

PROBABILITY AND STATISTICS

CHEAT SHEET

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This cheat sheet integrates a variety of topics in probability theory and statistics. It is based on literature [1, 6, 3] and in-class material from courses of the statistics department at the University of California in Berkeley but also influenced by other sources [4, 5]. If you find errors or have suggestions for further topics, I would appreciate if you send me an [email](mailto:). The most recent version of this document is available at <http://bit.ly/probstat>. To reproduce, please contact me.

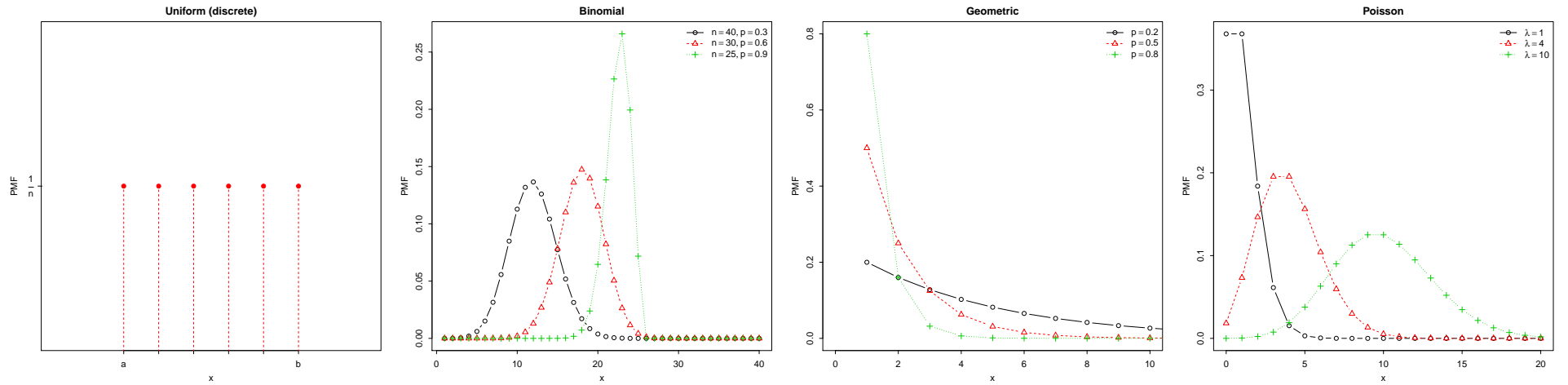
Contents

1 Distribution Overview	3	12 Parametric Inference	11	20 Stochastic Processes	22
1.1 Discrete Distributions	3	12.1 Method of Moments	11	20.1 Markov Chains	22
1.2 Continuous Distributions	4	12.2 Maximum Likelihood	12	20.2 Poisson Processes	22
2 Probability Theory	6	12.2.1 Delta Method	12	21 Time Series	23
3 Random Variables	6	12.3 Multiparameter Models	12	21.1 Stationary Time Series	23
3.1 Transformations	7	12.3.1 Multiparameter Delta Method	13	21.2 Estimation of Correlation	24
4 Expectation	7	12.4 Parametric Bootstrap	13	21.3 Non-Stationary Time Series	24
5 Variance	7	13 Hypothesis Testing	13	21.3.1 Detrending	24
6 Inequalities	8	14 Bayesian Inference	14	21.4 ARIMA models	24
7 Distribution Relationships	8	14.1 Credible Intervals	14	21.4.1 Causality and Invertibility	25
8 Probability and Moment Generating Functions	9	14.2 Function of Parameters	14	21.5 Spectral Analysis	25
9 Multivariate Distributions	9	14.3 Priors	15	22 Math	26
9.1 Standard Bivariate Normal	9	14.3.1 Conjugate Priors	15	22.1 Gamma Function	26
9.2 Bivariate Normal	9	14.4 Bayesian Testing	15	22.2 Beta Function	26
9.3 Multivariate Normal	9	15 Exponential Family	16	22.3 Series	27
10 Convergence	9	16 Sampling Methods	16	22.4 Combinatorics	27
10.1 Law of Large Numbers (LLN)	10	16.1 The Bootstrap	16		
10.2 Central Limit Theorem (CLT)	10	16.1.1 Bootstrap Confidence Intervals	16		
11 Statistical Inference	10	16.2 Rejection Sampling	17		
11.1 Point Estimation	10	16.3 Importance Sampling	17		
11.2 Normal-based Confidence Interval	11	17 Decision Theory	17		
11.3 Empirical Distribution Function	11	17.1 Risk	17		
11.4 Statistical Functionals	11	17.2 Admissibility	17		
		17.3 Bayes Rule	18		
		17.4 Minimax Rules	18		
		18 Linear Regression	18		
		18.1 Simple Linear Regression	18		
		18.2 Prediction	19		
		18.3 Multiple Regression	19		
		18.4 Model Selection	19		
		19 Non-parametric Function Estimation	20		
		19.1 Density Estimation	20		
		19.1.1 Histograms	20		
		19.1.2 Kernel Density Estimator (KDE)	21		
		19.2 Non-parametric Regression	21		
		19.3 Smoothing Using Orthogonal Functions	21		

Distribution Overview

1.1 Discrete Distributions

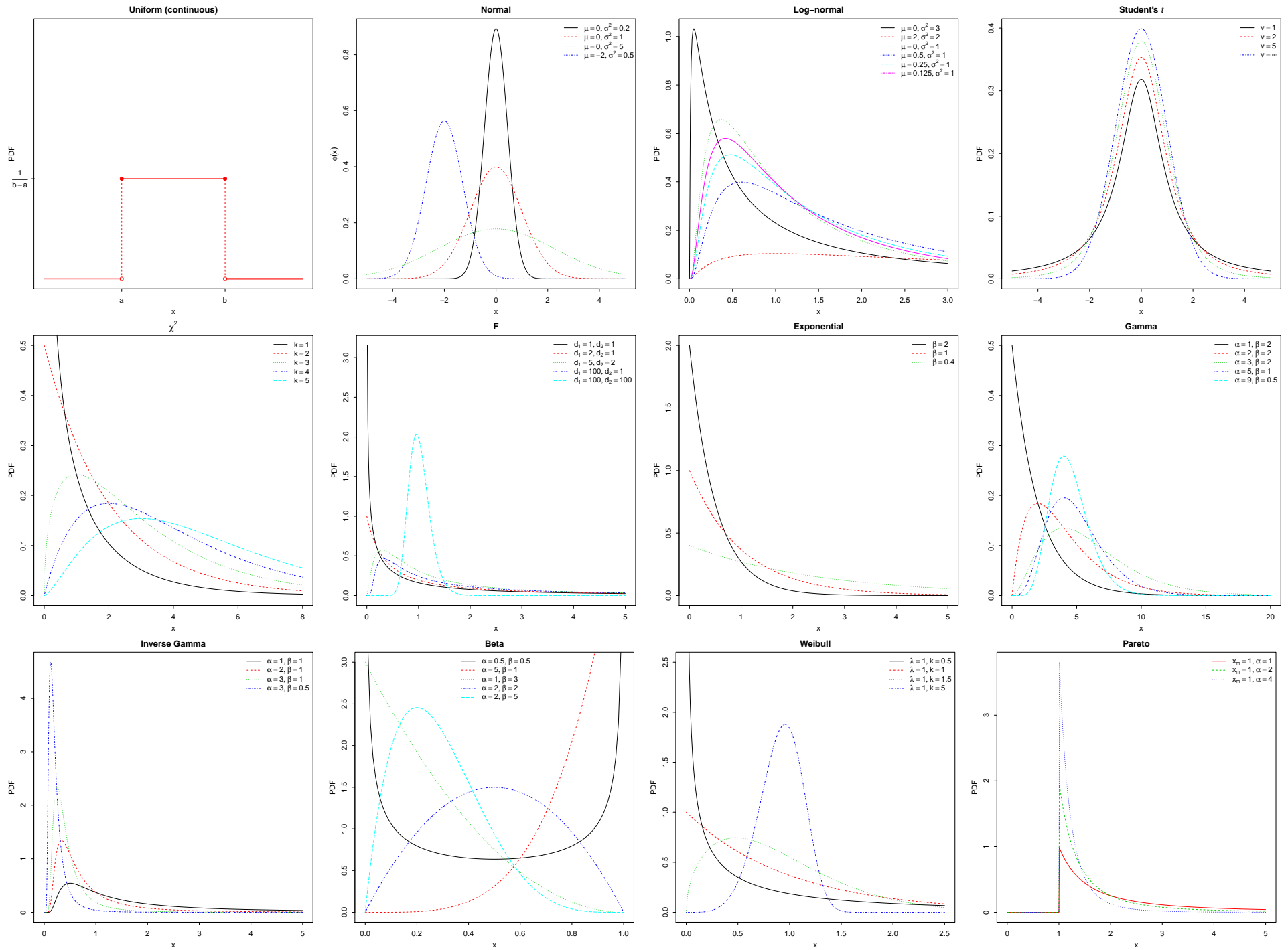
	Notation ¹	$F_X(x)$	$f_X(x)$	$\mathbb{E}[X]$	$\mathbb{V}[X]$	$M_X(s)$
Uniform	$\text{Unif}\{a, \dots, b\}$	$\begin{cases} 0 & x < a \\ \frac{\lfloor x \rfloor - a + 1}{b - a} & a \leq x \leq b \\ 1 & x > b \end{cases}$	$\frac{I(a < x < b)}{b - a + 1}$	$\frac{a + b}{2}$	$\frac{(b - a + 1)^2 - 1}{12}$	$\frac{e^{as} - e^{-(b+1)s}}{s(b - a)}$
Bernoulli	$\text{Bern}(p)$	$(1 - p)^{1-x}$	$p^x (1 - p)^{1-x}$	p	$p(1 - p)$	$1 - p + pe^s$
Binomial	$\text{Bin}(n, p)$	$I_{1-p}(n - x, x + 1)$	$\binom{n}{x} p^x (1 - p)^{n-x}$	np	$np(1 - p)$	$(1 - p + pe^s)^n$
Multinomial	$\text{Mult}(n, p)$		$\frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k} \quad \sum_{i=1}^k x_i = n$	np_i	$np_i(1 - p_i)$	$\left(\sum_{i=0}^k p_i e^{s_i} \right)^n$
Hypergeometric	$\text{Hyp}(N, m, n)$	$\approx \Phi\left(\frac{x - np}{\sqrt{np(1 - p)}}\right)$	$\frac{\binom{m}{x} \binom{m-x}{n-x}}{\binom{N}{n}}$	$\frac{nm}{N}$	$\frac{nm(N - n)(N - m)}{N^2(N - 1)}$	N/A
Negative Binomial	$\text{NBin}(n, p)$	$I_p(r, x + 1)$	$\binom{x + r - 1}{r - 1} p^r (1 - p)^x$	$r \frac{1 - p}{p}$	$r \frac{1 - p}{p^2}$	$\left(\frac{p}{1 - (1 - p)e^s} \right)^r$
Geometric	$\text{Geo}(p)$	$1 - (1 - p)^x \quad x \in \mathbb{N}^+$	$p(1 - p)^{x-1} \quad x \in \mathbb{N}^+$	$\frac{1}{p}$	$\frac{1 - p}{p^2}$	$\frac{p}{1 - (1 - p)e^s}$
Poisson	$\text{Po}(\lambda)$	$e^{-\lambda} \sum_{i=0}^x \frac{\lambda^i}{i!}$	$\frac{\lambda^x e^{-\lambda}}{x!}$	λ	λ	$e^{\lambda(e^s - 1)}$



¹We use the notation $\gamma(s, x)$ and $\Gamma(x)$ to refer to the Gamma functions (see §22.1), and use $B(x, y)$ and I_x to refer to the Beta functions (see §22.2).

1.2 Continuous Distributions

	Notation	$F_X(x)$	$f_X(x)$	$\mathbb{E}[X]$	$\mathbb{V}[X]$	$M_X(s)$
Uniform	$\text{Unif}(a, b)$	$\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x > b \end{cases}$	$\frac{I(a < x < b)}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{sb} - e^{sa}}{s(b-a)}$
Normal	$\mathcal{N}(\mu, \sigma^2)$	$\Phi(x) = \int_{-\infty}^x \phi(t) dt$	$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$	μ	σ^2	$\exp\left\{\mu s + \frac{\sigma^2 s^2}{2}\right\}$
Log-Normal	$\ln \mathcal{N}(\mu, \sigma^2)$	$\frac{1}{2} + \frac{1}{2} \operatorname{erf}\left[\frac{\ln x - \mu}{\sqrt{2\sigma^2}}\right]$	$\frac{1}{x\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\right\}$	$e^{\mu + \sigma^2/2}$	$(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$	
Multivariate Normal	$\text{MVN}(\mu, \Sigma)$		$(2\pi)^{-k/2} \Sigma ^{-1/2} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$	μ	Σ	$\exp\left\{\mu^T s + \frac{1}{2} s^T \Sigma s\right\}$
Student's t	$\text{Student}(\nu)$	$I_x\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$	$\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$	0	0	
Chi-square	χ_k^2	$\frac{1}{\Gamma(k/2)} \gamma\left(\frac{k}{2}, \frac{x}{2}\right)$	$\frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2} e^{-x/2}$	k	$2k$	$(1-2s)^{-k/2} \quad s < 1/2$
F	$F(d_1, d_2)$	$I_{\frac{d_1 x}{d_1 x + d_2}}\left(\frac{d_1}{2}, \frac{d_1}{2}\right)$	$\frac{\sqrt{\frac{(d_1 x)^{d_1} d_2^{d_2}}{(d_1 x + d_2)^{d_1 + d_2}}}}{x B\left(\frac{d_1}{2}, \frac{d_1}{2}\right)}$	$\frac{d_2}{d_2 - 2}$	$\frac{2d_2^2(d_1 + d_2 - 2)}{d_1(d_2 - 2)^2(d_2 - 4)}$	
Exponential	$\text{Exp}(\beta)$	$1 - e^{-x/\beta}$	$\frac{1}{\beta} e^{-x/\beta}$	β	β^2	$\frac{1}{1 - \beta s} \quad (s < 1/\beta)$
Gamma	$\text{Gamma}(\alpha, \beta)$	$\frac{\gamma(\alpha, x/\beta)}{\Gamma(\alpha)}$	$\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1 - \beta s}\right)^\alpha \quad (s < 1/\beta)$
Inverse Gamma	$\text{InvGamma}(\alpha, \beta)$	$\frac{\Gamma(\alpha, \frac{\beta}{x})}{\Gamma(\alpha)}$	$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}$	$\frac{\beta}{\alpha - 1} \quad \alpha > 1$	$\frac{\beta^2}{(\alpha - 1)^2(\alpha - 2)^2} \quad \alpha > 2$	$\frac{2(-\beta s)^{\alpha/2}}{\Gamma(\alpha)} K_\alpha\left(\sqrt{-4\beta s}\right)$
Dirichlet	$\text{Dir}(\alpha)$		$\frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$	$\frac{\alpha_i}{\sum_{i=1}^k \alpha_i}$	$\frac{\mathbb{E}[X_i](1 - \mathbb{E}[X_i])}{\sum_{i=1}^k \alpha_i + 1}$	
Beta	$\text{Beta}(\alpha, \beta)$	$I_x(\alpha, \beta)$	$\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	$1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha + r}{\alpha + \beta + r} \right) \frac{s^k}{k!}$
Weibull	$\text{Weibull}(\lambda, k)$	$1 - e^{-(x/\lambda)^k}$	$\frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$	$\lambda \Gamma\left(1 + \frac{1}{k}\right)$	$\lambda^2 \Gamma\left(1 + \frac{2}{k}\right) - \mu^2$	$\sum_{n=0}^{\infty} \frac{s^n \lambda^n}{n!} \Gamma\left(1 + \frac{n}{k}\right)$
Pareto	$\text{Pareto}(x_m, \alpha)$	$1 - \left(\frac{x_m}{x}\right)^\alpha \quad x \geq x_m$	$\alpha \frac{x_m^\alpha}{x^{\alpha+1}} \quad x \geq x_m$	$\frac{\alpha x_m}{\alpha - 1} \quad \alpha > 1$	$\frac{x_m^\alpha}{(\alpha - 1)^2(\alpha - 2)} \quad \alpha > 2$	$\alpha(-x_m s)^\alpha \Gamma(-\alpha, -x_m s) \quad s < 0$



