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- Conductors, dielectrics, and capacitance

*Current and current density

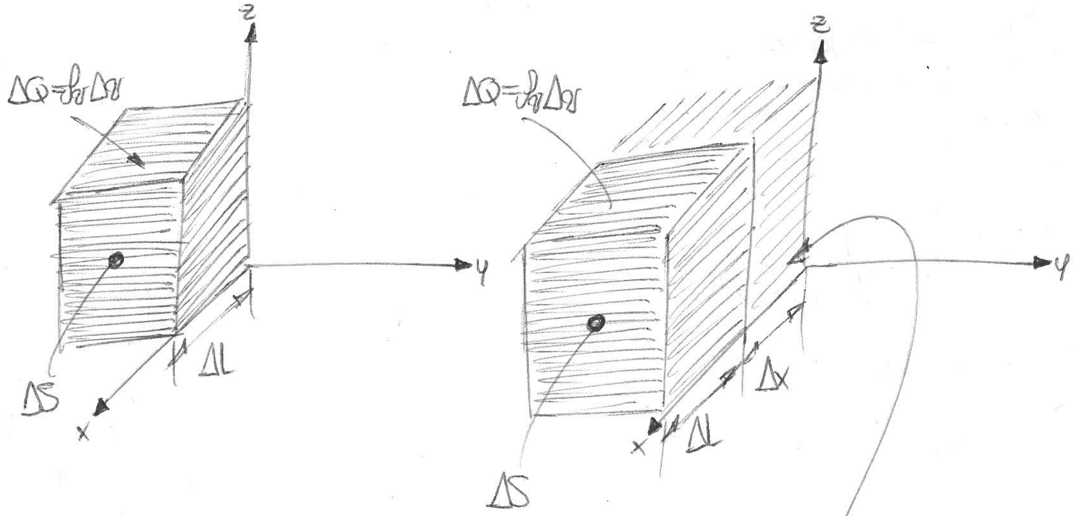
$I = \frac{dQ}{dt}$ [A]

$\vec{J}$ [A/m^2] current density

$\Delta I = J_N \Delta S = \vec{J} \cdot \vec{\Delta S}$

↳ Normal current density

$I = \int_S \vec{J} \cdot \vec{dS}$



total element of charge
$\Delta Q = \rho_v \Delta v = \rho_v \Delta S \Delta L$

in time Δt
$\Delta Q = \rho_v \Delta S \Delta x$

$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \Delta S \frac{\Delta x}{\Delta t} = \rho_v \Delta S v_x$

$J_x = \frac{\Delta I}{\Delta S} = \rho_v v_x$

$\underbrace{\vec{J} = \rho_v \vec{v}}$ //

↳ convection current density

*Continuity of current

Previously

$$I = \int_s \vec{J} \cdot d\vec{S}$$

However, in a region bounded by a closed surface

$$I = \oint_s \vec{J} \cdot d\vec{S} = -\frac{dQ_i}{dt}$$ // outward flowing current

Integral form of the continuity equation

$$\oint_s \vec{A} \cdot d\vec{S} = \int_{vol} (\vec{\nabla} \cdot \vec{A}) dv$$ — divergence theorem

$$\therefore \oint_s \vec{J} \cdot d\vec{S} = \int_{vol} (\vec{\nabla} \cdot \vec{J}) dv$$

$$I = \int_{vol} (\vec{\nabla} \cdot \vec{J}) dv = -\frac{dQ_i}{dt} = -\frac{d}{dt} \int_{vol} \rho_v dv$$

if we keep the surface constant

$$-\frac{d}{dt} \int_{vol} \rho_v dv = \int_{vol} -\frac{\partial \rho_v}{\partial t} dv$$

