities and

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Then, working with the densities and without loss of generality...

$$f_F(v_1, v_2) = \frac{v_2}{v_1} \cdot \frac{f_{Ga}(v_1, \sigma)}{f_{Ga}(v_2, \sigma)}$$

$$\Rightarrow \underbrace{\frac{v_1}{v_2}} \cdot f_F(v_1, v_2) = \frac{f_{Ga}(v_1, \sigma)}{f_{Ga}(v_2, \sigma)}$$

we can multiply this term, as it is just a constant.

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Shorthanding $f_{Ga}(V_1, \sigma)$ and $f_{Ga}(V_2, \sigma)$ by $g1$ and $g2$

$$\frac{V_1}{V_2} \cdot f_F(V_1, V_2) + f_F(V_1, V_1)$$

$$= \frac{g1}{g_2} + \frac{g_1}{g1} = \frac{g_1^2 + g_1 g_2}{g_1 \cdot g_2} = \frac{\cancel{g_1}(g_1 + g_2)}{\cancel{g_1} \cdot g_2}$$

$$= \frac{(g_1 + g_2)}{g_2}$$

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Thus $\left[\frac{v_1}{v_2} \cdot f_F(v_1, v_2) + f_F(v_1, v_1)\right]^{-1} = \frac{g_2}{g_1 + g_2}$

which is a $Be(v_1, v_2)$ density.

we conclude.

$\underline{X \sim Be_{v_1, v_2}}$ iff

$$X \sim \frac{1}{\frac{v_1}{v_2} \cdot F_{v_1, v_2} + F_{v_1, v_1}} \quad \blacksquare$$

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V2

(3) See Jupyter Notebook.

(4) $x \sim N_p(\mu, \Sigma)$

We start with <u>5.26</u>

$$I_\mu = E_\mu\left\{ \dot{l}_x(\mu) \cdot \dot{l}_x(\mu)^T \right\}$$

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First,

$$f_\mu(\underline{x}) = \frac{(2\pi)^{-p/2}}{|\Sigma|^{1/2}} \cdot e^{-\frac{1}{2}(x-\mu)^T \cdot \Sigma \cdot (x-\mu)}$$

(pdf of MVNormal)

then

$$\log f_\mu(\underline{x}) = -\frac{p}{2} \cdot \log(2\pi) - \frac{1}{2}\log(|\Sigma|)$$

$$- \frac{1}{2}(x-\mu)^T \cdot \Sigma^{-1} \cdot (x-\mu)$$

- We know that $\dot{\ell}_x(\mu)$ is defined as a <u>gradient</u> in this multidimensional context.

... the way in which we got $\underline{\ddot{\ell}_x(\mu)}$ to

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- We know ... gradient in this multidimensional context.

- Analogously to the way in which we got $\ddot{\ell}_x(\mu)$ to more easily compute the variance of $\dot{\ell}_x(\mu)$, we can compute the Hessian (matrix of second derivatives) to more easily compute

$$E\left[\dot{\ell}_x(\mu)\,\dot{\ell}_x(\mu)\right] = I_\mu$$

- Thus, let's calculate generic element $I_\mu(ij)$.

One key aspect to bear in mind is that only

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Thus, let's calculate generic element $I_\mu(ij)$.

One key aspect to bear in mind is that only terms containing μ_i and μ_j will be relevant to our computation of $\frac{\partial}{\partial \mu_i \mu_j} \log f_\mu(\underline{x})$

Thus

$$\frac{\partial}{\partial \mu_i \mu_j} \log f_\mu(\underline{x}) = \frac{\partial}{\partial \mu_i \mu_j} \left\{ -\frac{1}{2} \cdot \left[(x_i - \mu_i) \cdot \Sigma^{-1}_{ij} (x_j - \mu_j) + (x_j - \mu_j) \cdot \Sigma^{-1}_{ji} (x_i - \mu_i) \right] \right\}$$

