Forces

a) Body forces (Volume forces)
b) Surface forces (stresses)
c) Line forces (or ~~surface tension,~~ must say surface energy)

Modeling Approaches in Fluid Mechanics [1]

- Lattice-Boltzman method and molecular dynamics
- Integral Approach (Macorscopic balances)
- **Differential Approach (Microscopic balances)**
- Phenomenological Approach (Empirical correlations and Dimensional analysis)

[1] Kleinstreuer Clement. *Modern Fluid Dynamics*. CRC Press, 2018

a) Body forces
b) Surface forces
c) Line forces or surface tension

Body forces : this type of force acts upon the whole material volume at a distance without contact, e.g. gravitational force

$$\underset{\rightarrow}{f_v} = force/volume \qquad \underset{\rightarrow}{f_m} = force/mass$$

$$d\underset{\rightarrow}{F_V} = \underset{\rightarrow}{f_V}\, dV = \underset{\rightarrow}{f_m} dm = \rho \underset{\rightarrow}{f_m}\, dV$$

$$\underset{\rightarrow}{f_v} = \rho \underset{\rightarrow}{g}$$ Gravity effect (specific weight)

$$\underset{\rightarrow}{f_v} = \rho_e \underset{\rightarrow}{E} + \underset{\rightarrow}{J} \times \underset{\rightarrow}{B}$$ Electromagnetic

Lorentz force:
$\rho_e \underline{E}$

$$\underline{f_v} = -\rho\left[\underline{a}_0 + \underline{\dot{\Omega}} \times \underline{r} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) + 2\underline{\Omega} \times \underline{v}\right]$$ Inertial

Note: should not surprise us to express these forces in terms of force per unit volume

$$\underline{a} = \underline{a}_0 + \underline{\dot{\Omega}} \times \underline{r} + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right) + 2\underline{\Omega} \times \underline{v}$$

Origin acceleration

Angular acceleration

Centripetal acceleration

Coriolis acceleration

$$\underline{f_v} = -\rho\left[\underline{a}_0 + \underline{\dot{\Omega}} \times \underline{r} + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right) + 2\underline{\Omega} \times \underline{v}\right]$$

$$\underline{f_m} = -\left[\underline{a}_0 + \underline{\dot{\Omega}} \times \underline{r} + \underline{\Omega} \times \left(\underline{\Omega} \times \underline{r}\right) + 2\underline{\Omega} \times \underline{v}\right]$$

$$\underrightarrow{F_V} = \int_V \underrightarrow{f_V}\, dV = \int_V \rho\, \underrightarrow{f_m} dV = \int_V \underrightarrow{f_m} dm$$

Volume force or mass force is a construct that will be used to quantify the total force in a system, when that particular force (either mass force or volume force) is acting on each portion of the system.

a) Body forces
b) Surface forces
c) Line forces or surface tension

Surface forces : forces act upon the surface of the fluid particle or upon the surface of the considered fluid domain.(e.g. pressure , friction, shear stress, normal stress, etc)

$$\underrightarrow{f_s} = force / surface$$

Forces are computed from stress

$$d\,\underrightarrow{F_S} = \underrightarrow{f_S}\, dA \qquad \underrightarrow{F_S} = \int_A \underrightarrow{f_S}\, dA$$

$$\underrightarrow{f_S} = -p\,\underrightarrow{n} = -p\underline{\underline{I}} \cdot \underrightarrow{n} = -p\underrightarrow{n} \cdot \underline{\underline{I}}$$

Pressure surface force

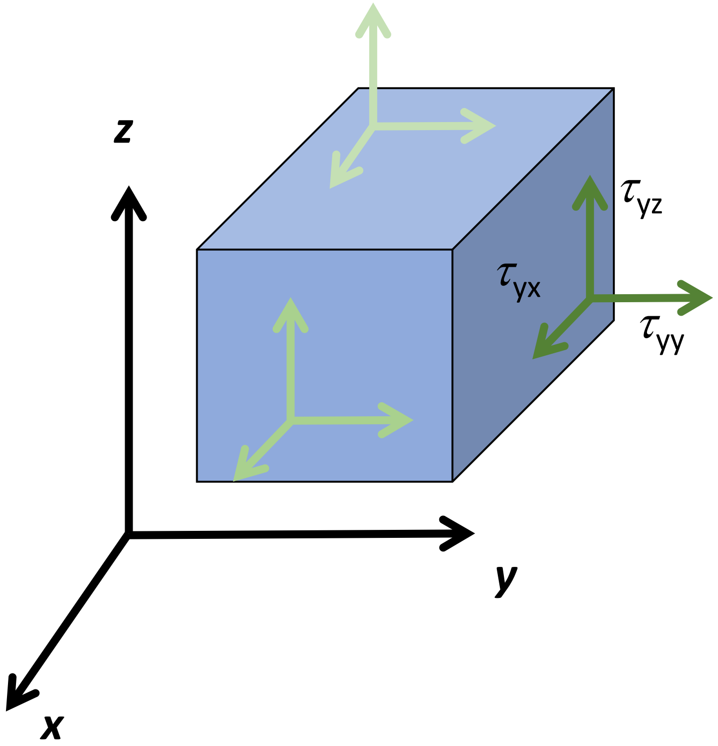
surface force is a construct that will be used to quantify the force acting over the boundaries of the system, the boundary can be permeable or impermeable.

$\underline{\underline{I}}$ is the identity tensor

$\underline{n}$ is the normal vector always pointing outward the surface

Note: should not surprise us to express these forces in terms of force per unit area

Stress tensor is another construct to simplify the analysis



$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

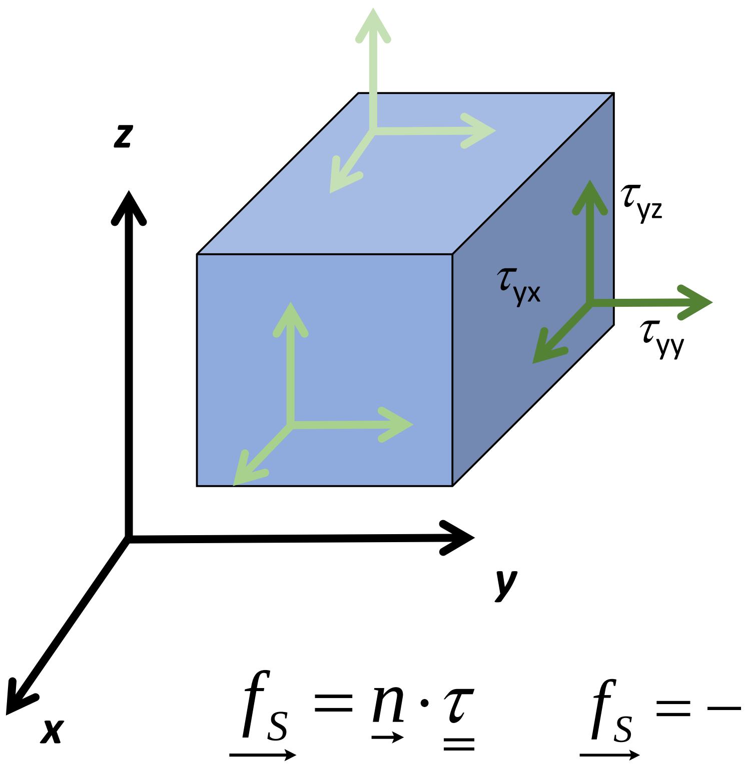
$$\underset{\rightarrow}{f_S} = \underset{\rightarrow}{n} \cdot \underline{\underline{\tau}}$$

$$\underset{\rightarrow}{f_S} = -p\underset{\rightarrow}{n} \cdot \underline{\underline{I}}$$

τ_{yx}=Is seen to be the force per unit area on a plane perpendicular to the *y* axis, acting in the ***x*** direction and exerted by the fluid at greater ***y***.

n is the normal vector always pointing outward the surface

Stress tensor is another construct to simplify the analysis



$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx}\hat{i}\hat{i} & + \tau_{xy}\hat{i}\hat{j} & + \tau_{xz}\hat{i}\hat{k} & + \\ \tau_{yx}\hat{j}\hat{i} & + \tau_{yy}\hat{j}\hat{j} & + \tau_{yz}\hat{j}\hat{k} & + \\ \tau_{zx}\hat{k}\hat{i} & + \tau_{zy}\hat{k}\hat{j} & + \tau_{zz}\hat{k}\hat{k} & \end{bmatrix}$$

$$\underline{\underline{I}} = \begin{bmatrix} 1\,\hat{i}\hat{i} & + 0\,\hat{i}\hat{j} & + 0\,\hat{i}\hat{k} & + \\ 0\,\hat{j}\hat{i} & + 1\,\hat{j}\hat{j} & + 0\,\hat{j}\hat{k} & + \\ 0\,\hat{k}\hat{i} & + 0\,\hat{k}\hat{j} & + 1\,\hat{k}\hat{k} & \end{bmatrix}$$

$$\underset{\rightarrow}{f_S} = \underset{\rightarrow}{n} \cdot \underline{\underline{\tau}} \qquad \underset{\rightarrow}{f_S} = -p\underset{\rightarrow}{n} \cdot \underline{\underline{I}}$$

τ_{yx}=Is seen to be the force per unit area on a plane perpendicular to the *y* axis, acting in the ***x*** direction and exerted by the fluid at greater ***y***.

n is the normal vector always pointing outward the surface

a) Body forces

b) Surface forces

c) Line forces or surface tension

Line forces: forces acting in the three phases contact line (For instance surface tension acts where liquid, gas and solid meet. For interfacial tension where liquid A, Liquid B and solid meet)

$$\underset{\longrightarrow}{f_s} = force/line$$

$$\underset{\longrightarrow}{F_L} = \int_L \underset{\longrightarrow}{f_L}\, dL$$

$$d\underset{\longrightarrow}{F_L} = \underset{\longrightarrow}{f_L}\, dL$$

Surface tension can be analyzed as force per unit length (also is the energy that must be supplied to increase the surface area by one unit, surface tension contributes to the thermodynamic work, and can be defined as the increase in internal energy *U* or in the Helmholtz free energy *A* that accompanies an increase in surface area, *a*)

Line Force or Surface energy

$$\gamma = \left[\frac{\partial A}{\partial a}\right]_{T,V,n}$$

Hydrostatics

$$\underset{\rightarrow}{\nabla}\, p = \rho \underset{\rightarrow}{f_m} = \underset{\rightarrow}{f_v}$$

$$\underset{\rightarrow}{\nabla}\, p = \rho\left(\underset{\rightarrow}{g} - \underset{\rightarrow}{a}\right)$$

Surface forces are balanced with body forces

$$\rho \underset{\rightarrow}{a} = \rho \underset{\rightarrow}{g} - \underset{\rightarrow}{\nabla}\, p$$

Recasting we have

Where this equation comes from ?

Change in momentum

Body forces (specific weigh)

Surface forces (pressure)

$$\frac{d}{dt}[m\underline{v}] = \iiint \frac{d[\rho \underline{v}]}{dt} dV = \iiint \rho \underline{a} dV = \iiint \rho \underline{g}\, dV - \oiint \underline{n} \cdot \underline{\underline{I}}\, p\, dA = \iiint \rho \underline{g}\, dV - \iiint \underline{\nabla} \cdot [p\underline{\underline{I}}] dV$$

Divergence theorem

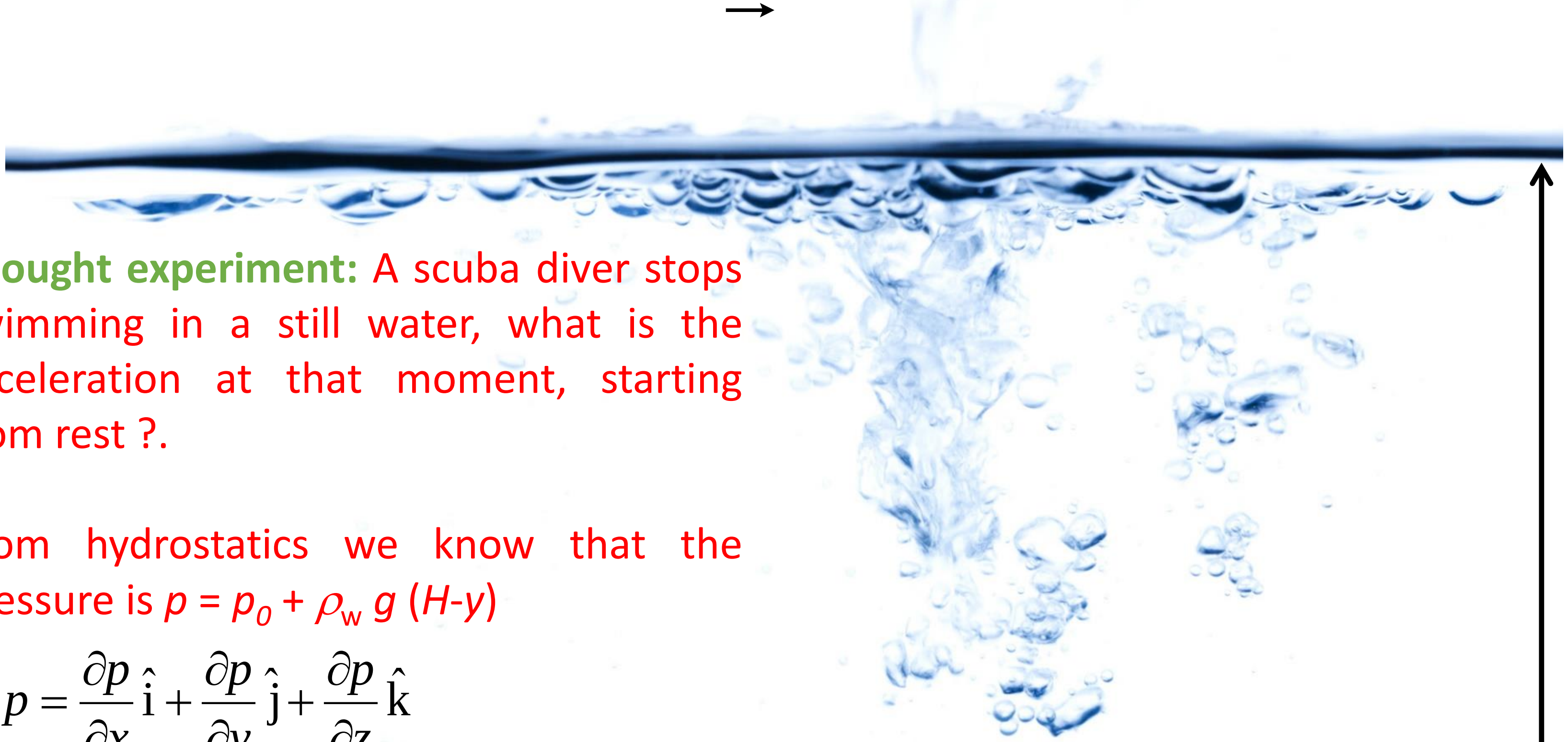
$$\iiint [\rho \underline{a} - \rho \underline{g} + \underline{\nabla} \cdot [p\underline{\underline{I}}]] dV = 0 \qquad \rho \underline{a} - \rho \underline{g} + \underline{\nabla} \cdot [p\underline{\underline{I}}] = 0$$

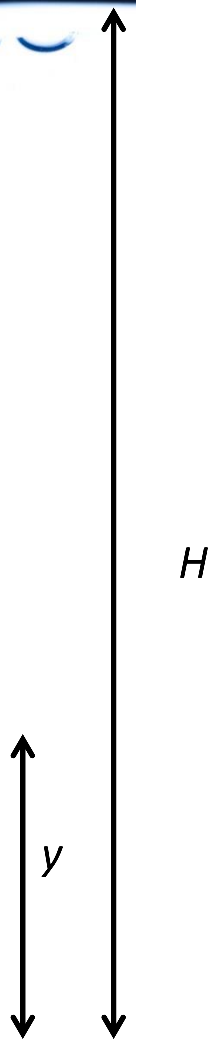
$$\rho \underline{a} - \rho \underline{g} + \underline{\nabla} p = 0$$

The surface integral was transformed into volume integral by the use of the Divergence theorem

$$\oiint \underset{\rightarrow}{\Psi} \cdot \underset{\rightarrow}{n}\, dA = \iiint \underset{\rightarrow}{\nabla} \cdot \underset{\rightarrow}{\Psi} dV \qquad \oint_A \underset{\rightarrow}{n} \cdot \underline{\underline{\tau}} dA = \int_V \underset{\rightarrow}{\nabla} \cdot \underline{\underline{\tau}} dV$$

$$\rho \underset{\rightarrow}{a} dV = \rho \underset{\rightarrow}{g}\, dV - \underset{\rightarrow}{\nabla}\, p dV$$





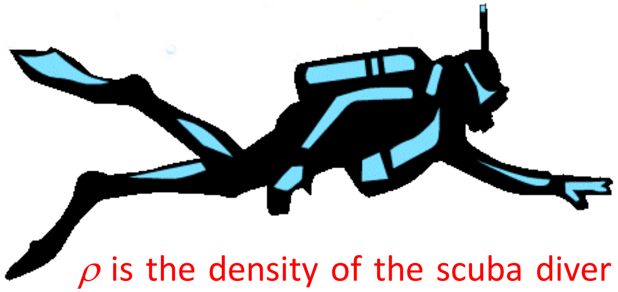
Thought experiment: A scuba diver stops swimming in a still water, what is the acceleration at that moment, starting from rest ?.

From hydrostatics we know that the pressure is $p = p_0 + \rho_w\, g\, (H-y)$

$$\underline{\nabla}\, p = \frac{\partial p}{\partial x}\hat{\mathrm{i}} + \frac{\partial p}{\partial y}\hat{\mathrm{j}} + \frac{\partial p}{\partial z}\hat{\mathrm{k}}$$

$$\underline{\nabla}\, p = -\rho_w g\, \hat{\mathrm{j}}$$

$$\rho \underset{\rightarrow}{a}\, dV = -\rho\, g\, dV\, \hat{\mathrm{j}} + \rho_w\, g\, dV\, \hat{\mathrm{j}}$$



ρ is the density of the scuba diver (i.e. guy + tanks + O_2 + He + dry suit + hoses + mask + valves + gages + clock + fins + speedos) , m is the mass of the SD and , V is the volume of the SD

Integrating:

$$\rho \underset{\rightarrow}{a}\, V = m \underset{\rightarrow}{a} = \left[\rho_w - \rho\right] g\, V\, \hat{\mathrm{j}}$$

Velocity vector

Orthogonal unit vectors

$$\underline{V} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$$

$$\underline{V} = u\,\underline{i} + v\,\underline{j} + w\,\underline{k}$$

Both nomenclatures for designate unit vectors will be accepted in this document

Viscosity and shear stress

$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

Velocity vector

$$\underset{\rightarrow}{v} = \hat{\mathrm{i}}\, u + \hat{\mathrm{j}}\, v + \hat{\mathrm{k}}\, w$$

Tensors

Velocity gradient tensor

$$\underset{\rightarrow}{\nabla}\, \underset{\rightarrow}{v} = \begin{bmatrix} \hat{\mathrm{i}}\hat{\mathrm{i}}\, \partial u/\partial x & \hat{\mathrm{i}}\hat{\mathrm{j}}\, \partial v/\partial x & \hat{\mathrm{i}}\hat{\mathrm{k}}\, \partial w/\partial x \\ \hat{\mathrm{j}}\hat{\mathrm{i}}\, \partial u/\partial y & \hat{\mathrm{j}}\hat{\mathrm{j}}\, \partial v/\partial y & \hat{\mathrm{j}}\hat{\mathrm{k}}\, \partial w/\partial y \\ \hat{\mathrm{k}}\hat{\mathrm{i}}\, \partial u/\partial z & \hat{\mathrm{k}}\hat{\mathrm{j}}\, \partial v/\partial z & \hat{\mathrm{k}}\hat{\mathrm{k}}\, \partial w/\partial z \end{bmatrix}$$

Rate of strain tensor

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underset{\rightarrow}{\nabla}\, \underset{\rightarrow}{v} + \left(\underset{\rightarrow}{\nabla}\, \underset{\rightarrow}{v} \right)^T \right]$$

Vorticity tensor

$$\underline{\underline{\Omega}} = \frac{1}{2}\left[\underset{\rightarrow}{\nabla}\, \underset{\rightarrow}{v} - \left(\underset{\rightarrow}{\nabla}\, \underset{\rightarrow}{v} \right)^T \right]$$

$$\underline{\nabla}\, \underline{V} = \underline{\underline{\Gamma}} + \underline{\underline{\Omega}}$$

Symmetric + Skew-symmetric

$$\underline{\underline{\Gamma}} = \underline{\underline{\Gamma}}^T$$

$$\underline{\underline{\Omega}} = -\underline{\underline{\Omega}}^T$$

Viscosity and shear stress relationship

Relationship between shear stress tensor and strain rate tensor

$$\underline{\underline{\tau}} = 2\mu\left[\underline{\underline{\Gamma}} - \frac{1}{3}\left(\underset{\rightarrow}{\nabla} \cdot \underset{\rightarrow}{v}\right)\underline{\underline{I}}\right] + 3\kappa\left[\frac{1}{3}\left(\underset{\rightarrow}{\nabla} \cdot \underset{\rightarrow}{v}\right)\underline{\underline{I}}\right]$$

$$\underline{\underline{\tau}} = 2\mu\left[\underline{\underline{\Gamma}} - \frac{1}{3}Tr\left(\underline{\underline{\Gamma}}\right)\underline{\underline{I}}\right] + 3\kappa\left[\frac{1}{3}Tr\left(\underline{\underline{\Gamma}}\right)\underline{\underline{I}}\right]$$

For incompressible Newtonian fluids

$$\tau_{xy} = \tau_{yx} = \mu\left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y}\right]$$ Cartesian coordinates

$$\tau_{rz} = \tau_{zr} = \mu\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]$$ Polar coordinates

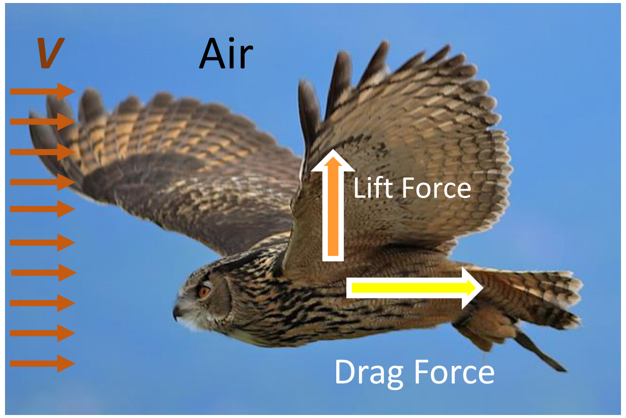
$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

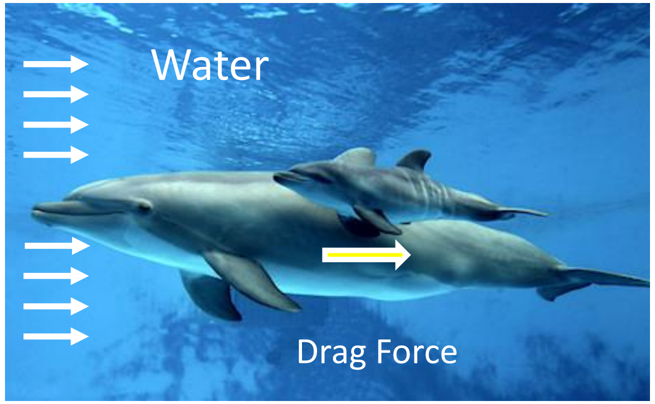
$$Tr\left(\underline{\underline{\Gamma}}\right) = Trace\left(\underline{\underline{\Gamma}}\right) = \Gamma_{xx} + \Gamma_{yy} + \Gamma_{zz}$$

VISCOSITY

Viscosity: A property that represents the internal resistance of a fluid to motion or the “fluidity”. Is the ratio between the shear stress and the strain rate.

Drag force: The force a flowing fluid exerts on a body in the flow direction (if a solid is moving in a fluid, is the force in the direction of the resultant relative motion between solid and fluid). The magnitude of this force depends, in part, on viscosity, velocity, size, geometry and hydrodynamics/aerodynamics.





The viscosity of a fluid is a measure of its “*resistance to deformation*.”

Viscosity is due to the internal frictional force that develops between different layers of fluids as they are forced to move relative to each other.

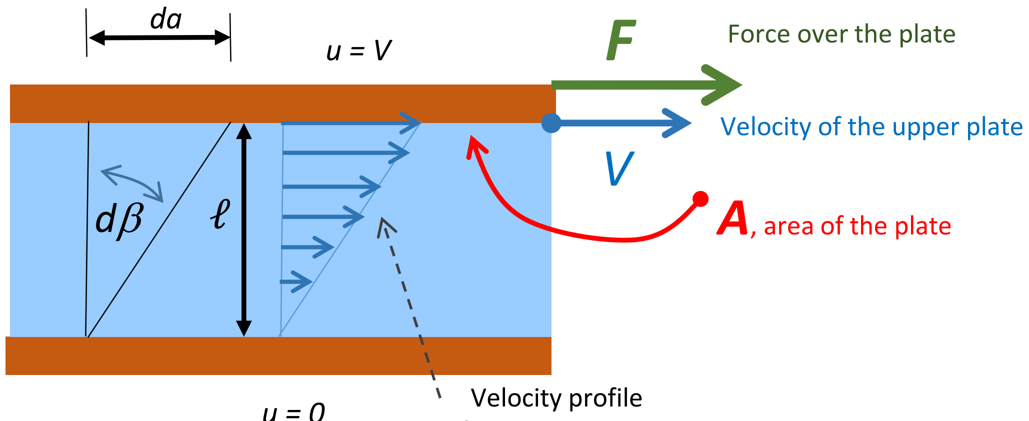
A fluid moving relative to a body exerts a drag force on the body, partly because of friction caused by viscosity.

Here hydrodynamics/aerodynamics is in the context of flow conditions, flow regime, e.g. Laminar-Turbulent, Sonic-subsonic, etc (i.e. relationship of the magnitude of the different forces involved in the process)

Challenge:

I) Explain the silent flight of owls

II) Explain drafting in nature (lobster train, shoaling and schooling of fish, vortex drafting, and dolphins swimming in groups)



$$\underline{v} = \hat{\imath}\,\frac{y}{\ell}\,V$$

The behavior of a fluid in laminar flow between two parallel plates when the upper plate moves with a constant velocity.

Shear stress is a tensor, and by multiplication with the normal vector produces the ratio between force and area

$$\underline{n}\cdot\underline{\underline{\tau}} = (\underline{F})/A$$

For a linear velocity field (1-D flow)

$$\underline{v} = \hat{\imath}v_x = \hat{\imath}\,u(y) = \hat{\imath}\,\frac{y}{\ell}\,V$$

The velocity gradient tensor is a tensor

$$\underline{\nabla}\,\underline{v} = \hat{\jmath}\,\frac{\partial}{\partial y}\left[\hat{\imath}\,\frac{y}{\ell}\,V\right] = \hat{\jmath}\hat{\imath}\,\frac{V}{\ell}$$

Strain and strain rate are, that are in nature tensors as well can be calculated as:

$$\gamma = d\beta \approx \frac{da}{\ell} = \frac{V\,dt}{\ell} = \frac{du}{dy}\,dt \qquad \dot{\gamma} = \frac{d\beta}{dt} = \frac{du}{dy}$$

Newtonian fluids: Fluids for which the rate of deformation is proportional to the shear stress.

$$\tau_{yx} \propto \dot{\gamma} \qquad \text{or} \qquad \tau_{yx} \propto = \frac{du}{dy}$$

$$\underline{\underline{\tau}}_{yx} = \mu\,\hat{\jmath}\left[\frac{\partial}{\partial y}\right][\hat{\imath}\,v_x] = \hat{\jmath}\hat{\imath}\left[\mu\,\frac{\partial v_x}{\partial y}\right]$$

Shear stress

$$\underline{F} = [A\,\underline{n}]\cdot\left[\underline{\underline{\tau}}\right] = A\,[\hat{\jmath}]\cdot\left[\hat{\jmath}\hat{\imath}\,\mu\,\frac{\partial u}{\partial y}\right] = \hat{\imath}\,A\,\mu\,\frac{\partial u}{\partial y}$$

Shear force

Newtonian fluids: have a property to relate shear stress with strain rate, that is called viscosity, and depends only on the fluid characteristics.

μ coefficient of viscosity
Dynamic (absolute) viscosity
kg/m · s or N · s/m^2 or Pa · s
1 poise = 0.1 Pa · s

$$\underline{\underline{\tau}}_{yx} = \hat{\jmath}\hat{\imath}\left[\mu\,\frac{\partial v_x}{\partial y}\right] \approx \hat{\jmath}\hat{\imath}\,\mu\,\frac{\Delta v_x}{\Delta y} \approx \hat{\jmath}\hat{\imath}\,\mu\,\frac{u}{L}$$

$$\underline{\nabla} = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$$

Example No.1

In film condensation along a vertical plate in a vapor atmosphere, Nusselt found out that in laminar flow the velocity profile at a station *x* is.

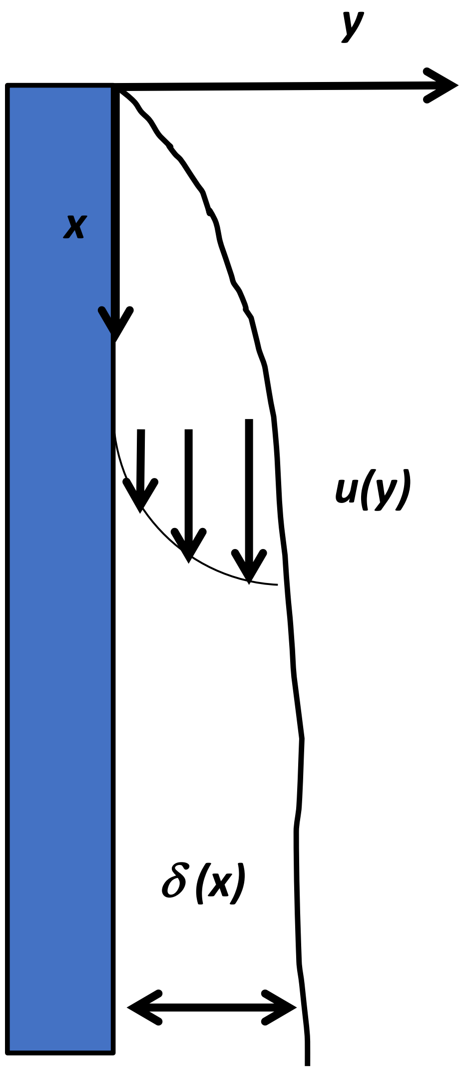
$$u(y) = \frac{[\rho_l - \rho_v]\, g\, \delta^2}{\mu} \left[\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2 \right]$$

Where ρ_l and ρ_v are the density of the fluid in the liquid and vapor phase, respectively, and μ the liquid viscosity. Find the flow rate per unit width at any value of *x*, and the shear stress.

The previous equation in dimensionless form is:

$$Re = Ar\left[\eta - \frac{\eta^2}{2}\right]$$

$$Re = \frac{\rho\, v\, \delta}{\mu} \qquad Ar = \frac{\frac{\Delta\rho}{\rho}\, g\, \delta^3}{\left(\frac{\mu}{\rho}\right)^2} \qquad \eta = \frac{y}{\delta}$$



To calculate volumetric flow, the velocity profile is integrated through the cross sectional area, perpendicular to the velocity

$$\dot{V} = \int_0^W \int_0^\delta u\, dz dy = \int_0^\delta W\, u\, dy$$

$$\dot{V} = \int_0^\delta W\, u\, dy = \int_0^\delta W \frac{[\rho_l - \rho_v]\, g\, \delta^2}{\mu} \left[\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right] dy = \frac{W \Delta\rho\, g\, \delta^2}{\mu} \int_0^1 \left[\eta - \frac{1}{2}\eta^2\right] d\eta$$

$$\dot{V} = \frac{W \Delta\rho g \delta^3}{\mu} \left[\frac{\eta^2}{2} - \frac{\eta^3}{6}\right]\Bigg|_0^1 = \frac{W \Delta\rho g \delta^3}{3\mu}$$

To calculate shear rate over the wall, we use Newton's viscosity law

$$\tau_{yx} = \mu \frac{du}{dy}$$

$$\tau_{yx} = \mu \frac{du}{dy} = [\rho_l - \rho_v]\, g\, \delta^2 \left[\frac{1}{\delta} - \frac{y}{\delta^2}\right]$$

Then we evaluate the stress in the wall, i.e y=0

$$\tau_w = \mu \frac{du}{dy}\Bigg|_{y=0} = [\rho_l - \rho_v]\, g\, \delta$$

In the Math refresher, we explain the Divergence theorem, and we discussed about flow and flux.

In the next slide you will find how to express flow and flux for different properties.

Physical meaning of different fluxes and expression of flow

Flux	property	Flow symbol	Units of flow	Expression for flow
$\underline{v}$	Volume	$\dot{V}$	m^3/s	$\int \underline{v} \cdot \underline{n}\, dA$
$\rho\, \underline{v}$	mass	$\dot{m}$	kg/s	$\int \rho\, \underline{v} \cdot \underline{n}\, dA$
$\phi\, \rho\, \underline{v}$	Φ	$\dot{\Phi}$	prop/s	$\int \rho\, \phi\, \underline{v} \cdot \underline{n}\, dA$
$\underline{q} = \underline{\dot{Q}}_A$	heat	$\dot{Q}$	J/s	$\int \underline{q} \cdot \underline{n}\, dA = \int \underline{\dot{Q}}_A \cdot \underline{n}\, dA$
$\rho \underline{v}\, \underline{v}$	momentum	$\dot{m}\langle \underline{v} \rangle \beta$	N	$\int \rho \underline{v}\, \underline{v} \cdot \underline{n}\, dA$
$\rho \hat{E}\, \underline{v}$	Energy, *E*	$\dot{m}\hat{E}$	J/s	$\int \rho \hat{E} \underline{v} \cdot \underline{n}\, dA$

$\phi = \hat{\Phi}$ $\qquad \beta = \dfrac{\langle v^2 \rangle}{\langle v \rangle^2}$ $\qquad \hat{E} = \hat{U} + gz + \dfrac{\langle v^3 \rangle}{2\langle v \rangle}$

This is expression for flow not flux, in math jargon the term flux and flow are interchangeable

$$\dot{\Phi} = \frac{d\Phi}{dt}$$

$$\hat{\Phi} = \frac{d\Phi}{dm}$$

$$\langle \phi \rangle = \frac{\int \phi\, dA}{\int dA}$$

Kinematics

Types of Motion or Deformation Rates of Fluid Elements

In fluid mechanics, an element may undergo four fundamental types of motion or deformation:

(*a*) **translation**, (*b*) **rotation**, (*c*) **linear strain** (also called **extensional strain**), and (*d*) **shear strain**.

All four types of motion or deformation rates usually occur simultaneously.

In continuous media and in fluid dynamics the motion and deformation of fluid elements is done in terms of *rates* such as

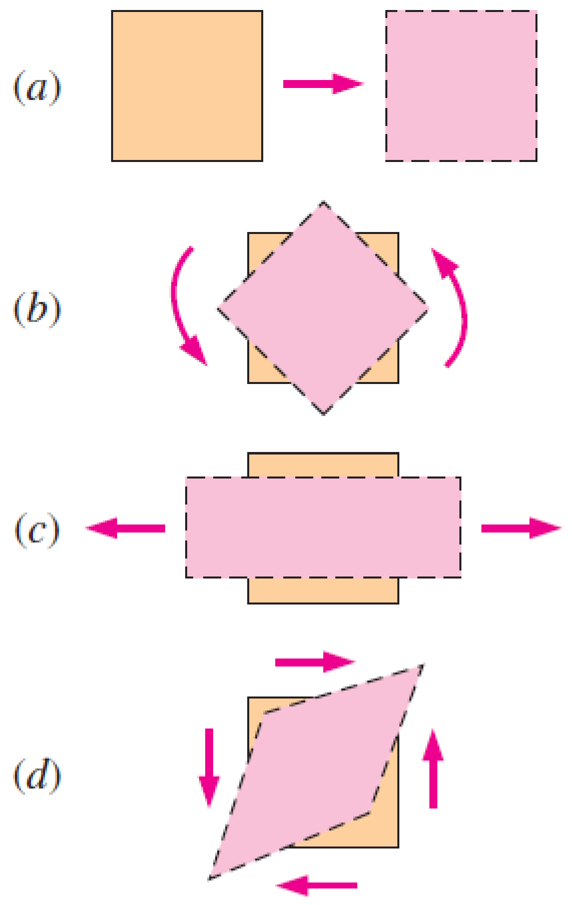
velocity (rate of translation),

angular velocity (rate of rotation),

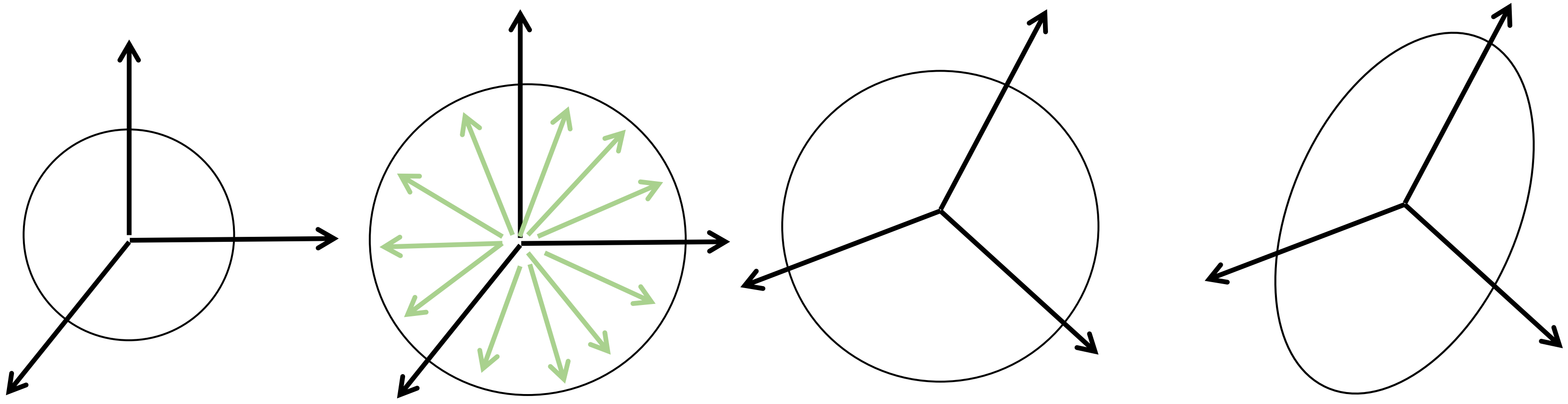
linear strain rate (rate of linear strain), and

shear strain rate (rate of shear strain).

In order for these **deformation rates** to be useful in the calculation of fluid flows, we must express them in terms of the velocity field and the partial derivatives of the velocity field.



Fundamental types of fluid element motion or deformation: (*a*) translation, (*b*) rotation, (*c*) linear strain, and (*d*) shear strain.



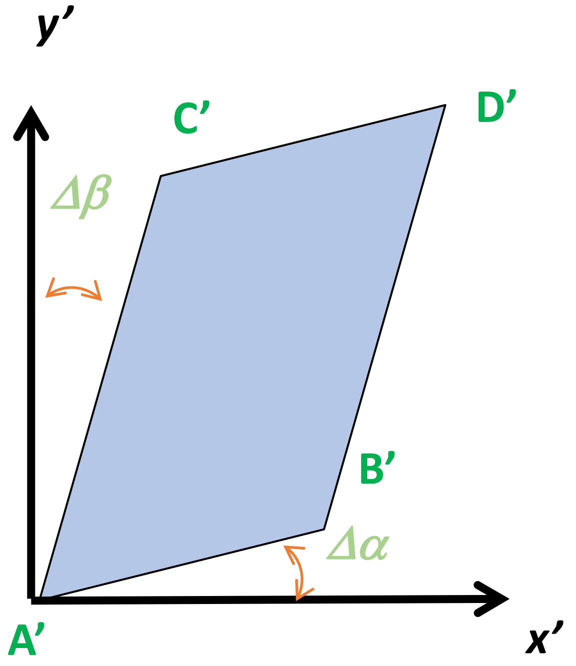
Expansion Rotation Deformation

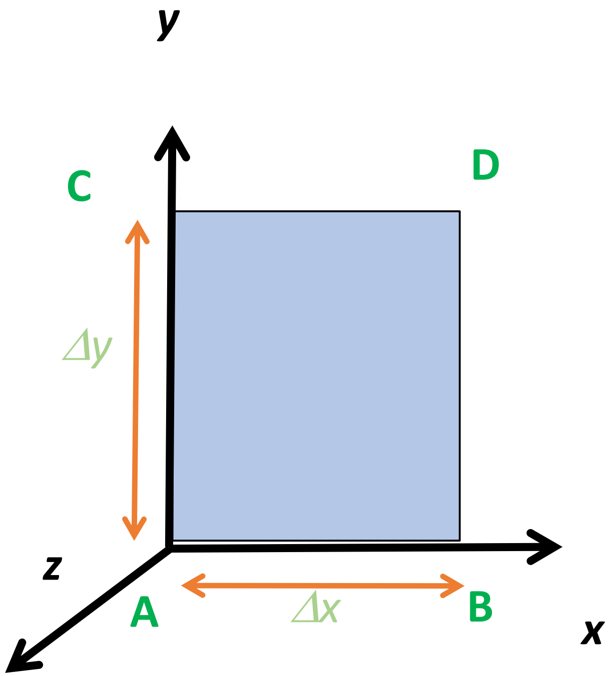
Introduction to kinematics:

All the derivations will be done in 2-D flow to simplify the definition of terms, and to visualize as simple as possible the kinematics of fluids

Quantification of rates…

Quantification of: Translation, Deformation and Rotation





$$\underline{v} = \hat{\imath}\, u + \hat{\jmath}\, v + \hat{k}\, w$$ Velocity vector

$$\underline{r} = \hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z$$ position vector

In this cartoon points **A,B,C** and **D** are the corners of a rectangle of fluid element that is moving, after a time $t+\Delta t$ these points have displaced and are labeled as **A',B',C'** and **D'**

Note: $\Delta\alpha$ is measured counter-clock wise form ***x***-axis, $\Delta\beta$ is measured clock wise sense respect to ***y***-axis.

Chain rule will be used to express the velocity of each point

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

Using the chain rule, if the velocity of point **A** is known, the velocities of the rest of points are calculated.

The components of velocity in *x* and *y* are:

Point	Component in *x*	Component in *y*
A	u	v
B	$u + \frac{\partial u}{\partial x}dx$	$v + \frac{\partial v}{\partial x}dx$
C	$u + \frac{\partial u}{\partial y}dy$	$v + \frac{\partial v}{\partial y}dy$
D	$u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$	$v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

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B	$u + \frac{\partial u}{\partial x}dx$	$v + \frac{\partial v}{\partial x}dx$
C	$u + \frac{\partial u}{\partial y}dy$	$v + \frac{\partial v}{\partial y}dy$
D	$u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$	$v + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$

Simplifying the symbols to represent partial derivatives

$$u_x = \frac{\partial u}{\partial x} \qquad v_x = \frac{\partial v}{\partial x}$$

$$u_y = \frac{\partial u}{\partial y} \qquad v_y = \frac{\partial v}{\partial y}$$

Point	Component in *x*	Component in *y*
A	u	v
B	$u + u_x dx$	$v + v_x dx$
C	$u + u_y dy$	$v + v_y dy$
D	$u + u_x dx + u_y dy$	$v + v_x dx + v_y dy$

In this slide, index notation is used to indicate the partial derivative of a variable respect to an independent variable indicated in the sub index.

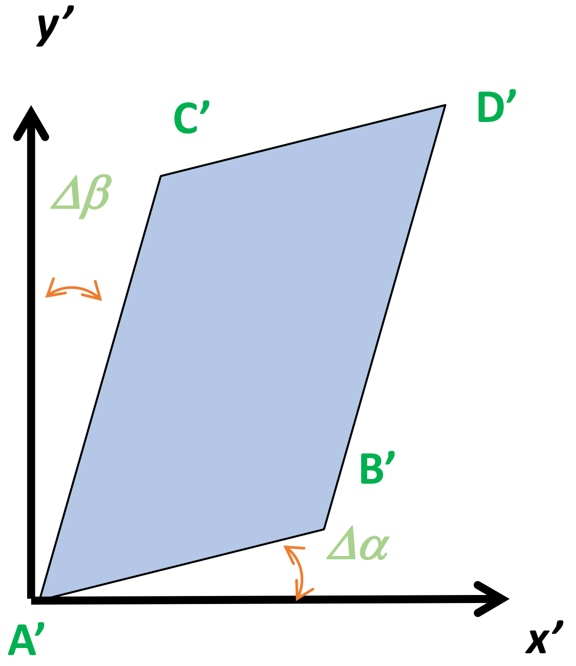
Using the velocity of each particle, its position will be tracked. The position at the beginning of the process and at the end is given in the next table.

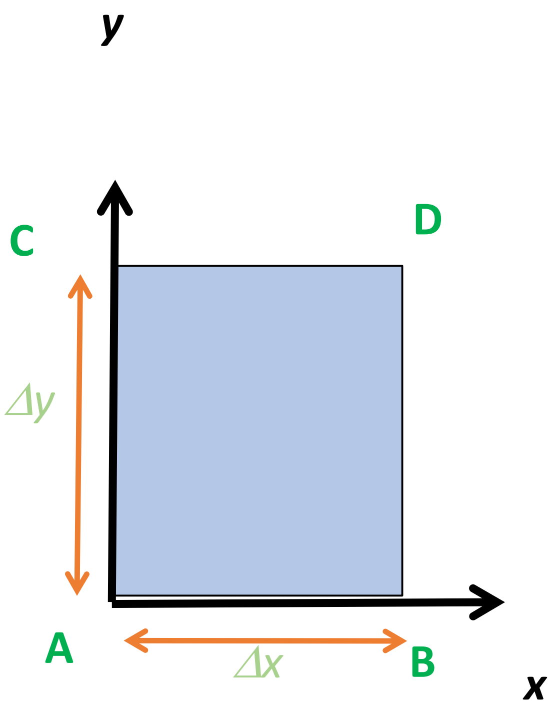
Using the information of the table, the translation, rotation, expansion and shear are calculated.

The position of the points **r** and **r’** are:

	x-component	y-component
A	0	0
B	Δx	0
C	0	Δy
D	Δx	Δy
A’	$u\,\Delta t$	$v\Delta t$
B’	$\Delta x+[u+u_x dx]\Delta t$	$[v+v_x dx\,]\Delta t$
C’	$[u+u_y dy]\Delta t$	$\Delta y+[v+v_y dy]\Delta t$
D’	$\Delta x+[u+u_x dx+u_y dy]\Delta t$	$\Delta y+[v+v_x dx+v_y dy]\Delta t$

Dilatation or Extensional strain rates





$$\dot{\varepsilon}_{xx} = \frac{\left(\overline{\mathrm{A'B'}_x}\right) - \left(\overline{\mathrm{AB}_x}\right)}{\left(\overline{\mathrm{AB}_x}\right)\Delta t}$$

$$\dot{\varepsilon}_{yy} = \frac{\left(\overline{\mathrm{A'C'}_y}\right) - \left(\overline{\mathrm{AC}_y}\right)}{\left(\overline{\mathrm{AC}_y}\right)\Delta t}$$

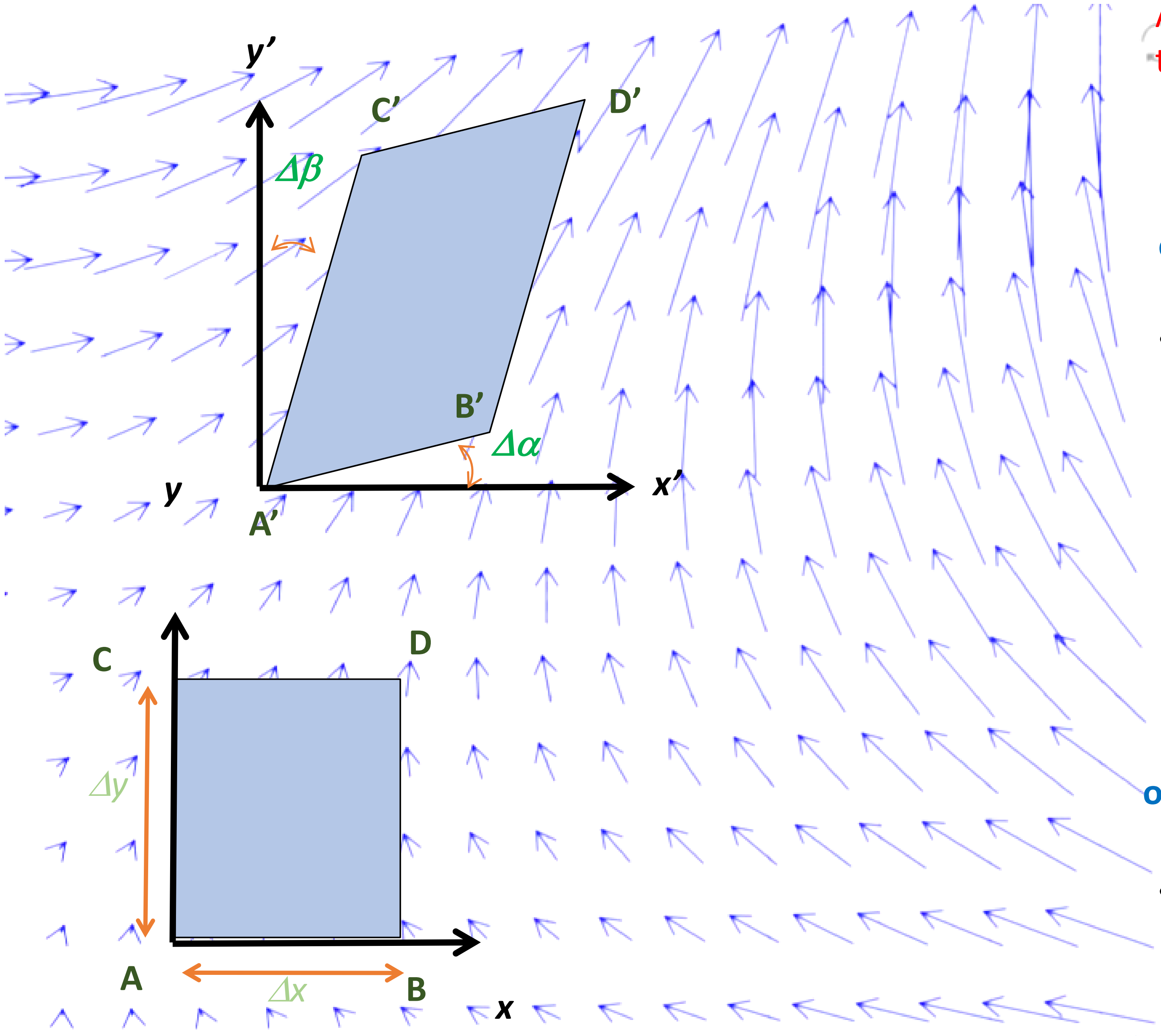
$$\dot{\varepsilon}_{xx} = \frac{\left(\overline{\mathrm{C'D'}_x}\right) - \left(\overline{\mathrm{CD}_x}\right)}{\left(\overline{\mathrm{CD}_x}\right)\Delta t}$$

or

$$\dot{\varepsilon}_{yy} = \frac{\left(\overline{\mathrm{B'D'}_y}\right) - \left(\overline{\mathrm{BD}_y}\right)}{\left(\overline{\mathrm{BD}_y}\right)\Delta t}$$

Any definition will lead you to the same answer

Dilatation or Extensional strain rates



Any definition will lead you to the same answer

$$\dot{\varepsilon}_{xx} = \frac{(\overline{A'B'_x}) - (\overline{AB_x})}{(\overline{AB_x})\Delta t}$$

or

$$\dot{\varepsilon}_{xx} = \frac{(\overline{C'D'_x}) - (\overline{CD_x})}{(\overline{CD_x})\Delta t}$$

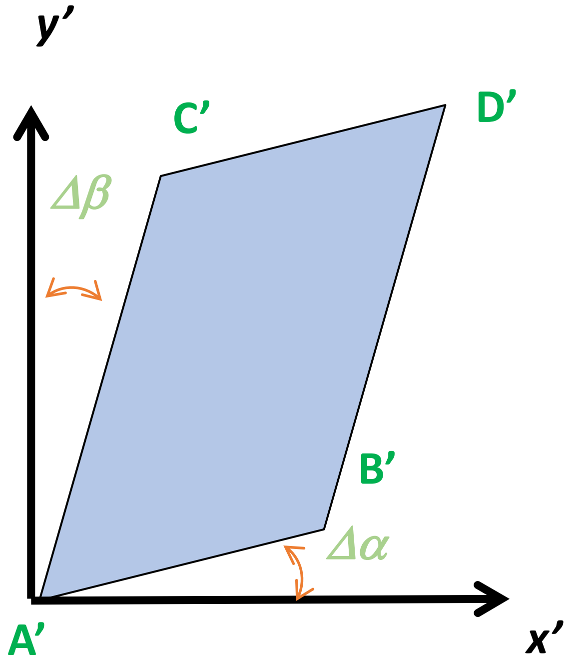
$$\dot{\varepsilon}_{yy} = \frac{(\overline{A'C'_y}) - (\overline{AC_y})}{(\overline{AC_y})\Delta t}$$

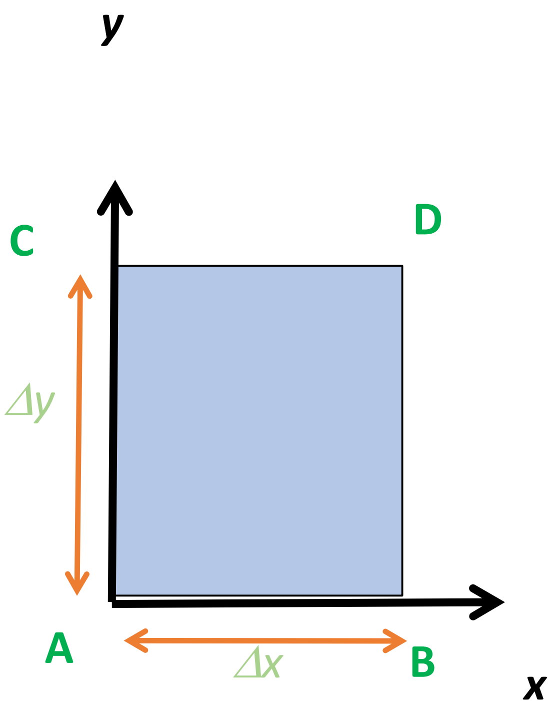
or

$$\dot{\varepsilon}_{yy} = \frac{(\overline{B'D'_y}) - (\overline{BD_y})}{(\overline{BD_y})\Delta t}$$

To make plots of vector fields you can use https://www.geogebra.org/m/QPE4PaDZ

Dilatation or Extensional strain rates





$$\dot{\varepsilon}_{xx} = \frac{\left(\overline{\mathrm{A'B'}_x}\right) - \left(\overline{\mathrm{AB}_x}\right)}{\left(\overline{\mathrm{AB}_x}\right)\Delta t}$$

$$\dot{\varepsilon}_{yy} = \frac{\left(\overline{\mathrm{A'C'}_y}\right) - \left(\overline{\mathrm{AC}_y}\right)}{\left(\overline{\mathrm{AC}_y}\right)\Delta t}$$

$$\dot{\varepsilon}_{xx} = \frac{\left(\overline{\mathrm{C'D'}_x}\right) - \left(\overline{\mathrm{CD}_x}\right)}{\left(\overline{\mathrm{CD}_x}\right)\Delta t}$$

or

$$\dot{\varepsilon}_{yy} = \frac{\left(\overline{\mathrm{B'D'}_y}\right) - \left(\overline{\mathrm{BD}_y}\right)}{\left(\overline{\mathrm{BD}_y}\right)\Delta t}$$

Any definition will lead you to the same answer

Dilatation or Extensional strain rates

For instance we can use the second definition, and the result is: $\dot{\varepsilon}_{xx} = \dfrac{(\overline{\mathrm{C'D'}_x}) - (\overline{\mathrm{CD}_x})}{(\overline{\mathrm{CD}_x})\Delta t}$

$$\dot{\varepsilon}_{xx} = \frac{\left[\Delta x + u\Delta t + u_x \Delta x \Delta t + u_y \Delta y \Delta t - \left(u\Delta t + u_y \Delta y \Delta t\right)\right] - \Delta x}{\Delta x \Delta t} = u_x = \frac{\partial u}{\partial x}$$