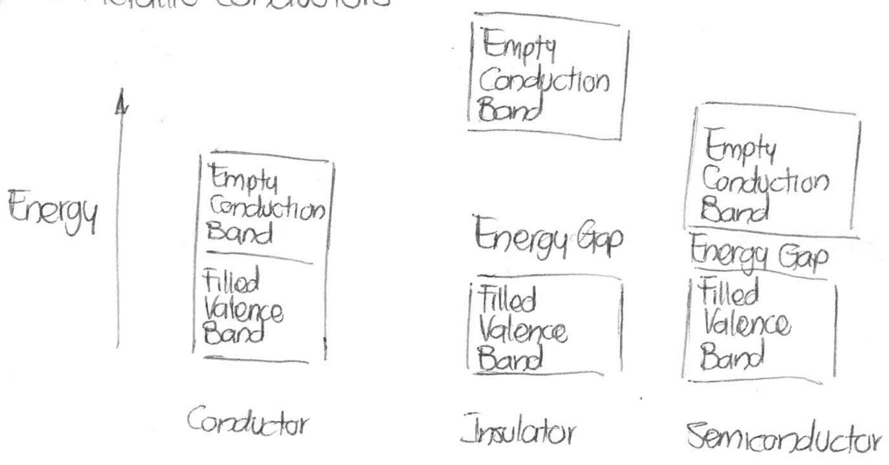
$$\therefore (\vec{\nabla} \cdot \vec{J}) \Delta v = -\frac{\partial \rho_v}{\partial t} \Delta v$$

$$(\vec{\nabla} \cdot \vec{J}) = -\frac{\partial \rho_v}{\partial t}$$ //

Point form of the continuity equation

this equation indicates that the current, or charge per second, diverging from a small volume per unit volume is equal to the time rate of decrease of charge per unit volume at every point.

* Metallic Conductors



For an electron

$Q = -e$

$\vec{F}_e = -e\vec{E}$

$\vec{V}_d = -\mu_e \vec{E}$

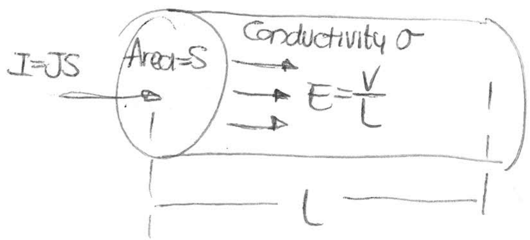
mobility of an electron (positive by definition)

drift velocity

$\vec{J} = \rho_e \vec{V} = -\rho_e \mu_e \vec{E} = \sigma \vec{E}$

$\therefore \sigma = -\rho_e \mu_e$

$\vec{J} = \sigma \vec{E}$



Assuming that $\vec{J}$ and $\vec{E}$ are uniform

$I = \int_S \vec{J} \cdot d\vec{S} = JS \Rightarrow J = \frac{I}{S}$

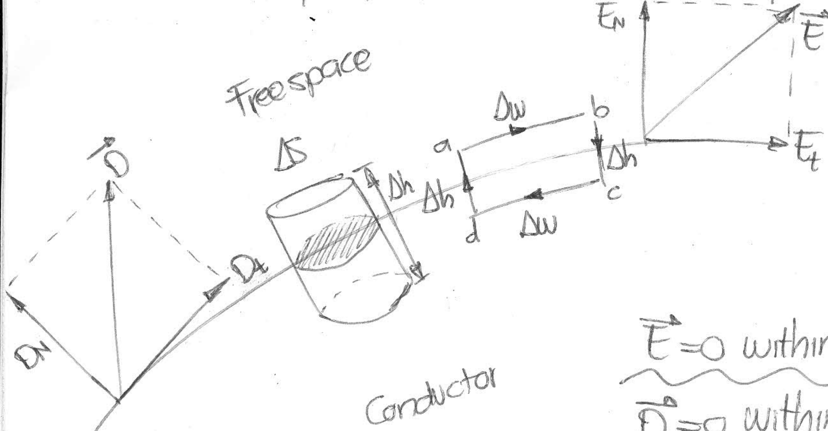
$V_{ab} = -\int_b^a \vec{E} \cdot d\vec{L} = -\vec{E} \cdot \int_b^a d\vec{L} = -\vec{E} \cdot \vec{L}_{ba} = \vec{E} \cdot \vec{L}_{ab}$

