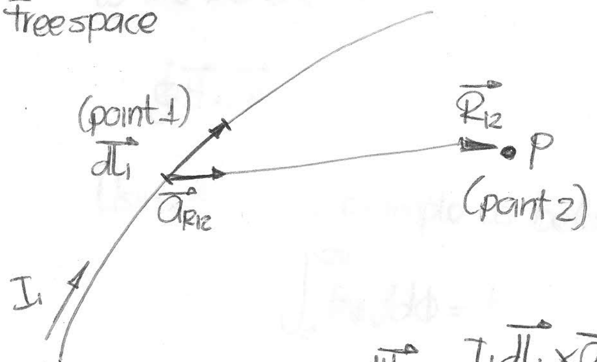
* The steady magnetic field

- Biot-Savart law

Freespace



$$d\vec{H}_2 = \frac{I_1 \vec{dL_1} \times \vec{a}_{R12}}{4\pi R_{12}^2}$$

in DC. the charge density is not a function of time

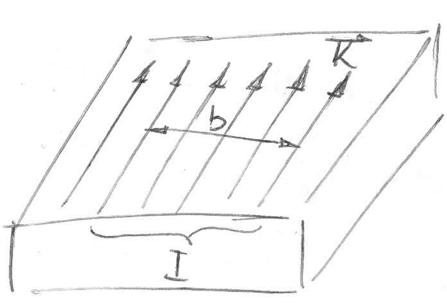
$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} = 0$$

$$\oint_S \vec{A} \cdot \vec{dS} = \int_{vol} (\vec{\nabla} \cdot \vec{A}) dv \quad \text{(Divergence theorem)}$$

$$\int_{vol} (\vec{\nabla} \cdot \vec{J}) dv = \oint_S \vec{J} \cdot \vec{dS} = 0$$

the total current crossing any closed surface is zero. and this condition may be satisfied only by assuming a current flow around a closed path, then

$$\vec{H} = \oint \frac{I \vec{dL} \times \vec{a}_R}{4\pi R^2}$$



$\vec{K}$ = surface current density [A/m]

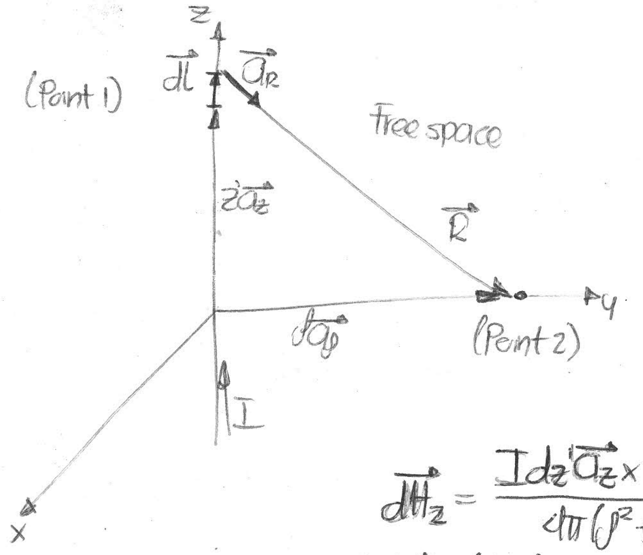
$I = Kb$

$I = \int K dN$

$$\therefore I\vec{dl} = \vec{K}dS = \vec{J}dv$$

$$\vec{H} = \int_S \frac{\vec{K} \times \vec{a}_R \, dS}{4\pi R^2}$$

$$\vec{H} = \int_{vol} \frac{\vec{J} \times \vec{a}_R \, dv}{4\pi R^2}$$



$$\vec{R}_{12} = \vec{r} - \vec{r}\,' = \rho\vec{a}_\rho - z'\vec{a}_z$$

$$\vec{a}_{R_{12}} = \frac{\rho\vec{a}_\rho - z'\vec{a}_z}{\sqrt{\rho^2 + z'^2}}$$