Abscissa Extensional strain rate

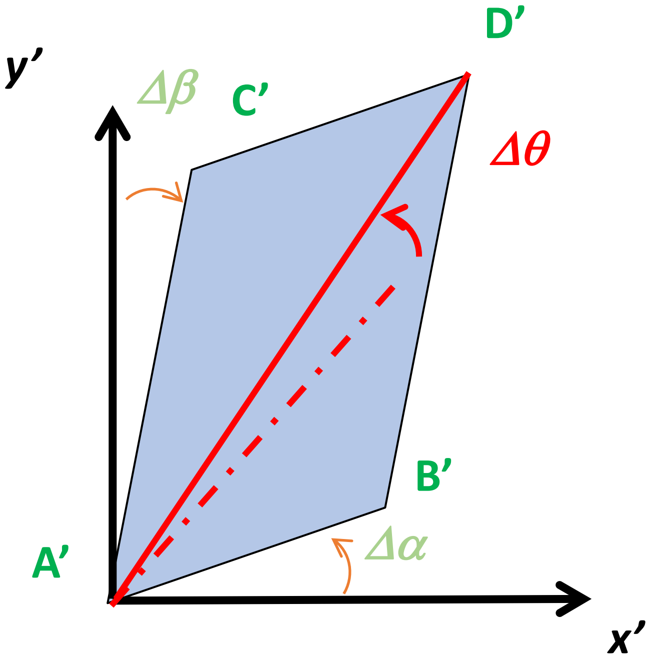
$$\boxed{\dot{\varepsilon}_{xx} = u_x = \frac{\partial u}{\partial x}}$$

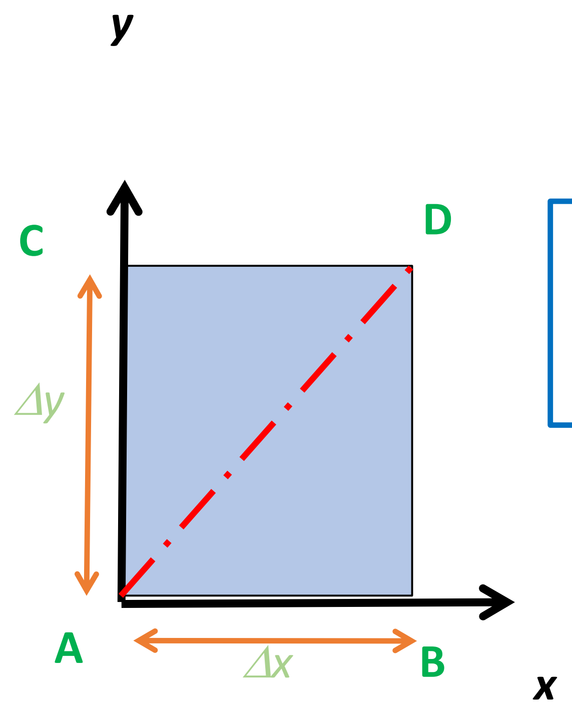
The same procedure is done for the **ordinate extensional strain rate,** leading you to the equation:

$$\dot{\varepsilon}_{yy} = \frac{(\overline{\mathrm{A'C'}_y}) - (\overline{\mathrm{AC}_y})}{(\overline{\mathrm{AC}_y})\Delta t}$$

$$\boxed{\dot{\varepsilon}_{yy} = v_y = \frac{\partial v}{\partial y}}$$

Angular velocity and vorticity





Rotation will be measured counterclockwise respect to a reference

$$\Delta\theta = \frac{1}{2}\left(\Delta\alpha - \Delta\beta\right)$$

Look out: $\Delta\alpha$ is measured **counterclockwise** from the positive ***x*** axis

And $\Delta\beta$ is measured **clockwise** from the positive ***y*** axis

Using trigonometry

$$\tan(\Delta\alpha) = \frac{y_{B'} - y_{A'}}{x_{B'} - x_{A'}} = \frac{(v + v_x \Delta x)\Delta t - v\Delta t}{\Delta x + u\Delta t + u_x \Delta x \Delta t - u\Delta t} = \frac{v_x \Delta t}{1 + u_x \Delta t}$$

If $\Delta t \to 0$ and $\Delta\alpha \to 0$ then

$$\Delta\alpha \approx v_x \Delta t$$

$$\tan(\Delta\beta) = \frac{x_{C'} - x_{A'}}{y_{C'} - y_{A'}} = \frac{u_y \Delta t}{1 + v_y \Delta t}$$

If $\Delta t \to 0$ and $\Delta\beta \to 0$ then

$$\Delta\beta \approx u_y \Delta t$$

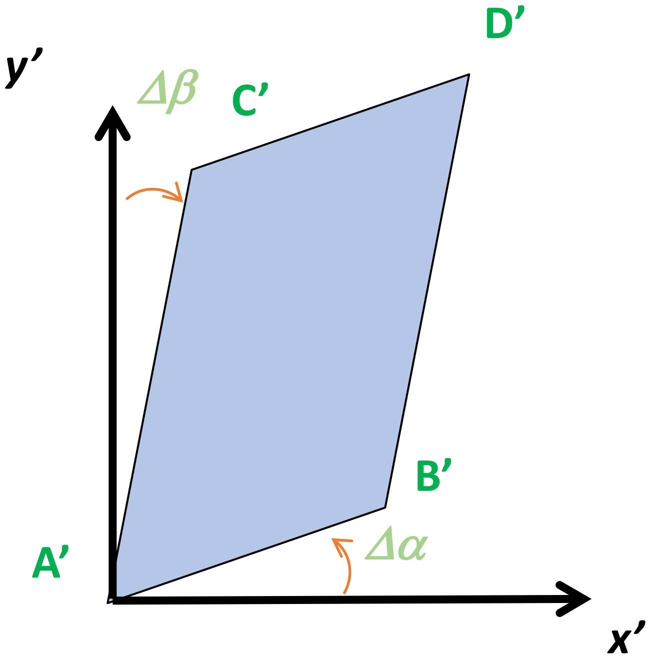
Angular velocity about the z axis

$$\boxed{\dot{\theta}_z = \frac{\Delta\theta}{\Delta t} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$

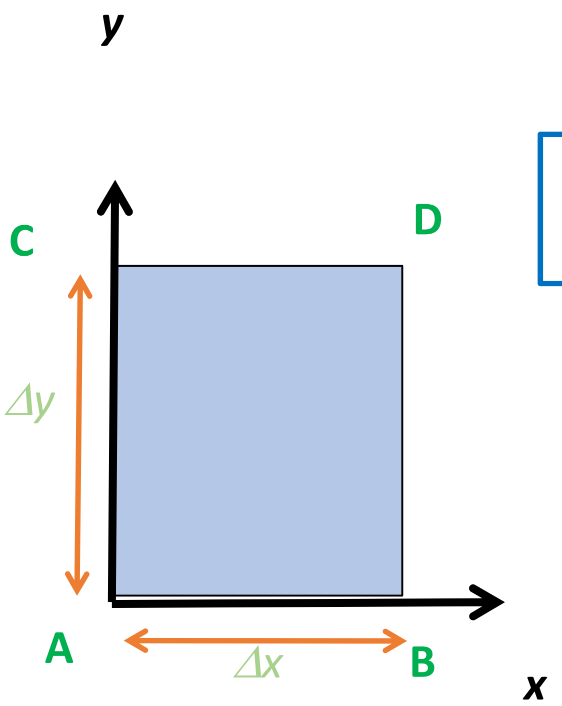
$$\underline{\dot{\theta}} = \frac{1}{2}\underline{\omega}$$

$$\tan(\varepsilon) = \varepsilon + \frac{\varepsilon^3}{3} + \frac{2\varepsilon^5}{15} + \frac{17\varepsilon^7}{315} + \ldots$$

Shear-strain rate



$$\Delta\alpha \approx v_x \Delta t$$
$$\Delta\beta \approx u_y \Delta t$$



Shear-strain rate $\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \dfrac{1}{2}\dfrac{(\Delta\alpha + \Delta\beta)}{\Delta t}$

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \frac{1}{2}\frac{\left(v_x\Delta t + u_y\Delta t\right)}{\Delta t}$$

$$\boxed{\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \frac{1}{2}\left(v_x + u_y\right) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}$$

Generalizing in 3-D

See the appendix for more details about 3-D

Rate of rotation

$$\dot{\theta}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \dot{\theta}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \dot{\theta}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

About z-axis x-axis y-axis

$$\underline{\dot{\theta}} = \frac{1}{2}\left[\underline{\nabla} \times \underline{V}\right] = \frac{1}{2}\underline{\omega}$$

Rate of rotation vector

$$\underline{\underline{\Omega}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{V} - \left(\underline{\nabla}\,\underline{V}\right)^{T}\right]$$

Vorticity tensor

$$\underset{\rightarrow}{\nabla}(\ ) = \frac{\partial(\ )}{\partial x}\underset{\rightarrow}{i} + \frac{\partial(\ )}{\partial y}\underset{\rightarrow}{j} + \frac{\partial(\ )}{\partial z}\underset{\rightarrow}{k}$$

$$\underset{\rightarrow}{V} = u\,\underset{\rightarrow}{i} + v\,\underset{\rightarrow}{j} + w\,\underset{\rightarrow}{k}$$

Shear-strain rate

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \dot{\varepsilon}_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \dot{\varepsilon}_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\dot{\varepsilon}_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \dot{\varepsilon}_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \dot{\varepsilon}_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{V} + \left(\underline{\nabla}\,\underline{V}\right)^T\right]$$

Rate of strain tensor

$$\underset{\rightarrow}{\nabla}(\ ) = \frac{\partial(\ )}{\partial x}\underset{\rightarrow}{i} + \frac{\partial(\ )}{\partial y}\underset{\rightarrow}{j} + \frac{\partial(\ )}{\partial z}\underset{\rightarrow}{k}$$

$$\underset{\rightarrow}{V} = u\,\underset{\rightarrow}{i} + v\,\underset{\rightarrow}{j} + w\,\underset{\rightarrow}{k}$$

Extensional strain rate

$$\dot{\varepsilon}_{xx} = u_x = \frac{\partial u}{\partial x} \qquad \dot{\varepsilon}_{yy} = v_y = \frac{\partial v}{\partial y} \qquad \dot{\varepsilon}_{zz} = w_z = \frac{\partial w}{\partial w}$$

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\,\underline{V} + \left(\underline{\nabla}\,\underline{V}\right)^T\right]$$

Rate of strain tensor

$$\underset{\rightarrow}{\nabla}(\ ) = \frac{\partial(\ )}{\partial x}\underset{\rightarrow}{i} + \frac{\partial(\ )}{\partial y}\underset{\rightarrow}{j} + \frac{\partial(\ )}{\partial z}\underset{\rightarrow}{k}$$

$$\underset{\rightarrow}{V} = u\,\underset{\rightarrow}{i} + v\,\underset{\rightarrow}{j} + w\,\underset{\rightarrow}{k}$$

Shear-strain rate and Extensional strain rate

$$\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v} = \begin{bmatrix} \hat{i}\hat{i}\;\partial u/\partial x & \hat{i}\hat{j}\;\partial v/\partial x & \hat{i}\hat{k}\;\partial w/\partial x \\ \hat{j}\hat{i}\;\partial u/\partial y & \hat{j}\hat{j}\;\partial v/\partial y & \hat{j}\hat{k}\;\partial w/\partial y \\ \hat{k}\hat{i}\;\partial u/\partial z & \hat{k}\hat{j}\;\partial v/\partial z & \hat{k}\hat{k}\;\partial w/\partial z \end{bmatrix}$$

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \hat{i}\hat{i}\;\dot{\varepsilon}_{xx} & \hat{i}\hat{j}\;\dot{\varepsilon}_{xy} & \hat{i}\hat{k}\;\dot{\varepsilon}_{xz} \\ \hat{j}\hat{i}\;\dot{\varepsilon}_{yx} & \hat{j}\hat{j}\;\dot{\varepsilon}_{yy} & \hat{j}\hat{k}\;\dot{\varepsilon}_{yz} \\ \hat{k}\hat{i}\;\dot{\varepsilon}_{zx} & \hat{k}\hat{j}\;\dot{\varepsilon}_{zy} & \hat{k}\hat{k}\;\dot{\varepsilon}_{zz} \end{bmatrix}$$

$$\left(\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v}\right)^T = \begin{bmatrix} \hat{i}\hat{i}\;\partial u/\partial x & \hat{i}\hat{j}\;\partial u/\partial y & \hat{i}\hat{k}\;\partial u/\partial z \\ \hat{j}\hat{i}\;\partial v/\partial x & \hat{j}\hat{j}\;\partial v/\partial y & \hat{j}\hat{k}\;\partial v/\partial z \\ \hat{k}\hat{i}\;\partial w/\partial x & \hat{k}\hat{j}\;\partial w/\partial y & \hat{k}\hat{k}\;\partial w/\partial z \end{bmatrix}$$

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \hat{i}\hat{i}\;\partial u/\partial x & \hat{i}\hat{j}\,(\partial v/\partial x + \partial u/\partial y)/2 & \hat{i}\hat{k}\,(\partial w/\partial x + \partial u/\partial z)/2 \\ \hat{j}\hat{i}\,(\partial u/\partial y + \partial v/\partial x)/2 & \hat{j}\hat{j}\;\partial v/\partial y & \hat{j}\hat{k}\,(\partial w/\partial y + \partial v/\partial z)/2 \\ \hat{k}\hat{i}\,(\partial u/\partial z + \partial w/\partial x)/2 & \hat{k}\hat{j}\,(\partial v/\partial z + \partial w/\partial y)/2 & \hat{k}\hat{k}\;\partial w/\partial z \end{bmatrix}$$

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v} + \left(\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v}\right)^T\right]$$

Rate of rotation tensor

$$\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v} = \begin{bmatrix} \hat{i}\hat{i}\,\partial u/\partial x & \hat{i}\hat{j}\,\partial v/\partial x & \hat{i}\hat{k}\,\partial w/\partial x \\ \hat{j}\hat{i}\,\partial u/\partial y & \hat{j}\hat{j}\,\partial v/\partial y & \hat{j}\hat{k}\,\partial w/\partial y \\ \hat{k}\hat{i}\,\partial u/\partial z & \hat{k}\hat{j}\,\partial v/\partial z & \hat{k}\hat{k}\,\partial w/\partial z \end{bmatrix}$$

$$\underline{\underline{\Omega}} = \begin{bmatrix} 0\ \hat{i}\hat{i} & \dot{\theta}_z\hat{i}\hat{j} & -\dot{\theta}_y\hat{i}\hat{k} \\ -\dot{\theta}_z\hat{j}\hat{i} & 0\ \hat{j}\hat{j} & \dot{\theta}_x\hat{j}\hat{k} \\ \dot{\theta}_y\hat{k}\hat{i} & -\dot{\theta}_x\hat{k}\hat{j} & 0\ \hat{k}\hat{k} \end{bmatrix}$$

$$\left(\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v}\right)^T = \begin{bmatrix} \hat{i}\hat{i}\,\partial u/\partial x & \hat{i}\hat{j}\,\partial u/\partial y & \hat{i}\hat{k}\,\partial u/\partial z \\ \hat{j}\hat{i}\,\partial v/\partial x & \hat{j}\hat{j}\,\partial v/\partial y & \hat{j}\hat{k}\,\partial v/\partial z \\ \hat{k}\hat{i}\,\partial w/\partial x & \hat{k}\hat{j}\,\partial w/\partial y & \hat{k}\hat{k}\,\partial w/\partial z \end{bmatrix}$$

$$\underline{\underline{\Omega}} = \begin{bmatrix} \hat{i}\hat{i}\,0 & \hat{i}\hat{j}(\partial v/\partial x - \partial u/\partial y)/2 & \hat{i}\hat{k}(\partial w/\partial x - \partial u/\partial z)/2 \\ \hat{j}\hat{i}(\partial u/\partial y - \partial v/\partial x)/2 & \hat{j}\hat{j}\,0 & \hat{j}\hat{k}(\partial w/\partial y - \partial v/\partial z)/2 \\ \hat{k}\hat{i}(\partial u/\partial z - \partial w/\partial x)/2 & \hat{k}\hat{j}(\partial v/\partial z - \partial w/\partial y)/2 & \hat{k}\hat{k}\,0 \end{bmatrix}$$

$$\underline{\underline{\Omega}} = \frac{1}{2}\left[\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v} - \left(\underset{\rightarrow}{\nabla}\,\underset{\rightarrow}{v}\right)^T\right]$$

Vectors and tensors in flow analysis

Velocity vector

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Rate of rotation tensor

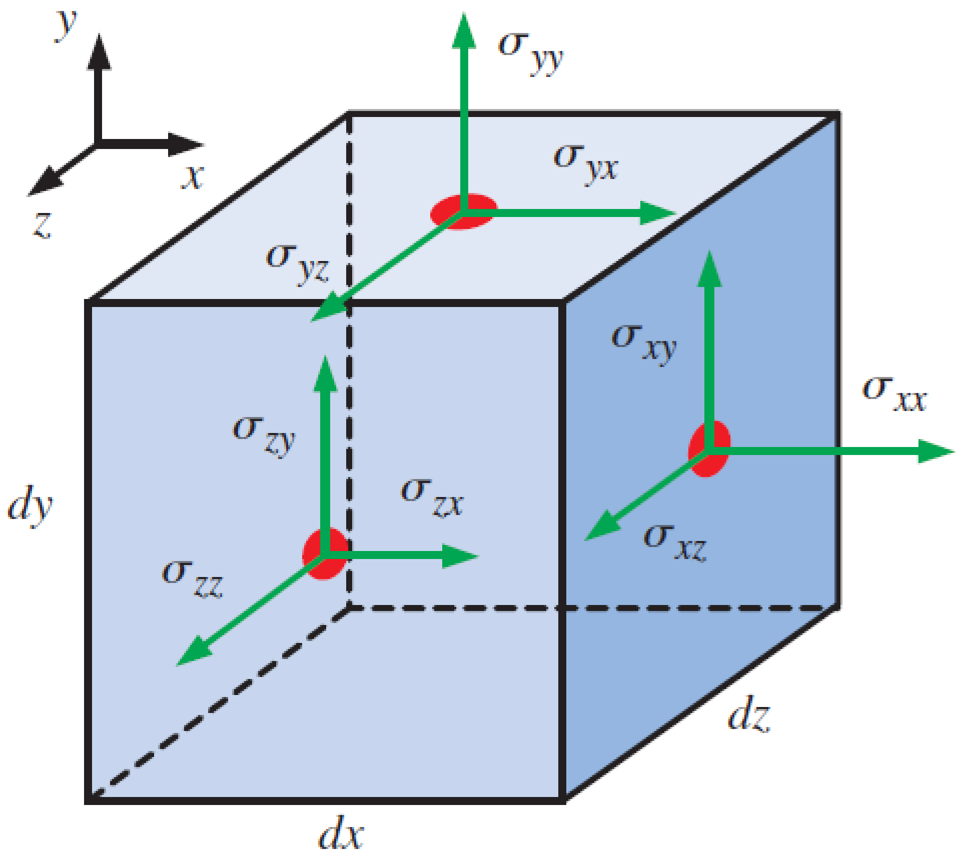
$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & \dot{\theta}_z & -\dot{\theta}_y \\ -\dot{\theta}_z & 0 & \dot{\theta}_x \\ \dot{\theta}_y & -\dot{\theta}_x & 0 \end{bmatrix}$$

Rate strain tensor

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{bmatrix}$$

acceleration vector

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix} + \begin{bmatrix} uu_x + vu_y + wu_z \\ uv_x + vv_y + wv_z \\ uw_x + vw_y + ww_z \end{bmatrix}$$



Components of the stress tensor in Cartesian coordinates on the right, top, and front faces.

σ_{ij} =Stress in the cross sectional area perpendicular to "i" axis acting in "j" direction (The normal vector is in "i" direction)

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$\underline{s}$ ($\underline{n}$)= Stress exerted by fluid on a particular side of the surface

$$\underline{s}(\underline{n}) = \underline{n} \cdot \underline{\underline{\sigma}}$$

$$\underline{s}(\underline{n}) = s(e_j) = \sigma_{yx}\underline{e_x} + \sigma_{yy}\underline{e_y} + \sigma_{yz}\underline{e_z}$$

σ_{yx}=Is seen to be the force per unit area on a plane perpendicular to the y axis, acting in the ***x*** direction and exerted by the fluid at greater ***y***.

σ_{yx} has three parts, reference plane, a direction and a sign convention. The orientation of the reference plane is specified by the first subscript, and the direction of the force is indicated by the second subscript. Positive values of corresponds to transfer of ***x*** momentum in the –***y*** direction

Caution !

$$\underline{s}\left(\underline{n}\right) = \underline{n} \cdot \underline{\underline{\sigma}}$$

Other authors define the stress tensor using subscripts in reverse order, on that case the stress vector is:

$$\cancel{\underline{s}\left(\underline{n}\right) = \underline{\underline{\sigma}} \cdot \underline{n}}$$

Some other authors define the stress tensor inverting the sign to extend the analogy of heat and mass transfer. (i.e. Fick’s and Fourier)

Use your own words to define and give a physical meaning to the following operators of velocity

$$\underline{v}$$

$$\underline{\nabla} \cdot \underline{v}$$

$$\underline{\nabla} \times \underline{v}$$

$$\underline{\nabla}\ \underline{v}$$

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\ \underline{v} + \left(\underline{\nabla}\ \underline{v}\right)^T\right]$$

$$\underline{\underline{\Omega}} = \frac{1}{2}\left[\underline{\nabla}\ \underline{v} - \left(\underline{\nabla}\ \underline{v}\right)^T\right]$$

$$v = \sqrt{\underline{v} \cdot \underline{v}}$$

$$\langle v \rangle$$

$$\langle \underline{v} \rangle$$

$$\frac{1}{2}\underline{v} \cdot \underline{v}$$

$$\underline{v} \cdot \left[\underline{\nabla} \times \underline{v}\right]$$

$$\langle v^n \rangle$$

$$\langle v^n \rangle / \langle v \rangle^n$$

$$\frac{d}{dt}\left[\underline{v}\right]$$

$$\frac{D}{Dt}\left[\underline{v}\right]$$

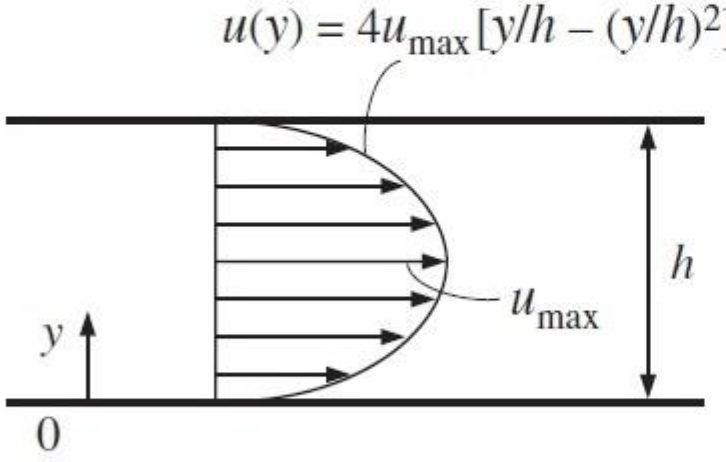
$$\frac{\partial}{\partial t}\left[\underline{v}\right]$$

Volume flux, Divergence of velocity, Vorticity, Velocity Gradient, Strain rate tensor, Vorticity tensor, Speed, Average speed, Average velocity, Specific kinetic energy, Helicity, Average powered-n speed, Dynamic correction factor (discuss just for the cases ***n***=2, and ***n***=3), Velocity time rate, Material velocity time rate.

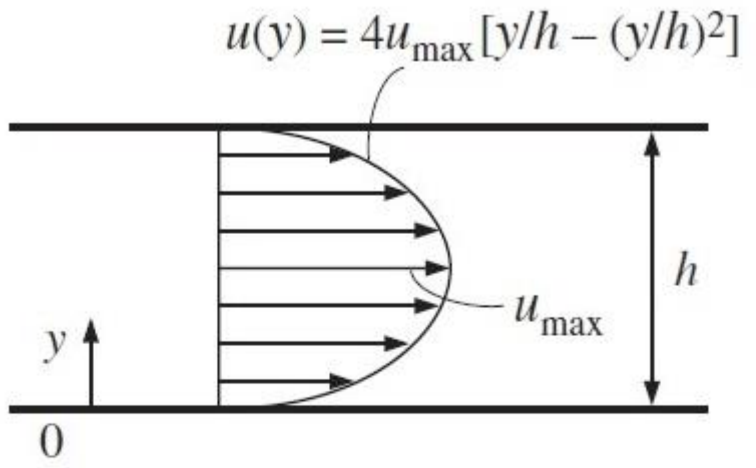
Example No.1

Consider laminar flow of a Newtonian fluid of viscosity μ between two parallel plates. The flow is one-dimensional, and the velocity profile is given as $u(y) = 4u_{max} [y/h - (y/h)^2]$, where y is the vertical coordinate from the bottom surface, h is the distance between the two plates, and u_{max} is the maximum flow velocity that occurs at midplane.

Develop a relation for the drag force exerted on both plates by the fluid in the flow direction per unit area of the plates.



Before solving the problem, some flow properties are going to be analyzed , like strain rate, rate of rotation, rate of deformation, rate of extensional strain rate, and continuity criterion.



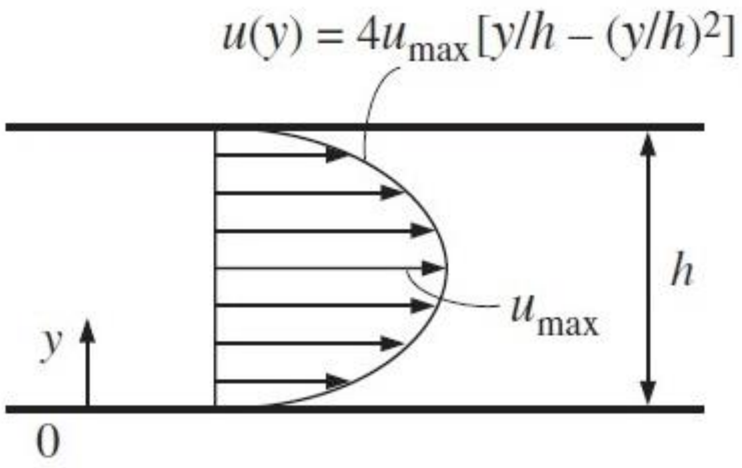
$$u = 4u_{\max}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$
$$v = 0$$
$$w = 0$$

Rate of rotation

$$\dot{\theta}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \dot{\theta}_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \dot{\theta}_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{4u_{\max}}{h}\left[1 - 2\left(\frac{y}{h}\right)\right]$$

$$\boxed{\dot{\theta}_z = -\frac{2u_{max}}{h}\left[1 - 2\left(\frac{y}{h}\right)\right] \qquad \dot{\theta}_x = 0 \qquad \dot{\theta}_y = 0}$$



$$u = 4u_{max}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$

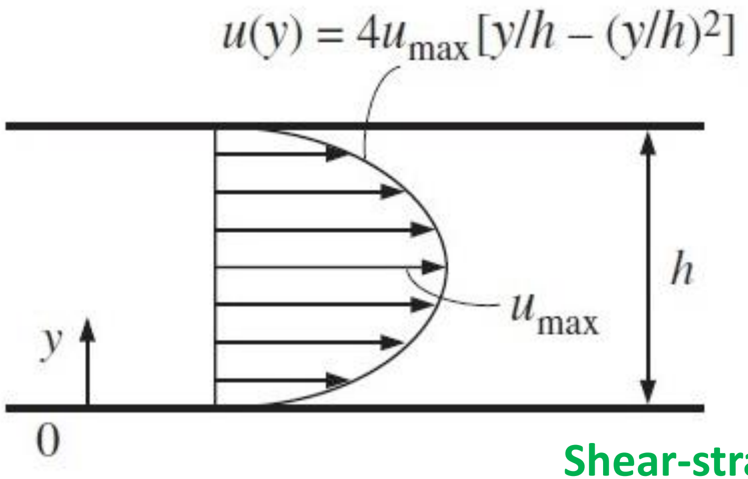
$$v = 0$$

$$w = 0$$

Extensional strain rate

$$\dot{\varepsilon}_{xx} = u_x = \frac{\partial u}{\partial x} \qquad \dot{\varepsilon}_{yy} = v_y = \frac{\partial v}{\partial y} \qquad \dot{\varepsilon}_{zz} = w_z = \frac{\partial w}{\partial w}$$

$$\boxed{\dot{\varepsilon}_{xx} = 0 \qquad \dot{\varepsilon}_{yy} = 0 \qquad \dot{\varepsilon}_{zz} = 0}$$



$$u = 4u_{max}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$

$$v = 0$$

$$w = 0$$

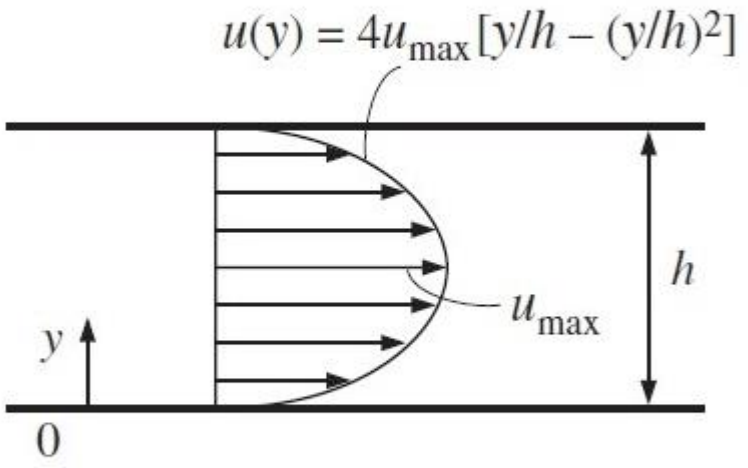
$$\frac{\partial u}{\partial y} = \frac{4u_{max}}{h}\left[1 - 2\left(\frac{y}{h}\right)\right]$$

Shear-strain rate

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \dot{\varepsilon}_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \dot{\varepsilon}_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\dot{\varepsilon}_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \dot{\varepsilon}_{zy} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \qquad \dot{\varepsilon}_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\boxed{\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \frac{2u_{max}}{h}\left[1 - 2\left(\frac{y}{h}\right)\right] \qquad \dot{\varepsilon}_{zy} = \dot{\varepsilon}_{yz} = 0 \qquad \dot{\varepsilon}_{zx} = \dot{\varepsilon}_{xz} = 0}$$



$$u = 4u_{\max}\left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]$$

$$v = 0$$

$$w = 0$$

$$\frac{\partial u}{\partial y} = \frac{4u_{\max}}{h}\left[1 - 2\left(\frac{y}{h}\right)\right]$$

Shear-strain stress, and drag force per unit area at the boundaries

$$\tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]\hat{j}\hat{i}$$

$$\tau_{yx} = \frac{4\,\mu\,u_{\max}}{h}\left[1 - 2\left(\frac{y}{h}\right)\right]\hat{j}\hat{i}$$

$$\tau_{yx}\Big|_{y=0} = \frac{4\,\mu\,u_{\max}}{h}\,\hat{j}\hat{i}$$

$$\underline{n}\Big|_{y=0} = -\hat{j}$$

$$\tau_{yx}\Big|_{y=h} = -\frac{4\,\mu\,u_{\max}}{h}\,\hat{j}\hat{i}$$

$$\underline{n}\Big|_{y=h} = +\hat{j}$$

$$\underline{s}(\underline{n}) = \underline{n}\cdot\underline{\underline{\tau}}$$

$$\boxed{\underline{s}\Big|_{y=0} = -\frac{4\,\mu\,u_{\max}}{h}\,\hat{i} \qquad \underline{s}\Big|_{y=h} = -\frac{4\,\mu\,u_{\max}}{h}\,\hat{i}}$$

Note: $\underline{\underline{\tau}}$ is a stress tensor, while $\underline{s}$ is the stress vector

$$\underline{s}(\underline{n}) = \underline{n} \cdot \underline{\underline{\tau}}$$

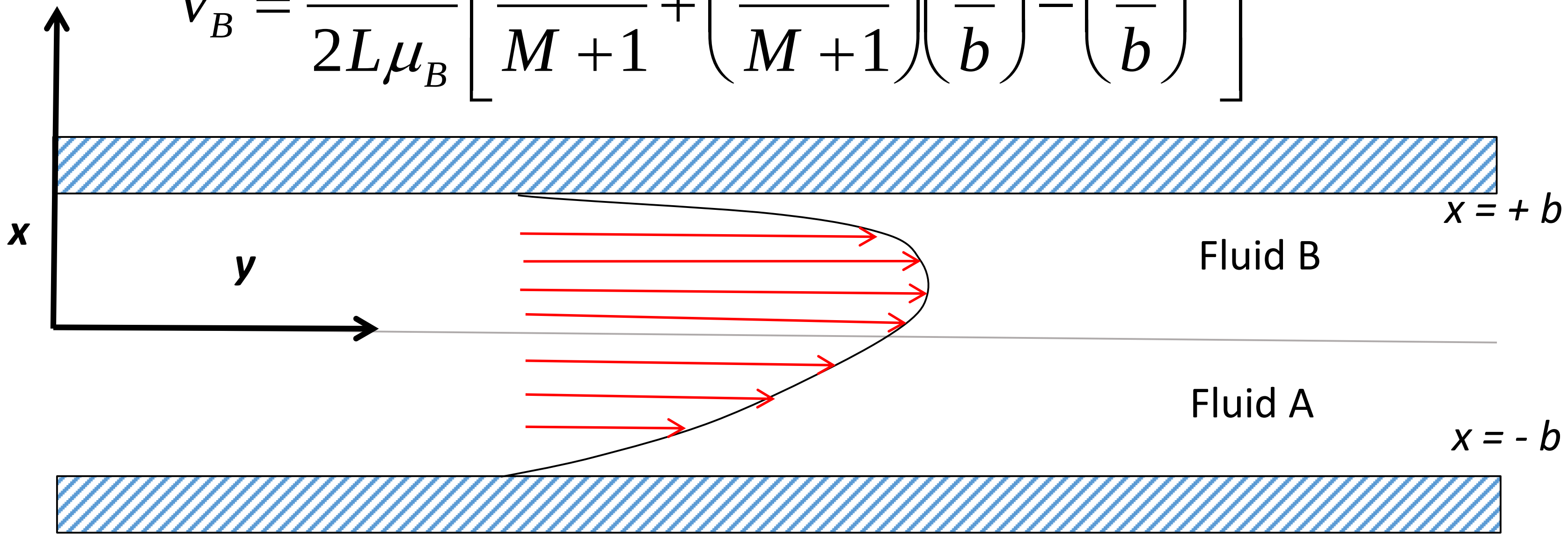
Example No.2

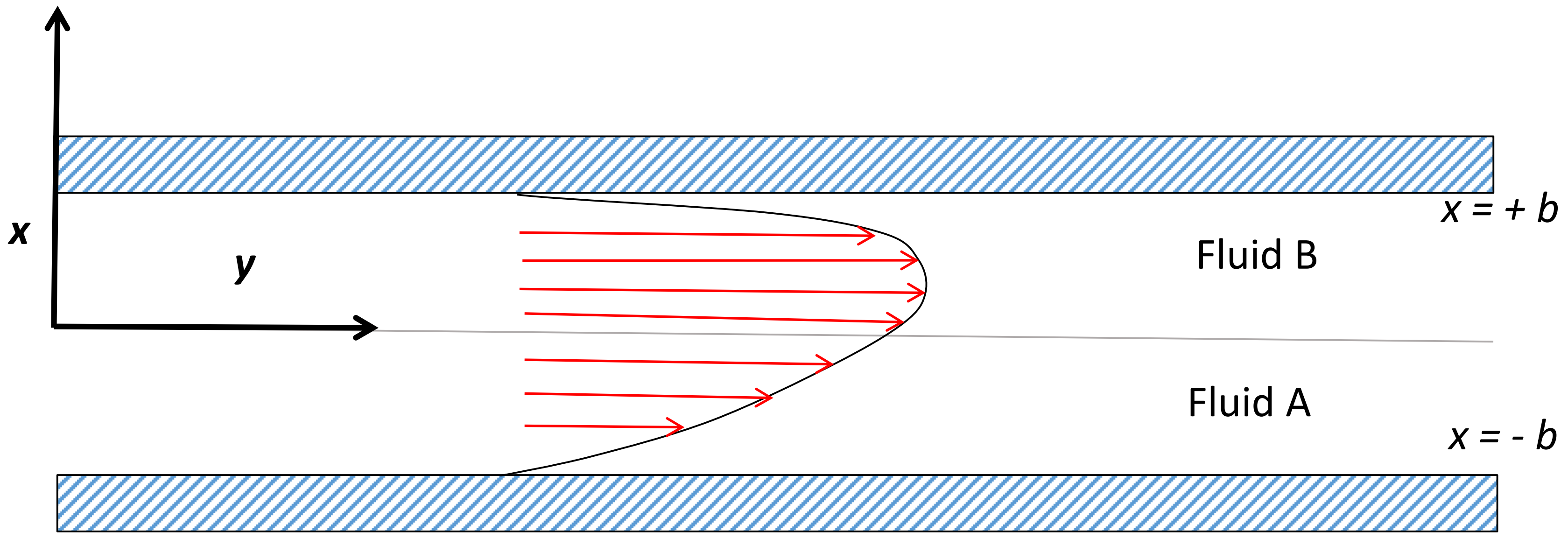
Immiscible Liquids in a Parallel-Plate cannel. Two phase flow occurs when immiscible liquids from distinct layers in parallel-plate cannel, as depicted in the figure. The viscosity and the density of liquid 1 may differ from these of liquid 2. It is desired to fill half of the cannel with fluid 1, and the rest with fluid 2. The velocity profiles for each phase are given:

$$v_A = \frac{\Delta p b^2}{2L\mu_A}\left[\frac{2M}{M+1} + \left(\frac{M-1}{M+1}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right]$$

$M=\mu_A/\mu_B$

$$v_B = \frac{\Delta p b^2}{2L\mu_B}\left[\frac{2}{M+1} + \left(\frac{M-1}{M+1}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right]$$





x
y
x = + b
Fluid B
Fluid A
x = - b

a) Demonstrate that at the boundary, the following relations prevail:

$$\underline{v}_A = \underline{v}_B$$

$$\underline{s}_A = -\underline{s}_B \qquad \underline{s} = \underline{n} \cdot \underline{\underline{\tau}}$$

b) Why the previous boundary conditions are valid?

c) Calculate the shear stress over the plates:

$$\underline{v}_A = \frac{\Delta p b^2}{2L\mu_A}\left[\frac{2M}{M+1} + \left(\frac{M-1}{M+1}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right]\hat{\mathrm{j}} \qquad \underline{v}_B = \frac{\Delta p b^2}{2L\mu_B}\left[\frac{2}{M+1} + \left(\frac{M-1}{M+1}\right)\left(\frac{x}{b}\right) - \left(\frac{x}{b}\right)^2\right]\hat{\mathrm{j}}$$

$$\underline{\underline{\tau}} = \mu_A \frac{dV_A}{dx} = \frac{\Delta p b}{2L}\left[\left(\frac{M-1}{M+1}\right) - 2\left(\frac{x}{b}\right)\right]\hat{\mathrm{i}}\,\hat{\mathrm{j}} \qquad \underline{\underline{\tau}} = \mu_B \frac{dV_B}{dx} = \frac{\Delta p b}{2L}\left[\left(\frac{M-1}{M+1}\right) - 2\left(\frac{x}{b}\right)\right]\hat{\mathrm{i}}\,\hat{\mathrm{j}}$$

$$\underline{n}_A\Big|_{x=0} = -\underline{n}_B\Big|_{x=0} = +\hat{\mathrm{i}} \qquad \underline{n}_B\Big|_{x=0} = -\hat{\mathrm{i}}$$

$$\underline{v}_A\Big|_{x=0} = \frac{\Delta p b^2}{L\mu_B(M+1)}\hat{\mathrm{j}} \qquad \underline{v}_B\Big|_{x=0} = \frac{\Delta p b^2}{L\mu_B(M+1)}\hat{\mathrm{j}}$$

$$\underline{s}_A\Big|_{x=0} = \frac{\Delta p b}{2L}\left(\frac{M-1}{M+1}\right)\hat{\mathrm{j}} \qquad \underline{s}_B\Big|_{x=0} = -\frac{\Delta p b}{2L}\left(\frac{M-1}{M+1}\right)\hat{\mathrm{j}}$$

Stress over the walls

$$\underline{\underline{\tau}}\Big|_{x=-b} = \mu_A \left.\frac{dV_A}{dx}\right|_{x=-b} = \frac{\Delta pb}{2L}\left[2+\left(\frac{M-1}{M+1}\right)\right]\hat{\mathrm{i}}\,\hat{\mathrm{j}} \qquad \underline{\underline{\tau}}\Big|_{x=b} = \mu_B \left.\frac{dV_B}{dx}\right|_{x=b} = \frac{\Delta pb}{2L}\left[\left(\frac{M-1}{M+1}\right)-2\right]\hat{\mathrm{i}}\,\hat{\mathrm{j}}$$

$$\underline{n}_A\Big|_{x=b} = -\underline{n}_B\Big|_{x=b} = -\hat{\mathrm{i}} \qquad \underline{n}_B\Big|_{x=b} = +\hat{\mathrm{i}}$$

$$\underline{s}_A\Big|_{x=-b} = -\frac{\Delta pb}{L}\left[1+\frac{1}{2}\left(\frac{M-1}{M+1}\right)\right]\hat{\mathrm{j}} \qquad \underline{s}_B\Big|_{x=b} = \mu_B \left.\frac{dV_B}{dx}\right|_{x=b} = -\frac{\Delta pb}{L}\left[1-\frac{1}{2}\left(\frac{M-1}{M+1}\right)\right]\hat{\mathrm{j}}$$

Just for the sake of science, total stress over the "A" control volume, and over the "B" control volume respectively:

$$\underline{s}_A\Big|_{x=-b} = -\frac{\Delta pb}{L}\left[1+\frac{1}{2}\left(\frac{M-1}{M+1}\right)\right]\hat{\mathrm{j}} \qquad \underline{s}_B\Big|_{x=b} = \mu_B \left.\frac{dV_B}{dx}\right|_{x=b} = -\frac{\Delta pb}{L}\left[1-\frac{1}{2}\left(\frac{M-1}{M+1}\right)\right]\hat{\mathrm{j}}$$

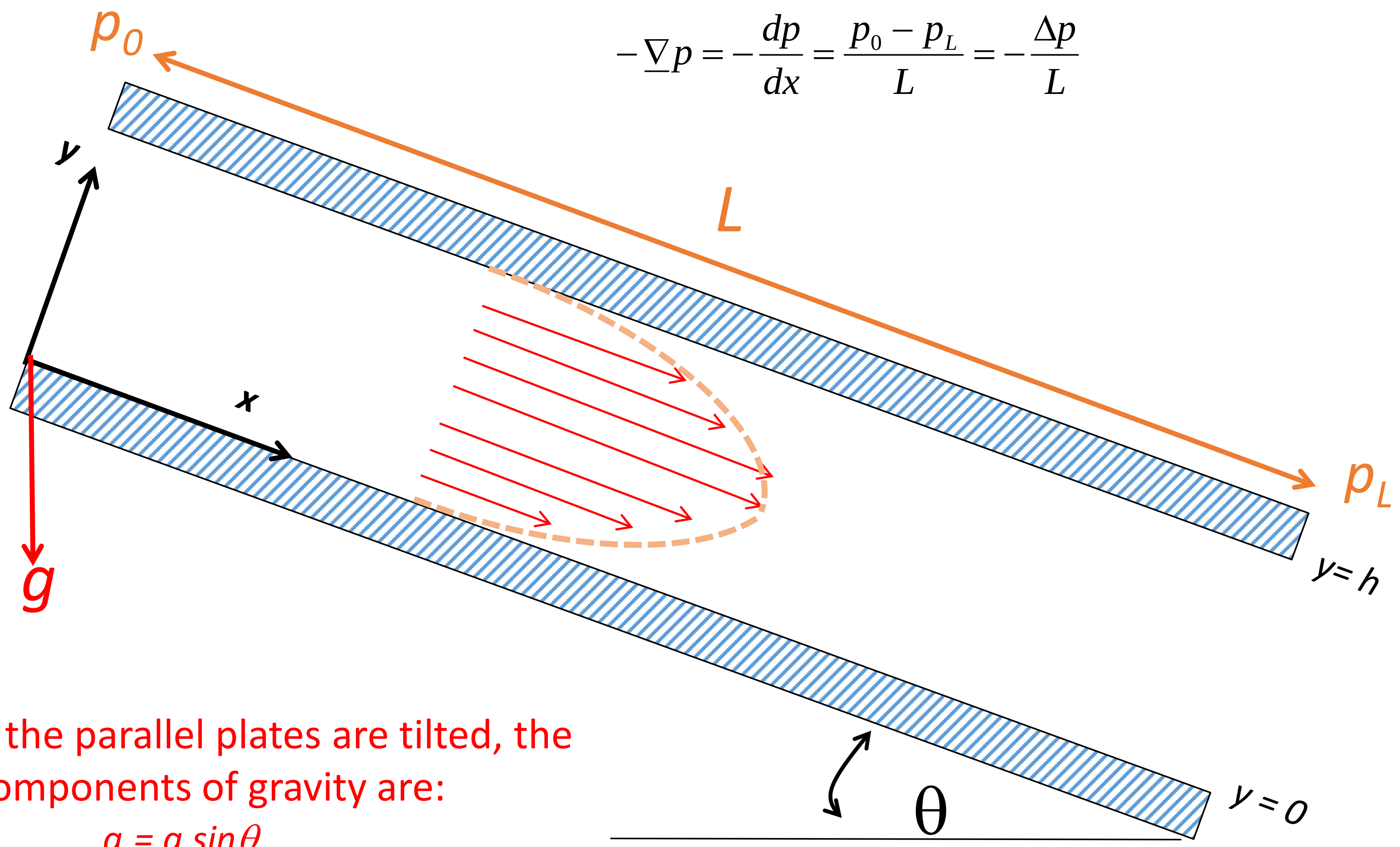
$$\underline{s}_A\Big|_{x=0} = \frac{\Delta pb}{2L}\left(\frac{M-1}{M+1}\right)\hat{\mathrm{j}} \qquad \underline{s}_B\Big|_{x=0} = -\frac{\Delta pb}{2L}\left(\frac{M-1}{M+1}\right)\hat{\mathrm{j}}$$

$$\underline{s}_A\Big|_{x=-b} + \underline{s}_A\Big|_{x=0} = -\frac{\Delta pb}{L}\,\hat{\mathrm{j}} \qquad \underline{s}_B\Big|_{x=b} + \underline{s}_B\Big|_{x=0} = -\frac{\Delta pb}{L}\,\hat{\mathrm{j}}$$

Example No.3

Characterize the flow between parallel plates, under a pressure gradient

$$-\underline{\nabla} p = -\frac{dp}{dx} = \frac{p_0 - p_L}{L} = -\frac{\Delta p}{L}$$



If the parallel plates are tilted, the components of gravity are:

$g_x = g \sin\theta$

$g_y = - g \cos\theta$

$$\frac{\partial \rho}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\underline{v}\right)=0$$

For incompressible flow, the density is constant

$$\rho\underline{\nabla}\cdot\left(\underline{v}\right)=0 \;\Rightarrow\; \underline{\nabla}\cdot\left(\underline{v}\right)=0 \;\Rightarrow\; \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

From the sketch, the velocity is unidirectional in *x*-axis, then *v* = *w* = 0

$$\frac{\partial u}{\partial x}=0 \;\Rightarrow$$

$$u=u\left(y\right)$$

The continuity equation helps to conclude that the velocity in the *x*-axis is only function of *y*, if the plate is wide and extend to infinity, the end effects are negligible

The linear momentum equation or Cauchy equation of motion will be used to describe the motion:

$$\frac{\partial\left(\rho\underline{v}\right)}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\underline{v}\,\underline{v}\right)=\rho\underline{g}+\underline{\nabla}\cdot\underline{\underline{\tau}}-\underline{\nabla}p$$

The linear momentum for *x*-axis, and assuming incompressible flow, and steady flow:

$$\frac{\partial\left(\rho u\right)}{\partial t}+\frac{\partial\left(\rho uu\right)}{\partial x}=\rho g_x+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}-\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial x} = 0$$

Coupling continuity and momentum:

$$\rho \frac{\partial u}{\partial t} + \rho \frac{\partial (uu)}{\partial x} = \rho g_x + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} - \frac{\partial p}{\partial x}$$

And understanding *u* is only function of *y*, then: $\tau_{xx} = \tau_{zx} = 0$

$$0 = \rho g_x + \frac{\partial \tau_{yx}}{\partial y} - \frac{\partial p}{\partial x}$$

Lets simplify for integration purposes and use "α" to collect the driving potential of the flow:

$$\alpha = \rho g_x - \frac{\partial p}{\partial x}$$

$$\frac{d\tau_{yx}}{dy} = -\alpha$$

After integration: $\tau_{yx} = -\alpha\, y + C_1$

Use a boundary condition to get the value of C_1: We know that at the center there is no stress: $\tau_{yx} = 0$ at $y = h/2$. This is also called symmetry boundary condition, equivalent to state *du/dy* =0 at the center

$$0 = -\alpha \frac{h}{2} + C_1$$

Replacing the value of C_1

$$\tau_{yx} = \alpha\left(\frac{h}{2} - y\right)$$

$$\tau_{yx} = \frac{\alpha}{2}(h - 2y)$$

For a Newtonian fluid

$$\tau_{yx} = \mu \frac{du}{dy}$$

$$\mu \frac{du}{dy} = \frac{\alpha}{2}(h - 2y)$$

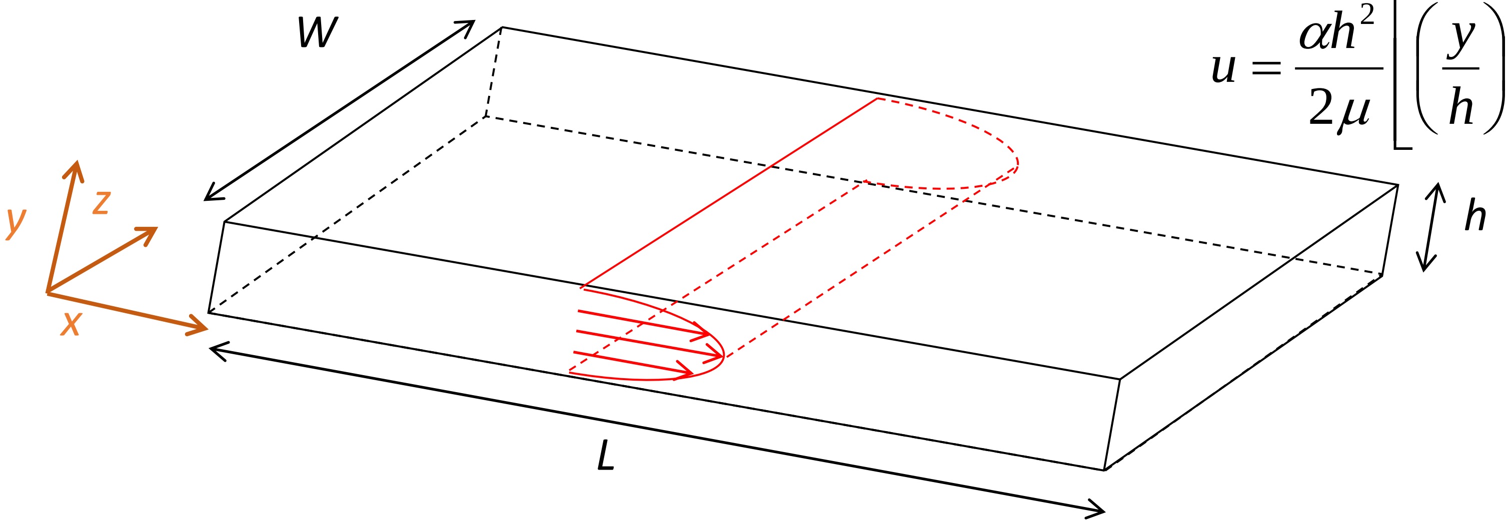
After integration

$$du = \frac{\alpha h^2}{2\mu}\left(1 - 2\left(\frac{y}{h}\right)\right) d\left(\frac{y}{h}\right) = \frac{\alpha h^2}{2\mu} d\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right]$$

$$u = \frac{\alpha h^2}{2\mu}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right] + C_2$$

Boundary condition: The velocity at the walls is zero, u=0 at y=0

$$u = \frac{\alpha h^2}{2\mu}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right]$$



$$u = \frac{\alpha h^2}{2\mu}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right]$$

With the velocity profile, the volumetric flow rate can be calculated

$$\dot{V} = \int_{y=0}^{y=h}\int_{z=0}^{z=W} u\,dz\,dy = \int\int udzdy = \int_{y=0}^{y=h} \frac{\alpha h^2}{2\mu}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right] W dy$$

$$\dot{V} = \frac{\alpha h^3 W}{2\,\mu}\left[\frac{1}{2}\left(\frac{y}{h}\right)^2 - \frac{1}{3}\left(\frac{y}{h}\right)^3\right]\Bigg|_{y=0}^{y=h} = \frac{\alpha h^3 W}{12\,\mu}$$

In terms of average velocity

$$\langle u \rangle hW = \frac{\alpha h^3 W}{12\mu}$$

$$\langle u\rangle hW=\frac{\alpha h^3W}{12\mu} \qquad \langle u\rangle=\frac{\alpha h^2}{12\mu}$$

$$\alpha=\rho g_x-\frac{\partial p}{\partial x}$$

α can be written in terms of measurable variables

$$\alpha=\rho\, g\sin\theta-\frac{\Delta p}{L}$$

Recasting in terms of energy, and coupling with the expression of average velocity

$$\frac{12\mu\langle u\rangle}{h^2}=\rho g\sin\theta-\frac{\Delta p}{L}$$

The equation is divided by density and multiplied by length

$$\frac{12\mu L\langle u\rangle}{h^2\rho}=Lg\sin\theta-\frac{\Delta p}{\rho}$$

$$\frac{24\mu L}{h^2\rho\langle u\rangle}\left(\frac{\langle u\rangle^2}{2}\right)=Lg\sin\theta-\frac{\Delta p}{\rho}$$

Characteristic specific kinetic energy

Specific potential energy **difference**

Specific flow energy **difference**

$$\frac{24\mu L}{h^2\rho\langle u\rangle}\left(\frac{\langle u\rangle^2}{2}\right) = Lg\sin\theta - \frac{\Delta p}{\rho}$$

The coefficient of the specific characteristic kinetic energy must be dimensionless

In Hydraulics, there is something called hydraulic diameter and is defined as fourfold the Flow cross sectional area divided by the wetted perimeter, or the perimeter of solid boundaries in contact with fluid.

$$D_H = 4\frac{A}{w_p} \qquad D_H = 4\frac{W\,h}{2(W+h)} = \frac{2\,h}{\left(1+\dfrac{h}{W}\right)}$$

If *W*>>*h* $\qquad D_H \approx 2\,h$

$$\frac{96}{\left[\dfrac{\rho\langle u\rangle D_H}{\mu}\right]}\left(\frac{L}{D_H}\right)\left(\frac{\langle u\rangle^2}{2}\right) = Lg\sin\theta - \frac{\Delta p}{\rho}$$

$$\frac{96}{\left[\frac{\rho\langle u\rangle D_H}{\mu}\right]}\left(\frac{L}{D_H}\right)\left(\frac{\langle u\rangle^2}{2}\right)=Lg\sin\theta-\frac{\Delta p}{\rho}$$

The first term can be lumped into something called friction factor

$$f_D\left(\frac{L}{D_H}\right)\left(\frac{\langle u\rangle^2}{2}\right)=Lg\sin\theta-\frac{\Delta p}{\rho}$$