$V = EL \Rightarrow E = \frac{V}{L}$

$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L} \Rightarrow V = \frac{L}{\sigma S} I \; , \; \frac{L}{\sigma S} = R$

$\therefore V = IR$ Ohm's law

* Conductor properties and boundary conditions



Electrostatic

$\vec{E}=0$ within the conductor

$\vec{D}=0$ within the conductor

$$\oint \vec{E}\cdot d\vec{l}=0$$

$$\int_a^b+\int_b^c+\int_c^d+\int_d^a=0$$

$$E_t\Delta w-\frac{1}{2}E_{N_{atb}}\Delta h+0\left(\tfrac{1}{2}\Delta h\right)+0(\Delta w)+0\left(\tfrac{1}{2}\Delta h\right)+\frac{1}{2}E_{N_{ata}}\Delta h=0$$

$$\lim_{\Delta h\to 0}\left(E_t\Delta w-\frac{1}{2}E_{N_{atb}}\Delta h+\frac{1}{2}E_{N_{ata}}\Delta h\right)=E_t\Delta w=0$$

$\therefore E_t=0$ // and $D_t=0$ //

to calculate the tangential component of the field

$$\oint_S \vec{D}\cdot d\vec{S}=Q$$

$$\int_{top}+\int_{bottom}+\int_{side}=Q$$

$$\int_{top}D_N dS+0\int_{bottom}dS+0\int_{side}dS=Q$$

$$D_N\cancel{\Delta S}=Q=\rho_S\cancel{\Delta S}$$

$D_N=\rho_S$ // and $E_N=\rho_S/\epsilon_0$ //

to calculate the normal component of the field

because $E_t=0$ $\vec{E}\cdot d\vec{l}=0$ between any two points on the surface

1) The static electric field inside a conductor is zero
2) The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface
3) The conductor surface is an equipotential surface

* Semiconductors

in metallic conductors

$$\sigma = -\rho_e \mu_e$$

in semiconductors

$$\sigma = -\rho_e \mu_e + \rho_h \mu_h$$

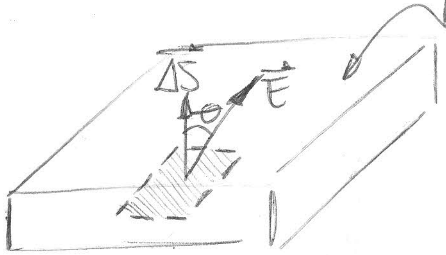
* The nature of dielectric materials

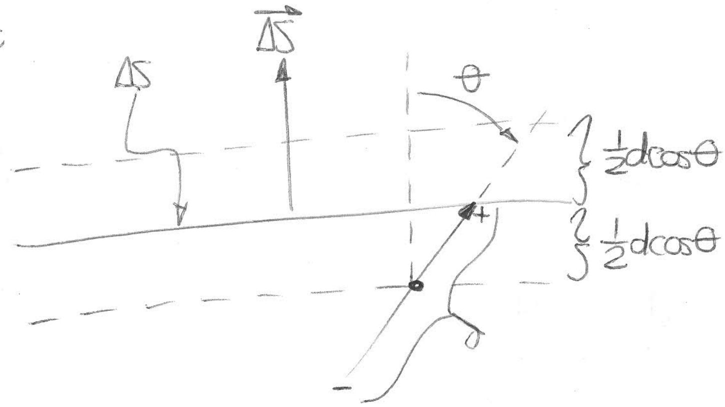
$$\vec{p} = Q\vec{d} \qquad [C \cdot m]$$

if there are n dipoles per unit volume and we deal with a volume Δv

$$\vec{p}_{total} = \sum_{i=1}^{n\Delta v} \vec{p}_i \qquad [C \cdot m]$$

$$\vec{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \vec{p}_i \qquad [C/m^2]$$





$$\Delta Q_b = nQ\vec{d} \cdot \vec{\Delta S} = \vec{P} \cdot \vec{\Delta S}$$

$Q_b = -\oint_S \vec{P} \cdot \vec{dS}$ (the dot product is gonna be negative)

