16. $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \ (x^2 < \infty)$.

17. $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \ (x^2 < \infty)$.

APPENDIX D. A SHORT TABLE OF DERIVATIVES[2]

In the following list, a, b, and c are constants, and e is the base of natural logarithms:

1. $\frac{\mathrm{d}}{\mathrm{d}x}(au) = a\frac{\mathrm{d}u}{\mathrm{d}x}$.

2. $\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$.

3. $\frac{\mathrm{d}}{\mathrm{d}x}(uvw) = uv\frac{dw}{\mathrm{d}x} + uw\frac{\mathrm{d}v}{\mathrm{d}x} + vw\frac{\mathrm{d}u}{\mathrm{d}x}$.

4. $\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = nx^{n-1}$.

5. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{u}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{v^2}\left(v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}\right)$.

6. $\frac{\mathrm{d}}{\mathrm{d}x}f(u) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$, where f is some differentiable function of u and u is some differentiable function of x (the *chain rule*).

7. $\frac{\mathrm{d}^2}{\mathrm{d}x^2}f(u) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{\mathrm{d}^2f}{\mathrm{d}u^2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2$.

8. $\frac{\mathrm{d}}{\mathrm{d}x}\sin(ax) = a\cos(ax)$.

9. $\frac{\mathrm{d}}{\mathrm{d}x}\cos(ax) = -a\sin(ax)$.

10. $\frac{\mathrm{d}}{\mathrm{d}x}\tan(ax) = a\sec^2(ax)$.

11. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{ctn}(ax) = -a\csc^2(ax)$.

12. $\frac{\mathrm{d}}{\mathrm{d}x}\sec(ax) = a\sec(ax)\tan(ax)$.

13. $\frac{\mathrm{d}}{\mathrm{d}x}\csc(ax) = -a\csc(ax)\mathrm{ctn}(ax)$.

14. $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\left(\frac{x}{a}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\arcsin\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$ if x/a is in the first or fourth quadrant $= \frac{-1}{\sqrt{a^2 - x^2}}$ if x/a is in the second or third quadrant.

15. $\frac{\mathrm{d}}{\mathrm{d}x}\cos^{-1}\left(\frac{x}{a}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\arccos\left(\frac{x}{a}\right) = \frac{-1}{\sqrt{a^2 - x^2}}$ if x/a is in the first or second quadrant $= \frac{1}{\sqrt{a^2 - x^2}}$ if x/a is in the third or fourth quadrant.

16. $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(\frac{x}{a}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\arctan\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$.

17. $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{ctn}^{-1}\left(\frac{x}{a}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{arcctn}\left(\frac{x}{a}\right) = \frac{-a}{a^2 + x^2}$.

18. $\frac{\mathrm{d}}{\mathrm{d}x}e^{ax} = ae^{ax}$.

19. $\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x\ln(a)$.

20. $\frac{\mathrm{d}}{\mathrm{d}x}a^{cx} = ca^{cx}\ln(a)$.

21. $\frac{\mathrm{d}}{\mathrm{d}x}u^y = yu^{y-1}\frac{\mathrm{d}}{\mathrm{d}x} + u^y\ln(u)\frac{\mathrm{d}}{\mathrm{d}x}$.

22. $\frac{\mathrm{d}}{\mathrm{d}x}x^X = x^x[1 + \ln(x)]$.

23. $\frac{\mathrm{d}}{\mathrm{d}x}\ln(ax) = \frac{1}{x}$.

24. $\frac{\mathrm{d}}{\mathrm{d}x}\log_a(x) = \frac{\log_a(a)}{x}$.

25. $\frac{\mathrm{d}}{\mathrm{d}q}\int_p^q f(x)\mathrm{d}x = f(q)$ if p is independent of q.

26. $\frac{\mathrm{d}}{\mathrm{d}q}\int_p^q f(x)\mathrm{d}x = -f(p)$ if q is independent of p.

APPENDIX E. A SHORT TABLE OF INDEFINITE INTEGRALS

In the following, an arbitrary constant of integration is to be added to each equation. a, b, c, g, and n are constants.

1. $\int \mathrm{d}x = x$.

2. $\int x\,\mathrm{d}x = \frac{x^2}{2}$.