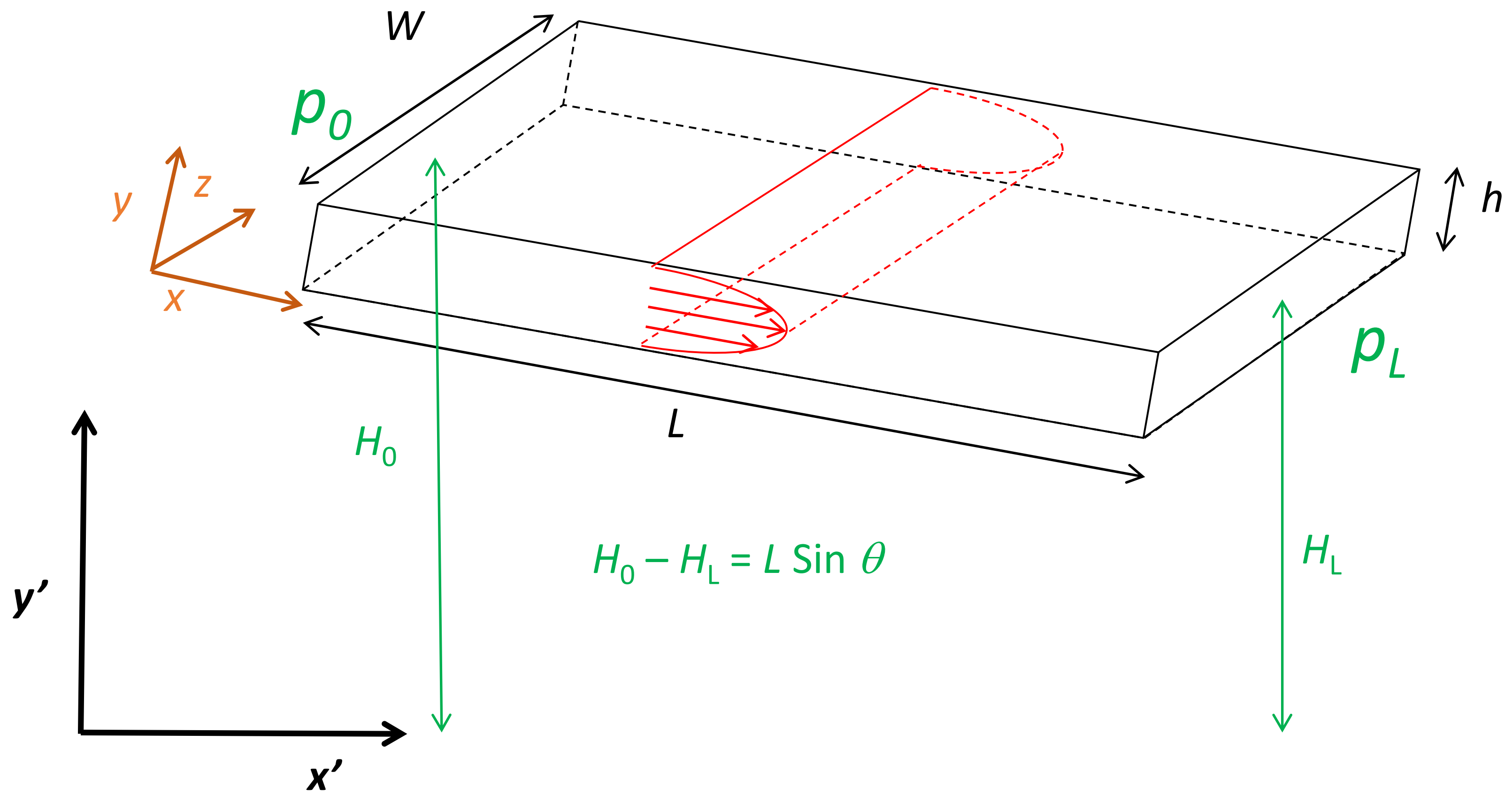
$$f_D=\frac{96}{\left[\frac{\rho\langle u\rangle D_H}{\mu}\right]}=\frac{96}{Re}$$

Darcy friction factor

$$Re=\frac{\rho\langle u\rangle D_H}{\mu}$$

Reynolds Number

If a Bernoulli equation is written over the previous system and considering friction loss term, and compare both equations, we obtain:

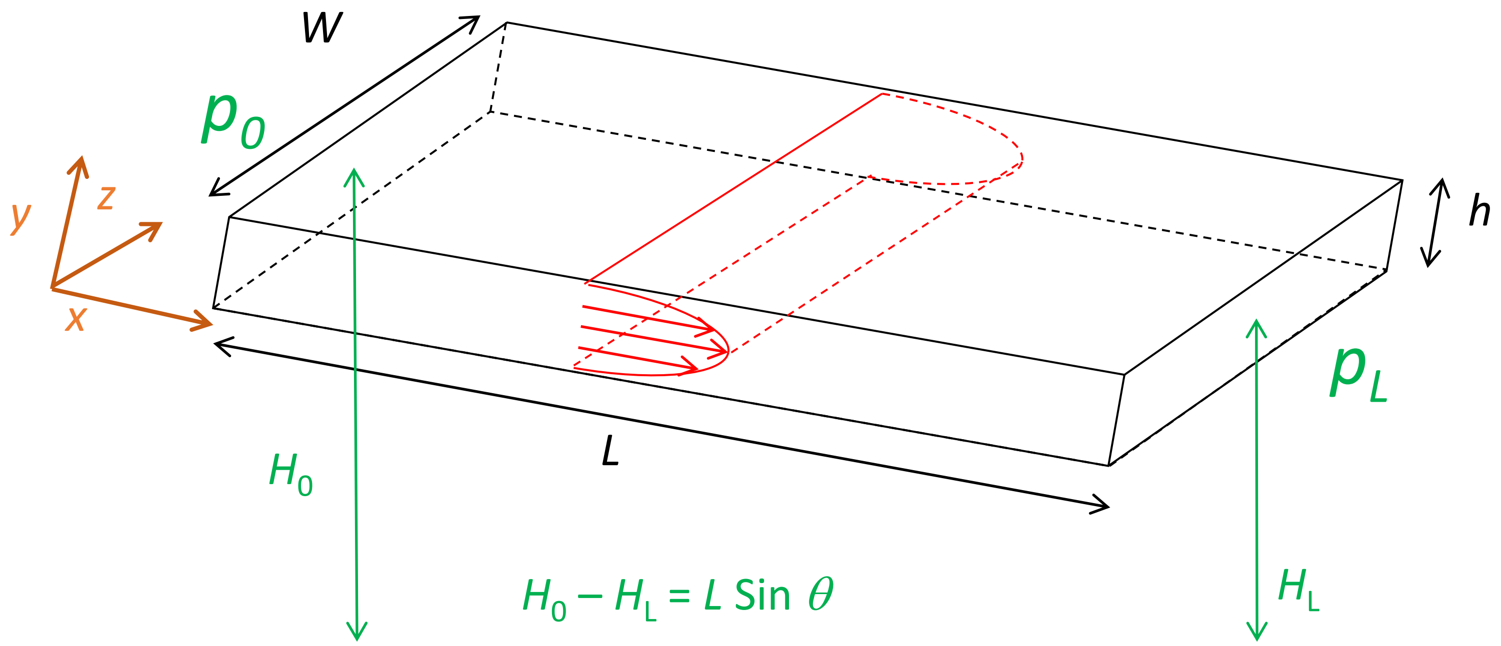


$$\frac{p_0}{\rho} + \frac{\langle v_0 \rangle^2}{2} + g\, H_0 = \frac{p_L}{\rho} + \frac{\langle v_L \rangle^2}{2} + g\, H_L + \hat{E}_{friction}$$

Bernoulli Equation

$$\rho \langle v_0 \rangle W\, h = \rho \langle v_L \rangle W\, h$$

Continuity Equation



$H_0 - H_L = L \sin\theta$

$$-\left(\frac{p_L}{\rho}-\frac{p_0}{\rho}\right)+\frac{\langle v_0\rangle^2}{2}-\frac{\langle v_L\rangle^2}{2}+g\left(H_0-H_L\right)=\hat{E}_{friction}$$

$$-\frac{\Delta p}{\rho}+g\,L\sin\theta=\hat{E}_{friction}$$

Comparing with the previous flow analysis

$$-\frac{\Delta p}{\rho}+Lg\sin\theta=f_D\left(\frac{L}{D_H}\right)\left(\frac{\langle u\rangle^2}{2}\right)$$

Then energy losses are computed as:

$$\hat{E}_{friction}=f_D\left(\frac{L}{D_H}\right)\left(\frac{\langle u\rangle^2}{2}\right)$$

Challenge: Calculate the inertial force of the flow of liquid, and the flow of kinetic energy. Write them both in terms of the form used in the macroscopic energy and linear momentum balances.

$$\widehat{K} = \alpha \left[\frac{1}{2}\langle \underline{u} \rangle \cdot \langle \underline{u} \rangle\right]$$

$$\underline{F}_{inertial} = \beta \, \dot{m} \, \langle \underline{u} \rangle$$

$$\dot{m} = \rho \, \langle \underline{u} \rangle \cdot \underline{n} \, [h \, W]$$

$$\dot{K} = \alpha \, \rho \, \langle \underline{u} \rangle \cdot \underline{n} \, [h \, W] \left[\frac{1}{2}\langle \underline{u} \rangle \cdot \langle \underline{u} \rangle\right]$$

Prove that:

$$\alpha = \frac{54}{35}$$

$$\beta = \frac{6}{5}$$

$$u = \frac{\alpha h^2}{2\mu}\left[\left(\frac{y}{h}\right) - \left(\frac{y}{h}\right)^2\right] \qquad \langle u\rangle = \frac{\alpha h^2}{12\mu}$$

$$\tau_{yx} = \frac{\alpha}{2}(h - 2y) \qquad \tau_{yx}\big|_{y=0} = \tau_w = \frac{\alpha h}{2}$$

$$\dot{K} = \int \rho \langle \underline{u}\rangle \cdot \underline{n} \left[\frac{1}{2}\langle \underline{u}\rangle \cdot \langle \underline{u}\rangle\right] dA = \frac{\rho W \alpha^3 h^7}{8\mu^3} \int \left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]^3 d\left(\frac{y}{h}\right) = \frac{\rho W \alpha^3 h^7}{(140)8\mu^3}$$

$$F_{inertial} = \int \rho\, u\,(u) dA = \frac{\rho W \alpha^2 h^5}{4\mu^2} \int \left[\frac{y}{h} - \left(\frac{y}{h}\right)^2\right]^2 d\left(\frac{y}{h}\right) = \frac{\rho W \alpha^2 h^5}{120\mu^2}$$

For the sake of science, Darcy friction factor is defined as:

$$f_D = \frac{8\tau_w}{\rho\langle u\rangle^2}$$

Fourfold the stress at the walls divided by the characteristic kinetic energy

$$\tau_w = \frac{\alpha h}{2} = \frac{12\langle u\rangle\mu}{2h} \qquad f_D = \frac{8\tau_w}{\rho\langle u\rangle^2} = \frac{8}{\rho\langle u\rangle^2}\left(\frac{12\langle u\rangle\mu}{2h}\right) = \frac{96}{Re}$$

Coincide with the previous equation for friction factor

Example No.6

Human blood is a no Newtonian fluid, and the constitutive equation between shear strain rate and shear stress has the form :

$$\tau = K\left(\left|-\frac{du}{dr}\right|\right)^{n}$$

For human blood n=0.89 and K=0.00384 Pa-s^n:

Velocity profile has the form:

$$u = u_{\max}\left[1-\left(\frac{r}{R}\right)^{1+\frac{1}{n}}\right]$$

a) Give an expression for volumetric flow, and an equation for the shear stress in a circular duct.

Balancing friction with pressure drop across the capillary conduit:

$$\tau[(2\pi R)L] = (\Delta p)[\pi R^2]$$

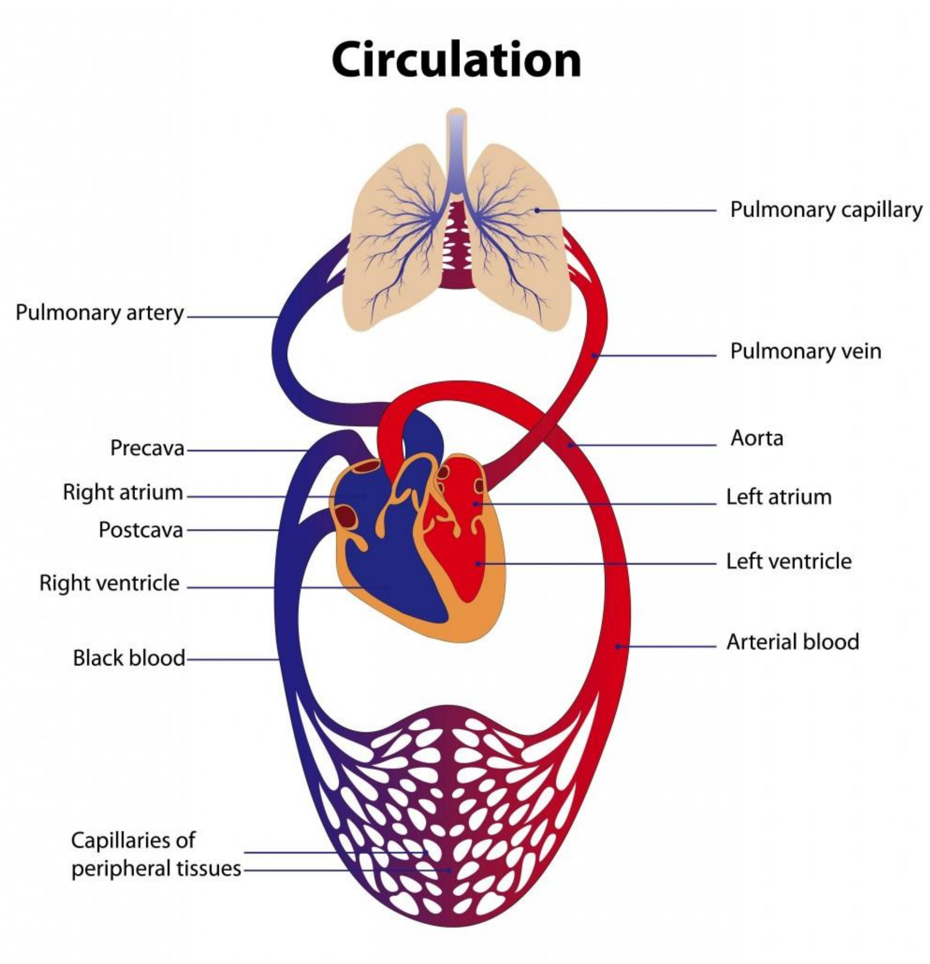
b) Give an equation for pressure drop in terms of the conduit radius, flow rate, consistency coefficient (*K*) and flow index (*n*).

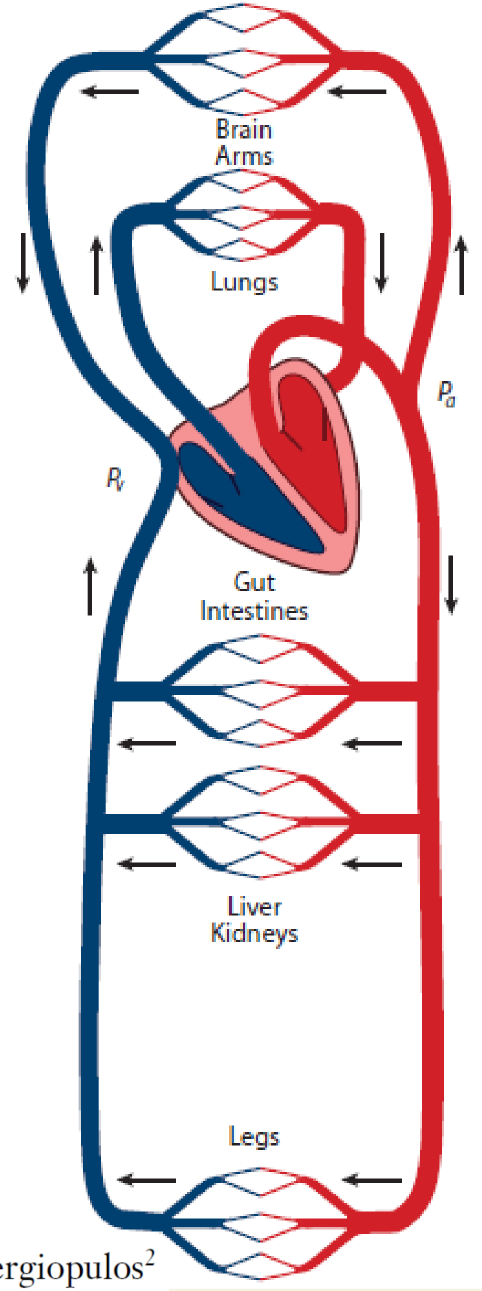
c) A70 kg person, with an average heart rate of 70 bpm, and pumping volume capacity of 70 ml /beat has a pressure of 120 mmHg/80 mmHg in the heart discharge, and a pressure of 5 mmHg in the suction. Determine the reduction in size of the capillaries if the pressure has a value of 140 mmHg/90 mmHg.

Mean arterial pressure can be calculated as:

$$p = \frac{2}{3} p_{diastolic} + \frac{1}{3}(p_{systolic})$$

Jump to slide 133 to continue this problem





Frans N. van de Vosse[1] and Nikos Stergiopulos[2]

Annu. Rev. Fluid Mech. 2011. 43:467–99

Stream lines and Potential lines

In steady flow, stream lines, path lines and streak lines coincide.

Lets follow the path of a particle in a 2-D flow

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt}$$

Dividing this two equations, if velocity field is known the stream line can be sketched

$$\frac{u}{v} = \frac{dx}{dy}$$

Writing this equation in different form:

$$-vdx + udy = 0$$

And using Cauchy-Riemann approach:

$$-vdx + udy = 0$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \qquad v = -\frac{\partial \psi}{\partial x} \qquad u = \frac{\partial \psi}{\partial y}$$

Then following the stream function or constant ψ will give you the streamline.

If we conceptualize the flow given by a potential, this potential lines should be perpendicular to the streamlines, then the potential lines are defined as:

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \qquad \frac{dy}{dx} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} = -\frac{1}{\left(\frac{v}{u}\right)}$$

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

If we define velocity in terms of a potential gradient:

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

Vector form: $\underline{v} = \underline{\nabla} \phi$

In terms of vorticity: $\underline{\zeta} = \underline{\nabla} \times \underline{v} = \underline{\nabla} \times \underline{\nabla} \phi$

$$\underline{\zeta} = \underline{\nabla} \times \underline{v} = \left[\hat{\mathrm{i}}\left(\partial[\ \ ]/\partial x\right) + \hat{\mathrm{j}}\left(\partial[\ \ ]/\partial y\right)\right] \times \left[\hat{\mathrm{i}}\, u + \hat{\mathrm{j}}\, v\right] = \hat{\mathrm{k}}\left[\partial v/\partial x - \partial u/\partial y\right]$$

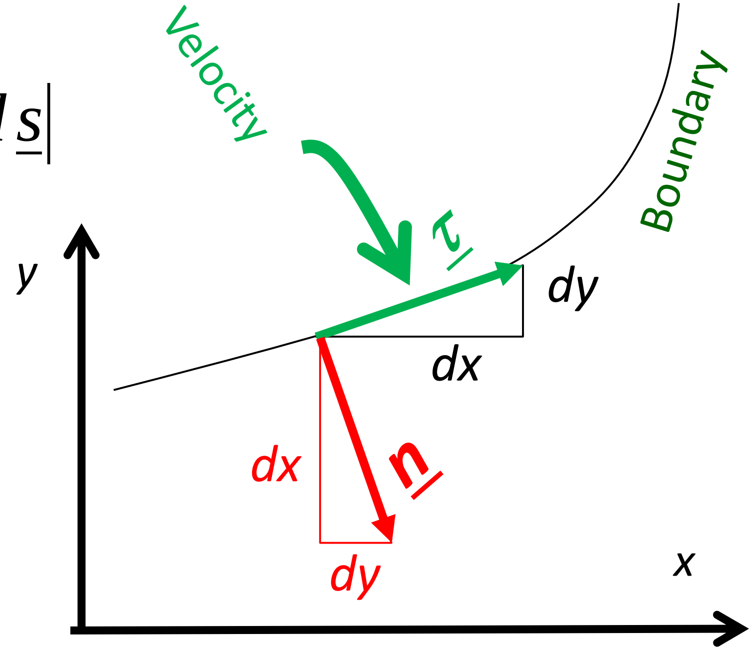
$$\underline{\zeta} = \underline{\nabla} \times \underline{v} = \hat{\mathrm{k}}\left[\partial^2 \phi/\partial y \partial x - \partial^2 \phi/\partial x \partial y\right] = 0\, \hat{\mathrm{k}}$$

If this proposal for flow description is valid, then the flow has to be irrotational:

Another important characteristic of streamline is the relationship with flow rate.

Displacement vector.

$$d\underline{s} = \hat{\mathrm{i}}\, dx + \hat{\mathrm{j}}\, dy \qquad d\ell = |d\underline{s}|$$



Unit Tangent Vector.

$$\underline{\tau}\, d\ell = \hat{\mathrm{i}}\, dx + \hat{\mathrm{j}}\, dy$$

Unit Normal Vector.

$$\underline{n}\, d\ell = \hat{\mathrm{i}}\, dy - \hat{\mathrm{j}}\, dx$$

Volumetric flow rate.

$$d\dot{V} = \underline{V} \cdot \underline{n}\, W\, d\lambda = W(u\,\hat{\imath} + v\,\hat{\jmath}) \cdot (dy\,\hat{\imath} - dx\,\hat{\jmath}) = W[u\,dy - v\,dx]$$

$$d\dot{V} = W[u\,dy - v\,dx] = W\,d\psi = W\left[\frac{\partial\psi}{\partial y}dy + \frac{\partial\psi}{\partial x}dx\right]$$

In this slide, $\underline{\tau}$ is unit tangent vector not stress (by the way stress is a tensor not vector)

Volumetric flow rate per unit width between stream lines.

$$\dot{V}_W = \int \frac{d\dot{V}}{W} = \int \underline{V} \cdot \underline{n}\, d\lambda = \int \left[\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx\right] = \int d\psi = \psi_2 - \psi_1$$

$$d\dot{V} = \underline{V} \cdot \underline{n}\, W\, d\lambda = W(u\,\hat{\imath} + v\,\hat{\jmath}) \cdot (dy\,\hat{\imath} - dx\,\hat{\jmath}) = W[u\,dy - v\,dx]$$

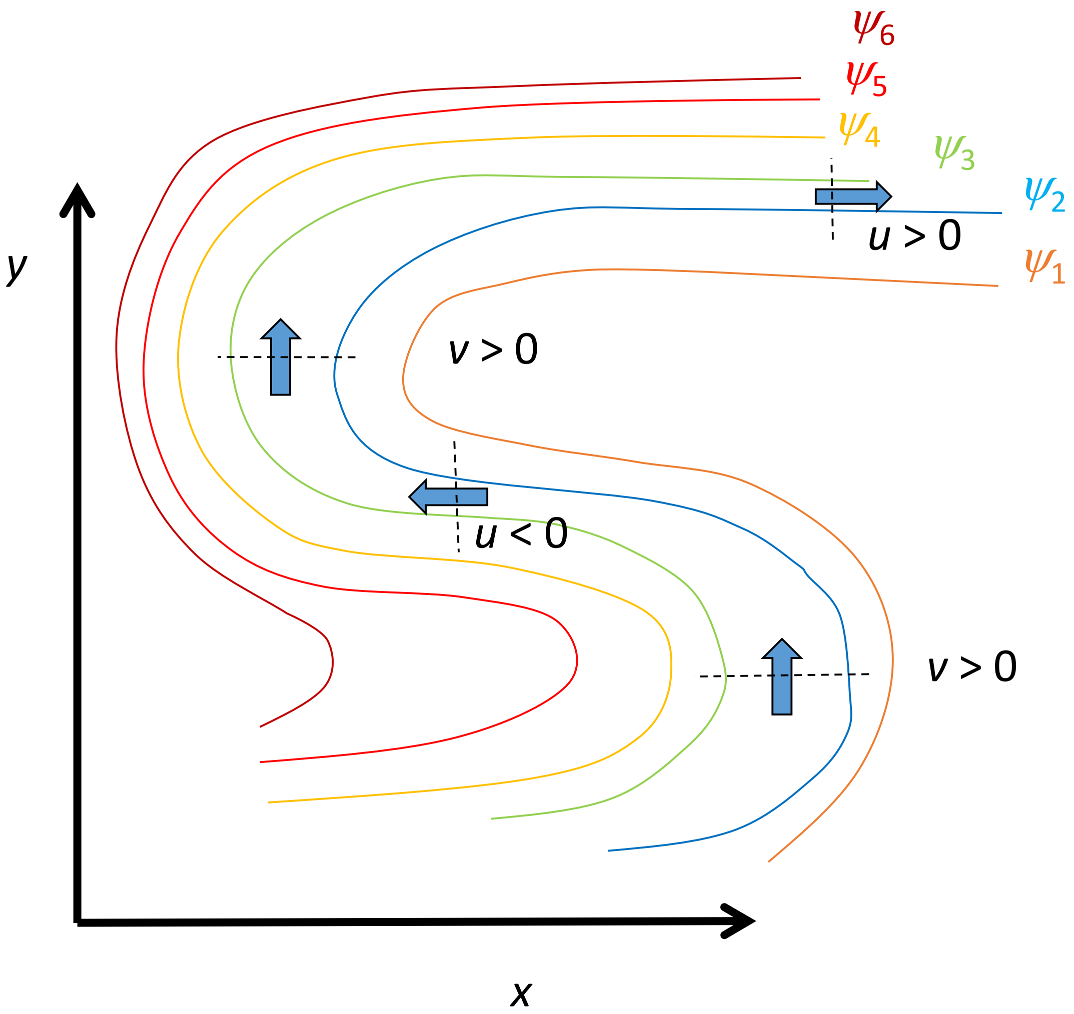
$$d\dot{V} = W[u\,dy - v\,dx] = W\,d\psi = W\left[\frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx\right]$$

Velocity direction using the ψ construct.

If : $\psi_6 > \psi_5 > \psi_4 > \psi_3 > \psi_2 > \psi_1$

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$



2-D Flow

Cartesian coordinates

$$v_x = \frac{\partial \phi}{\partial x} \qquad v_y = \frac{\partial \phi}{\partial y} \qquad\qquad v_x = \frac{\partial \psi}{\partial y} \qquad v_y = -\frac{\partial \psi}{\partial x}$$

Polar coordinates

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad\qquad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

Cylindrical coordinates

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_z = \frac{\partial \phi}{\partial z} \qquad\qquad v_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \qquad v_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$

Example No.7

Imagine a 2-D flow field given by the equations:

$$u = a + bx \qquad\qquad v = \alpha + \beta\, y$$

For a = 0.5 , b = 0.8 , α = 1.5 and β = -0.8

Determine

a) if the flow obeys continuity
b) if the flow is inviscid
c) if the flow is irrotational
d) Potential lines
e) Stream lines

$$u = a + bx \qquad v = \alpha + \beta y$$

Determine

a) if the flow obeys continuity

b) if the flow is inviscid Only if $b=-\beta$

c) if the flow is irrotational yes

d) Potential lines

e) Stream lines yes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = b + \beta$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 + 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 - 0$$

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\dot{\theta}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Potential lines

Integrating both equations for potential lines:

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y}$$

$$u = a + bx \qquad v = \alpha + \beta\, y$$

$$\phi = ax + \frac{bx^2}{2} + r(y) \qquad \phi = \alpha y + \frac{\beta\, y^2}{2} + s(x)$$

By comparison:

$$\phi = \alpha y + \frac{\beta\, y^2}{2} + ax + \frac{b\, x^2}{2}$$

Stream lines

$$v = -\frac{\partial \psi}{\partial x} \qquad u = \frac{\partial \psi}{\partial y}$$

$$u = a + bx \qquad v = \alpha + \beta\ y$$

$$v = -\frac{\partial \psi}{\partial x} = \alpha + \beta\ \ y \qquad u = \frac{\partial \psi}{\partial y} = a + b\ x$$

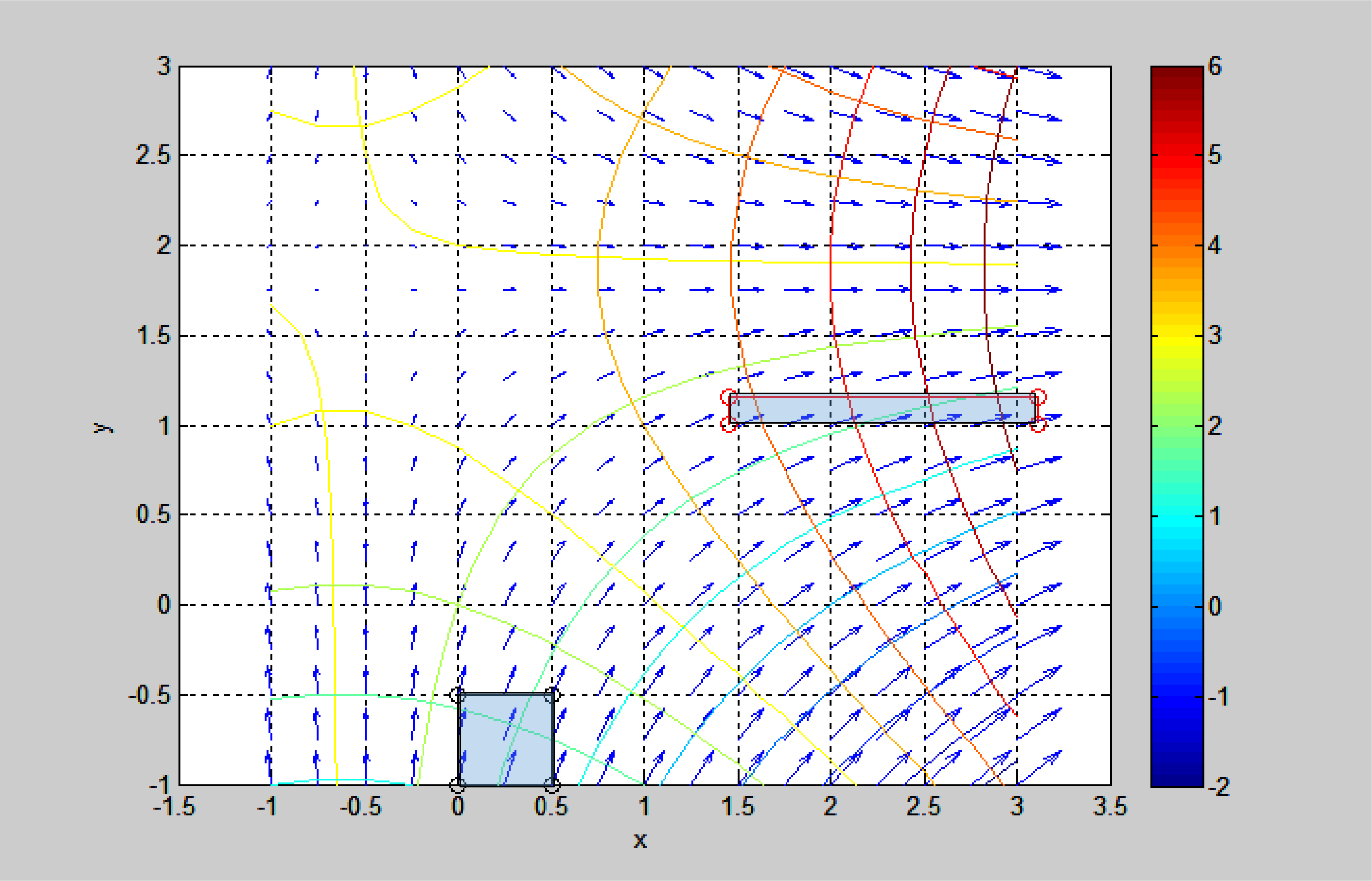
Integrating both equations:

$$\psi = -(\alpha + \beta\ \ y)x + f(y) \qquad \psi = (a + b\ x)y + g(x)$$

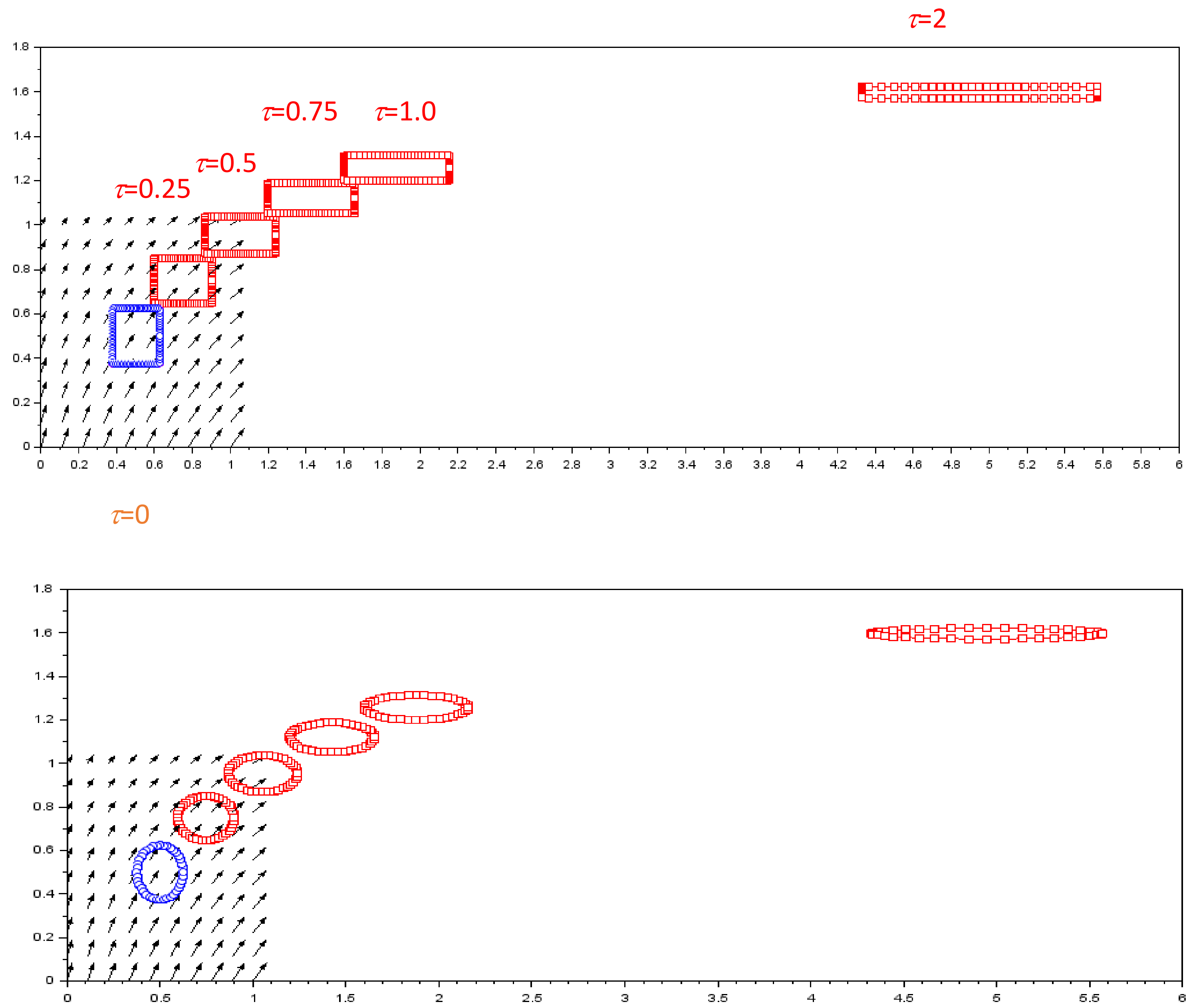
By comparison:

$$f(y) = ay \qquad g(x) = -\alpha\ x$$

$$\psi = -(\alpha + \beta\ \ y)x + a\ y$$



y
x
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
6
5
4
3
2
1
0
-1
-2

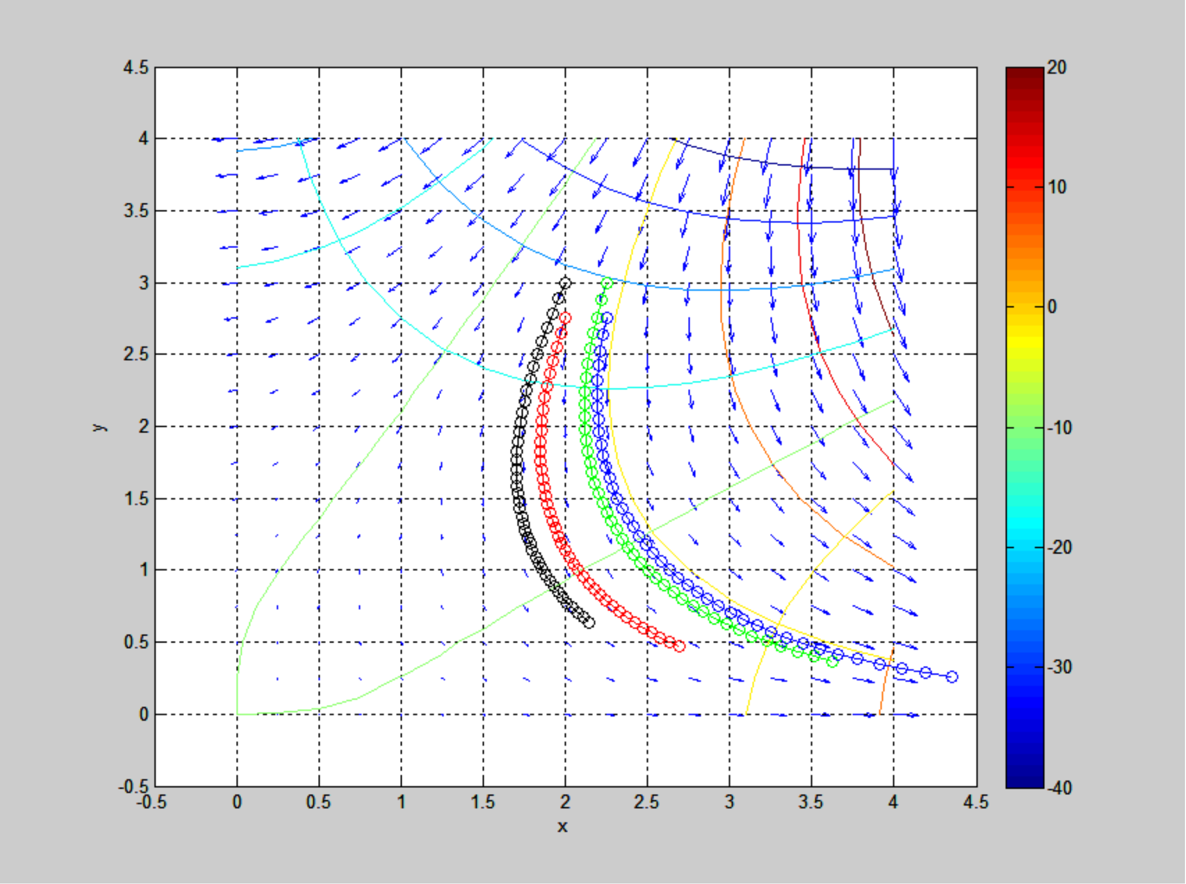


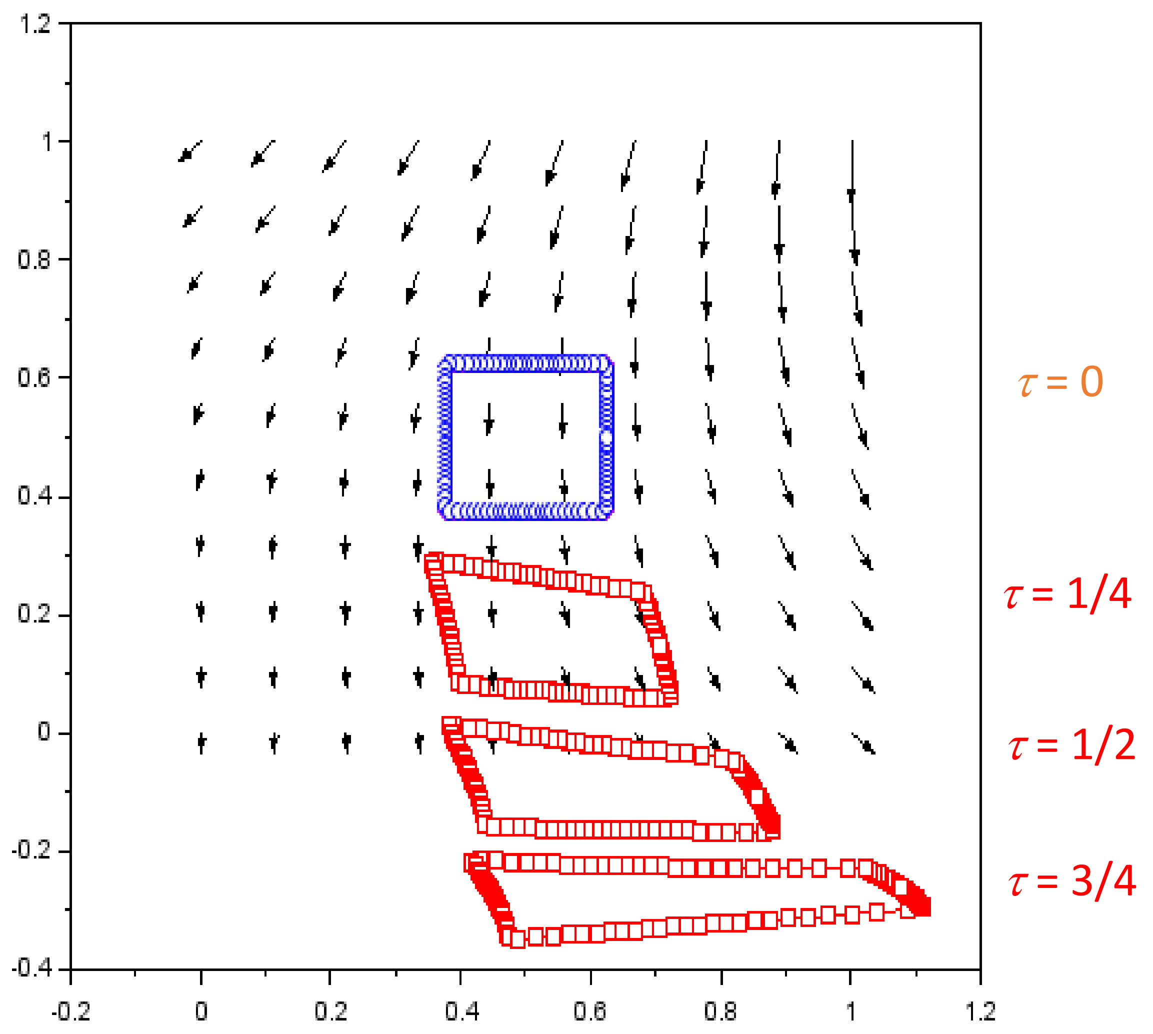
τ=2
τ=0.75
τ=1.0
τ=0.5
τ=0.25
τ=0

Example No.8

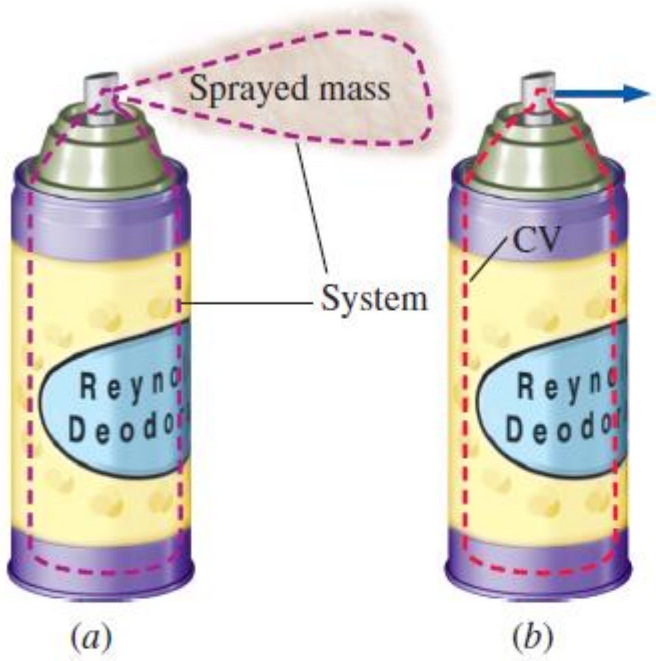
Imagine a 2-D flow field given by the equations:

$$u = x^2 - y^2 \qquad v = -1 - 2x\,y$$




1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
$\tau = 0$
$\tau = 1/4$
$\tau = 1/2$
$\tau = 3/4$

The Reynolds transport theorem

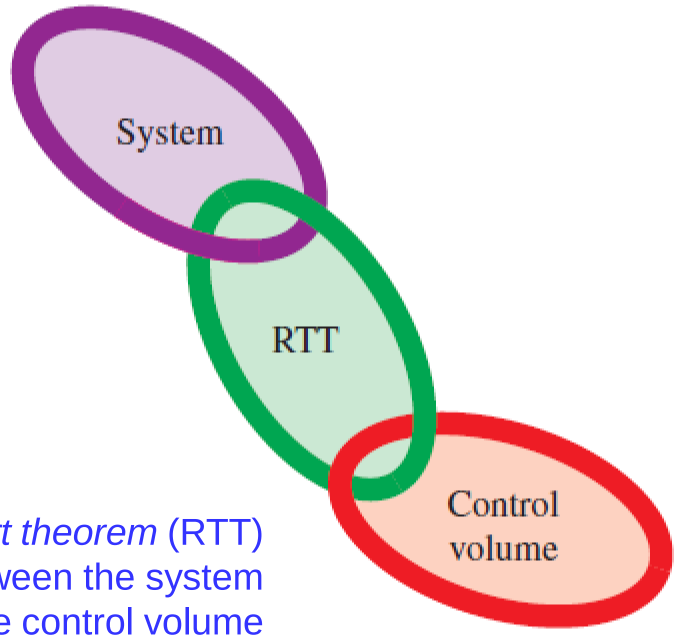


Two methods of analyzing the spraying of deodorant from a spray can:

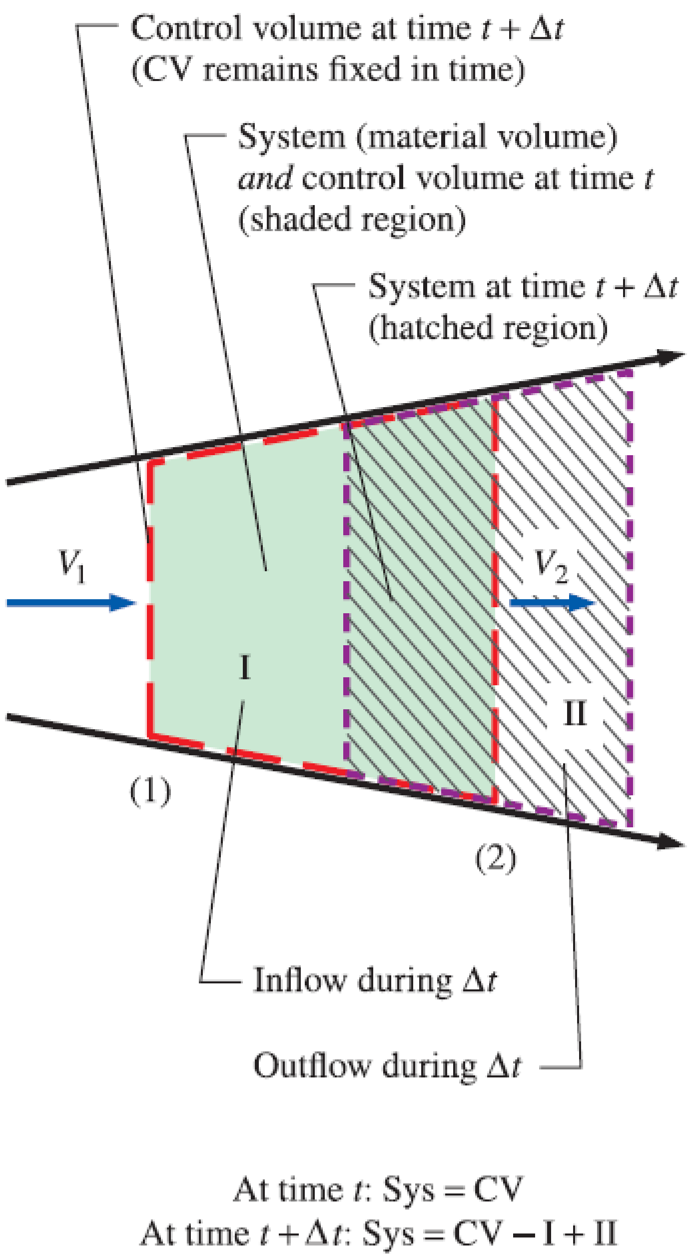
(***a***) We follow the fluid as it moves and deforms. This is the *system approach*—no mass crosses the boundary, and the total mass of the system remains fixed.

(***b***) We consider a fixed interior volume of the can. This is the *control volume approach*—mass crosses the boundary.

The relationship between the time rates of change of an extensive property for a system and for a control volume is expressed by the **Reynolds transport theorem (RTT)**.



The *Reynolds transport theorem* (RTT) provides a link between the system approach and the control volume approach.



$$\frac{dB_{sys}}{dt} = \frac{dB_{CV}}{dt} - \dot{B}_{in} + \dot{B}_{out}$$

The time rate of change of the property *B* of the system is equal to the time rate of change of *B* of the control volume plus the net flow of *B* out of the control volume by mass crossing the control surface.

This equation applies at any instant in time, where it is assumed that the system and the control volume occupy the same space at that particular instant in time.

A moving *system* (hatched region) and a fixed *control volume* (shaded region) in a diverging portion of a flow field at times *t* and *t*+Δ*t*. The upper and lower bounds are streamlines of the flow.

Let B represent any **extensive property** (such as mass, energy, or momentum), and let $b = B/m$ represent the corresponding **intensive property**. Noting that extensive properties are additive, the extensive property B of the system at times t and $t + \Delta t$ is expressed as

$$B_{\text{sys},\,t} = B_{\text{CV},\,t} \qquad \text{(the system and CV concide at time } t\text{)}$$

$$B_{\text{sys},\,t+\Delta t} = B_{\text{CV},\,t+\Delta t} - B_{\text{I},\,t+\Delta t} + B_{\text{II},\,t+\Delta t}$$

Subtracting the first equation from the second one and dividing by Δt gives

$$\frac{B_{\text{sys},\,t+\Delta t} - B_{\text{sys},\,t}}{\Delta t} = \frac{B_{\text{CV},\,t+\Delta t} - B_{\text{CV},\,t}}{\Delta t} - \frac{B_{\text{I},\,t+\Delta t}}{\Delta t} + \frac{B_{\text{II},\,t+\Delta t}}{\Delta t}$$

Reynolds Transport Theorem

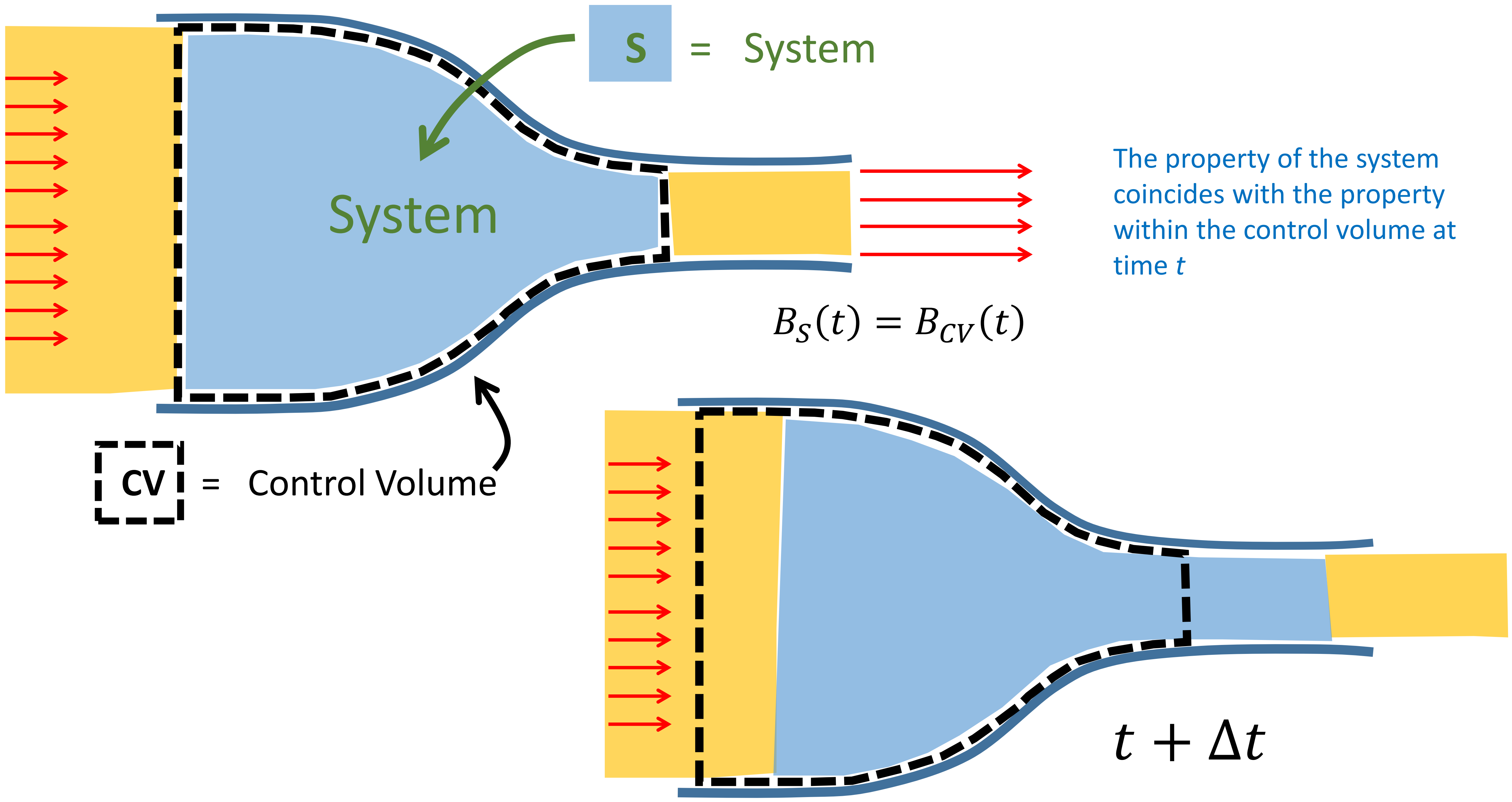
The relationship between the time rates of change of an extensive property for a system and for a control volume is expressed by the **Reynolds transport theorem (RTT)**.

Eulerian frame, is basically an open system (i.e. Mass, momentum and energy may readily cross boundaries, this is , being advected (convected) across the control volume surface or boundary , and local fluid flow change or exchange may occur within the control volume, or across portions of the boundary)

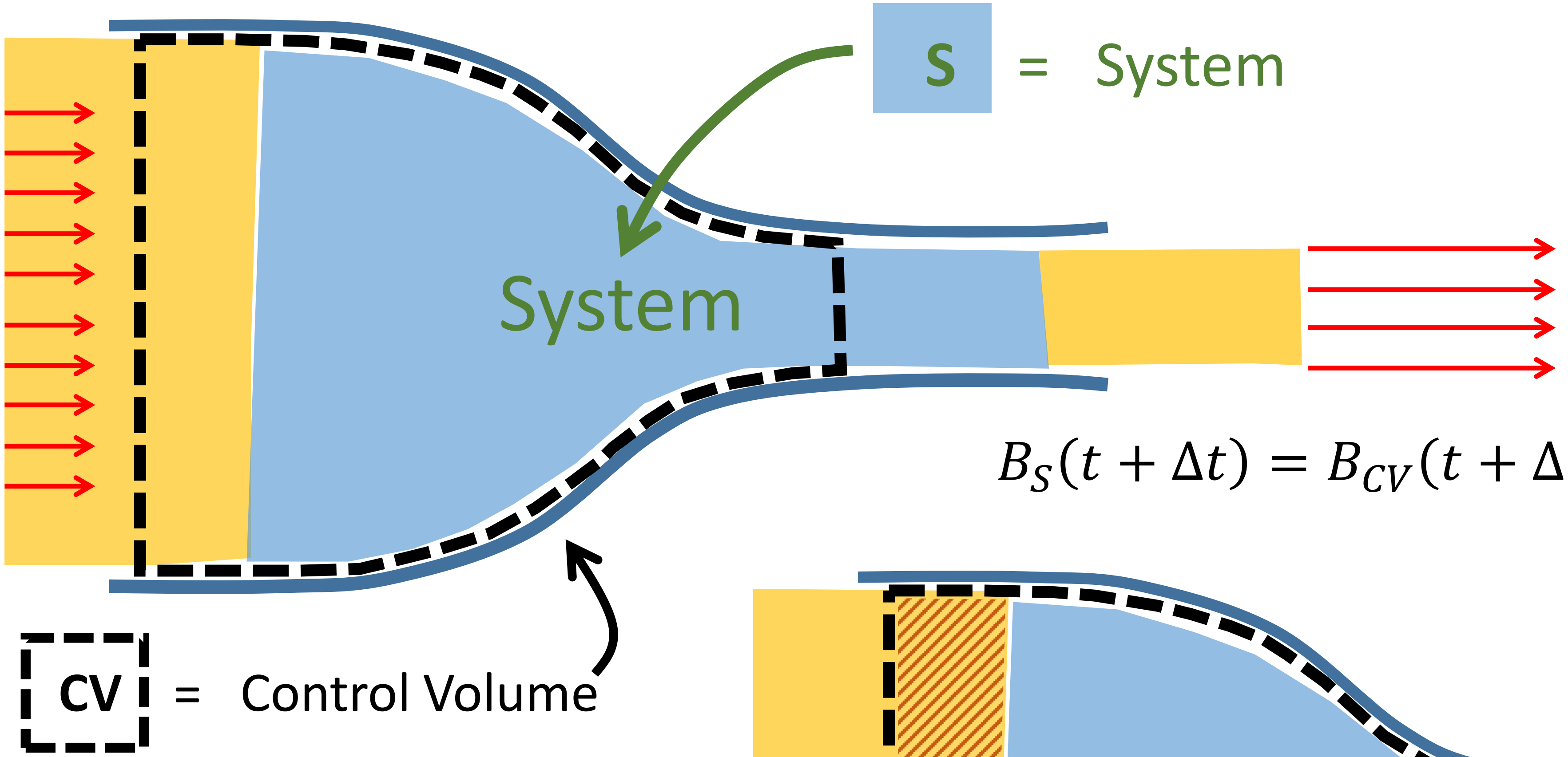
This fixed or moving control volume may be a system, an engine, a device with inlet and outlet ports, ducts , etc.

