Partial Exam 3

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$\boxed{\text{Problem 1}}$ Para los siguientes incisos, calcula el trabajo hecho por la fuerza F en el desplazamiento a lo largo de C. Muestra los detalles de tu cálculo.

$\boxed{a}$ $F = [y^2, -x^2]$, $C: y = 4x^2$ de $(0,0)$ a $(1,4)$

- Let's calculate

$\int_C F(r)\cdot dr$; $F_1 = y^2$ & $F_2 = -x^2$

$= \int_C F_1 dx + F_2 dy = \int_C y^2 dx - x^2 dy$

➤ From $C: y = 4x^2$ from $(0,0)$ to $(1,4)$, x varies from 0 to 1 as $y = 4x^2$
$\longrightarrow dy = 8x\,dx$

$= \int_0^1 (4x^2)^2 dx - x^2 8x\,dx$

$= \int_0^1 (16x^4 - 8x^3)dx$

$= \left(\frac{16}{5}x^5 - \frac{8}{4}x^4\right)_0^1$

$= \left(\frac{16}{5}(1)^5 - \frac{8}{4}(1)^4\right) - 0$

$= \frac{16}{5} - \frac{8}{4}$

$= \frac{6}{5}$ //

$\boxed{b}$ $F = [xy, x^2y^2]$, C de $(2,0)$ en línea recta a $(0,2)$

- Let's get x & y from

$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} = t$ ➤ where $(x_1, y_1) = (2,0)$
$(x_2, y_2) = (0,2)$

$\frac{y-0}{2-0} = \frac{x-2}{0-2} = t$

$\frac{y}{2} = \frac{x-2}{-2} = t$ ➤ So $y = 2t$ & $x = -2t+2$
$dy = 2dt$ $dx = -2dt$

- Let's calculate

$$\int_C F(r)\cdot dr = \int_{x=2}^{0} F(r(t))\cdot r'(t)\,dt$$

➤ So first, let's get $F(r(t))$ & $r'(t)$

$$r(t) = [\,x(t),\ y(t)\,] \qquad \Rightarrow \quad r'(t) = [-2,\ 2]$$
$$= [-2t-2,\ 2t\,]$$

$$F = \begin{bmatrix} xy \\ x^2y^2 \end{bmatrix} \quad \Rightarrow \quad F(r(t)) = \begin{bmatrix} (-2t-2)(2t) \\ (-2t-2)^2(2t)^2 \end{bmatrix}$$
$$= \begin{bmatrix} -4t^2-4t \\ (4t^2+4-8t)(4t^2) \end{bmatrix}$$
$$= \begin{bmatrix} -4t^2-4t \\ 16t^4+16t^2-32t^3 \end{bmatrix}$$

➤ And, let's get the integral limits from

$x = -2t+2 \quad \rightarrow$ for $x=2 \qquad 2 = -2t+2$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad t = 0$

$\qquad\qquad\qquad \rightarrow$ for $x=0 \qquad 0 = -2t+2$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad t = 1$

➤ Now let's evaluate the integral

$$\int_{x=2}^{0} F(r(t))\cdot r'(t)\,dt = \int_{t=0}^{1} \begin{bmatrix} -4t^2-4t \\ 16t^4+16t^2-32t^3 \end{bmatrix}\cdot\begin{bmatrix} -2 \\ 2 \end{bmatrix} dt$$

$$= \int_0^1 (8t^2-8t+32t^4+32t^2-64t^3)\,dt$$

$$= \int_0^1 (32t^4-64t^3+40t^2-8t)\,dt$$

$$= \left(\frac{32}{5}t^5 - \frac{64}{4}t^4 + \frac{40}{3}t^3 - \frac{8}{2}t^2\right)\Big|_0^1$$

$$= \left(\frac{32}{5} - 16 + \frac{40}{3} - 4\right) - 0$$

$$= -\frac{4}{15}$$

$\boxed{c}$ $F=[x, -z, 2y]$ de $(0,0,0)$ en línea recta a $(1,1,0)$ luego a $(1,1,1)$ y de vuelta a $(0,0,0)$.