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Since $\Sigma^{-1}_{ij} = \Sigma^{-1}_{ji}$ (Covariance matrices are symmetrical)

$$\frac{\partial}{\partial \mu_i \mu_j} \log f_\mu(\underline{x}) = \frac{\partial}{\partial \mu_i \mu_j} -(x_i - \mu_i) \cdot \Sigma^{-1}_{ij} \cdot (x_j - \mu_j)$$

$$= -\Sigma^{-1}_{ij}$$

Now, taking $-E_\mu\left[\frac{\partial}{\partial \mu_i \mu_j} \log f_\mu(\underline{x})\right] = -E_\mu\left[-\Sigma^{-1}_{ij}\right]$

Thus $\underline{I_\mu(ij)} = -E_\mu\left[-\Sigma^{-1}_{ij}\right] = \underline{\Sigma^{-1}_{ij}}$

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Thus $\underline{I_{\mu}(ij)} = -E_{\mu}\left[-\Sigma^{-1}_{ij}\right] = \underline{\Sigma^{-1}_{ij}}$

and $\underline{I_{\mu} = \Sigma^{-1}}$.

$n \times p$ - dimensional matrix X.

- Under repeated sampling, following our same procedure.

$$\frac{\partial}{\partial \mu_i \mu_j} \log f_{\mu}(X) = \frac{\partial}{\partial \mu_i \mu_j} - \sum_{s=1}^{n} (x_{is} - \mu_i) \cdot \Sigma^{-1}_{ij} (x_{js} - \mu_j)$$

sample #, our usual "i"

$$n \cdot \Sigma^{-1}_{ii}$$

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sample #, our usual "i"

$$= -n \cdot \Sigma_{ij}^{-1}$$

and $I_{\mu} = n \cdot \Sigma^{-1}$

For large n, we can connect this result with 5.27

and say $\hat{\mu}^{MLE} \sim N(\mu, (n \cdot \Sigma^{-1})^{-1})$

$$\sim N(\mu, \Sigma / n)$$

[illegible] is a generalization of the

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$$= -n \cdot \Sigma^{-1}$$

and $I_{\mu\mu} = n \cdot \Sigma^{-1}$

For <u>large n</u>, we can connect this result with 5.27

and say $\hat{\mu}^{MLE} \sim N(\mu, (n \cdot \Sigma^{-1})^{-1})$

$$\sim N(\mu, \Sigma / n)$$

which is a generalization of the <u>univariate</u> result.

3 outcomes 5 samples

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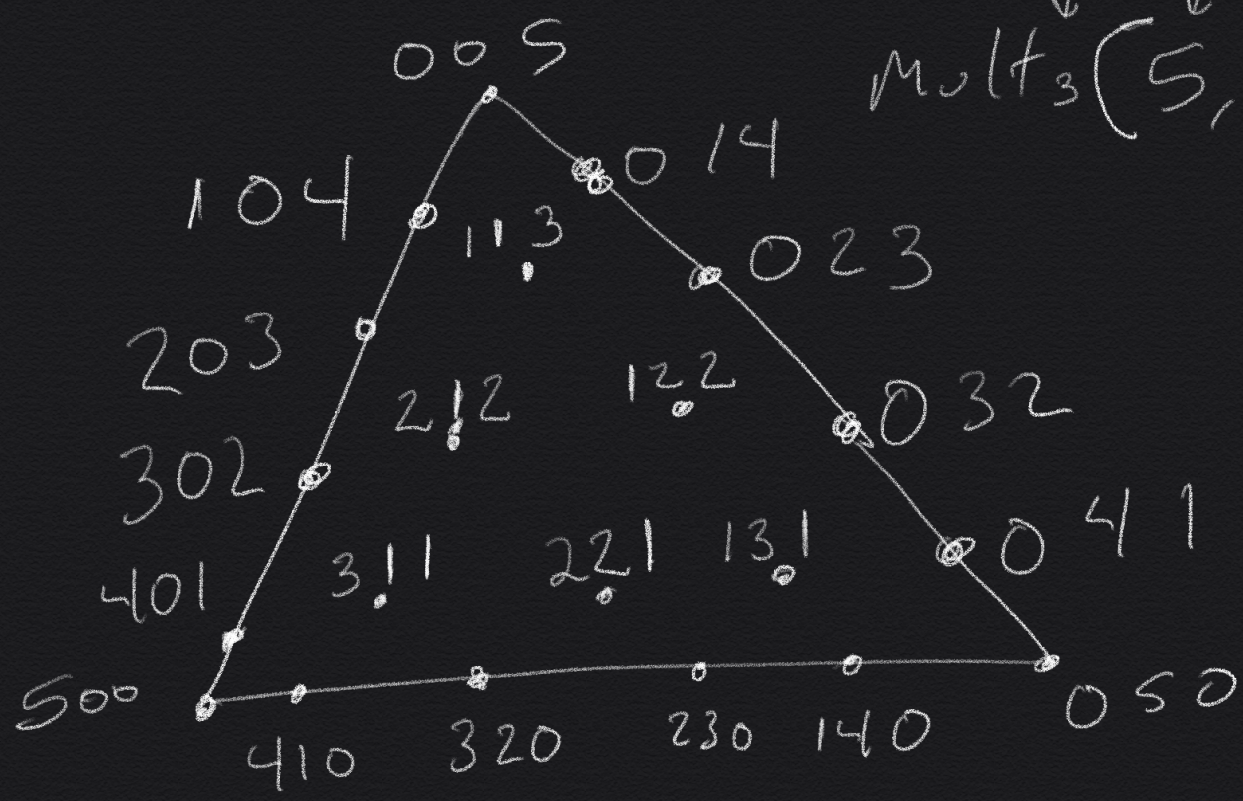
univariate result.

