Mathematical Physical Modelling - Homework 03
Eigenvalues Problems

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February 28, 2019

- **PROBLEMAS DE EIGENVALORES**

1. Encuentra los eigenvalores y los eigenvectores correspondientes.

(a) $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

- **TRANSFORMACIONES LINEALES Y EIGENVALORES**

2. Encuentra la matriz $\mathbf{A}$ en la transformación lineal $\mathbf{y} = \mathbf{Ax}$, donde $\mathbf{x}$ son las coordenadas Cartesianas. Encuentra los eigenvalores y eigenvectores y explica su significado geométrico.

(a) Reflexión alrededor del eje x_1 en R^2.

(b) Proyección ortogonal de R^3 en el plano $x_2 = x_1$.

- **APLICACIONES: DEFORMACIONES ELÁSTICAS Y MODELOS DE POBLACIÓN**

3. Dada $\mathbf{A}$ en una deformación $\mathbf{y} = \mathbf{Ax}$, encuentra las direcciones principales y los correspondientes factores de extensión o contracción. Muestra los detalles de tu cálculo.

(a) $\begin{bmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 \\ 2 & 13 \end{bmatrix}$ (c) $\begin{bmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{bmatrix}$

4. Encuentra el ritmo de crecimiento en el modelo de Leslie con la matriz dada. Muestra los detalles, manito.

(a) $\begin{bmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}$

Homework 3 Eigenvalue problems

3 Deformaciones elásticas

Dada A en una deformación $y = Ax$, encuentra las direcciones principales y los correspondientes factores de extensión o contracción. Muestra los detalles de tu cálculo.

a

$$A = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix}$$

Note: The principal directions are the directions of the position vector x for which the direction of the position vector y is the same or exact opposite.

$$y = Ax$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 2.0 & 0.4 \\ 0.4 & 2.0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{aligned} y_1 &= 2.0x_1 + 0.4x_2 \\ y_2 &= 0.4x_1 + 2.0x_2 \end{aligned}$$

▷ Let's solve the eigenvalue problem...

$$Ax = \lambda x \qquad Ax - \lambda x = 0$$

$$\begin{cases} 2.0x_1 + 0.4x_2 = \lambda x_1 \\ 0.4x_1 + 2.0x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (2.0-\lambda)x_1 + 0.4x_2 = 0 \\ 0.4x_1 + (2.0-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det\begin{pmatrix} 2.0-\lambda & 0.4 \\ 0.4 & 2.0-\lambda \end{pmatrix} = 0$$ is the characteristic equation

$$(2.0-\lambda)^2 - 0.4^2 = 0$$
$$\lambda^2 - 4.0\lambda + 3.84 = 0$$

the eigenvalues are:

$$\lambda_1 = -\frac{8}{5}$$

$$\lambda_2 = -\frac{12}{5}$$; where λ is the strech factor

for $\lambda = \lambda_1$, then

$$\begin{cases} 3.6x_1 + 0.4x_2 = 0 \\ 0.4x_1 + 3.6x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda = \lambda_2$, then

$$\begin{cases} 4.4x_1 + 0.4x_2 = 0 \\ 0.4x_1 + 4.4x_2 = 0 \end{cases}$$

$$x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$; where x is the eigen vector

▷ The eigenvalues show a contraction, with no direction

b $A = \begin{pmatrix} 5 & 2 \\ 2 & 13 \end{pmatrix}$ The eigenvalues show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2, respectively.

▷ Let's solve the eigen problem...