- Let's name the points
  $O=(0,0,0)$
  $A=(1,1,0)$
  $B=(1,1,1)$

- Let's evaluate:

$$\int_C F(r)\cdot dr = \int_{C_1(OA)} F(r)\cdot dr + \int_{C_2(AB)} F(r)\cdot dr - \int_{C_3(BO)} F(r)\cdot dr$$

➤ For $C_1(OA)$

$$r(t) = (1-t)\,O + At$$
$$= (1-t)(0,0,0) + t(1,1,0)$$
$$= (t,t,0) \Longrightarrow x(t)=t, \quad y(t)=t, \quad z(t)=0$$

$F=[x, -z, 2y]$

$\hookrightarrow F(r(t)) = [t, 0, 2t] \quad \& \quad r'(t) = (1,1,0)$

- From O to A, x varies from 0 to 1 ; $x(t)=t$

$$\int_{C_1(OA)} F(r(t))\cdot r'(t)\,dt = \int_{t=0}^{1} [t, 0, 2t]\cdot[1,1,0]\,dt$$
$$= \int_0^1 t\,dt$$
$$= \left(\frac{1}{2}t^2\right)\Big|_0^1$$
$$= \frac{1}{2}$$

➤ For $C_2(AB)$

$$r(t) = (1-t)\,A + Bt$$
$$= (1-t)(1,1,0) + t(1,1,1)$$
$$= (1,1,t) \Longrightarrow x(t)=1, \quad y(t)=1, \quad z(t)=t$$

$F = [x, -z, 2y]$

$\hookrightarrow F(r(t)) = [1, -t, 2]$ & $r'(t) = (0, 0, 1)$

- From A to B, z varies from 0 to 1 ; $z(t) = t$

$$\int_{C_2(AB)} F(r(t)) \cdot r'(t)\, dt = \int_{t=0}^{1} (1, -t, 2) \cdot (0, 0, 1)\, dt$$

$$= \int_0^1 2\, dt$$

$$= 2$$

➤ For $C_3(BO)$

$$r(t) = (1-t)B + Ot$$
$$= (1-t)(1,1,1) + t(0,0,0)$$
$$= (1-t, 1-t, 1-t) \longrightarrow x(t) = y(t) = z(t) = 1-t$$

$F = [x, -z, 2y]$

$\hookrightarrow F(r(t)) = [1-t, t-1, 2-2t]$ & $r'(t) = [-1, -1, -1]$

- From B to O, z varies from 1 to 0

$z(t) = 1-t \longrightarrow$ for $z = 1$ $\quad 1 = 1-t$, $t = 0$

$\longrightarrow$ for $z = 0$ $\quad 0 = 1-t$, $t = 1$

$$\int_{C_3(BO)} F(r(t)) \cdot r'(t)\, dt = \int_{t=0}^{1} (1-t, t-1, 2-2t) \cdot (-1, -1, -1)\, dt$$

$$= \int_0^1 (-2+2t)\, dt$$

$$= \left(-2t + t^2\right)\Big|_0^1$$

$$= 0 - (-2+1) = 1$$

➤ Now, let's evaluate:

$$\int_{C_1(OA)} F(r)\cdot dr + \int_{C_2(AB)} F(r)\cdot dr - \int_{C_3(BO)} F(r)\cdot dr = \frac{1}{2} + 2 - 1$$

$$= \frac{3}{2}$$

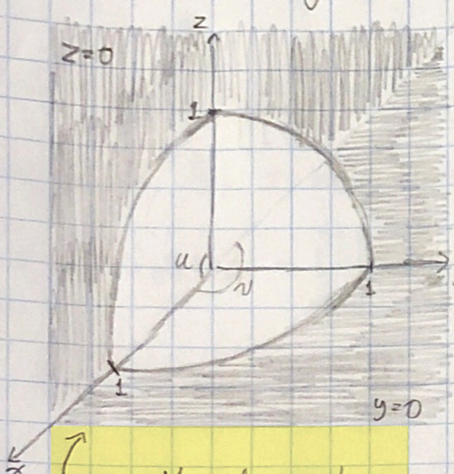
Problem 2 Evalúa la siguente integral e indica el tipo de superficie. Muestra los detalles.