$$\vec{dl} = dz'\vec{a}_z$$

$$\vec{dH}_2 = \frac{I\,dz'\vec{a}_z \times (\rho\vec{a}_\rho - z'\vec{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$\vec{H}_2 = \int_{-\infty}^{\infty} \frac{I\,dz'\vec{a}_z \times (\rho\vec{a}_\rho - z'\vec{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi}\int_{-\infty}^{\infty} \frac{\rho\,dz'\vec{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I\rho\vec{a}_\phi}{4\pi}\int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I\rho\vec{a}_\phi}{4\pi} \frac{z'}{\rho^2\sqrt{\rho^2 + z'^2}}\Bigg|_{-\infty}^{\infty}$$

$$= \frac{I}{2\pi\rho}\vec{a}_\phi$$

- Ampère's Circuital law

States that the line integral of $\vec{H}$ about any closed path is exactly equal to the DC enclosed by that path

$$\oint \vec{H} \cdot \vec{dl} = I$$

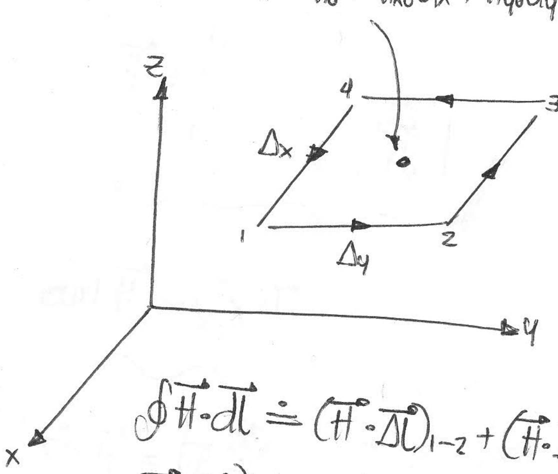
Using the same example as before

$$\oint \vec{H} \cdot \vec{dl} = \int_0^{2\pi} H_\phi \rho \, d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

$$\therefore H_\phi = \frac{I}{2\pi \rho}$$

- Curl

$$\vec{H} = \vec{H_0} = H_{x0} \vec{a_x} + H_{y0} \vec{a_y} + H_{z0} \vec{a_z}$$



$$\oint \vec{H} \cdot \vec{dl} \doteq (\vec{H} \cdot \vec{\Delta l})_{1-2} + (\vec{H} \cdot \vec{\Delta l})_{2-3} + (\vec{H} \cdot \vec{\Delta l})_{3-4} + (\vec{H} \cdot \vec{\Delta l})_{4-1}$$

$$(\vec{H} \cdot \vec{\Delta l})_{1-2} = H_{y,1-2} \Delta y$$

$$H_{y,1,2} \doteq H_{y0} + \frac{\partial H_y}{\partial x} \left(\frac{1}{2} \Delta x\right)$$

$$(\vec{H} \cdot \vec{\Delta l})_{1-2} \doteq \left(H_{y0} + \frac{1}{2} \frac{\partial H_y}{\partial x} \Delta x\right) \Delta y$$

$$(\vec{H} \cdot \vec{\Delta l})_{2-3} = H_{x,2-3} (-\Delta x)$$

$$H_{x,2,3} \doteq H_{x0} + \frac{\partial H_x}{\partial y} \left(\frac{1}{2} \Delta y\right)$$

$$(\vec{H} \cdot \vec{\Delta l})_{2-3} \doteq \left(H_{x0} + \frac{1}{2} \frac{\partial H_x}{\partial y} \Delta y\right) (-\Delta x)$$

$$(\vec{H}\cdot\vec{\Delta L})_{3-4} = H_{y,3-4}(-\Delta_y)$$

$$H_{y,3-4} \doteq H_{y_0} + \frac{\partial H_y}{\partial x}\left(-\frac{1}{2}\Delta x\right)$$