$Ax = \lambda x$ $\quad\quad Ax - \lambda x = 0$

$$\begin{cases} 5x_1 + 2x_2 = \lambda x_1 \\ 2x_1 + 13x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (5-\lambda)x_1 + 2x_2 = 0 \\ 2x_1 + (13-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 13-\lambda \end{pmatrix} = 0$$

$$(5-\lambda)(13-\lambda) - 4 = 0$$

$$\lambda^2 - 18\lambda + 61 = 0$$

the eigenvalues are:

$\lambda_1 = 9 + 2\sqrt{5}$

for $\lambda = \lambda_1$, then

$$\begin{cases} (-4-2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4-2\sqrt{5})x_2 = 0 \end{cases}$$

$$x_1 = \frac{-(4-2\sqrt{5})x_2}{2}$$

$$x_1 = (-2+\sqrt{5})x_2$$

$$-2x_2 + 2x_2 = 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-2+\sqrt{5})x_2 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -2+\sqrt{5} \\ 1 \end{pmatrix}$$

$\lambda_2 = 9 - 2\sqrt{5}$

for $\lambda = \lambda_2$

$$\begin{cases} (-4+2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4+2\sqrt{5})x_2 = 0 \end{cases}$$

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$$x = \begin{pmatrix} -(4+2\sqrt{5})x_2 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -4-2\sqrt{5} \\ 1 \end{pmatrix}$$

▷ Let's compute the principal directions

$$\theta_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(\frac{1}{-2+\sqrt{5}}\right)$$

$= 76°43'2.91''$ with the positive x_1 direction

$$\theta_2 = \tan^{-1}\left(\frac{1}{-4-2\sqrt{5}}\right)$$

$= 173°16'5.87''$ with the positive x_1 direction

c $A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$

▷ Let's solve the eigen problem...

$Ax = \lambda x \qquad Ax - \lambda x = 0$

$$\begin{cases} 1.25x_1 + 0.75x_2 = \lambda x_1 \\ 0.75x_1 + 1.25x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} (1.25-\lambda)x_1 + 0.75x_2 = 0 \\ 0.75x_1 + (1.25-\lambda)x_2 = 0 \end{cases}$$

so...

$$\det\begin{pmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 2.5\lambda + 1 = 0$$

the eigen values are:

$\lambda_1 = 2$

for $\lambda = \lambda_1$, then

$$\begin{cases} -0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 - 0.75x_2 = 0 \end{cases}$$

$x_1 = x_2$

$x = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 0.5$

for $\lambda = \lambda_2$, then

$$\begin{cases} 0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 + 0.75x_2 = 0 \end{cases}$$

$x_1 = -x_2$

$x = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

▷ Let's compute the principal directions.

$\theta_1 = \mathrm{Tan}^{-1}\left(\frac{1}{1}\right)$

$= 45°0'0''$ with the positive x_1 direction

$\theta_2 = \mathrm{Tan}^{-1}\left(\frac{1}{-1}\right)$

$= 135°0'0''$ with the positive x_1 direction

The eigen values show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2, respectively.

Homework 3 Eigenvalue problems.

4 Modelos de población.

a

$$L = \begin{pmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}$$

Note: The Leslie model describes age-specified population growth.

Assumptions:
- i) Let the oldest age attained be 99 years.
- ii) Since the provided matrix L has 3 columns, divide the population into three age classes of 33 years each.

Note: Given a Leslie matrix $L = [l_{jk}]$,
1) l_{1k} is the average number of new borns to a single member durig the time he/she is in age class k.
2) $l_{j,j-1}$ is the fraction of members in age class $j-1$ that will survive and pass into class j.

Note: Proportional change means that we are looking for a distribution vector x such that $Lx = \lambda x$, where λ is the rate of change (growth if $\lambda > 1$, decrease if $\lambda < 1$).

Encuentra el ritmo de crecimiento en el modelo de Leslie con la matrix dada.

▷ Let's find the λ's

$$\det(L - \lambda I)x = 0$$

$$\det\begin{pmatrix} -\lambda & 9.0 & 5.0 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{pmatrix} = 0$$

$$-0.4 \det\begin{pmatrix} -\lambda & 5.0 \\ 0.4 & -\lambda \end{pmatrix} + (-\lambda)\det\begin{pmatrix} -\lambda & 9.0 \\ 0.4 & -\lambda \end{pmatrix} = 0$$

$$(-0.4)(\lambda^2 - 2) + (-\lambda)(\lambda^2 - 3.6) = 0$$

$$-0.4\lambda^2 + 0.8 - \lambda^3 + 3.6\lambda = 0$$

$$-\lambda^3 - 0.4\lambda^2 + 3.6\lambda + 0.8 = 0$$

$$\lambda_1 = 1.82, \quad \lambda_2 = -0.23, \quad \lambda_3 = -2$$

▷ A positive root is found to be $\lambda = 1.82$

The growth rate will be 1.82 per 33 years.

