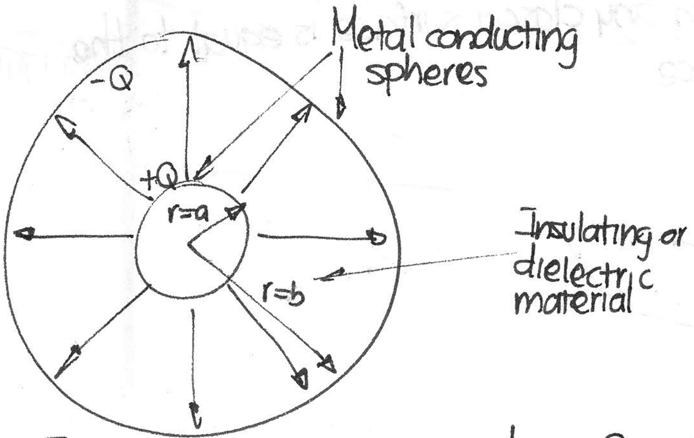
- Electric Flux density, Gauss's Law, and divergence

* Electric flux density



Faraday's Experiment

$\psi \rightarrow$ Electric Flux

$\psi = Q$

$\vec{D} \rightarrow$ Electric Flux Density

$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\vec{a_r} \quad \text{(inner sphere)}$$

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\vec{a_r} \quad \text{(outer sphere)}$$

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}$$ Also applies for a point charge

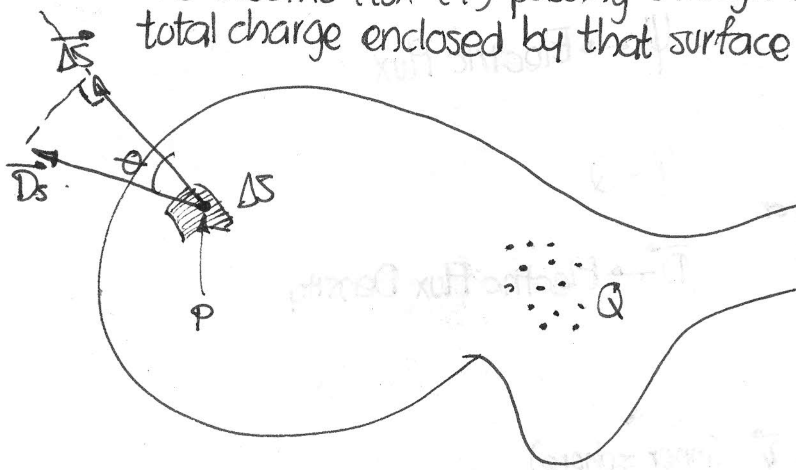
$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r} \qquad \therefore \ \underline{D = \varepsilon_0 E} \ \text{(free space only)}$$

$$\vec{E} = \int_{Vol} \frac{\rho_v\,dv}{4\pi\varepsilon_0 R^2}\vec{a_r} \quad \text{(free space only)}$$

$$\vec{D} = \int_{Vol} \frac{\rho_v\,dv}{4\pi R^2}\vec{a_r}$$

* Gauss's Law

The electric flux (Ψ) passing through any closed surface is equal to the total charge enclosed by that surface



$$\Delta\Psi = \text{flux crossing } \overrightarrow{\Delta S}$$
$$= D_{s,norm}\Delta S = D_s\cos\theta\,\Delta S = \overrightarrow{D_s}\cdot\overrightarrow{\Delta S}$$

$$\Psi = \int d\Psi = \oint_s \overrightarrow{D_s}\cdot\overrightarrow{dS} = Q$$

where

$$Q = \sum Q_n,\quad Q = \int \rho_L\, dL,\quad Q = \int_s \rho_s\, dS,\quad Q = \int_{vol} \rho_v\, dv$$