2 Probability Theory

Definitions

- **Sample space Ω**
- Outcome (point or element) $\omega \in \Omega$
- Event $A \subseteq \Omega$
- σ -algebra \mathcal{A}
 1. $\emptyset \in \mathcal{A}$
 2. $A_1, A_2, \dots \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$
 3. $A \in \mathcal{A} \implies \neg A \in \mathcal{A}$
- Probability distribution \mathbb{P}
 1. $\mathbb{P}[A] \geq 0$ for every A
 2. $\mathbb{P}[\Omega] = 1$
 3. $\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$
- Probability space $(\Omega, \mathcal{A}, \mathbb{P})$

Properties

- $\mathbb{P}[\emptyset] = 0$
- $B = \Omega \cap B = (A \cup \neg A) \cap B = (A \cap B) \cup (\neg A \cap B)$
- $\mathbb{P}[\neg A] = 1 - \mathbb{P}[A]$
- $\mathbb{P}[B] = \mathbb{P}[A \cap B] + \mathbb{P}[\neg A \cap B]$
- $\mathbb{P}[\Omega] = 1 \quad \mathbb{P}[\emptyset] = 0$
- $\neg(\bigcup_n A_n) = \bigcap_n \neg A_n \quad \neg(\bigcap_n A_n) = \bigcup_n \neg A_n \quad \text{DEMORGAN}$
- $\mathbb{P}[\bigcup_n A_n] = 1 - \mathbb{P}[\bigcap_n \neg A_n]$
- $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$
 $\implies \mathbb{P}[A \cup B] \leq \mathbb{P}[A] + \mathbb{P}[B]$
- $\mathbb{P}[A \cup B] = \mathbb{P}[A \cap \neg B] + \mathbb{P}[\neg A \cap B] + \mathbb{P}[A \cap B]$
- $\mathbb{P}[A \cap \neg B] = \mathbb{P}[A] - \mathbb{P}[A \cap B]$

Continuity of Probabilities

- $A_1 \subset A_2 \subset \dots \implies \lim_{n \rightarrow \infty} \mathbb{P}[A_n] = \mathbb{P}[A] \quad \text{where } A = \bigcup_{i=1}^{\infty} A_i$
- $A_1 \supset A_2 \supset \dots \implies \lim_{n \rightarrow \infty} \mathbb{P}[A_n] = \mathbb{P}[A] \quad \text{where } A = \bigcap_{i=1}^{\infty} A_i$

Independence $\perp\!\!\!\perp$

$$A \perp\!\!\!\perp B \iff \mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$$

Conditional Probability

$$\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \quad \text{if } \mathbb{P}[B] > 0$$

Law of Total Probability

$$\mathbb{P}[B] = \sum_{i=1}^n \mathbb{P}[B|A_i] \mathbb{P}[A_i] \quad \Omega = \bigcup_{i=1}^n A_i$$

Bayes' Theorem

$$\mathbb{P}[A_i|B] = \frac{\mathbb{P}[B|A_i] \mathbb{P}[A_i]}{\sum_{j=1}^n \mathbb{P}[B|A_j] \mathbb{P}[A_j]} \quad \Omega = \bigcup_{i=1}^n A_i$$

Inclusion-Exclusion Principle

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{r=1}^n (-1)^{r-1} \sum_{i_1 < \dots < i_r \leq n} \left| \bigcap_{j=1}^r A_{i_j} \right|$$

3 Random Variables

Random Variable

$$X : \Omega \rightarrow \mathbb{R}$$

Probability Mass Function (PMF)

$$f_X(x) = \mathbb{P}[X = x] = \mathbb{P}[\{\omega \in \Omega : X(\omega) = x\}]$$

Probability Density Function (PDF)

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f(x) dx$$

Cumulative Distribution Function (CDF):

$$F_X : \mathbb{R} \rightarrow [0, 1] \quad F_X(x) = \mathbb{P}[X \leq x]$$

1. Nondecreasing: $x_1 < x_2 \implies F(x_1) \leq F(x_2)$
2. Normalized: $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$
3. Right-continuous: $\lim_{y \downarrow x} F(y) = F(x)$

$$\mathbb{P}[a \leq Y \leq b | X = x] = \int_a^b f_{Y|X}(y|x) dy \quad a \leq b$$

$$f_{Y|X}(y|x) = \frac{f(x, y)}{f_X(x)}$$

Independence

1. $\mathbb{P}[X \leq x, Y \leq y] = \mathbb{P}[X \leq x] \mathbb{P}[Y \leq y]$
2. $f_{X,Y}(x, y) = f_X(x) f_Y(y)$

3.1 Transformations

Transformation function

$$Z = \varphi(X)$$

Discrete

$$f_Z(z) = \mathbb{P}[\varphi(X) = z] = \mathbb{P}[\{x : \varphi(x) = z\}] = \mathbb{P}[X \in \varphi^{-1}(z)] = \sum_{x \in \varphi^{-1}(z)} f(x)$$

Continuous

$$F_Z(z) = \mathbb{P}[\varphi(X) \leq z] = \int_{A_z} f(x) dx \quad \text{with } A_z = \{x : \varphi(x) \leq z\}$$

Special case if φ strictly monotone

$$f_Z(z) = f_X(\varphi^{-1}(z)) \left| \frac{d}{dz} \varphi^{-1}(z) \right| = f_X(x) \left| \frac{dx}{dz} \right| = f_X(x) \frac{1}{|J|}$$

The Rule of the Lazy Statistician

$$\mathbb{E}[Z] = \int \varphi(x) dF_X(x)$$

$$\mathbb{E}[I_A(x)] = \int I_A(x) dF_X(x) = \int_A dF_X(x) = \mathbb{P}[X \in A]$$

Convolution

- $Z := X + Y \quad f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x) dx \stackrel{X,Y \geq 0}{=} \int_0^z f_{X,Y}(x, z-x) dx$
- $Z := |X - Y| \quad f_Z(z) = 2 \int_0^{\infty} f_{X,Y}(x, z+x) dx$
- $Z := \frac{X}{Y} \quad f_Z(z) = \int_{-\infty}^{\infty} |x| f_{X,Y}(x, xz) dx \stackrel{!}{=} \int_{-\infty}^{\infty} x f_X(x) f_Y(xz) dx$

4 Expectation

Expectation

- $\mathbb{E}[X] = \mu_X = \int x dF_X(x) = \begin{cases} \sum_x x f_X(x) & X \text{ discrete} \\ \int x f_X(x) & X \text{ continuous} \end{cases}$
- $\mathbb{P}[X = c] = 1 \implies \mathbb{E}[c] = c$
- $\mathbb{E}[cX] = c \mathbb{E}[X]$
- $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

- $\mathbb{E}[XY] = \int_{X,Y} xy f_{X,Y}(x, y) dF_X(x) dF_Y(y)$
- $\mathbb{E}[\varphi(Y)] \neq \varphi(\mathbb{E}[X])$ (cf. **JENSEN inequality**)
- $\mathbb{P}[X \geq Y] = 0 \implies \mathbb{E}[X] \geq \mathbb{E}[Y] \wedge \mathbb{P}[X = Y] = 1 \implies \mathbb{E}[X] = \mathbb{E}[Y]$
- $\mathbb{E}[X] = \sum_{x=1}^{\infty} \mathbb{P}[X \geq x]$

Sample mean

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

Conditional Expectation

- $\mathbb{E}[Y | X = x] = \int y f(y | x) dy$
- $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X | Y]]$
- $E[\varphi(X, Y) | X = x] = \int_{-\infty}^{\infty} \varphi(x, y) f_{Y|X}(y | x) dx$
- $\mathbb{E}[\varphi(Y, Z) | X = x] = \int_{-\infty}^{\infty} \varphi(y, z) f_{(Y,Z)|X}(y, z | x) dy dz$
- $\mathbb{E}[Y + Z | X] = \mathbb{E}[Y | X] + \mathbb{E}[Z | X]$
- $\mathbb{E}[\varphi(X)Y | X] = \varphi(X)\mathbb{E}[Y | X]$
- $E[Y | X] = c \implies \text{Cov}[X, Y] = 0$

5 Variance

Variance

- $\mathbb{V}[X] = \sigma_X^2 = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$
- $\mathbb{V}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{V}[X_i] + 2 \sum_{i \neq j} \text{Cov}[X_i, X_j]$
- $\mathbb{V}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{V}[X_i] \quad \text{iff } X_i \perp\!\!\!\perp X_j$

Standard deviation

$$\text{sd}[X] = \sqrt{\mathbb{V}[X]} = \sigma_X$$

Covariance

- $\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$
- $\text{Cov}[X, a] = 0$
- $\text{Cov}[X, X] = \mathbb{V}[X]$
- $\text{Cov}[X, Y] = \text{Cov}[Y, X]$
- $\text{Cov}[aX, bY] = ab \text{Cov}[X, Y]$
- $\text{Cov}[X + a, Y + b] = \text{Cov}[X, Y]$

- $\text{Cov} \left[\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j \right] = \sum_{i=1}^n \sum_{j=1}^m \text{Cov} [X_i, Y_j]$

Correlation

$$\rho [X, Y] = \frac{\text{Cov} [X, Y]}{\sqrt{\mathbb{V} [X] \mathbb{V} [Y]}}$$

Independence

$$X \perp\!\!\!\perp Y \implies \rho [X, Y] = 0 \iff \text{Cov} [X, Y] = 0 \iff \mathbb{E} [XY] = \mathbb{E} [X] \mathbb{E} [Y]$$

Sample variance

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$$

Conditional Variance

- $\mathbb{V} [Y | X] = \mathbb{E} [(Y - \mathbb{E} [Y | X])^2 | X] = \mathbb{E} [Y^2 | X] - \mathbb{E} [Y | X]^2$
- $\mathbb{V} [Y] = \mathbb{E} [\mathbb{V} [Y | X]] + \mathbb{V} [\mathbb{E} [Y | X]]$

6 Inequalities

CAUCHY-SCHWARZ

$$\mathbb{E} [XY]^2 \leq \mathbb{E} [X^2] \mathbb{E} [Y^2]$$

MARKOV

$$\mathbb{P} [\varphi(X) \geq t] \leq \frac{\mathbb{E} [\varphi(X)]}{t}$$

CHEBYSHEV

$$\mathbb{P} [|X - \mathbb{E} [X]| \geq t] \leq \frac{\mathbb{V} [X]}{t^2}$$

CHERNOFF

$$\mathbb{P} [X \geq (1 + \delta)\mu] \leq \left(\frac{e^\delta}{(1 + \delta)^{1+\delta}} \right) \quad \delta > -1$$

JENSEN

$$\mathbb{E} [\varphi(X)] \geq \varphi(\mathbb{E} [X]) \quad \varphi \text{ convex}$$

7 Distribution Relationships

Binomial

- $X_i \sim \text{Bern} (p) \implies \sum_{i=1}^n X_i \sim \text{Bin} (n, p)$
- $X \sim \text{Bin} (n, p), Y \sim \text{Bin} (m, p) \implies X + Y \sim \text{Bin} (n + m, p)$
- $\lim_{n \rightarrow \infty} \text{Bin} (n, p) = \text{Po} (np) \quad (n \text{ large, } p \text{ small})$

- $\lim_{n \rightarrow \infty} \text{Bin} (n, p) = \mathcal{N} (np, np(1 - p)) \quad (n \text{ large, } p \text{ far from 0 and 1})$

Negative Binomial

- $X \sim \text{NBin} (1, p) = \text{Geo} (p)$
- $X \sim \text{NBin} (r, p) = \sum_{i=1}^r \text{Geo} (p)$
- $X_i \sim \text{NBin} (r_i, p) \implies \sum X_i \sim \text{NBin} (\sum r_i, p)$
- $X \sim \text{NBin} (r, p) \cdot Y \sim \text{Bin} (s + r, p) \implies \mathbb{P} [X \leq s] = \mathbb{P} [Y \geq r]$

Poisson

- $X_i \sim \text{Po} (\lambda_i) \wedge X_i \perp\!\!\!\perp X_j \implies \sum_{i=1}^n X_i \sim \text{Po} \left(\sum_{i=1}^n \lambda_i \right)$
- $X_i \sim \text{Po} (\lambda_i) \wedge X_i \perp\!\!\!\perp X_j \implies X_i \left| \sum_{j=1}^n X_j \sim \text{Bin} \left(\sum_{j=1}^n X_j, \frac{\lambda_i}{\sum_{j=1}^n \lambda_j} \right)$

Exponential

- $X_i \sim \text{Exp} (\beta) \wedge X_i \perp\!\!\!\perp X_j \implies \sum_{i=1}^n X_i \sim \text{Gamma} (n, \beta)$
- Memoryless property: $\mathbb{P} [X > x + y | X > y] = \mathbb{P} [X > x]$