$\boxed{b}$

$$L = \begin{pmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}$$

Assumptions:

i) Let the oldest age attained be 100 years

ii) Since the provided matrix L has four columns, let's divide the population into four age classes of 25 years each.

▷ let's find the λ's

$$\det(L - \lambda I)x = 0$$

$$\det\begin{pmatrix} -\lambda & 3.0 & 2.0 & 2.0 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{pmatrix} = +(-\lambda)\begin{vmatrix} -\lambda & 0 & 0 \\ 0.5 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{vmatrix} - (3.0)\begin{vmatrix} 0.5 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{vmatrix} + (2.0)\begin{vmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -\lambda \end{vmatrix} - (2.0)\begin{vmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \\ 0 & 0 & 0.1 \end{vmatrix}$$

$$= -\lambda\left[+(-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{vmatrix} - 0 + 0\right] - 3.0\left[+(0.5)\begin{vmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{vmatrix} - 0 + 0\right] + 2.0\left[+(0.5)\begin{vmatrix} 0.5 & 0 \\ 0 & -\lambda \end{vmatrix} \cancel{-(-\lambda)\begin{vmatrix} 0 & 0 \\ 0 & -\lambda \end{vmatrix}} + 0\right] - 2.0\left[+(0.5)\begin{vmatrix} 0.5 & -\lambda \\ 0 & 0.1 \end{vmatrix} \cancel{-(-\lambda)\begin{vmatrix} 0 & -\lambda \\ 0 & 0.1 \end{vmatrix}} + 0\right]$$

$$= -\lambda(-\lambda(\lambda^2)) - 3.0(0.5(\lambda^2)) + 2.0(0.5(-0.5\lambda)) - 2.0(0.5(0.5(0.1)))$$

$$= \lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05$$

$\lambda_1 = 1.37$ $\quad \lambda_2 = -1.03$ $\quad \lambda_3 = -0.17 + 0.07i$ $\quad \lambda_4 = -0.17 - 0.07i$

▷ A positive root is found to be $\lambda = 1.37$

The growth rate will be 1.37 per 25 years.

Homework 3 Eigenvalue problems

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$$(-0.4)(\lambda^2 - 2) + (-\lambda)(\lambda^2 - 3.6) = 0$$

$$-0.4\lambda^2 + 0.8 - \lambda^3 + 3.6\lambda = 0$$

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$$\lambda_1 = 1.82 \quad , \quad \lambda_2 = -0.23 \quad , \quad \lambda_3 = -2$$

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Assumptions:

i) Let the oldest age attained be 100 years

ii) Since the provided matrix L has four columns, let's divide the population into four age classes of 25 years each.

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$$\det(L-\lambda I)x=0$$

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$$-(3.0)\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix}$$

$$+(2.0)\begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -\lambda \end{pmatrix}$$

$$-(2.0)\begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \\ 0 & 0 & 0.1 \end{pmatrix}$$

$$= -\lambda\left[+(-\lambda)\begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0\right]$$

$$-3.0\left[+(0.5)\begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0\right]$$

$$+2.0\left[+(0.5)\begin{pmatrix} 0.5 & 0 \\ 0 & -\lambda \end{pmatrix} \cancel{-(-\lambda)\begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix}}^{0} + 0\right]$$

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$$= -\lambda(-\lambda(\lambda^2)) - 3.0(0.5(\lambda^2)) + 2.0(0.5(-0.5\lambda))$$

$$-2.0(0.5(0.5(0.1)))$$

$$= \lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05$$

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$$\lambda^2 - 18\lambda + 61 = 0$$

the eigenvalues are:

$\lambda_1 = 9 + 2\sqrt{5}$

for $\lambda = \lambda_1$, then

$$\begin{cases} (-4-2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4-2\sqrt{5})x_2 = 0 \end{cases}$$

$$x_1 = \frac{-(4-2\sqrt{5})x_2}{2}$$
$$x_1 = (-2+\sqrt{5})x_2$$

$$-2x_2 + 2x_2 = 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (-2+\sqrt{5})x_2 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -2+\sqrt{5} \\ 1 \end{pmatrix}$$

$\lambda_2 = 9 - 2\sqrt{5}$

for $\lambda = \lambda_2$

$$\begin{cases} (-4+2\sqrt{5})x_1 + 2x_2 = 0 \\ 2x_1 + (4+2\sqrt{5})x_2 = 0 \end{cases}$$

$$x_1 = \frac{-(4+2\sqrt{5})x_2}{2} = (-2-\sqrt{5})x_2$$

$$-2x_2 + 2x_2 = 0$$

$$x = \begin{pmatrix} -(4+2\sqrt{5})x_2 \\ x_2 \end{pmatrix} = t\begin{pmatrix} -4-2\sqrt{5} \\ 1 \end{pmatrix}$$

▷ Let's compute the principal directions

$$\theta_1 = \tan^{-1}\left(\frac{x_2}{x_1}\right) = \tan^{-1}\left(\frac{1}{-2+\sqrt{5}}\right)$$

$= 76°43'2.91''$ with the positive x_1 direction

$$\theta_2 = \tan^{-1}\left(\frac{1}{-4-2\sqrt{5}}\right)$$

$= 173°16'5.87''$ with the positive x_1 direction

$\boxed{c}$ $A = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix}$

▷ Let's solve the eigen problem...

$Ax = \lambda x \qquad Ax - \lambda x = 0$

$$\begin{cases} 1.25x_1 + 0.75x_2 = \lambda x_1 \\ 0.75x_1 + 1.25x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{matrix} (1.25-\lambda)x_1 + 0.75x_2 = 0 \\ 0.75x_1 + (1.25-\lambda)x_2 = 0 \end{matrix}$$

so...

$$\det\begin{pmatrix} 1.25-\lambda & 0.75 \\ 0.75 & 1.25-\lambda \end{pmatrix} = 0$$

$$\lambda^2 - 2.5\lambda + 1 = 0$$

the eigen values are:

$\lambda_1 = 2$

for $\lambda = \lambda_1$, then

$$\begin{cases} -0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 - 0.75x_2 = 0 \end{cases}$$

$x_1 = x_2$

$x = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 0.5$

for $\lambda = \lambda_2$, then

$$\begin{cases} 0.75x_1 + 0.75x_2 = 0 \\ 0.75x_1 + 0.75x_2 = 0 \end{cases}$$

$x_1 = -x_2$

$x = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

▷ Let's compute the principal directions.

$\theta_1 = \text{Tan}^{-1}\left(\frac{1}{1}\right)$
$= 45°0'0''$ with the positive x_1 direction

$\theta_2 = \text{Tan}^{-1}\left(\frac{1}{-1}\right)$
$= 135°0'0''$ with the positive x_1 direction

The eigen values show that in the principal directions θ_1 and θ_2 the deformation is stretched by factors λ_1 and λ_2, respectively.

Homework 3 Eigenvalue problems.

$\boxed{4}$ Modelos de población.

$\boxed{a}$

$$L = \begin{pmatrix} 0 & 9.0 & 5.0 \\ 0.4 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}$$

Note: The Leslie model describes age-specified population growth.

Assumptions:

i) Let the oldest age attained be 99 years.

ii) Since the provided matrix L has 3 columns, divide the population into three age classes of 33 years each.

Note: Given a Leslie matrix $L = [l_{jk}]$,

1) l_{1k} is the average number of new borns to a single member durig the time he/she is in age class k.

2) $l_{j,j-1}$ is the fraction of members in age class $j-1$ that will survive and pass into class j.