3. $\int \frac{1}{x}\mathrm{d}x = \ln(|x|)$ Do not integrate from negative to positive values of x.

4. $\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1}$, where $n \neq -1$.

5. $\int (a + bx)^n\mathrm{d}x = \frac{(a + bx)^{n+1}}{b(n+1)}$.

6. $\int \frac{1}{(a + bx)}\mathrm{d}x = \frac{1}{b}\ln(|a + bx|)$.

7. $\int \frac{1}{(a + bx)^n}\mathrm{d}x = \frac{-1}{(n-1)b(a + bx)^{n-1}}$.

8. $\int \frac{x}{(a + bx)}\mathrm{d}x = \frac{1}{b^2}\left[(a + bx) - a\ln(|a + bx|)\right]$.

9. $\int \frac{a + bx}{c + gx}\mathrm{d}x = \frac{bx}{g} + \frac{ag - bc}{g^2}\ln(|c + gx|)$.

10. $\int \frac{1}{(a + bx)(c + gx)}\mathrm{d}x = \frac{1}{ag - bc}\ln\left(\left|\frac{c + gx}{cx + bx}\right|\right)$.

11. $\int \frac{1}{a^2 + x^2}\mathrm{d}x = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$.

[2] These formulas, and the other material in Appendices 3–7, are from H.B. Dwight, *Tables of Integrals and other Mathematical Data*, 4th ed., Macmillan, New York, 1961.

12. $\int \frac{x}{a^2+x^2}\mathrm{d}x = \frac{1}{2}\ln(a^2+x^2).$

13. $\int \frac{x^2}{a^2+x^2}\mathrm{d}x = x - a\tan^{-1}\left(\frac{x}{a}\right) = x - a\arctan\left(\frac{x}{a}\right).$

14. $\int \frac{x}{(a^2+x^2)^2}\mathrm{d}x = \frac{-1}{2(a^2+x^2)}.$

15. $\int \frac{1}{(a^2-b^2x^2)}\mathrm{d}x = \frac{1}{2ab}\ln\left(\left|\frac{a+bx}{a-bx}\right|\right).$

16. $\int \frac{x}{(a^2-x^2)}\mathrm{d}x = -\frac{1}{2}\ln(|a^2-x^2|).$

17. $\int \frac{x^{1/2}}{(a^2+b^2x)}\mathrm{d}x = \frac{2x^{1/2}}{b^2} - \frac{2a}{b^3}\tan^{-1}\left(\frac{bx^{1/2}}{a}\right).$

18. $\int \frac{1}{(a+bx^2)^{\rho/2}}\mathrm{d}x = \frac{-2}{(p-2)b(a+bx)^{(\rho-2)/2}}.$

19. $\int \frac{1}{(x^2+a^2)^{1/2}}\mathrm{d}x = \ln(x+(x^2+a^2)^{1/2}).$

20. $\int \frac{x}{(x^2+a^2)^{1/2}}\mathrm{d}x = (x^2+a^2)^{1/2}.$

21. $\int \frac{1}{(x^2-a^2)^{1/2}}\mathrm{d}x = \ln(x+(x^2-a^2)^{1/2}).$

22. $\int \frac{x}{(x^2-a^2)^{1/2}}\mathrm{d}x = (x^2-a^2)^{1/2}.$

23. $\int \sin(ax)\mathrm{d}x = -\frac{1}{a}\cos(ax).$

24. $\int \sin(a+bx)\mathrm{d}x = -\frac{1}{b}\cos(a+bx).$

25. $\int x\sin(x)\mathrm{d}x = \sin(x) - x\cos(x).$

26. $\int x^2\sin(x)\mathrm{d}x = 2x\sin(x) - (x^2-2)\cos(x).$

27. $\int \sin^2(x)\mathrm{d}x = \frac{x}{2} - \frac{\sin(2x)}{4} = \frac{x}{2} - \frac{\sin(x)\cos(x)}{2}.$

28. $\int x\sin^2(x)\mathrm{d}x = \frac{x^2}{4} - \frac{x\sin(2x)}{4} - \frac{\cos(2x)}{8}.$

29. $\int \frac{1}{1+\sin(x)}\mathrm{d}x = -\tan\left(\frac{\pi}{4} - \frac{x}{2}\right).$

