Simulation – Basics & Integrals

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Outline

- Taylor Polynomials
- Approximating integrals with Taylor polynomials
- Random and Pseudo Random Numbers
- Riemann Sum

Taylor Polynomials

- The derivatives are the instantaneous rate of change of a function f(x) at a given point c
- Therefore $f'(x)$ gives us a linear approximation of f(x) near c_i for small values of $\epsilon \in \mathbb{R}$, we have:

$$f(c+\epsilon) \approx f(c) + \epsilon f'(c)$$

- If f(x) has higher derivatives, why stop at the first derivative?

Taylor Polynomials

Taylor series

- Let f(x) be a C^n polynomial. F is n-times continuously differentiable
- The n-th order Taylor polynomial off(x) about c is:

$$T_n(f)(x) = \sum_{k=0}^{n} \frac{f^k(c)}{k!}(x-c)^k$$

- As Taylor polynomials are approximations of f(x), there will be residuals R_n
- We want $R_n(f)(x) \to 0$

Taylor series

❖ **Theorem:** Suppose f(x) is (n+1)-times continuously differentiable. Then,

$$R_n(f)(x) = \int_c^x \frac{f^{(n+1)}(y)}{n!}(x-y)^{n+1}\ \mathrm{d}y$$

❖ This says how much $Tn(f)(x)$ is off the true value of f(x).

Example

- Taylor series for:

$$f(x) = e^x \text{ about } 0$$

$$\forall k, f^{(k)}(x) = e^x$$

$$e^x = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}(x-0)^k \quad f^{(0)}(0) = e^0 = 1$$

$$f^{(0)}(0) = e^0 = 1 \qquad \frac{f^{(2)}(0)}{2!}x^2 = \frac{x^2}{2}$$

$$\frac{f^{(1)}(0)}{1!}x^1 = \frac{x^1}{1} \qquad \frac{f^{(3)}(0)}{3!}x^3 = \frac{x^3}{3!}$$

$$\therefore e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example

- Function $f(x) = \cos(x)$ about 0

Approximating Integrals

- Approximate the Gaussian curve $\mu = 0$ and $\sigma = 1$:

$$\int_0^{1/3} e^{-x^2} dx$$

Approximating Integrals

Riemann Sum

- Let a closed interval [a,b] be partitioned by points:

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

- Where the length of the points are denoted by:

$$\Delta x_1 < \Delta x_2 < \cdots < \Delta x_n$$

- Let X_n^* be an arbitrary point in the k^{th}subinterval. Then:

$$\sum_{k=1}^{n} f(X_k^*)\Delta \mathrm{x_k}$$

- Is called a Riemann sum for a given function f(x)

Riemann Sum

Riemann sum

- The value $\max(\Delta x_k)$ is called the mesh size

Riemann sum 2D

Using random numbers

- Let g(x) be a function and suppose we wanted θ where

$$\theta = \int_0^1 g(x)dx$$

- To compute the value of θ, note that if U is uniformly distributed over (0,1) then we can express θ as

$$\theta = E[\, g\, (\, U\,)]$$

Using random numbers

- Independent and identically distributed (iid) random variables have mean θ $\rightarrow$Strong law of large numbers

Random Numbers

- The building block of a simulation study is the ability to generate random numbers
- The generated random number will represent an observation from the measured system
- A random number represents the value of a random variable uniformly distributed an (0,1)

Pseudo Random Number Generation

- Pseudo Random Numbers (PRN) is a sequence of values
- They are deterministically generated
- Have the appearances of being an independent uniform (0,1) random variables

Pseudo Random Number Generation

- To generate a PRN starts with an initial value X_0 called the seed
- The most common approach uses recursion to compute successive values where $X_n, n \geq 1$, by letting:

$$X_n = aX_{n-1} \ \ modulo\ m$$

Where $a, m \in \mathbb{N}^+$

- The quantity X_n/m is an approximation to the uniform (0,1)
- The method is called Multiplicative congruential method

Pseudo Random Number Generation

- Criteria to choose a and m:
  1. For any “seed”, the result must “appear” to be a uniform random variable
  2. For any seed, the number of values generated before repetition must be large
  3. The values can be computed efficiently on a digital computer

Multiplicative congruential method

Pseudo Random Number Generation

- Another PRN Generator

$$X_n = (aX_{n-1} + C)\ modulo\ m$$

- This is called mixed congruential method
- Mixed = addition + multiplication
- Quiet efficient
  - m bigger than in Multiplicative Congruential method

Mixed congruential method

Pseudo Random Number Generation

- Transforming the uniform (0,1) to (a,b)

Monte Carlo approach

$$\sum_{i=1}^{N} \frac{g(u_i)}{N} = E[g(u)] = \theta$$

$$as\ N \to \infty$$

- This approach is called the Monte Carlo approach

Monte Carlo approach

- What happens if the integral goes from a to b, instead of 0,1

Monte Carlo approach

Monte Carlo approach

- Ejemplo

Monte Carlo approach

- What happens if we have a multivariate function?

Monte Carlo approach

- Hence, if we generate k independent sets, each consisting of n independent uniform (0,1) random variables

Monte Carlo approach

- Since $g\left(U_1^i, U_2^i, \dots, U_P^i\right)$ for all i are iid

HW

- Taylor series: (25 points)
  - Handwritten until the 6th-term to get the formula
  - Function 1

$$f(x) = \sin(x)$$

  - Function 2

If $i^2 = -1$ compute e^{ix} about 0

HW

- Code: (25 points)
  - Riemann sums function
- Function (25 points)

$$\int_{-2}^{2} e^{x+x^2}\,dx$$

  - Handwritten : Apply Taylor series
  - R Code: Riemann sum, and Monte Carlo approach to:

HW

- Monte Carlo approach to: (25 points)

$$\int_0^1 \int_0^1 e^{(x+y)^2}\,dy\,dx$$