$Q_T = \oint_S \varepsilon_0 \vec{E} \cdot \vec{dS} = \oint \vec{D} \cdot \vec{dS}$ as previously defined

However... $Q_T = Q_b + Q \Rightarrow Q = Q_T - Q_b = \oint_S \varepsilon_0 \vec{E} \cdot \vec{dS} + \oint_S \vec{P} \cdot \vec{dS} = \oint_S (\varepsilon_0 \vec{E} + \vec{P}) \cdot \vec{dS}$

$$\therefore \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$Q = \oint_S \vec{D} \cdot d\vec{S} = \int_v \rho_v \, dv \quad , \quad \rho_v = \vec{\nabla} \cdot \vec{D}$$

$$Q_b = \int_v \rho_b \, dv \quad , \quad \rho_b = -\vec{\nabla} \cdot \vec{P}$$

$$Q_T = \int_v \rho_T \, dv , \quad \rho_T = \vec{\nabla} \cdot \varepsilon_0 \vec{E}$$

The linear relationship between $\vec{P}$ and $\vec{E}$ is

$\vec{P} = \chi_e \varepsilon_0 \vec{E}$ electric susceptibility

$$\therefore \vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E} = (1+\chi_e)\varepsilon_0 \vec{E}$$

$$\varepsilon_R = \chi_e + 1$$

$$\vec{D} = \varepsilon_0 \varepsilon_R \vec{E} = \varepsilon \vec{E} , \quad \varepsilon = \varepsilon_0 \varepsilon_R$$

In anisotropic materials

$$D_x = \varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z$$
$$D_y = \varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z$$
$$D_z = \varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z$$

In summary, for isotropic materials

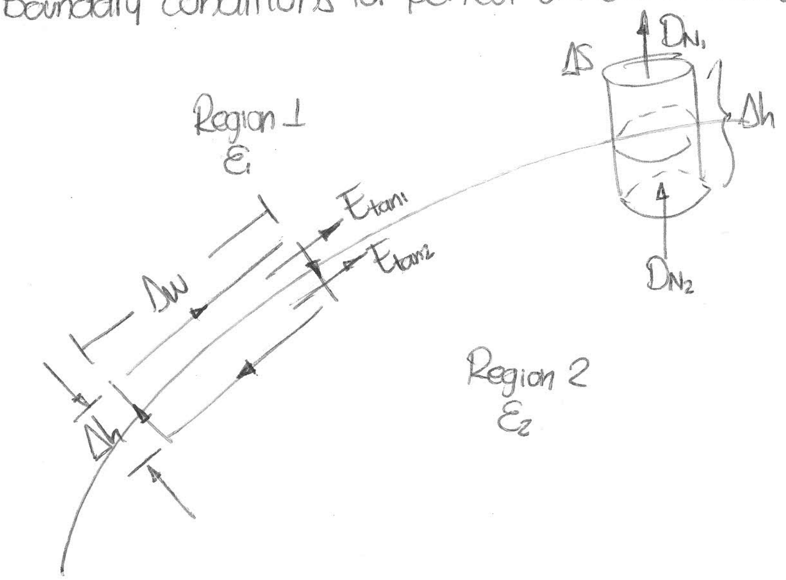
$$\vec{D} = \varepsilon \vec{E}$$

$$\varepsilon = \varepsilon_0 \varepsilon_R$$

$$\vec{\nabla} \cdot \vec{D} = \rho_v$$

$$\oint_S \vec{D} \cdot d\vec{S} = Q$$

* Boundary conditions for perfect dielectric materials



$$\oint \vec{E} \cdot \vec{dl} = 0$$

$$E_{tan1}\Delta w - E_N\Delta h - E_{tan2}\Delta w + E_N\Delta h = 0$$

$$\lim_{\Delta h \to 0} (E_{tan1}\Delta w - E_N\Delta h - E_{tan2}\Delta w + E_N\Delta h) = E_{tan1}\Delta w - E_{tan2}\Delta w = 0$$