3 outcomes 5 samples

$\text{Mult}_3(5, \underline{\pi})$

5



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6 $X \sim Mult_L(n, \underline{\pi})$

Using $Mult_L(N, \underline{\pi}) \sim Poi(n\underline{\pi})$ Approximation.

Want <u>mean</u> & variance of X_1/X_2

$$\frac{X_1}{X_2} \approx \frac{n \cdot \pi_1}{n \cdot \pi_2} \cdot \left(1 - \frac{X_1 - n \cdot \pi_1}{n \cdot \pi_1} - \frac{X_2 - n \cdot \pi_2}{n \cdot \pi_2}\right)$$

$E(X_1) =$

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$$E\left(\frac{X_1}{X_2}\right) =$$

$$E\left[\frac{n \cdot \pi_1}{n \cdot \pi_2} \cdot \left(1 - \frac{X_1 - n \cdot \pi_1}{n \cdot \pi_1} - \frac{X_2 - n\pi_2}{n \cdot \pi_2}\right)\right]$$

$$= \frac{n \cdot \pi_1}{n \cdot \pi_2} \cdot \left[E(1) - \cancel{E\left[\frac{X_1 - n\pi_1}{n \cdot \pi_1}\right]}^{0} - \cancel{E\left[\frac{X_2 - n\pi_2}{n\pi_2}\right]}^{0}\right]$$

$$= \underline{\pi_1 / \pi_2}$$

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Constants

Variance

$$V\left(\frac{x_1}{x_2}\right) = \frac{n^2 \cdot \pi_1^2}{n^2 \cdot \pi_2^2} \cdot V\left(1 - \frac{x_1 - n\pi_1}{n \cdot \pi_1} - \frac{x_2 - n\pi_2}{n \cdot \pi_2}\right)$$

$$= \frac{\pi_1^2}{\pi_2^2} \cdot \cancel{(-1)^2} \cdot V\left(\frac{\pi_2 \cdot x_1 + \pi_1 \cdot x_2 + C}{n \cdot \pi_1 \pi_2}\right)$$

$$= \frac{\pi_1^2}{\pi_2^2} \cdot \frac{1}{n^2 \cdot \pi_1^2} \cdot V\left(\pi_2 \cdot x_1 + \pi_1 \cdot x_2\right)$$

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$$= \frac{\pi_1^2}{\pi_2^2} \cdot \frac{1}{(n \cdot \pi_1 \pi_2)^2} \cdot V\left(\pi_2 \cdot X_1 + \pi_1 \cdot X_2\right)$$