The Eulerian observed, fixed to an inertial frame of reference, records temporal and spatial changes of the **flow field** at all points, or in the case of control volume , transient mass, momentum, and/or energy changes inside and fluxes across its control surfaces.

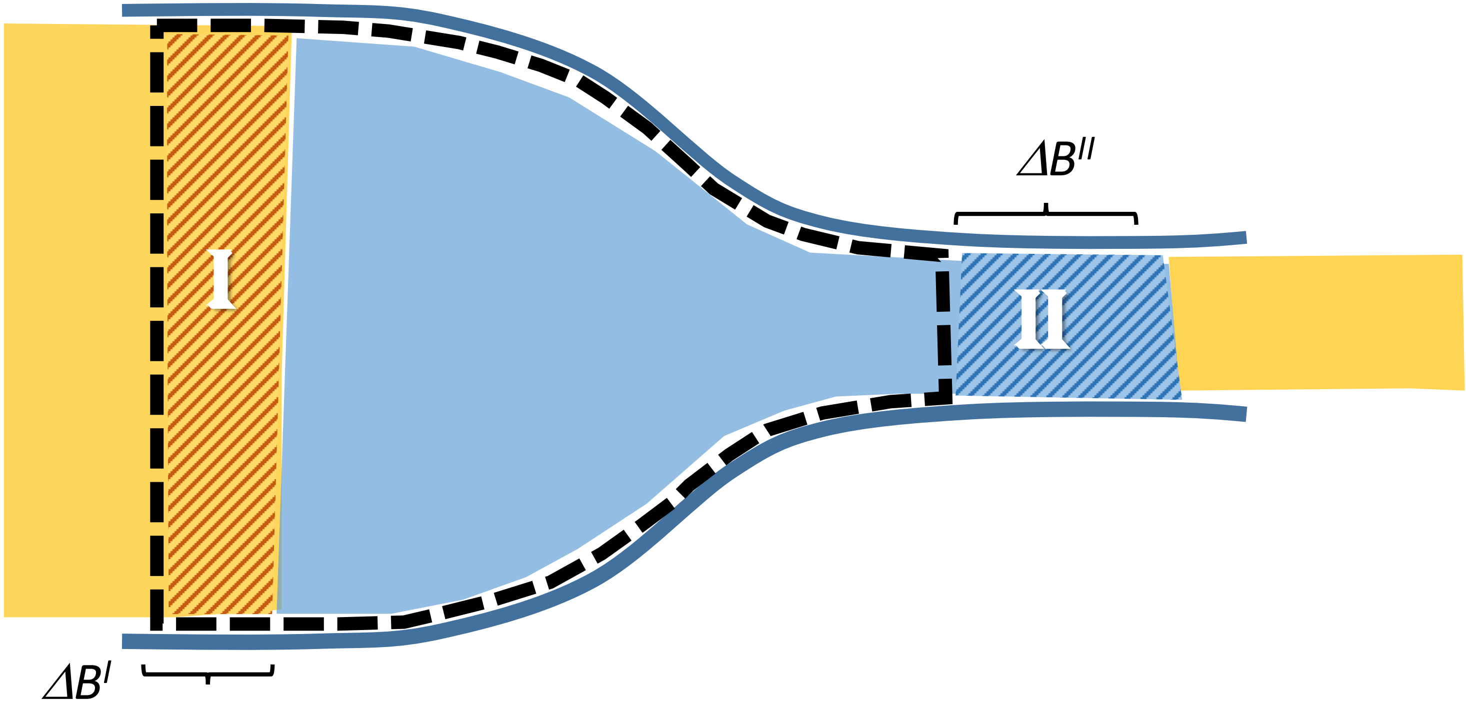
Lagrangian observer stays with each (closed) fluid element or material volume (entity, blob, parcel, molecule, etc.) and records its basic changes while moving through space



S = System
System
The property of the system coincides with the property within the control volume at time t
$B_S(t) = B_{CV}(t)$
CV = Control Volume
$t + \Delta t$



$$B_S(t+\Delta t) = B_{CV}(t+\Delta t) + \Delta B^{II} - \Delta B^{I}$$



As the time progresses to $t+\Delta t$, the property of the system need to be tracked because a portion has left the control volume (**II**) , and property of the pushing "subsystem" has entered to the control volume (**I**). The leaving portion needs to be added (**II**) and the entering portion needs to be subtracted (**I**)

$$B_S(t+\Delta t) = B_{CV}(t+\Delta t) + \Delta B^{II} - \Delta B^{I}$$

Extensive quantity at time $t+\Delta t$

$$B_S(t) = B_{CV}(t)$$

Extensive quantity at time t

$$B_S(t+\Delta t) - B_S(t) = B_{CV}(t+\Delta t) - B_{CV}(t) + \Delta B^{II} - \Delta B^{I}$$

Subtracting them gives the expression to track the change

$$\frac{B_S(t+\Delta t) - B_S(t)}{\Delta t} = \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} + \frac{\Delta B^{II}}{\Delta t} - \frac{\Delta B^{I}}{\Delta t}$$

Dividing the change by the time increment, results the equation for the rate of the extensive property

The rate of the leaving extensive property is re written in terms of the intensive property times the mass flow rate

$$\frac{\Delta B^{II}}{\Delta t} = \frac{\Delta B^{II}}{\Delta m^{II}}\frac{\Delta m^{II}}{\Delta t} = \hat{B}^{II}\frac{\Delta m^{II}}{\Delta t} = \hat{B}^{II}\,\dot{m}^{II}$$

The mass flow rate in terms of the volume flow rate times the density, and the volume flow rate in terms of the cross sectional area and the speed

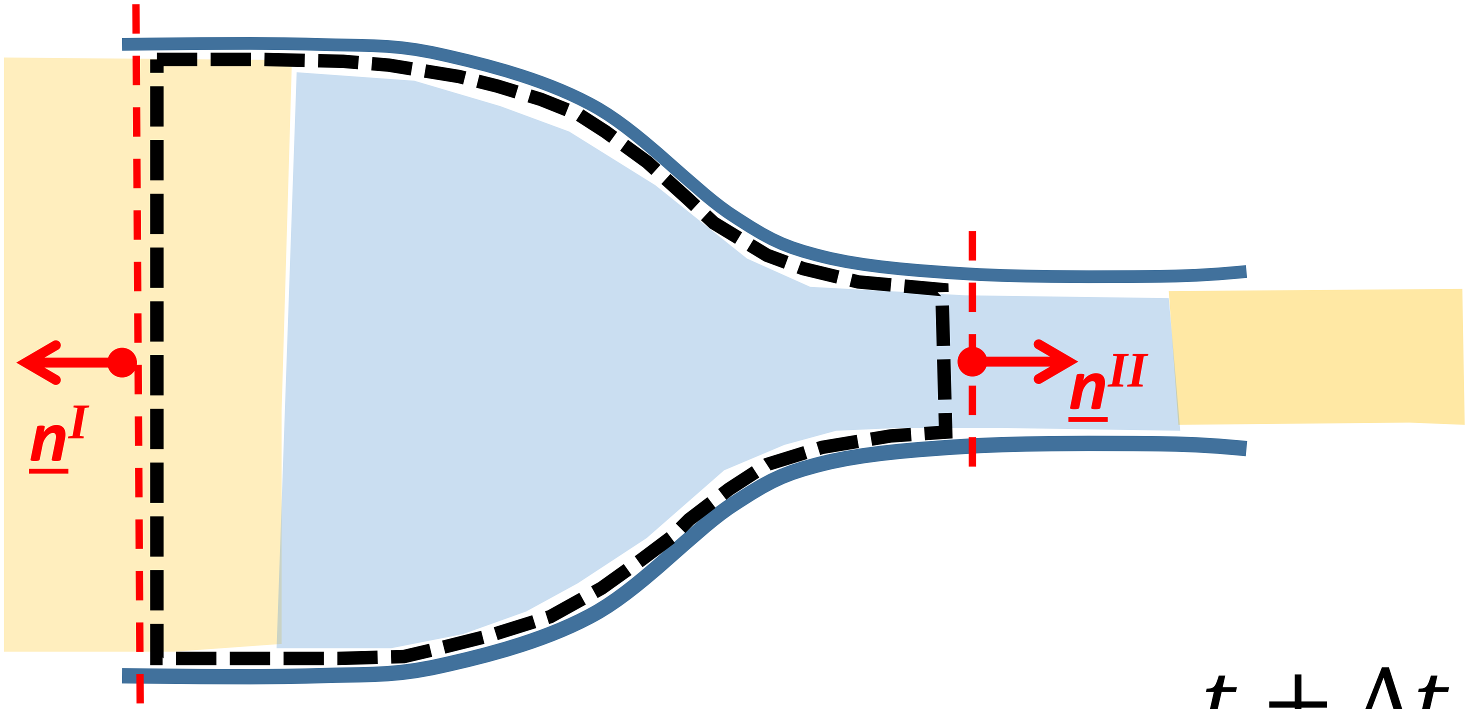
$$\hat{B}^{II}\,\dot{m}^{II} = \hat{B}^{II}\rho^{II}A^{II}\,\Delta x^{II}/\Delta t$$

The flow rate of the property in terms of the product of the intensive property, the density, the cross sectional area and the dot product of the velocity and the normal vector

$$\hat{B}^{II}\,\dot{m}^{II} = \hat{B}^{II}\rho^{II}A^{II}\underline{v}^{II}\cdot\underline{n}^{II}$$

The same is done with the inflowing flow of the property

$$-\frac{\Delta B^{I}}{\Delta t} = \hat{B}^{I}\rho^{I}A^{I}\underline{v}^{I}\cdot\underline{n}^{I}$$



$t+\Delta t$

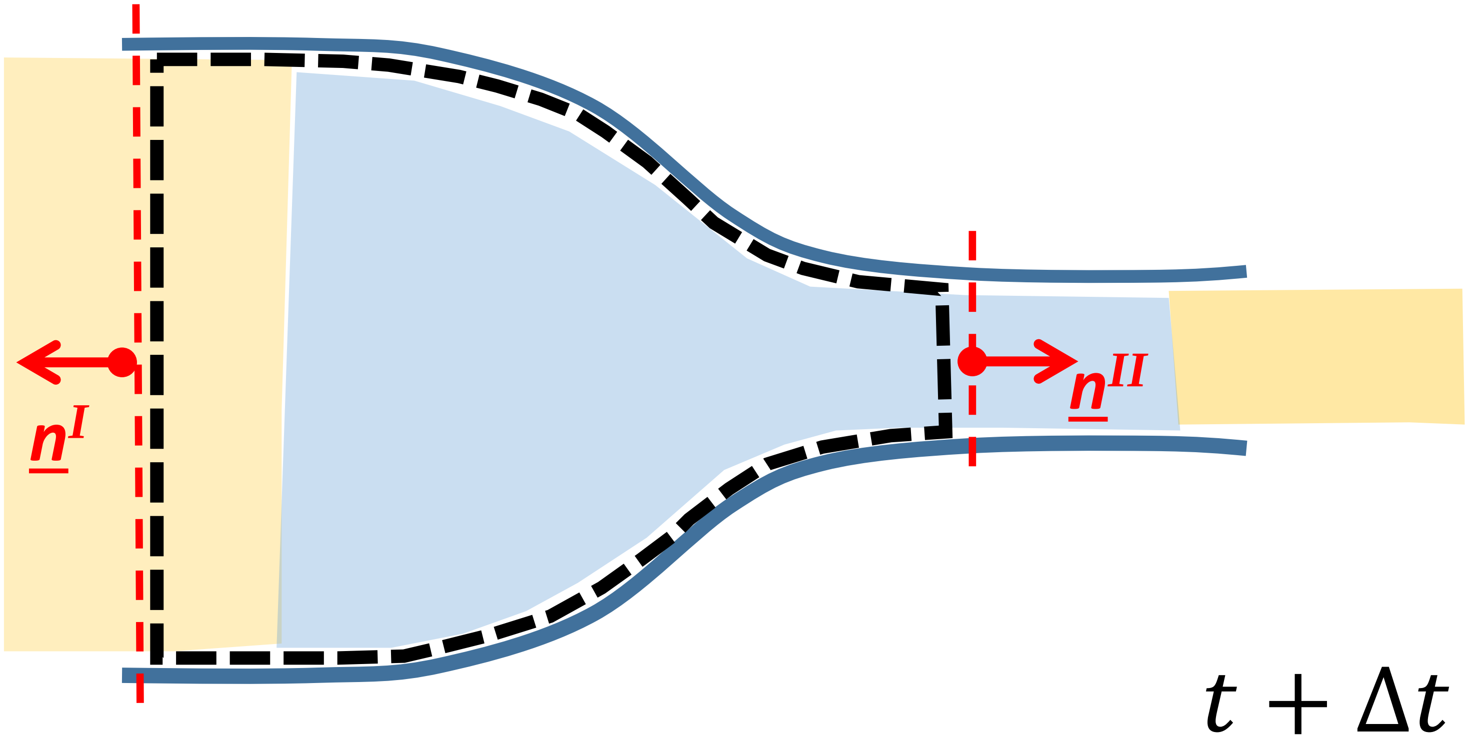
$$\frac{B_S(t+\Delta t) - B_S(t)}{\Delta t} = \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} + \frac{\Delta B^{II}}{\Delta t} - \frac{\Delta B^{I}}{\Delta t}$$

$$\frac{\Delta B^{II}}{\Delta t} = \hat{B}^{II} \rho^{II} A^{II} \underline{v}^{II} \cdot \underline{n}^{II}$$

$$-\frac{\Delta B^{I}}{\Delta t} = \hat{B}^{I} \rho^{I} A^{I} \underline{v}^{I} \cdot \underline{n}^{I}$$

Applying the limit when time increment goes to zero, and considering the boundary may be no uniform in fluxes, then the flow rates of property is expressed as the closed surface integral, to consider any exchange of property associated with a mass flow rate.

$$\frac{dB_S}{dt} = \frac{dB_{CV}}{dt} + \oint \rho \, \hat{B} \, \underline{v} \cdot \underline{n} \, dA$$



The Reynolds transport theorem can be applied for mass analysis

$$B = m \qquad \hat{B} = \frac{\Delta m}{\Delta m} = 1$$

$$\frac{dB_S}{dt} = \frac{dB_{CV}}{dt} + \oint \rho\, \hat{B}\, \underline{v} \cdot \underline{n}\, dA$$

The mass is conserved, then

$$\frac{d}{dt}\int \rho dV = \frac{\partial}{\partial t}\int \rho dV + \oint \underline{n} \cdot \rho\, \underline{v}\; dA$$

Using dot product properties of vectors, in this case the commutative property

Transforming the surface integral into the volume integral by the Gauss divergence theorem, the equation takes the form

$$\frac{d}{dt}\int \rho dV = 0 = \frac{\partial}{\partial t}\int \rho dV + \int \underline{\nabla} \cdot [\rho\, \underline{v}]\, dV$$

Removing the integral

$$\int \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v})\right] dv = 0$$

Results the continuity equation in differential form or microscopic mass balance

$$\boxed{\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}) = 0}$$

The Reynolds transport theorem can be applied for linear momentum

$$\frac{dB_s}{dt} = \frac{\partial B_{cv}}{\partial t} + \oint \underline{n} \cdot (\rho\, \underline{v}\, \hat{B})\, dA$$

Reynolds transport theorem for an extensive property

$$B = \int \rho\ \hat{B}\ dV$$

$$\frac{d}{dt}\int \rho\, \hat{B}\, dV = \frac{\partial}{\partial t}\int \rho\, \hat{B}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \hat{B})\, dA$$

Reynolds transport theorem in terms of an intensive property

$$\hat{B} = \frac{dB}{dm}$$

For linear momentum, the rate change in linear momentum is the sum of all the forces acting in the system (body forces and surface forces)

$$\frac{\partial}{\partial t}\int \rho\, \underline{v}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \underline{v})\, dA = \int \underline{f_v}\ dV + \oint \underline{f_s}\ dA$$

For body forces we have the weight, and for surface forces we have that caused by pressure, and the one caused by viscous stress

$$\frac{d}{dt}\int \rho\underline{v}\, dV = \frac{\partial}{\partial t}\int \rho\, \underline{v}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \underline{v})\, dA = \int \rho\, \underline{g}\ dV + \oint \underline{n} \cdot \underline{\underline{\tau}}\, dA - \oint \underline{n}\, p\, dA$$

Pressure can be recast in form of stress as well

$$\frac{d}{dt}\int \rho\underline{v}\, dV = \frac{\partial}{\partial t}\int \rho\, \underline{v}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \underline{v})\, dA = \int \rho\, \underline{g}\ dV + \oint \underline{n} \cdot \underline{\underline{\tau}}\, dA - \oint \underline{n} \cdot \underline{\underline{I}}\, p\, dA$$

Gauss divergence theorem is used to transform the surface integrals to volume integrals

$$\frac{d}{dt}\int \rho\underline{v}\, dV = \frac{\partial}{\partial t}\int \rho\, \underline{v}\, dV + \int \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v})\, dV = \int \rho\, \underline{g}\ dV + \int \underline{\nabla} \cdot \underline{\underline{\tau}}\, dV - \int \underline{\nabla} \cdot p\, \underline{\underline{I}}\, dV$$

Pressure is a scalar, even though represents a stress which is a tensor (symmetric and isotropic tensor)

$$\frac{d}{dt}\int \rho\underline{v}\, dV = \frac{\partial}{\partial t}\int \rho\, \underline{v}\, dV + \int \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v})\, dV = \int \rho\, \underline{g}\ dV + \int \underline{\nabla} \cdot \underline{\underline{\tau}}\, dV - \int \underline{\nabla}\, p\, dV$$

Removing the integral and volume differential we obtain:

Cauchy equation of Linear Momentum

$$\boxed{\frac{\partial}{\partial t}(\rho\, \underline{v}) + \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v}) = \rho\, \underline{g} + \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p}$$

Some derivative operators of velocity will help to calculate viscous stress

For cylindrical coordinates

Velocity Gradient tensor in Cylindrical coordinates

$$\underline{\nabla}\ \underline{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{e}_r\hat{e}_r & \frac{\partial v_\theta}{\partial r}\hat{e}_r\hat{e}_\theta & \frac{\partial v_z}{\partial r}\hat{e}_r\hat{e}_z \\ \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{e}_\theta\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta\hat{e}_\theta & \frac{1}{r}\frac{\partial v_z}{\partial \theta}\hat{e}_\theta\hat{e}_z \\ \frac{\partial v_r}{\partial z}\hat{e}_z\hat{e}_r & \frac{\partial v_\theta}{\partial z}\hat{e}_z\hat{e}_\theta & \frac{\partial v_z}{\partial z}\hat{e}_z\hat{e}_z \end{bmatrix}$$

Transpose of velocity Gradient tensor

$$[\underline{\nabla}\ \underline{v}]' = \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{e}_r\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\hat{e}_r\hat{e}_\theta & \frac{\partial v_r}{\partial z}\hat{e}_r\hat{e}_z \\ \frac{\partial v_\theta}{\partial r}\hat{e}_\theta\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta\hat{e}_\theta & \frac{\partial v_\theta}{\partial z}\hat{e}_\theta\hat{e}_z \\ \frac{\partial v_z}{\partial r}\hat{e}_z\hat{e}_r & \frac{1}{r}\frac{\partial v_z}{\partial \theta}\hat{e}_z\hat{e}_\theta & \frac{\partial v_z}{\partial z}\hat{e}_z\hat{e}_z \end{bmatrix}$$

Strain rate tensor

$$\underline{\underline{\Gamma}} = \frac{1}{2}\left[\underline{\nabla}\ \underline{v} + [\underline{\nabla}\ \underline{v}]'\right] = \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{e}_r\hat{e}_r & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]\hat{e}_r\hat{e}_\theta & \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{e}_r\hat{e}_z \\ \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]\hat{e}_\theta\hat{e}_r & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{e}_\theta\hat{e}_\theta & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]\hat{e}_\theta\hat{e}_z \\ \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{e}_z\hat{e}_r & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]\hat{e}_z\hat{e}_\theta & \frac{\partial v_z}{\partial z}\hat{e}_z\hat{e}_z \end{bmatrix}$$

Divergence of viscous stress tensor

$$\underline{\nabla} \cdot \underline{\underline{\tau}} = \begin{matrix} \left[\frac{\partial(\tau_{rr})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta r})}{\partial \theta} + \frac{(\tau_{rr} - \tau_{\theta\theta})}{r} + \frac{\partial \tau_{rz}}{\partial z}\right] \hat{\mathrm{e}}_{\mathrm{r}} \\ \left[\frac{\partial(\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta\theta})}{\partial \theta} + \frac{(\tau_{r\theta} + \tau_{\theta r})}{r} + \frac{\partial \tau_{z\theta}}{\partial z}\right] \hat{\mathrm{e}}_{\theta} \\ \left[\frac{\partial(\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(\tau_{\theta z})}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{rz}}{r}\right] \hat{\mathrm{e}}_{\mathrm{z}} \end{matrix}$$

Velocity divergence

$$\underline{\nabla} \cdot \underline{v} = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(v_\theta)}{\partial \theta} + \frac{\partial(v_z)}{\partial z}$$

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{\Gamma}} + \left[\kappa - \frac{2}{3}\mu\right]\left[\,\underline{\nabla} \cdot \underline{v}\,\right]\underline{\underline{I}}$$

Relationship among viscous stress, strain rate tensor, and velocity divergence

$\boldsymbol{\mu}$ is shear viscosity, dynamic viscosity. $\boldsymbol{\kappa}$ volume viscosity coefficient, bulk viscosity or dilatation viscosity or second coefficient.

Viscous stress tensor

$$\underline{\underline{\tau}} = 2\,\mu \begin{bmatrix} \frac{\partial v_r}{\partial r}\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\theta} & \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{z}} \\ \frac{1}{2}\left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right)\right]\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{r}} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right)\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\theta} & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{z}} \\ \frac{1}{2}\left[\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z}\right]\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{r}} & \frac{1}{2}\left[\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right]\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\theta} & \frac{\partial v_z}{\partial z}\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{z}} \end{bmatrix} + \left[\kappa - \frac{2}{3}\mu\right]\left[\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(v_\theta)}{\partial \theta} + \frac{\partial(v_z)}{\partial z}\right]\begin{bmatrix} 1\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{r}} & 0\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\theta} & 0\hat{\mathrm{e}}_{\mathrm{r}}\hat{\mathrm{e}}_{\mathrm{z}} \\ 0\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{r}} & 1\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\theta} & 0\hat{\mathrm{e}}_{\theta}\hat{\mathrm{e}}_{\mathrm{z}} \\ 0\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{r}} & 0\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\theta} & 1\hat{\mathrm{e}}_{\mathrm{z}}\hat{\mathrm{e}}_{\mathrm{z}} \end{bmatrix}$$

$$\frac{\partial}{\partial t}(\rho\,\underline{v}) + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p$$

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}) = 0$$

Using the chain rule, you can include continuity equation, and simplify the Cauchy equation

$$\frac{\partial}{\partial t}(\rho\,\underline{v}) + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \underline{v}\frac{\partial}{\partial t}(\rho) + \rho\frac{\partial}{\partial t}(\underline{v}) + \underline{v}\;\underline{\nabla}\cdot(\rho\,\underline{v}) + \rho\,\underline{v}\cdot\underline{\nabla}(\underline{v}) = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p$$

$$\underline{v}\left[\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v})\right] + \rho\frac{\partial}{\partial t}(\underline{v}) + \rho\,\underline{v}\cdot\underline{\nabla}(\underline{v}) = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p$$

$$\rho\left[\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}\right]\underline{v} = \rho\frac{D\underline{v}}{Dt} = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p$$

$$\underline{\underline{\tau}} = 2\,\mu\,\underline{\underline{\Gamma}} + \left[\kappa - \frac{2}{3}\mu\right][\underline{\nabla}\cdot\underline{v}]\,\underline{\underline{I}}$$

$$\underline{\underline{\tau}} = 2\,\mu\left[\underline{\underline{\Gamma}} - \frac{1}{3}[\underline{\nabla}\cdot\underline{v}]\underline{\underline{I}}\right] + 3\,[\kappa]\left(\frac{1}{3}\right)[\underline{\nabla}\cdot\underline{v}]\,\underline{\underline{I}}$$

$$\underline{\underline{S}} = \left[\underline{\underline{\Gamma}} - \frac{1}{3}[\underline{\nabla}\cdot\underline{v}]\underline{\underline{I}}\right] \qquad \underline{\underline{C}} = \left(\frac{1}{3}\right)[\underline{\nabla}\cdot\underline{v}]\,\underline{\underline{I}}$$

Navier-Stokes equation

$$\rho\frac{D\underline{v}}{Dt} = \rho\,\underline{g} + \underline{\nabla}\cdot\left[2\,\mu\,\underline{\underline{S}}\right] + \underline{\nabla}\cdot\left[3\,\kappa\,\underline{\underline{C}}\right] - \underline{\nabla}\,p$$

$$\frac{D}{Dt} = \left[\frac{\partial}{\partial t} + \underline{v}\cdot\underline{\nabla}\right]$$

$$\frac{D\Psi}{Dt} = \left[\frac{\partial\Psi}{\partial t} + \underline{v}\cdot\underline{\nabla}\Psi\right]$$

This new derivative is called material derivative or substantive derivative. a.k.a. Stokes derivative

For simplification in the writing, we have used short notation for integrals

$$\frac{\partial}{\partial t}\int \rho\,\underline{v}\,dV + \oint \underline{n}\cdot(\rho\,\underline{v}\,\underline{v})\,dA = \int \underline{f_v}\,dV + \oint \underline{f_s}\,dA$$

Short notation

The actual representation should be the following

$$\frac{\partial}{\partial t}\iiint \rho\,\underline{v}\,dV + \oiint \underline{n}\cdot(\rho\,\underline{v}\,\underline{v})\,dA = \iiint \underline{f_v}\,dV + \oiint \underline{f_s}dA + \oint \underline{f_L}\,dL$$

$$\oiint \underline{n}\cdot\underline{\psi}\,dA = \iiint \underline{\nabla}\cdot\underline{\psi}\,dV$$

Gauss Divergence Theorem
(Derivation of this theorem can be found in the appendix of this presentation)

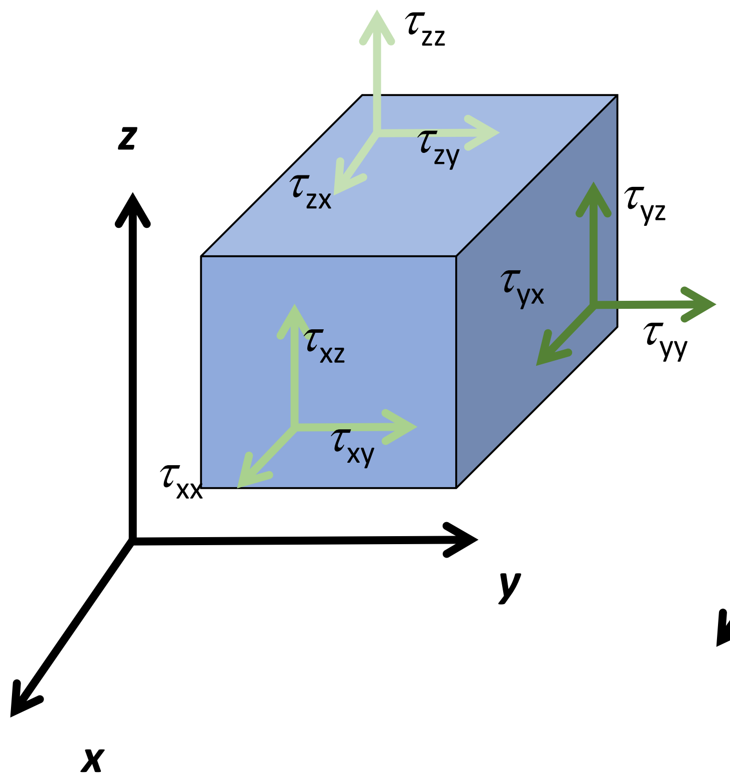
All the equations are written for a single phase flow, and we are not considering line forces, the reason is because we do not have contact lines where three phases meet.

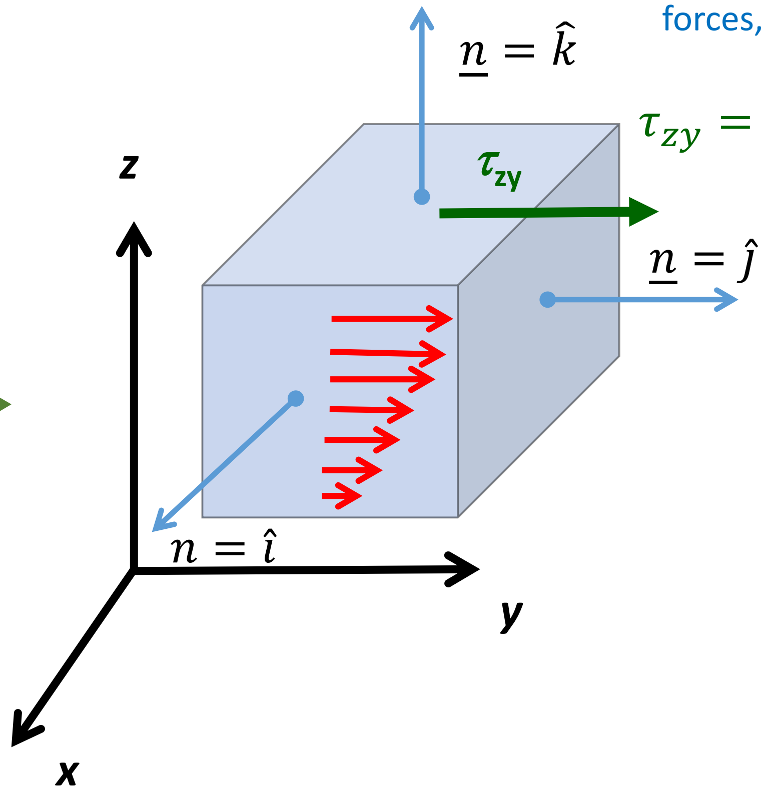
$$\frac{\partial}{\partial t}(\rho\,\underline{v}) + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p + \sigma\,\kappa\,\delta_s\underline{n}$$

accumulation advection gravitational viscous pressure line

Reflect about stress tensor

τ_{yx} = Stress in the face perpendicular to *y*-axis in the x-direction





Understanding the relation of viscous stress with viscous stress forces, which are surface forces:

$$\tau_{zy} = \mu \frac{dv_y}{dz} \hat{k}\, \hat{j}$$

$$F_{zy} = \left[\Delta x\, \Delta y\, \hat{k}\right] \cdot \left[\mu \frac{dv_y}{dz} \hat{k}\, \hat{j}\right]$$

The viscous force then is calculated with: $d\underline{F_v} = \underline{n} \cdot \underline{\underline{\tau}}\, dA$ $\qquad \underline{F_v} = \oint \underline{n} \cdot \underline{\underline{\tau}}\, dA$

The Reynolds transport theorem can be applied for energy

$$\frac{dB_s}{dt} = \frac{\partial B_{cv}}{\partial t} + \oint \underline{n} \cdot (\rho\, \underline{v}\, \hat{B})\, dA \qquad \frac{d}{dt}\int \rho\, \hat{B}\, dV = \frac{\partial}{\partial t}\int \rho\, \hat{B}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \hat{B})\, dA$$

For Energy the Reynolds transport theorem will consider all the energy interactions, like flow energy which happens either by interaction at the boundaries exchanging mass or impermeable moving boundaries resulting from pressure displacing mass or moving parts, viscous dissipation, heat conduction and heat generation by other factors different from viscous dissipation (electrical current, chemical reaction, etc.)

$$\frac{d}{dt}\int \rho\, \hat{E}\, dV = \frac{\partial}{\partial t}\int \rho\, \hat{E}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \hat{E})\, dA = -\oint \underline{n} \cdot \left[p\, \underline{\underline{I}} \cdot \underline{v}\right] dA + \oint \underline{n} \cdot \left[\underline{\underline{\tau}} \cdot \underline{v}\right] dA - \oint \underline{n} \cdot \underline{q}\, dA + \int \dot{Q}_V dV$$

Energy is conserved, this is either accumulated within the control volume or transported across boundaries

$$\frac{\partial}{\partial t}\int \rho\, \hat{E}\, dV + \oint \underline{n} \cdot (\rho\, \underline{v}\, \hat{E})\, dA = -\oint \underline{n} \cdot \left[p\, \underline{\underline{I}} \cdot \underline{v}\right] dA + \oint \underline{n} \cdot \left[\underline{\underline{\tau}} \cdot \underline{v}\right] dA - \oint \underline{n} \cdot \underline{q}\, dA + \int \dot{Q}_V dV$$

Using the Gauss divergence theorem, all the surface energy interactions are transformed into volume integrals

$$\frac{\partial}{\partial t}\int \rho\, \hat{E}\, dV + \int \underline{\nabla} \cdot (\rho\, \underline{v}\, \hat{E})\, dV = -\int \underline{\nabla} \cdot \left(p\, \underline{\underline{I}} \cdot \underline{v}\right) dV + \int \underline{\nabla} \cdot \left(\underline{\underline{\tau}} \cdot \underline{v}\right) dV - \int \underline{\nabla} \cdot \underline{q}\, dV + \int \dot{Q}_V dV$$

Gering rid of the integral and the volume differential, the differential equation of conservation of energy is obtained.

$$\frac{\partial(\rho\, \hat{E})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \hat{E}) = -\underline{\nabla} \cdot \left(p\, \underline{\underline{I}} \cdot \underline{v}\right) + \underline{\nabla} \cdot \left(\underline{\underline{\tau}} \cdot \underline{v}\right) - \underline{\nabla} \cdot \underline{q} + \dot{Q}_V$$

$$\frac{\partial(\rho\, \hat{E})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \hat{E}) = -\underline{\nabla} \cdot (p\, \underline{v}) + \underline{\nabla} \cdot \left(\underline{\underline{\tau}} \cdot \underline{v}\right) - \underline{\nabla} \cdot \underline{q} + \dot{Q}_V$$

Understanding pressure as scalar instead of an isotropic and symmetric stress tensor.

$$\frac{\partial(\rho\, \hat{E})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \hat{E}) = -\underline{\nabla} \cdot \left(\rho\, \underline{v}\, \frac{p}{\rho}\right) + \underline{\nabla} \cdot \left(\underline{\underline{\tau}} \cdot \underline{v}\right) - \underline{\nabla} \cdot \underline{q} + \dot{Q}_V$$

The pressure advective term is also known as pressure energy, by dividing by density (or flow energy) .

$$\frac{\partial(\rho\,\hat{E})}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}\,\hat{E}) + \underline{\nabla}\cdot\left(\rho\,\underline{v}\,\frac{p}{\rho}\right) = \underline{\nabla}\cdot(\underline{\underline{\tau}}\cdot\underline{v}) - \underline{\nabla}\cdot\underline{q} + \dot{Q}_V$$

The pressure advective term typically is added to the internal energy term in the known property of enthalpy.

$$\boxed{\frac{\partial(\rho\,\hat{E})}{\partial t} + \underline{\nabla}\cdot\left(\rho\,\underline{v}\left[\hat{E} + \frac{p}{\rho}\right]\right) = \underline{\nabla}\cdot(\underline{\underline{\tau}}\cdot\underline{v}) - \underline{\nabla}\cdot\underline{q} + \dot{Q}_V}$$

$$\hat{E} = \frac{1}{2}\underline{v}\cdot\underline{v} + g\,z + \hat{U}$$

Energy can be stored as kinetic energy, potential energy or internal energy.

$$\hat{E} + \frac{p}{\rho} = \frac{1}{2}\underline{v}\cdot\underline{v} + g\,z + \hat{H}$$

Flowing energy can be kinetic energy, potential or enthalpy.

The pressure energy term when integrated across a control volume, has to done considering two scenarios, when there is mass exchange of flow, that will be associated with the internal energy with the so called enthalpy, and when there is a moving boundary but no mass exchange , and it is recognized as work .

Additional notes on linear momentum balance

Applying the Reynolds transport theorem for linear momentum

$$B = m\underline{v} \qquad \hat{B} = \underline{v}$$

$$\frac{d(\rho\underline{v})}{dt} = \frac{\partial(\rho\underline{v})}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v}\,\underline{v})$$

And using the chain rule

$$\frac{d(\rho\underline{v})}{dt} = \frac{\partial(\rho)}{\partial t}\underline{v} + \rho\frac{\partial(\underline{v})}{\partial t} + [\underline{\nabla}\cdot(\rho\underline{v})]\underline{v} + \rho\underline{v}\cdot\underline{\nabla}(\underline{v})$$

$$\frac{d(\rho\underline{v})}{dt} = \rho\left[\frac{\partial(\underline{v})}{\partial t} + \underline{v}\cdot\underline{\nabla}(\underline{v})\right] + \left[\frac{\partial(\rho)}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v})\right]\underline{v}$$

Continuity equation

$$\frac{\partial(\rho)}{\partial t} + \underline{\nabla}\cdot(\rho\underline{v}) = 0$$

If the density is constant, then

$$\frac{d(\rho \underline{v})}{dt} = \rho \left[\frac{\partial(\underline{v})}{\partial t} + \underline{v} \cdot \underline{\nabla}(\underline{v})\right]$$

Acceleration analysis

$$\frac{d\underline{v}}{dt} = \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}(\underline{v})\right]$$

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}(\underline{v})$$

For incompressible fluids

acceleration

Local acceleration

advective acceleration

This acceleration term is called substantial acceleration

$$\frac{\partial(\underline{v})}{\partial t} + \underline{v} \cdot \underline{\nabla}(\underline{v}) = \frac{D\underline{v}}{Dt}$$

This material derivative, or substantial derivative is so common in fluid dynamics, that is a term adopted for future analysis

$$\frac{\partial(\quad)}{\partial t} + \underline{v} \cdot \underline{\nabla}(\quad) = \frac{D(\quad)}{Dt}$$

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}(\underline{v}) - \frac{\underline{v}}{\rho}\frac{d\rho}{dt}$$

For compressible fluids, the acceleration must have to consider changes in density as well

Some properties of the substantial derivative, material derivative , or substantial derivative operator

$$\frac{\partial(\ \ )}{\partial t} + \underline{v} \cdot \underline{\nabla}(\ \ ) = \frac{D(\ \ )}{Dt}$$

Differential operator

$$\left[\frac{\partial(\ \ )}{\partial t} + \underline{v} \cdot \underline{\nabla}(\ \ )\right]\psi = \left[\frac{D(\ \ )}{Dt}\right]\psi = \frac{D\psi}{Dt}$$

Differential operator over any variable ψ

$$\frac{\partial\psi}{\partial t} + \underline{v} \cdot \underline{\nabla}\psi = \frac{D\psi}{Dt}$$

Material derivative of ψ

$$\rho\frac{\partial\psi}{\partial t} + \rho\underline{v} \cdot \underline{\nabla}\psi = \rho\frac{D\psi}{Dt}$$

Both sides are multiplied by density

$$\rho\frac{\partial\psi}{\partial t} + \rho\underline{v} \cdot \underline{\nabla}\psi + \psi\left[\frac{\partial\rho}{\partial t} + \underline{\nabla} \cdot (\rho\,\underline{v})\right] = \rho\frac{D\psi}{Dt}$$

Adding zero to complete the product of a derivative

$$\boxed{\frac{\partial(\psi\,\rho)}{\partial t} + \underline{\nabla} \cdot (\psi\,\rho\,\underline{v}) = \rho\frac{D\psi}{Dt}}$$

Identity of substantial derivative

Continuity equation used to obtain the properties of the substantial derivative

$$\frac{\partial\rho}{\partial t} + \underline{\nabla} \cdot (\rho\,\underline{v}) = 0$$

The Reynolds transport theorem can be applied for mechanical energy (from power coming from linear momentum)

$$\frac{\partial}{\partial t}(\rho\,\underline{v}) + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p$$

Linear momentum equation

$$\rho\frac{\partial}{\partial t}(\,\underline{v}) + \rho\,\underline{v}\;\underline{\nabla}\cdot(\,\underline{v}) = \rho\frac{D(\underline{v})}{Dt} = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p$$

Linear momentum equation in terms of substantial derivative

$$\underline{v}\cdot\left[\rho\frac{D(\underline{v})}{Dt} = \rho\,\underline{g} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p\right]$$

Dot product of velocity and the linear momentum equation

$$\rho\frac{D\left(\frac{\underline{v}\cdot\underline{v}}{2}\right)}{Dt} = \rho\underline{v}\cdot\;\underline{g} + \underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{v}\cdot\underline{\nabla}\,p$$

In terms of specific kinetic energy

$$\underline{g} = -\underline{\nabla}\widehat{\Phi}$$

Gravitational field in terms of gravitational potential gradient

$$\rho\frac{D\left(\frac{\underline{v}\cdot\underline{v}}{2}\right)}{Dt} = -\,\rho\underline{v}\cdot\underline{\nabla}\widehat{\Phi} + \;\underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{v}\cdot\underline{\nabla}\,p$$

Specific kinetic energy substantial derivative in terms of potential gradient

$$\rho\frac{D\widehat{K}}{Dt} = -\,\rho\underline{v}\cdot\underline{\nabla}\widehat{\Phi} + \;\underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{v}\cdot\underline{\nabla}\,p$$

Specific kinetic energy substantial derivative

$$\rho \frac{D\hat{K}}{Dt} = -\rho \underline{v} \cdot \underline{\nabla} \hat{\Phi} + \underline{v} \cdot \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{v} \cdot \underline{\nabla}\, p$$

Conservation of mechanical energy

$$\left[\frac{\partial(\ \ )}{\partial t} + \underline{v} \cdot \underline{\nabla} (\ \ ) \right] \hat{\Phi} = \frac{D\hat{\Phi}}{Dt}$$

Differential operator of the specific potential energy

$$\frac{D\hat{\Phi}}{Dt} = \frac{\partial \hat{\Phi}}{\partial t} + \underline{v} \cdot \underline{\nabla} \hat{\Phi}$$

Material derivative of specific potential energy

$$\rho \frac{D\hat{\Phi}}{Dt} = \rho \frac{\partial \hat{\Phi}}{\partial t} + \rho\, \underline{v} \cdot \underline{\nabla} \hat{\Phi}$$

Material derivative of specific potential energy multiplied by density

$$\rho \frac{D\hat{\Phi}}{Dt} = \frac{\partial\left(\hat{\Phi}\, \rho\right)}{\partial t} + \underline{\nabla} \cdot \left(\hat{\Phi}\, \rho\, \underline{v}\right)$$

$$\frac{\partial \hat{\Phi}}{\partial t} = 0$$

Potential energy depends on coordinates, but the time

$$-\rho \frac{D\hat{\Phi}}{Dt} = -\rho\, \underline{v} \cdot \underline{\nabla} \hat{\Phi}$$

$$\rho \frac{D\left[\hat{K} + \hat{\Phi}\right]}{Dt} = \underline{v} \cdot \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{v} \cdot \underline{\nabla}\, p$$

Conservation of mechanical energy

$$\rho \frac{D[\hat{K}+\hat{\Phi}]}{Dt} = \underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{v}\cdot\underline{\nabla}\, p$$ Conservation of mechanical energy

$$\frac{\partial(\rho[\hat{\Phi}+\hat{K}])}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}[\hat{\Phi}+\hat{K}]) = \underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{v}\cdot\underline{\nabla}\, p$$ expanding

$$\underline{\nabla}\,(p/\rho) = \frac{1}{\rho}\underline{\nabla}\,(p) - \frac{p}{\rho^2}\underline{\nabla}\rho$$ Calculating gradient of flow energy

$$\rho\underline{\nabla}\,(p/\rho) = \underline{\nabla}\,(p) - \frac{p}{\rho}\underline{\nabla}\rho$$ Multiplied by density

$$\rho\underline{v}\cdot\underline{\nabla}\,(p/\rho) = \underline{v}\cdot\underline{\nabla}\,(p) - \frac{p}{\rho}\underline{v}\cdot\underline{\nabla}\rho$$ Calculating dot product of velocity and the previous equation

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}) = \frac{\partial\rho}{\partial t} + \rho\,\underline{\nabla}\cdot(\underline{v}) + \underline{v}\cdot\underline{\nabla}\rho = 0$$ Chain rule over the continuity

$$\frac{p}{\rho}\frac{\partial\rho}{\partial t} + p\,\underline{\nabla}\cdot(\underline{v}) + \frac{p}{\rho}\underline{v}\cdot\underline{\nabla}\rho = 0$$ Multiplying by flow energy by the continuity

$$\rho\underline{v}\cdot\underline{\nabla}\,(p/\rho) = \underline{v}\cdot\underline{\nabla}\,(p) + \frac{p}{\rho}\frac{\partial\rho}{\partial t} + p\,\underline{\nabla}\cdot(\underline{v})$$ Calculating dot product of density, velocity and the gradient of flow energy

$$\rho\underline{v}\cdot\underline{\nabla}\,(p/\rho) + (p/\rho)\underline{\nabla}\cdot(\rho\underline{v}) = \underline{v}\cdot\underline{\nabla}\,(p) + \frac{p}{\rho}\frac{\partial\rho}{\partial t} + (p/\rho)\underline{\nabla}\cdot(\rho\underline{v}) + p\,\underline{\nabla}\cdot(\underline{v})$$ Completing the divergence of the flux of pressure energy

$$\underline{\nabla}\cdot\left(\frac{p}{\rho}\rho\underline{v}\right) = \underline{v}\cdot\underline{\nabla}\,(p) + p\,\underline{\nabla}\cdot(\underline{v})$$ Simplified form of the divergence of advective pressure energy flux

Other forms of continuity

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}) = \frac{\partial\rho}{\partial t} + \rho\,\underline{\nabla}\cdot(\underline{v}) + \underline{v}\cdot\underline{\nabla}\rho$$

$$\frac{\partial\rho}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}) = \frac{D\rho}{Dt} + \rho\,\underline{\nabla}\cdot(\underline{v}) = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \underline{\nabla}\cdot(\underline{v}) = 0$$

$$\frac{p}{\rho}\frac{D\rho}{Dt} + p\,\underline{\nabla}\cdot(\underline{v}) = 0$$

$$\frac{\partial(\rho[\hat{\Phi}+\hat{K}])}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v}[\hat{\Phi}+\hat{K}])=\underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}}-\underline{v}\cdot\underline{\nabla}\,p$$

Mechanical energy balance

$$\underline{\nabla}\cdot\left(\frac{p}{\rho}\rho\underline{v}\right)-p\,\underline{\nabla}\cdot(\underline{v})=\underline{v}\cdot\underline{\nabla}\,(p)$$

Divergence of pressure energy flux

$$\frac{\partial(\rho[\hat{\Phi}+\hat{K}])}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v}[\hat{\Phi}+\hat{K}+p/\rho])=\underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}}+p\,\underline{\nabla}\cdot(\underline{v})$$

Coupling the previous equations

$$\frac{\partial(\rho[\hat{\Phi}+\hat{K}])}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v}[\hat{\Phi}+\hat{K}+p/\rho])=\underline{v}\cdot\underline{\nabla}\cdot\underline{\underline{\tau}}-\frac{p}{\rho}\frac{D\rho}{Dt}$$

Differential form of the mechanical energy balance

For a frictionless system, no viscous dissipation

$$\frac{d(m[\hat{K}+\hat{\Phi}])}{dt}=\dot{m}_i\left[\hat{K}_i+\hat{\Phi}_i+\frac{p_i}{\rho_i}\right]-\dot{m}_o\left[\hat{K}_o+\hat{\Phi}_o+\frac{p_o}{\rho_o}\right]-\dot{W}$$

Bernoulli equation or mechanical energy balance

$$\frac{\partial\left[\int \rho \underline{v} dV\right]}{\partial t} + \int \underline{\nabla} \cdot \left(\rho\, \underline{v}\, \underline{v}\right) dV = \int \rho \underline{g} dV + \oint \underline{n} \cdot \left[\underline{\underline{\tau}} - p \underline{\underline{I}}\right] dA$$

$$\frac{\partial\left[\int \rho \underline{v} dV\right]}{\partial t} + \oint \underline{n} \cdot \left[\rho\, \underline{v}\, \underline{v}\right] dA = \int \rho \underline{g} dV + \oint \underline{n} \cdot \left[\underline{\underline{\tau}} - p \underline{\underline{I}}\right] dA$$

$$\frac{d\left[m\langle \underline{V}\rangle\right]}{dt} + \sum_{j=out} \beta_j\, \dot{m}_j \langle \underline{V_j}\rangle - \sum_{i=in} \beta_i\, \dot{m}_i \langle \underline{V_i}\rangle = m\underline{g} - \sum_{all} p_k A_k\, \underline{n}_k - \underline{F_f}$$

$$\beta_i = \frac{\left\langle v^2 \right\rangle}{\left\langle v \right\rangle^2}$$

Energy

$$\frac{d\left[\rho\hat{B}_{sys}\right]}{dt}=\frac{\partial\left[\rho\hat{B}_{CV}\right]}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\hat{B}\,\underline{v}\right)$$

$$\frac{d\left[\int\rho\hat{E}dV\right]}{dt}=\frac{\partial\left[\int\rho\hat{E}dV\right]}{\partial t}+\int\underline{\nabla}\cdot\left(\rho\,\underline{v}\,\hat{E}\right)dV=\int\rho e_m dV+\oint e_s dA$$

$$\frac{\partial\left[\int\rho\hat{E}dV\right]}{\partial t}+\int\underline{\nabla}\cdot\left(\rho\,\hat{E}\,\underline{v}\right)dV=\oint\underline{n}\cdot\left[\left(\underline{\underline{\tau}}-p\underline{\underline{I}}\right)\cdot\underline{v}\right]dA-\oint\underline{n}\cdot\underline{q}dA$$

$$\frac{\partial\left[\int\rho\hat{E}dV\right]}{\partial t}+\int\underline{\nabla}\cdot\left(\rho\,\hat{E}\,\underline{v}\right)dV=\int\underline{\nabla}\cdot\left[\left(\underline{\underline{\tau}}-p\underline{\underline{I}}\right)\cdot\underline{v}\right]dV-\int\underline{\nabla}\cdot\left[\underline{q}\right]dV$$

$$\frac{\partial\left[\rho\hat{E}\right]}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\underline{v}\left[\hat{E}+\frac{p}{\rho}\right]\right)=\underline{\nabla}\cdot\left[\underline{\underline{\tau}}\cdot\underline{v}\right]-\underline{\nabla}\cdot\underline{q}$$

$$\frac{\partial\left[\rho\hat{E}\right]}{\partial t}+\underline{\nabla}\cdot\left(\rho\,\underline{v}\left[\frac{1}{2}\underline{v}\cdot\underline{v}+gz+\hat{U}+\frac{p}{\rho}\right]\right)=\underline{\nabla}\cdot\left[\underline{\underline{\tau}}\cdot\underline{v}\right]-\underline{\nabla}\cdot\underline{q}$$

$$\frac{d\left(m\hat{E}\right)}{dt}=\dot{m}_i\left[\frac{1}{2}\alpha_{in}\langle V_{in}\rangle^2+\hat{U}_{in}+g\,z_{in}+\frac{p_{in}}{\rho_{in}}\right]-\dot{m}_{out}\left[\frac{1}{2}\alpha_{out}\langle V_{out}\rangle^2+\hat{U}_{out}+g\,z_{out}+\frac{p_{out}}{\rho_{out}}\right]+\dot{Q}-\dot{W}$$

$$\alpha_i=\frac{\left\langle v_i^3\right\rangle}{\left\langle v_i\right\rangle^3}$$

Example No.9

For the 2-D flow verify if the velocity field: $\boxed{u = 2xy \quad v = -y^2}$

Symmetric

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{bmatrix}$$

a) Satisfies continuity equation.

b) Is rotational, if does, calculate angular velocity.

c) Describes a viscous flow, if does, calculate the strain rate.

d) Calculate the acceleration of a parcel.

$$\underline{\nabla}\,\underline{V} = \begin{bmatrix} \partial u/\partial x & \partial v/\partial x \\ \partial u/\partial y & \partial v/\partial y \end{bmatrix} \qquad \underline{\nabla}\,\underline{V} = \begin{bmatrix} 2y & 0 \\ 2x & -2y \end{bmatrix} \qquad [\underline{\nabla}\,\underline{V}]^T = \begin{bmatrix} 2y & 2x \\ 0 & -2y \end{bmatrix}$$

$$\dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = x \qquad \dot{\theta}_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -x$$

$$\underline{\nabla} \cdot \underline{v} = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} = 2y - 2y + 0 = 0$$

2-D

Shear-strain rate tensor

yes

$$\underline{\underline{\Gamma}} = \begin{bmatrix} 2y & x \\ x & -2y \end{bmatrix}$$

yes

Vorticity tensor

yes

$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$$

3-D

$$\underline{\underline{\Gamma}} = \begin{bmatrix} 2y & x & 0 \\ x & -2y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Skew-Symmetric

$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & \dot{\theta}_z & -\dot{\theta}_y \\ -\dot{\theta}_z & 0 & \dot{\theta}_x \\ \dot{\theta}_y & -\dot{\theta}_x & 0 \end{bmatrix}$$

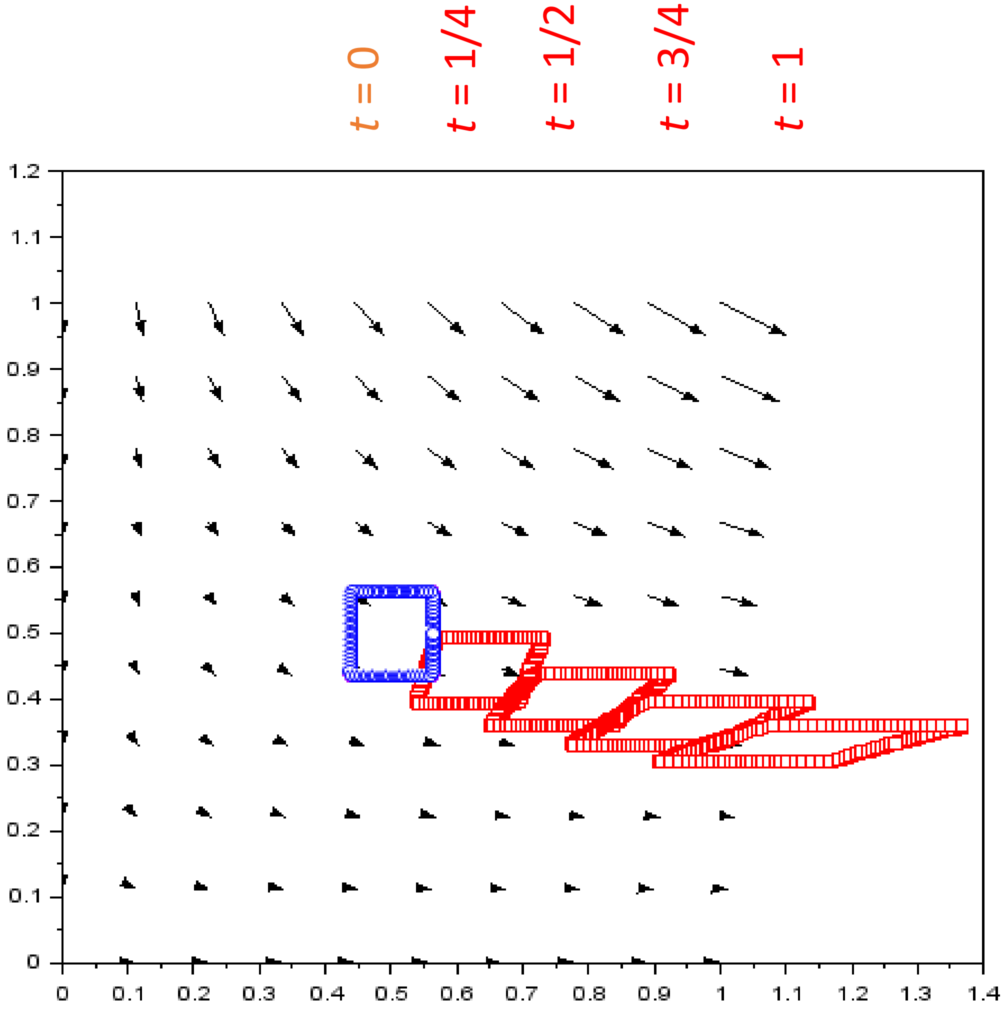
Jargon note:

$$\underline{\zeta} = \underline{\nabla} \times \underline{V}$$

Vorticity vector:

$$\underline{\zeta} = 2\underline{\omega} = 2\underline{\dot{\theta}}$$

$$\underline{\underline{\Omega}} = \underline{\underline{\underline{\varepsilon}}} \cdot \underline{\zeta}/2$$



$$\underline{a} = \underline{v} \cdot \underline{\nabla}(\underline{v}) = 2xy^2\hat{\imath} + 2y^3\hat{\jmath}$$

$$\frac{d\underline{v}}{dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}(\underline{v})$$

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

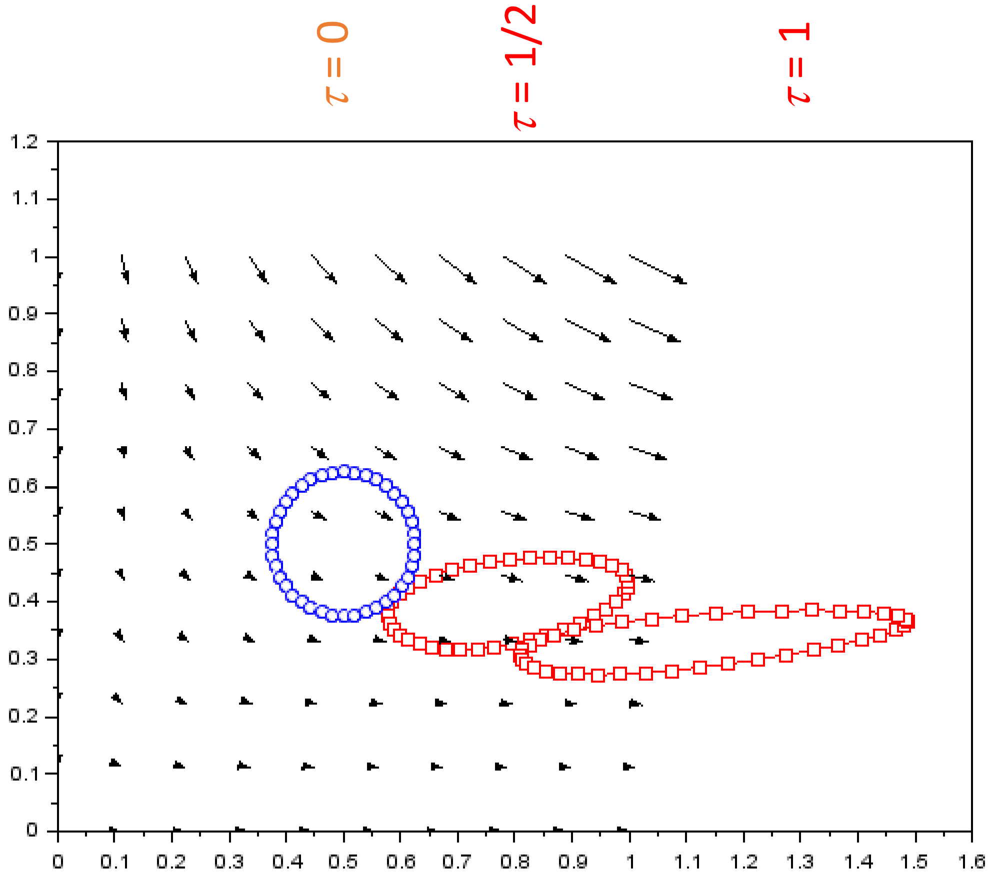
$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

$$a_x = 0 + 2xy(2y) - y^2(2x) + 0(0)$$

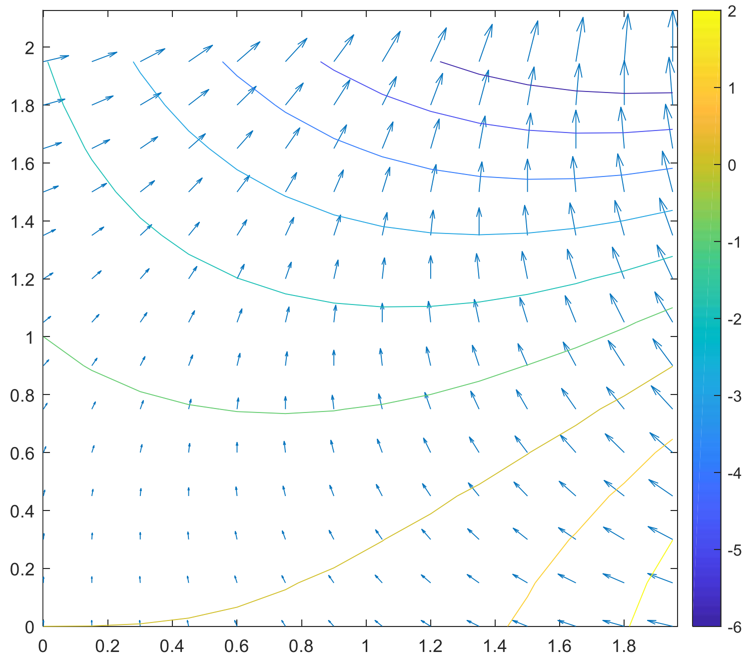
$$a_y = 0 + 2xy(0) - y^2(-2y) + 0(0)$$

$$a_z = 0 + 0(0) + 0(0) + 0(0)$$


$\tau = 0$
$\tau = 1/2$
$\tau = 1$

Trivial example to use matlab

```
[x,y] = meshgrid(0:0.15:2,0:0.15:2) ; %to create the points I want
to take a look at
 VectorX = [-x.^2+y.^2]; %is the Vector in the "x" direction
 VectorY = [+1+2*x.*y]; %is the Vector in the "y" direction
 % size(x)
 % size(y)
 % size(VectorX)
 % size(VectorY)
 V=sqrt(VectorX.^2+VectorY.^2);
 PHI=x.^3/3-y.^2.*x-y;
 [Dx,Dy]=gradient(V,0.2,0.2);
 quiver(x,y,VectorX,VectorY)
 %quiver(x,y,Dx,Dy)
 hold on
 %Look out this is not the potential is just the magnitud of the
vector
 contour(x,y,PHI);
 hold off
```



$$v_x = -\frac{\partial \Phi}{\partial x} \qquad v_y = -\frac{\partial \Phi}{\partial y}$$

Disclaimer: I strongly recommend to define velocity in terms of gradient consistent with the concept of driving potential

$$\underline{v} = -\underline{\nabla}\Phi$$

Disclaimer: for a very unreasonable cause fluid dynamics literature uses the opposite.

$$\underline{v} = \underline{\nabla}\Phi$$

Example No.10

Fluye aire a través de una tobera, y entra a una velocidad de 1 m/s donde el diámetro es de 0.1 m. Calcule:

a)El flujo volumétrico en m^3/s y másico en kg/s a las condiciones de entrada de 200 kPa y 300 K, donde la densidad es de 2.4 kg/m^3
b)A que presión deberé aumentar en la alimentación a la tobera, si se quiere duplicar el gasto másico (kg/s) manteniendo la temperatura y la velocidad a la entrada de la tobera.