$$(E_{tan1} - E_{tan2})\Delta w = 0$$

$$E_{tan1} = E_{tan2}$$

$$E_{tan1} = \frac{D_{tan1}}{\varepsilon_1}, \quad E_{tan2} = \frac{D_{tan2}}{\varepsilon_2} \quad \Rightarrow \quad \frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

$$\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}$$

to calculate the tangential component of the field

$\oint \vec{D} \cdot d\vec{S} = Q$

$\int_{top} + \int_{bottom} + \cancel{\int_{side}} = Q$ $\quad {}^{0\ (\Delta h \to 0)}$

$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_s \Delta S$

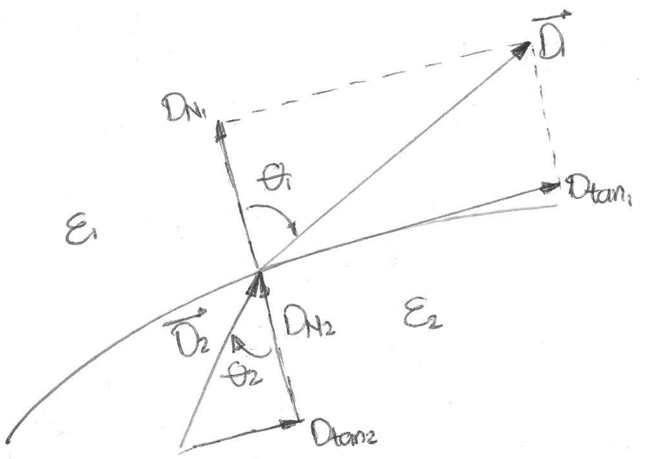
$(D_{N1} - D_{N2})\cancel{\Delta S} = \rho_s \cancel{\Delta S}$

$D_{N1} - D_{N2} = \rho_s$ this is a special case

Generally

$D_{N1} = D_{N2}$ // and $\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$ //

to calculate the normal component of the field



$D_{N1} = D_1\cos\theta_1$

$D_{N2} = D_2\cos\theta_2$

$D_{tan1} = D_1\sin\theta_1$

$D_{tan2} = D_2\sin\theta_2$

$\frac{D_{tan_1}}{D_{tan2}} = \frac{D_1\sin\theta_1}{D_2\sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} \Rightarrow \varepsilon_2 D_1 \sin\theta_1 = \varepsilon_1 D_2 \sin\theta_2$

$D_{N1} = D_{N2}$

$D_1\cos\theta_1 = D_2\cos\theta_2$

$\therefore \frac{\varepsilon_2 \cancel{D_1}\sin\theta_1}{\cancel{D_1}\cos\theta_1} = \frac{\varepsilon_1 \cancel{D_2}\sin\theta_2}{\cancel{D_2}\cos\theta_2}$

$\varepsilon_2\tan\theta_1 = \varepsilon_1\tan\theta_2 \Rightarrow \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$

and $\theta_1 > \theta_2$ if $\varepsilon_1 > \varepsilon_2$ //

$$D_2 = \sqrt{D_{N_2}^2 + D_{tan_2}^2}$$

$$D_{N_2}^2 = D_2^2 \cos^2\theta_2 = D_1^2 \cos^2\theta_1 = D_{N_1}^2$$

$$D_{tan_2} = D_{tan_1}\left(\frac{\varepsilon_2}{\varepsilon_1}\right) = D_1 \sin\theta_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)$$

$$D_{tan_2}^2 = D_1^2 \sin^2\theta_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2$$

$$D_2 = \sqrt{D_1^2\cos^2\theta_1 + D_1^2 \sin^2\theta_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2}$$