Normal

- $X \sim \mathcal{N} (\mu, \sigma^2) \implies \left(\frac{X - \mu}{\sigma} \right) \sim \mathcal{N} (0, 1)$
- $X \sim \mathcal{N} (\mu, \sigma^2) \wedge Z = aX + b \implies Z \sim \mathcal{N} (a\mu + b, a^2\sigma^2)$
- $X \sim \mathcal{N} (\mu_1, \sigma_1^2) \wedge Y \sim \mathcal{N} (\mu_2, \sigma_2^2) \implies X + Y \sim \mathcal{N} (\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
- $X_i \sim \mathcal{N} (\mu_i, \sigma_i^2) \implies \sum_i X_i \sim \mathcal{N} (\sum_i \mu_i, \sum_i \sigma_i^2)$
- $\mathbb{P} [a < X \leq b] = \Phi \left(\frac{b - \mu}{\sigma} \right) - \Phi \left(\frac{a - \mu}{\sigma} \right)$
- $\Phi(-x) = 1 - \Phi(x) \quad \phi'(x) = -x\phi(x) \quad \phi''(x) = (x^2 - 1)\phi(x)$
- Upper quantile of $\mathcal{N} (0, 1)$: $z_\alpha = \Phi^{-1}(1 - \alpha)$

Gamma

- $X \sim \text{Gamma} (\alpha, \beta) \iff X/\beta \sim \text{Gamma} (\alpha, 1)$
- $\text{Gamma} (\alpha, \beta) \sim \sum_{i=1}^\alpha \text{Exp} (\beta)$
- $X_i \sim \text{Gamma} (\alpha_i, \beta) \wedge X_i \perp\!\!\!\perp X_j \implies \sum_i X_i \sim \text{Gamma} (\sum_i \alpha_i, \beta)$
- $\frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty x^{\alpha-1} e^{-\lambda x} dx$

Beta

- $\frac{1}{\text{B}(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$
- $\mathbb{E} [X^k] = \frac{\text{B}(\alpha + k, \beta)}{\text{B}(\alpha, \beta)} = \frac{\alpha + k - 1}{\alpha + \beta + k - 1} \mathbb{E} [X^{k-1}]$
- $\text{Beta} (1, 1) \sim \text{Unif} (0, 1)$

8 Probability and Moment Generating Functions

- $G_X(t) = \mathbb{E} [t^X] \quad |t| < 1$
- $M_X(t) = G_X(e^t) = \mathbb{E} [e^{Xt}] = \mathbb{E} \left[\sum_{i=0}^{\infty} \frac{(Xt)^i}{i!} \right] = \sum_{i=0}^{\infty} \frac{\mathbb{E} [X^i]}{i!} \cdot t^i$
- $\mathbb{P} [X = 0] = G_X(0)$
- $\mathbb{P} [X = 1] = G'_X(0)$
- $\mathbb{P} [X = i] = \frac{G_X^{(i)}(0)}{i!}$
- $\mathbb{E} [X] = G'_X(1^-)$
- $\mathbb{E} [X^k] = M_X^{(k)}(0)$
- $\mathbb{E} \left[\frac{X!}{(X-k)!} \right] = G_X^{(k)}(1^-)$
- $\mathbb{V} [X] = G''_X(1^-) + G'_X(1^-) - (G'_X(1^-))^2$
- $G_X(t) = G_Y(t) \implies X \stackrel{d}{=} Y$

9 Multivariate Distributions

9.1 Standard Bivariate Normal

Let $X, Y \sim \mathcal{N}(0, 1) \wedge X \perp\!\!\!\perp Z$ with $Y = \rho X + \sqrt{1 - \rho^2} Z$

Joint density

$$f(x, y) = \frac{1}{2\pi\sqrt{1 - \rho^2}} \exp \left\{ -\frac{x^2 + y^2 - 2\rho xy}{2(1 - \rho^2)} \right\}$$

Conditionals

$$(Y | X = x) \sim \mathcal{N}(\rho x, 1 - \rho^2) \quad \text{and} \quad (X | Y = y) \sim \mathcal{N}(\rho y, 1 - \rho^2)$$

Independence

$$X \perp\!\!\!\perp Y \iff \rho = 0$$

9.2 Bivariate Normal

Let $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$.

$$f(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - \rho^2}} \exp \left\{ -\frac{z}{2(1 - \rho^2)} \right\}$$

$$z = \left[\left(\frac{x - \mu_x}{\sigma_x} \right)^2 + \left(\frac{y - \mu_y}{\sigma_y} \right)^2 - 2\rho \left(\frac{x - \mu_x}{\sigma_x} \right) \left(\frac{y - \mu_y}{\sigma_y} \right) \right]$$

Conditional mean and variance

$$\mathbb{E} [X | Y] = \mathbb{E} [X] + \rho \frac{\sigma_X}{\sigma_Y} (Y - \mathbb{E} [Y])$$

$$\mathbb{V} [X | Y] = \sigma_X \sqrt{1 - \rho^2}$$

9.3 Multivariate Normal

Covariance Matrix Σ (Precision Matrix Σ^{-1})

$$\Sigma = \begin{pmatrix} \mathbb{V} [X_1] & \cdots & \text{Cov} [X_1, X_k] \\ \vdots & \ddots & \vdots \\ \text{Cov} [X_k, X_1] & \cdots & \mathbb{V} [X_k] \end{pmatrix}$$

If $X \sim \mathcal{N}(\mu, \Sigma)$,

$$f_X(x) = (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}$$

Properties

- $Z \sim \mathcal{N}(0, 1) \wedge X = \mu + \Sigma^{1/2} Z \implies X \sim \mathcal{N}(\mu, \Sigma)$
- $X \sim \mathcal{N}(\mu, \Sigma) \implies \Sigma^{-1/2}(X - \mu) \sim \mathcal{N}(0, 1)$
- $X \sim \mathcal{N}(\mu, \Sigma) \implies AX \sim \mathcal{N}(A\mu, A\Sigma A^T)$
- $X \sim \mathcal{N}(\mu, \Sigma) \wedge a$ is vector of length $k \implies a^T X \sim \mathcal{N}(a^T \mu, a^T \Sigma a)$

10 Convergence

Let $\{X_1, X_2, \dots\}$ be a sequence of RV's and let X be another RV. Let F_n denote the CDF of X_n and let F denote the CDF of X .

Types of Convergence

1. In distribution (weakly, in law): $X_n \xrightarrow{D} X$

$$\lim_{n \rightarrow \infty} F_n(t) = F(t) \quad \forall t \text{ where } F \text{ continuous}$$

2. In probability: $X_n \xrightarrow{P} X$

$$(\forall \varepsilon > 0) \lim_{n \rightarrow \infty} \mathbb{P} [|X_n - X| > \varepsilon] = 0$$

3. Almost surely (strongly): $X_n \xrightarrow{\text{as}} X$

$$\mathbb{P} \left[\lim_{n \rightarrow \infty} X_n = X \right] = \mathbb{P} \left[\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega) \right] = 1$$

4. In quadratic mean (L_2): $X_n \xrightarrow{qm} X$

$$\lim_{n \rightarrow \infty} \mathbb{E} [(X_n - X)^2] = 0$$

Relationships

- $X_n \xrightarrow{qm} X \implies X_n \xrightarrow{P} X \implies X_n \xrightarrow{D} X$
- $X_n \xrightarrow{as} X \implies X_n \xrightarrow{P} X$
- $X_n \xrightarrow{D} X \wedge (\exists c \in \mathbb{R}) \mathbb{P}[X = c] = 1 \implies X_n \xrightarrow{P} X$
- $X_n \xrightarrow{P} X \wedge Y_n \xrightarrow{P} Y \implies X_n + Y_n \xrightarrow{P} X + Y$
- $X_n \xrightarrow{qm} X \wedge Y_n \xrightarrow{qm} Y \implies X_n + Y_n \xrightarrow{qm} X + Y$
- $X_n \xrightarrow{P} X \wedge Y_n \xrightarrow{P} Y \implies X_n Y_n \xrightarrow{P} XY$
- $X_n \xrightarrow{P} X \implies \varphi(X_n) \xrightarrow{P} \varphi(X)$
- $X_n \xrightarrow{D} X \implies \varphi(X_n) \xrightarrow{D} \varphi(X)$
- $X_n \xrightarrow{qm} b \iff \lim_{n \rightarrow \infty} \mathbb{E}[X_n] = b \wedge \lim_{n \rightarrow \infty} \mathbb{V}[X_n] = 0$
- $X_1, \dots, X_n \text{ IID} \wedge \mathbb{E}[X] = \mu \wedge \mathbb{V}[X] < \infty \iff \bar{X}_n \xrightarrow{qm} \mu$

Slutzky's Theorem

- $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c \implies X_n + Y_n \xrightarrow{D} X + c$
- $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c \implies X_n Y_n \xrightarrow{D} cX$
- In general: $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y \not\implies X_n + Y_n \xrightarrow{D} X + Y$

10.1 Law of Large Numbers (LLN)

Let $\{X_1, \dots, X_n\}$ be a sequence of IID RV's, $\mathbb{E}[X_1] = \mu$, and $\mathbb{V}[X_1] < \infty$.

Weak (WLLN)

$$\bar{X}_n \xrightarrow{P} \mu \quad \text{as } n \rightarrow \infty$$

Strong (SLLN)

$$\bar{X}_n \xrightarrow{as} \mu \quad \text{as } n \rightarrow \infty$$

10.2 Central Limit Theorem (CLT)

Let $\{X_1, \dots, X_n\}$ be a sequence of IID RV's, $\mathbb{E}[X_1] = \mu$, and $\mathbb{V}[X_1] = \sigma^2$.

$$Z_n := \frac{\bar{X}_n - \mu}{\sqrt{\mathbb{V}[\bar{X}_n]}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{D} Z \quad \text{where } Z \sim \mathcal{N}(0, 1)$$

$$\lim_{n \rightarrow \infty} \mathbb{P}[Z_n \leq z] = \Phi(z) \quad z \in \mathbb{R}$$

CLT Notations

$$Z_n \approx \mathcal{N}(0, 1)$$

$$\bar{X}_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\bar{X}_n - \mu \approx \mathcal{N}\left(0, \frac{\sigma^2}{n}\right)$$

$$\sqrt{n}(\bar{X}_n - \mu) \approx \mathcal{N}(0, \sigma^2)$$

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{n} \approx \mathcal{N}(0, 1)$$

Continuity Correction

$$\mathbb{P}[\bar{X}_n \leq x] \approx \Phi\left(\frac{x + \frac{1}{2} - \mu}{\sigma/\sqrt{n}}\right)$$

$$\mathbb{P}[\bar{X}_n \geq x] \approx 1 - \Phi\left(\frac{x - \frac{1}{2} - \mu}{\sigma/\sqrt{n}}\right)$$

Delta Method

$$Y_n \approx \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \implies \varphi(Y_n) \approx \mathcal{N}\left(\varphi(\mu), (\varphi'(\mu))^2 \frac{\sigma^2}{n}\right)$$

11 Statistical Inference

Let $X_1, \dots, X_n \stackrel{iid}{\sim} F$ if not otherwise noted.

11.1 Point Estimation

- Point estimator $\hat{\theta}_n$ of θ is a RV: $\hat{\theta}_n = g(X_1, \dots, X_n)$

$$\text{bias}(\hat{\theta}_n) = \mathbb{E}[\hat{\theta}_n] - \theta$$

- Consistency: $\hat{\theta}_n \xrightarrow{P} \theta$

- Sampling distribution: $F(\hat{\theta}_n)$

$$\text{Standard error: } \text{se}(\hat{\theta}_n) = \sqrt{\mathbb{V}[\hat{\theta}_n]}$$

- Mean squared error: $\text{MSE} = \mathbb{E}[(\hat{\theta}_n - \theta)^2] = \text{bias}(\hat{\theta}_n)^2 + \mathbb{V}[\hat{\theta}_n]$

- $\lim_{n \rightarrow \infty} \text{bias}(\hat{\theta}_n) = 0 \wedge \lim_{n \rightarrow \infty} \text{se}(\hat{\theta}_n) = 0 \implies \hat{\theta}_n$ is consistent

- Asymptotic normality: $\frac{\hat{\theta}_n - \theta}{\text{se}} \xrightarrow{D} \mathcal{N}(0, 1)$

- SLUTZKY'S THEOREM often lets us replace $\text{se}(\hat{\theta}_n)$ by some (weakly) consistent estimator $\hat{\sigma}_n$.

11.2 Normal-based Confidence Interval

Suppose $\hat{\theta}_n \approx \mathcal{N}(\theta, \hat{\mathbf{se}}^2)$. Let $z_{\alpha/2} = \Phi^{-1}(1 - (\alpha/2))$, i.e., $\mathbb{P}[Z > z_{\alpha/2}] = \alpha/2$ and $\mathbb{P}[-z_{\alpha/2} < Z < z_{\alpha/2}] = 1 - \alpha$ where $Z \sim \mathcal{N}(0, 1)$. Then

$$C_n = \hat{\theta}_n \pm z_{\alpha/2} \hat{\mathbf{se}}$$

11.3 Empirical Distribution Function

Empirical Distribution Function (ECDF)

$$\hat{F}_n(x) = \frac{\sum_{i=1}^n I(X_i \leq x)}{n}$$

$$I(X_i \leq x) = \begin{cases} 1 & X_i \leq x \\ 0 & X_i > x \end{cases}$$

Properties (for any fixed x)

- $\mathbb{E}[\hat{F}_n] = F(x)$
- $\mathbb{V}[\hat{F}_n] = \frac{F(x)(1 - F(x))}{n}$
- $\text{MSE} = \frac{F(x)(1 - F(x))}{n} \xrightarrow{p} 0$
- $\hat{F}_n \xrightarrow{p} F(x)$

DVORETZKY-KIEFER-WOLFOWITZ (DKW) Inequality ($X_1, \dots, X_n \sim F$)

$$\mathbb{P}\left[\sup_x |F(x) - \hat{F}_n(x)| > \varepsilon\right] = 2e^{-2n\varepsilon^2}$$

Nonparametric $1 - \alpha$ confidence band for F

$$L(x) = \max\{\hat{F}_n - \epsilon_n, 0\}$$

$$U(x) = \min\{\hat{F}_n + \epsilon_n, 1\}$$

$$\epsilon = \sqrt{\frac{1}{2n} \log\left(\frac{2}{\alpha}\right)}$$

$$\mathbb{P}[L(x) \leq F(x) \leq U(x) \forall x] \geq 1 - \alpha$$

11.4 Statistical Functionals

- Statistical functional: $T(F)$
- Plug-in estimator of $\theta = T(F) : \hat{\theta}_n = T(\hat{F}_n)$
- Linear functional: $T(F) = \int \varphi(x) dF_X(x)$
- Plug-in estimator for linear functional:

$$T(\hat{F}_n) = \int \varphi(x) d\hat{F}_n(x) = \frac{1}{n} \sum_{i=1}^n \varphi(X_i)$$

- Often: $T(\hat{F}_n) \approx \mathcal{N}(T(F), \hat{\mathbf{se}}^2) \implies T(\hat{F}_n) \pm z_{\alpha/2} \hat{\mathbf{se}}$
- p^{th} quantile: $F^{-1}(p) = \inf\{x : F(x) \geq p\}$
- $\hat{\mu} = \bar{X}_n$
- $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$
- $\hat{\kappa} = \frac{\frac{1}{n} \sum_{i=1}^n (X_i - \hat{\mu})^3}{\hat{\sigma}^3 j}$
- $\hat{\rho} = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \sqrt{\sum_{i=1}^n (Y_i - \bar{Y}_n)^2}}$

12 Parametric Inference

Let $\mathfrak{F} = \{f(x; \theta) : \theta \in \Theta\}$ be a parametric model with parameter space $\Theta \subset \mathbb{R}^k$ and parameter $\theta = (\theta_1, \dots, \theta_k)$.