$A = \pi D^2/4$

$A = 7.853e\text{-}3\ m^2 = 7853\ cm^2$

$(V)\ (A) = 7.853e\text{-}3\ m^3/s = 7.853\ dm^3/s$

$(\rho)\ (V)\ (A) = 1.885e\text{-}2\ kg/s = 18.85\ g/s$

$(\rho)\ (V)\ (A) = 2\ (\rho_0)\ (V)\ (A)$

Si deseamos duplicar el gasto manteniendo el área y velocidad constante, entonces la densidad deberá duplicarse. Si la densidad es proporcional a la presión entonces la presión deberá ser el doble.

$\rho = p / RT$ $\qquad \rho = 2\ \rho_0$

$p = 2\ p_0$ $\qquad p = 400\ kPa$

Example No.11

Una aeronave produce un flujo tipo remolino en la parte externa de las alas. Bajo ciertas circunstancias el flujo se puede aproximar por un campo de velocidad de la forma $u = - K\, y/ (x^2+y^2)$ y $v = K\, x / (x^2+y^2)$, donde K es una constante que depende de varios parámetros asociados con el aeronave (i.e. velocidad y peso), aquí "x" y "y" son medidos desde el centro del remolino. Demuestre:

a) Que la velocidad es inversamente proporcional a la distancia desde el origen, esto es $V= K/(x^2+y^2)^{1/2}$

b) Si el flujo obedece la ecuación de continuidad.

c) Si el flujo es viscoso.

d) Si este flujo es rotacional.

y
x

$u = - K\, y / (x^2+y^2)$ y $v = K\, x / (x^2+y^2)$

a) Que la velocidad es inversamente proporcional a la distancia desde el origen, esto es $V= K/(x^2+y^2)^{1/2}$

$V^2= (u^2+v^2)$ Velocidad absoluta es la suma vectorial de las velocidades en "x" y en "y", mismas que son ortogonales

$V^2 = K^2 x^2/ (x^2+y^2)^2 + K^2 y^2/ (x^2+y^2)^2= K^2 / (x^2+y^2)$

Entonces tenemos $V= K/(x^2+y^2)^{1/2}$

b) Si el flujo obedece la ecuación de continuidad.

c) Si el flujo es viscoso.

d) Si este flujo es rotacional.

$u = -K\,y/(x^2+y^2)$ y $v = K\,x/(x^2+y^2)$

a) Que la velocidad es inversamente proporcional a la distancia desde el origen, esto es $V = K/(x^2+y^2)^{1/2}$

Sacando las derivadas parciales facilitara el evaluar b) ,c) y d)

$\partial u / \partial x = [-2Kyx]/(x^2+y^2)^2$

$\partial v / \partial y = [2Kyx]/(x^2+y^2)^2$

$\partial u / \partial y = [(x^2+y^2)(-K)-(-Ky)(2y)]/(x^2+y^2)^2 = K(y^2-x^2)/(x^2+y^2)^2$

$\partial v / \partial x = [(x^2+y^2)(K)-(Kx)(2x)]/(x^2+y^2)^2 = K(y^2-x^2)/(x^2+y^2)^2$

b) Si el flujo obedece la ecuación de continuidad. (Si)

Continuidad: $\underline{\nabla} \bullet \underline{v} = 0$

$\partial u / \partial x + \partial v / \partial y = [-2Kyx]/(x^2+y^2)^2 + [2Kyx]/(x^2+y^2)^2 = 0$

Si cumple con ecuación de continuidad

$u = -K y / (x^2+y^2)$ y $v = K x / (x^2+y^2)$

a) Que la velocidad es inversamente proporcional a la distancia desde el origen, esto es $V = K/(x^2+y^2)^{1/2}$

Sacando las derivadas parciales facilitara el evaluar b) ,c) y d)

$\partial u / \partial x = [-2Kyx] / (x^2+y^2)^2$
$\partial v / \partial y = [2Kyx] / (x^2+y^2)^2$

$\partial u / \partial y = [(x^2+y^2)(-K) - (-Ky)(2y)] / (x^2+y^2)^2 = K(y^2-x^2) / (x^2+y^2)^2$
$\partial v / \partial x = [(x^2+y^2)(K) - (Kx)(2x)] / (x^2+y^2)^2 = K(y^2-x^2) / (x^2+y^2)^2$

c) Si el flujo es viscoso. (Si)
$\varepsilon_{xy} = (1/2)[\partial u / \partial y + \partial v / \partial x] = K(y^2-x^2) / (x^2+y^2)^2$

d) Si este flujo es rotacional. (No)
$d\theta_z/dt = (1/2)[\partial v / \partial x - \partial u / \partial y] = 0$

Basic Flow assumptions and Their Mathematical Statements

Flow assumption:	Consequence:
Time-dependence	$\frac{\partial B}{\partial t} = 0$ i.e. steady state; $\underline{v} = \underline{v}(t)$ i.e. transient flow
Dimensionality	Required number of space variables $\underline{x} = (x, y, z)$
Directionality	Required number of velocity components $\underline{v} = (u, v, w)$
Unidirectional flow	Special case when all but one velocity component are zero
Development phase	$\frac{\partial v}{\partial s} = 0$ i.e. fully developed flow, where s is the axial coordinate
Symmetry	$\left.\frac{\partial B}{\partial n}\right\rvert_{n=0} = 0$ Mid plane (n is the normal coordinate); $\frac{\partial B}{\partial \theta} = 0$ axisymmetric

Eulerian point of view = Control volume approach
Lagrangian point of view = System concept, system approach
(Robert A Granger, Fluid Mech)

Stationing ourselves at the origin of an inertial frame of reference we use the control volume approach

To observe the motion of fluid following the collection of particles or a single particle (or parcel, fluid element is a concept used in continuum theory) you identify the “system ” attaching a coordinate system to its center of mass and locating the observer at the origin of the moving frame of reference (system approach)

- Step 1: Problem Statement
- Step 2: Sketch (Diagram, process flow chart)
- Step 3: Assumptions and Approximations
- Step 4: Physical Laws (Fundamental Laws)
- Step 5: Properties (Thermodynamic relations)
- Step 6: Calculations
- Step 7: Reasoning (Sensitivity analysis, what if), Verification (context), and Discussion

1. Read **problem statement**, and collect the information that may be needed.
2. Make a **Sketch** (Diagram, process flow chart), indicating mass, linear or angular momentum (i.e. forces and torques) and energy interaction, and label each stream and boundaries as well.
3. List **Assumptions** and **Approximations** (sometimes they may be inferred by the sketch, but make them explicit) supported by equations if possible (geometric relationships, or fundamental equations).
4. **Physical Laws** (Fundamental Laws) must be written in full form, and terms can be dropped by the right selection of frame of reference, operating conditions, assumptions, simplifications or constraints.
5. **Physical constants** should be obtained from a reliable source (knowing this information by heart is always helpful), geometric relations and formulae must be included as part of your analysis.
6. **Physical** transport or thermodynamic **properties** (Thermodynamic relations) should be evaluated, approximated, calculated or obtained from a reliable source.
7. **Calculations** are done including units. Any algebraic manipulation is recommended in few cases, because limits the step 8, but if needed should be done before using numerical values of constants, properties or variables.
8. **Reasoning** (Sensitivity analysis, what if), Verification (context), and Discussion should always be part of your answer to any problem, regardless the task requested.

The engineer has to defend the idealizations he is imposing on the solution, and thus he must be part mathematician and part applied scientist. The engineer also has the added responsibility of ensuring that his design functions. It can not work only on paper. To make his design work, the engineer may have to relax some of the physical constraints of the real problem, or bypass some of the mathematical rigor, and of course, whatever idealizations are chosen, they must be applied to any solution with considerable caution.

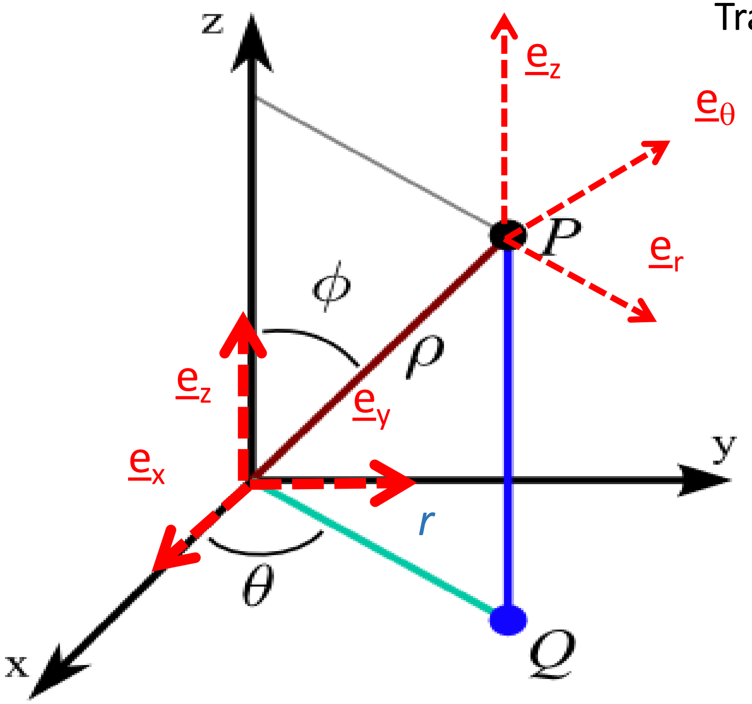
Cylindrical coordinates

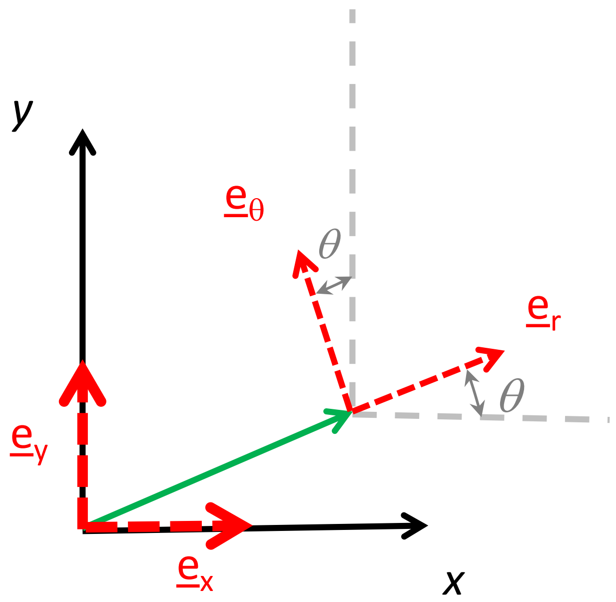
Cylindrical coordinates analysis for scalar, vector and tensors

Transformation of Gradient, Curl, Divergence and Laplacean from Cartesian Coordinates to cylindrical or polar coordinates.

Unit vector conversion

Transformation from polar to Cartesian coordinates





$$\rho^2 = r^2 + z^2 \qquad x = r\,Cos\theta$$

$$r^2 = x^2 + y^2 \qquad y = r\,Sin\theta$$

$$\underline{e}_r = Cos\theta\,\underline{e}_x + Sin\theta\,\underline{e}_y$$

$$\underline{e}_\theta = -Sin\theta\,\underline{e}_x + Cos\theta\,\underline{e}_y$$

$$\left\{ \underline{e}_r = Cos\theta\,\underline{e}_x + Sin\theta\,\underline{e}_y \right\} \text{ Sin } \theta \ \{\text{Cos } \theta\}$$

$$\left\{ \underline{e}_\theta = -Sin\theta\,\underline{e}_x + Cos\theta\,\underline{e}_y \right\} \text{ Cos } \theta \ \{\text{- Sin } \theta\}$$

$$\boxed{\underline{e}_x = Cos\theta\,\underline{e}_r - Sin\theta\,\underline{e}_\theta}$$

$$\boxed{\underline{e}_y = Sin\theta\,\underline{e}_r + Cos\theta\,\underline{e}_\theta}$$

Transformation from Cartesian to polar coordinates

From Trigonometry, some relations for ratio of change between polar and Cartesian coordinates

$$r^2 = x^2 + y^2 \qquad 2rdr = 2xdx + 2ydy$$

$$x = r\,Cos\theta \qquad \left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{x}{r} = Cos\theta$$

$$y = r\,Sin\theta \qquad \left(\frac{\partial r}{\partial y}\right)_{x,z} = \frac{y}{r} = Sin\theta$$

$$\theta = ArcTan\left(\frac{y}{x}\right) \qquad \left(\frac{\partial \theta}{\partial x}\right)_{y,z} = \frac{-y/x}{1+y^2/x^2} = -\frac{y}{r}\left(\frac{1}{r}\right) = -\left(\frac{Sin\theta}{r}\right)$$

$$\left(\frac{\partial \theta}{\partial y}\right)_{x,z} = \frac{1/x}{1+y^2/x^2} = \frac{x}{r}\left(\frac{1}{r}\right) = \frac{Cos\theta}{r}$$

For Cartesian coordinates (3D)

$$df = \left(\frac{\partial f}{\partial x}\right)_{y,z} dx + \left(\frac{\partial f}{\partial y}\right)_{x,z} dy + \left(\frac{\partial f}{\partial z}\right)_{x,y} dz$$

For polar coordinates

$$df = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} dr + \left(\frac{\partial f}{\partial \theta}\right)_{r,z} d\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} dz$$

Recasting the gradient from Cartesian to cylindrical coordinates

$$df = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} dr + \left(\frac{\partial f}{\partial \theta}\right)_{r,z} d\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} dz$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\right)_{y,z} \underline{e}_x + \left(\frac{\partial f}{\partial y}\right)_{x,z} \underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z}\left(\frac{\partial r}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{\partial \theta}{\partial x}\right)_{y,z} + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}\left(\frac{\partial z}{\partial x}\right)_{y,z}$$

$$\left(\frac{\partial f}{\partial x}\right)_{y,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} Cos\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(-\frac{Sin\theta}{r}\right) + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}(0)$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z}\left(\frac{\partial r}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{\partial \theta}{\partial y}\right)_{x,z} + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}\left(\frac{\partial z}{\partial y}\right)_{x,z}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,z} = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} Sin\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{Cos\theta}{r}\right) + \left(\frac{\partial f}{\partial z}\right)_{r,\theta}(0)$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\right)_{y,z} \underline{e}_x + \left(\frac{\partial f}{\partial y}\right)_{x,z} \underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z = \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Cos\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(-\frac{Sin\theta}{r}\right)\right]\underline{e}_x + \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Sin\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{Cos\theta}{r}\right)\right]\underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\underline{\nabla} f = \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Cos\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(-\frac{Sin\theta}{r}\right)\right]\left[Cos\theta\,\underline{e}_r - Sin\theta\,\underline{e}_\theta\right] + \left[\left(\frac{\partial f}{\partial r}\right)_{\theta,z} Sin\theta + \left(\frac{\partial f}{\partial \theta}\right)_{r,z}\left(\frac{Cos\theta}{r}\right)\right]\left[Sin\theta\,\underline{e}_r + Cos\theta\,\underline{e}_\theta\right] + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial x}\right)_{y,z} \underline{e}_x + \left(\frac{\partial f}{\partial y}\right)_{x,z} \underline{e}_y + \left(\frac{\partial f}{\partial z}\right)_{x,y} \underline{e}_z$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} \underline{e}_z$$

Unit vectors in Cartesian coordinates remain constant, but for cylindrical coordinates, they will depend on the evolution of the flow

$$\frac{d\,\underline{e}_x}{dt} = \frac{d\,\underline{e}_y}{dt} = \frac{d\,\underline{e}_z}{dt} = 0$$

$$\underline{e}_r = Cos\theta\,\underline{e}_x + Sin\theta\,\underline{e}_y$$

$$\frac{d\underline{e}_r}{dt} = -Sin\theta\,\underline{e}_x\frac{d\theta}{dt} + Cos\theta\,\underline{e}_y\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_r}{dt} = \underline{e}_\theta\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_r}{d\theta} = \underline{e}_\theta$$

$$\underline{e}_\theta = -Sin\theta\,\underline{e}_x + Cos\theta\,\underline{e}_y$$

$$\frac{d\underline{e}_\theta}{dt} = -Cos\theta\,\underline{e}_x\frac{d\theta}{dt} - Sin\theta\,\underline{e}_y\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_\theta}{dt} = -\underline{e}_r\frac{d\theta}{dt}$$

$$\frac{d\underline{e}_\theta}{d\theta} = -\underline{e}_r$$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} \underline{e}_z$$

Gradient

$$\underline{a} = a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_z \underline{e}_z$$

$$\underline{e}_r \cdot \underline{e}_r = \underline{e}_z \cdot \underline{e}_z = \underline{e}_\theta \cdot \underline{e}_\theta = 1$$

$$\underline{e}_r \cdot \underline{e}_\theta = \underline{e}_\theta \cdot \underline{e}_z = \underline{e}_z \cdot \underline{e}_r = 0$$

$$\frac{\partial \underline{e}_r}{\partial \theta} = \underline{e}_\theta \qquad \frac{\partial \underline{e}_\theta}{\partial \theta} = -\underline{e}_r$$

$$\frac{\partial(\underline{e}_r \underline{e}_\theta)}{\partial \theta} = \underline{e}_\theta \underline{e}_\theta - \underline{e}_r \underline{e}_r$$

$$\underline{\nabla}(\ ) = \left(\frac{\partial(\ )}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial(\ )}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial(\ )}{\partial z}\right)_{r,\theta} \underline{e}_z$$

$$\underline{\nabla} \cdot \underline{a} = \left(\frac{\partial a_r}{\partial r}\right)_{\theta,z} + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right)_{r,z} + \frac{a_r}{r} + \left(\frac{\partial a_z}{\partial z}\right)_{r,\theta}$$

Divergence

$$\frac{d(\underline{F}\,\underline{G})}{d\eta} = \frac{d(\underline{F})}{d\eta}\underline{G} + \underline{F}\frac{d(\underline{G})}{d\eta}$$

$$\underline{e}_r \times \underline{e}_\theta = \underline{e}_z \qquad \underline{e}_\theta \times \underline{e}_z = \underline{e}_r \qquad \underline{e}_z \times \underline{e}_r = \underline{e}_\theta$$

$$\underline{e}_r \times \underline{e}_r = \underline{e}_z \times \underline{e}_z = \underline{e}_\theta \times \underline{e}_\theta = 0 \qquad \underline{a} \times \underline{b} = -\underline{b} \times \underline{a}$$

$$\underline{\nabla} \times \underline{a} = \left[\underline{e}_r \left(\frac{\partial (\;)}{\partial r} \right)_{\theta,z} + \underline{e}_\theta \frac{1}{r} \left(\frac{\partial (\;)}{\partial \theta} \right)_{r,z} + \underline{e}_z \left(\frac{\partial (\;)}{\partial z} \right)_{r,\theta} \right] \times \left[a_r \underline{e}_r + a_\theta \underline{e}_\theta + a_z \underline{e}_z \right]$$

$$\underline{\nabla} \times \underline{a} = \left[\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \right] \underline{e}_r + \left[\frac{\partial a_r}{\partial z} - \frac{\partial a_z}{\partial r} \right] \underline{e}_\theta + \left[\frac{\partial a_\theta}{\partial r} - \frac{1}{r} \frac{\partial a_r}{\partial \theta} + \frac{a_\theta}{r} \right] \underline{e}_z$$

Curl of vector $\underline{a}$

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right)_{\theta,z} \underline{e}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)_{r,z} \underline{e}_\theta + \left(\frac{\partial f}{\partial z}\right)_{r,\theta} \underline{e}_z$$

$$\underline{\nabla} \cdot \underline{a} = \left(\frac{\partial a_r}{\partial r}\right)_{\theta,z} + \frac{1}{r}\left(\frac{\partial a_\theta}{\partial \theta}\right)_{r,z} + \frac{a_r}{r} + \left(\frac{\partial a_z}{\partial z}\right)_{r,\theta}$$

$$\underline{\nabla} \cdot \underline{\nabla} f = \nabla^2 f = \left(\frac{\partial^2 f}{\partial r^2}\right)_{\theta,z} + \frac{1}{r}\left(\frac{\partial f}{\partial r}\right)_{\theta,z} + \frac{1}{r^2}\left(\frac{\partial^2 f}{\partial \theta^2}\right)_{r,z} + \left(\frac{\partial^2 f}{\partial z^2}\right)_{r,\theta}$$

Divergence of the Gradient is called Laplacian

Divergence of Tensor

$$\underline{\nabla}\cdot\underline{\underline{T}}=\left[\underline{e}_r\left(\frac{\partial(\;)}{\partial r}\right)_{\theta,z}+\underline{e}_\theta\frac{1}{r}\left(\frac{\partial(\;)}{\partial\theta}\right)_{r,z}+\underline{e}_z\left(\frac{\partial(\;)}{\partial z}\right)_{r,\theta}\right]\cdot\begin{bmatrix}\underline{T_{rr}\underline{e}_r\underline{e}_r}+\underline{T_{r\theta}\underline{e}_r\underline{e}_\theta}+\underline{T_{rz}\underline{e}_r\underline{e}_z}\\ \underline{T_{\theta r}\underline{e}_\theta\underline{e}_r}+\underline{T_{\theta\theta}\underline{e}_\theta\underline{e}_\theta}+\underline{T_{\theta z}\underline{e}_\theta\underline{e}_z}\\ T_{zr}\underline{e}_z\underline{e}_r+T_{z\theta}\underline{e}_z\underline{e}_\theta+T_{zz}\underline{e}_z\underline{e}_z\end{bmatrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}}=\begin{bmatrix}\frac{\partial(T_{rr})}{\partial r}\underline{e}_r+\frac{\partial(T_{r\theta})}{\partial r}\underline{e}_\theta+\frac{\partial(T_{rz})}{\partial r}\underline{e}_z\\ \frac{1}{r}\frac{\partial(T_{\theta r})}{\partial\theta}\underline{e}_r+\frac{1}{r}\frac{\partial(T_{\theta\theta})}{\partial\theta}\underline{e}_\theta+\frac{1}{r}\frac{\partial(T_{\theta z})}{\partial\theta}\underline{e}_z\\ \frac{\partial(T_{zr})}{\partial z}\underline{e}_r+\frac{\partial(T_{z\theta})}{\partial z}\underline{e}_\theta+\frac{\partial(T_{zz})}{\partial z}\underline{e}_z\end{bmatrix}+\begin{matrix}\frac{\underline{e}_\theta}{r}\cdot T_{rr}\frac{\partial(\underline{e}_r\underline{e}_r)}{\partial\theta}+\frac{\underline{e}_\theta}{r}\cdot T_{r\theta}\frac{\partial(\underline{e}_r\underline{e}_\theta)}{\partial\theta}+\frac{\underline{e}_\theta}{r}\cdot T_{rz}\frac{\partial(\underline{e}_r\underline{e}_z)}{\partial\theta}\\ \frac{\underline{e}_\theta}{r}\cdot T_{\theta r}\frac{\partial(\underline{e}_\theta\underline{e}_r)}{\partial\theta}+\frac{\underline{e}_\theta}{r}\cdot T_{\theta\theta}\frac{\partial(\underline{e}_\theta\underline{e}_\theta)}{\partial\theta}+\frac{\underline{e}_\theta}{r}\cdot T_{\theta z}\frac{\partial(\underline{e}_\theta\underline{e}_z)}{\partial\theta}\end{matrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}}=\begin{bmatrix}\frac{\partial(T_{rr})}{\partial r}\underline{e}_r+\frac{\partial(T_{r\theta})}{\partial r}\underline{e}_\theta+\frac{\partial(T_{rz})}{\partial r}\underline{e}_z\\ \frac{1}{r}\frac{\partial(T_{\theta r})}{\partial\theta}\underline{e}_r+\frac{1}{r}\frac{\partial(T_{\theta\theta})}{\partial\theta}\underline{e}_\theta+\frac{1}{r}\frac{\partial(T_{\theta z})}{\partial\theta}\underline{e}_z\\ \frac{\partial(T_{zr})}{\partial z}\underline{e}_r+\frac{\partial(T_{z\theta})}{\partial z}\underline{e}_\theta+\frac{\partial(T_{zz})}{\partial z}\underline{e}_z\end{bmatrix}+\begin{matrix}\frac{\underline{e}_r}{r}T_{rr}+\frac{\underline{e}_\theta}{r}T_{r\theta}+\frac{\underline{e}_z}{r}T_{rz}\\ \frac{\underline{e}_\theta}{r}\cdot T_{\theta r}-\frac{\underline{e}_r}{r}\cdot T_{\theta\theta}\end{matrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}} = \begin{bmatrix} \dfrac{\partial(T_{rr})}{\partial r}\underline{e}_r + \dfrac{\partial(T_{r\theta})}{\partial r}\underline{e}_\theta + \dfrac{\partial(T_{rz})}{\partial r}\underline{e}_z \\ \dfrac{1}{r}\dfrac{\partial(T_{\theta r})}{\partial \theta}\underline{e}_r + \dfrac{1}{r}\dfrac{\partial(T_{\theta\theta})}{\partial \theta}\underline{e}_\theta + \dfrac{1}{r}\dfrac{\partial(T_{\theta z})}{\partial \theta}\underline{e}_z \\ \dfrac{\partial(T_{zr})}{\partial z}\underline{e}_r + \dfrac{\partial(T_{z\theta})}{\partial z}\underline{e}_\theta + \dfrac{\partial(T_{zz})}{\partial z}\underline{e}_z \end{bmatrix} + \begin{matrix} \dfrac{\underline{e}_r}{r}T_{rr} + \dfrac{\underline{e}_\theta}{r}T_{r\theta} + \dfrac{\underline{e}_z}{r}T_{rz} \\ \dfrac{\underline{e}_\theta}{r}\cdot T_{\theta r} - \dfrac{\underline{e}_r}{r}\cdot T_{\theta\theta} \end{matrix}$$

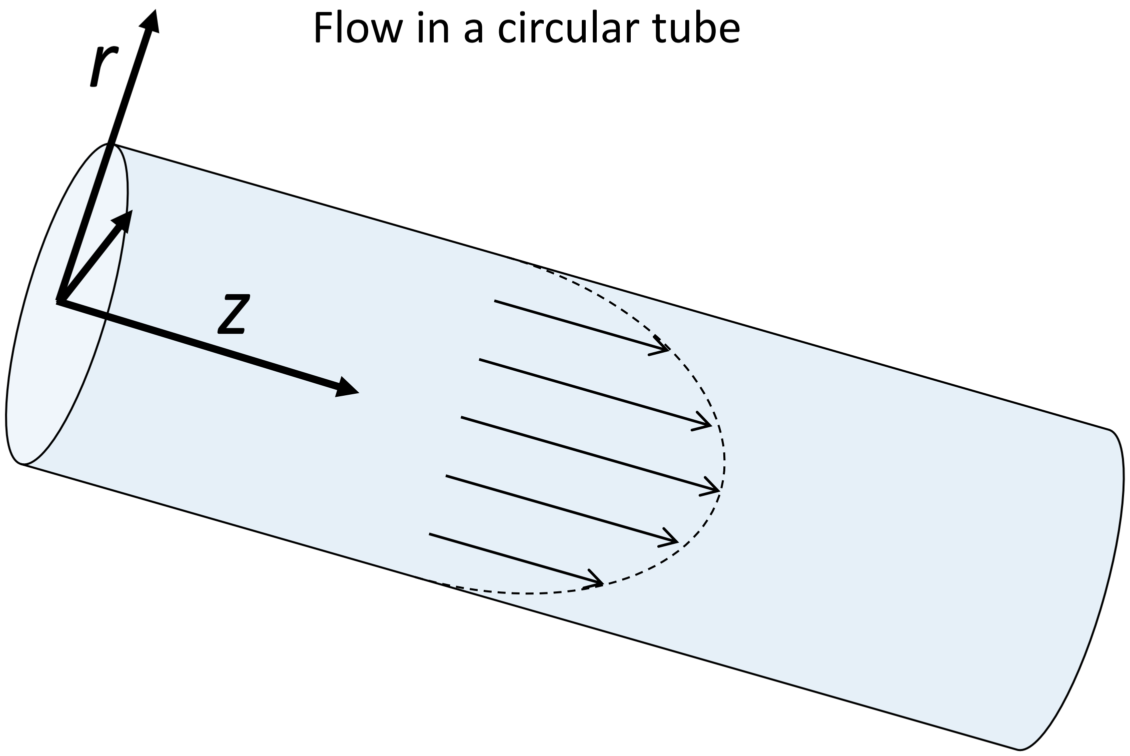
$$\underline{\nabla}\cdot\underline{\underline{T}} = \begin{matrix} \left[\dfrac{\partial(T_{rr})}{\partial r} + \dfrac{1}{r}\dfrac{\partial(T_{\theta r})}{\partial \theta} + \dfrac{\partial(T_{zr})}{\partial z} + \dfrac{1}{r}T_{rr} - \dfrac{1}{r}\cdot T_{\theta\theta}\right]\underline{e}_r \\ \left[\dfrac{\partial(T_{r\theta})}{\partial r} + \dfrac{1}{r}\dfrac{\partial(T_{\theta\theta})}{\partial \theta} + \dfrac{\partial(T_{z\theta})}{\partial z} + \dfrac{1}{r}T_{r\theta} + \dfrac{1}{r}\cdot T_{\theta r}\right]\underline{e}_\theta \\ \left[\dfrac{\partial(T_{rz})}{\partial r} + \dfrac{1}{r}\dfrac{\partial(T_{\theta z})}{\partial \theta} + \dfrac{\partial(T_{zz})}{\partial z} + \dfrac{1}{r}T_{rz}\right]\underline{e}_z \end{matrix}$$

Divergence of Tensor

Operator	Name
$\underline{\nabla} f$	Grad of f, Gradient
$\underline{\nabla} \cdot \underline{a}$	Div of a, Divergence
$\underline{\nabla} \times \underline{a}$	Curl of a, rotor or rotational
$\underline{\nabla}\underline{a}$	Dyad of a
$\underline{\nabla} \cdot \underline{\nabla} f = \nabla^2 f$	Laplacian of f
$\underline{\nabla} \cdot \underline{\underline{T}}$	Div of a tensor

$$
\begin{aligned}
\underline{\nabla}\,\underline{a} = &\quad \underline{e}_r\underline{e}_r\,\frac{\partial a_r}{\partial r} + \qquad \underline{e}_r\underline{e}_\theta\,\frac{\partial a_\theta}{\partial r} \qquad + \underline{e}_r\underline{e}_z\,\frac{\partial a_z}{\partial r} \\
&+ \underline{e}_\theta\underline{e}_r\left(\frac{1}{r}\frac{\partial a_r}{\partial \theta} - \frac{a_\theta}{r}\right) + \quad \underline{e}_\theta\underline{e}_\theta\left(\frac{1}{r}\frac{\partial a_\theta}{\partial \theta} - \frac{a_\theta}{r}\right) + \quad \underline{e}_\theta\underline{e}_z\left(\frac{1}{r}\frac{\partial a_z}{\partial \theta}\right) \\
&\quad + \underline{e}_z\underline{e}_r\left(\frac{\partial a_r}{\partial z}\right) \qquad + \underline{e}_z\underline{e}_\theta\left(\frac{\partial a_\theta}{\partial z}\right) + \qquad \underline{e}_z\underline{e}_z\left(\frac{\partial a_z}{\partial z}\right)
\end{aligned}
$$

Flow in a circular tube



Analysis will be done using steady flow, and steady state condition for an incompressible fluid

Continuity and linear momentum equation

$$\frac{\partial(\rho)}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v})=0$$

$$\frac{\partial(\rho)}{\partial t}+\frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta}+\frac{\partial(\rho V_z)}{\partial z}=0$$

Incompressible

No radial flow

Axis-symmetric

$$\frac{\partial(\rho V_z)}{\partial z}=\rho\frac{\partial(V_z)}{\partial z}=0$$

Axial flow

From continuity equation, we conclude that the velocity in axial direction is not function of the axial-coordinate

linear momentum analysis in axial direction

$$\frac{\partial(\rho\underline{v})}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v})=\rho\underline{g}+\underline{\nabla}\cdot\underline{\underline{\tau}}-\underline{\nabla}p$$

$$\begin{bmatrix}\frac{\partial(\rho V_r)}{\partial t}\underline{e}_r\\ \frac{\partial(\rho V_\theta)}{\partial t}\underline{e}_\theta\\ \frac{\partial(\rho V_z)}{\partial t}\underline{e}_z\end{bmatrix}+\underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v})=\begin{bmatrix}\rho g_r\underline{e}_r\\ \rho g_\theta\underline{e}_\theta\\ \rho g_z\underline{e}_z\end{bmatrix}+\underline{\nabla}\cdot\underline{\underline{\tau}}-\begin{bmatrix}\frac{\partial p}{\partial r}\underline{e}_r\\ \frac{1}{r}\frac{\partial p}{\partial \theta}\underline{e}_\theta\\ \frac{\partial p}{\partial z}\underline{e}_z\end{bmatrix}$$

$$\underline{\nabla}\cdot(\rho\underline{V}\underline{V})=\begin{matrix}\left[\frac{\partial(\rho V_rV_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta V_r)}{\partial\theta}+\frac{\partial(\rho V_zV_r)}{\partial z}+\frac{\rho V_rV_r}{r}-\frac{\rho V_\theta V_\theta}{r}\right]\underline{e}_r\\ \left[\frac{\partial(\rho V_rV_\theta)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta V_\theta)}{\partial\theta}+\frac{\partial(\rho V_zV_\theta)}{\partial z}+\frac{\rho V_rV_\theta}{r}+\frac{\rho V_\theta V_r}{r}\right]\underline{e}_\theta\\ \left[\frac{\partial(\rho V_rV_z)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta V_z)}{\partial\theta}+\frac{\partial(\rho V_zV_z)}{\partial z}+\frac{\rho V_rV_z}{r}\right]\underline{e}_z\end{matrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}}=\begin{matrix}\left[\frac{\partial(T_{rr})}{\partial r}+\frac{1}{r}\frac{\partial(T_{\theta r})}{\partial\theta}+\frac{\partial(T_{zr})}{\partial z}+\frac{1}{r}T_{rr}-\frac{1}{r}\cdot T_{\theta\theta}\right]\underline{e}_r\\ \left[\frac{\partial(T_{r\theta})}{\partial r}+\frac{1}{r}\frac{\partial(T_{\theta\theta})}{\partial\theta}+\frac{\partial(T_{z\theta})}{\partial z}+\frac{1}{r}T_{r\theta}+\frac{1}{r}\cdot T_{\theta r}\right]\underline{e}_\theta\\ \left[\frac{\partial(T_{rz})}{\partial r}+\frac{1}{r}\frac{\partial(T_{\theta z})}{\partial\theta}+\frac{\partial(T_{zz})}{\partial z}+\frac{1}{r}T_{rz}\right]\underline{e}_z\end{matrix}$$

$$\begin{bmatrix} \frac{\partial(\rho V_r)}{\partial t}\underline{e}_r \\ \frac{\partial(\rho V_\theta)}{\partial t}\underline{e}_\theta \\ \frac{\partial(\rho V_z)}{\partial t}\underline{e}_z \end{bmatrix} + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \begin{bmatrix} \rho g_r \underline{e}_r \\ \rho g_\theta \underline{e}_\theta \\ \rho g_z \underline{e}_z \end{bmatrix} + \underline{\nabla}\cdot\underline{\underline{\tau}} - \begin{bmatrix} \frac{\partial p}{\partial r}\underline{e}_r \\ \frac{1}{r}\frac{\partial p}{\partial \theta}\underline{e}_\theta \\ \frac{\partial p}{\partial z}\underline{e}_z \end{bmatrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}} = \begin{matrix} \left[\frac{\partial(T_{rr})}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta r})}{\partial \theta} + \frac{\partial(T_{zr})}{\partial z} + \frac{1}{r}T_{rr} - \frac{1}{r}\cdot T_{\theta\theta}\right]\underline{e}_r \\ \left[\frac{\partial(T_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta\theta})}{\partial \theta} + \frac{\partial(T_{z\theta})}{\partial z} + \frac{1}{r}T_{r\theta} + \frac{1}{r}\cdot T_{\theta r}\right]\underline{e}_\theta \\ \left[\frac{\partial(T_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta z})}{\partial \theta} + \frac{\partial(T_{zz})}{\partial z} + \frac{1}{r}T_{rz}\right]\underline{e}_z \end{matrix}$$

For axial direction , knowing that radial and angular velocity is zero, for steady flow, steady state and incompressible fluid.

$$\frac{\partial(\rho V_z)}{\partial t} + \left[\frac{\partial(\rho V_r V_z)}{\partial r} + \frac{1}{r}\frac{\partial(\rho V_\theta V_z)}{\partial \theta} + \frac{\partial(\rho V_z V_z)}{\partial z} + \frac{\rho V_r V_z}{r}\right] = \rho g_z + \frac{\partial(T_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta z})}{\partial \theta} + \frac{\partial(T_{zz})}{\partial z} + \frac{1}{r}T_{rz} - \frac{\partial p}{\partial z}$$

$$\frac{\partial(\rho V_z)}{\partial t} + \left[\frac{\partial(\rho V_r V_z)}{\partial r} + \frac{1}{r}\frac{\partial(\rho V_\theta V_z)}{\partial \theta} + \frac{\partial(\rho V_z V_z)}{\partial z} + \frac{\rho V_r V_z}{r}\right] = \rho g_z + \frac{1}{r}\frac{\partial(rT_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta z})}{\partial \theta} + \frac{\partial(T_{zz})}{\partial z} - \frac{\partial p}{\partial z}$$

$$0 = \rho g_z + \frac{1}{r}\frac{\partial(rT_{rz})}{\partial r} + \frac{1}{r}\frac{\partial(T_{\theta z})}{\partial \theta} + \frac{\partial(T_{zz})}{\partial z} - \frac{\partial p}{\partial z}$$

$$0 = \rho g_z + \frac{1}{r}\frac{\partial(rT_{rz})}{\partial r} - \frac{\partial p}{\partial z}$$

Only three terms are present under these conditions

Integrating in radial direction

$$0 = \rho g_z + \frac{1}{r}\frac{\partial(rT_{rz})}{\partial r} - \frac{\partial p}{\partial z}$$

The flow potential is expressed in terms of α

$$\alpha = \rho g_z - \frac{\partial p}{\partial z} \qquad \alpha = \frac{\Delta p}{L}$$

If gravity is negligible

$$\frac{1}{r}\frac{\partial(rT_{rz})}{\partial r} = -\alpha$$

Integrating for stress

At the center the stress is zero, this is also called symmetric boundary condition

$$\frac{d(rT_{rz})}{dr} = -\alpha\, r \qquad rT_{rz} = -\frac{\alpha\, r^2}{2} + C_1$$

for r=0 C_1=0

Using power law constitutive equation for shear stress, the velocity profile can be obtained

$$rT_{rz} = -\frac{\alpha\, r^2}{2} \qquad T_{rz} = -\frac{\alpha\, r}{2} \qquad T_{rz} = -\frac{\alpha\, r}{2} = K\left(-\frac{dV_z}{dr}\right)\left(\left|-\frac{dV_z}{dr}\right|\right)^{n-1}$$

Integrating once more to get an expression for velocity

$$\left(\frac{\alpha\, r}{2K}\right)^{1/n} = -\frac{dV_z}{dr} \qquad -\left(\frac{\alpha\, r}{2K}\right)^{1/n} dr = dV_z \qquad V_z = -\left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} r^{1+1/n} + C_2$$

Note: See that is not trivial the use of the correct sign in the relationship between shear rate and stress

The boundary condition at the wall, known as no slip boundary condition is used to estimate the integration constant..

$$V_z = -\left(\frac{n}{n+1}\right)\left(\frac{\alpha\, r}{2K}\right)^{1/n} r^{1+1/n} + C_2$$

$$V_z = \left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} (R)^{1+1/n}\left[1-\left(\frac{r}{R}\right)^{1+1/n}\right]$$

Integrating the velocity to express the result in terms of average velocity or measurable quantities.

$$\langle V_z\rangle \pi R^2 = \int_0^R \left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} R^{1+1/n}\left[1-\left(\frac{r}{R}\right)^{1+1/n}\right](2\pi r)dr$$

$$\frac{\langle V_z\rangle}{2} = \int_0^1 \left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} R^{1+1/n}\left[1-(\chi)^{1+1/n}\right](\chi)d\chi$$

$$\frac{\langle V_z\rangle}{2} = \left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} R^{1+1/n}\int_0^1 \left[\chi-(\chi)^{2+1/n}\right]d\chi$$

$$\frac{\langle V_z\rangle}{2} = \left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} R^{1+1/n}\left[\frac{\chi^2}{2}-\frac{(\chi)^{3+1/n}}{3+1/n}\right]_0^1$$

$$\frac{\langle V_z\rangle}{2} = \left(\frac{n}{n+1}\right)\left(\frac{\alpha}{2K}\right)^{1/n} R^{1+1/n}\left[\frac{n+1}{2(3n+1)}\right]$$

$$\langle V_z\rangle = \left(\frac{\alpha}{2K}\right)^{1/n} R^{1+1/n}\left[\frac{n}{(3n+1)}\right]$$

Equation is re casted in terms of pressure drop

$$\langle V_z\rangle = \left(\frac{\alpha R}{2K}\right)^{1/n} R\left[\frac{n}{(3n+1)}\right]$$

$$\left[\frac{(3n+1)}{nR}\langle V_z\rangle\right]^n = \left(\frac{\alpha\, R}{2K}\right)$$

$$\frac{\Delta P}{L} = \frac{2K}{R}\left[\frac{(3n+1)}{nR}\langle V_z\rangle\right]^n$$

$$\frac{\Delta P}{\rho} = 4L\frac{K}{2\rho R}\left[\frac{(3n+1)}{nR}\langle V_z\rangle\right]^n$$

$$\frac{\Delta P}{\rho} = 4\left(\frac{L}{D}\right)\frac{K}{\rho}\left[\frac{(3n+1)}{nR}\langle V_z\rangle\right]^n$$

Written the result in a general form

$$\frac{\Delta P}{\rho} = 4\left(\frac{L}{D}\right)\frac{K}{\rho}\left[\frac{(3n+1)}{nR}\langle V_z\rangle\right]^n$$

$$\frac{\Delta P}{\rho} = 4\left(\frac{L}{D}\right)\frac{K}{\rho}\left[2\left[\frac{(3n+1)}{nR}\right]^n\right]\langle V_z\rangle^{n-2}\left[\frac{\langle V_z\rangle^2}{2}\right]$$

$$\frac{\Delta P}{\rho} = 4\left(\frac{L}{D}\right)\frac{16K}{2^3\rho\langle V_z\rangle^{2-n}R^n}\left[\left[\frac{(3n+1)}{n}\right]^n\right]\left[\frac{\langle V_z\rangle^2}{2}\right]$$

This expression is known as generalized Reynolds Number.

$$\frac{\Delta P}{\rho} = 4\left(\frac{L}{D}\right)\left[\frac{16}{\mathrm{Re}_{GEN}}\right]\left[\frac{\langle V_z\rangle^2}{2}\right]$$

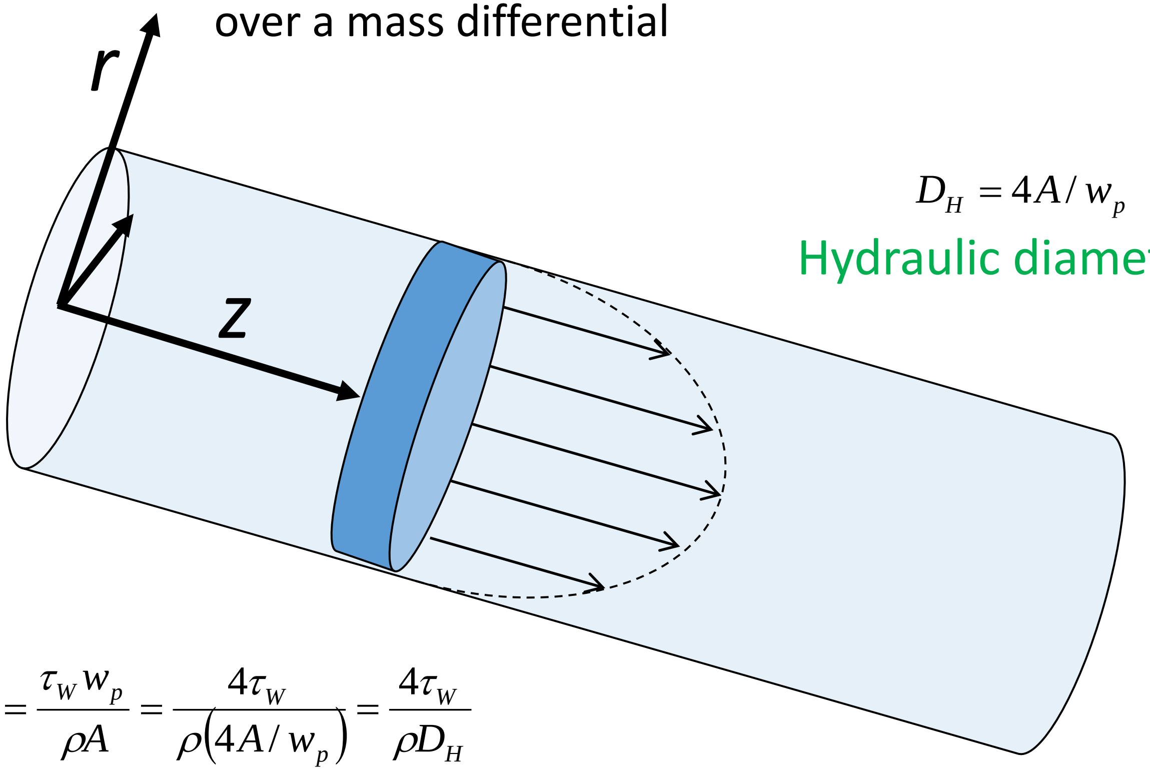
$$\boxed{\mathrm{Re}_{GEN} = \frac{2^3\rho\langle V_z\rangle^{2-n}R^n}{K}\left[\frac{n}{(3n+1)}\right]^n}$$

$$\boxed{\frac{\Delta P}{\rho} = 4\left(\frac{L}{D}\right)f_F\left[\frac{\langle V_z\rangle^2}{2}\right]}$$

$$f_F = \left[\frac{16}{\mathrm{Re}_{GEN}}\right]\Psi \qquad \Psi = 1$$

The multiplicand accounts for the hydrodynamics while the multiplier for geometry.