30. $\int \cos(ax)\mathrm{d}x = \frac{1}{a}\sin(ax).$

31. $\int \cos(a+bx)\mathrm{d}x = \frac{1}{b}\sin(a+bx).$

32. $\int x\cos(x)\mathrm{d}x = \cos(x) + x\sin(x).$

33. $\int x^2\cos(x)\mathrm{d}x = 2x\cos(x) + (x^2-2)\sin(x).$

34. $\int \cos^2(x)\mathrm{d}x = \frac{x}{2} + \frac{\sin(2x)}{4} = \frac{x}{2} + \frac{\sin(x)\cos(x)}{2}.$

35. $\int x\cos^2(x)\mathrm{d}x = \frac{x^2}{4} + \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8}.$

36. $\int \frac{1}{1+\cos(x)}\mathrm{d}x = \tan\left(\frac{x}{2}\right).$

37. $\int \sin(x)\cos(x)\mathrm{d}x = \frac{\sin^2(x)}{2}.$

38. $\int \sin^2(x)\cos^2(x)\mathrm{d}x = \frac{1}{8}\left[x - \frac{\sin(4x)}{4}\right].$

39. $\int \sin^{-1}\left(\frac{x}{a}\right)\mathrm{d}x = x\sin^{-1}\left(\frac{x}{a}\right) + (a^2-x^2)^{1/2}.$

40. $\int \left[\sin^{-1}\left(\frac{x}{a}\right)\right]^2\mathrm{d}x = x\left[\sin^{-1}\left(\frac{x}{a}\right)\right]^2 - 2x + 2(a^2-x^2)^{1/2}\sin^{-1}\left(\frac{x}{a}\right).$

41. $\int \cos^{-1}\left(\frac{x}{a}\right)\mathrm{d}x = x\cos^{-1}\left(\frac{x}{a}\right) - (a^2-x^2)^{1/2}.$

42. $\int \left[\cos^{-1}\left(\frac{x}{a}\right)\right]^2\mathrm{d}x = x\left[\cos^{-1}\left(\frac{x}{a}\right)\right]^2 - 2x - 2(a^2-x^2)^{1/2}\cos^{-1}\left(\frac{x}{a}\right).$

43. $\int \tan^{-1}\left(\frac{x}{a}\right)\mathrm{d}x = x\tan^{-1}\left(\frac{x}{a}\right) - \frac{a}{2}\ln(a^2+x^2).$

44. $\int x\tan^{-1}\left(\frac{x}{a}\right)\mathrm{d}x = \frac{1}{2}(x^2+a^2)\tan^{-1}\left(\frac{x}{a}\right) - \frac{ax}{2}.$

45. $\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax}.$

46. $\int a^x\,\mathrm{d}x = \frac{a^x}{\ln(a)}.$

47. $\int x\,e^{ax}\,\mathrm{d}x = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$

48. $\int x^2\,e^{ax}\,\mathrm{d}x = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right].$

49. $\int e^{ax}\sin(x)\mathrm{d}x = \frac{e^{ax}}{a^2+1}[a\sin(x) - \cos(x)].$

50. $\int e^{ax}\cos(x)\mathrm{d}x = \frac{e^{ax}}{a^2+1}[a\sin(x) + \sin(x)].$

51. $\int e^{ax}\sin^2(x)\mathrm{d}x = \frac{e^{ax}}{a^2+4}\left[a\sin^2(x) - 2\sin(x)\cos(x) + \frac{2}{a}\right].$

52. $\int \ln(ax)\mathrm{d}x = x\ln(ax) - x.$

53. $\int x\ln(x)\mathrm{d}x = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}.$

54. $\int \frac{\ln(ax)}{x}\mathrm{d}x = \frac{1}{2}[\ln(ax)]^2.$

55. $\int \frac{1}{x\ln(x)}\mathrm{d}x = \ln(|\ln(x)|).$

56. $\int \tan(ax)\mathrm{d}x = \frac{1}{a}\ln(|\sec(ax)|) = -\frac{1}{a}\ln(|\cos(ax)|).$

57. $\int \cot(ax)\mathrm{d}x = \frac{1}{a}\ln(|\sin(ax)|).$