12.1 Method of Moments

j^{th} moment

$$\alpha_j(\theta) = \mathbb{E}[X^j] = \int x^j dF_X(x)$$

j^{th} sample moment

$$\hat{\alpha}_j = \frac{1}{n} \sum_{i=1}^n X_i^j$$

Method of Moments Estimator (MoM)

$$\alpha_1(\theta) = \hat{\alpha}_1$$

$$\alpha_2(\theta) = \hat{\alpha}_2$$

$$\vdots = \vdots$$

$$\alpha_k(\theta) = \hat{\alpha}_k$$

Properties of the MoM estimator

- $\hat{\theta}_n$ exists with probability tending to 1
- Consistency: $\hat{\theta}_n \xrightarrow{P} \theta$
- Asymptotic normality:

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} \mathcal{N}(0, \Sigma)$$

where $\Sigma = g\mathbb{E}[YY^T]g^T$, $Y = (X, X^2, \dots, X^k)^T$,
 $g = (g_1, \dots, g_k)$ and $g_j = \frac{\partial}{\partial \theta} \alpha_j^{-1}(\theta)$

12.2 Maximum Likelihood

Likelihood: $\mathcal{L}_n : \Theta \rightarrow [0, \infty)$

$$\mathcal{L}_n(\theta) = \prod_{i=1}^n f(X_i; \theta)$$

Log-likelihood

$$\ell_n(\theta) = \log \mathcal{L}_n(\theta) = \sum_{i=1}^n \log f(X_i; \theta)$$

Maximum Likelihood Estimator (MLE)

$$\mathcal{L}_n(\hat{\theta}_n) = \sup_{\theta} \mathcal{L}_n(\theta)$$

Score Function

$$s(X; \theta) = \frac{\partial}{\partial \theta} \log f(X; \theta)$$

Fisher Information

$$I(\theta) = \mathbb{V}_{\theta}[s(X; \theta)]$$

$$I_n(\theta) = nI(\theta)$$

Fisher Information (exponential family)

$$I(\theta) = \mathbb{E}_{\theta} \left[-\frac{\partial}{\partial \theta} s(X; \theta) \right]$$

Observed Fisher Information

$$I_n^{obs}(\theta) = -\frac{\partial^2}{\partial \theta^2} \sum_{i=1}^n \log f(X_i; \theta)$$

Properties of the MLE

- Consistency: $\hat{\theta}_n \xrightarrow{P} \theta$

- Equivariance: $\hat{\theta}_n$ is the MLE $\implies \varphi(\hat{\theta}_n)$ is the MLE of $\varphi(\theta)$
- Asymptotic normality:

$$1. \text{ se} \approx \sqrt{1/I_n(\theta)}$$

$$\frac{(\hat{\theta}_n - \theta)}{\text{se}} \xrightarrow{D} \mathcal{N}(0, 1)$$

$$2. \text{ se} \approx \sqrt{1/I_n(\hat{\theta}_n)}$$

$$\frac{(\hat{\theta}_n - \theta)}{\widehat{\text{se}}} \xrightarrow{D} \mathcal{N}(0, 1)$$

- Asymptotic optimality (or efficiency), i.e., smallest variance for large samples. If $\tilde{\theta}_n$ is any other estimator, the asymptotic relative efficiency is

$$\text{ARE}(\tilde{\theta}_n, \hat{\theta}_n) = \frac{\mathbb{V}[\hat{\theta}_n]}{\mathbb{V}[\tilde{\theta}_n]} \leq 1$$

- Approximately the Bayes estimator

12.2.1 Delta Method

If $\tau = \varphi(\hat{\theta})$ where φ is differentiable and $\varphi'(\theta) \neq 0$:

$$\frac{(\hat{\tau}_n - \tau)}{\widehat{\text{se}}(\hat{\tau})} \xrightarrow{D} \mathcal{N}(0, 1)$$

where $\hat{\tau} = \varphi(\hat{\theta})$ is the MLE of τ and

$$\widehat{\text{se}} = \left| \varphi'(\hat{\theta}) \right| \widehat{\text{se}}(\hat{\theta}_n)$$

12.3 Multiparameter Models

Let $\theta = (\theta_1, \dots, \theta_k)$ and $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_k)$ be the MLE.

$$H_{jj} = \frac{\partial^2 \ell_n}{\partial \theta_j^2} \quad H_{jk} = \frac{\partial^2 \ell_n}{\partial \theta_j \partial \theta_k}$$

Fisher Information Matrix

$$I_n(\theta) = - \begin{bmatrix} \mathbb{E}_{\theta}[H_{11}] & \cdots & \mathbb{E}_{\theta}[H_{1k}] \\ \vdots & \ddots & \vdots \\ \mathbb{E}_{\theta}[H_{k1}] & \cdots & \mathbb{E}_{\theta}[H_{kk}] \end{bmatrix}$$

Under appropriate regularity conditions

$$(\hat{\theta} - \theta) \approx \mathcal{N}(0, J_n)$$

with $J_n(\theta) = I_n^{-1}$. Further, if $\hat{\theta}_j$ is the j^{th} component of θ , then

$$\frac{(\hat{\theta}_j - \theta_j)}{\widehat{\text{se}}_j} \xrightarrow{D} \mathcal{N}(0, 1)$$

where $\widehat{\text{se}}_j^2 = J_n(j, j)$ and $\text{Cov}[\hat{\theta}_j, \hat{\theta}_k] = J_n(j, k)$

12.3.1 Multiparameter Delta Method

Let $\tau = \varphi(\theta_1, \dots, \theta_k)$ be a function and let the gradient of φ be

$$\nabla \varphi = \begin{pmatrix} \frac{\partial \varphi}{\partial \theta_1} \\ \vdots \\ \frac{\partial \varphi}{\partial \theta_k} \end{pmatrix}$$

Suppose $\nabla \varphi|_{\theta=\hat{\theta}} \neq 0$ and $\hat{\tau} = \varphi(\hat{\theta})$. Then,

$$\frac{(\hat{\tau} - \tau)}{\widehat{\text{se}}(\hat{\tau})} \xrightarrow{D} \mathcal{N}(0, 1)$$

where

$$\widehat{\text{se}}(\hat{\tau}) = \sqrt{(\hat{\nabla} \varphi)^T \hat{J}_n(\hat{\nabla} \varphi)}$$

and $\hat{J}_n = J_n(\hat{\theta})$ and $\hat{\nabla} \varphi = \nabla \varphi|_{\theta=\hat{\theta}}$.

12.4 Parametric Bootstrap

Sample from $f(x; \hat{\theta}_n)$ instead of from \hat{F}_n , where $\hat{\theta}_n$ could be the MLE or method of moments estimator.

13 Hypothesis Testing

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_1$$

Definitions

- Null hypothesis H_0
- Alternative hypothesis H_1
- Simple hypothesis $\theta = \theta_0$
- Composite hypothesis $\theta > \theta_0$ or $\theta < \theta_0$
- Two-sided test: $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$
- One-sided test: $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$
- Critical value c
- Test statistic T
- Rejection Region $R = \{x : T(x) > c\}$
- Power function $\beta(\theta) = \mathbb{P}[X \in R]$
- Power of a test: $1 - \mathbb{P}[\text{Type II error}] = 1 - \beta = \inf_{\theta \in \Theta_1} \beta(\theta)$
- Test size: $\alpha = \mathbb{P}[\text{Type I error}] = \sup_{\theta \in \Theta_0} \beta(\theta)$

	Retain H_0	Reject H_0
H_0 true	✓	Type I error (α)
H_1 true	Type II error (β)	✓ (power)

p-value

- p-value = $\sup_{\theta \in \Theta_0} \mathbb{P}_\theta [T(X) \geq T(x)] = \inf \{ \alpha : T(x) \in R_\alpha \}$
- p-value = $\sup_{\theta \in \Theta_0} \underbrace{\mathbb{P}_\theta [T(X^*) \geq T(X)]}_{1 - F_\theta(T(X)) \text{ since } T(X^*) \sim F_\theta} = \inf \{ \alpha : T(X) \in R_\alpha \}$

p-value	evidence
< 0.01	very strong evidence against H_0
0.01 – 0.05	strong evidence against H_0
0.05 – 0.1	weak evidence against H_0
> 0.1	little or no evidence against H_0

Wald Test

- Two-sided test
- Reject H_0 when $|W| > z_{\alpha/2}$ where $W = \frac{\hat{\theta} - \theta_0}{\widehat{\text{se}}}$
- $\mathbb{P}[|W| > z_{\alpha/2}] \rightarrow \alpha$
- p-value = $\mathbb{P}_{\theta_0}[|W| > |w|] \approx \mathbb{P}[|Z| > |w|] = 2\Phi(-|w|)$

Likelihood Ratio Test (LRT)

- $T(X) = \frac{\sup_{\theta \in \Theta} \mathcal{L}_n(\theta)}{\sup_{\theta \in \Theta_0} \mathcal{L}_n(\theta)} = \frac{\mathcal{L}_n(\hat{\theta}_n)}{\mathcal{L}_n(\hat{\theta}_{n,0})}$

- $\lambda(X) = 2 \log T(X) \xrightarrow{D} \chi^2_{r-q}$ where $\sum_{i=1}^k Z_i^2 \sim \chi_k^2$ with $Z_1, \dots, Z_k \stackrel{iid}{\sim} \mathcal{N}(0, 1)$
- p-value = $\mathbb{P}_{\theta_0} [\lambda(X) > \lambda(x)] \approx \mathbb{P} [\chi^2_{r-q} > \lambda(x)]$

Multinomial LRT

- Let $\hat{p}_n = \left(\frac{X_1}{n}, \dots, \frac{X_k}{n} \right)$ be the MLE
- $T(X) = \frac{\mathcal{L}_n(\hat{p}_n)}{\mathcal{L}_n(p_0)} = \prod_{j=1}^k \left(\frac{\hat{p}_j}{p_{0j}} \right)^{X_j}$
- $\lambda(X) = 2 \sum_{j=1}^k X_j \log \left(\frac{\hat{p}_j}{p_{0j}} \right) \xrightarrow{D} \chi^2_{k-1}$
- The approximate size α LRT rejects H_0 when $\lambda(X) \geq \chi^2_{k-1, \alpha}$

Pearson χ^2 Test

- $T = \sum_{j=1}^k \frac{(X_j - \mathbb{E}[X_j])^2}{\mathbb{E}[X_j]}$ where $\mathbb{E}[X_j] = np_{0j}$ under H_0
- $T \xrightarrow{D} \chi^2_{k-1}$
- p-value = $\mathbb{P} [\chi^2_{k-1} > T(x)]$
- Faster $\xrightarrow{D} \chi^2_{k-1}$ than LRT, hence preferable for small n

Independence Testing

- I rows, J columns, \mathbf{X} multinomial sample of size $n = I * J$
- MLEs unconstrained: $\hat{p}_{ij} = \frac{X_{ij}}{n}$
- MLEs under H_0 : $\hat{p}_{0ij} = \hat{p}_{i \cdot} \hat{p}_{\cdot j} = \frac{X_{i \cdot}}{n} \frac{X_{\cdot j}}{n}$
- LRT: $\lambda = 2 \sum_{i=1}^I \sum_{j=1}^J X_{ij} \log \left(\frac{n X_{ij}}{X_{i \cdot} X_{\cdot j}} \right)$
- Pearson χ^2 : $T = \sum_{i=1}^I \sum_{j=1}^J \frac{(X_{ij} - \mathbb{E}[X_{ij}])^2}{\mathbb{E}[X_{ij}]}$
- LRT and Pearson $\xrightarrow{D} \chi^2_k \nu$, where $\nu = (I - 1)(J - 1)$

14 Bayesian Inference

BAYES' THEOREM

$$f(\theta | x) = \frac{f(x | \theta) f(\theta)}{f(x^n)} = \frac{f(x | \theta) f(\theta)}{\int f(x | \theta) f(\theta) d\theta} \propto \mathcal{L}_n(\theta) f(\theta)$$

Definitions

- $X^n = (X_1, \dots, X_n)$

- $x^n = (x_1, \dots, x_n)$
- Prior density $f(\theta)$
- Likelihood $f(x^n | \theta)$: joint density of the data

$$\text{In particular, } X^n \text{ IID} \implies f(x^n | \theta) = \prod_{i=1}^n f(x_i | \theta) = \mathcal{L}_n(\theta)$$

- Posterior density $f(\theta | x^n)$
- Normalizing constant $c_n = f(x^n) = \int f(x | \theta) f(\theta) d\theta$
- Kernel: part of a density that depends on θ
- Posterior Mean $\bar{\theta}_n = \int \theta f(\theta | x^n) d\theta = \frac{\int \theta \mathcal{L}_n(\theta) f(\theta) d\theta}{\int \mathcal{L}_n(\theta) f(\theta) d\theta}$

14.1 Credible Intervals

$1 - \alpha$ Posterior Interval

$$\mathbb{P} [\theta \in (a, b) | x^n] = \int_a^b f(\theta | x^n) d\theta = 1 - \alpha$$

$1 - \alpha$ Equal-tail Credible Interval

$$\int_{-\infty}^a f(\theta | x^n) d\theta = \int_b^{\infty} f(\theta | x^n) d\theta = \alpha/2$$

$1 - \alpha$ Highest Posterior Density (HPD) region R_n

1. $\mathbb{P} [\theta \in R_n] = 1 - \alpha$
2. $R_n = \{\theta : f(\theta | x^n) > k\}$ for some k

R_n is unimodal $\implies R_n$ is an interval

14.2 Function of Parameters

Let $\tau = \varphi(\theta)$ and $A = \{\theta : \varphi(\theta) \leq \tau\}$.