To quantify the energy loss, consider the friction force exerted over a mass differential



$$D_H = 4A / w_p$$

Hydraulic diameter

$$dF_F = \tau_W w_p dL$$

$$\hat{F}_F = \frac{dF_F}{dm} = \frac{\tau_W w_p dL}{\rho A dL} = \frac{\tau_W w_p}{\rho A} = \frac{4\tau_W}{\rho (4A / w_p)} = \frac{4\tau_W}{\rho D_H}$$

Energy loss per unit mass:

$$d\hat{E}_f = \hat{F}_F dl = \frac{4\tau_W}{\rho}\left(\frac{dl}{D_H}\right)$$

$$f_F = \frac{\tau_W}{\frac{1}{2}\rho \langle V \rangle^2}$$

Fanning Friction factor

$$f_D = 4 f_F = \frac{8\tau_W}{\rho \langle V \rangle^2}$$

Darcy Friction factor

Specific Energy loss

$$\boxed{\hat{E}_f = 4 f_F \left(\frac{1}{2}\langle V \rangle^2\right)\left(\frac{L}{D_H}\right)}$$

For compressible Fluid in micro-channels

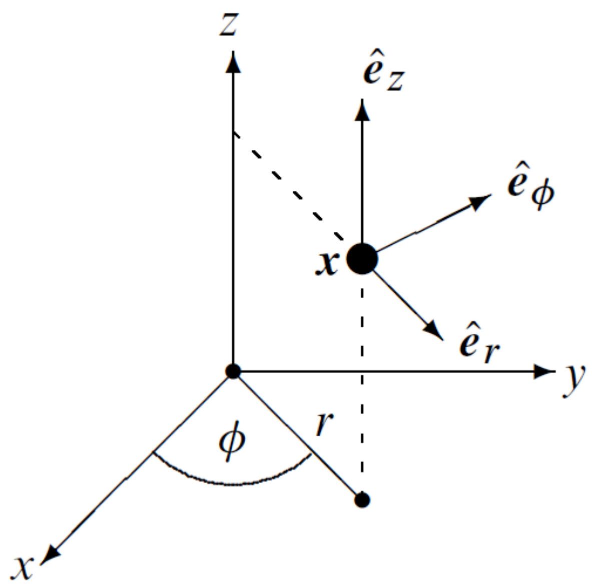
$$\frac{Re}{16\,\psi\,(L/D)\,Eu} = \frac{2}{1 + \sqrt{1 + 2\,\alpha}}$$

$$\alpha = \frac{16\,\psi\,(L/D)\,[\kappa_T\,\rho\,v^2/2]}{Re}$$

$$Re = \frac{\rho\,v\,D}{\mu} \qquad \frac{1}{Eu} = 2\,\Delta p\,/(\rho\,v^2)$$

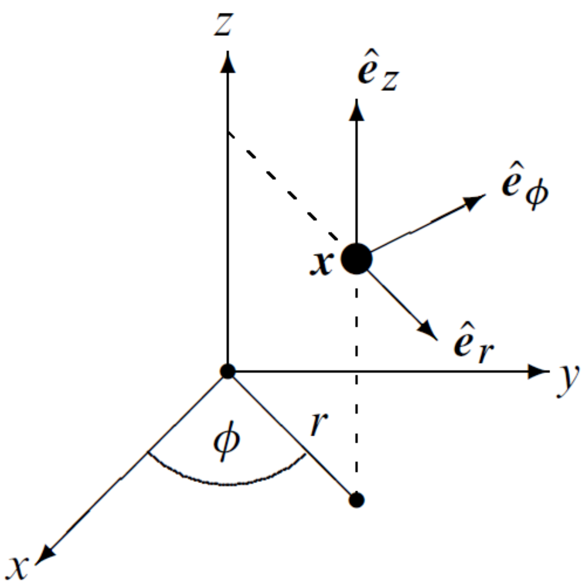
Math refresher revisited

If you want to use ϕ instead of θ



$$\nabla \boldsymbol{U} = \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_r\frac{\partial U_r}{\partial r} + \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_\phi\frac{\partial U_\phi}{\partial r} + \hat{\boldsymbol{e}}_r\hat{\boldsymbol{e}}_z\frac{\partial U_z}{\partial r}$$
$$+ \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_r\left(\frac{1}{r}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r}\right) + \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_\phi\left(\frac{1}{r}\frac{\partial U_\phi}{\partial \phi} + \frac{U_r}{r}\right) + \hat{\boldsymbol{e}}_\phi\hat{\boldsymbol{e}}_z\frac{1}{r}\frac{\partial U_z}{\partial \phi}$$
$$+ \hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_r\frac{\partial U_r}{\partial z} + \hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_\phi\frac{\partial U_\phi}{\partial z} + \hat{\boldsymbol{e}}_z\hat{\boldsymbol{e}}_z\frac{\partial U_z}{\partial z}$$

If you want to use ϕ instead of θ



$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} = (\cos\phi, \sin\phi, 0) \ ,$$
$$\boldsymbol{e}_\phi = \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0) \ ,$$
$$\boldsymbol{e}_z = \frac{\partial \boldsymbol{x}}{\partial z} = (0, 0, 1) \ .$$

$$\frac{\partial \boldsymbol{e}_r}{\partial \phi} = \boldsymbol{e}_\phi \ ,$$
$$\frac{\partial \boldsymbol{e}_\phi}{\partial \phi} = -\boldsymbol{e}_r \ ,$$

$$\boldsymbol{V} = \boldsymbol{e}_r V_r + \boldsymbol{e}_\phi V_\phi + \boldsymbol{e}_z V_z \ ,$$

$$\boldsymbol{\nabla} = \boldsymbol{e}_r \nabla_r + \boldsymbol{e}_\phi \nabla_\phi + \boldsymbol{e}_z \nabla_z = \boldsymbol{e}_r \frac{\partial}{\partial r} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial \phi} + \boldsymbol{e}_z \frac{\partial}{\partial z}$$

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

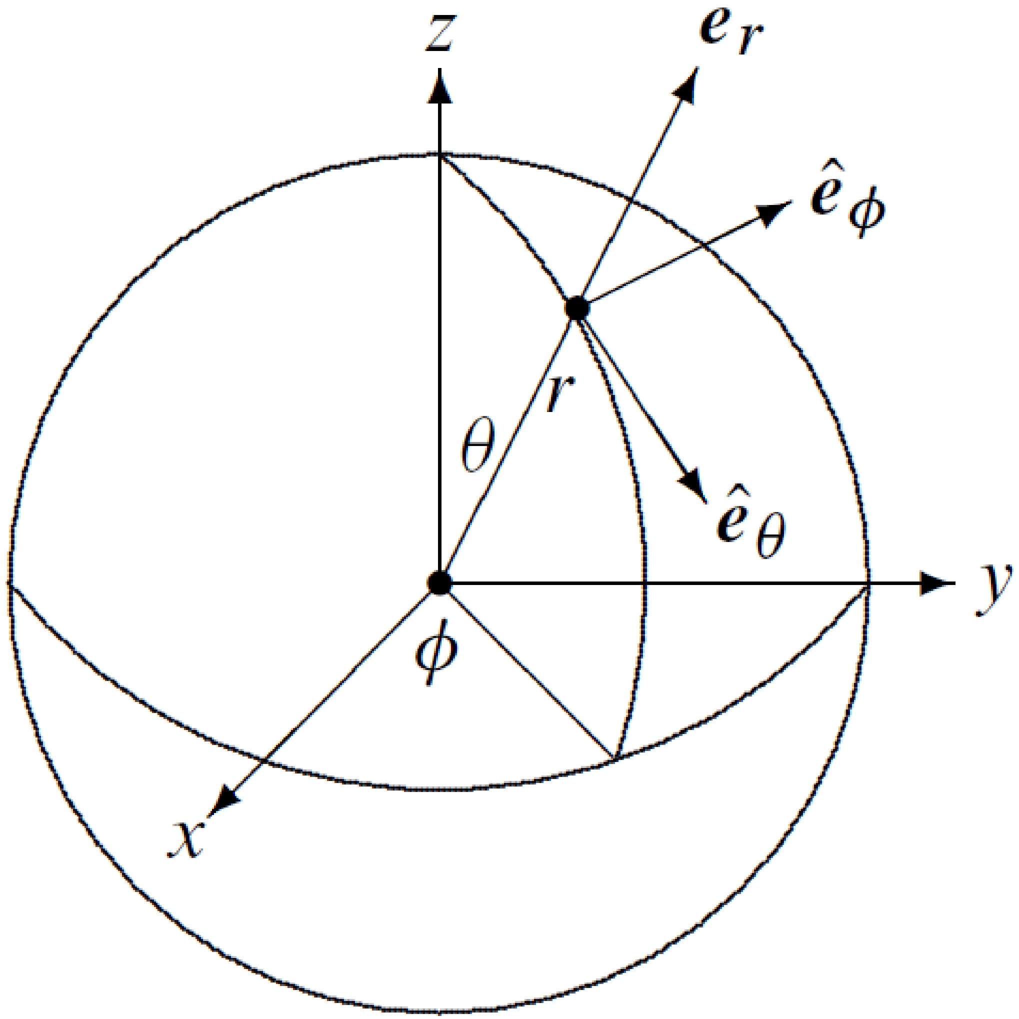
$$
\begin{aligned}
(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = \hat{\boldsymbol{e}}_r &\left(U_r\frac{\partial U_r}{\partial r} + \frac{U_\phi}{r}\frac{\partial U_r}{\partial \phi} + U_z\frac{\partial U_r}{\partial z} - \frac{U_\phi^2}{r}\right) \\
+\, \hat{\boldsymbol{e}}_\phi &\left(U_r\frac{\partial U_\phi}{\partial r} + \frac{U_\phi}{r}\frac{\partial U_\phi}{\partial \phi} + U_z\frac{\partial U_\phi}{\partial z} + \frac{U_r U_\phi}{r}\right) \\
+\, \hat{\boldsymbol{e}}_z &\left(U_z\frac{\partial U_z}{\partial r} + \frac{U_\phi}{r}\frac{\partial U_z}{\partial \phi} + U_z\frac{\partial U_z}{\partial z}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\cdot\boldsymbol{T} = \hat{\boldsymbol{e}}_r &\left(\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi r}}{\partial \phi} + \frac{\partial T_{zr}}{\partial z} + \frac{T_{rr}}{r} - \frac{T_{\phi\phi}}{r}\right) \\
+\, \hat{\boldsymbol{e}}_\phi &\left(\frac{\partial T_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\partial T_{z\phi}}{\partial z} + \frac{T_{r\phi}}{r} + \frac{T_{\phi r}}{r}\right) \\
+\, \hat{\boldsymbol{e}}_z &\left(\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{rz}}{r}\right)
\end{aligned}
$$

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \phi^2} + \frac{\partial^2 S}{\partial z^2} + \frac{1}{r}\frac{\partial S}{\partial r}.$$

$$\begin{aligned}
\nabla^2 \boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r \left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_r}{\partial \phi^2} + \frac{\partial^2 U_r}{\partial z^2} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{2}{r^2}\frac{\partial U_\phi}{\partial \phi} - \frac{U_r}{r^2} \right) \\
& + \hat{\boldsymbol{e}}_\phi \left(\frac{\partial^2 U_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \phi^2} + \frac{\partial^2 U_\phi}{\partial z^2} + \frac{1}{r}\frac{\partial U_\phi}{\partial r} + \frac{2}{r^2}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r^2} \right) \\
& + \hat{\boldsymbol{e}}_z \left(\frac{\partial^2 U_z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_z}{\partial \phi^2} + \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r}\frac{\partial U_z}{\partial r} \right)
\end{aligned}$$

$$
\begin{aligned}
\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{U}) = \hat{\boldsymbol{e}}_r &\left(\frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U_\phi}{\partial r\partial\phi} + \frac{\partial^2 U_z}{\partial r\partial z} + \frac{1}{r}\frac{\partial U_r}{\partial r} - \frac{1}{r^2}\frac{\partial U_\phi}{\partial\phi} - \frac{U_r}{r^2}\right)\\
+ \hat{\boldsymbol{e}}_\phi &\left(\frac{1}{r}\frac{\partial^2 U_r}{\partial\phi\partial r} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial\phi^2} + \frac{1}{r}\frac{\partial^2 U_z}{\partial\phi\partial z} + \frac{1}{r^2}\frac{\partial U_r}{\partial\phi}\right)\\
+ \hat{\boldsymbol{e}}_z &\left(\frac{\partial^2 U_r}{\partial z\partial r} + \frac{1}{r}\frac{\partial^2 U_\phi}{\partial z\partial\phi} + \frac{\partial^2 U_z}{\partial z^2} + \frac{1}{r}\frac{\partial U_r}{\partial z}\right)
\end{aligned}
$$



z
$\hat{e}_r$
$\hat{e}_\phi$
r
θ
$\hat{e}_\theta$
y
ϕ
x

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

$$r = \sqrt{x^2+y^2+z^2}, \qquad \theta = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}, \qquad \phi = \arctan\frac{y}{x}$$

$$\begin{aligned}
\hat{\boldsymbol{e}}_r &= \frac{\partial \boldsymbol{x}}{\partial r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\
\hat{\boldsymbol{e}}_\theta &= \frac{1}{r}\frac{\partial \boldsymbol{x}}{\partial \theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \\
\hat{\boldsymbol{e}}_\phi &= \frac{1}{r\sin\theta}\frac{\partial \boldsymbol{x}}{\partial \phi} = (-\sin\phi, \cos\phi, 0).
\end{aligned}$$

$$\boldsymbol{U} = \hat{\boldsymbol{e}}_r U_r + \hat{\boldsymbol{e}}_\theta U_\theta + \hat{\boldsymbol{e}}_\phi U_\phi$$

$$\begin{aligned}
d\boldsymbol{\ell} &= \hat{\boldsymbol{e}}_r\, dr + \hat{\boldsymbol{e}}_\theta\, r d\theta + \hat{\boldsymbol{e}}_\phi\, r \sin\theta\, d\phi, \\
d\boldsymbol{S} &= \hat{\boldsymbol{e}}_r\, r^2 \sin\theta\, d\theta\, d\phi + \hat{\boldsymbol{e}}_\theta r \sin\theta d\phi dr + \hat{\boldsymbol{e}}_\phi\, r dr d\theta, \\
dV &= r^2 \sin\theta\, dr d\theta d\phi.
\end{aligned}$$

$$\nabla = \hat{\boldsymbol{e}}_r \nabla_r + \hat{\boldsymbol{e}}_\theta \nabla_\theta + \hat{\boldsymbol{e}}_\phi \nabla_\phi = \hat{\boldsymbol{e}}_r \frac{\partial}{\partial r} + \hat{\boldsymbol{e}}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{e}}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi},$$

$$\frac{\partial \hat{\boldsymbol{e}}_r}{\partial \theta} = \hat{\boldsymbol{e}}_\theta, \qquad \frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial \theta} = -\hat{\boldsymbol{e}}_r,$$

$$\frac{\partial \hat{\boldsymbol{e}}_\theta}{\partial \phi} = \cos\theta\, \hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial \hat{\boldsymbol{e}}_r}{\partial \phi} = \sin\theta\, \hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial \hat{\boldsymbol{e}}_\phi}{\partial \phi} = -\sin\theta\, \hat{\boldsymbol{e}}_r - \cos\theta\, \hat{\boldsymbol{e}}_\theta.$$

$$\underline{\nabla} S = \hat{e}_r \frac{\partial S}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial S}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}$$

$$\underline{\nabla} \cdot \underline{U} = \frac{\partial U_r}{\partial r} + \frac{1}{r}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi} + \frac{2\,U_r}{r} + \frac{U_\theta}{r\tan\theta}$$

$$
\begin{aligned}
\nabla \times \boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r \left(\frac{1}{r} \frac{\partial U_\theta}{\partial \phi} - \frac{1}{r \sin\theta} \frac{\partial U_\theta}{\partial \theta} + \frac{U_\phi}{r \tan\theta} \right) \\
& + \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r \sin\theta} \frac{\partial U_r}{\partial \phi} - \frac{\partial U_\phi}{\partial r} - \frac{U_\phi}{r} \right) \\
& + \hat{\boldsymbol{e}}_\phi \left(\frac{\partial U_\theta}{\partial r} - \frac{1}{r} \frac{\partial U_r}{\partial \theta} + \frac{U_\theta}{r} \right).
\end{aligned}
$$

$$
\begin{aligned}
\nabla \boldsymbol{U} =\ & \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_r \frac{\partial U_r}{\partial r} + \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_\theta \frac{\partial U_\theta}{\partial r} + \hat{\boldsymbol{e}}_r \hat{\boldsymbol{e}}_\phi \frac{\partial U_\phi}{\partial r} \\
& + \hat{\boldsymbol{e}}_\theta \hat{\boldsymbol{e}}_r \left(\frac{1}{r} \frac{\partial U_r}{\partial \theta} - \frac{U_\theta}{r} \right) + \hat{\boldsymbol{e}}_\theta \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r} \frac{\partial U_\theta}{\partial \theta} + \frac{U_r}{r} \right) + \hat{\boldsymbol{e}}_\theta \hat{\boldsymbol{e}}_\phi \frac{1}{r} \frac{\partial U_\phi}{\partial \theta} \\
& + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_r \left(\frac{1}{r \sin\theta} \frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r} \right) + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_\theta \left(\frac{1}{r \sin\theta} \frac{\partial U_\theta}{\partial \phi} - \frac{U_\phi}{r \tan\theta} \right) \\
& + \hat{\boldsymbol{e}}_\phi \hat{\boldsymbol{e}}_\phi \left(\frac{1}{r \sin\theta} \frac{\partial U_\phi}{\partial \phi} + \frac{U_\theta}{r \tan\theta} + \frac{U_r}{r} \right).
\end{aligned}
$$

$$
\begin{aligned}
(\boldsymbol{U}\cdot\nabla)\boldsymbol{U} = {} & \hat{\boldsymbol{e}}_r\left(U_r\frac{\partial U_r}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_r}{\partial \theta}+\frac{U_\phi}{r\sin\theta}\frac{\partial U_r}{\partial \phi}-\frac{U_\theta^2}{r}-\frac{U_\phi^2}{r}\right) \\
& +\hat{\boldsymbol{e}}_\theta\left(U_r\frac{\partial U_\theta}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_\theta}{\partial \theta}+\frac{U_\phi}{r\sin\theta}\frac{\partial U_\theta}{\partial \phi}+\frac{U_rU_\theta}{r}-\frac{U_\phi^2}{r\tan\theta}\right) \\
& +\hat{\boldsymbol{e}}_\phi\left(U_r\frac{\partial U_\phi}{\partial r}+\frac{U_\theta}{r}\frac{\partial U_\phi}{\partial \theta}+\frac{U_\phi}{r\sin\theta}\frac{\partial U_\phi}{\partial \phi}+\frac{U_rU_\phi}{r}+\frac{U_\theta U_\phi}{r\tan\theta}\right)
\end{aligned}
$$

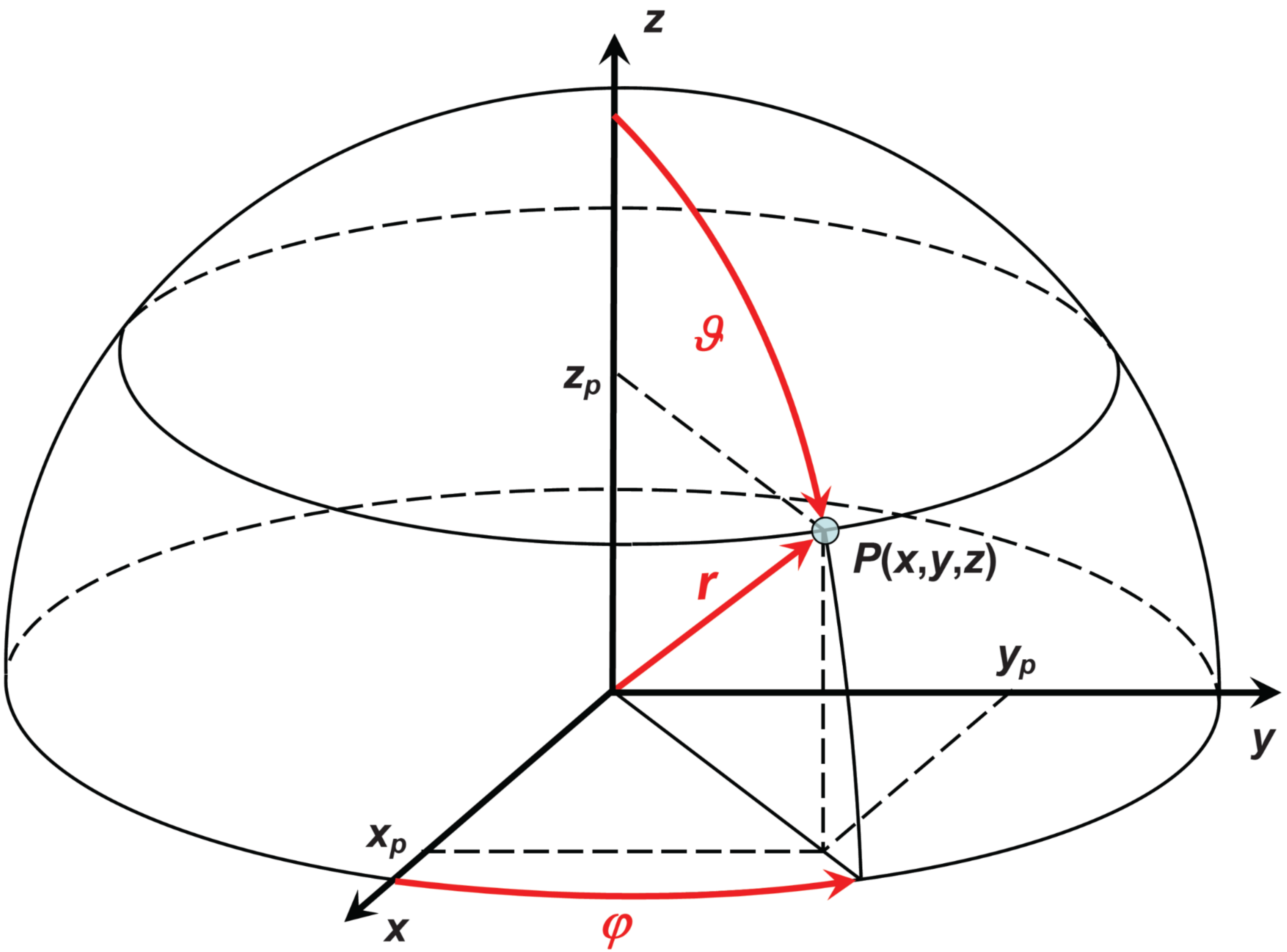
$$
\begin{aligned}
\nabla\cdot\boldsymbol{T} = {} & \hat{\boldsymbol{e}}_r\left(\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta r}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi r}}{\partial \phi}+\frac{2T_{rr}}{r}+\frac{T_{\theta r}}{r\tan\theta}-\frac{T_{\theta\theta}}{r}-\frac{T_{\phi\phi}}{r}\right) \\
& +\hat{\boldsymbol{e}}_\theta\left(\frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\theta}}{\partial \phi}+\frac{2T_{r\theta}}{r}+\frac{T_{\theta r}}{r}+\frac{T_{\theta\theta}}{r\tan\theta}-\frac{T_{\phi\phi}}{r\tan\theta}\right) \\
& +\hat{\boldsymbol{e}}_\phi\left(\frac{\partial T_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\phi}}{\partial \theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi}+\frac{2T_{r\phi}}{r}+\frac{T_{\phi r}}{r}+\frac{T_{\theta\phi}}{r\tan\theta}+\frac{T_{\phi\theta}}{r\tan\theta}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \left[\frac{1}{r^2}\frac{\partial(r^2B_{rr})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(B_{\theta r}\sin\theta)}{\partial\theta}-\frac{(B_{\theta\theta}+B_{\phi\phi})}{r}+\frac{1}{r\sin\theta}\frac{\partial B_{\phi r}}{\partial\phi}\right]\hat{e}_r \\
\underline{\nabla}\cdot\underline{\underline{B}} = {} & \left[\frac{1}{r^3}\frac{\partial(r^3B_{r\theta})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(B_{\theta\theta}\sin\theta)}{\partial\theta}+\frac{(B_{\theta r}-B_{r\theta}-B_{\phi\phi}\cot\theta)}{r}+\frac{1}{r\sin\theta}\frac{\partial B_{\phi\theta}}{\partial\phi}\right]\hat{e}_\theta \\
& \left[\frac{1}{r^3}\frac{\partial(r^3B_{r\phi})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(B_{\theta\phi}\sin\theta)}{\partial\theta}+\frac{(B_{\phi r}-B_{r\phi}+B_{\phi\theta}\cot\theta)}{r}+\frac{1}{r\sin\theta}\frac{\partial B_{\phi\phi}}{\partial\phi}\right]\hat{e}_\phi
\end{aligned}
$$

$$\nabla^2 S = \frac{\partial^2 S}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 S}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 S}{\partial \phi^2} + \frac{2}{r}\frac{\partial S}{\partial r} + \frac{1}{r^2 \tan\theta}\frac{\partial S}{\partial \theta}.$$

$$\begin{aligned}
\nabla^2 \boldsymbol{U} = \hat{\boldsymbol{e}}_r \Bigg(& \frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_r}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_r}{\partial \phi^2} \\
& + \frac{2}{r}\frac{\partial U_r}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2}{r^2}\frac{\partial U_\theta}{\partial \theta} - \frac{2}{r^2 \sin\theta}\frac{\partial U_\phi}{\partial \phi} - 2\frac{U_r}{r^2} - \frac{U_\theta}{r^2 \tan\theta} \Bigg) \\
+ \hat{\boldsymbol{e}}_\theta \Bigg(& \frac{\partial^2 U_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\theta}{\partial \phi^2} \\
& + \frac{2}{r}\frac{\partial U_\theta}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_\theta}{\partial \theta} + \frac{1}{r^2}\frac{\partial U_r}{\partial \theta} - \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\phi}{\partial \phi} - \frac{U_\theta}{r^2 \sin^2\theta} \Bigg) \\
+ \hat{\boldsymbol{e}}_\phi \Bigg(& \frac{\partial^2 U_\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U_\phi}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 U_\phi}{\partial \phi^2} \\
& + \frac{2}{r}\frac{\partial U_\phi}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial U_\phi}{\partial \theta} + \frac{2\cos\theta}{r^2 \sin^2\theta}\frac{\partial U_\theta}{\partial \phi} + \frac{2}{r^2 \sin\theta}\frac{\partial U_r}{\partial \phi} - \frac{U_\phi}{r^2 \sin^2\theta} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
\nabla(\nabla\cdot U) = \hat{e}_r \Bigg(& \frac{\partial^2 U_r}{\partial r^2} + \frac{1}{r}\frac{\partial^2 U_\theta}{\partial r \partial\theta} + \frac{1}{r\sin\theta}\frac{\partial^2 U_\phi}{\partial r\partial\phi} \\
& + \frac{2}{r}\frac{\partial U_r}{\partial r} + \frac{\cot\theta}{r}\frac{\partial U_\theta}{\partial r} - \frac{1}{r^2}\frac{\partial U_\theta}{\partial\theta} - \frac{1}{r^2\sin\theta}\frac{\partial U_\phi}{\partial\phi} - \frac{2U_r}{r^2} - \frac{U_\theta}{r^2\tan\theta}\Bigg) \\
+ \hat{e}_\theta \Bigg(& \frac{1}{r}\frac{\partial^2 U_r}{\partial\theta\partial r} + \frac{1}{r^2}\frac{\partial^2 U_\theta}{\partial\theta^2} + \frac{1}{r^2\sin\theta}\frac{\partial^2 U_\phi}{\partial\theta\partial\phi} \\
& + \frac{2}{r^2}\frac{\partial U_r}{\partial\theta} + \frac{\cot\theta}{r^2}\frac{\partial U_\theta}{\partial\theta} - \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\phi}{\partial\phi} - \frac{U_\theta}{r^2\sin^2\theta}\Bigg) \\
+ \hat{e}_\phi \Bigg(& \frac{1}{r\sin\theta}\frac{\partial^2 U_r}{\partial\phi\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial^2 U_\theta}{\partial\phi^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U_\phi}{\partial\phi\partial\theta} \\
& + \frac{2}{r^2\sin\theta}\frac{\partial U_r}{\partial\phi} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial U_\theta}{\partial\theta}\Bigg)
\end{aligned}
$$



z
ϑ
z_p
$P(x,y,z)$
r
y_p
y
x_p
x
φ

Example No.13

2. Get acquainted with Matlab and the functions (plot, contour & quiver) and/or Scilab and the functions (champ, contour2d & contour)

3. Generate a circle and use these points as tracer (o-shape), in a 2-D velocity field, track how the tracer particles evolve in time, and rework the analysis using a square figure instead the circle.

I) The velocity field has the form: $V_x = -\Omega y$, $V_y = +\Omega x$

II) The velocity field has the form: $V_x = +G x$, $V_y = -G y$

III) The velocity field has the form: $V_x = +\alpha x$, $V_y = +\alpha y$

Functions to create the ring and the square

```
function [x, y]=fsquarex()
  n=101;
  for i=1:n
    theta(i)=2*%pi*(i-1)/(n-1)

  x0=0.5;y0=0.5;a=0.5;
  if theta(i)<=%pi/4 then
    x(i)=x0+a;
    y(i)=y0+a*tan(theta(i));
  end
  if theta(i)>%pi/4&theta(i)<=3*%pi/4 then
    y(i)=y0+a;
    x(i)=x0+a*cotg(theta(i));
  end
  if theta(i)>3*%pi/4&theta(i)<=5*%pi/4 then
    x(i)=x0-a;
    y(i)=y0+a*tan(%pi-theta(i));
  end
  if theta(i)>5*%pi/4&theta(i)<=7*%pi/4 then
    y(i)=y0-a;
    x(i)=x0-a*cotg(theta(i)-%pi);
  end
    if theta(i)>7*%pi/4 then
    x(i)=x0+a;
    y(i)=y0+a*tan(theta(i));
  end
end
plot(x,y,'ro-')
endfunction
```

```
function [x, y]=fcircle()
  n=40;
  m=n+1;
  pi=%pi;
  beta1=2*pi/1;
  rho=0.25
  x0=0.5;y0=0.5;
  //rho=1.0;x0=0;y0=0;
  for i=1:m
    theta(i)=(i-1)*beta1/n;
    x(i)=rho*cos(theta(i));
    y(i)=rho*sin(theta(i));
  end
  for i=1:m
    x(i)=x0+x(i);
    y(i)=y0+y(i);
  end
  plot(x,y,'bo-')
endfunction
```

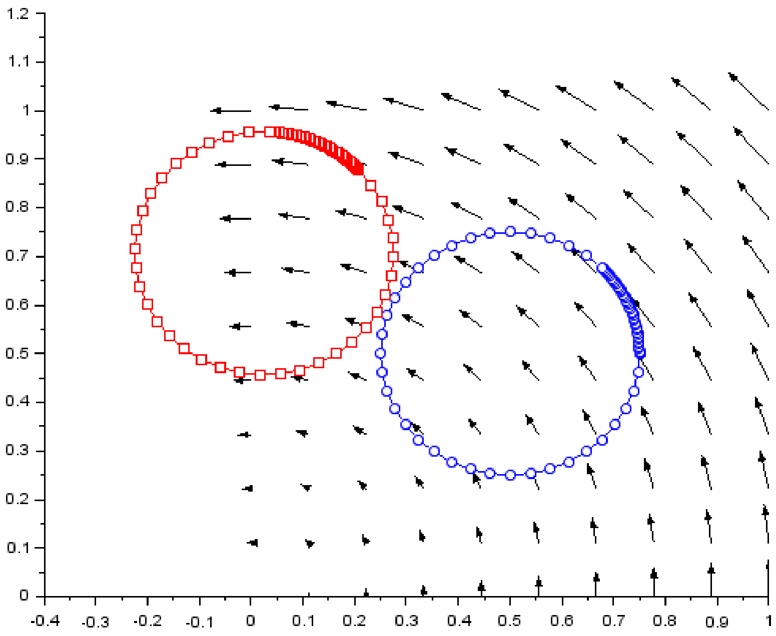
Main program

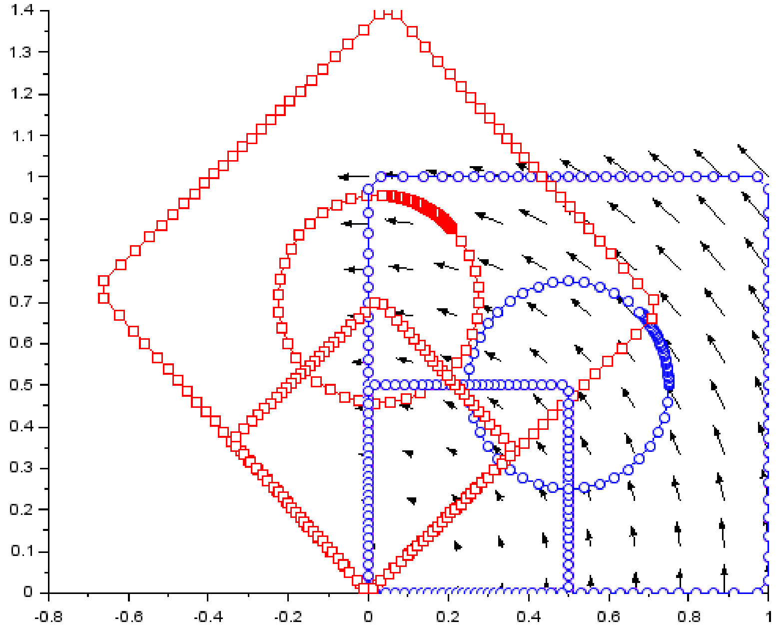
```
function [x, y, z]=mainFD()
 // for rotational use tf=1.5
// For deformation use tf=0.5
[x,y]=fcircle()
//[x,y]=fsquarex()
[n,m1]=size(x);
m=n;    to=0;tf=0.75;
for i=1:m
      z(2*i-1)=x(i);
      z(2*i)=y(i);
end
zo=z;
[zf]=ode(zo,to,tf,Vfield);
 [n,m]=size(zf);
for i=1:n/2
      xf(i)=zf(2*i-1);
      yf(i)=zf(2*i)
end
plot(x,y,'bo-',xf,yf,'rs-');
endfunction
```

Function to generate the velocity field

```
function [zprime]=Vfield(t, z)
 [n,m]=size(z);
for i=1:n/2
      x(i)=z(2*i-1);
      y(i)=z(2*i)
end
for i=1:n/2
//Vx(i)=-y(i);
 //Vy(i)=x(i);
//==============
      Vx(i)=x(i);
      Vy(i)=-y(i);
end
for i=1:n/2
      zprime(2*i-1)=Vx(i);
      zprime(2*i)=Vy(i);
end
endfunction
```

I) $V_x = -\Omega y\,,\; V_y = +\Omega x$



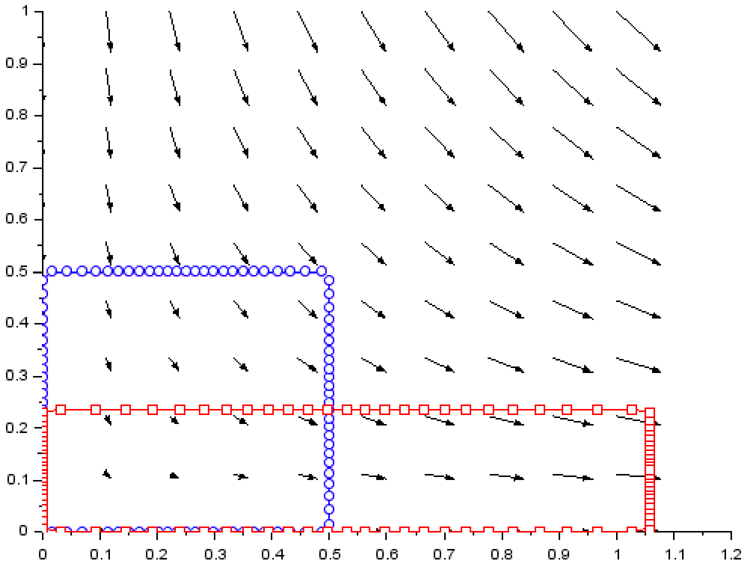


$$\underline{\underline{\Gamma}} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{bmatrix} \quad \underline{\underline{\Gamma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

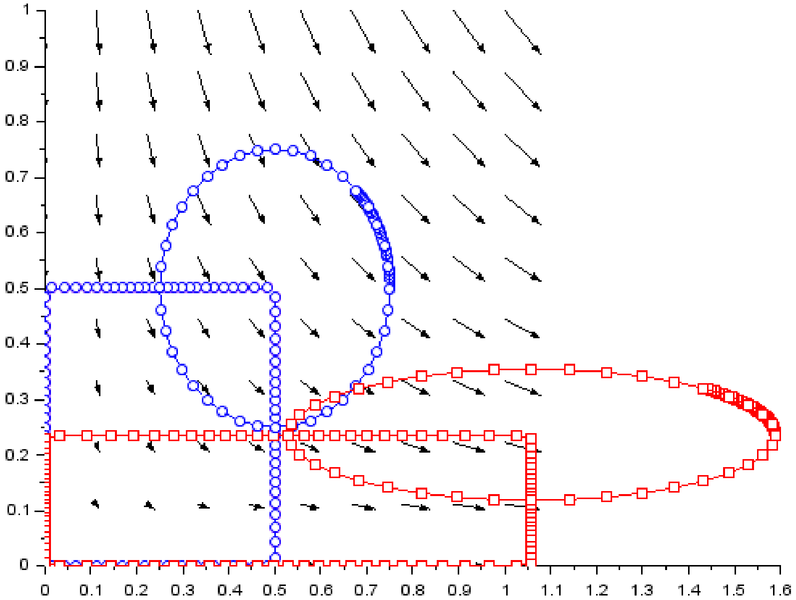
$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\underline{\Omega}} = \begin{bmatrix} 0 & \dot{\theta}_z & -\dot{\theta}_y \\ -\dot{\theta}_z & 0 & \dot{\theta}_x \\ \dot{\theta}_y & -\dot{\theta}_x & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\Omega^2 x \\ -\Omega^2 y \\ 0 \end{bmatrix}$$

$$V_x = +G\,x\,,\; V_y = -G\,y$$



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
1.5
1.6

Vectors and tensors in for this problem

Velocity vector

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$u = \ G\,x\,,\ v = -\,G\,y$

Rate of rotation tensor

$$\underline{\underline{\Omega}} = \begin{bmatrix} 0 & \dot{\theta}_z & -\dot{\theta}_y \\ -\dot{\theta}_z & 0 & \dot{\theta}_x \\ \dot{\theta}_y & -\dot{\theta}_x & 0 \end{bmatrix}$$

Rate strain tensor

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{bmatrix}$$

acceleration vector

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial t} \end{bmatrix} + \begin{bmatrix} uu_x + vu_y + wu_z \\ uv_x + vv_y + wv_z \\ uw_x + vw_y + ww_z \end{bmatrix}$$

$u = G x$, $v = -G y$

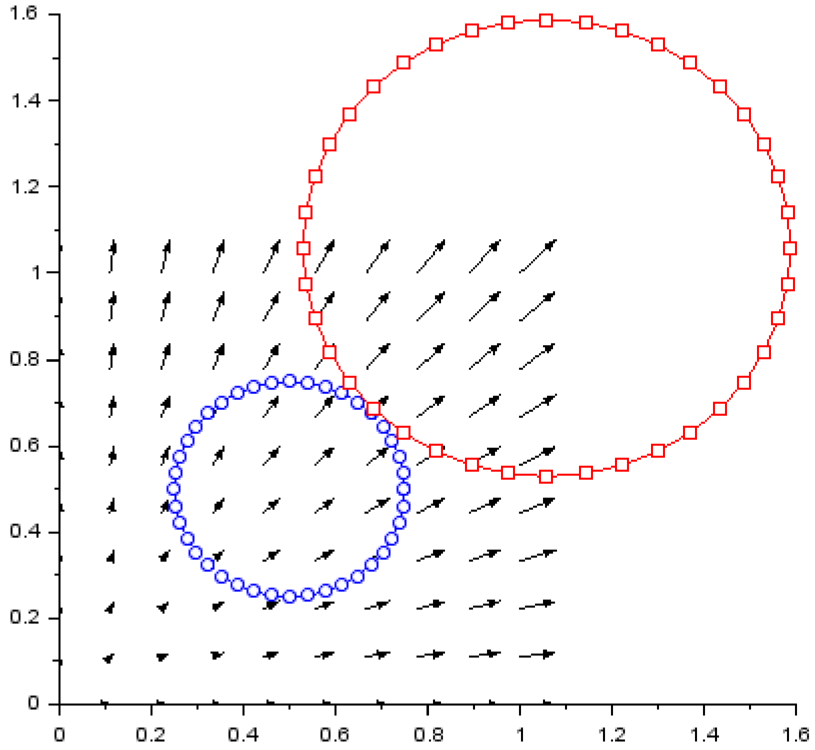
$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} \partial u/\partial x & \partial v/\partial x & \partial w/\partial x \\ \partial u/\partial y & \partial v/\partial y & \partial w/\partial y \\ \partial u/\partial z & \partial v/\partial z & \partial w/\partial z \end{bmatrix}$$

$$\underline{\nabla}\,\underline{v} = \begin{bmatrix} G & 0 & 0 \\ 0 & -G & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad [\underline{\nabla}\,\underline{v}]^T = \begin{bmatrix} G & 0 & 0 \\ 0 & -G & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\underline{\Gamma}} = \begin{bmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_{zz} \end{bmatrix} \quad \underline{\underline{\Gamma}} = \begin{bmatrix} G & 0 & 0 \\ 0 & -G & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \underline{\underline{\Omega}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{\underline{\Omega}} = \begin{bmatrix} 0 & \dot{\theta}_z & -\dot{\theta}_y \\ -\dot{\theta}_z & 0 & \dot{\theta}_x \\ \dot{\theta}_y & -\dot{\theta}_x & 0 \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} G^2 x \\ G^2 y \\ 0 \end{bmatrix}$$

$$V_x = +\alpha x\ ,\ V_y = +\alpha y$$



$$\frac{d\left(\underline{F}\,\underline{G}\right)}{d\eta}=\lim_{\eta\to 0}\frac{\left[\underline{F}\left(\eta+\Delta\eta\right)\underline{G}\left(\eta+\Delta\eta\right)-\underline{F}\left(\eta\right)\underline{G}\left(\eta\right)\right]}{\Delta\eta}$$

$$\frac{d\left(\underline{F}\,\underline{G}\right)}{d\eta}=\lim_{\eta\to 0}\frac{\left[\underline{F}\left(\eta+\Delta\eta\right)\underline{G}\left(\eta+\Delta\eta\right)-\underline{F}\left(\eta\right)\underline{G}\left(\eta+\Delta\eta\right)+\underline{F}\left(\eta\right)\underline{G}\left(\eta+\Delta\eta\right)-\underline{F}\left(\eta\right)\underline{G}\left(\eta\right)\right]}{\Delta\eta}$$

$$\frac{d\left(\underline{F}\,\underline{G}\right)}{d\eta}=\lim_{\eta\to 0}\frac{\left[\underline{F}\left(\eta+\Delta\eta\right)-\underline{F}\left(\eta\right)\right]}{\Delta\eta}\underline{G}\left(\eta+\Delta\eta\right)+\lim_{\eta\to 0}\underline{F}\left(\eta\right)\frac{\left[\underline{G}\left(\eta+\Delta\eta\right)-\underline{G}\left(\eta\right)\right]}{\Delta\eta}$$

$$\frac{d\left(\underline{F}\,\underline{G}\right)}{d\eta}=\frac{d\left(\underline{F}\right)}{d\eta}\underline{G}+\underline{F}\,\frac{d\left(\underline{G}\right)}{d\eta}$$

$$\frac{d\left(\underline{F}\,\underline{G}\right)}{d\eta}=\frac{d\left(\underline{F}\right)}{d\eta}\underline{G}+\underline{F}\,\frac{d\left(\underline{G}\right)}{d\eta}$$

Method of Similarity Transform and Laplace Transform

fluid at rest is disturbed while a plate is set in motion (x-direction)

$$\frac{\partial(\rho V_x)}{\partial t}+\frac{\partial(\rho V_x V_x)}{\partial x}+\frac{\partial(\rho V_y V_x)}{\partial y}+\frac{\partial(\rho V_z V_x)}{\partial z}=-\frac{\partial p}{\partial x}+\rho g_x+\frac{\partial(\tau_{xx})}{\partial x}+\frac{\partial(\tau_{yx})}{\partial y}+\frac{\partial(\tau_{zx})}{\partial z}$$

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho V_x)}{\partial x}+\frac{\partial(\rho V_y)}{\partial y}=0$$

$$\frac{\partial(\rho V_x)}{\partial t}=\rho\left[V_x\frac{\partial(V_x)}{\partial x}+V_y\frac{\partial(V_x)}{\partial y}\right]+V_x\left[\frac{\partial(\rho V_x)}{\partial x}+\frac{\partial(\rho V_y)}{\partial y}\right]=-\frac{\partial p}{\partial x}+\rho g_x+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}$$

1-D flow.

$$\rho\frac{\partial V_x}{\partial t}=-\frac{\partial p}{\partial x}+\rho g_x+\frac{\partial \tau_{yx}}{\partial y}$$

The pressure is uniform, and gravity is only in y axis.

$$\rho\frac{\partial V_x}{\partial t}=\frac{\partial \tau_{yx}}{\partial y}$$

Two scenarios are analyzed. a) A constant stress is applied, b) The plate is set in motion at constant velocity

$$\rho \frac{\partial V_x}{\partial t} = \frac{\partial \tau_{yx}}{\partial y} \qquad \tau_{yx} = \mu \frac{\partial V_x}{\partial y}$$

To solve the first case, is better to express the momentum equation in terms of the stress, then the equation of momentum is differentiated respect to "*y*", and multiplied by viscosity to transform velocity into stress.

$$\mu \frac{\partial}{\partial y}\left[\rho \frac{\partial V_x}{\partial t} = \frac{\partial \tau_{yx}}{\partial y}\right] \Rightarrow \rho \frac{\partial}{\partial t}\left(\mu \frac{\partial V_x}{\partial y}\right) = \mu \frac{\partial^2 \tau_{yx}}{\partial y^2} \Rightarrow \boxed{\rho \frac{\partial \tau_{yx}}{\partial t} = \mu \frac{\partial^2 \tau_{yx}}{\partial y^2}}$$

This PDE is transformed in Laplace domain

$$\mathcal{L}\left[\frac{\partial \tau_{yx}}{\partial t}\right] = \mathcal{L}\left[\nu \frac{\partial^2 \tau_{yx}}{\partial y^2}\right]$$

$$\frac{\partial \tau_{yx}}{\partial t} = \nu \frac{\partial^2 \tau_{yx}}{\partial y^2}$$

$$\boxed{T = \mathcal{L}\,[\tau] = \int_0^{\infty} e^{-st}\,\tau\, dt}$$

This PDE is now an ODE

$$sT - \tau(0) = \nu \frac{d^2T}{dy^2} \qquad sT - \tau(0) = \nu \frac{d^2T}{dy^2} \qquad \frac{d^2T}{dy^2} - \frac{s}{\nu}T = 0$$

This second order ODE has a solution in the form

$$\tau = \tau_o\left[1 - erf\left(\frac{y}{\sqrt{4\nu}}\right)\right]$$

$$\left(\frac{d^2}{dy^2} - \frac{s}{\nu}\right)T = 0$$

$$T = Ae^{-y\sqrt{s/\nu}} + Be^{y\sqrt{s/\nu}}$$

$$\mathcal{L}^{-1}T = \tau$$

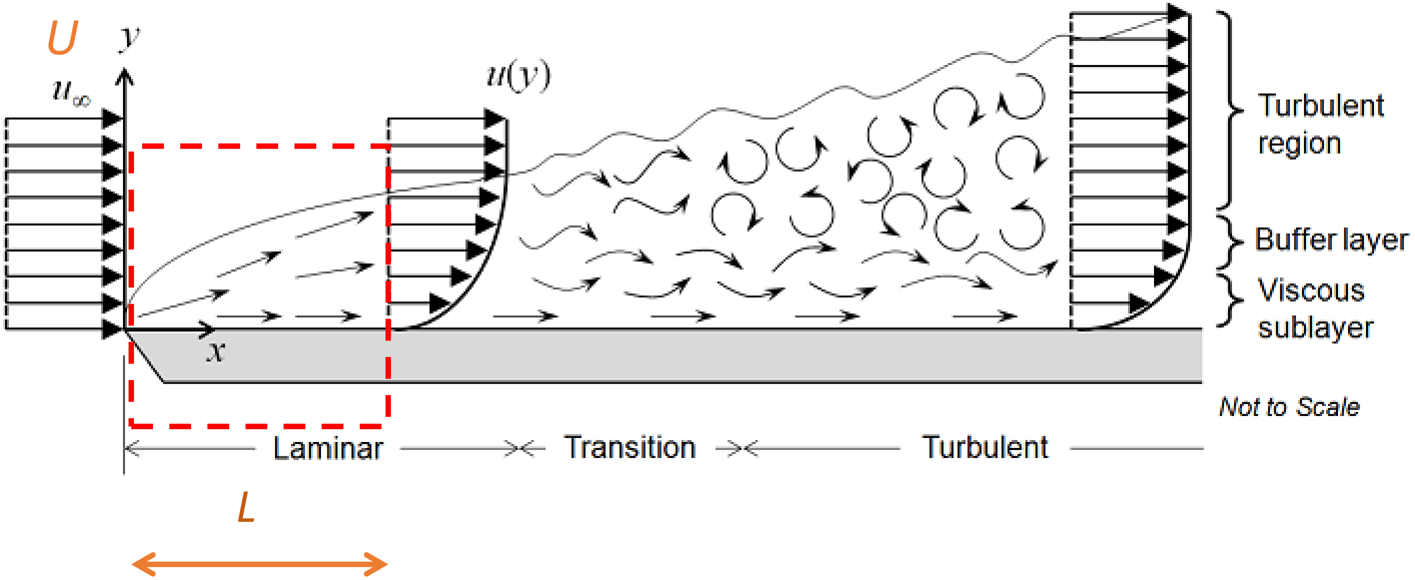
As y $\longrightarrow\infty$ there is a finite solution , then *B*=0

If the stress is fixed, then the transform of the perturbation is a step function

$$\tau = \tau_o\, erfc\left(\frac{y}{\sqrt{4\nu}}\right)$$

$$T = T_o e^{-y\sqrt{s/\nu}} \qquad T = \frac{\tau_o}{s} e^{-y\sqrt{s/\nu}}$$

Prediction of the drag coefficient over a flat surface exposed to a parallel flow (laminar Boundary layer)



Continuity and linear momentum equation

$$\frac{\partial(\rho)}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v}\,)=0$$

For incompressible fluid

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0$$

For a 2-D flow, or assuming width in z direction extends to infinity

x and y are the tangential and the normal coordinates

linear momentum analysis in tangential direction

$$\frac{\partial(\rho\underline{v})}{\partial t}+\underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v})=\rho\underline{g}+\underline{\nabla}\cdot\underline{\underline{\tau}}-\underline{\nabla}p$$

$$\begin{bmatrix}\dfrac{\partial(\rho V_x)}{\partial t}\underline{e}_x\\ \dfrac{\partial(\rho V_y)}{\partial t}\underline{e}_y\\ \dfrac{\partial(\rho V_z)}{\partial t}\underline{e}_z\end{bmatrix}+\underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v})=\begin{bmatrix}\rho g_x\underline{e}_x\\ \rho g_y\underline{e}_y\\ \rho g_z\underline{e}_z\end{bmatrix}+\underline{\nabla}\cdot\underline{\underline{\tau}}-\begin{bmatrix}\dfrac{\partial p}{\partial x}\underline{e}_x\\ \dfrac{\partial p}{\partial y}\underline{e}_y\\ \dfrac{\partial p}{\partial z}\underline{e}_z\end{bmatrix}$$

$$\underline{\nabla}\cdot(\rho\underline{V}\underline{V})=\begin{matrix}\left[\dfrac{\partial(\rho V_xV_x)}{\partial x}+\dfrac{\partial(\rho V_yV_x)}{\partial y}+\dfrac{\partial(\rho V_zV_x)}{\partial z}\right]\underline{e}_x\\ \left[\dfrac{\partial(\rho V_xV_y)}{\partial x}+\dfrac{\partial(\rho V_yV_y)}{\partial y}+\dfrac{\partial(\rho V_zV_y)}{\partial z}\right]\underline{e}_y\\ \left[\dfrac{\partial(\rho V_xV_z)}{\partial x}+\dfrac{\partial(\rho V_yV_z)}{\partial y}+\dfrac{\partial(\rho V_zV_z)}{\partial z}\right]\underline{e}_z\end{matrix}$$

$$\underline{\nabla}\cdot\underline{\underline{T}}=\begin{matrix}\left[\dfrac{\partial(T_{xx})}{\partial x}+\dfrac{\partial(T_{yx})}{\partial y}+\dfrac{\partial(T_{zx})}{\partial z}\right]\underline{e}_x\\ \left[\dfrac{\partial(T_{xy})}{\partial x}+\dfrac{\partial(T_{yy})}{\partial y}+\dfrac{\partial(T_{xy})}{\partial z}\right]\underline{e}_y\\ \left[\dfrac{\partial(T_{xz})}{\partial x}+\dfrac{\partial(T_{yz})}{\partial y}+\dfrac{\partial(T_{zz})}{\partial z}\right]\underline{e}_z\end{matrix}$$

Under Steady state condition and for Newtonian fluid

$$\frac{\partial(\rho V_x)}{\partial t}+\frac{\partial(\rho V_x V_x)}{\partial x}+\frac{\partial(\rho V_y V_x)}{\partial y}+\frac{\partial(\rho V_z V_x)}{\partial z}=-\frac{\partial p}{\partial x}+\rho g_x+\frac{\partial(\tau_{xx})}{\partial x}+\frac{\partial(\tau_{yx})}{\partial y}+\frac{\partial(\tau_{zx})}{\partial z}$$

$$\rho\left[V_x\frac{\partial(V_x)}{\partial x}+V_y\frac{\partial(V_x)}{\partial y}\right]+V_x\left[\frac{\partial(\rho V_x)}{\partial x}+\frac{\partial(\rho V_y)}{\partial y}\right]=-\frac{\partial p}{\partial x}+\rho g_x+\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}$$

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}=0$$

Continuity equation

To recast the momentum balance in dimensionless form, divide by momentum per unit volume:

$$\left[\rho\left[V_x\frac{\partial(V_x)}{\partial x}+V_y\frac{\partial(V_x)}{\partial y}\right]=-\frac{\partial p}{\partial x}+\rho g_x+\mu\frac{\partial^2 V_x}{\partial x^2}+\mu\frac{\partial^2 V_x}{\partial y^2}\right]\left(\frac{L}{\rho U^2}\right)$$

$$\left[\left[\hat{V}_x\frac{\partial(\hat{V}_x)}{\partial\hat{x}}+\left(\frac{L}{U}\right)V_y\frac{\partial(\hat{V}_x)}{\partial y}\right]=\alpha+\left(\frac{\mu}{\rho UL}\right)\frac{\partial^2\hat{V}_x}{\partial\hat{x}^2}+\left(\frac{\mu L}{\rho U}\right)\frac{\partial^2\hat{V}_x}{\partial y^2}\right]$$

Tangential coordinate and tangential velocities are written in dimensionless form:

$$\hat{x}=\frac{x}{L}\qquad \hat{V}_x=\frac{V_x}{U}\qquad \alpha=\left[-\frac{\partial p}{\partial x}+\rho g_x\right]\left(\frac{L}{\rho U^2}\right)$$

Collecting terms, new definitions are required for dimensionless normal coordinate and normal component of velocity, which will be defined in a way to simplify the PDE:

$$\left[\hat{V}_x \frac{\partial\left(\hat{V}_x\right)}{\partial\hat{x}} + \left(\frac{L}{U}\right)V_y \frac{\partial\left(\hat{V}_x\right)}{\partial y}\right] = \alpha + \left(\frac{1}{Re}\right)\frac{\partial^2\hat{V}_x}{\partial\hat{x}^2} + \left(\frac{\mu L^2}{\rho UL}\right)\frac{\partial^2\hat{V}_x}{\partial y^2}$$

$$\left[\hat{V}_x \frac{\partial\left(\hat{V}_x\right)}{\partial\hat{x}} + \left(\frac{L}{U}\right)V_y \frac{\partial\left(\hat{V}_x\right)}{\partial y}\right] = \alpha + \left(\frac{1}{Re}\right)\frac{\partial^2\hat{V}_x}{\partial\hat{x}^2} + \frac{\partial^2\hat{V}_x}{\partial\hat{y}^2}$$

$$\hat{y}^2 = \frac{y^2}{L^2}\left(\frac{\rho UL}{\mu}\right) \qquad \hat{y} = \frac{y}{L}Re^{1/2}$$

Dimensionless normal component

$$\left[\rho\left[V_x \frac{\partial\left(V_x\right)}{\partial x} + V_y \frac{\partial\left(V_x\right)}{\partial y}\right] = -\frac{\partial p}{\partial x} + \rho g_x + \mu\frac{\partial^2 V_x}{\partial x^2} + \mu\frac{\partial^2 V_x}{\partial y^2}\right]\left(\frac{L}{\rho U^2}\right)$$

$$\hat{V}_y = \left(\frac{Re^{1/2}}{U}\right)V_y$$

Dimensionless velocity normal to the plate

$$\left[\hat{V}_x \frac{\partial\hat{V}_x}{\partial\hat{x}} + \hat{V}_y \frac{\partial\hat{V}_x}{\partial\hat{y}}\right] = \alpha + \left(\frac{1}{Re}\right)\frac{\partial^2\hat{V}_x}{\partial\hat{x}^2} + \frac{\partial^2\hat{V}_x}{\partial\hat{y}^2}$$

$$\alpha = \left[-\frac{\partial p}{\partial x} + \rho g_x\right]\left(\frac{L}{\rho U^2}\right) = -\frac{\partial Eu}{\partial\hat{x}} + \frac{1}{Fr}$$

Linear momentum balance in dimensionless form coupled with continuity

$$\hat{V}_x \frac{\partial \hat{V}_x}{\partial \hat{x}} + \hat{V}_y \frac{\partial \hat{V}_x}{\partial \hat{y}} = \alpha + \left(\frac{1}{\mathrm{Re}}\right)\frac{\partial^2 \hat{V}_x}{\partial \hat{x}^2} + \frac{\partial^2 \hat{V}_x}{\partial \hat{y}^2}$$

$$\boxed{\alpha = \left[-\frac{\partial p}{\partial x} + \rho g_x\right]\left(\frac{L}{\rho U^2}\right) = -\frac{\partial Eu}{\partial \hat{x}} + \frac{1}{Fr}} \quad \boxed{\hat{y} = \frac{y}{L}\mathrm{Re}^{1/2}} \quad \boxed{\hat{x} = \frac{x}{L}} \quad \boxed{\hat{V}_x = \frac{V_x}{U}} \quad \boxed{\hat{V}_y = \left(\frac{\mathrm{Re}^{1/2}}{U}\right)V_y}$$

For high values of ***Re*** numbers, high Freud number (***Fr***) and constant ***Eu*** number the equation is simplified to the form

$$\hat{V}_x \frac{\partial \hat{V}_x}{\partial \hat{x}} + \hat{V}_y \frac{\partial \hat{V}_x}{\partial \hat{y}} = \frac{\partial^2 \hat{V}_x}{\partial \hat{y}^2}$$

Using the stream functions in dimensionless form:

$$\hat{V}_x = \frac{\partial \Psi}{\partial \hat{y}} \qquad \hat{V}_y = -\frac{\partial \Psi}{\partial \hat{x}}$$

The tangential component of linear momentum is written as:

$$\hat{V}_x \frac{\partial \hat{V}_x}{\partial \hat{x}} + \hat{V}_y \frac{\partial \hat{V}_x}{\partial \hat{y}} = \frac{\partial^2 \hat{V}_x}{\partial \hat{y}^2} \quad \Longrightarrow \quad \frac{\partial \Psi}{\partial \hat{y}}\frac{\partial^2 \Psi}{\partial \hat{x}\partial \hat{y}} - \frac{\partial \Psi}{\partial \hat{x}}\frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{\partial^3 \Psi}{\partial \hat{y}^3}$$

$$\frac{\partial\Psi}{\partial\hat{y}}\frac{\partial^2\Psi}{\partial\hat{x}\partial\hat{y}}-\frac{\partial\Psi}{\partial\hat{x}}\frac{\partial^2\Psi}{\partial\hat{y}^2}=\frac{\partial^3\Psi}{\partial\hat{y}^3}$$