$G = ax + by + cz$, $\quad S: x^2+y^2+z^2=1$, $\quad y=0$, $\quad z=0$

➤ Let's use the surface integral

$$\int_S G(r)\,dA = \int_R G(r(u,v))\,|r_u \times r_v|\,du\,dv$$

- Let's sketch $S: x^2+y^2+z^2=1$



one 1/4 of a sphere...

$v \in [0, \pi]$ & $u \in [0, 0.5\pi]$

$S: (\cos u \sin v, \sin u \sin v, \cos v) = r(u,v)$

$r_u(u,v) = (-\sin u \sin v, \cos u \sin v, 0)$

$r_v(u,v) = (\cos u \cos v, \sin u \cos v, -\sin v)$

$$r_u \times r_v = \begin{bmatrix} i & j & k \\ -\sin u \sin v & \cos u \sin v & 0 \\ \cos u \cos v & \sin u \cos v & -\sin v \end{bmatrix}$$

$$= \begin{bmatrix} -\cos u \sin^2 v \\ \sin u \sin^2 v \\ -\sin^2 u \sin v \cos v - \cos^2 u \sin v \cos v \end{bmatrix}$$

$$|r_u \times r_v| = \sqrt{\cos^2 u \sin^4 v + \sin^2 u \sin^4 v + \sin^4 u \sin^2 v \cos^2 v + \cos^4 u \sin^2 v \cos^2 v - 2\sin^2 u \cos^2 u \sin^2 v \cos^2 v}$$

$$= \sqrt{\sin^4 v(\cos^2 u + \sin^2 u) + (\sin^4 u + \cos^4 u)(\sin^2 v \cos^2 v) - 2\sin^2 u \cos^2 u \sin^2 v \cos^2 v}$$

$$= \sqrt{\sin^4 v + [(\sin^2 u - \cos^2 u)^2 + 2\sin^2 u \cos^2 u]\sin^2 v \cos^2 v - 2\sin^2 u \cos^2 u \sin^2 v \cos^2 v}$$

$$= \sqrt{\sin^4 v + (\sin^2 u - \cos^2 u)^2 \sin^2 v \cos^2 v + \underline{2\sin^2 u \cos^2 u \sin^2 v \cos^2 v} - \underline{2\sin^2 u \cos^2 u \sin^2 v \cos^2 v}}$$

$$= \sqrt{\sin^4 v + \cos 2u \sin^2 v \cos^2 v}$$

$$= \sin u$$

$G = ax + by + cz$

$\hookrightarrow G(r(u,v)) = a\cos u \sin v + b\sin u \sin v + c\cos v$

➤ So,

$$\int_R G(r(u,v))|r_u \times r_v|\,du\,dv = \int_{u=0}^{\frac{1}{2}\pi}\int_{v=0}^{\pi}(a\cos u \sin v + b\sin u \sin v + c\cos v)\sin u\,dv\,du$$

$$= \int_0^{\frac{1}{2}\pi}(2\sin u(a\cos u + b\sin u))\,du$$

$$= a + b\frac{\pi}{2} \,//$$

★ If we ignore $y=0$ & $z=0$ from the problem istructions, then S shall be a (complete sphere) with

$v \in [0, 2\pi]$ & $u \in [0, 2\pi]$, in that case:

$$\int_R G(r(u,v))|r_u \times r_v|\,du\,dv = \int_{u=0}^{2\pi}\int_{v=0}^{2\pi}(a\cos u \sin v + b\sin u \sin v + c\cos v)\,dv\,du$$

$$= \int_0^{2\pi} 0\,du$$

$$= 0 \,//$$