Posterior CDF for τ

$$H(\tau | x^n) = \mathbb{P} [\varphi(\theta) \leq \tau | x^n] = \int_A f(\theta | x^n) d\theta$$

Posterior Density

$$h(\tau | x^n) = H'(\tau | x^n)$$

Bayesian Delta Method

$$\tau | X^n \approx \mathcal{N} \left(\varphi(\hat{\theta}), \widehat{\text{se}} \left| \varphi'(\hat{\theta}) \right| \right)$$

14.3 Priors

Choice

- Subjective Bayesianism: prior should incorporate as much detail as possible the research's a priori knowledge — via *prior elicitation*.
- Objective Bayesianism: prior should incorporate as little detail as possible (*non-informative* prior).
- Robust Bayesianism: consider various priors and determine *sensitivity* of our inferences to changes in the prior.

Types

- Flat: $f(\theta) \propto \text{constant}$
- Proper: $\int_{-\infty}^{\infty} f(\theta) d\theta = 1$
- Improper: $\int_{-\infty}^{\infty} f(\theta) d\theta = \infty$
- JEFFREYS' prior (transformation-invariant):

$$f(\theta) \propto \sqrt{I(\theta)} \quad f(\theta) \propto \sqrt{\det(I(\theta))}$$

- Conjugate: $f(\theta)$ and $f(\theta | x^n)$ belong to the same parametric family

14.3.1 Conjugate Priors

Discrete likelihood		
Likelihood	Conjugate Prior	Posterior hyperparameters
Bernoulli(p)	Beta(α, β)	$\alpha + \sum_{i=1}^n x_i, \beta + n - \sum_{i=1}^n x_i$
Binomial(p)	Beta(α, β)	$\alpha + \sum_{i=1}^n x_i, \beta + \sum_{i=1}^n N_i - \sum_{i=1}^n x_i$
Negative Binomial(p)	Beta(α, β)	$\alpha + rn, \beta + \sum_{i=1}^n x_i$
Poisson(λ)	Gamma(α, β)	$\alpha + \sum_{i=1}^n x_i, \beta + n$
Multinomial(\mathbf{p})	Dirichlet(α)	$\alpha + \sum_{i=1}^n \mathbf{x}^{(i)}$
Geometric(p)	Beta(α, β)	$\alpha + n, \beta + \sum_{i=1}^n x_i$

Continuous likelihood (subscript c denotes constant)		
Likelihood	Conjugate Prior	Posterior hyperparameters
Uniform($0, \theta$)	Pareto(x_m, k)	$\max\{x_{(n)}, x_m\}, k + n$
Exponential(λ)	Gamma(α, β)	$\alpha + n, \beta + \sum_{i=1}^n x_i$
Normal(μ, σ_c^2)	Normal(μ_0, σ_0^2)	$\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^n x_i}{\sigma_c^2}\right) / \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma_c^2}\right),$ $\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma_c^2}\right)^{-1}$
Normal(μ_c, σ^2)	Scaled Inverse Chi-square(ν, σ_0^2)	$\nu + n, \frac{\nu \sigma_0^2 + \sum_{i=1}^n (x_i - \mu)^2}{\nu + n}$
Normal(μ, σ^2)	Normal-scaled Inverse Gamma($\lambda, \nu, \alpha, \beta$)	$\frac{\nu \lambda + n \bar{x}}{\nu + n}, \quad \nu + n, \quad \alpha + \frac{n}{2},$ $\beta + \frac{1}{2} \sum_{i=1}^n (x_i - \bar{x})^2 + \frac{\gamma(\bar{x} - \lambda)^2}{2(n + \gamma)}$
MVN(μ, Σ_c)	MVN(μ_0, Σ_0)	$(\Sigma_0^{-1} + n \Sigma_c^{-1})^{-1} (\Sigma_0^{-1} \mu_0 + n \Sigma_c^{-1} \bar{x}),$ $(\Sigma_0^{-1} + n \Sigma_c^{-1})^{-1}$
MVN(μ_c, Σ)	Inverse-Wishart(κ, Ψ)	$n + \kappa, \Psi + \sum_{i=1}^n (x_i - \mu_c)(x_i - \mu_c)^T$
Pareto(x_{m_c}, k)	Gamma(α, β)	$\alpha + n, \beta + \sum_{i=1}^n \log \frac{x_i}{x_{m_c}}$
Pareto(x_m, k_c)	Pareto(x_0, k_0)	$x_0, k_0 - kn$ where $k_0 > kn$
Gamma(α_c, β)	Gamma(α_0, β_0)	$\alpha_0 + n \alpha_c, \beta_0 + \sum_{i=1}^n x_i$

14.4 Bayesian Testing

If $H_0 : \theta \in \Theta_0$:

$$\text{Prior probability } \mathbb{P}[H_0] = \int_{\Theta_0} f(\theta) d\theta$$

$$\text{Posterior probability } \mathbb{P}[H_0 | x^n] = \int_{\Theta_0} f(\theta | x^n) d\theta$$

Let H_0, \dots, H_{K-1} be K hypotheses. Suppose $\theta \sim f(\theta | H_k)$,

$$\mathbb{P}[H_k | x^n] = \frac{f(x^n | H_k) \mathbb{P}[H_k]}{\sum_{k=1}^K f(x^n | H_k) \mathbb{P}[H_k]},$$

Marginal Likelihood

$$f(x^n | H_i) = \int_{\Theta} f(x^n | \theta, H_i) f(\theta | H_i) d\theta$$

Posterior Odds (of H_i relative to H_j)

$$\frac{\mathbb{P}[H_i | x^n]}{\mathbb{P}[H_j | x^n]} = \underbrace{\frac{f(x^n | H_i)}{f(x^n | H_j)}}_{\text{Bayes Factor } BF_{ij}} \times \underbrace{\frac{\mathbb{P}[H_i]}{\mathbb{P}[H_j]}}_{\text{prior odds}}$$

Bayes Factor

$\log_{10} BF_{10}$	BF_{10}	evidence
0 – 0.5	1 – 1.5	Weak
0.5 – 1	1.5 – 10	Moderate
1 – 2	10 – 100	Strong
> 2	> 100	Decisive

$$p^* = \frac{\frac{p}{1-p} BF_{10}}{1 + \frac{p}{1-p} BF_{10}} \text{ where } p = \mathbb{P}[H_1] \text{ and } p^* = \mathbb{P}[H_1 | x^n]$$

15 Exponential Family

Scalar parameter

$$\begin{aligned} f_X(x | \theta) &= h(x) \exp \{ \eta(\theta) T(x) - A(\theta) \} \\ &= h(x) g(\theta) \exp \{ \eta(\theta) T(x) \} \end{aligned}$$

Vector parameter

$$\begin{aligned} f_X(x | \theta) &= h(x) \exp \left\{ \sum_{i=1}^s \eta_i(\theta) T_i(x) - A(\theta) \right\} \\ &= h(x) \exp \{ \eta(\theta) \cdot T(x) - A(\theta) \} \\ &= h(x) g(\theta) \exp \{ \eta(\theta) \cdot T(x) \} \end{aligned}$$

Natural form

$$\begin{aligned} f_X(x | \eta) &= h(x) \exp \{ \eta \cdot \mathbf{T}(x) - A(\eta) \} \\ &= h(x) g(\eta) \exp \{ \eta \cdot \mathbf{T}(x) \} \\ &= h(x) g(\eta) \exp \{ \eta^T \mathbf{T}(x) \} \end{aligned}$$

16 Sampling Methods

16.1 The Bootstrap

Let $T_n = g(X_1, \dots, X_n)$ be a statistic.

1. Estimate $\mathbb{V}_F[T_n]$ with $\mathbb{V}_{\hat{F}_n}[T_n]$.

2. Approximate $\mathbb{V}_{\hat{F}_n}[T_n]$ using simulation:

(a) Repeat the following B times to get $T_{n,1}^*, \dots, T_{n,B}^*$, an IID sample from the sampling distribution implied by \hat{F}_n

i. Sample uniformly $X_1^*, \dots, X_n^* \sim \hat{F}_n$.

ii. Compute $T_n^* = g(X_1^*, \dots, X_n^*)$.

(b) Then

$$v_{boot} = \hat{\mathbb{V}}_{\hat{F}_n} = \frac{1}{B} \sum_{b=1}^B \left(T_{n,b}^* - \frac{1}{B} \sum_{r=1}^B T_{n,r}^* \right)^2$$

16.1.1 Bootstrap Confidence Intervals

Normal-based Interval

$$T_n \pm z_{\alpha/2} \hat{se}_{boot}$$

Pivotal Interval

1. Location parameter $\theta = T(F)$

2. Pivot $R_n = \hat{\theta}_n - \theta$

3. Let $H(r) = \mathbb{P}[R_n \leq r]$ be the CDF of R_n

4. Let $R_{n,b}^* = \hat{\theta}_{n,b}^* - \hat{\theta}_n$. Approximate H using bootstrap:

$$\hat{H}(r) = \frac{1}{B} \sum_{b=1}^B I(R_{n,b}^* \leq r)$$

5. Let θ_β^* denote the β sample quantile of $(\hat{\theta}_{n,1}^*, \dots, \hat{\theta}_{n,B}^*)$

6. Let r_β^* denote the β sample quantile of $(R_{n,1}^*, \dots, R_{n,B}^*)$, i.e., $r_\beta^* = \theta_\beta^* - \hat{\theta}_n$

7. Then, an approximate $1 - \alpha$ confidence interval is $C_n = (\hat{a}, \hat{b})$ with

$$\begin{aligned} \hat{a} &= \hat{\theta}_n - \hat{H}^{-1} \left(1 - \frac{\alpha}{2} \right) = \hat{\theta}_n - r_{1-\alpha/2}^* = 2\hat{\theta}_n - \theta_{1-\alpha/2}^* \\ \hat{b} &= \hat{\theta}_n - \hat{H}^{-1} \left(\frac{\alpha}{2} \right) = \hat{\theta}_n - r_{\alpha/2}^* = 2\hat{\theta}_n - \theta_{\alpha/2}^* \end{aligned}$$

Percentile Interval

$$C_n = (\theta_{\alpha/2}^*, \theta_{1-\alpha/2}^*)$$

16.2 Rejection Sampling

Setup

- We can easily sample from $g(\theta)$
- We want to sample from $h(\theta)$, but it is difficult
- We know $h(\theta)$ up to proportional constant: $h(\theta) = \frac{k(\theta)}{\int k(\theta) d\theta}$
- Envelope condition: we can find $M > 0$ such that $k(\theta) \leq Mg(\theta) \quad \forall \theta$

Algorithm

1. Draw $\theta^{cand} \sim g(\theta)$
2. Generate $u \sim \text{Unif}(0, 1)$
3. Accept θ^{cand} if $u \leq \frac{k(\theta^{cand})}{Mg(\theta^{cand})}$
4. Repeat until B values of θ^{cand} have been accepted

Example

- We can easily sample from the prior $g(\theta) = f(\theta)$
- Target is the posterior with $h(\theta) \propto k(\theta) = f(x^n | \theta)f(\theta)$
- Envelope condition: $f(x^n | \theta) \leq f(x^n | \hat{\theta}_n) = \mathcal{L}_n(\hat{\theta}_n) \equiv M$
- Algorithm
 1. Draw $\theta^{cand} \sim f(\theta)$
 2. Generate $u \sim \text{Unif}(0, 1)$
 3. Accept θ^{cand} if $u \leq \frac{\mathcal{L}_n(\theta^{cand})}{\mathcal{L}_n(\hat{\theta}_n)}$

16.3 Importance Sampling

Sample from an importance function g rather than target density h .

Algorithm to obtain an approximation to $\mathbb{E}[q(\theta) | x^n]$:

1. Sample from the prior $\theta_1, \dots, \theta_n \stackrel{iid}{\sim} f(\theta)$
2. For each $i = 1, \dots, B$, calculate $w_i = \frac{\mathcal{L}_n(\theta_i)}{\sum_{i=1}^B \mathcal{L}_n(\theta_i)}$
3. $\mathbb{E}[q(\theta) | x^n] \approx \sum_{i=1}^B q(\theta_i)w_i$

17 Decision Theory

Definitions

- Unknown quantity affecting our decision: $\theta \in \Theta$

- Decision rule: synonymous for an estimator $\hat{\theta}$
- Action $a \in \mathcal{A}$: possible value of the decision rule. In the estimation context, the action is just an estimate of θ , $\hat{\theta}(x)$.
- Loss function L : consequences of taking action a when true state is θ or discrepancy between θ and $\hat{\theta}$, $L : \Theta \times \mathcal{A} \rightarrow [-k, \infty)$.

Loss functions

- Squared error loss: $L(\theta, a) = (\theta - a)^2$
- Linear loss: $L(\theta, a) = \begin{cases} K_1(\theta - a) & a - \theta < 0 \\ K_2(a - \theta) & a - \theta \geq 0 \end{cases}$
- Absolute error loss: $L(\theta, a) = |\theta - a|$ (linear loss with $K_1 = K_2$)
- L_p loss: $L(\theta, a) = |\theta - a|^p$
- Zero-one loss: $L(\theta, a) = \begin{cases} 0 & a = \theta \\ 1 & a \neq \theta \end{cases}$

17.1 Risk

Posterior Risk

$$r(\hat{\theta} | x) = \int L(\theta, \hat{\theta}(x))f(\theta | x) d\theta = \mathbb{E}_{\theta | x} [L(\theta, \hat{\theta}(x))]$$

(Frequentist) Risk

$$R(\theta, \hat{\theta}) = \int L(\theta, \hat{\theta}(x))f(x | \theta) dx = \mathbb{E}_{X | \theta} [L(\theta, \hat{\theta}(X))]$$

Bayes Risk

$$r(f, \hat{\theta}) = \iint L(\theta, \hat{\theta}(x))f(x, \theta) dx d\theta = \mathbb{E}_{\theta, X} [L(\theta, \hat{\theta}(X))]$$

$$r(f, \hat{\theta}) = \mathbb{E}_{\theta} [\mathbb{E}_{X | \theta} [L(\theta, \hat{\theta}(X))]] = \mathbb{E}_{\theta} [R(\theta, \hat{\theta})]$$

$$r(f, \hat{\theta}) = \mathbb{E}_X [\mathbb{E}_{\theta | X} [L(\theta, \hat{\theta}(X))]] = \mathbb{E}_X [r(\hat{\theta} | X)]$$

17.2 Admissibility

- $\hat{\theta}'$ dominates $\hat{\theta}$ if

$$\forall \theta : R(\theta, \hat{\theta}') \leq R(\theta, \hat{\theta})$$

$$\exists \theta : R(\theta, \hat{\theta}') < R(\theta, \hat{\theta})$$

- $\hat{\theta}$ is inadmissible if there is at least one other estimator $\hat{\theta}'$ that dominates it. Otherwise it is called admissible.