Lets assume that stream function can be expressed in terms of the product of two functions in the form:

$$\Psi(\hat{x},\hat{y})=f(\eta)g(\hat{x})$$

And the new similarity transformation variable is proportional to dimensionless normal component:

$$\eta=\frac{\hat{y}}{g(\hat{x})}$$

Estimating, first and second partial derivatives of the similarity transformation variable η:

$$\frac{\partial\eta}{\partial\hat{y}}=\frac{1}{g(\hat{x})}=\frac{\eta}{\hat{y}} \qquad \frac{\partial^2\eta}{\partial\hat{y}^2}=0$$

$$\frac{\partial\eta}{\partial\hat{x}}=-\frac{\hat{y}g'}{g^2}=-\frac{\eta g'}{g}$$

Prime notation will be used to simplify expressions, prime over f refers derivatives respect to the similarity transformation η, and prime on g is respect to the tangential coordinate x

$$f'=\frac{df}{d\eta} \qquad g'=\frac{dg}{dx}$$

Using chain rule, differentials of Ψ are:

$$\Psi=f\ g \qquad d\Psi=f\ dg+g\ df$$

$$\frac{\partial\Psi}{\partial x}=f\ g'+g\frac{df}{d\eta}\left(\frac{\partial\eta}{\partial x}\right)=f\ g'+g\ f'\left(-\frac{\eta\ g'}{g}\right)=g'(f-\eta\ f')$$

$$\frac{\partial\Psi}{\partial y}=g\frac{df}{d\eta}\left(\frac{\partial\eta}{\partial y}\right)=g\ f'\left(\frac{1}{g}\right)=f'$$

Using chain rule, differentials of high order of Ψ are:

$$\left(\frac{\partial \Psi}{\partial y}\right) = f' \qquad d\left(\frac{\partial \Psi}{\partial y}\right) = \frac{df'}{d\eta} d\eta = f'' d\eta$$

$$\frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{\partial}{\partial \hat{y}}\left(\frac{\partial \Psi}{\partial \hat{y}}\right) = \frac{df'}{d\eta} d\eta = \frac{f''}{g}$$

$$\frac{\partial}{\partial \hat{x}}\left(\frac{\partial \Psi}{\partial \hat{y}}\right) = \frac{df'}{d\eta}\left(\frac{\partial \eta}{\partial x}\right) = f''\left(-\frac{\eta g'}{g}\right) = -\eta\, f''\left(\frac{g'}{g}\right)$$

$$d\,\frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{g\, df'' - f'' dg}{g^2} \qquad \frac{\partial^3 \Psi}{\partial \hat{y}^3} = \frac{\partial}{\partial \hat{y}}\left(\frac{\partial^2 \Psi}{\partial \hat{y}^2}\right) = \frac{g\left(\frac{df''}{d\eta}\right)\left(\frac{\partial \eta}{\partial \hat{y}}\right)}{g^2} = \frac{g\,(f''')\left(\frac{1}{g}\right)}{g^2} = \frac{f'''}{g^2}$$

Partial derivatives of Ψ are given in terms of ***f*** and ***g*** functions:

$$\boxed{\frac{\partial \Psi}{\partial x} = g'(f - \eta\, f')} \quad \boxed{\frac{\partial \Psi}{\partial y} = f'} \quad \boxed{\frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{f''}{g}} \quad \boxed{\frac{\partial^2 \Psi}{\partial \hat{x}\partial \hat{y}} = -\eta\, f''\left(\frac{g'}{g}\right)} \quad \boxed{\frac{\partial^3 \Psi}{\partial \hat{y}^3} = \frac{f'''}{g^2}}$$

$$\boxed{\frac{\partial \Psi}{\partial x} = g'\left(f - \eta\, f'\right)} \quad \boxed{\frac{\partial \Psi}{\partial y} = f'} \quad \boxed{\frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{f''}{g}} \quad \boxed{\frac{\partial^2 \Psi}{\partial \hat{x} \partial \hat{y}} = -\eta\, f''\left(\frac{g'}{g}\right)} \quad \boxed{\frac{\partial^3 \Psi}{\partial \hat{y}^3} = \frac{f'''}{g^2}}$$

Partial derivatives of Ψ in terms of ***f*** and ***g*** functions are plug in the linear momentum equation:

$$\frac{\partial \Psi}{\partial \hat{y}} \frac{\partial^2 \Psi}{\partial \hat{x} \partial \hat{y}} - \frac{\partial \Psi}{\partial \hat{x}} \frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{\partial^3 \Psi}{\partial \hat{y}^3}$$

$$[f']\left(\frac{-\eta\, f''\, g'}{g}\right) - [g'(f - \eta\, f')]\left(\frac{f''}{g}\right) = \frac{f'''}{g^2}$$

$$[-\eta\, f''\, g'\, g\, f'] - [g'(fgf'' - \eta\, f'\, g\, f'')] = f'''$$

$$-\eta\, f''\, g'\, g\, f' - g'\, fgf'' + \eta\, f'\, g'g\, f'' = -g'\, fgf'' = f'''$$

To be consistent with the original assumption, function f and g must be independent of each other, then:

$$g\, g' = 1 \qquad\qquad f''' + f\; f''(g\, g') = 0$$

The term *g g'* can be selected to be equal to 1, in order to simplify the solution of the equation:

$$f''' + f\; f'' = 0 \qquad g\frac{dg}{d\hat{x}} = 1 \qquad g\, dg = d\hat{x} = d\left(\frac{g^2}{2}\right) \qquad g = \sqrt{2\hat{x}} \qquad \eta = \frac{\hat{y}}{\sqrt{2\hat{x}}}$$

In a nutshell , the final differential equation, and the similarity transformation variables are:

$$f''' + f\ f'' = 0 \qquad \eta = \frac{\hat{y}}{\sqrt{2\hat{x}}} \qquad \Psi(\hat{x},\hat{y}) = \left(\sqrt{2\hat{x}}\right) f(\eta)$$

Velocities in tangential and normal direction expressed in terms of ***f*** and ***g*** functions are used to define boundary conditions of the transformed problem.

$$\hat{V}_x = \frac{\partial \Psi}{\partial \hat{y}} \qquad \hat{V}_y = -\frac{\partial \Psi}{\partial \hat{x}} \qquad \frac{\partial \Psi}{\partial x} = g'(f - \eta\ f') \qquad \frac{\partial \Psi}{\partial y} = f'$$

$$g = \sqrt{2\hat{x}} \qquad g' = 1/g$$

$$\hat{V}_y = -\frac{(f - \eta\ f')}{g} \qquad \hat{V}_x = f' \qquad \hat{V}_x = \frac{V_x}{U}$$

Boundary conditions using tangential and normal velocities at the interface with the plate (no slip boundary condition):

$$V_x(x, y=0) = 0 \qquad \eta = \frac{\hat{y}}{\sqrt{2\hat{x}}} \qquad y \to 0 = \eta \to 0$$

$$\hat{V}_x = f' \qquad \hat{V}_x \to 0 = f' \to 0$$

$V_x(x, y=0) = 0$ Is equivalent to: $\boxed{f'(0) = 0}$

Boundary conditions:

$$V_y(x, y=0)=0 \qquad \eta=\frac{\hat{y}}{\sqrt{2\hat{x}}} \qquad y \to 0 = \eta \to 0$$

$$\hat{V}_y=-\frac{(f-\eta f')}{g} \qquad \hat{V}_y \to 0 = f \to 0$$

$V_y(x, y=0)=0$ Is equivalent to: $\boxed{f(0)=0}$

Boundary condition far from the interface fluid-solid, the upstream velocity will be unperturbed far away from the surface

$$V_x(x, y \to \infty)=U \qquad \eta=\frac{\hat{y}}{\sqrt{2\hat{x}}} \qquad y \to \infty = \eta \to \infty \qquad \hat{V}_x=\frac{V_x}{U}$$

$$\hat{V}_x(\hat{x}, \eta \to \infty)=\frac{U}{U}=1 \qquad \hat{V}_x = f' \qquad \boxed{f'(\infty)=1}$$

The problem can be written in the form of a high order differential equation and a set of boundary conditions

$$f''' + f\ f'' = 0$$

$$f(0) = 0$$
$$f'(0) = 0$$
$$f'(\infty) = 1$$

Canonical form:

$$w_1 = f$$

$$\frac{df}{d\eta} = \frac{dw_1}{d\eta} = w_2$$

$$\frac{d^2 f}{d\eta^2} = \frac{dw_2}{d\eta} = w_3$$

$$\frac{d^3 f}{d\eta^3} = \frac{dw_3}{d\eta} = -w_1\ w_3$$

$$\frac{dw_1}{d\eta} = w_2$$

$$\frac{dw_2}{d\eta} = w_3$$

$$\frac{dw_3}{d\eta} = -w_1\ w_3$$

$$w_1(0) = 0$$
$$w_2(0) = 0$$
$$w_2(\infty) = 1$$

Parametric equations to solve using shooting method

$$\frac{dw_1}{d\eta} = w_2$$

$$\frac{dw_2}{d\eta} = w_3$$

$$\frac{dw_3}{d\eta} = -w_1\, w_3 = 0$$

$$w_1(0) = 0 \qquad w_1(\infty) = \sigma_1$$

$$w_2(0) = 0 \qquad w_2(\infty) = 1$$

$$w_3(0) = \zeta_3 \qquad w_3(\infty) = \sigma_3$$

The system of equations can be solved using shooting method. If shooting method is used the parametric equations should be included

$$\frac{dw_1}{d\eta} = z_1$$

$$\frac{dw_2}{d\eta} = z_2$$

$$\frac{dw_3}{d\eta} = z_3$$

$$dz_1 = \left(\frac{\partial z_1}{\partial w_1}\right)dw_1 + \left(\frac{\partial z_1}{\partial w_2}\right)dw_2 + \left(\frac{\partial z_1}{\partial w_3}\right)dw_3$$

$$\frac{dz_1}{d\zeta_3} = \frac{d}{d\eta}\left(\frac{dw_1}{d\zeta_3}\right) = \frac{dw_4}{d\eta} = \left(\frac{\partial z_1}{\partial w_1}\right)\frac{dw_1}{d\zeta_3} + \left(\frac{\partial z_1}{\partial w_2}\right)\frac{dw_2}{d\zeta_3} + \left(\frac{\partial z_1}{\partial w_3}\right)\frac{dw_3}{d\zeta_3}$$

$$\frac{dz_2}{d\zeta_3} = \frac{d}{d\eta}\left(\frac{dw_2}{d\zeta_3}\right) = \frac{dw_5}{d\eta} = \left(\frac{\partial z_2}{\partial w_1}\right)\frac{dw_1}{d\zeta_3} + \left(\frac{\partial z_2}{\partial w_2}\right)\frac{dw_2}{d\zeta_3} + \left(\frac{\partial z_2}{\partial w_3}\right)\frac{dw_3}{d\zeta_3}$$

$$\frac{dz_3}{d\zeta_3} = \frac{d}{d\eta}\left(\frac{dw_3}{d\zeta_3}\right) = \frac{dw_6}{d\eta} = \left(\frac{\partial z_3}{\partial w_1}\right)\frac{dw_1}{d\zeta_3} + \left(\frac{\partial z_3}{\partial w_2}\right)\frac{dw_2}{d\zeta_3} + \left(\frac{\partial z_3}{\partial w_3}\right)\frac{dw_3}{d\zeta_3}$$

Final set of ODEs, boundary conditions and residual
Augmented set of ODEs

$$\frac{dw_1}{d\eta} = z_1 \qquad w_1(0)=0 \qquad w_1(\infty)=\sigma_1$$

$$\frac{dw_2}{d\eta} = z_2 \qquad w_2(0)=0 \qquad w_2(\infty)=1$$

$$\frac{dw_3}{d\eta} = z_3 \qquad w_3(0)=\zeta_3 \qquad w_3(\infty)=\sigma_3$$

$$\frac{dw_4}{d\eta} = \left(\frac{\partial z_1}{\partial w_1}\right)w_4 + \left(\frac{\partial z_1}{\partial w_2}\right)w_5 + \left(\frac{\partial z_1}{\partial w_3}\right)w_6 \qquad w_4(0)=0 \qquad w_4(\infty)=\sigma_4$$

$$\frac{dw_5}{d\eta} = \left(\frac{\partial z_2}{\partial w_1}\right)w_4 + \left(\frac{\partial z_2}{\partial w_2}\right)w_5 + \left(\frac{\partial z_2}{\partial w_3}\right)w_6 \qquad w_5(0)=0 \qquad w_5(\infty)=\sigma_5$$

$$\frac{dw_6}{d\eta} = \left(\frac{\partial z_3}{\partial w_1}\right)w_4 + \left(\frac{\partial z_3}{\partial w_2}\right)w_5 + \left(\frac{\partial z_3}{\partial w_3}\right)w_6 \qquad w_6(0)=1 \qquad w_6(\infty)=\sigma_6$$

$$R(\zeta_3) = w_2(\infty) - 1 \qquad \left.\frac{dR(\zeta_3)}{d\zeta_3}\right|_{\eta\to\infty} = \left.\frac{dw_2}{d\zeta_3}\right|_{\eta\to\infty} = \left.w_5\right|_{\eta\to\infty} \qquad \zeta_{3,n+1} = \zeta_{3,n} - \frac{R(\zeta_{3,n})}{\left(\dfrac{dR}{d\zeta_3}\right)}$$

$$\zeta_{3,n+1} = \zeta_{3,n} - \frac{w_2(\infty)-1}{w_5(\infty)}$$

To estimate drag coefficient , the shear stress over the plate has to be calculated

$$\tau_w = \tau_{yx} = \mu \left.\frac{dV_x}{dy}\right|_{y=0}$$

Recasting this equation in terms of the dimensionless derivatives

$$\tau_w = \tau_{yx} = \mu \left.\frac{dV_x}{dy}\right|_{y=0} \qquad \hat{V}_x = \frac{V_x}{U} \qquad \hat{y} = \frac{y}{L}\mathrm{Re}^{1/2}$$

$$\tau_w = \tau_{yx} = \mu \left.\frac{dV_x}{dy}\right|_{y=0} = \frac{\mu U}{L\,\mathrm{Re}^{-1/2}} \left.\frac{d\hat{V}_x}{d\hat{y}}\right|_{y=0}$$

$$\frac{d\hat{V}_x}{d\hat{y}} = \frac{\partial}{\partial \hat{y}}\left(\frac{\partial \Psi}{\partial \hat{y}}\right) = \frac{\partial^2 \Psi}{\partial \hat{y}^2} \qquad \frac{\partial^2 \Psi}{\partial \hat{y}^2} = \frac{f''}{g} \qquad \eta = \frac{\hat{y}}{\sqrt{2\hat{x}}}$$

$$\tau_w = \frac{\mu U}{L\,\mathrm{Re}^{-1/2}} \left.\frac{d\hat{V}_x}{d\hat{y}}\right|_{y=0} = \frac{\mu U}{L\,\mathrm{Re}^{-1/2}} \left.\frac{\partial^2 \Psi}{\partial \hat{y}^2}\right|_{\eta=0} = \frac{\mu U\,\mathrm{Re}^{1/2}}{L}\frac{f''(0)}{g} = \frac{\mu U\,\mathrm{Re}^{1/2}}{L}\frac{f''(0)}{\sqrt{2\hat{x}}}$$

The stress is function of the tangential component, then the average stress is calculated

$$\tau_w = \frac{\mu U \,\mathrm{Re}^{1/2}}{L} \frac{f''(0)}{\sqrt{2\hat{x}}}$$

$$\langle \tau_w \rangle = \frac{1}{L}\int_0^L \tau_w dx = \int_0^1 \tau_w d\hat{x} = \frac{\mu U \,\mathrm{Re}^{1/2}}{L} f''(0) \int_0^1 \frac{d\hat{x}}{\sqrt{2\hat{x}}} = \frac{2\mu U \,\mathrm{Re}^{1/2}}{L\sqrt{2}} f''(0)\sqrt{\hat{x}}\Big|_0^1 = \frac{\sqrt{2}\mu U \,\mathrm{Re}^{1/2}}{L} f''(0)$$

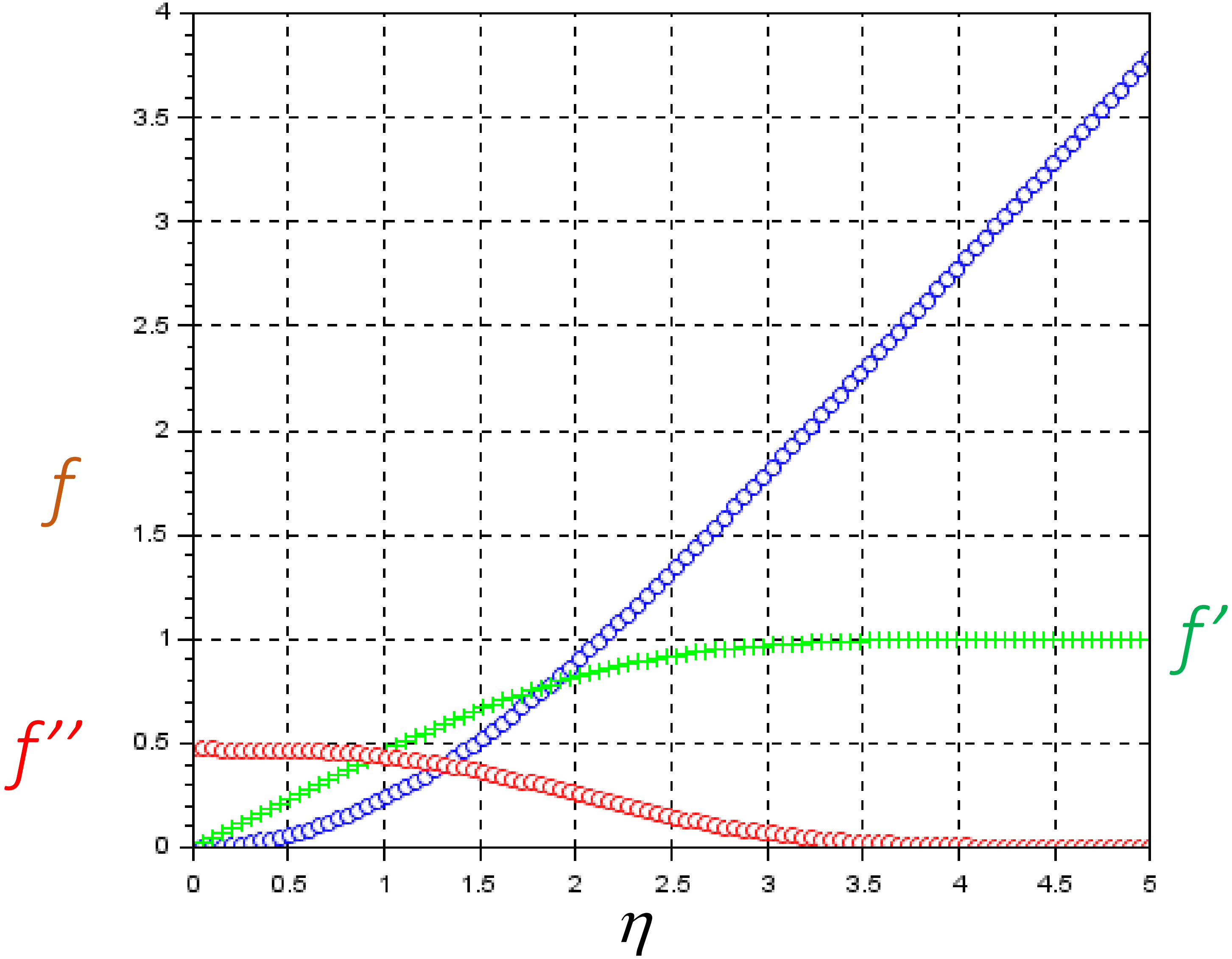
$$C_D = \frac{\langle \tau_w \rangle}{\frac{1}{2}\rho U^2} = \frac{2\sqrt{2}\mu U \,\mathrm{Re}^{1/2}}{\rho L U^2} f''(0) = \frac{2\sqrt{2}}{\mathrm{Re}^{1/2}} f''(0)$$

Drag coefficient is function of Reynolds number, and the value of *f ''(0)* is estimated numerically , using the shooting method.

f ''(0) =0.4696452

$$C_D = \frac{1.328357}{\mathrm{Re}^{1/2}}$$

Numerical solution



Scilab algorithm Numerical solution

```
function [Y, res]=FBlasius2N(id)
zetha=1
 y0=[0,0,zetha,0,0,1]'
t0=0;
tspan=linspace(0,5)';
NI=5 //Number of iterations (replace by while loop)
 for i=1:NI
y0=[0,0,zetha,0,0,1]'
[Y]=ode(y0,t0,tspan,BlasiusDE);
plot(tspan,Y(1,:)',tspan,Y(2,:)',tspan,Y(3,:)')
res=Y(2,$)-1
zetha=zetha-res/Y(5,$)
End
halt('press any key')
clf
plot(tspan,Y(1,:)','b.o',tspan,Y(2,:)','g+',tspan,Y(3,:)','ro-')
xgrid
endfunction

function [yprime]=BlasiusDE(t, y)
 // Dimensionless problem
// f*f''+f'''=0 // f(0)=f'(0)=0 and f'(eta=%inf)=0
 yprime(1)=y(2)
yprime(2)=y(3)
 yprime(3)=-y(1)*y(3)
RJ(1,1)=0;RJ(1,2)=1;RJ(1,3)=0;
RJ(2,1)=0;RJ(2,2)=0;RJ(2,3)=1;
RJ(3,1)=-y(3);RJ(3,2)=0;RJ(3,3)=-y(1);
yprime(4)=RJ(1,1)*y(4)+RJ(1,2)*y(5)+RJ(1,3)*y(6);
yprime(5)=RJ(2,1)*y(4)+RJ(2,2)*y(5)+RJ(2,3)*y(6);
yprime(6)=RJ(3,1)*y(4)+RJ(3,2)*y(5)+RJ(3,3)*y(6);
endfunction
```

Homework, Problem 1

1. The velocity profile for the boundary layer in the flow of a convergent channel (Hamel Flow) can be transformed into the form:

$$f''' - (f')^2 + 1 = 0$$

$$f'(0) = 0$$

$$f'(\infty) = 1$$

$$f''(\infty) = 0$$

a. Use the shooting method to solve this problem:
b. Verify your numerical solution with the analytical solution given for the first derivative:
c. Calculate the analytical solution for *f*, and compare it with the numerical solution

$$f'(\eta) = 3\tanh^2\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right] - 2$$

Hint: $\int \tanh^2 z \, dz = z - \tanh z$

A. Show that the differential equation:

$$f''' - (f')^2 + 1 = 0$$

Can be written as:

$$f'' = \sqrt{\frac{2}{3}}(1 - f')\sqrt{2 + f'}$$

$$f''(\eta) = \frac{6}{\sqrt{2}}\left\{1 - \tanh^2\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right]\right\}\tanh\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right]$$

$$f(\eta) = 3\sqrt{2}\left\{\left[\frac{\eta}{\sqrt{2}}\right] - \tanh\left[\frac{\eta}{\sqrt{2}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right] + \tanh\left[\tanh^{-1}\sqrt{\frac{2}{3}}\right]\right\} - 2\eta$$

$$f''' - \left(f'\right)^2 + 1 = 0$$

$$f'(0) = 0$$

$$f'(\infty) = 1$$

$$f''(\infty) = 0$$

$$f = w_1$$

$$\frac{df}{d\eta} = \frac{dw_1}{d\eta} = w_2$$

$$\frac{d^2 f}{d\eta^2} = \frac{d^2 w_1}{d\eta^2} = \frac{dw_2}{d\eta} = w_3$$

$$\frac{d^3 f}{d\eta^3} = \frac{d^3 w_1}{d\eta^3} = \frac{d^2 w_2}{d\eta^2} = \frac{dw_3}{d\eta}$$

$$\frac{dw_1}{d\eta} = F_1 = w_2$$

$$\frac{dw_2}{d\eta} = F_2 = w_3$$

$$\frac{dw_3}{d\eta} = F_3 = -1 + w_2^2$$

$$w_1(0) = \zeta_1 \qquad w_1(\infty) = \sigma_1$$

$$w_2(0) = 0 \qquad w_2(\infty) = 1$$

$$w_3(0) = \zeta_3 \qquad w_3(\infty) = 0$$

$$R_2 = w_2(\infty) - 1 = 0$$

$$R_3 = w_3(\infty) - 0 = 0$$

$$\frac{dF_1}{d\zeta_1}=\frac{d}{d\zeta_1}\left(\frac{dw_1}{d\eta}\right)=\frac{d}{d\eta}\left(\frac{dw_1}{d\zeta_1}\right)=\frac{\partial F_1}{\partial w_1}\frac{dw_1}{d\zeta_1}+\frac{\partial F_1}{\partial w_2}\frac{dw_2}{d\zeta_1}+\frac{\partial F_1}{\partial w_3}\frac{dw_3}{d\zeta_1}$$

$$dF_1=\frac{\partial F_1}{\partial w_1}dw_1+\frac{\partial F_1}{\partial w_2}dw_2+\frac{\partial F_1}{\partial w_3}dw_3$$

$$dF_2=\frac{\partial F_2}{\partial w_1}dw_1+\frac{\partial F_2}{\partial w_2}dw_2+\frac{\partial F_2}{\partial w_3}dw_3$$

$$dF_3=\frac{\partial F_3}{\partial w_1}dw_1+\frac{\partial F_3}{\partial w_2}dw_2+\frac{\partial F_3}{\partial w_3}dw_3$$

$$\frac{d}{d\eta}\left(\frac{dw_1}{d\zeta_1}\right)=\frac{\partial F_1}{\partial w_1}\overbrace{\frac{dw_1}{d\zeta_1}}^{w_4}+\frac{\partial F_1}{\partial w_2}\overbrace{\frac{dw_2}{d\zeta_1}}^{w_5}+\frac{\partial F_1}{\partial w_3}\overbrace{\frac{dw_3}{d\zeta_1}}^{w_6}$$

$$\frac{d}{d\eta}\left(\frac{dw_2}{d\zeta_1}\right)=\frac{\partial F_2}{\partial w_1}\frac{dw_1}{d\zeta_1}+\frac{\partial F_2}{\partial w_2}\frac{dw_2}{d\zeta_1}+\frac{\partial F_2}{\partial w_3}\frac{dw_3}{d\zeta_1}$$

$$\frac{d}{d\eta}\left(\frac{dw_3}{d\zeta_1}\right)=\frac{\partial F_3}{\partial w_1}\frac{dw_1}{d\zeta_1}+\frac{\partial F_3}{\partial w_2}\frac{dw_2}{d\zeta_1}+\frac{\partial F_3}{\partial w_3}\frac{dw_3}{d\zeta_1}$$

$$\frac{d}{d\eta}\left(\frac{dw_1}{d\zeta_3}\right)=\frac{\partial F_1}{\partial w_1}\overbrace{\frac{dw_1}{d\zeta_3}}^{w_7}+\frac{\partial F_1}{\partial w_2}\overbrace{\frac{dw_2}{d\zeta_3}}^{w_8}+\frac{\partial F_1}{\partial w_3}\overbrace{\frac{dw_3}{d\zeta_3}}^{w_9}$$

$$\frac{d}{d\eta}\left(\frac{dw_2}{d\zeta_3}\right)=\frac{\partial F_2}{\partial w_1}\frac{dw_1}{d\zeta_3}+\frac{\partial F_2}{\partial w_2}\frac{dw_2}{d\zeta_3}+\frac{\partial F_2}{\partial w_3}\frac{dw_3}{d\zeta_3}$$

$$\frac{d}{d\eta}\left(\frac{dw_3}{d\zeta_3}\right)=\frac{\partial F_3}{\partial w_1}\frac{dw_1}{d\zeta_3}+\frac{\partial F_3}{\partial w_2}\frac{dw_2}{d\zeta_3}+\frac{\partial F_3}{\partial w_3}\frac{dw_3}{d\zeta_3}$$

$$\frac{d}{d\eta}\left(\frac{dw_1}{d\zeta_1}\right)=\frac{\partial F_1}{\partial w_1}\frac{dw_1}{d\zeta_1}+\frac{\partial F_1}{\partial w_2}\frac{dw_2}{d\zeta_1}+\frac{\partial F_1}{\partial w_3}\frac{dw_3}{d\zeta_1}$$

$$\frac{d}{d\eta}\left(\frac{dw_2}{d\zeta_1}\right)=\frac{\partial F_2}{\partial w_1}\frac{dw_1}{d\zeta_1}+\frac{\partial F_2}{\partial w_2}\frac{dw_2}{d\zeta_1}+\frac{\partial F_2}{\partial w_3}\frac{dw_3}{d\zeta_1}$$

$$\frac{d}{d\eta}\left(\frac{dw_3}{d\zeta_1}\right)=\frac{\partial F_3}{\partial w_1}\frac{dw_1}{d\zeta_1}+\frac{\partial F_3}{\partial w_2}\frac{dw_2}{d\zeta_1}+\frac{\partial F_3}{\partial w_3}\frac{dw_3}{d\zeta_1}$$

$$\frac{d}{d\eta}\left(\frac{dw_1}{d\zeta_3}\right)=\frac{\partial F_1}{\partial w_1}\frac{dw_1}{d\zeta_3}+\frac{\partial F_1}{\partial w_2}\frac{dw_2}{d\zeta_3}+\frac{\partial F_1}{\partial w_3}\frac{dw_3}{d\zeta_3}$$

$$\frac{d}{d\eta}\left(\frac{dw_2}{d\zeta_3}\right)=\frac{\partial F_2}{\partial w_1}\frac{dw_1}{d\zeta_3}+\frac{\partial F_2}{\partial w_2}\frac{dw_2}{d\zeta_3}+\frac{\partial F_2}{\partial w_3}\frac{dw_3}{d\zeta_3}$$

$$\frac{d}{d\eta}\left(\frac{dw_3}{d\zeta_3}\right)=\frac{\partial F_3}{\partial w_1}\frac{dw_1}{d\zeta_3}+\frac{\partial F_3}{\partial w_2}\frac{dw_2}{d\zeta_3}+\frac{\partial F_3}{\partial w_3}\frac{dw_3}{d\zeta_3}$$

$$\frac{dw_4}{d\eta}=\frac{\partial F_1}{\partial w_1}w_4+\frac{\partial F_1}{\partial w_2}w_5+\frac{\partial F_1}{\partial w_3}w_6$$

$$\frac{dw_5}{d\eta}=\frac{\partial F_2}{\partial w_1}w_4+\frac{\partial F_2}{\partial w_2}w_5+\frac{\partial F_2}{\partial w_3}w_6$$

$$\frac{dw_6}{d\eta}=\frac{\partial F_3}{\partial w_1}w_4+\frac{\partial F_3}{\partial w_2}w_5+\frac{\partial F_3}{\partial w_3}w_6$$

$$\frac{dw_7}{d\eta}=\frac{\partial F_1}{\partial w_1}w_7+\frac{\partial F_1}{\partial w_2}w_8+\frac{\partial F_1}{\partial w_3}w_9$$

$$\frac{dw_8}{d\eta}=\frac{\partial F_2}{\partial w_1}w_7+\frac{\partial F_2}{\partial w_2}w_8+\frac{\partial F_2}{\partial w_3}w_9$$

$$\frac{dw_9}{d\eta}=\frac{\partial F_3}{\partial w_1}w_7+\frac{\partial F_3}{\partial w_2}w_8+\frac{\partial F_3}{\partial w_3}w_9$$

$$\frac{dw_1}{d\eta} = F_1 = w_2 \qquad w_1(0) = \zeta_1$$

$$\frac{dw_2}{d\eta} = F_2 = w_3 \qquad w_2(0) = 0$$

$$\frac{dw_3}{d\eta} = F_3 = -1 + w_2^2 \qquad w_3(0) = \zeta_3$$

$$\frac{dw_4}{d\eta} = J_{11}w_4 + J_{12}w_5 + J_{13}w_6 \qquad w_4(0) = 1$$

$$\frac{dw_5}{d\eta} = J_{21}w_4 + J_{22}w_5 + J_{23}w_6 \qquad w_5(0) = 0$$

$$\frac{dw_6}{d\eta} = J_{31}w_4 + J_{32}w_5 + J_{33}w_6 \qquad w_6(0) = 0$$

$$\frac{dw_7}{d\eta} = J_{11}w_7 + J_{12}w_8 + J_{13}w_9 \qquad w_6(0) = 0$$

$$\frac{dw_8}{d\eta} = J_{21}w_7 + J_{22}w_8 + J_{23}w_9 \qquad w_7(0) = 0$$

$$\frac{dw_9}{d\eta} = J_{31}w_7 + J_{32}w_8 + J_{33}w_9 \qquad w_8(0) = 1$$

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2w_2 & 0 \end{bmatrix}$$

$$R_2 = w_2(\infty) - 1 = 0$$

$$R_3 = w_3(\infty) - 0 = 0$$

$$R_2(\zeta_1, \zeta_3) = 0$$

$$R_3(\zeta_1, \zeta_3) = 0$$

$$\begin{bmatrix} R_2(\zeta_{1,n+1}, \zeta_{3,n+1}) \\ R_3(\zeta_{1,n+1}, \zeta_{3,n+1}) \end{bmatrix} = \begin{bmatrix} R_2(\zeta_1, \zeta_3) \\ R_3(\zeta_1, \zeta_3) \end{bmatrix} + \begin{bmatrix} \frac{dR_2}{d\zeta_1} & \frac{dR_2}{d\zeta_3} \\ \frac{dR_3}{d\zeta_1} & \frac{dR_3}{d\zeta_3} \end{bmatrix} \begin{bmatrix} \zeta_{1,n+1} - \zeta_1 \\ \zeta_{3,n+1} - \zeta_3 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} R_2(\zeta_1, \zeta_3) \\ R_3(\zeta_1, \zeta_3) \end{bmatrix} + \begin{bmatrix} w_5 & w_8 \\ w_6 & w_9 \end{bmatrix} \begin{bmatrix} \zeta_{1,n+1} - \zeta_1 \\ \zeta_{3,n+1} - \zeta_3 \end{bmatrix}$$

Homework, Problem 2

1. The velocity profile for the boundary layer in the flow past a wedge an be written in the form:

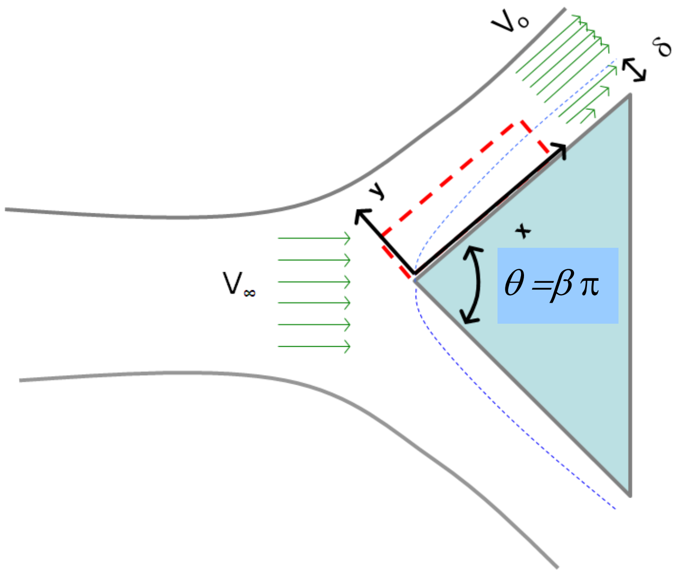
$$f''' + ff'' + \beta\left(1-\left(f'\right)^2\right) = 0$$

$$f(0)=0$$

$$f'(0)=0$$

$$f'(\infty)=1$$

a. Use the shooting method to solve this problem and draw the plot of f' vs η for different values of β (-0.2,0,0.5,1.0,1.5)



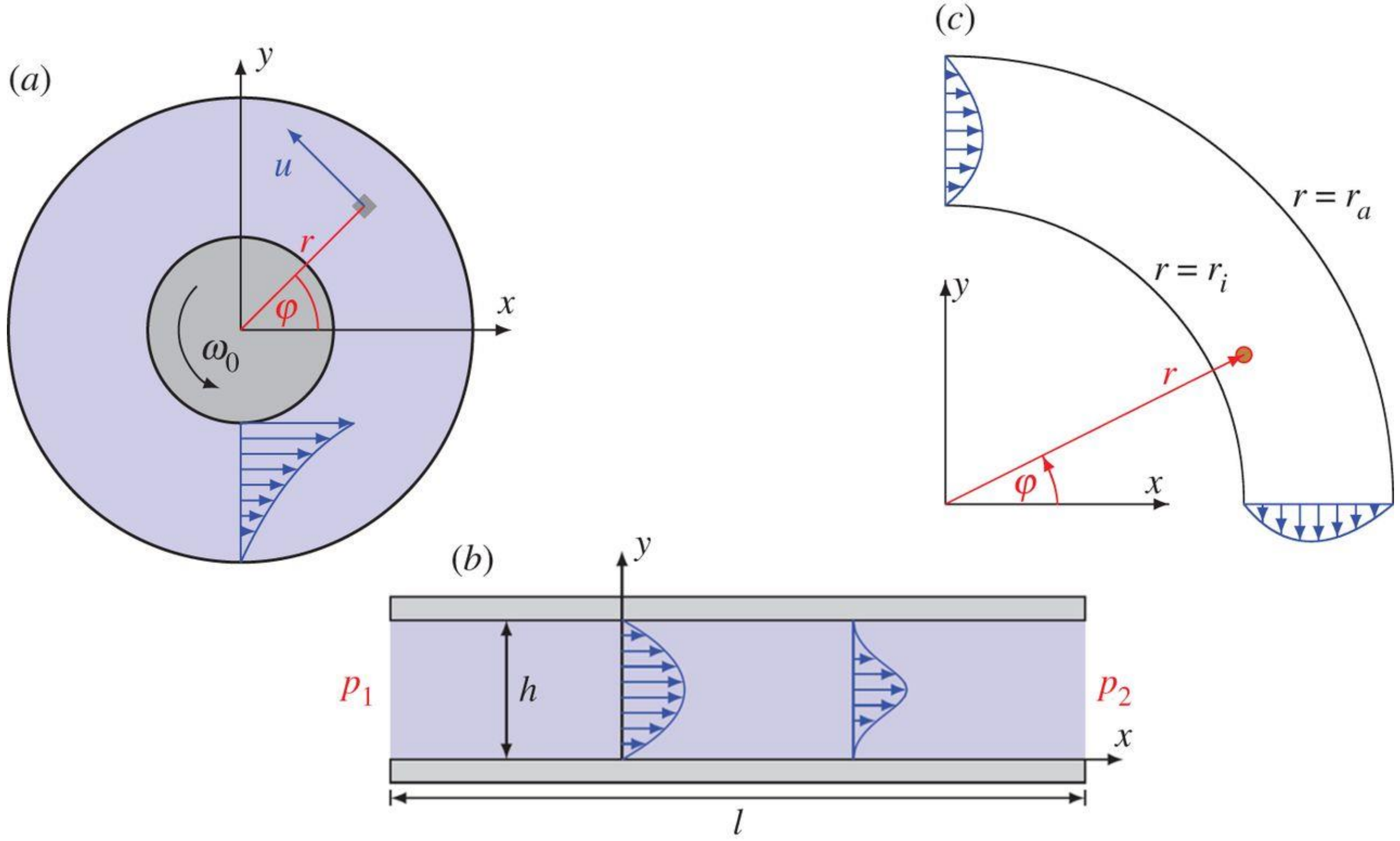
Internal energy relationship for complex systems

$$dU = TdS - pdV + \sum \overline{G}_i dn_i + \sigma\, da + \Psi dq + BdI$$

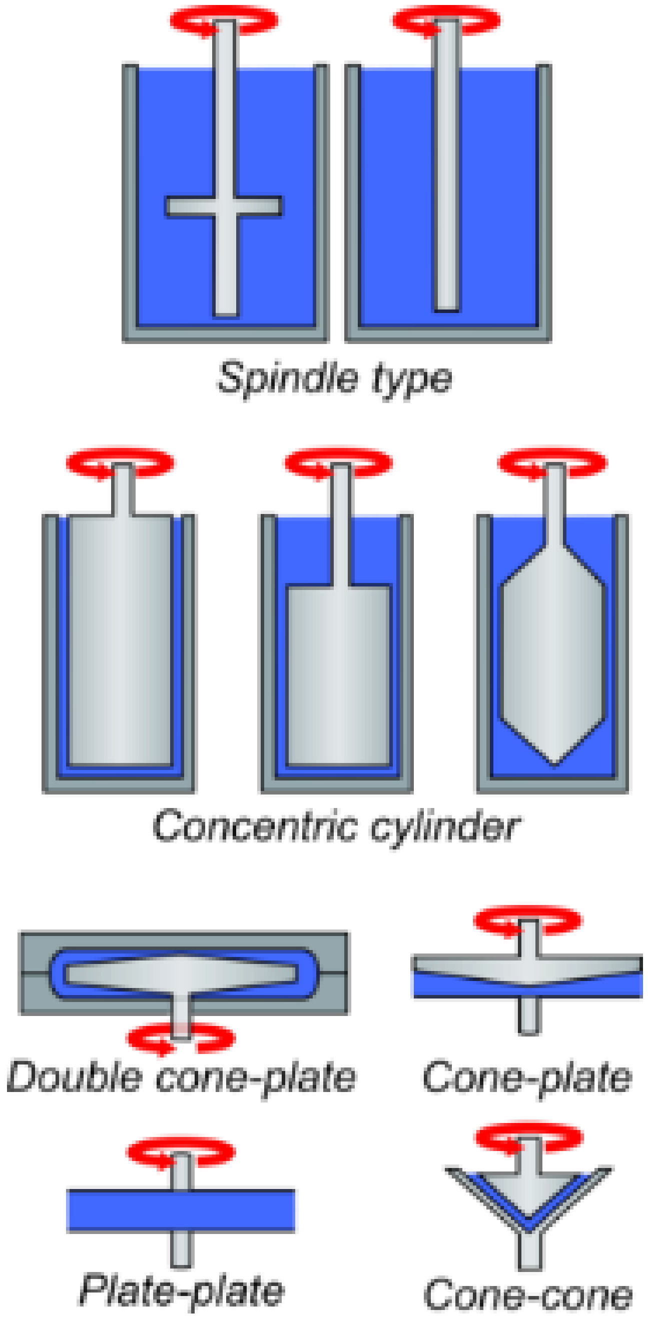
σ=Surface tensión, a= área

Ψ=Electrical potential, q=charge

B=Magnetic field, I= Magnetic moment



(a)
y
x
u
r
φ
ω0
(b)
y
x
h
p1
p2
l
(c)
r = ra
r = ri
y
x
r
φ


Spindle type
Concentric cylinder
Double cone-plate
Cone-plate
Plate-plate
Cone-cone

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\ \underline{v}) = 0$$

$$\frac{\partial(\rho\ \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho\ \underline{v}\ \underline{v}) = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\ p + \rho\ \underline{g}$$

$$\frac{\partial(\rho\ \underline{v})}{\partial t} = \begin{matrix} \left[\frac{\partial[\rho v_r]}{\partial t}\right] \hat{e}_r \\ \left[\frac{\partial[\rho v_\theta]}{\partial t}\right] \hat{e}_\theta \\ \left[\frac{\partial[\rho v_z]}{\partial t}\right] \hat{e}_z \end{matrix}$$

$$\underline{\nabla} p = \begin{matrix} \left[\frac{\partial p}{\partial r}\right] \hat{e}_r \\ \left[\frac{1}{r}\frac{\partial p}{\partial \theta}\right] \hat{e}_\theta \\ \left[\frac{\partial p}{\partial z}\right] \hat{e}_z \end{matrix}$$

$$\rho\ \underline{g} = \begin{matrix} [0]\hat{e}_r \\ [0]\hat{e}_\theta \\ [-g]\hat{e}_z \end{matrix}$$

$$\underline{\nabla} \cdot \underline{\underline{\tau}} = \begin{matrix} \left[\dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\partial \tau_{zr}}{\partial z} + \dfrac{\tau_{rr}}{r} - \dfrac{\tau_{\theta\theta}}{r}\right] \hat{e}_r \\ \left[\dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + \dfrac{\partial \tau_{z\theta}}{\partial z} + \dfrac{\tau_{r\theta}}{r} + \dfrac{\tau_{\theta r}}{r}\right] \hat{e}_\theta \\ \left[\dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \tau_{zz}}{\partial z} + \dfrac{\tau_{rz}}{r}\right] \hat{e}_z \end{matrix}$$

$$\boxed{\tau_{r\theta} = \tau_{\theta r}}$$

$$\underline{\nabla} \cdot \underline{\underline{\tau}} = \begin{matrix} \left[\dfrac{\partial \tau_{rr}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta r}}{\partial \theta} + \dfrac{\partial \tau_{zr}}{\partial z} + \dfrac{\tau_{rr}}{r} - \dfrac{\tau_{\theta\theta}}{r}\right] \hat{e}_r \\ \left[\dfrac{1}{r^2}\dfrac{\partial [r^2\, \tau_{r\theta}]}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta\theta}}{\partial \theta} + \dfrac{\partial \tau_{z\theta}}{\partial z}\right] \hat{e}_\theta \\ \left[\dfrac{\partial \tau_{rz}}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{\theta z}}{\partial \theta} + \dfrac{\partial \tau_{zz}}{\partial z} + \dfrac{\tau_{rz}}{r}\right] \hat{e}_z \end{matrix}$$

$$\underline{\nabla} \cdot [\rho\, \underline{v}\, \underline{v}\,] = \begin{matrix} \left[\dfrac{\partial[\rho\, v_r\, v_r]}{\partial r} + \dfrac{1}{r}\dfrac{\partial[\rho\, v_\theta\, v_r]}{\partial \theta} + \dfrac{\partial[\rho\, v_z\, v_r]}{\partial z} + \dfrac{\rho\, v_r\, v_r}{r} - \dfrac{\rho\, v_\theta\, v_\theta}{r}\right] \hat{e}_r \\ \left[\dfrac{\partial[\rho\, v_r\, v_\theta]}{\partial r} + \dfrac{1}{r}\dfrac{\partial[\rho\, v_\theta\, v_\theta]}{\partial \theta} + \dfrac{\partial[\rho\, v_z\, v_\theta]}{\partial z} + \dfrac{\rho\, v_r\, v_\theta}{r} + \dfrac{\rho\, v_\theta\, v_r}{r}\right] \hat{e}_\theta \\ \left[\dfrac{\partial[\rho\, v_r\, v_z]}{\partial r} + \dfrac{1}{r}\dfrac{\partial[\rho\, v_\theta\, v_z]}{\partial \theta} + \dfrac{\partial[\rho\, v_z\, v_z]}{\partial z} + \dfrac{\rho\, v_r\, v_z}{r}\right] \hat{e}_z \end{matrix}$$

$$\underline{\nabla} \cdot [\rho\, \underline{v}\, \underline{v}\,] = \begin{matrix} \left[\dfrac{\partial[\rho\, v_r\, v_r]}{\partial r} + \dfrac{1}{r}\dfrac{\partial[\rho\, v_\theta\, v_r]}{\partial \theta} + \dfrac{\partial[\rho\, v_z\, v_r]}{\partial z} + \dfrac{\rho\, v_r\, v_r}{r} - \dfrac{\rho\, v_\theta\, v_\theta}{r}\right] \hat{e}_r \\ \left[\dfrac{1}{r^2}\dfrac{\partial[r^2\, \rho\, v_r\, v_\theta]}{\partial r} + \dfrac{1}{r}\dfrac{\partial[\rho\, v_\theta\, v_\theta]}{\partial \theta} + \dfrac{\partial[\rho\, v_z\, v_\theta]}{\partial z}\right] \hat{e}_\theta \\ \left[\dfrac{\partial[\rho\, v_r\, v_z]}{\partial r} + \dfrac{1}{r}\dfrac{\partial[\rho\, v_\theta\, v_z]}{\partial \theta} + \dfrac{\partial[\rho\, v_z\, v_z]}{\partial z} + \dfrac{\rho\, v_r\, v_z}{r}\right] \hat{e}_z \end{matrix}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial [r\, \rho v_r]}{\partial r} + \frac{1}{r}\frac{\partial [\rho v_\theta]}{\partial \theta} + \frac{\partial [\rho v_z]}{\partial z} = 0$$

For steady state flow, and without radial flow, and without axial flow

$$\frac{\partial B}{\partial t} = 0$$

$v_r = 0$

$v_z = 0$

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial [r\, \rho v_r]}{\partial r} + \frac{1}{r}\frac{\partial [\rho v_\theta]}{\partial \theta} + \frac{\partial [\rho v_z]}{\partial z} = 0$$

$$\frac{1}{r}\frac{\partial [\rho v_\theta]}{\partial \theta} = 0$$

For incompressible flow

$$\frac{\rho}{r}\frac{\partial [v_\theta]}{\partial \theta} = 0$$

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

$$\frac{\partial(\rho\,\underline{v})}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \underline{\nabla}\cdot\underline{\underline{\tau}} - \underline{\nabla}\,p + \rho\,\underline{g}$$

For azimuthal direction

$$\left[\frac{\partial[\rho v_\theta]}{\partial t}\right] + \left[\frac{1}{r^2}\frac{\partial[r^2\,\rho\,v_r\,v_\theta]}{\partial r} + \frac{1}{r}\frac{\partial[\rho\,v_\theta\,v_\theta]}{\partial \theta} + \frac{\partial[\rho\,v_z\,v_\theta]}{\partial z}\right] =$$

$$\left[\frac{1}{r^2}\frac{\partial[r^2\,\tau_{r\theta}]}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right] + \left[\frac{1}{r}\frac{\partial p}{\partial \theta}\right] + [0]$$

For steady state flow, and without radial flow, and without axial flow

$$\frac{\partial B}{\partial t} = 0$$

$v_r = 0$

$v_z = 0$

$$\left[\frac{1}{r}\frac{\partial[\rho\,v_\theta\,v_\theta]}{\partial \theta}\right] = \left[\frac{1}{r^2}\frac{\partial[r^2\,\tau_{r\theta}]}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z}\right] + \left[\frac{1}{r}\frac{\partial p}{\partial \theta}\right] + [0]$$

Continuity equation gives…

$$\frac{\partial v_\theta}{\partial \theta} = 0$$

$$0 = \left[\frac{1}{r^2}\frac{\partial[r^2\,\tau_{r\theta}]}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial z}\right] + \left[\frac{1}{r}\frac{\partial p}{\partial\theta}\right]$$

For axis symmetric flow…

$$\left[\frac{1}{r}\frac{\partial p}{\partial\theta}\right] = 0$$

$$0 = \left[\frac{1}{r^2}\frac{\partial[r^2\,\tau_{r\theta}]}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{z\theta}}{\partial z}\right]$$

Viscous stress analysis

$$\underline{\underline{\tau}} = \mu\left[\underline{\nabla}\,\underline{v} + \left(\underline{\nabla}\,\underline{v}\right)^{\mathrm{T}}\right] - \left(\frac{2}{3}\mu - \kappa\right)\left(\underline{\nabla}\cdot\underline{v}\right)\underline{I}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial[v_\theta/r]}{\partial r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] = \mu\left[r\frac{\partial[v_\theta/r]}{\partial r}\right]$$

$$\tau_{z\theta} = \tau_{\theta z} = \mu\left[\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right] = 0$$

$$\tau_{\theta\theta} = \tau_{\theta\theta} = \mu\left[\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right] - \left(\frac{2}{3}\mu - \kappa\right)\left(\underline{\nabla}\cdot\underline{v}\right) = 0$$

$$0 = \left[\frac{1}{r^2}\frac{\partial[r^2\,\tau_{r\theta}]}{\partial r}\right]$$

$$\tau_{r\theta} = \mu\left[r\frac{\partial[v_\theta/r]}{\partial r}\right]$$

$$r^2\,\tau_{r\theta} = C_1$$

$$r^2\tau_{r\theta} = C_1 = \mu\,r^3\frac{\partial[v_\theta/r]}{\partial r}$$

$$\frac{C_1}{\mu\,r^3}dr = d[v_\theta/r]$$

$$\frac{C_1}{\mu}d\left(-\frac{1}{2r^2}\right) = d[v_\theta/r]$$

$$\left(-\frac{C_1}{2\mu r^2}\right) + C_2 = [v_\theta/r]$$

$$\left(-\frac{C_1}{2\,\mu\,r}\right) + C_2\,r = v_\theta$$

$$v_\theta = \left(-\frac{C_1}{2\,\mu\, r}\right) + C_2\, r$$

$$\omega_i R_i = \left(-\frac{C_1}{2\,\mu R_i}\right) + C_2 R_i$$

$$\omega_o R_o = \left(-\frac{C_1}{2\,\mu R_o}\right) + C_2 R_o$$

$$\left(-\frac{C_1}{2\,\mu}\right) = \frac{(\omega_i - \omega_o) R_o^2}{[(R_o/R_i)^2 - 1]}$$

$$C_2 = \omega_o - \frac{(\omega_i - \omega_o)}{[(R_o/R_i)^2 - 1]}$$

$$r^2\, \tau_{r\theta} = C_1$$

$$R_i^2 \tau_{iw} = C_1$$

$$R_i^2 \tau_{iw} = C_1$$

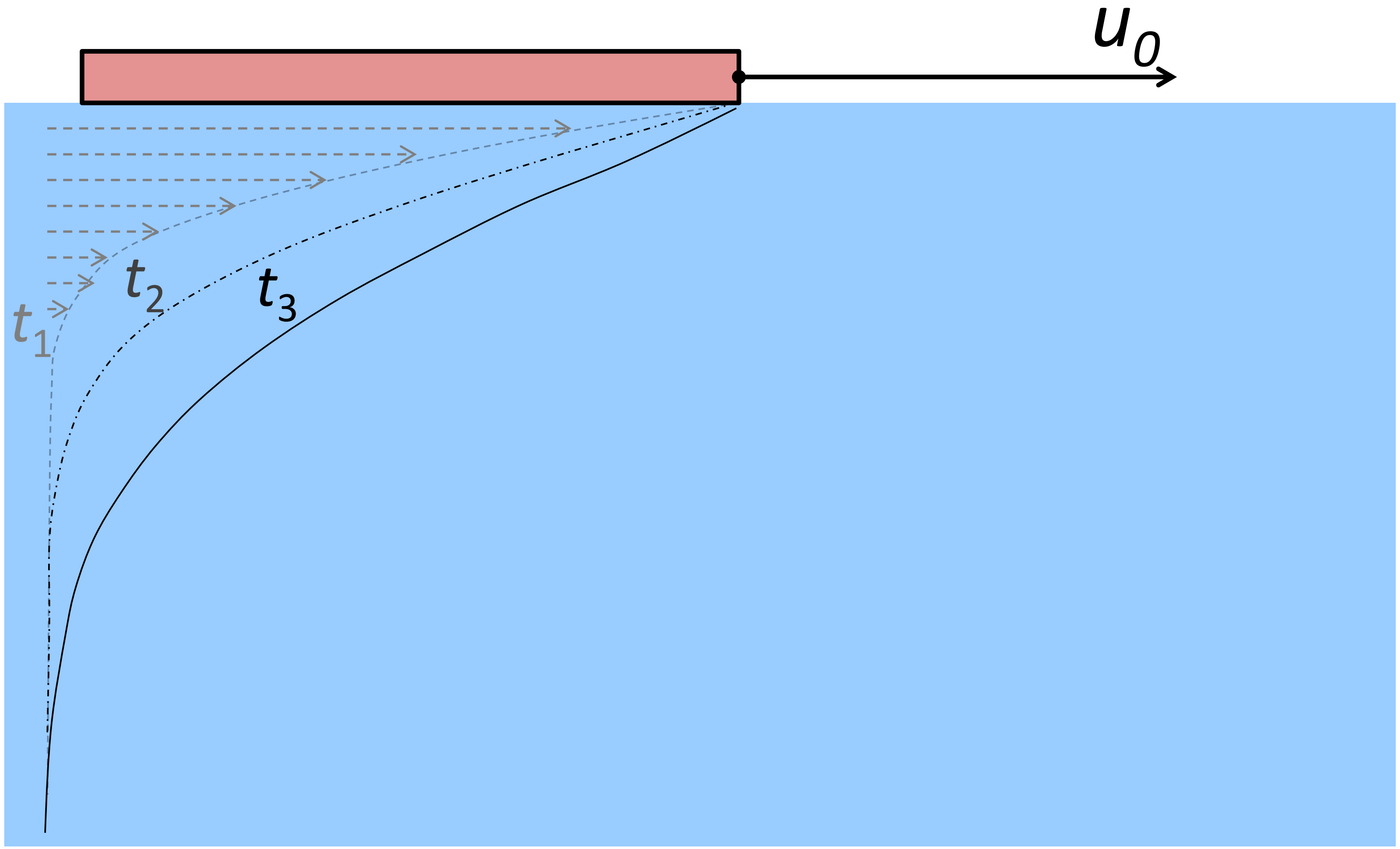
$$C_1 = -\frac{2\,\mu\,(\omega_i - \omega_o) R_o^2}{[(R_o/R_i)^2 - 1]}$$

$$\boxed{\tau_{iw} = -\frac{2\,\mu\,(\omega_i - \omega_o) R_o^2}{[(R_o/R_i)^2 - 1] R_i^2}}$$

