I.1 $x - \pi \mathrm{y} + \sqrt[3]{5}z = 0$ is a linear equation
I.2 $x^2 + y^2 + z^2 = 1$ is NOT a linear equation
I.3 $x^{-1} + 7y + z = \left(\mathrm{Sin}\left[\frac{\pi}{9}\right]\right)^2$ is NOT a linear equation
I.4 $x + 7y + z = \mathrm{Sin}\left[\frac{\pi}{9}\right]$ is a linear equation
I.5 $3\mathrm{Cos}[x] - 4y + z = \sqrt{3}$ is NOT a linear equation
I.6 $\mathrm{Cos}[3]x - 4y + z = \sqrt{3}$ is a linear equation
II.7

```
ContourPlot[{x + y==0, 2 * x + y==3}, {x, 2, 4}, {y, -4, -2}, ContourStyle → {Blue, Orange}]
```

(*

$$\left\{ \begin{array}{ll} x + y & = 0 \\ 2x + y & = 3 \end{array} \right. \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix} \Rightarrow R_2 - 2R_1 \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \end{pmatrix} \Rightarrow \left\{ \begin{array}{ll} x + y & = 0 \\ -y & = 3 \end{array} \right.$$

$y = -3$

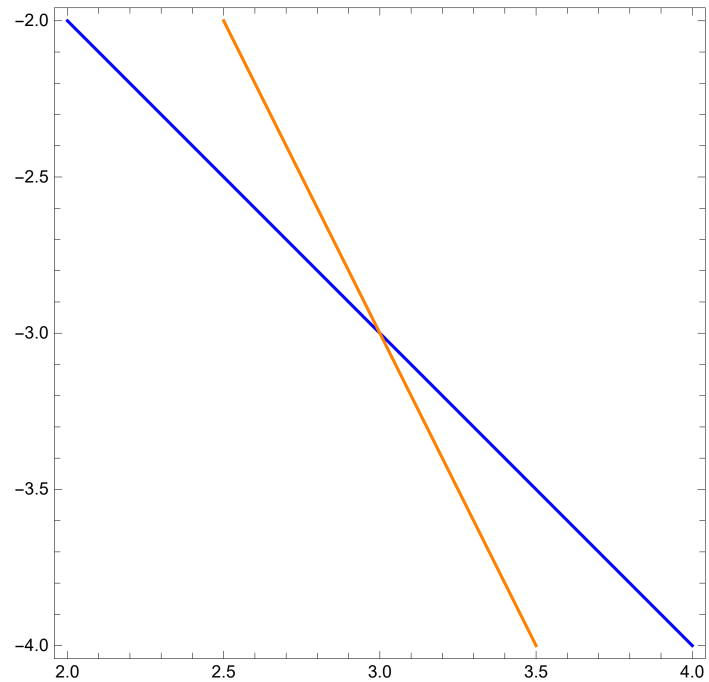
$x + y = 0$

$x = -(-3)$

$x = 3$

*)

```
Solve[{x + y==0, 2 * x + y==3}, {x, y}]
```

Out[ ]= 

$\{\{x \to 3, y \to -3\}\}$

II.8

ContourPlot[$\{x - 2 * y == 7, 3 * x + y == 7\}, \{x, 2, 4\}, \{y, -3, -1\}$, ContourStyle $\to$ {Blue, Orange}]

(*

$$\left\{ \begin{array}{ll} x - 2y & = 7 \\ 3x + y & = 7 \end{array} \right. \Rightarrow \begin{pmatrix} 1 & -2 & 7 \\ 3 & 1 & 7 \end{pmatrix} \Rightarrow R_2 - 3R_1 \Rightarrow \begin{pmatrix} 1 & -2 & 7 \\ 0 & 7 & -14 \end{pmatrix} \Rightarrow \left\{ \begin{array}{ll} x - 2y & = 7 \\ 7y & = -14 \end{array} \right.$$

$y = -2$

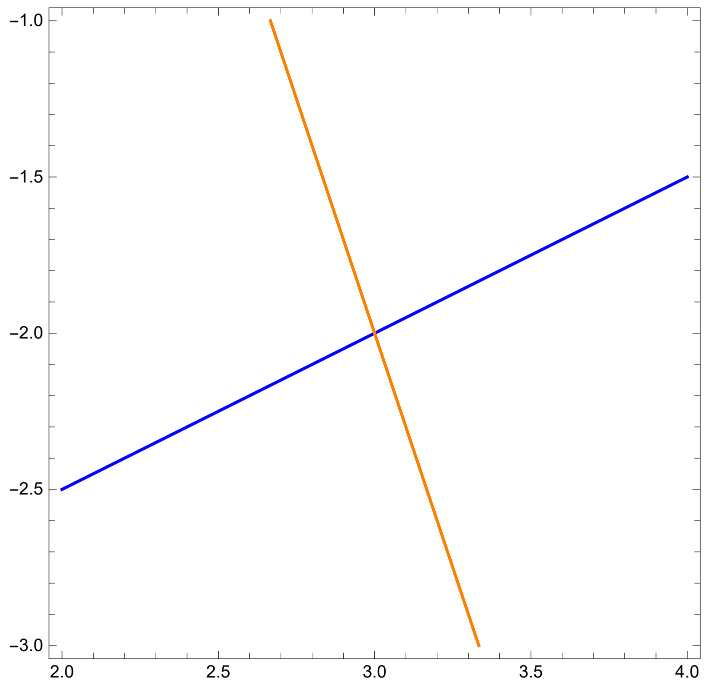
$x - 2y = 7$

$x = 7 + 2(-2)$

$x = 3$

*)

Solve[$\{x - 2 * y == 7, 3 * x + y == 7\}, \{x, y\}$]

Out[]=


$\{\{x \to 3, y \to -2\}\}$

II.9

ContourPlot[{3 ∗ x − 6 ∗ y==3, −x + 2 ∗ y==1}, {x, −3.125, 4.5}, {y, −1.75, 1.75}, ContourStyle → {Blue, Oran

(*

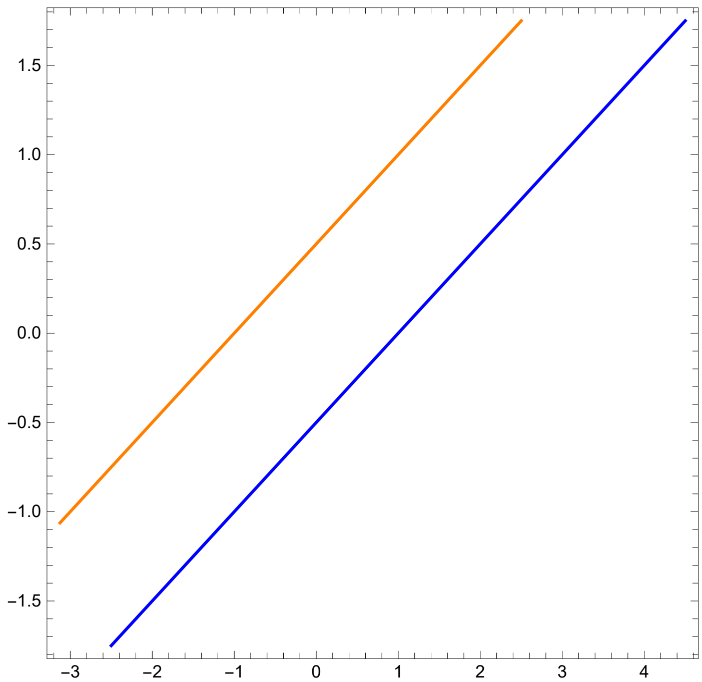
$$\left\{ \begin{array}{ll} 3x-6y & =3 \\ -x+2y & =1 \end{array} \right. \Rightarrow \begin{pmatrix} 3 & -6 & 3 \\ -1 & 2 & 1 \end{pmatrix} \Rightarrow R_2+\tfrac{1}{3}R_1 \Rightarrow \begin{pmatrix} 3 & -6 & 3 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \left\{ \begin{array}{ll} 3x-6y & =3 \\ 0 & =2 \end{array} \right.$$

nosolution

***)**

Solve[{3 ∗ x − 6 ∗ y==3, − x + 2 ∗ y==1}, {x, y}]

Out[∘]=



{}

VIII

$$\left\{\begin{array}{ll} \text{kx}+y & =-2 \\ 2x-2y & =4 \end{array}\right. \Rightarrow \begin{pmatrix} k & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} \Rightarrow \frac{1}{k}R_1 \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & -\frac{2}{k} \\ 2 & -2 & 4 \end{pmatrix} \Rightarrow R_2-2R_1 \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & - \\ 0 & -\frac{2(1+k)}{k} & 4 \end{pmatrix}$$

$$x=0$$

$$y=-2;\text{forall}k$$

$$A=\begin{pmatrix} k & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix};$$

RowReduce$[A]$;

MatrixForm[%]

Solve$[\{k*x+y==-2, 2x-2y==4\},\{x,y\}]$

xMin $=-3$;

xMax $=2$;

ContourPlot3D$[\{k*x+y==-2, 2x-2y==4\},\{x,\text{xMin},\text{xMax}\},\{y,\text{xMin},\text{xMax}\},\{k,\text{xMin},\text{xMax}\},\text{Axes}\rightarrow$

PlotLegends → "Expressions"]

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

$\{\{x \to 0, y \to -2\}\}$

Optional 1

Consider the following matrix

A= $\begin{pmatrix} \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi^2 \\ \pi^3 & \pi^3 & \pi^3 \end{pmatrix}$

1. Find the reduced row echelon from of A; then find the rank of A.
2. How can you enter in Mathenatica (in one line) the matrix A? (without typing every entry!). Hint: Consider the Table command.
3. Now generalize the result as follows: let X be the following arbitrary square matrix of size n, where c is any non-zero number. Compute the rank of X.

X= $\begin{pmatrix} c & c & \cdots & c \\ c^2 & c^2 & \cdots & c^2 \\ \vdots & \vdots & \vdots & \vdots \\ c^n & c^n & \cdots & c^n \end{pmatrix}$

Optional 1.1

***A* =;**

Print["RowReduce[A]="MatrixForm[RowReduce[*A*]]]

Print["MatrixRank[A]="]

MatrixRank[*A*]

RowReduce[A]= $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

MatrixRank[A]=

1

Optional 1.2

***A* = Table[Pi^*i*, {*i*, 1, 3}, {*j*, 1, 3}];**

MatrixForm[*A*]

$$\begin{pmatrix} \pi & \pi & \pi \\ \pi^2 & \pi^2 & \pi^2 \\ \pi^3 & \pi^3 & \pi^3 \end{pmatrix}$$

Optional 1.3

$$\text{X=}\begin{pmatrix} c & c & \cdots & c \\ c^2 & c^2 & \cdots & c^2 \\ \vdots & \vdots & \vdots & \vdots \\ c^n & c^n & \cdots & c^n \end{pmatrix} \Rightarrow \begin{matrix} \frac{1}{X_{11}}R_1 \to R_1 \\ \frac{1}{X_{21}}R_2 \to R_2 \\ \vdots \\ \frac{1}{X_{\text{n}1}}R_n \to R_n \end{matrix} \Rightarrow \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \Rightarrow \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ \vdots \\ R_n - R_1 \to R_n \end{matrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Optional 2

For what value(s) of k, if any, will the following system:

$$\{ \quad \begin{matrix} x + y + \text{kz} & = 1 \\ x + \text{ky} + z & = 1 \\ \text{kx} + y + z & = -2 \end{matrix}$$

have

1. No solution
2. A unique solution
3. Infinitely many solutions

Hint. Find the reduced echelon form of the augmented matrix, then analyze different cases (beware of division by zero!).

$$\{ \quad \begin{matrix} x + y + \text{kz} & = 1 \\ x + \text{ky} + z & = 1 \\ \text{kx} + y + z & = -2 \end{matrix} \Rightarrow \begin{pmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{pmatrix} \Rightarrow R_1 \leftrightarrow R_3 \Rightarrow \begin{pmatrix} k & 1 & 1 & -2 \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \Rightarrow$$

$$\frac{1}{k}R_1 \to R_1 \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & \frac{1}{k} & -\frac{2}{k} \\ 1 & k & 1 & 1 \\ 1 & 1 & k & 1 \end{pmatrix} \Rightarrow \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix} \Rightarrow \begin{pmatrix} 1 & \frac{1}{k} & \frac{1}{k} & -\frac{2}{k} \\ 0 & k - \frac{1}{k} & 1 - \frac{1}{k} & 1 + \frac{2}{k} \\ 0 & 1 - \frac{1}{k} & k - \frac{1}{k} & 1 + \frac{2}{k} \end{pmatrix} \Rightarrow$$

$$R_1 - \frac{1}{-1+k^2}R_2 \to R_1 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & k - \frac{1}{k} & 1 - \frac{1}{k} & 1 + \frac{2}{k} \\ 0 & 1 - \frac{1}{k} & k - \frac{1}{k} & 1 + \frac{2}{k} \end{pmatrix} \Rightarrow \frac{k}{-1+k^2}R_2 \to R_2 \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & 1 & \frac{1}{1+k} & \frac{2+k}{-1+k^2} \\ 0 & 1 - \frac{1}{k} & k - \frac{1}{k} & 1 + \frac{2}{k} \end{pmatrix} \Rightarrow R_3 - \left(1 - \frac{1}{k}\right) R_2 \to R_3 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & 1 & \frac{1}{1+k} & \frac{2+k}{-1+k^2} \\ 0 & 0 & k - \frac{2}{1+k} & 1 + \frac{1}{1+k} \end{pmatrix} \Rightarrow$$

$$\frac{1+k}{-2+k+k^2}R_3 \to R_3 \Rightarrow \begin{pmatrix} 1 & 0 & \frac{1}{1+k} & \frac{1+2k}{1-k^2} \\ 0 & 1 & \frac{1}{1+k} & \frac{2+k}{-1+k^2} \\ 0 & 0 & 1 & \frac{1}{-1+k} \end{pmatrix} \Rightarrow \begin{matrix} R_2 - \frac{1}{1+k}R_3 \to R_2 \\ R_1 - \frac{1}{1+k}R_3 \to R_1 \end{matrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{-1+k} \\ 0 & 1 & 0 & \frac{1}{-1+k} \\ 0 & 0 & 1 & \frac{1}{-1+k} \end{pmatrix}$$

$z = \frac{1}{-1+k}$

$y = \frac{1}{-1+k}$

$x = -\frac{2}{-1+k}$

Optional 2.1

The system does not have a solution for k = 1.

Optional 2.2 & 2.3

Since y = z, the system does not have a unique solution. Therefore the system has infinitely many solutions for k $\neq$ 1.

$$\boldsymbol{A = \begin{pmatrix} 1 & 1 & k & 1 \\ 1 & k & 1 & 1 \\ k & 1 & 1 & -2 \end{pmatrix};}$$

RowReduce[*A*];

MatrixForm[%]

Solve[{$x + y + k * z == 1, x + k * y + z == 1, k * x + y + z == -2$}, {$x, y, z$}]

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{2}{-1+k} \\ 0 & 1 & 0 & \frac{1}{-1+k} \\ 0 & 0 & 1 & \frac{1}{-1+k} \end{pmatrix}$$

$$\left\{\left\{x \to -\frac{2}{-1+k}, y \to \frac{1}{-1+k}, z \to \frac{1}{-1+k}\right\}\right\}$$