17.3 Bayes Rule

Bayes Rule (or Bayes Estimator)

- $r(f, \hat{\theta}) = \inf_{\tilde{\theta}} r(f, \tilde{\theta})$
- $\hat{\theta}(x) = \inf_{\tilde{\theta}} r(\tilde{\theta} | x) \forall x \implies r(f, \hat{\theta}) = \int r(\hat{\theta} | x) f(x) dx$

Theorems

- Squared error loss: posterior mean
- Absolute error loss: posterior median
- Zero-one loss: posterior mode

17.4 Minimax Rules

Maximum Risk

$$\bar{R}(\hat{\theta}) = \sup_{\theta} R(\theta, \hat{\theta}) \quad \bar{R}(a) = \sup_{\theta} R(\theta, a)$$

Minimax Rule

$$\sup_{\theta} R(\theta, \hat{\theta}) = \inf_{\tilde{\theta}} \bar{R}(\tilde{\theta}) = \inf_{\tilde{\theta}} \sup_{\theta} R(\theta, \tilde{\theta})$$

$$\hat{\theta} = \text{Bayes rule} \wedge \exists c : R(\theta, \hat{\theta}) = c$$

Least Favorable Prior

$$\hat{\theta}^f = \text{Bayes rule} \wedge R(\theta, \hat{\theta}^f) \leq r(f, \hat{\theta}^f) \forall \theta$$

18 Linear Regression

Definitions

- Response variable Y
- Covariate X (aka predictor variable or feature)

18.1 Simple Linear Regression

Model

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \quad \mathbb{E}[\epsilon_i | X_i] = 0, \mathbb{V}[\epsilon_i | X_i] = \sigma^2$$

Fitted Line

$$\hat{r}(x) = \hat{\beta}_0 + \hat{\beta}_1 x$$

Predicted (Fitted) Values

$$\hat{Y}_i = \hat{r}(X_i)$$

Residuals

$$\hat{\epsilon}_i = Y_i - \hat{Y}_i = Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i)$$

Residual Sums of Squares (RSS)

$$\text{RSS}(\hat{\beta}_0, \hat{\beta}_1) = \sum_{i=1}^n \hat{\epsilon}_i^2$$

Least Square Estimates

$$\hat{\beta}^T = (\hat{\beta}_0, \hat{\beta}_1)^T : \min_{\hat{\beta}_0, \hat{\beta}_1} \text{RSS}$$

$$\hat{\beta}_0 = \bar{Y}_n - \hat{\beta}_1 \bar{X}_n$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} = \frac{\sum_{i=1}^n X_i Y_i - n \bar{X} \bar{Y}}{\sum_{i=1}^n X_i^2 - n \bar{X}^2}$$

$$\mathbb{E}[\hat{\beta} | X^n] = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

$$\mathbb{V}[\hat{\beta} | X^n] = \frac{\sigma^2}{n s_X} \begin{pmatrix} n^{-1} \sum_{i=1}^n X_i^2 & -\bar{X}_n \\ -\bar{X}_n & 1 \end{pmatrix}$$

$$\widehat{\text{se}}(\hat{\beta}_0) = \frac{\hat{\sigma}}{s_X \sqrt{n}} \sqrt{\frac{\sum_{i=1}^n X_i^2}{n}}$$

$$\widehat{\text{se}}(\hat{\beta}_1) = \frac{\hat{\sigma}}{s_X \sqrt{n}}$$

where $s_X^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $\hat{\sigma}^2 = \frac{1}{n-2} \sum_{i=1}^n \hat{\epsilon}_i^2$ an (unbiased) estimate of σ . Further properties:

- Consistency: $\hat{\beta}_0 \xrightarrow{P} \beta_0$ and $\hat{\beta}_1 \xrightarrow{P} \beta_1$
- Asymptotic normality:

$$\frac{\hat{\beta}_0 - \beta_0}{\widehat{\text{se}}(\hat{\beta}_0)} \xrightarrow{D} \mathcal{N}(0, 1) \quad \text{and} \quad \frac{\hat{\beta}_1 - \beta_1}{\widehat{\text{se}}(\hat{\beta}_1)} \xrightarrow{D} \mathcal{N}(0, 1)$$

- Approximate $1 - \alpha$ confidence intervals for β_0 and β_1 are

$$\hat{\beta}_0 \pm z_{\alpha/2} \widehat{\text{se}}(\hat{\beta}_0) \quad \text{and} \quad \hat{\beta}_1 \pm z_{\alpha/2} \widehat{\text{se}}(\hat{\beta}_1)$$

- The Wald test for testing $H_0 : \beta_1 = 0$ vs. $H_1 : \beta_1 \neq 0$ is: reject H_0 if $|W| > z_{\alpha/2}$ where $W = \hat{\beta}_1 / \widehat{\text{se}}(\hat{\beta}_1)$.

R^2

$$R^2 = \frac{\sum_{i=1}^n (\hat{Y}_i - \bar{Y})^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2} = 1 - \frac{\sum_{i=1}^n \hat{\epsilon}_i^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2} = 1 - \frac{\text{RSS}}{\text{TSS}}$$

Likelihood

$$\begin{aligned}\mathcal{L} &= \prod_{i=1}^n f(X_i, Y_i) = \prod_{i=1}^n f_X(X_i) \times \prod_{i=1}^n f_{Y|X}(Y_i | X_i) = \mathcal{L}_1 \times \mathcal{L}_2 \\ \mathcal{L}_1 &= \prod_{i=1}^n f_X(X_i) \\ \mathcal{L}_2 &= \prod_{i=1}^n f_{Y|X}(Y_i | X_i) \propto \sigma^{-n} \exp \left\{ -\frac{1}{2\sigma^2} \sum_i \left(Y_i - (\beta_0 - \beta_1 X_i) \right)^2 \right\}\end{aligned}$$

Under the assumption of Normality, the least squares estimator is also the MLE

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \hat{\epsilon}_i^2$$

18.2 Prediction

Observe $X = x_*$ of the covariate and want to predict their outcome Y_* .

$$\begin{aligned}\hat{Y}_* &= \hat{\beta}_0 + \hat{\beta}_1 x_* \\ \mathbb{V} [\hat{Y}_*] &= \mathbb{V} [\hat{\beta}_0] + x_*^2 \mathbb{V} [\hat{\beta}_1] + 2x_* \text{Cov} [\hat{\beta}_0, \hat{\beta}_1]\end{aligned}$$

Prediction Interval

$$\begin{aligned}\hat{\xi}_n^2 &= \hat{\sigma}^2 \left(\frac{\sum_{i=1}^n (X_i - X_*)^2}{n \sum_i (X_i - \bar{X})^2} + 1 \right) \\ \hat{Y}_* &\pm z_{\alpha/2} \hat{\xi}_n\end{aligned}$$

18.3 Multiple Regression

$$Y = X\beta + \epsilon$$

where

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1k} \\ \vdots & \ddots & \vdots \\ X_{n1} & \cdots & X_{nk} \end{pmatrix} \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

Likelihood

$$\mathcal{L}(\mu, \Sigma) = (2\pi\sigma^2)^{-n/2} \exp \left\{ -\frac{1}{2\sigma^2} \text{RSS} \right\}$$

$$\text{RSS} = (y - X\beta)^T (y - X\beta) = \|Y - X\beta\|^2 = \sum_{i=1}^N (Y_i - x_i^T \beta)^2$$

If the $(k \times k)$ matrix $X^T X$ is invertible,

$$\begin{aligned}\hat{\beta} &= (X^T X)^{-1} X^T Y \\ \mathbb{V} [\hat{\beta} | X^n] &= \sigma^2 (X^T X)^{-1} \\ \hat{\beta} &\approx \mathcal{N}(\beta, \sigma^2 (X^T X)^{-1})\end{aligned}$$

Estimate regression function

$$\hat{r}(x) = \sum_{j=1}^k \hat{\beta}_j x_j$$

Unbiased estimate for σ^2

$$\hat{\sigma}^2 = \frac{1}{n-k} \sum_{i=1}^n \hat{\epsilon}_i^2 \quad \hat{\epsilon} = X\hat{\beta} - Y$$

MLE

$$\hat{\mu} = \bar{X} \quad \hat{\sigma}^2 = \frac{n-k}{n} \sigma^2$$

$1 - \alpha$ Confidence Interval

$$\hat{\beta}_j \pm z_{\alpha/2} \hat{\text{se}}(\hat{\beta}_j)$$

18.4 Model Selection

Consider predicting a new observation Y^* for covariates X^* and let $S \subset J$ denote a subset of the covariates in the model, where $|S| = k$ and $|J| = n$.
Issues

- Underfitting: too few covariates yields high bias
- Overfitting: too many covariates yields high variance

Procedure

1. Assign a score to each model
2. Search through all models to find the one with the highest score

Hypothesis Testing

$$H_0 : \beta_j = 0 \text{ vs. } H_1 : \beta_j \neq 0 \quad \forall j \in J$$

Mean Squared Prediction Error (MSPE)

$$\text{MSPE} = \mathbb{E} [(\hat{Y}(S) - Y^*)^2]$$

Prediction Risk

$$R(S) = \sum_{i=1}^n \text{MSPE}_i = \sum_{i=1}^n \mathbb{E} [(\hat{Y}_i(S) - Y_i^*)^2]$$

Training Error

$$\widehat{R}_{tr}(S) = \sum_{i=1}^n (\widehat{Y}_i(S) - Y_i)^2$$

R^2

$$R^2(S) = 1 - \frac{\text{RSS}(S)}{\text{TSS}} = 1 - \frac{\widehat{R}_{tr}(S)}{\text{TSS}} = 1 - \frac{\sum_{i=1}^n (\widehat{Y}_i(S) - \bar{Y})^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2}$$

The training error is a downward-biased estimate of the prediction risk.

$$\mathbb{E} [\widehat{R}_{tr}(S)] < R(S)$$

$$\text{bias}(\widehat{R}_{tr}(S)) = \mathbb{E} [\widehat{R}_{tr}(S)] - R(S) = -2 \sum_{i=1}^n \text{Cov} [\widehat{Y}_i, Y_i]$$

Adjusted R^2

$$R^2(S) = 1 - \frac{n-1}{n-k} \frac{\text{RSS}}{\text{TSS}}$$

MALLOW'S C_p statistic

$$\widehat{R}(S) = \widehat{R}_{tr}(S) + 2k\widehat{\sigma}^2 = \text{lack of fit} + \text{complexity penalty}$$

Akaike Information Criterion (AIC)

$$AIC(S) = \ell_n(\widehat{\beta}_S, \widehat{\sigma}_S^2) - k$$

Bayesian Information Criterion (BIC)

$$BIC(S) = \ell_n(\widehat{\beta}_S, \widehat{\sigma}_S^2) - \frac{k}{2} \log n$$

Validation and Training

$$\widehat{R}_V(S) = \sum_{i=1}^m (\widehat{Y}_i^*(S) - Y_i^*)^2 \quad m = |\{\text{validation data}\}|, \text{ often } \frac{n}{4} \text{ or } \frac{n}{2}$$

Leave-one-out Cross-validation

$$\widehat{R}_{CV}(S) = \sum_{i=1}^n (Y_i - \widehat{Y}_{(i)})^2 = \sum_{i=1}^n \left(\frac{Y_i - \widehat{Y}_i(S)}{1 - U_{ii}(S)} \right)^2$$

$$U(S) = X_S (X_S^T X_S)^{-1} X_S \text{ ("hat matrix")}$$

19 Non-parametric Function Estimation

19.1 Density Estimation

Estimate $f(x)$, where $f(x) = \mathbb{P}[X \in A] = \int_A f(x) dx$.

Integrated Square Error (ISE)

$$L(f, \widehat{f}_n) = \int \left(f(x) - \widehat{f}_n(x) \right)^2 dx = J(h) + \int f^2(x) dx$$

Frequentist Risk

$$R(f, \widehat{f}_n) = \mathbb{E} [L(f, \widehat{f}_n)] = \int b^2(x) dx + \int v(x) dx$$

$$b(x) = \mathbb{E} [\widehat{f}_n(x)] - f(x)$$

$$v(x) = \mathbb{V} [\widehat{f}_n(x)]$$

19.1.1 Histograms

Definitions

- Number of bins m
- Binwidth $h = \frac{1}{m}$
- Bin B_j has ν_j observations
- Define $\widehat{p}_j = \nu_j/n$ and $p_j = \int_{B_j} f(u) du$

Histogram Estimator

$$\widehat{f}_n(x) = \sum_{j=1}^m \frac{\widehat{p}_j}{h} I(x \in B_j)$$

$$\mathbb{E} [\widehat{f}_n(x)] = \frac{p_j}{h}$$

$$\mathbb{V} [\widehat{f}_n(x)] = \frac{p_j(1-p_j)}{nh^2}$$

$$R(\widehat{f}_n, f) \approx \frac{h^2}{12} \int (f'(u))^2 du + \frac{1}{nh}$$

$$h^* = \frac{1}{n^{1/3}} \left(\frac{6}{\int (f'(u))^2 du} \right)^{1/3}$$

$$R^*(\widehat{f}_n, f) \approx \frac{C}{n^{2/3}} \quad C = \left(\frac{3}{4} \right)^{2/3} \left(\int (f'(u))^2 du \right)^{1/3}$$

Cross-validation estimate of $\mathbb{E}[J(h)]$

$$\widehat{J}_{CV}(h) = \int \widehat{f}_n^2(x) dx - \frac{2}{n} \sum_{i=1}^n \widehat{f}_{(-i)}(X_i) = \frac{2}{(n-1)h} - \frac{n+1}{(n-1)h} \sum_{j=1}^m \widehat{p}_j^2$$

19.1.2 Kernel Density Estimator (KDE)

Kernel K

- $K(x) \geq 0$
- $\int K(x) dx = 1$
- $\int xK(x) dx = 0$
- $\int x^2 K(x) dx \equiv \sigma_K^2 > 0$

KDE

$$\begin{aligned}\hat{f}_n(x) &= \frac{1}{n} \sum_{i=1}^n \frac{1}{h} K\left(\frac{x - X_i}{h}\right) \\ R(f, \hat{f}_n) &\approx \frac{1}{4} (h\sigma_K)^4 \int (f''(x))^2 dx + \frac{1}{nh} \int K^2(x) dx \\ h^* &= \frac{c_1^{-2/5} c_2^{-1/5} c_3^{-1/5}}{n^{1/5}} \quad c_1 = \sigma_K^2, \quad c_2 = \int K^2(x) dx, \quad c_3 = \int (f''(x))^2 dx \\ R^*(f, \hat{f}_n) &= \frac{c_4}{n^{4/5}} \quad c_4 = \underbrace{\frac{5}{4} (\sigma_K^2)^{2/5} \left(\int K^2(x) dx \right)^{4/5}}_{C(K)} \left(\int (f'')^2 dx \right)^{1/5}\end{aligned}$$

Epanechnikov Kernel

$$K(x) = \begin{cases} \frac{3}{4\sqrt{5}(1-x^2/5)} & |x| < \sqrt{5} \\ 0 & \text{otherwise} \end{cases}$$