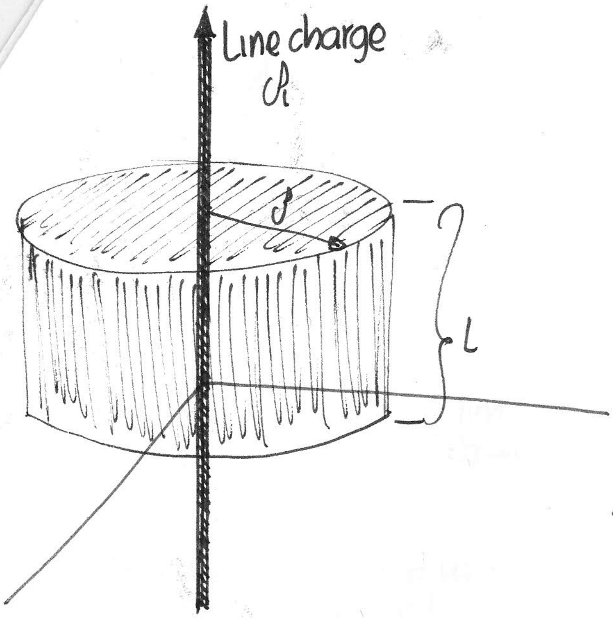
$$\oint_s \overrightarrow{D_s}\cdot\overrightarrow{dS} = \int_{vol} \rho_v\, dv$$

* Application of Gauss's law: Some symmetrical charge distributions

- Point charge

$$Q = \oint_s \overrightarrow{D_s}\cdot\overrightarrow{dS} = \oint_{sph} D_s\, dS = D_s\oint_{sph} dS = D_s\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} r^2\sin\theta\, d\theta\, d\phi = D_s r^2\int_{\phi=0}^{\phi=2\pi}\left(-\cos\theta\Big|_0^{\pi}\right)d\phi$$
$$= D_s r^2(2)\,\phi\Big|_0^{2\pi} = 4\pi r^2 D_s$$

$$D_s = \frac{Q}{4\pi r^2} \qquad \vec{D} = \frac{Q}{4\pi r^2}\vec{a_r} \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}$$



$$Q = \oint_{cyl} \vec{D_S} \cdot d\vec{S}$$

$$= D_S \int_{sides} dS + \cancel{0 \int_{top} dS} + \cancel{0 \int_{bottom} dS}$$

$$= D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho \, d\phi \, dz = D_S \rho (2\pi) L$$

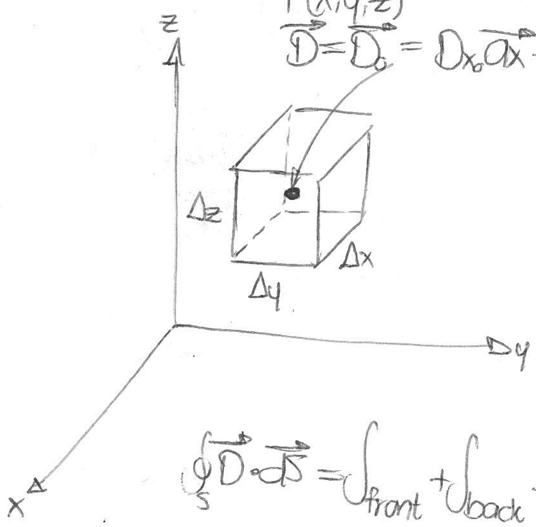
$$\therefore D_S = D_\rho = \frac{Q}{2\pi \rho L}, \quad Q = \rho_L L$$

$$D_\rho = \frac{\rho_L}{2\pi \rho} \text{ and } E_\rho = \frac{\rho_L}{2\pi \varepsilon_0 \rho}$$

* Application of Gauss's law: Differential volume element

$P(x,y,z)$

$$\vec{D} = \vec{D_0} = D_{x0} \vec{a_x} + D_{y0} \vec{a_y} + D_{z0} \vec{a_z}$$



$$\oint_S \vec{D} \cdot d\vec{S} = Q$$

$$\oint_S \vec{D} \cdot d\vec{S} = \int_{front} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

$$\int_{front} \doteq \vec{D}_{front} \cdot \vec{\Delta S}_{front}$$

$$\doteq \vec{D}_{front} \cdot \Delta y \Delta z \, \vec{a_x}$$

$$\doteq D_{x,front} \Delta y \Delta z$$

$$D_{x,front} \doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{ with } x$$