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \sin^2\theta_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} \quad //$$

$$\varepsilon_2 E_2 = \varepsilon_1 E_1 \sqrt{\cos^2\theta_1 + \sin^2\theta_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2}$$

$$E_2 = E_1 \left(\frac{\varepsilon_1}{\varepsilon_2}\right) \sqrt{\cos^2\theta_1 + \sin^2\theta_1 \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2}$$

$$E_2 = E_1 \sqrt{\cos^2\theta_1 \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + \sin^2\theta_1 \cancel{\left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2} \cancel{\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2}}$$

$$E_2 = E_1 \sqrt{\cos^2\theta_1 \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 + \sin^2\theta_1} \quad //$$

* Boundary conditions between conductor and dielectric

$\vec{D}=0$
$\vec{E}=0$ } inside the conductor

$E_t=0$
$D_t=0$ } to satisfy $\oint \vec{E}\cdot d\vec{l}=0$

$\vec{D}=\varepsilon\vec{E}$

$\oint_s \vec{D}\cdot d\vec{S}=Q$ leads to $D_N=\rho_s$ and $E_N=\rho_s/\varepsilon$

- Any charge that is introduced internally within a conducting material arrives at the surface as a surface charge

$\vec{J}=\sigma\vec{E}$ Ohm's Law

$\vec{\nabla}\cdot\vec{J}=-\frac{\partial\rho_v}{\partial t}$ Continuity equation

$\vec{\nabla}\cdot\sigma\vec{E}=-\frac{\partial\rho_v}{\partial t}$, $\vec{E}=\frac{\vec{D}}{\varepsilon}$

$\vec{\nabla}\cdot\frac{\sigma}{\varepsilon}\vec{D}=-\frac{\partial\rho_v}{\partial t}$

$\vec{\nabla}\cdot\vec{D}=-\frac{\varepsilon}{\sigma}\frac{\partial\rho_v}{\partial t}$

$\rho_v=-\frac{\varepsilon}{\sigma}\frac{\partial\rho_v}{\partial t}$

$-\frac{\sigma}{\varepsilon}dt=\frac{d\rho_v}{\rho_v}$

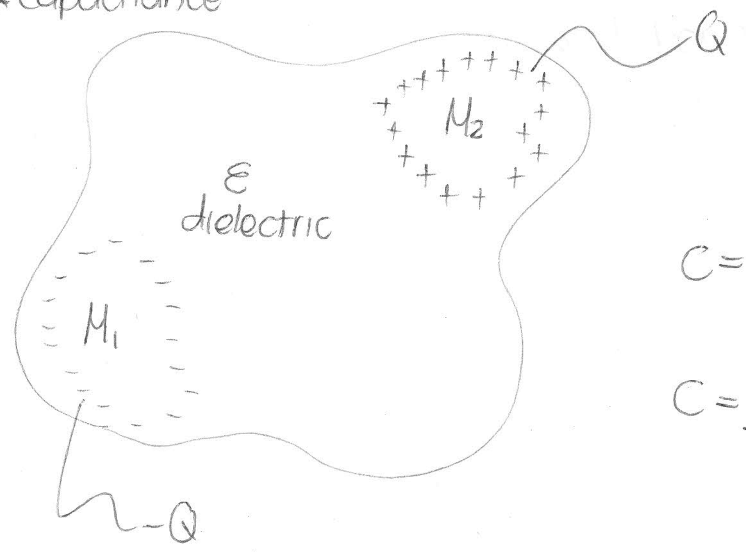
$-\frac{\sigma}{\varepsilon}\int_0^t dt=\int_{\rho_0}^{\rho_v}\frac{d\rho_v}{\rho_v}$

$-\frac{\sigma}{\varepsilon}t=\ln\left(\frac{\rho_v}{\rho_0}\right)$

$e^{-(\sigma/\varepsilon)t}=\rho_v/\rho_0$

$\rho_v=\rho_0 e^{-(\sigma/\varepsilon)t}$

* Capacitance



$$C = \frac{Q}{V_0}$$

$$C = \frac{\oint_S \varepsilon \vec{E} \cdot d\vec{S}}{-\int_-^+ \vec{E} \cdot d\vec{l}} \quad [F]$$