Cross-validation estimate of $\mathbb{E}[J(h)]$

$$\hat{J}_{CV}(h) = \int \hat{f}_n^2(x) dx - \frac{2}{n} \sum_{i=1}^n \hat{f}_{(-i)}(X_i) \approx \frac{1}{hn^2} \sum_{i=1}^n \sum_{j=1}^n K^*\left(\frac{X_i - X_j}{h}\right) + \frac{2}{nh} K(0)$$

$$K^*(x) = K^{(2)}(x) - 2K(x) \quad K^{(2)}(x) = \int K(x-y)K(y) dy$$

19.2 Non-parametric Regression

Estimate $f(x)$, where $f(x) = \mathbb{E}[Y | X = x]$. Consider pairs of points $(x_1, Y_1), \dots, (x_n, Y_n)$ related by

$$\begin{aligned}Y_i &= r(x_i) + \epsilon_i \\ \mathbb{E}[\epsilon_i] &= 0 \\ \mathbb{V}[\epsilon_i] &= \sigma^2\end{aligned}$$

k -nearest Neighbor Estimator

$$\hat{r}(x) = \frac{1}{k} \sum_{i: x_i \in N_k(x)} Y_i \quad \text{where } N_k(x) = \{k \text{ values of } x_1, \dots, x_n \text{ closest to } x\}$$

Nadaraya-Watson Kernel Estimator

$$\begin{aligned}\hat{r}(x) &= \sum_{i=1}^n w_i(x) Y_i \\ w_i(x) &= \frac{K\left(\frac{x-x_i}{h}\right)}{\sum_{j=1}^n K\left(\frac{x-x_j}{h}\right)} \in [0, 1] \\ R(\hat{r}_n, r) &\approx \frac{h^4}{4} \left(\int x^2 K^2(x) dx \right)^4 \int \left(r''(x) + 2r'(x) \frac{f'(x)}{f(x)} \right)^2 dx \\ &\quad + \int \frac{\sigma^2 \int K^2(x) dx}{nhf(x)} dx \\ h^* &\approx \frac{c_1}{n^{1/5}} \\ R^*(\hat{r}_n, r) &\approx \frac{c_2}{n^{4/5}}\end{aligned}$$

Cross-validation estimate of $\mathbb{E}[J(h)]$

$$\hat{J}_{CV}(h) = \sum_{i=1}^n (Y_i - \hat{r}_{(-i)}(x_i))^2 = \sum_{i=1}^n \frac{(Y_i - \hat{r}(x_i))^2}{\left(1 - \frac{K(0)}{\sum_{j=1}^n K\left(\frac{x-x_j}{h}\right)}\right)^2}$$

19.3 Smoothing Using Orthogonal Functions

Approximation

$$r(x) = \sum_{j=1}^{\infty} \beta_j \phi_j(x) \approx \sum_{j=1}^J \beta_j \phi_j(x)$$

Multivariate Regression

$$Y = \Phi\beta + \eta$$

$$\text{where } \eta_i = \epsilon_i \quad \text{and} \quad \Phi = \begin{pmatrix} \phi_0(x_1) & \cdots & \phi_J(x_1) \\ \vdots & \ddots & \vdots \\ \phi_0(x_n) & \cdots & \phi_J(x_n) \end{pmatrix}$$

Least Squares Estimator

$$\begin{aligned}\hat{\beta} &= (\Phi^T \Phi)^{-1} \Phi^T Y \\ &\approx \frac{1}{n} \Phi^T Y \quad (\text{for equally spaced observations only})\end{aligned}$$

Cross-validation estimate of $\mathbb{E}[J(h)]$

$$\hat{R}_{CV}(J) = \sum_{i=1}^n \left(Y_i - \sum_{j=1}^J \phi_j(x_i) \hat{\beta}_{j,(-i)} \right)^2$$

20 Stochastic Processes

Stochastic Process

$$\{X_t : t \in T\} \quad T = \begin{cases} \{0, \pm 1, \dots\} = \mathbb{Z} & \text{discrete} \\ [0, \infty) & \text{continuous} \end{cases}$$

- Notations: $X_t, X(t)$
- State space \mathcal{X}
- Index set T

20.1 Markov Chains

Markov Chain

$$\mathbb{P}[X_n = x \mid X_0, \dots, X_{n-1}] = \mathbb{P}[X_n = x \mid X_{n-1}] \quad \forall n \in T, x \in \mathcal{X}$$

Transition probabilities

$$\begin{aligned} p_{ij} &\equiv \mathbb{P}[X_{n+1} = j \mid X_n = i] \\ p_{ij}(n) &\equiv \mathbb{P}[X_{m+n} = j \mid X_m = i] \quad \text{n-step} \end{aligned}$$

Transition matrix \mathbf{P} (n-step: \mathbf{P}_n)

- (i, j) element is p_{ij}
- $p_{ij} > 0$
- $\sum_i p_{ij} = 1$

CHAPMAN-KOLMOGOROV

$$p_{ij}(m+n) = \sum_k p_{ik}(m) p_{kj}(n)$$

$$\mathbf{P}_{m+n} = \mathbf{P}_m \mathbf{P}_n$$

$$\mathbf{P}_n = \mathbf{P} \times \dots \times \mathbf{P} = \mathbf{P}^n$$

Marginal probability

$$\mu_n = (\mu_n(1), \dots, \mu_n(N)) \quad \text{where} \quad \mu_i(i) = \mathbb{P}[X_n = i]$$

$$\mu_0 \triangleq \text{initial distribution}$$

$$\mu_n = \mu_0 \mathbf{P}^n$$

20.2 Poisson Processes

Poisson Process

- $\{X_t : t \in [0, \infty)\}$ – number of events up to and including time t
- $X_0 = 0$
- Independent increments:

$$\forall t_0 < \dots < t_n : X_{t_1} - X_{t_0} \perp\!\!\!\perp \dots \perp\!\!\!\perp X_{t_n} - X_{t_{n-1}}$$

- Intensity function $\lambda(t)$

$$\begin{aligned} - \mathbb{P}[X_{t+h} - X_t = 1] &= \lambda(t)h + o(h) \\ - \mathbb{P}[X_{t+h} - X_t = 2] &= o(h) \end{aligned}$$

- $X_{s+t} - X_s \sim \text{Po}(m(s+t) - m(s))$ where $m(t) = \int_0^t \lambda(s) ds$

Homogeneous Poisson Process

$$\lambda(t) \equiv \lambda \implies X_t \sim \text{Po}(\lambda t) \quad \lambda > 0$$

Waiting Times

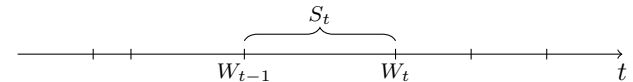
W_t := time at which X_t occurs

$$W_t \sim \text{Gamma}\left(t, \frac{1}{\lambda}\right)$$

Interarrival Times

$$S_t = W_{t+1} - W_t$$

$$S_t \sim \text{Exp}\left(\frac{1}{\lambda}\right)$$



21 Time Series

Mean function

$$\mu_{x_t} = \mathbb{E}[x_t] = \int_{-\infty}^{\infty} x f_t(x) dx$$

Autocovariance function

$$\gamma_x(s, t) = \mathbb{E}[(x_s - \mu_s)(x_t - \mu_t)] = \mathbb{E}[x_s x_t] - \mu_s \mu_t$$

$$\gamma_x(t, t) = \mathbb{E}[(x_t - \mu_t)^2] = \mathbb{V}[x_t]$$

Autocorrelation function (ACF)

$$\rho(s, t) = \frac{\text{Cov}[x_s, x_t]}{\sqrt{\mathbb{V}[x_s] \mathbb{V}[x_t]}} = \frac{\gamma(s, t)}{\sqrt{\gamma(s, s) \gamma(t, t)}}$$

Cross-covariance function (CCV)

$$\gamma_{xy}(s, t) = \mathbb{E}[(x_s - \mu_{x_s})(y_t - \mu_{y_t})]$$

Cross-correlation function (CCF)

$$\rho_{xy}(s, t) = \frac{\gamma_{xy}(s, t)}{\sqrt{\gamma_x(s, s) \gamma_y(t, t)}}$$

Backshift operator

$$B^k(x_t) = x_{t-k}$$

Difference operator

$$\nabla^d = (1 - B)^d$$

White Noise

- $w_t \sim wn(0, \sigma_w^2)$
- Gaussian: $w_t \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_w^2)$
- $\mathbb{E}[w_t] = 0 \quad t \in T$
- $\mathbb{V}[w_t] = \sigma^2 \quad t \in T$
- $\gamma_w(s, t) = 0 \quad s \neq t \wedge s, t \in T$

Random Walk

- Drift δ
- $x_t = \delta t + \sum_{j=1}^t w_j$
- $\mathbb{E}[x_t] = \delta t$

Symmetric Moving Average

$$m_t = \sum_{j=-k}^k a_j x_{t-j} \quad \text{where } a_j = a_{-j} \geq 0 \text{ and } \sum_{j=-k}^k a_j = 1$$

21.1 Stationary Time Series

Strictly stationary

$$\mathbb{P}[x_{t_1} \leq c_1, \dots, x_{t_k} \leq c_k] = \mathbb{P}[x_{t_1+h} \leq c_1, \dots, x_{t_k+h} \leq c_k]$$

$$\forall k \in \mathbb{N}, t_k, c_k, h \in \mathbb{Z}$$

Weakly stationary

- $\mathbb{E}[x_t^2] < \infty \quad \forall t \in \mathbb{Z}$
- $\mathbb{E}[x_t^2] = m \quad \forall t \in \mathbb{Z}$
- $\gamma_x(s, t) = \gamma_x(s+r, t+r) \quad \forall r, s, t \in \mathbb{Z}$

Autocovariance function

- $\gamma(h) = \mathbb{E}[(x_{t+h} - \mu)(x_t - \mu)] \quad \forall h \in \mathbb{Z}$
- $\gamma(0) = \mathbb{E}[(x_t - \mu)^2]$
- $\gamma(0) \geq 0$
- $\gamma(0) \geq |\gamma(h)|$
- $\gamma(h) = \gamma(-h)$

Autocorrelation function (ACF)

$$\rho_x(h) = \frac{\text{Cov}[x_{t+h}, x_t]}{\sqrt{\mathbb{V}[x_{t+h}] \mathbb{V}[x_t]}} = \frac{\gamma(t+h, t)}{\sqrt{\gamma(t+h, t+h) \gamma(t, t)}} = \frac{\gamma(h)}{\gamma(0)}$$

Jointly stationary time series

$$\gamma_{xy}(h) = \mathbb{E}[(x_{t+h} - \mu_x)(y_t - \mu_y)]$$

$$\rho_{xy}(h) = \frac{\gamma_{xy}(h)}{\sqrt{\gamma_x(0) \gamma_y(h)}}$$

Linear Process

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j} \quad \text{where} \quad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j$$

21.2 Estimation of Correlation

Sample mean

$$\bar{x} = \frac{1}{n} \sum_{t=1}^n x_t$$

Sample variance

$$\mathbb{V}[\bar{x}] = \frac{1}{n} \sum_{h=-n}^n \left(1 - \frac{|h|}{n}\right) \gamma_x(h)$$

Sample autocovariance function

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(x_t - \bar{x})$$

Sample autocorrelation function

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

Sample cross-variance function

$$\hat{\gamma}_{xy}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(y_t - \bar{y})$$

Sample cross-correlation function

$$\hat{\rho}_{xy}(h) = \frac{\hat{\gamma}_{xy}(h)}{\sqrt{\hat{\gamma}_x(0)\hat{\gamma}_y(0)}}$$

Properties

- $\sigma_{\hat{\rho}_x(h)} = \frac{1}{\sqrt{n}}$ if x_t is white noise
- $\sigma_{\hat{\rho}_{xy}(h)} = \frac{1}{\sqrt{n}}$ if x_t or y_t is white noise

21.3 Non-Stationary Time Series

Classical decomposition model

$$x_t = \mu_t + s_t + w_t$$

- μ_t = trend
- s_t = seasonal component
- w_t = random noise term

21.3.1 Detrending

Least Squares

1. Choose trend model, e.g., $\mu_t = \beta_0 + \beta_1 t + \beta_2 t^2$
2. Minimize RSS to obtain trend estimate $\hat{\mu}_t = \hat{\beta}_0 + \hat{\beta}_1 t + \hat{\beta}_2 t^2$
3. Residuals \triangleq noise w_t

Moving average

- The *low-pass* filter v_t is a symmetric moving average m_t with $a_j = \frac{1}{2k+1}$:

$$v_t = \frac{1}{2k+1} \sum_{i=-k}^k x_{t-i}$$

- If $\frac{1}{2k+1} \sum_{i=-k}^k w_{t-i} \approx 0$, a linear trend function $\mu_t = \beta_0 + \beta_1 t$ passes without distortion

Differencing

- $\mu_t = \beta_0 + \beta_1 t \implies \nabla x_t = \beta_1$

21.4 ARIMA models

Autoregressive polynomial

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p \quad z \in \mathbb{C} \wedge \phi_p \neq 0$$

Autoregressive operator

$$\phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$$

Autoregressive model order p , AR(p)

$$x_t = \phi_1 x_{t-1} + \dots + \phi_p x_{t-p} + w_t \iff \phi(B)x_t = w_t$$

AR(1)

$$x_t = \phi^k(x_{t-k}) + \sum_{j=0}^{k-1} \phi^j(w_{t-j}) \stackrel{k \rightarrow \infty, |\phi| < 1}{=} \underbrace{\sum_{j=0}^{\infty} \phi^j(w_{t-j})}_{\text{linear process}}$$

- $\mathbb{E}[x_t] = \sum_{j=0}^{\infty} \phi^j(\mathbb{E}[w_{t-j}]) = 0$
- $\gamma(h) = \text{Cov}[x_{t+h}, x_t] = \frac{\sigma_w^2 \phi^h}{1 - \phi^2}$
- $\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h$
- $\rho(h) = \phi \rho(h-1) \quad h = 1, 2, \dots$

Moving average polynomial

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q \quad z \in \mathbb{C} \wedge \theta_q \neq 0$$

Moving average operator

$$\theta(B) = 1 + \theta_1 B + \cdots + \theta_p B^p$$

MA(q) (moving average model order q)

$$x_t = w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} \iff x_t = \theta(B)w_t$$

$$\mathbb{E}[x_t] = \sum_{j=0}^q \theta_j \mathbb{E}[w_{t-j}] = 0$$

$$\gamma(h) = \text{Cov}[x_{t+h}, x_t] = \begin{cases} \sigma_w^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h} & 0 \leq h \leq q \\ 0 & h > q \end{cases}$$