Note: Proportional change means that we are looking for a distribution vector x such that $Lx = \lambda x$, where λ is the rate of change (growth if $\lambda > 1$, decrease if $\lambda < 1$).

Encuentra el ritmo de crecimiento en el modelo de Leslie con la matrix dada.

▷ Let's find the λ's

$$\det(L - \lambda I)x = 0$$

$$\det\begin{pmatrix} -\lambda & 9.0 & 5.0 \\ 0.4 & -\lambda & 0 \\ 0 & 0.4 & -\lambda \end{pmatrix} = 0$$

$$-0.4 \det\begin{pmatrix} -\lambda & 5.0 \\ 0.4 & -\lambda \end{pmatrix} + (-\lambda)\det\begin{pmatrix} -\lambda & 9.0 \\ 0.4 & -\lambda \end{pmatrix} = 0$$

$$(-0.4)(\lambda^2 - 2) + (-\lambda)(\lambda^2 - 3.6) = 0$$

$$-0.4\lambda^2 + 0.8 - \lambda^3 + 3.6\lambda = 0$$

$$-\lambda^3 - 0.4\lambda^2 + 3.6\lambda + 0.8 = 0$$

$$\lambda_1 = 1.82, \quad \lambda_2 = -0.23, \quad \lambda_3 = -2$$

▷ A positive root is found to be $\lambda = 1.82$

The growth rate will be 1.82 per 33 years.

$\boxed{b}$ $L = \begin{pmatrix} 0 & 3.0 & 2.0 & 2.0 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}$

Assumptions:

i) Let the oldest age attained be 100 years

ii) Since the provided matrix L has four columns, let's divide the population into four age classes of 25 years each.

▷ let's find the λ's

$$\det(L - \lambda I)x = 0$$

$$\det\begin{pmatrix} -\lambda & 3.0 & 2.0 & 2.0 \\ 0.5 & -\lambda & 0 & 0 \\ 0 & 0.5 & -\lambda & 0 \\ 0 & 0 & 0.1 & -\lambda \end{pmatrix} = +(-\lambda)\begin{pmatrix} -\lambda & 0 & 0 \\ 0.5 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix} - (3.0)\begin{pmatrix} 0.5 & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0.1 & -\lambda \end{pmatrix} + (2.0)\begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & -\lambda \end{pmatrix} - (2.0)\begin{pmatrix} 0.5 & -\lambda & 0 \\ 0 & 0.5 & -\lambda \\ 0 & 0 & 0.1 \end{pmatrix}$$

$$= -\lambda\left[+(-\lambda)\begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0\right]$$
$$-3.0\left[+(0.5)\begin{pmatrix} -\lambda & 0 \\ 0.1 & -\lambda \end{pmatrix} - 0 + 0\right]$$
$$+2.0\left[+(0.5)\begin{pmatrix} 0.5 & 0 \\ 0 & -\lambda \end{pmatrix} - \cancel{(-\lambda)\begin{pmatrix} 0 & 0 \\ 0 & -\lambda \end{pmatrix}}^{0} + 0\right]$$
$$-2.0\left[+(0.5)\begin{pmatrix} 0.5 & -\lambda \\ 0 & 0.1 \end{pmatrix} - \cancel{(-\lambda)\begin{pmatrix} 0 & -\lambda \\ 0 & 0.1 \end{pmatrix}}^{0} + 0\right]$$
$$= -\lambda(-\lambda(\lambda^2)) - 3.0(0.5(\lambda^2)) + 2.0(0.5(-0.5\lambda))$$
$$-2.0(0.5(0.5(0.1)))$$
$$= \lambda^4 - 1.5\lambda^2 - 0.5\lambda - 0.05$$

$\lambda_1 = 1.37$ $\lambda_2 = -1.03$ $\lambda_3 = -0.17 + 0.07i$ $\lambda_4 = -0.17 - 0.07i$

▷ A positive root is found to be $\lambda = 1.37$

The growth rate will be 1.37 per 25 years.