$$(\vec{H}\cdot\vec{\Delta L})_{3-4} \doteq \left(H_{y_0} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)(-\Delta_y) \,/\!/$$

$$(\vec{H}\cdot\vec{\Delta L})_{4-1} = H_{x,4-1}(\Delta x)$$

$$H_{x,4-1} \doteq H_{x_0} + \frac{\partial H_x}{\partial y}\left(-\frac{1}{2}\Delta_y\right)$$

$$(\vec{H}\cdot\vec{\Delta L})_{4-1} \doteq \left(H_{x_0} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta_y\right)\Delta x \,/\!/$$

$$\oint \vec{H}\cdot\vec{dL} \doteq \cancel{H_{y_0}\Delta_y} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\Delta_y + \cancel{H_{x_0}(-\Delta x)} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta_y(-\Delta x) + \cancel{H_{y_0}(-\Delta_y)} - \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x(-\Delta_y)$$
$$+ \cancel{H_{x_0}\Delta x} - \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta_y\Delta x$$
$$\doteq \frac{\partial H_y}{\partial x}\Delta x\Delta_y - \frac{\partial H_x}{\partial y}\Delta x\Delta_y$$
$$\doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\Delta_y$$

Assuming a general current density $\vec{J}$

$$\oint \vec{H}\cdot\vec{dL} = \Delta I \ , \ \Delta I \doteq J_z\Delta x\Delta_y$$

$$\therefore \oint \vec{H}\cdot\vec{dL} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\Delta_y \doteq J_z\Delta x\Delta_y$$

$$\frac{\oint \vec{H}\cdot\vec{dL}}{\Delta x\Delta_y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

$$\lim_{\Delta x,\Delta_y\to 0}\frac{\oint \vec{H}\cdot\vec{dL}}{\Delta x\Delta_y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \,/\!/$$

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

$$(\mathrm{curl}\ \vec{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta S_N}$$

$$\mathrm{curl}\ \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{a_x} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{a_y} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{a_z}$$

$$\mathrm{curl}\ \vec{H} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$\mathrm{curl}\ \vec{H} = \vec{\nabla} \times \vec{H}$$

$$\vec{\nabla} \times \vec{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\vec{a_\rho} + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\vec{a_\phi} + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\vec{a_z}$$ cylindrical

$$\vec{\nabla} \times \vec{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\vec{a_r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(r H_\phi)}{\partial r}\right)\vec{a_\theta} + \frac{1}{r}\left(\frac{\partial(r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\vec{a_\phi}$$ sphe

$$\text{curl } \vec{H} = \vec{\nabla}\times\vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{a_x} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{a_y} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{a_z} = \vec{J}$$

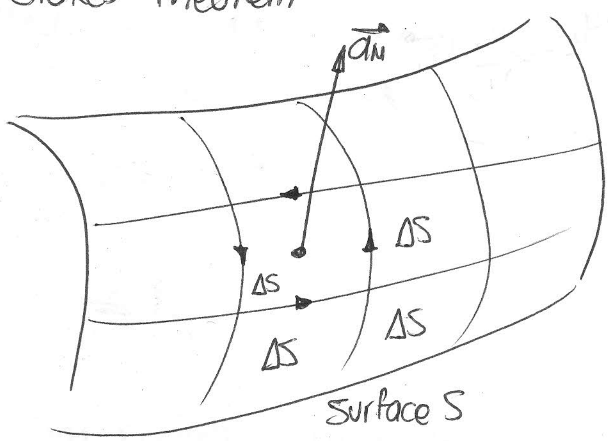
and the point form of Ampère's circuital law is

$$\vec{\nabla}\times\vec{H} = \vec{J}$$ (2nd Maxwell equation)

point form of $\oint \vec{E}\cdot\vec{dl} = 0$

$$\vec{\nabla}\times\vec{E} = 0$$ (3rd Maxwell equation)