- Since, <u>given π</u>, X_1 and X_2 are independent.

$$V\left(\frac{X_1}{X_2}\right) = \frac{\cancel{\pi_1^2}}{n^2 \cancel{\pi_1^2} \cdot \pi_2^4} \left[\pi_2^2 \cdot V(X_1) + \pi_1^2 \cdot V(X_2)\right]$$

Using the Variance of a Poisson r.v. $X \sim Poi(\mu)$

$\Rightarrow$ <u>$V(X) = \mu$</u>

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Using the Variance of a Poisson r.v. $X \sim Poi(\mu)$

$\Rightarrow \underline{V(X) = \mu}$

$\Rightarrow$

$$\underline{V\left(\frac{X_1}{X_2}\right)} = \frac{1}{n^2 \cdot \pi_2^4} \cdot \left[\pi_2^2 \cdot n \cdot \pi_1 + \pi_1^2 \cdot n \cdot \pi_2\right]$$

$$= \underline{\frac{\pi_1}{n\pi_2^2} + \frac{\pi_1^2}{n \cdot \pi_2^3}}.$$

7 We start with the <u>Binomial</u> density

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7

We start with the Binomial density

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Now, we write

$$f_1(x) = \binom{12}{x} (p_1)^x (1-p_1)^{n-x} \quad \text{and}$$

$$f_2(x) = \binom{12}{x} (p_2)^x (1-p_2)^{n-x}$$

Let's now rewrite them in terms of their logits.

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Let's now rewrite them in terms of their logits..

$$f_1(x) = \binom{12}{x} \cdot \exp\left\{ x \log(p_1) + (n-x)\log(1-p_1) \right\}$$

$$= \binom{12}{x} \cdot \exp\left\{ x \cdot \left[\log(p_1) - \log(1-p_1)\right] + n \cdot \log(1-p_1) \right\}$$

$$= \binom{12}{x} \cdot \exp\left\{ x \cdot \log\left(\frac{p_1}{1-p_1}\right) + n \log(1-p_1) \right\}$$

Likewise

$$f_2(x) = \binom{12}{x} \cdot \exp\left\{ x \cdot \log\left(\frac{p_2}{1-p_2}\right) + n \log(1-p_2) \right\}$$

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Then $$\frac{f_1(x)}{f_2(x)} = \exp\left\{x \cdot \left[\log\left(\frac{p_1}{1-p_1}\right) - \log\left(\frac{p_2}{1-p_2}\right)\right] + n \cdot \left[\log(1-p_1) - \log(1-p_2)\right]\right\}$$

$$= \exp\left\{x \cdot \left[\log\left(\frac{p_1(1-p_2)}{p_2(1-p_1)}\right)\right] + n \cdot \left[\log\left(\frac{1-p_1}{1-p_2}\right)\right]\right\}$$

So we let $\alpha = \log\left(\frac{p_1(1-p_2)}{p_2(1-p_1)}\right)$

$$\psi(\alpha) = -n \log\left(\frac{1-p_1}{1-p_2}\right)$$

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So we let $\alpha = \log\left(\frac{p_1(1-p_2)}{p_2(1-p_1)}\right)$

$$\psi(\alpha) = -n \log\left(\frac{1-p_1}{1-p_2}\right)$$

$$\Rightarrow \frac{f_1(x)}{f_2(x)} = e^{\alpha \cdot x - \psi(\alpha)}$$

$$f_1(x) = e^{\alpha \cdot x - \psi(\alpha)} \cdot f_2(x)$$

To solve example just let $p_1 = 0.3$

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$$\psi(\alpha) = -n \log\left(\frac{\quad}{1-p_2}\right)$$

$$\Rightarrow \frac{f_1(x)}{f_2(x)} = e^{\alpha \cdot x - \psi(\alpha)}$$

$$f_1(x) = e^{\alpha \cdot x - \psi(\alpha)} \cdot f_2(x)$$

To solve example just let $p_1 = 0.3$
$p_2 = 0.6$