MA(1)

$$\begin{aligned} x_t &= w_t + \theta w_{t-1} \\ \gamma(h) &= \begin{cases} (1 + \theta^2) \sigma_w^2 & h = 0 \\ \theta \sigma_w^2 & h = 1 \\ 0 & h > 1 \end{cases} \\ \rho(h) &= \begin{cases} \frac{\theta}{(1 + \theta^2)} & h = 1 \\ 0 & h > 1 \end{cases} \end{aligned}$$

ARMA(p, q)

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q}$$

$$\phi(B)x_t = \theta(B)w_t$$

Partial autocorrelation function (PACF)

- $x_i^{h-1} \triangleq$ regression of x_i on $\{x_{h-1}, x_{h-2}, \dots, x_1\}$
- $\phi_{hh} = \text{corr}(x_h - x_h^{h-1}, x_0 - x_0^{h-1}) \quad h \geq 2$
- E.g., $\phi_{11} = \text{corr}(x_1, x_0) = \rho(1)$

ARIMA(p, d, q)

$$\nabla^d x_t = (1 - B)^d x_t \text{ is ARMA}(p, q)$$

$$\phi(B)(1 - B)^d x_t = \theta(B)w_t$$

Exponentially Weighted Moving Average (EWMA)

$$x_t = x_{t-1} + w_t - \lambda w_{t-1}$$

$$x_t = \sum_{j=1}^{\infty} (1 - \lambda) \lambda^{j-1} x_{t-j} + w_t \quad \text{when } |\lambda| < 1$$

$$\tilde{x}_{n+1} = (1 - \lambda)x_n + \lambda \tilde{x}_n$$

Seasonal ARIMA

- Denoted by $\text{ARIMA}(p, d, q) \times (P, D, Q)_s$
- $\Phi_P(B^s)\phi(B)\nabla_s^D \nabla^d x_t = \delta + \Theta_Q(B^s)\theta(B)w_t$

21.4.1 Causality and Invertibility

ARMA(p, q) is causal (future-independent) $\iff \exists \{\psi_j\} : \sum_{j=0}^{\infty} \psi_j < \infty$ such that

$$x_t = \sum_{j=0}^{\infty} w_{t-j} = \psi(B)w_t$$

ARMA(p, q) is invertible $\iff \exists \{\pi_j\} : \sum_{j=0}^{\infty} \pi_j < \infty$ such that

$$\pi(B)x_t = \sum_{j=0}^{\infty} X_{t-j} = w_t$$

Properties

- ARMA(p, q) causal \iff roots of $\phi(z)$ lie outside the unit circle

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)} \quad |z| \leq 1$$

- ARMA(p, q) invertible \iff roots of $\theta(z)$ lie outside the unit circle

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)} \quad |z| \leq 1$$

Behavior of the ACF and PACF for causal and invertible ARMA models

	AR(p)	MA(q)	ARMA(p, q)
ACF	tails off	cuts off after lag q	tails off
PACF	cuts off after lag p	tails off q	tails off

21.5 Spectral Analysis

Periodic process

$$\begin{aligned} x_t &= A \cos(2\pi\omega t + \phi) \\ &= U_1 \cos(2\pi\omega t) + U_2 \sin(2\pi\omega t) \end{aligned}$$

- Frequency index ω (cycles per unit time), period $1/\omega$

- Amplitude A
- Phase ϕ
- $U_1 = A \cos \phi$ and $U_2 = A \sin \phi$ often normally distributed RV's

Periodic mixture

$$x_t = \sum_{k=1}^q (U_{k1} \cos(2\pi\omega_k t) + U_{k2} \sin(2\pi\omega_k t))$$

- U_{k1}, U_{k2} , for $k = 1, \dots, q$, are independent zero-mean RV's with variances σ_k^2
- $\gamma(h) = \sum_{k=1}^q \sigma_k^2 \cos(2\pi\omega_k h)$
- $\gamma(0) = \mathbb{E}[x_t^2] = \sum_{k=1}^q \sigma_k^2$

Spectral representation of a periodic process

$$\begin{aligned} \gamma(h) &= \sigma^2 \cos(2\pi\omega_0 h) \\ &= \frac{\sigma^2}{2} e^{-2\pi i\omega_0 h} + \frac{\sigma^2}{2} e^{2\pi i\omega_0 h} \\ &= \int_{-1/2}^{1/2} e^{2\pi i\omega h} dF(\omega) \end{aligned}$$

Spectral distribution function

$$F(\omega) = \begin{cases} 0 & \omega < -\omega_0 \\ \sigma^2/2 & -\omega \leq \omega < \omega_0 \\ \sigma^2 & \omega \geq \omega_0 \end{cases}$$

- $F(-\infty) = F(-1/2) = 0$
- $F(\infty) = F(1/2) = \gamma(0)$

Spectral density

$$f(\omega) = \sum_{h=-\infty}^{\infty} \gamma(h) e^{-2\pi i\omega h} \quad -\frac{1}{2} \leq \omega \leq \frac{1}{2}$$

- Needs $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty \implies \gamma(h) = \int_{-1/2}^{1/2} e^{2\pi i\omega h} f(\omega) d\omega \quad h = 0, \pm 1, \dots$
- $f(\omega) \geq 0$
- $f(\omega) = f(-\omega)$
- $f(\omega) = f(1 - \omega)$
- $\gamma(0) = \mathbb{V}[x_t] = \int_{-1/2}^{1/2} f(\omega) d\omega$
- White noise: $f_w(\omega) = \sigma_w^2$
- ARMA (p, q) , $\phi(B)x_t = \theta(B)w_t$:

$$f_x(\omega) = \sigma_w^2 \frac{|\theta(e^{-2\pi i\omega})|^2}{|\phi(e^{-2\pi i\omega})|^2}$$

where $\phi(z) = 1 - \sum_{k=1}^p \phi_k z^k$ and $\theta(z) = 1 + \sum_{k=1}^q \theta_k z^k$

Discrete Fourier Transform (DFT)

$$d(\omega_j) = n^{-1/2} \sum_{t=1}^n x_t e^{-2\pi i\omega_j t}$$

Fourier/Fundamental frequencies

$$\omega_j = j/n$$

Inverse DFT

$$x_t = n^{-1/2} \sum_{j=0}^{n-1} d(\omega_j) e^{2\pi i\omega_j t}$$

Periodogram

$$I(j/n) = |d(j/n)|^2$$

Scaled Periodogram

$$\begin{aligned} P(j/n) &= \frac{4}{n} I(j/n) \\ &= \left(\frac{2}{n} \sum_{t=1}^n x_t \cos(2\pi t j/n) \right)^2 + \left(\frac{2}{n} \sum_{t=1}^n x_t \sin(2\pi t j/n) \right)^2 \end{aligned}$$

22 Math

22.1 Gamma Function

- Ordinary: $\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt$
- Upper incomplete: $\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} dt$
- Lower incomplete: $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$
- $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \quad \alpha > 1$
- $\Gamma(n) = (n-1)! \quad n \in \mathbb{N}$
- $\Gamma(1/2) = \sqrt{\pi}$

22.2 Beta Function

- Ordinary: $B(x, y) = B(y, x) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$
- Incomplete: $B(x; a, b) = \int_0^x t^{a-1} (1-t)^{b-1} dt$
- Regularized incomplete: $I_x(a, b) = \frac{B(x; a, b)}{B(a, b)} \stackrel{a, b \in \mathbb{N}}{=} \sum_{j=a}^{a+b-1} \frac{(a+b-1)!}{j!(a+b-1-j)!} x^j (1-x)^{a+b-1-j}$

- $I_0(a, b) = 0 \quad I_1(a, b) = 1$
- $I_x(a, b) = 1 - I_{1-x}(b, a)$

22.3 Series

Finite

- $\sum_{k=1}^n k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^n (2k-1) = n^2$
- $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2}\right)^2$
- $\sum_{k=0}^n c^k = \frac{c^{n+1} - 1}{c - 1} \quad c \neq 1$

Binomial

- $\sum_{k=0}^n \binom{n}{k} = 2^n$
- $\sum_{k=0}^n \binom{r+k}{k} = \binom{r+n+1}{n}$
- $\sum_{k=0}^n \binom{k}{m} = \binom{n+1}{m+1}$
- VANDERMONDE'S Identity:
 $\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$
- Binomial Theorem:
 $\sum_{k=0}^n \binom{n}{k} a^{n-k} b^k = (a+b)^n$

Infinite

- $\sum_{k=0}^{\infty} p^k = \frac{1}{1-p}, \quad \sum_{k=1}^{\infty} p^k = \frac{p}{1-p} \quad |p| < 1$
- $\sum_{k=0}^{\infty} k p^{k-1} = \frac{d}{dp} \left(\sum_{k=0}^{\infty} p^k \right) = \frac{d}{dp} \left(\frac{1}{1-p} \right) = \frac{1}{1-p^2} \quad |p| < 1$
- $\sum_{k=0}^{\infty} \binom{r+k-1}{k} x^k = (1-x)^{-r} \quad r \in \mathbb{N}^+$
- $\sum_{k=0}^{\infty} \binom{\alpha}{k} p^k = (1+p)^\alpha \quad |p| < 1, \alpha \in \mathbb{C}$

22.4 Combinatorics

Sampling

k out of n	w/o replacement	w/ replacement
ordered	$n^k = \prod_{i=0}^{k-1} (n-i) = \frac{n!}{(n-k)!}$	n^k
unordered	$\binom{n}{k} = \frac{n^k}{k!} = \frac{n!}{k!(n-k)!}$	$\binom{n-1+r}{r} = \binom{n-1+r}{n-1}$

Stirling numbers, 2nd kind

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\} \quad 1 \leq k \leq n \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \begin{cases} 1 & n=0 \\ 0 & \text{else} \end{cases}$$

Partitions

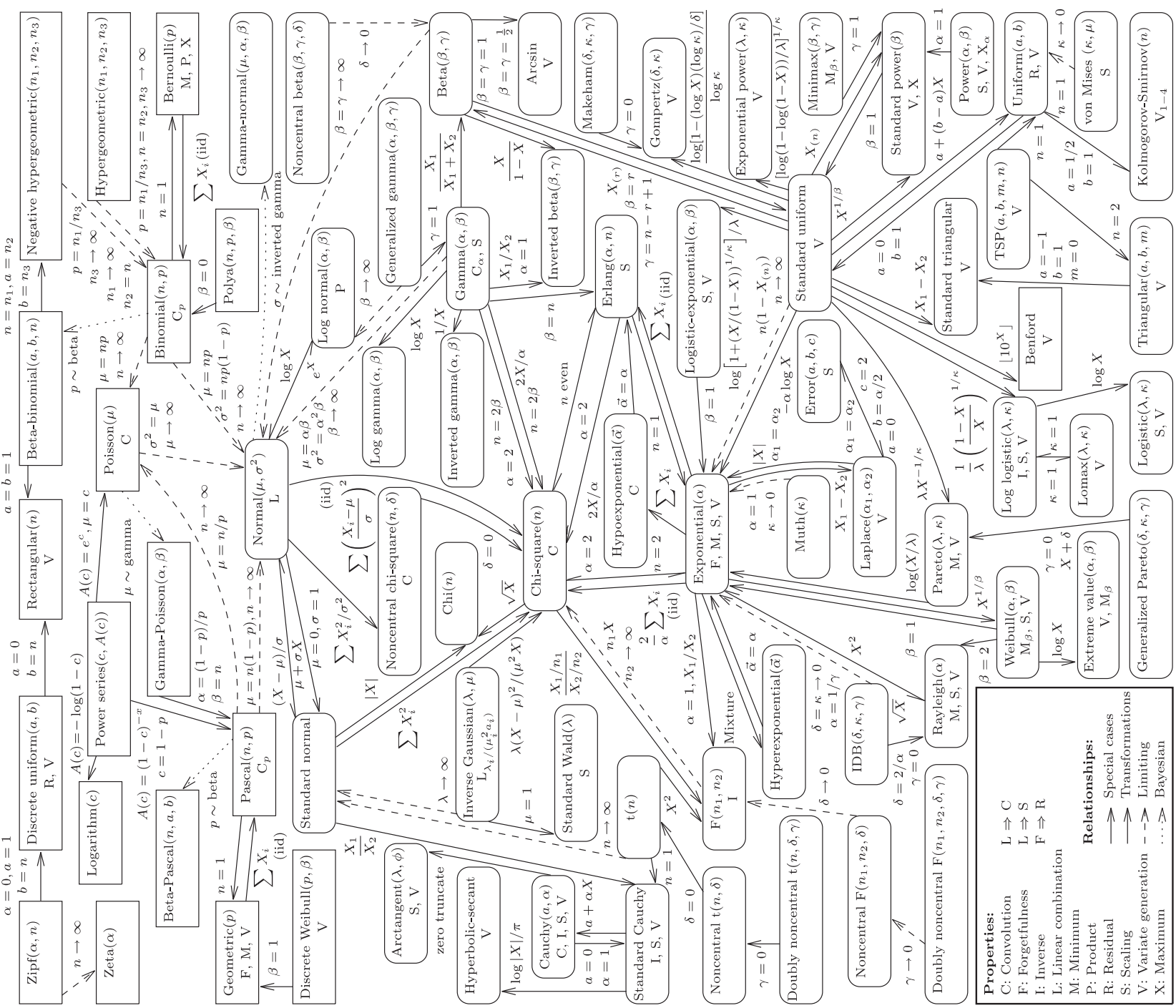
$$P_{n+k,k} = \sum_{i=1}^n P_{n,i} \quad k > n : P_{n,k} = 0 \quad n \geq 1 : P_{n,0} = 0, P_{0,0} = 1$$

Balls and Urns $f : B \rightarrow U$ $D = \text{distinguishable}, \neg D = \text{indistinguishable}.$

$ B = n, U = m$	f arbitrary	f injective	f surjective	f bijective
$B : D, U : \neg D$	m^n	$\begin{cases} m^{\underline{n}} & m \geq n \\ 0 & \text{else} \end{cases}$	$m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\}$	$\begin{cases} n! & m = n \\ 0 & \text{else} \end{cases}$
$B : \neg D, U : D$	$\binom{n+n-1}{n}$	$\binom{m}{n}$	$\binom{n-1}{m-1}$	$\begin{cases} 1 & m = n \\ 0 & \text{else} \end{cases}$
$B : D, U : \neg D$	$\sum_{k=1}^m \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$	$\begin{cases} 1 & m \geq n \\ 0 & \text{else} \end{cases}$	$\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$	$\begin{cases} 1 & m = n \\ 0 & \text{else} \end{cases}$
$B : \neg D, U : \neg D$	$\sum_{k=1}^m P_{n,k}$	$\begin{cases} 1 & m \geq n \\ 0 & \text{else} \end{cases}$	$P_{n,m}$	$\begin{cases} 1 & m = n \\ 0 & \text{else} \end{cases}$

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Univariate distribution relationships, courtesy of Leemis and McQueston [2].