Tecnológico de Monterrey

Linear Algebra

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Outline

- Vectorial spaces
- Linear equations
- Linear dependence and independence
- Lagrange polynomial
- The left division method
- Problems and examples

Vectorial spaces

- Let P and Q be two different points
- Let u and v be vectors that start at the origin and end at P and Q respectively
- Any point from P to Q can be found with the line equation:
- $x = u + tw$ where $w = v - u$ and $t \in \mathbb{R}$

Vectorial spaces

Vectorial spaces

- Let P, Q and R three no collinear points and u and v the vectors that start at P and end at Q and R respectively.
- The plane equation that contains P, Q and R is:
- $x = P + t_1 u + t_2 v \;\; where \; t_1, t_2 \in \mathbb{R}$

Vectorial spaces

Vectorial space

- A vectorial or linear space V is a collection of vectors that can be added or may be multiplied by a scalar value a

- Possible operations:
  - Addition
  - Multiplication

Linear combination

- A vector x is a linear combination of elements, if there is a finite number of elements $y_1, \dots, y_p$ and a set of scalars $\mathrm{a}_1, \dots, \mathrm{a}_\mathrm{p}$ such that:

$$x = a_1 y_1 + \cdots + a_p y_p$$

Linear equations

❖To solve a system of linear equations only three operations can be used:

1. Change the order of the linear equations in the system
2. Multiplication by a non-null scalar
3. Addition of vectors/equations

Linear equations

1. $3x + 2y + z = 1$
2. $5x + 3y + 4z = 2$
3. $x + y - z = 1$

Linear equations

- Change the order of the linear equations in the system

$3 \rightarrow 1$

$$\begin{aligned}
&1 \quad x + y - z = 1\\
&2 \quad 3x + 2y + z = 1\\
&3 \quad 5x + 3y + 4z = 2
\end{aligned}$$

Linear equations

- Multiplication by a non-null scalar

1 $* 3$

1 $3x + 3y - 3z = 3$

2 $3x + 2y + z = 1$

3 $5x + 3y + 4z = 2$

Linear equations

- Addition of vectors/equations

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Linear equations

$$
\begin{aligned}
&(1) \quad x + y - z = 1 \\
&(2) \quad 0 - y + 4z = -2 \\
&(3) \quad 0 - 2y + 9z = -3
\end{aligned}
$$

Linear equations

- Main objective:

1. The first non-null coefficient of any equation is 1
2. If the first non-null coefficient value is an algebraic symbol, then in the other equations it must have a null coefficient
3. The first non-null algebraic symbol of a linear equation has a bigger subindex than the precedent linear equation

Linear equations

$$
\begin{aligned}
&(1) \quad x + y - z = 1 \\
&(2) \quad 0 - y + 4z = -2 \\
&(3) \quad 0 - 2y + 9z = -3
\end{aligned}
$$

Linear equations

$$\begin{cases} 5x - 3y - z = 1 \\ x + 4y - 6z = -1 \\ 2x + 3y + 4z = 9 \end{cases} \equiv \overset{A}{\begin{bmatrix} 5 & -3 & -1 \\ 1 & 4 & -6 \\ 2 & 3 & 4 \end{bmatrix}} \overset{x}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{Y}{\begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}}$$

$$\equiv \quad Ax = Y$$

The left division method

- Matrix form

$$\begin{cases} 5x - 3y - z = 1 \\ x + 4y - 6z = -1 \\ 2x + 3y + 4z = 9 \end{cases} \equiv \overbrace{\begin{bmatrix} 5 & -3 & -1 \\ 1 & 4 & -6 \\ 2 & 3 & 4 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}^{x} = \overbrace{\begin{bmatrix} 1 \\ -1 \\ 9 \end{bmatrix}}^{Y}$$

$$\equiv \quad Ax = Y \rightarrow \underset{N\times 1}{x} = \underset{N\times N}{A^{-1}} \underset{N\times 1}{Y}$$

R → solve(A, Y)

Linear equations

❖ Underdetermined

$$\begin{bmatrix} 1 & -2 & 1 & 2 & 3 \\ 2 & -4 & 4 & 7 & 7 \\ 1 & -2 & 2 & 5 & 2 \\ 2 & -4 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 15 \\ 12 \end{bmatrix}$$

$$A \qquad x \qquad Y$$

Linear equations

- Overdetermined

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}_{y}$$

$$x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Linear dependence and independence

- What does it mean to be linearly independent?
  - Each vector is “unique”
  - The vectors cannot be decomposed in other vectors from the same set

- If one vector of a set can be decomposed in other vectors from the same set, it is called linearly dependent

- The rank of a matrix is equal to the number of unique vectors in the matrix. If the rank is equal to the number of variables, the matrix is linearly independent

Polynomials

- What happens if my dependent variable is a linear combination of the same variable, but with different exponents

- Example: A rock is thrown from a Cliff and I want to get the distance in y that has been travelled, but I only have 3 or 4 time measurements

- Formula to get:

$$y = y_i + v_i t + 0.5 g t^2$$

- How do we get it?
  - We need to interpolate

Lagrange polynomial

- We have n different points (c_i , f(x))
- We want to get the function f(x) using only the n different c points
- Let $c_o, c_1, \dots, c_n$ be different elements. We get the polynomials $f_0(x), f_1(x), \dots, f_n(x)$. Where $\mathrm{f_i(x)}$ is described as:

$$f_i(x) = \prod_{\substack{j=\emptyset \\ j\neq i}}^{n} \frac{x - c_j}{c_i - c_j} \qquad f_i(c_i) = \begin{cases} \emptyset & , f \quad i \neq j \\ 1 & , f \quad i = \underline{1} \end{cases}$$

- Are called Lagrange Polynomial associated with $c_o, c_1, \dots, c_n$

Lagrange polynomial

- Example:

- The polynomial that we want to model touches the following points : (1,8),(2,5)(3,-4).

- The Lagrange polynomial that is associated with $C_0 = 1, C_1 = 2\,, C_2 = 3$ are:

Lagrange polynomial

$$f_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - c_j}{c_i - c_j}$$

Lagrange interpolation

- To get the association with the $f(x)$ value we need to apply the next formula:

$$y = \sum_{i=0}^{n} b_i f_i$$

- Where b_i is the value of $f(x)$ at c_i

Lagrange interpolation

$f_\emptyset = \frac{1}{2}(x^2 - 5x + 6)$ $f_1 = -1(x^2 - 4x + 3)$ $f_2 = \frac{1}{2}(x^2 - 3x + 2)$

$\therefore$

$$g(x) = \sum_{i=\emptyset}^{2} b_i f_i(x) = 8f_\emptyset + 5f_1 - 4f_2$$

$$= -3x^2 + 6x + 5x$$