Homework 04 - Differential Vector Calculus

Curvas, Tangentes, Longitud de Curva, Curvas en Mecánica.

11 Dada una curva $C: r(t)$, encuentra un vector tangente $r'(t)$, un vector tangente unitario $u'(t)$ y la tangente de C en P. Esboza la curva y la tangente. (pp 410)

Let's solve the "Tangent to a Curve" problems...

Note: If $r'(t) \neq 0$, we call $r'(t)$ a tangent vector of C at P, and u the unit tangent vector.

a $r(t) = \left[t, \frac{1}{2}t^2, 1\right]$, $P: (2, 2, 1)$

$r'(t) = [1, t, 0]$

$$u = \frac{1}{|r'(t)|} r'(t) = \frac{1}{\sqrt{1^2 + t^2 + 0^2}}(1, t, 0)$$

$$= \frac{1}{\sqrt{1+t^2}}(1, t, 0)$$

at $t = 2$, $r(t) = P$...

$r'(2) = (1, 2, 0)$

$$u = \frac{1}{\sqrt{5}}(1, 2, 0)$$

b $r(t) = [\cos(t), \sin(t), 9t]$, $P: (1, 0, 18\pi)$

$r'(t) = [-\sin(t), \cos(t), 9]$

$$u = \frac{1}{\sqrt{(-\sin(t))^2 + (\cos(t))^2 + 9^2}}(-\sin(t), \cos(t), 9) \quad \Big\} \sin^2(t) + \cos^2(t) = 1$$

$$= \frac{1}{\sqrt{82}}(-\sin(t), \cos(t), 9)$$

at $t = 2\pi$, $r(t) = P$

$r'(2\pi) = (0, 1, 9)$

$u = \frac{1}{\sqrt{82}}(0, 1, 9)$ //

$\boxed{c}$ $r(t) = [t, t^2, t^3]$, $P: (1, 1, 1)$

$r'(t) = [1, 2t, 3t^2]$ //

$u = \frac{1}{\sqrt{1^2 + (2t)^2 + (3t^2)^2}}(1, 2t, 3t^2)$

$= \frac{1}{\sqrt{1 + 4t^2 + 9t^4}}(1, 2t, 3t^2)$ //

at $t = 1$, $r(t) = P$

$r'(1) = (1, 2, 3)$

$u = \frac{1}{\sqrt{1 + 4 + 9}}(1, 2, 3)$

$= \frac{1}{\sqrt{14}}(1, 2, 3)$ //

12 Encuentra la longitud total y haz un esbozo de la curva hipocicloide dada por $r(t) = [a\cos^3 t,\ a\sin^3 t]$ (pp 411)

Note: The length ℓ is given by:
$\ell = \int_a^b \sqrt{r'(t)\cdot r'(t)}\,dt$; where ℓ represents the limit of the lengths of broken lines of n chords. for each n, the interval $a \le t < b$ is subdivided by points.

Note: ℓ is called the length of C, and C is called rectifiable.

$$r'(t) = \frac{d}{dt} r(t)$$
$$= \frac{d}{dt}\left[a\cos^3(t), a\sin^3(t)\right]$$
$$= \left[-3a\cos^2(t)\sin(t),\ 3a\cos(t)\sin^2(t)\right]$$

$$r'(t)\cdot r'(t) = \left(9a^2\cos^4(t)\sin^2(t)\right) + \left(9a^2\cos^2(t)\sin^4(t)\right)$$
$$= 9a^2\cos^2(t)\sin^2(t)$$
$$= \frac{9a^2}{4}\sin^2(2t)$$

$\cos^2(t)\sin^2(t) = \frac{\sin^2(2t)}{4}$

$$\ell = \int_0^{2\pi} \sqrt{\frac{9a^2}{4}\sin^2(2t)}\,dt = \frac{3a}{2}\int_0^{2\pi} \sin(2t)\,dt$$
$$= \frac{3a}{2}\left[-\frac{1}{2}\cos(2t)\Big|_0^{2\pi}\right]$$
$$= \frac{3a}{2}\left[\cancel{\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)}\right]^{0}$$
$$= \frac{3a}{2}u$$

13 Para las trayectorias en los incisos (a) y (b), encuentra la aceleración tangencial, la aceleración normal, la velocidad y la rapidez. (pp. 412)

Note: The tangent vector of C is called the velocity vector v. (as the tangent points at the instantaneous directions of motion and its length gives the speed $|v|$. The second derivative of $r(t)$ is called the acceleration vector a. $|a|$ (a's length) is called the acceleration of the motion.

$$v(t) = r'(t)$$
$$|v| = |r'| = \sqrt{r' \cdot r'} = ds/dt \quad \& \quad a(t) = v'(t) = r''(t)$$

Note: The acceleration vector can be splited into directional components: a_{tan} & a_{norm} The tangential acceleration vector a_{tan} is tangent to the path and the normal acceleration vector a_{norm} is perpendicular (normal) to the path.

That is...

$$u(s) = \frac{dr}{ds} \; ; \; v(t) = u(s)\frac{ds}{dt} \; ; \; a(t) = \frac{du}{ds}\left(\frac{ds}{dt}\right)^2 + u(s)\frac{d^2s}{dt^2}$$

$$a_{tan} = \frac{a \cdot v}{v \cdot v} v \; ; \; a_{norm} = a - a_{tan}$$

a Línea recta $r(t) = [8t, 6t, 0]$

$$v(t) = r'(t) = [8, 6, 0]$$

$$|v| = \sqrt{r' \cdot r'} = \sqrt{8^2 + 6^2 + 0^2} = 10$$

$$a(t) = v'(t) = [0, 0, 0]$$

$$a_{tan} = \frac{a \cdot v}{v \cdot v} = \frac{0 + 0 + 0}{8^2 + 6^2 + 0^2}[8, 6, 0] = [0, 0, 0]$$

$$a_{norm} = a - a_{tan} = [0, 0, 0]$$

$\boxed{b}$ Elipse $r(t) = [\cos t, 2\sin t, 0]$

$$v(t) = r'(t) = [-\sin t, 2\cos t, 0]$$

$$|v| = \sqrt{(-\sin t)^2 + (2\cos t)^2 + 0}$$
$$= \sqrt{4\cos^2(t) + \sin^2(t)}$$

$$a(t) = v'(t) = [-\cos t, -2\sin t, 0]$$

$$a_{tan} = \frac{a \cdot v}{v \cdot v} = \frac{(-\cos t)(-\sin t) + (-2\sin t)(2\cos t) + 0}{\sin^2 t + 4\cos^2 t + 0}$$
$$= \frac{-3\cos(t)\sin(t)}{\sin^2(t) + 4\cos^2(t)}$$
$$= \frac{3\sin(2t)}{5 + 3\cos(2t)}[-\sin(t), 2\cos(t), 0]$$

$$a_{norm} = a - a_{tan}$$
$$= [-\cos(t), -2\sin(t), 0] - \underbrace{\frac{3\sin(2t)}{5+3\cos(2t)}}_{\alpha}[-\sin(t), 2\cos(t), 0]$$
$$= [(-\cos(t) - \alpha(-\sin(t)), (-2\sin(t) - \alpha(2\cos(t), 0]$$
$$= \left[-\cos(t) - \frac{6\sin(t)\sin(2t)}{5+3\cos(2t)}, -2\sin(t) + \frac{6\cos(t)\sin(2t)}{5+3\cos(2t)}, 0\right]$$
$$= \left[-\cos(t) - \frac{6\sin(t)\sin(2t)}{5+3\cos(2t)}, -\frac{4\sin(t)}{5+3\cos(2t)}, 0\right]$$