$$\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

$$\int_{front} \doteq \left(D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{back} \doteq \vec{D}_{back} \cdot \vec{\Delta S}_{back}$$

$$\doteq \vec{D}_{back} \cdot (-\Delta y \Delta z)\vec{a_x}$$

$$\doteq -D_{x,back}\Delta y \Delta z$$

$$D_{x,back} \doteq D_{x_0} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

$$\int_{back} \doteq \left(-D_{x_0} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}\right)\Delta y \Delta z$$

$$\int_{front} + \int_{back} \doteq \frac{\partial D_x}{\partial x}\Delta x \Delta y \Delta z$$

$$\int_{left} + \int_{right} \doteq \frac{\partial D_y}{\partial y}\Delta x \Delta y \Delta z$$

$$\int_{up} + \int_{bottom} \doteq \frac{\partial D_z}{\partial z}\Delta x \Delta y \Delta z$$

$$\therefore \oint_S \vec{D} \cdot \vec{dS} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

$$\oint_S \vec{D} \cdot \vec{dS} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta v = Q$$

*Divergence

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \doteq \frac{\oint_S \vec{D} \cdot \vec{dS}}{\Delta v} = \frac{Q}{\Delta v}$$

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \lim_{\Delta v \to 0} \frac{\oint_S \vec{D} \cdot \vec{dS}}{\Delta v} = \lim_{\Delta v \to 0} \frac{Q}{\Delta v} = \rho_v$$

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{S}}{\Delta v}$$

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\rho_v$$

$$\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \vec{A}\cdot d\vec{S}}{\Delta v}=\text{Divergence of }\vec{A}=\text{div }\vec{A}$$

The divergence of the vector flux density $\vec{A}$ is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero

$$\text{div }\vec{D}=\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)$$

$$\text{div }\vec{D}=\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho)+\frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi}+\frac{\partial D_z}{\partial z}$$

$$\text{div }\vec{D}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta)+\frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

* Maxwell's first equation (Electrostatics)

$$\text{div }\vec{D}=\lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{S}}{\Delta v}$$

$$\text{div }\vec{D}=\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}$$

$$\text{div }\vec{D}=\rho_v$$

$$\oint_S \vec{D}\cdot d\vec{S}=Q$$

the electric flux per unit volume leaving a vanishingly small volume unit is exactly equal to the volume charge density there.

$$\frac{\oint_S \vec{D}\cdot d\vec{S}}{\Delta v}=\frac{Q}{\Delta v}$$

$$\lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}$$

$$\text{div }\vec{D}=\rho_v$$

* The vector operator ∇ and the divergence theorem

$$\operatorname{div}\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}$$

$$\vec{D} = D_x\vec{a_x} + D_y\vec{a_y} + D_z\vec{a_z}$$

$$\vec{\nabla}\cdot\vec{D} = \left(\frac{\partial}{\partial x}\vec{a_x} + \frac{\partial}{\partial y}\vec{a_y} + \frac{\partial}{\partial z}\vec{a_z}\right)\cdot\left(D_x\vec{a_x} + D_y\vec{a_y} + D_z\vec{a_z}\right)$$

$$= \frac{\partial}{\partial x}(D_x) + \frac{\partial}{\partial y}(D_y) + \frac{\partial}{\partial z}(D_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\therefore \operatorname{div}\vec{D} = \vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \,//$$

$$\oint_S \vec{D}\cdot\vec{dS} = Q$$

$$Q = \int_{vol} \rho_v \, dv \,, \quad \rho_v = \operatorname{div}\vec{D} = \vec{\nabla}\cdot\vec{D}$$

$$\oint_S \vec{D}\cdot\vec{dS} = Q = \int_{vol} \rho_v \, dv = \int_{vol} \vec{\nabla}\cdot\vec{D}\, dv$$

$$\oint_S \vec{D}\cdot\vec{dS} = \int_{vol} \vec{\nabla}\cdot\vec{D}\, dv \,//$$ Divergence theorem

the integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface