The Navier Stokes Equation

The Navier-Stokes Equation

https://www.britannica.com/science/Navier-Stokes-equation

Navier-Stokes equation, in fluid mechanics, a partial differential equation that describes the flow of incompressible fluids.

The equation is a generalization of the equation devised by Swiss mathematician Leonhard Euler in the 18th century to describe the flow of incompressible and frictionless fluids.

In 1821 French engineer Claude-Louis Navier introduced the element of viscosity (friction) for the more realistic and vastly more difficult problem of viscous fluids.

Throughout the middle of the 19th century, British physicist and mathematician Sir George Gabriel Stokes improved on this work, though complete solutions were obtained only for the case of simple two-dimensional flows.

The complex vortices and turbulence, or chaos, that occur in three-dimensional fluid (including gas) flows as velocities increase have proven intractable to any but approximate numerical analysis methods.

Part 1: Continuity Equation

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Reflections

The **Navier-Stokes equation**, in fluid mechanics, a partial differential equation that describes the flow of incompressible fluids.

How can we get such differential equation?

- In order to do so, we need to imagne a flow of a liquid in a channel…
- Then "take" a differential volume element and…
- Make a mass balance on that differential element (CONTINUITY EQUATION) and
- Make a force balance on that differential element (MOMENTUM EQUATION)
- Apply the elements of the moentum equation appropriate for the system you have a channel, die, etc.
- Afterwards you use a Constitutive Equation to relate the Stresses to the Memory function data

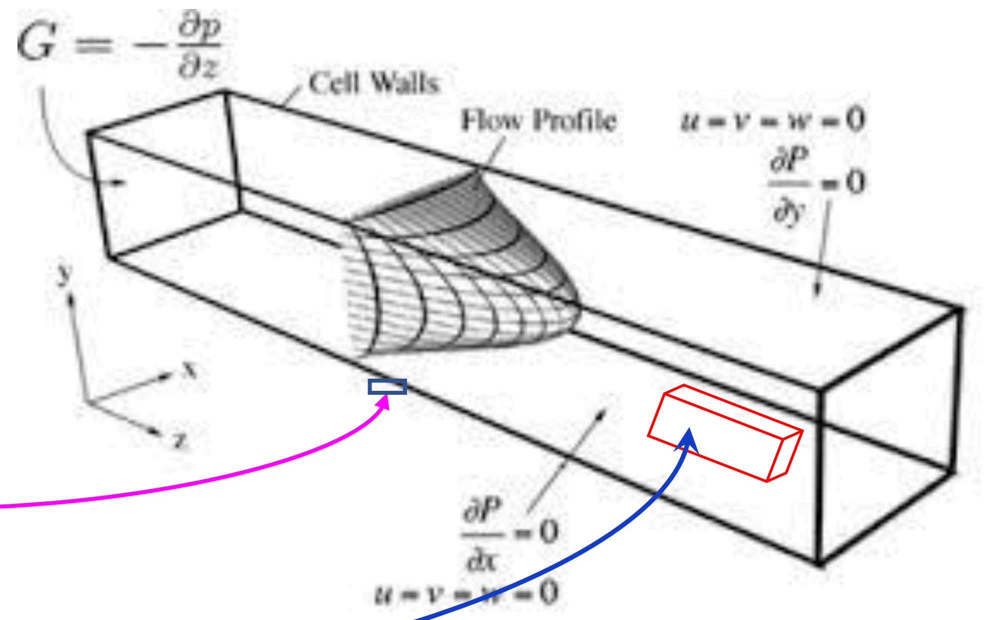


Image channel from: Fully developed flow of non-Newtonian fluids in a straight uniform square duct through porous medium. Journal of the Association of Arab Universities for Basic and Applied Sciences Volume 23, June 2017, Pages 66-74

Navier-Stokes Equation

Part 2: Momentum Equation

Conservation of Momentum

Temporary change of momentum inside a control volume (CV)

$$\text{momentum} = m\,v$$

$$\frac{d\ \text{momentum}}{d\ \text{time}} = \frac{d\,mv}{d\,t} = \frac{m\,d\,v}{d\,t} = m\,a = F$$

$$\frac{d\ \text{momentum}}{d\ \text{time}} = \frac{\partial\,(\rho\,\vec{v}\,)}{\partial\,t}\,dx\,dy\,dz$$

so

Summation of forces on the CV

Using the notation when we worked on the continuity equation

$$\frac{\textbf{d momentum}}{\textbf{d time}} = \frac{\partial\, (\rho\, \vec{v}\,)}{\partial\, t}\ \textbf{dx dy dz}$$

Conservation of Momentum

Therefore…

Temporary change of momentum inside the control volume (CV) $= \sum$ Momentum flow into the CV $-$ $\sum$ Momentum flow out the CV $+ \sum$ Body forces

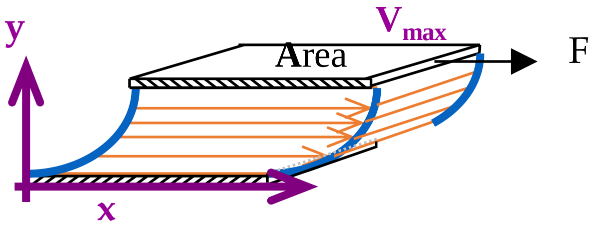
Where is the momentum coming from when you have fluid flowing in and out the CV?

We need to remember that…

The momentum can be transported by the mass moving in anad out (CONVECTION) and by transporting the momentum layer to layer wthin the CV (CONDUCTION)

This chapter is concerned only with *laminar flow*. "Laminar flow" is the orde
that is observed, for example, in tube flow at velocities sufficiently low that tiny
injected into the tube move along in a thin line. This is in sharp contrast with th
chaotic "turbulent flow" at sufficiently high velocities that the particles are flu
and dispersed throughout the entire cross section of the tube. Turbulent flow is

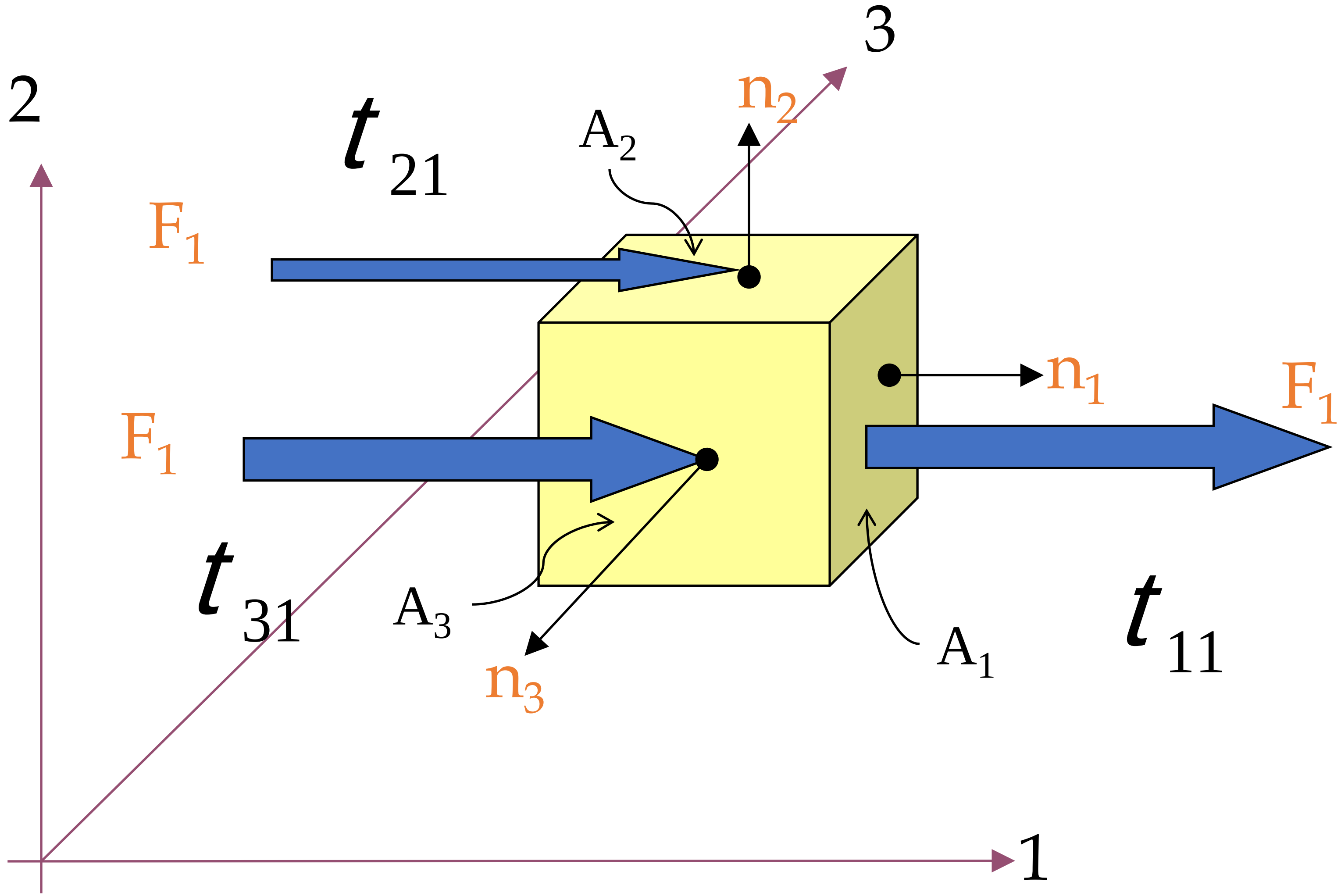
Tangential surface forces



Direction in which the movement is transmitted

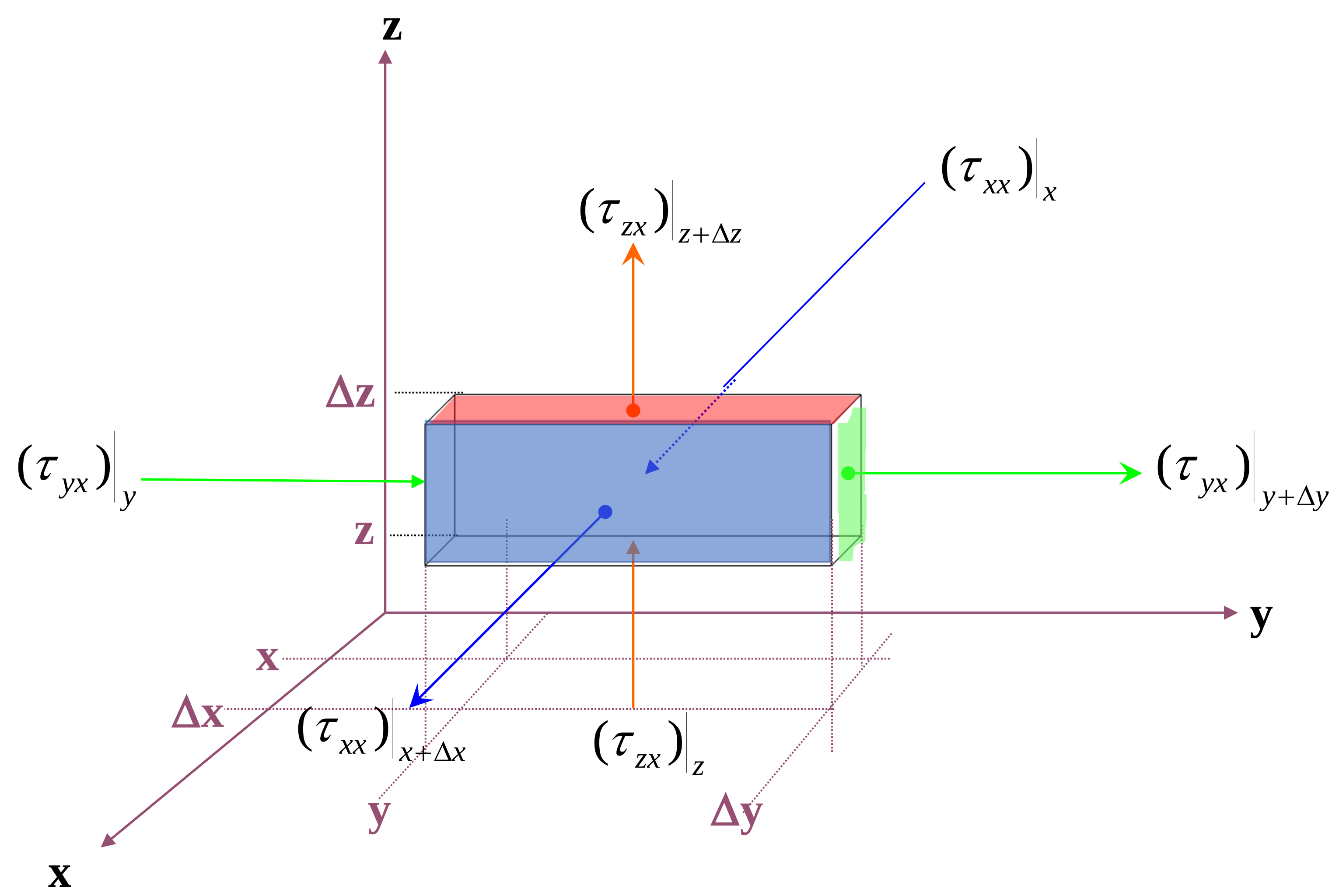
t_{yx}

Direction in which the material (macroscopic scale) is moving



Momentum Balance

due to a force in the x direction acting on each surface



Momentum Balance
due to forces in the x direction acting on each surface

A momentum balance over the diferential element:

Rate of momentum in - **Rate of momentum out** = **Rate of Momentum Acumulation**

=

If $dV = dxdydz \neq f(x,y,z,t)$

Momentum Balance

due to forces in the x direction acting on each surface

A momentum balance over the diferential element:

Rate of momentum input	-	Rate of momentum output	+	Sum of forces acting over the system	=	Rate of Momentum Acumulation

$$\Delta y\Delta z(\rho v_x v_x\big|_x - \rho v_x v_x\big|_{x+\Delta x}) + \Delta x\Delta z(\rho v_y v_x\big|_y - \rho v_y v_x\big|_{y+\Delta y}) +$$

$$\Delta x\Delta y(\rho v_z v_x\big|_z - \rho v_z v_x\big|_{z+\Delta z}) +$$

$$\Delta y\Delta z(\tau_{xx}\big|_x - \tau_{xx}\big|_{x+\Delta x}) + \Delta x\Delta z(\tau_{yx}\big|_y - \tau_{yx}\big|_{y+\Delta y}) +$$

$$\Delta x\Delta y(\tau_{zx}\big|_z - \tau_{zx}\big|_{z+\Delta z}) +$$

$$\Delta y\Delta z(p\big|_x - p\big|_{x+\Delta x}) + \rho g_x \Delta x\Delta y\Delta z = \frac{\partial}{\partial t}(\rho v_x \Delta x\Delta y\Delta z)$$

If $dV = dxdydz \neq f(x,y,z,t)$ Dividing by $\Delta x\Delta y\Delta z$ and making the differential approaching to zero

A momentum balance over the diferential element:

Rate of momentum in	-	Rate of momentum out	+	Sum of forces acting over the system	=	Rate of Momentum Acumulation

Convection

$$\Delta y \Delta z(\rho v_x v_x|_x - \rho v_x v_x|_{x+\Delta x}) + \Delta x \Delta z(\rho v_y v_x|_y - \rho v_y v_x|_{y+\Delta y}) +$$

$$\Delta x \Delta y(\rho v_z v_x|_z - \rho v_z v_x|_{z+\Delta z}) +$$

$$\boxed{V_y \frac{\partial V_x}{\partial x}} \quad \boxed{V_z \frac{\partial V_x}{\partial z}} \quad \boxed{V_y \frac{\partial V_x}{\partial y}}$$

Transmision

$$\Delta y \Delta z(\tau_{xx}|_x - \tau_{xx}|_{x+\Delta x}) + \Delta x \Delta z(\tau_{yx}|_y - \tau_{yx}|_{y+\Delta y}) +$$

$$\Delta x \Delta y(\tau_{zx}|_z - \tau_{zx}|_{z+\Delta z}) +$$

$$\boxed{\frac{\partial \tau_{xx}}{\partial y}} \quad \boxed{\frac{\partial \tau_{yx}}{\partial y}} \quad \boxed{\frac{\partial \tau_{zx}}{\partial z}}$$

$$\underbrace{\Delta y \Delta z(p|_x - p|_{x+\Delta x})}_{\text{pressure}} + \underbrace{\rho g_x \Delta x \Delta y \Delta z}_{\text{gravity}} = \underbrace{\frac{\partial}{\partial t}(\rho v_x \Delta x \Delta y \Delta z)}_{\text{Accumulation of Momentum}}$$

$$\boxed{\frac{\partial p}{\partial x}} \quad \boxed{\rho g_x} \quad \boxed{\rho \frac{\partial V_x}{\partial t}}$$

If $dV = dxdydz \neq f(x,y,z,t)$

Equation of Motion in Rectangular Coordinates (x, y, z)

Just for the x direction

$$r\left(\frac{\P V_x}{\P t}+V_x\frac{\P V_x}{\P x}+V_y\frac{\P V_x}{\P y}+V_z\frac{\P V_x}{\P z}\right)=$$

$$-\frac{\P p}{\P x}-\left(\frac{\P t_{xx}}{\P x}+\frac{\P t_{yx}}{\P y}+\frac{\P t_{zx}}{\P z}\right)+r\,g_x$$

We need to invite Nabla and have a dot product

For assingment to convert this into Nabla products and into Tensors

Equation of Motion in Rectangular Coordinates (x, y, z)

x- component)

$$\rho\left(\frac{\partial V_x}{\partial t}+V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}+V_z\frac{\partial V_x}{\partial z}\right)=$$

$$-\frac{\partial p}{\partial x}-\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)+\rho g_x$$

y- component)

$$\rho\left(\frac{\partial V_y}{\partial t}+V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}+V_z\frac{\partial V_y}{\partial z}\right)=$$

$$-\frac{\partial p}{\partial y}-\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right)+\rho g_y$$

z- component)

$$\rho\left(\frac{\partial V_z}{\partial t}+V_x\frac{\partial V_z}{\partial x}+V_y\frac{\partial V_z}{\partial y}+V_z\frac{\partial V_z}{\partial z}\right)=$$

$$-\frac{\partial p}{\partial z}-\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

For assingment to convert this into
Nabla products and into Tensors

Equation of Motion in Cylindrical Coordinates (r, θ, z)

r- component)

$$\rho\left(\frac{\partial V_r}{\partial t}+V_r\frac{\partial V_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}-\frac{V_\theta^2}{r}+V_z\frac{\partial V_r}{\partial z}\right)=$$

$$-\frac{\partial p}{\partial r}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr})+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}-\frac{\tau_{\theta\theta}}{r}+\frac{\partial \tau_{rz}}{\partial z}\right)+\rho g_r$$

θ- component)

$$\rho\left(\frac{\partial V_\theta}{\partial t}+V_r\frac{\partial V_\theta}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_r V_\theta}{r}+V_z\frac{\partial V_\theta}{\partial z}\right)=$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta}-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})+\frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta}+\frac{\partial \tau_{\theta rz}}{\partial z}\right)+\rho g_\theta$$

z- component)

$$\rho\left(\frac{\partial V_z}{\partial t}+V_r\frac{\partial V_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}\right)=$$

$$-\frac{\partial p}{\partial z}-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \tau_{zz}}{\partial z}\right)+\rho g_z$$

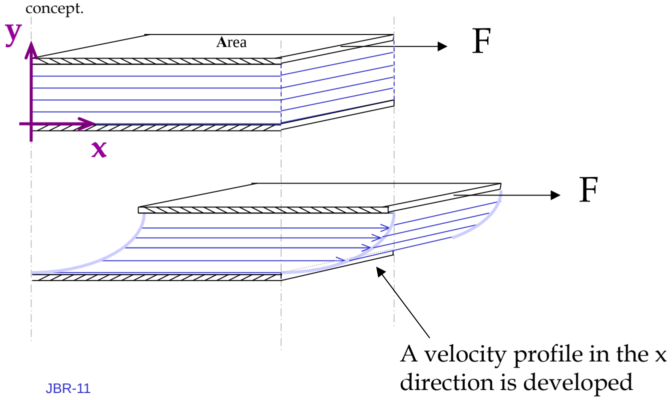
For assingment to convert this into Nabla products and into Tensors

ACTIVITY

TAKE 10 MINUTES TO WORK ON THE

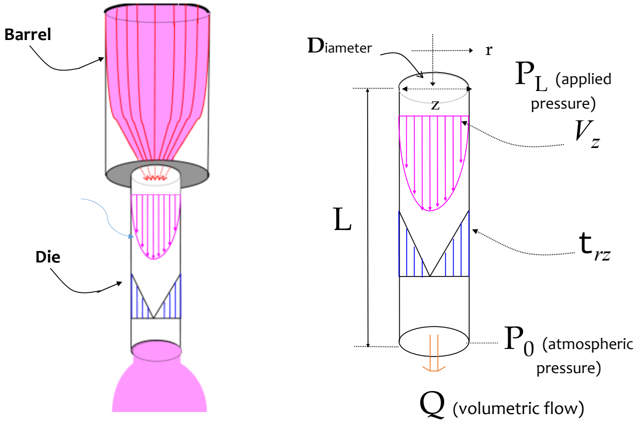
The shear phenomenon

Pictorial model: A set of parallel sheets, an imposed **F**orce, an **A**rea, a **V**elocity, a "new" concept.



Equation of Motion in Rectangular Coordinates (x, y, z)
and come back with theequations left after you decide what to drop from those equations

Capillary rheometer



Equation of Motion in Cylindrical Coordinates (x, y, z)
and come back with theequations left after you decide what to drop from those equations

For assingment apply this concept to fiber spinning and blown film processes

TABLE 3.4-3

THE EQUATION OF MOTION IN CYLINDRICAL COORDINATES (r, θ, z)

In terms of τ:

r-component[a]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r}$$

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z}\right) + \rho g_r \qquad \text{(A}$$

θ-component[b]

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$-\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z}\right) + \rho g_\theta \qquad \text{(B}$$

z-component

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z}$$

$$-\left(\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z}\right) + \rho g_z \qquad \text{(C}$$

In terms of velocity gradients for a Newtonian fluid with constant ρ and μ:

r-component[a]

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r}$$

$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] + \rho g_r \qquad \text{(D}$$

θ-component[b]

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] + \rho g_\theta \qquad \text{(E}$$

z-component

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z}$$

$$+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z \qquad \text{(F}$$