• Stokes' Theorem



$$\frac{\oint \vec{H}\cdot\vec{dl}_{\Delta S}}{\Delta S} \doteq (\vec{\nabla}\times\vec{H})_N$$

$$\frac{\oint \vec{H}\cdot\vec{dl}_{\Delta S}}{\Delta S} \doteq (\vec{\nabla}\times\vec{H})\cdot\vec{a_N}$$

$$\oint \vec{H}\cdot\vec{dl}_{\Delta S} \doteq (\vec{\nabla}\times\vec{H})\cdot\vec{a_N}\,\Delta S = (\vec{\nabla}\times\vec{H})\cdot\vec{\Delta S}$$

$$\oint \vec{H}\cdot\vec{dl} \equiv \int_S (\vec{\nabla}\times\vec{H})\cdot\vec{dS}$$ Stokes' theorem

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

$$(\vec{\nabla}\times\vec{H})\cdot\vec{\Delta S} = \vec{J}\cdot\vec{\Delta S}$$

$$\int_S (\vec{\nabla}\times\vec{H})\cdot\vec{dS} = \int_S \vec{J}\cdot\vec{dS} = I$$

$$\int_S (\vec{\nabla}\times\vec{H})\cdot\vec{dS} = \underbrace{\oint \vec{H}\cdot\vec{dl} = I}_{\text{Ampère's circuital law}}$$

- Using Stokes' Theorem

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = T$$

$$\int_{Vol}(\vec{\nabla}\cdot\vec{\nabla}\times\vec{A})\,dv = \int_{Vol} T\,dv$$

$$\int_{Vol}(\vec{\nabla}\cdot\vec{\nabla}\times\vec{A})\,dv = \oint_S(\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \int_{Vol} T\,dv$$

$$\oint_S(\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \cancel{\oint \vec{A}\cdot d\vec{l}} \to 0$$ (the path disappears)

$$\therefore \int_{Vol} T\,dv = 0$$

$$T\,dv = 0$$

$$T = 0$$

and $\vec{\nabla}\cdot\vec{\nabla}\times\vec{A} \equiv 0$ //

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

then $\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{H} = 0$ // as before

- Magnetic flux and magnetic flux density

$\vec{B} = \mu_0\vec{H}$ magnetic flux density

$\vec{B}$ $[Wb/m^2]$, $[T]$

$\mu_0 = 4\pi\times10^{-7}\,[H/m]$

$\Phi = \int_S \vec{B}\cdot d\vec{S}$ $[Wb]$ magnetic flux

$\Psi = \oint_S \vec{D}\cdot d\vec{S} = Q$ $[C]$ electric flux

Because magnetic flux lines are closed

$\oint_S \vec{B}\cdot d\vec{S} = 0$ //

$\oint \vec{B} \cdot d\vec{S} = 0$

Using the divergence theorem

$\oint \vec{B} \cdot d\vec{S} = \int_{Vol} (\vec{\nabla} \cdot \vec{B}) dv = 0$

$(\vec{\nabla} \cdot \vec{B}) \Delta v = 0$

$\vec{\nabla} \cdot \vec{B} = 0$ // (4th Maxwell equation)

Maxwell's equations

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho_v \\ \vec{\nabla} \times \vec{E} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J} \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned}\right\} \text{Point form} \iff \text{Integral form} \left\{\begin{aligned} &\oint_S \vec{D} \cdot d\vec{S} = Q = \int_{Vol} \rho_v dv \\ &\oint \vec{E} \cdot d\vec{L} = 0 \\ &\oint \vec{H} \cdot d\vec{L} = I = \int_S \vec{J} \cdot d\vec{S} \\ &\oint \vec{B} \cdot d\vec{S} = 0 \end{aligned}\right.$$

and, in freespace

$\vec{D} = \varepsilon_0 \vec{E}$

$\vec{B} = \mu_0 \vec{H}$

$\vec{E} = -\vec{\nabla} V$