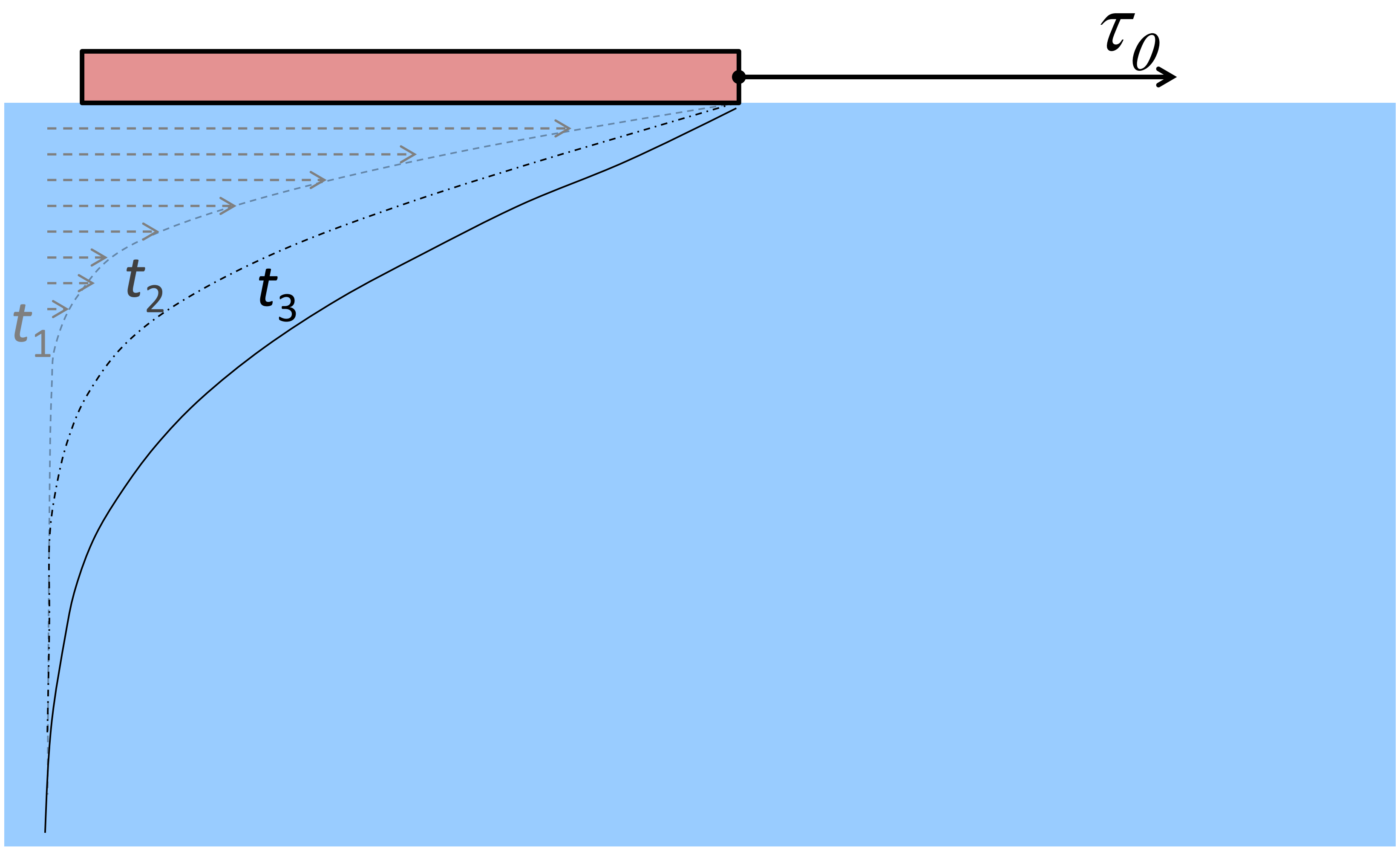
$$\frac{v_\theta}{r} = \omega_i - \frac{(\omega_i - \omega_o)[(R_o/R_i)^2 - (R_o/r)^2]}{[(R_o/R_i)^2 - 1]}$$

$$\frac{(T/R_i)}{2\pi R_i L} = \frac{2\,\mu\,(\omega_i - \omega_o)R_o^2}{[(R_o/R_i)^2 - 1]R_i^2}$$

Flow near a Wall suddenly set in motion



Flow near a Wall suddenly set in motion



$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}) = 0$$

$$\frac{\partial (\rho\, \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v}) = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g}$$

Simplified coupled continuity , linear momentum and newton viscosity law

$$\rho \frac{\partial v_x}{\partial t} = \mu \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2}$$

u will be the symbol to represent the velocity component in x-axis

The methodology called similarity method requires to understand the dimensions of each variable and properties, and group them in dimensionless new variables

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

Equation of dimensions

$$\frac{u}{t} [=] \frac{\nu\, u}{y^2}$$

Dimensionless group

$$\frac{y^2}{\nu\, t} [=] 1$$

$$\frac{y}{\sqrt{\nu\, t}} [=] 1$$

The root of a dimensionless variable is a dimensionless number as well

$$\eta = \frac{y}{\sqrt{4\, \nu\, t}}$$

A constant number is suggested to be included, just in order to simplify the resulting differential equation

The similarity method looks to simplify the number of independent variables

$$u = u(y, t)$$

Velocity as a function of two independent variables (y, t)

$$du = \left[\frac{\partial u}{\partial y}\right] dy + \left[\frac{\partial u}{\partial t}\right] dt$$

$$u = u(\eta)$$

$$du = \left[\frac{du}{d\eta}\right] d\eta$$

Velocity as a function of a single independent variable (η)

The model will be modified or simplified using the new independent variable

$$\eta = \frac{y}{\sqrt{4\,\nu\,t}}$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial y} = \left[\frac{du}{d\eta}\right] \frac{\partial \eta}{\partial y}$$

$$d\left[\frac{\partial u}{\partial y}\right] = \frac{\partial \eta}{\partial y} d\left[\frac{du}{d\eta}\right] + \left[\frac{du}{d\eta}\right] d\left[\frac{\partial \eta}{\partial y}\right]$$

$$d\left[\frac{\partial u}{\partial y}\right] = \frac{\partial \eta}{\partial y} \frac{d}{d\eta}\left[\frac{du}{d\eta}\right] d\eta + \left[\frac{du}{d\eta}\right] d\left[\frac{\partial \eta}{\partial y}\right]$$

$$\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial \eta}{\partial y} \frac{d}{d\eta}\left[\frac{du}{d\eta}\right] \frac{\partial \eta}{\partial y} + \left[\frac{du}{d\eta}\right] \frac{\partial}{\partial y}\left[\frac{\partial \eta}{\partial y}\right]$$

$$\boxed{\frac{\partial u}{\partial t} = \left[\frac{du}{d\eta}\right] \frac{\partial \eta}{\partial t}}$$

$$\boxed{\frac{\partial^2 u}{\partial y^2} = \left[\frac{d^2 u}{d\eta^2}\right] \left(\frac{\partial \eta}{\partial y}\right)^2 + \left[\frac{du}{d\eta}\right] \frac{\partial^2 \eta}{\partial y^2}}$$

The relationship between the previous independent variables, wit the new independent variable is calculated here

$$\eta = \frac{y}{\sqrt{4\,\nu\,t}}$$

$$\frac{\partial \eta}{\partial t} = -\frac{y}{2\,t^{3/2}\sqrt{4\nu}} = -\frac{\eta}{2\,t}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{4\,\nu\,t}}$$

$$\frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial t} = \left[\frac{du}{d\eta}\right]\frac{\partial \eta}{\partial t}$$

$$\frac{\partial^2 u}{\partial y^2} = \left[\frac{d^2 u}{d\eta^2}\right]\left(\frac{\partial \eta}{\partial y}\right)^2 + \left[\frac{du}{d\eta}\right]\frac{\partial^2 \eta}{\partial y^2}$$

$$\frac{\partial u}{\partial t} = -\frac{\eta}{2\,t}\left[\frac{du}{d\eta}\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{4\,\nu\,t}\left[\frac{d^2 u}{d\eta^2}\right]$$

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial t} = -\frac{\eta}{2\,t}\left[\frac{du}{d\eta}\right] = \nu\,\frac{1}{4\,\nu\,t}\left[\frac{d^2 u}{d\eta^2}\right] = \frac{\partial^2 u}{\partial y^2}$$

$$-\frac{\eta}{2\,t}\left[\frac{du}{d\eta}\right]=\nu\frac{1}{4\,\nu\,t}\left[\frac{d^2u}{d\eta^2}\right]$$

Original PDE

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}$$

$$\Rightarrow$$

Final DE

$$-2\eta\left[\frac{du}{d\eta}\right]=\left[\frac{d^2u}{d\eta^2}\right]=\frac{d}{d\eta}\left[\frac{du}{d\eta}\right]$$

$$-2\eta\,d\eta=\frac{d\left[\frac{du}{d\eta}\right]}{\left[\frac{du}{d\eta}\right]}$$

$$-\eta^2+\mathrm{c}=\ln\left\{\left[\frac{du}{d\eta}\right]\right\}$$

The new DE can be integrated

$$\frac{du}{d\eta}=ke^{-\eta^2}$$

$$du=ke^{-\eta^2}d\eta$$

The new DE is integrated again to obtain velocity

$$u-u_0=k\int_o^{\eta}e^{-\eta^2}d\eta$$

The boundary conditions may help to calculate the integration constant

$$u_\infty-u_0=k\int_o^{\infty}e^{-\eta^2}d\eta$$

$$\frac{u-u_0}{u_\infty-u_0}=\frac{\int_o^{\eta}e^{-\eta^2}d\eta}{\int_o^{\infty}e^{-\eta^2}d\eta}=\mathrm{erf}(\eta)$$

$$u=u_0[1-\mathrm{erf}(\eta)\,]$$

Gauss error function gives the probability of a random variable with normal distribution, and pops up here as well.

The velocity profile , and history can be tracked with the error function .

$$\boxed{u = u_0\left[1 - \operatorname{erf}\left(\frac{y}{\sqrt{4\,\nu\,t}}\right)\right]}$$

If this equation is derived the stress can be estimated as well.

$$\frac{du}{dy} = \frac{d}{dy}\left[u_0[1 - \operatorname{erf}(\eta)\,]\right] = -\frac{2u_0}{\sqrt{\pi}}e^{-\eta^2}\left(\frac{d\eta}{dy}\right)$$

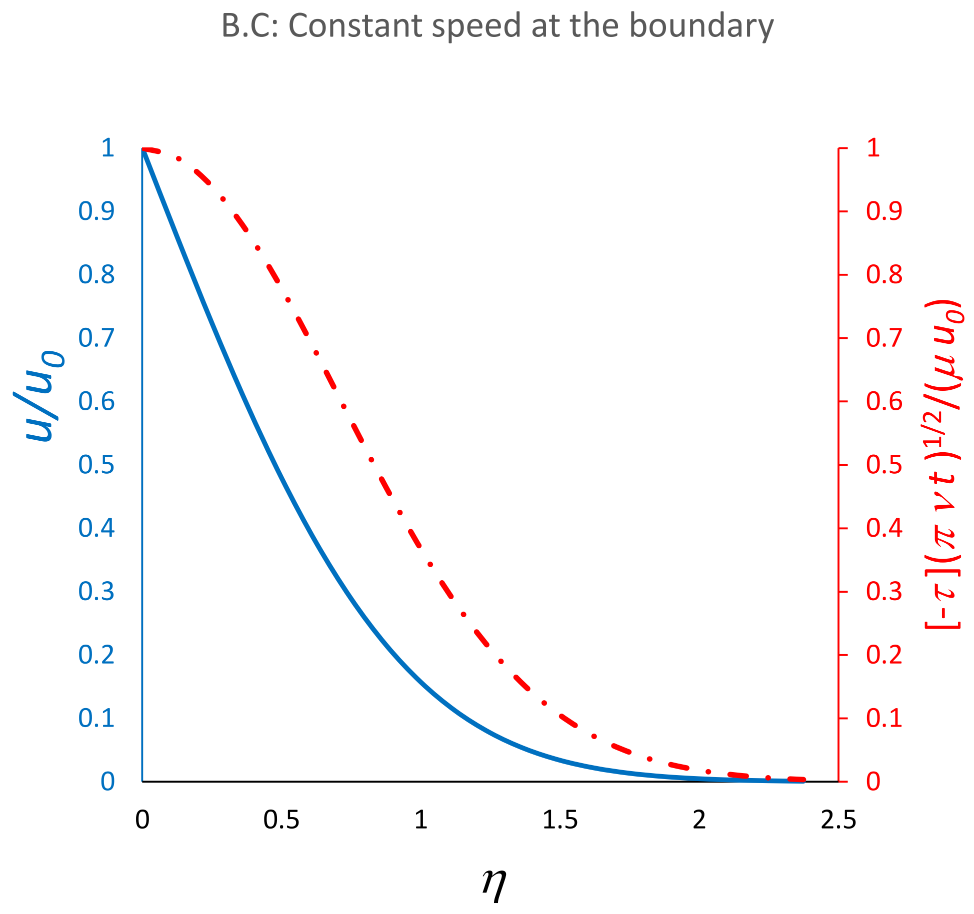
$$\frac{du}{dy} = -\frac{2u_0}{\sqrt{\pi}}e^{-\eta^2}\left(\frac{1}{\sqrt{4\,\nu\,t}}\right)$$

$$\mu\frac{du}{dy} = -\frac{u_0\,\mu}{\sqrt{\pi\nu\,t}}e^{-\eta^2}$$

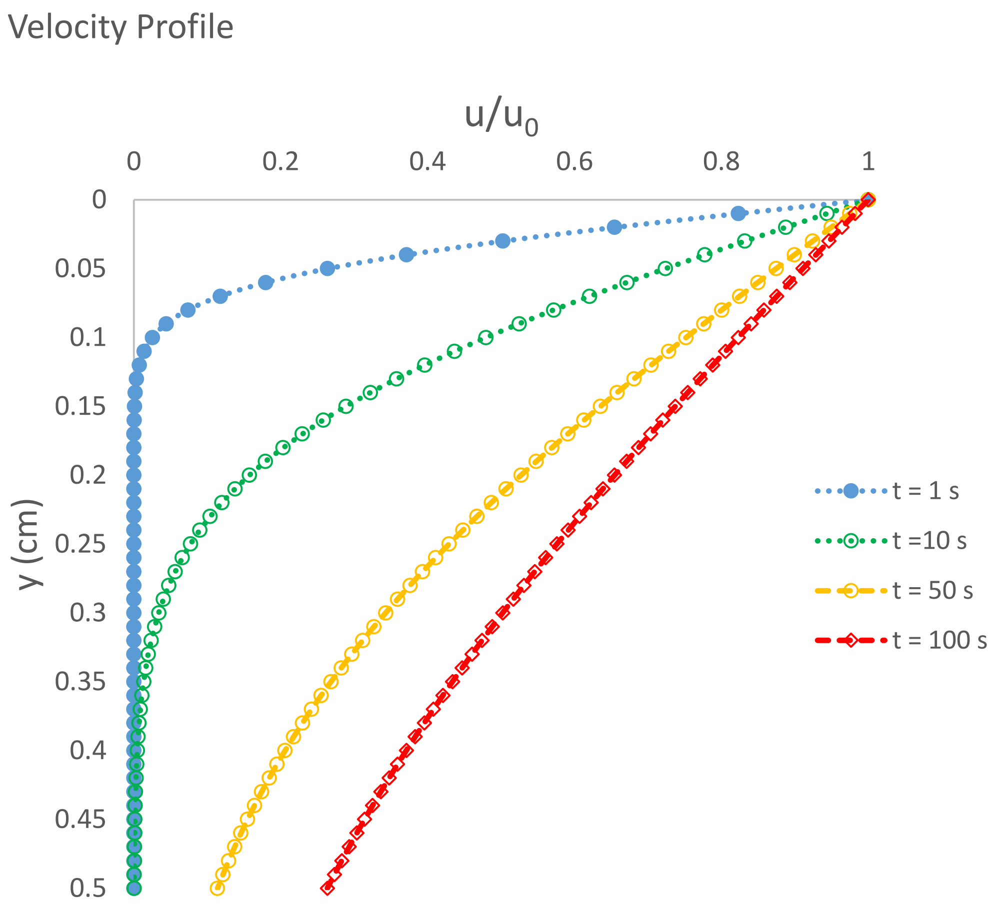
$$[-\tau] = \frac{u_0\,\mu}{\sqrt{\pi\nu\,t}}e^{-\eta^2} = \frac{u_0\,\mu}{\sqrt{\pi\nu\,t}}e^{-\frac{y^2}{4\,\nu\,t}}$$

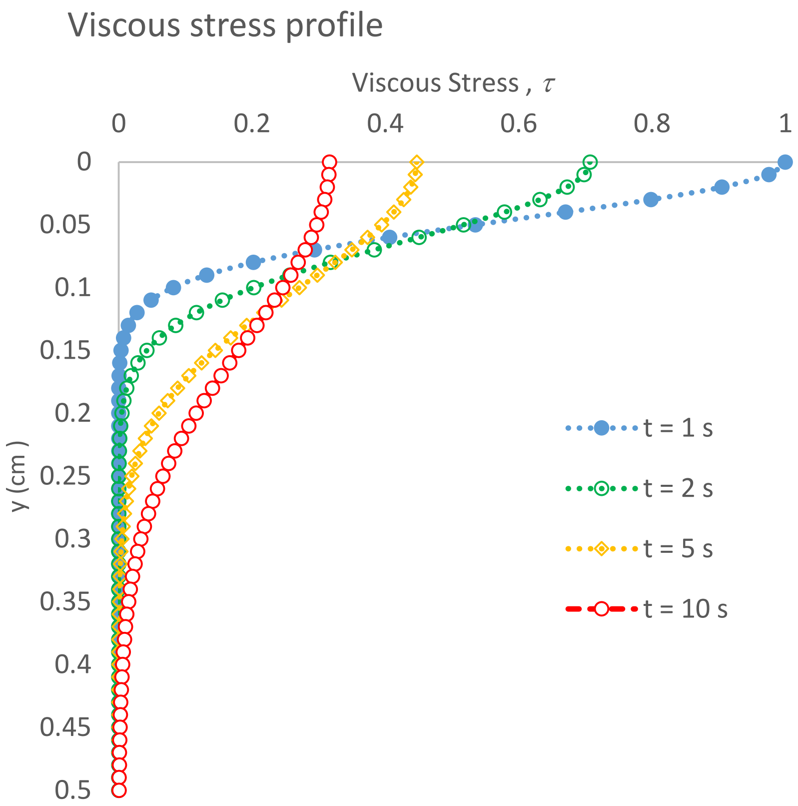
Equation of stress

$$\boxed{[-\tau] = \frac{u_0\,\mu}{\sqrt{\pi\nu\,t}}e^{-\frac{y^2}{4\,\nu\,t}}}$$



Velocity and stress as a function of space for different times





$$\mathrm{erf}(\eta) = \frac{\int_o^{\eta} e^{-\eta^2}\, d\eta}{\int_o^{\infty} e^{-\eta^2}\, d\eta} = \frac{2}{\sqrt{\pi}} \int_o^{\eta} e^{-\eta^2}\, d\eta \qquad \frac{d\ \mathrm{erf}(\eta)}{d\eta} = \frac{2}{\sqrt{\pi}} e^{-\eta^2}$$

$$u = u_0[1 - \mathrm{erf}(\eta)\ ]$$

$$u = u_0\left[1 - \mathrm{erf}(\frac{y}{\sqrt{4\nu t}})\ \right]$$

This is the velocity profile of the liquid when a constant velocity is imposed at the nearest boundary of the interface. What happen when the boundary condition changes to a constant stress.

Simplified coupled continuity and linear momentum

$$\rho\frac{\partial v_x}{\partial t} = \frac{\partial \tau_{yx}}{\partial y}$$

Rewrite for velocity

$$\frac{\partial v_x}{\partial t} = \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y}$$

multiplying by viscosity

$$\mu\frac{\partial v_x}{\partial t} = \frac{\mu}{\rho}\frac{\partial \tau_{yx}}{\partial y}$$

Differentiating by length

$$\frac{\partial}{\partial y}\left[\mu\frac{\partial v_x}{\partial t}\right] = \frac{\partial}{\partial y}\left[\nu\frac{\partial \tau_{yx}}{\partial y}\right]$$

Recasting for constant properties

$$\frac{\partial}{\partial t}\left[\mu\frac{\partial v_x}{\partial y}\right] = \nu\frac{\partial}{\partial y}\left[\frac{\partial \tau_{yx}}{\partial y}\right]$$

Using Newton viscosity law again

$$\frac{\partial \tau_{yx}}{\partial t} = \nu\frac{\partial}{\partial y}\left[\frac{\partial \tau_{yx}}{\partial y}\right]$$

$$\frac{\partial \tau_{yx}}{\partial t} = \nu\frac{\partial^2 \tau_{yx}}{\partial y^2}$$

A new diffusion equations shows up

$$\frac{\partial \tau}{\partial t} = \nu\frac{\partial^2 \tau}{\partial y^2}$$

The boundary conditions are the same as in the previous example, i.e. constant at origin and zero at infinity

$$\tau = \tau_0 \left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) \right]$$

The stress is used to integrate to obtain velocity

$$\mu \frac{du}{dy} = \tau_0 \left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) \right]$$

The velocity is known at infinity (e.g. zero velocity far away form the moving boundary), so integration will be to infinity to subside the integration constant.

$$u_\infty - u = \int_y^\infty \frac{\tau_0}{\mu} \left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) \right] dy = \int_y^\infty \frac{\tau_0 \sqrt{4\nu t}}{\mu} \left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) \right] d\left(\frac{y}{\sqrt{4\nu t}}\right)$$

From integration tables, of by integration by parts it can be proven to be:

$$\int_a^\infty [1 - \mathrm{erf}(w)]\, d\mathrm{w} = \frac{1}{\sqrt{\pi}} e^{-a^2} - a\left(1 - \mathrm{erf}(a)\right)$$

$$\int_a^\infty [1-\mathrm{erf}(w)]\,dw = \frac{1}{\sqrt{\pi}}e^{-a^2} - a\big(1-\mathrm{erf}(a)\big)$$

$$\int [1-\mathrm{erf}(w)]\,dw = w - \int \mathrm{erf}(w)\,dw$$

$$\int \mathrm{erf}(w)\,dw = w\,\mathrm{erf}(w) - \int w\,d\big(\mathrm{erf}(w)\big) = w\,\mathrm{erf}(w) - \int w\frac{d\big(\mathrm{erf}(w)\big)}{dw}\,dw$$

$$\int \mathrm{erf}(w)\,dw \;= w\,\mathrm{erf}(w) - \int w\frac{2e^{-w^2}}{\sqrt{\pi}}\,dw = w\,\mathrm{erf}(w) + \frac{1}{\sqrt{\pi}}\int e^{-w^2}\,d(-w^2)$$

$$\int \mathrm{erf}(w)\,dw \;= w\,\mathrm{erf}(w) + \frac{e^{-w^2}}{\sqrt{\pi}}$$

$$\int [1-\mathrm{erf}(w)]\,dw = w - \int \mathrm{erf}(w)\,dw = w[1-\mathrm{erf}(w)] \;- \frac{e^{-w^2}}{\sqrt{\pi}}$$

$$\frac{d\,\mathrm{erf}(\eta)}{d\eta} = \frac{2}{\sqrt{\pi}}e^{-\eta^2}$$

$$\int_{a}^{\infty} [1 - \mathrm{erf}(\mathrm{w})]\, \mathrm{dw} = \left\{ \mathrm{w}[1 - \mathrm{erf}(w)] - \frac{e^{-w^2}}{\sqrt{\pi}} \right\} \Bigg|_{w \to \infty} - \left[a[1 - \mathrm{erf}(a)] - \frac{e^{-a^2}}{\sqrt{\pi}} \right]$$

$$\int_{a}^{\infty} [1 - \mathrm{erf}(\mathrm{w})]\, \mathrm{dw} = \left[\frac{e^{-a^2}}{\sqrt{\pi}} - a[1 - \mathrm{erf}(a)] \right]$$

$$\frac{\mu(u_\infty - u)}{\tau_0\sqrt{4\nu t}} = \int_{y/\sqrt{4\nu t}}^{\infty} \left[1 - \mathrm{erf}\left(y/\sqrt{4\nu t}\right)\ \right] d\left(y/\sqrt{4\nu t}\right)$$

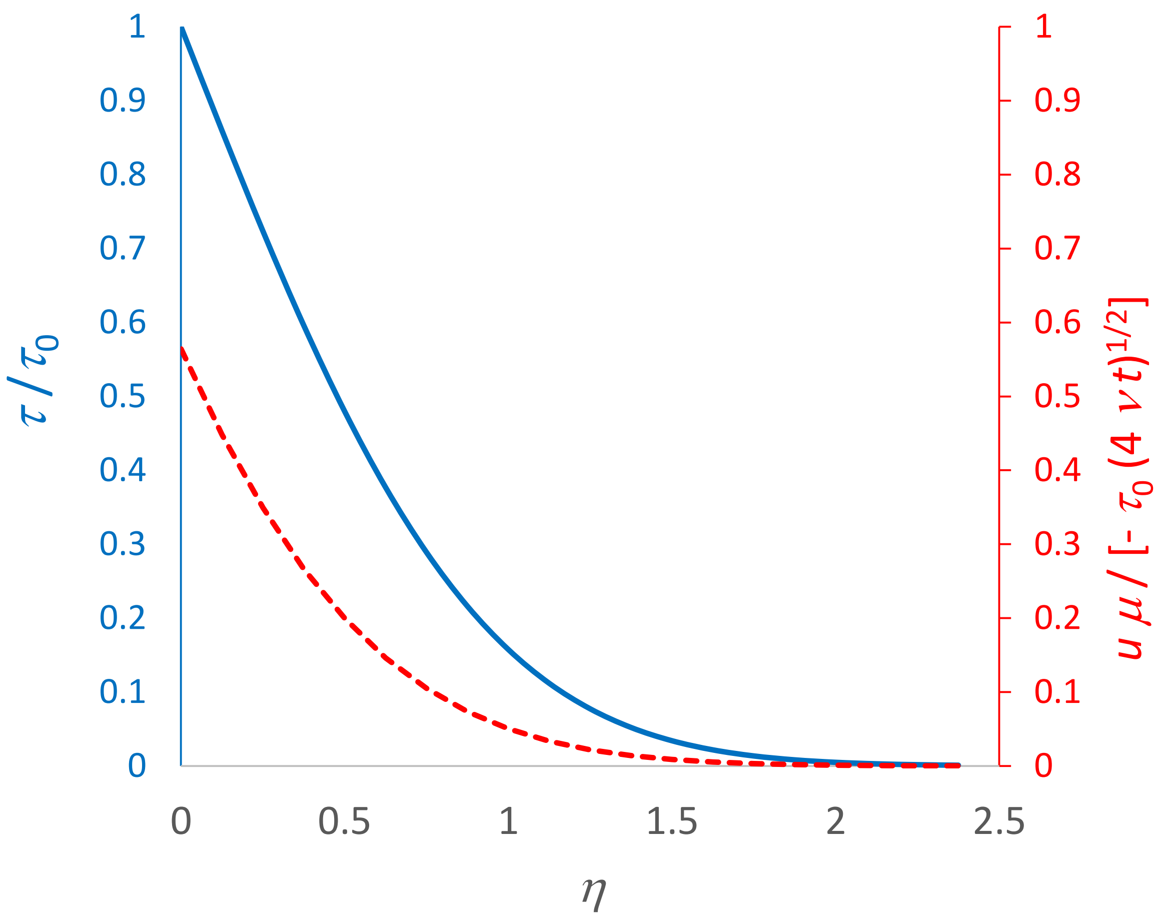
$$\frac{\mu(u_\infty - u)}{\tau_0\sqrt{4\nu t}} = \left[\frac{1}{\sqrt{\pi}} e^{-\frac{y^2}{4\nu t}}\right] - \left(y/\sqrt{4\nu t}\right)\left[1 - \mathrm{erf}\left(y/\sqrt{4\nu t}\right)\right]$$

$$\frac{\mu(u - u_\infty)}{[-\tau_0]\sqrt{4\nu t}} = \left[\frac{1}{\sqrt{\pi}} e^{-\frac{y^2}{4\nu t}}\right] - \left(y/\sqrt{4\nu t}\right)\left[1 - \mathrm{erf}\left(y/\sqrt{4\nu t}\right)\right]$$

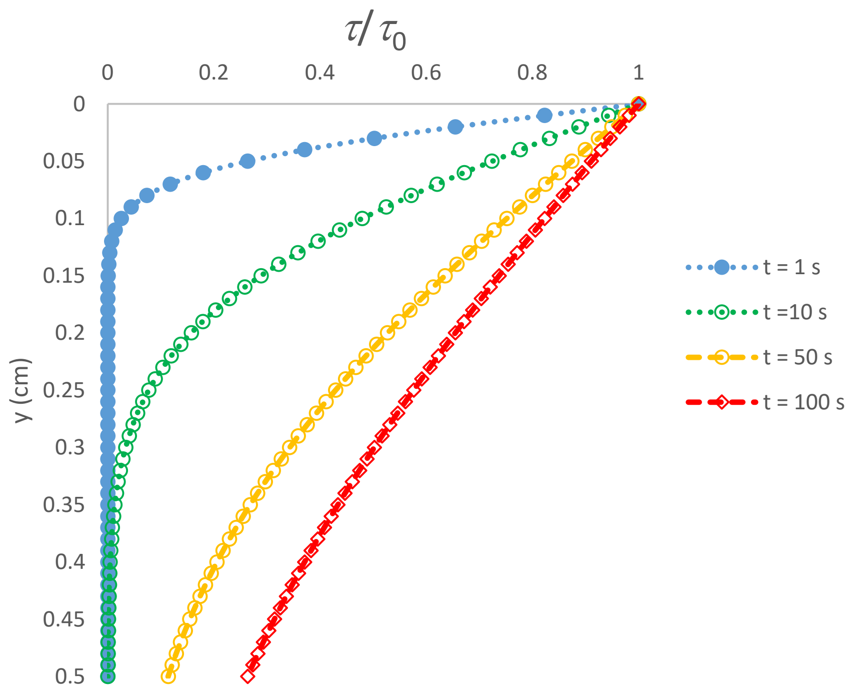
$$\frac{\mu(u - u_\infty)}{\tau_0\sqrt{4\nu t}} = \left(y/\sqrt{4\nu t}\right)\left[1 - \mathrm{erf}\left(y/\sqrt{4\nu t}\right)\right] - \left[\frac{1}{\sqrt{\pi}} e^{-\frac{y^2}{4\nu t}}\right]$$

$$\int_a^\infty [1 - \mathrm{erf}(w)]\, d\mathrm{w} = \frac{1}{\sqrt{\pi}} e^{-a^2} - a\left(1 - \mathrm{erf}(a)\right)$$

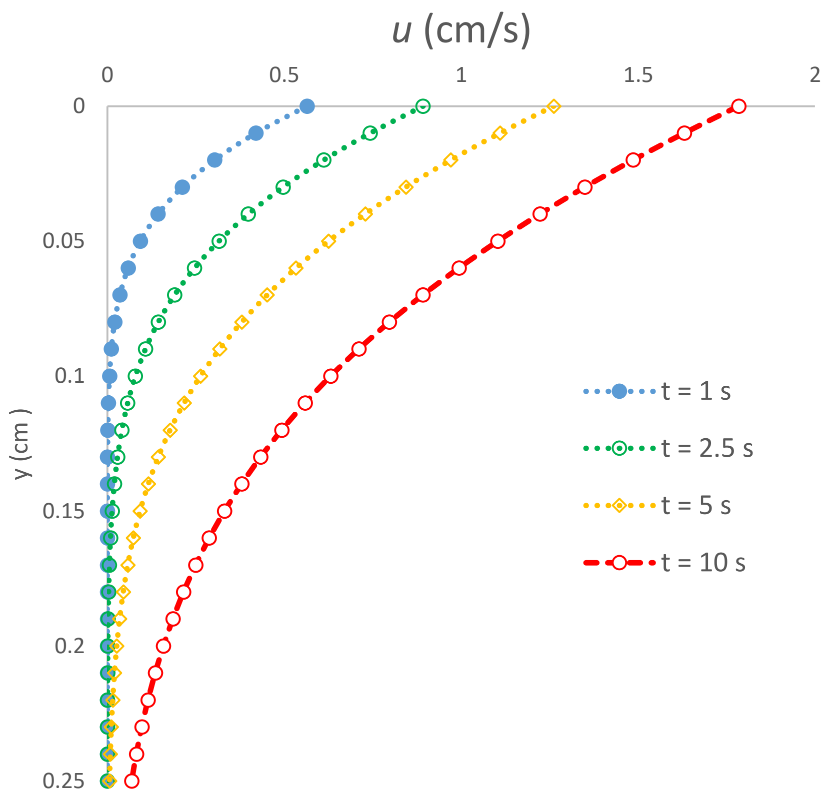
B.C: Constnt stress at the boundary



Viscous stress Profile



velocity profile

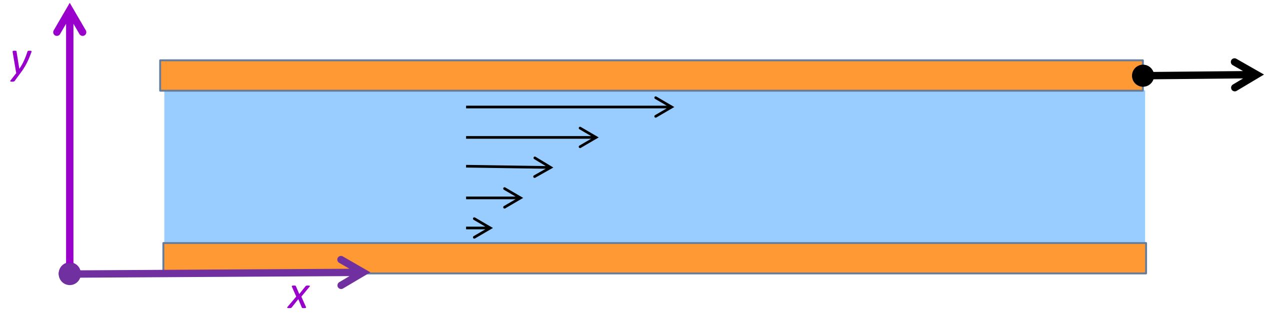


$$\tau = \tau_0 \left[1 - \mathrm{erf}\left(\frac{y}{\sqrt{4\nu t}}\right) \right]$$

$$\frac{\mu(u - u_\infty)}{[-\tau_0]\sqrt{4\nu t}} = \left[\frac{1}{\sqrt{\pi}} e^{-\frac{y^2}{4\nu t}} \right] - \left(y/\sqrt{4\nu t}\right)\left[1 - \mathrm{erf}\left(y/\sqrt{4\nu t}\right)\right]$$

The previous method works fine when both domains (time and space) extend to infinity, but if the space domain is limited, separation of variables method for solving differential equations may work, as long as we have homogeneous boundary conditions, but if not it may be possible to use some algebraic manipulations in order to satisfy the constraints

If we have a fluid within two parallel plates (separated by a distance *L*) and the upper one is set in motion calculate the velocity field as a function of time and space.



Using dimensionless variables to simplify the analysis

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial y^2} \qquad \theta = \frac{t\,\nu}{L^2} \qquad \xi = \frac{y}{L} \qquad u = \frac{v_x}{v_{xo}} \qquad \frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \xi^2}$$

We can split the contribution of steady state solution, and the transient contribution

$$\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \xi^2} \qquad u = u_{ss} - u_T$$

The sign of the transient contribution is trivial, you can use positive or negative

$$u(\xi,\theta) = u_{ss}(\xi) - u_T(\xi,\theta)$$

Then the differential equation under steady flow is written as

$$\frac{d^2 u_{ss}}{d\xi^2} = 0$$

And the solution is

$$\frac{du_{ss}}{d\xi} = k_1 \qquad u_{ss} = k_1\,\xi + k_2$$

Applying boundary conditions

$$u_{ss}\Big|_{\xi=0} = 0 \qquad u_{ss}\Big|_{\xi=1} = 1 \qquad u_{ss} = \xi$$

Subtract the steady from the original equation, and the same for the boundary and initial conditions

$$\frac{\partial u}{\partial \theta} = \frac{\partial^2 u}{\partial \xi^2} \qquad u\Big|_{\xi=0} = 0 \qquad u\Big|_{\xi=1,\theta>0} = 1$$

Boundary conditions

$$u\Big|_{\theta=0} = 0$$

Initial conditions

$$\frac{\partial u_T}{\partial \theta} = \frac{\partial^2 u_T}{\partial \xi^2} \qquad u_T\Big|_{\xi=0} = 0 \qquad u_T\Big|_{\xi=1,\theta>0} = 0$$

$$u(\xi,0) = u_{ss}(\xi) - u_T(\xi,0)$$

$$u(\xi,0) = u_{ss}(\xi) - u_T(\xi,0)$$ $$u_{ss} = \xi$$ $$u(\xi,0) = 0$$

$$u_T(\xi,0) = \xi$$

Then , this new differential equation can be solved by separation of variables

$$u_T(\xi,\theta) = W(\xi)\,T(\theta)$$ $$\frac{\partial u_T}{\partial \theta} = \frac{\partial^2 u_T}{\partial \xi^2}$$ $$W\frac{dT}{d\theta} = T\frac{d^2W}{d\xi^2}$$

$$\frac{1}{T}\frac{dT}{d\theta} = \frac{1}{W}\frac{d^2W}{d\xi^2} = -\lambda_n^2$$ $$\frac{1}{T}\frac{dT}{d\theta} = -\lambda_n^2$$ $$T = C_n\, e^{-\lambda_n^2\theta}$$

$$\frac{1}{W}\frac{d^2W}{d\xi^2} = -\lambda_n^2$$ $$W = A_n\,\cos(\lambda_n\xi) + B_n\sin(\lambda_n\xi)$$

$$u_T(\xi,\theta) = \sum e^{-\lambda_n^2\theta}[a_n\,\cos(\lambda_n\xi) + b_n\sin(\lambda_n\xi)\,]$$

$$u_T\Big|_{\xi=0} = 0$$

$$u_T(\xi,\theta) = \sum e^{-\lambda_n^2\theta}[a_n\ \cos(\lambda_n\xi) + b_n\sin(\lambda_n\xi)\ ]$$

From the boundary condition, we conclude that all the cosine coefficients are zero (a_n=0)

$$u_T(\xi,\theta) = \sum b_n e^{-\lambda_n^2\theta}\ \sin(\lambda_n\xi)$$

With the initial condition, we have

$$u_T(\xi,0) = \xi = \sum b_n \sin(\lambda_n\xi)$$

To satisfy the upper boundary condition the eigenvalues have to set as follows: $\lambda_n = n\,\pi$

Additionally to obtain the coefficients of the sum term, we can use the orthogonality principle for the sine function:

$$\xi \sin(\lambda_m\xi) = \sum b_n \sin(\lambda_n\xi)\sin(\lambda_m\xi)$$

$$\xi \sin(\lambda_m \xi) = \sum b_n \sin(\lambda_n \xi) \sin(\lambda_m \xi)$$

$$\int_0^1 \xi \sin(m\,\pi\,\xi)\, d\xi = \int_0^1 \sum b_n \sin(n\,\pi\,\xi) \sin(m\,\pi\,\xi)\, d\xi \qquad b_n = \frac{2(-1)^{n+1}}{n\,\pi}$$

$$u = \xi - \sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1}}{n\,\pi}\right] e^{-(n\pi)^2\theta} \sin(\,n\,\pi\,\xi)$$

$$\frac{v_x}{v_{xo}} = (y/L) - \sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1}}{n\,\pi}\right] e^{-\frac{t\,\nu}{L^2}(n\pi)^2} \sin(n\,\pi\,y/L)$$

Appendix

$$\int_0^1 \xi \sin(m\,\pi\,\xi)\, d\xi = \int_0^1 \sum b_n \sin(n\,\pi\,\xi)\sin(m\,\pi\,\xi)\, d\xi$$

$$\frac{1}{(m\,\pi\,)^2}\int_0^{m\pi} (m\,\pi\,\xi)\sin(m\,\pi\,\xi)\, d(m\,\pi\,\xi) = \frac{1}{(n\,\pi\,)}\sum b_n \int_0^{n\pi} \sin(n\,\pi\,\xi)\sin(m\,\pi\,\xi)\, d(n\,\pi\,\xi)$$

$$\frac{1}{(n\,\pi\,)^2}\int_0^{n\pi} (n\,\pi\,\xi)\sin(n\,\pi\,\xi)\, d(n\,\pi\,\xi) = \frac{1}{(n\,\pi\,)} b_n \int_0^{n\pi} \sin^2(n\,\pi\,\xi)\, d(n\,\pi\,\xi)$$

$$\frac{1}{(n\,\pi\,)^2}\int_0^{n\pi} \chi\sin(\chi)\, d\chi = \frac{1}{(n\,\pi\,)} b_n \int_0^{n\pi} \sin^2(\chi)\, d\chi$$

Orthogonality, for n≠ m the integral is zero

Appendix

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \qquad \cos^2(\alpha)+\sin^2(\alpha)=1$$

$$\cos(2\,\alpha)=\cos^2(\alpha)-\sin^2(\alpha) \qquad \cos(2\,\alpha)=1-2\sin^2(\alpha)$$

$$\int x\sin(x)\,dx=\int x\,d[-\cos(x)]=x[-\cos(x)]-\int[-\cos(x)]\,d\,x=-x\cos(x)+\int\cos(x)\,d\,x$$

$$\int x\sin(x)\,dx=\sin(x)-x\cos(x) \qquad \int_0^{n\pi}x\sin(x)\,dx=\sin(x)\Big|_0^{n\pi}-x\cos(x)\Big|_0^{n\pi}=-(n\pi)(-1)^n$$

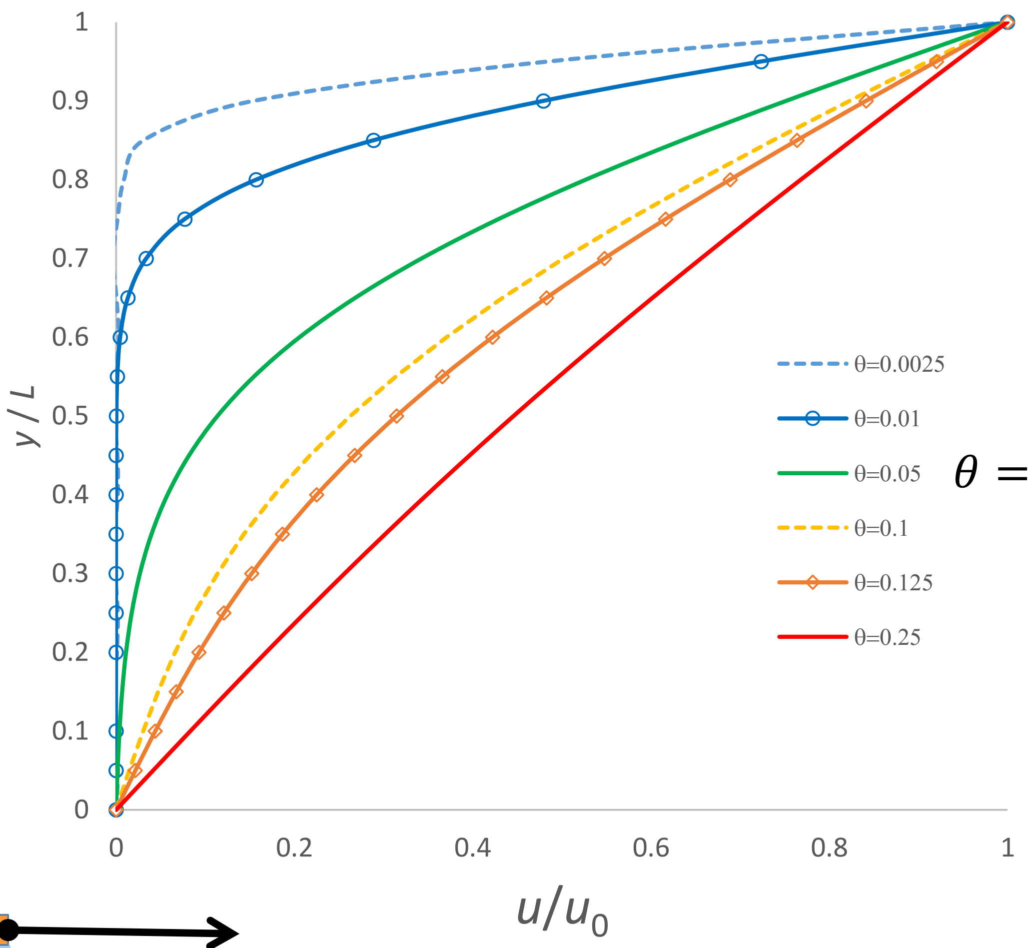
$$\int\sin^2(x)dx=\int\left[\frac{1-\cos(2x)}{2}\right]dx=\frac{x}{2}-\frac{1}{4}\sin(2\,x) \qquad \int_0^{n\pi}\sin^2(x)dx=\frac{x}{2}\Big|_0^{n\pi}-\frac{1}{4}\sin(2\,x)\Big|_0^{n\pi}=\frac{n\,\pi}{2}$$

$$\frac{1}{(n\,\pi\,)^2}\int_0^{n\pi}\chi\sin(\chi)\,d\chi=\frac{1}{(n\,\pi\,)}b_n\int_0^{n\pi}\sin^2(\chi)\,d\chi \qquad \frac{1}{(n\,\pi\,)^2}(n\pi)(-1)^{n+1}=\frac{1}{(n\,\pi\,)}b_n\frac{n\,\pi}{2} \qquad \boxed{b_n=\frac{2(-1)^{n+1}}{n\,\pi}}$$

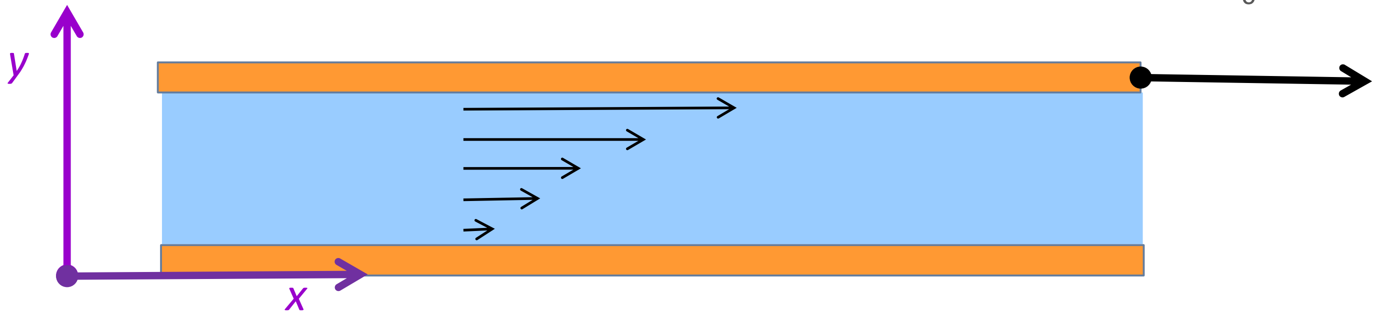
$$\frac{u}{u_o} = (y/L) - \sum_{n=1}^{\infty} 2\left[\frac{(-1)^{n+1}}{n\,\pi}\right] e^{-\frac{t\,\nu}{L^2}(n\pi)^2} \sin(n\,\pi\,y/L)$$

Time to reach steady flow

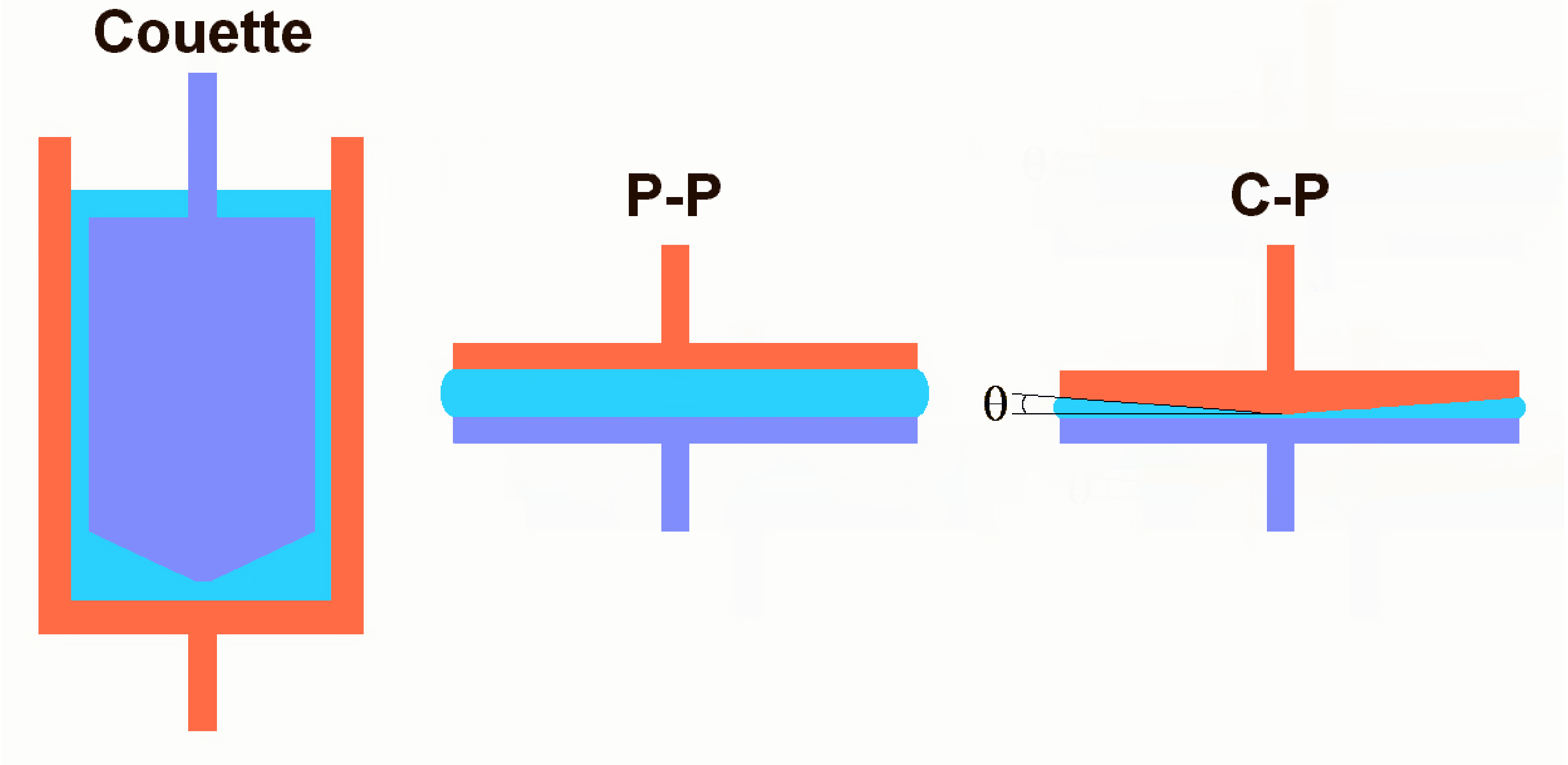
Fluid	Water	Water	Ethylen glycol	Glycerin	Galinstan
Density (kg/m³)	997	997	1113.2	1261	6400
Viscosity (cP)	0.89	0.89	16.10	1412.00	2.40
Thickness (mm)	10	100	10	10	10
time (s)	28.01	2800.56	1.73	0.02	66.67
ν (mm²/s)	0.89	0.89	14.46	1119.75	0.38

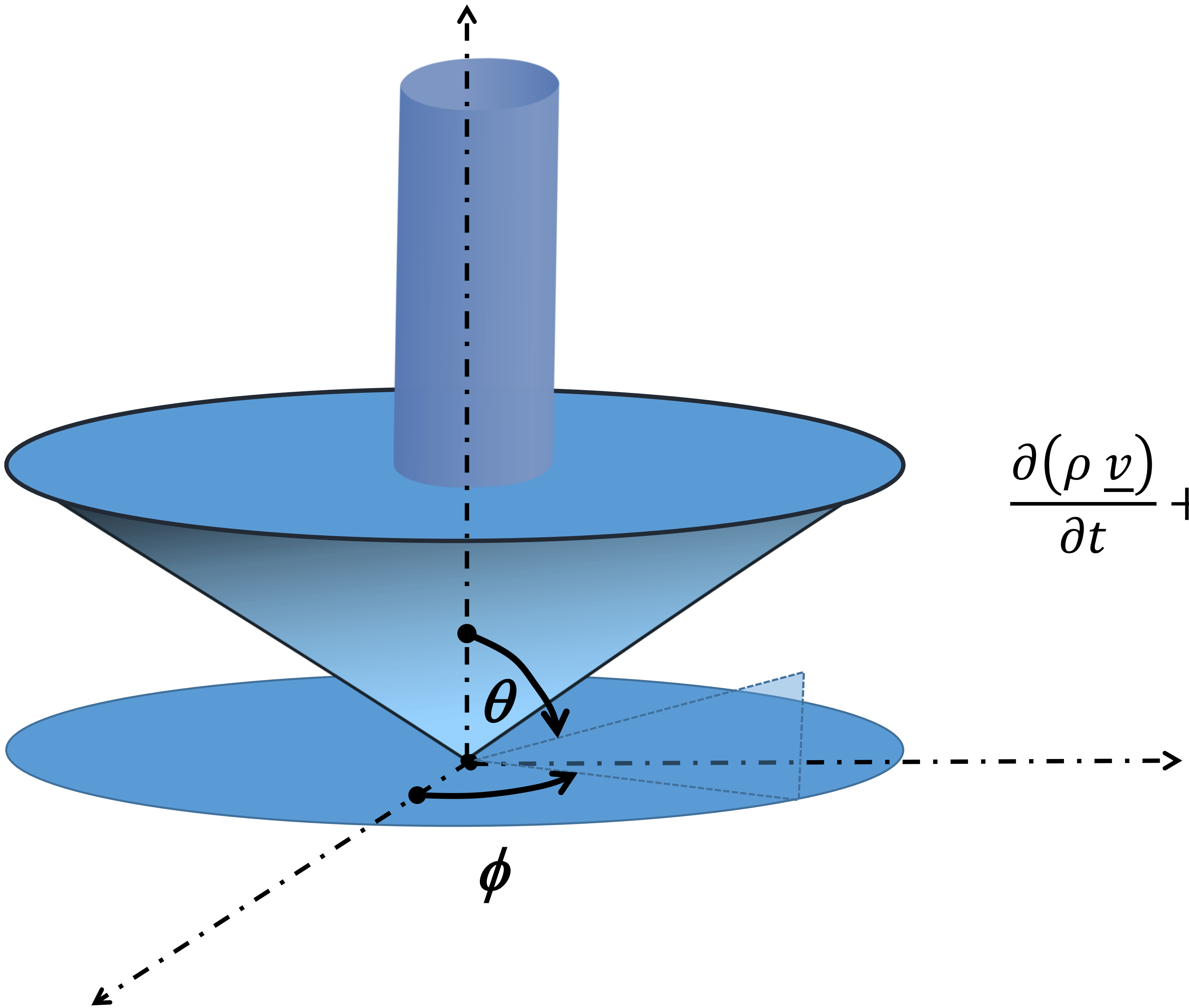


$$\theta = \frac{t\,\nu}{L^2}$$



Viscometer analysis





$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot \left(\rho\, \underline{v}\right) = 0$$

$$\frac{\partial\left(\rho\, \underline{v}\right)}{\partial t} + \underline{\nabla} \cdot \left(\rho\, \underline{v}\, \underline{v}\right) = \underline{\nabla} \cdot \underline{\underline{\tau}} - \underline{\nabla}\, p + \rho\, \underline{g}$$

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}) = 0$$

incompressible

No radial flow

No azimuthal velocity

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{1}{r^2}\frac{\partial[\rho\, r^2\, \cancel{v_r}]}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial[\rho\, \cancel{v_\theta}\sin\theta]}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial[\rho\, v_\phi]}{\partial \phi} = 0$$

$$\frac{\partial\, v_\phi}{\partial \phi} = 0$$

From continuity equation we conclude that velocity may depend on radius and azimuthal angle

$$v_\phi = v_\phi\,(r, \theta)$$

$$\frac{\partial(\rho\,\underline{v})}{\partial t} + \underline{\nabla}\cdot(\rho\,\underline{v}\,\underline{v}) = \underline{\nabla}\cdot\,\underline{\underline{\tau}} - \underline{\nabla}\,p + \rho\,\underline{g}$$

$$\underline{\nabla}\cdot\underline{\underline{\tau}} = \begin{matrix} \left[\frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi r}}{\partial\phi} + 2\frac{\tau_{rr}}{r} + \frac{\tau_{\theta r}}{r\tan\theta} - \frac{\tau_{\theta\theta}}{r} - \frac{\tau_{\phi\phi}}{r}\right]\hat{e}_r \\ \left[\frac{\partial\tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\theta}}{\partial\phi} + 2\frac{\tau_{r\theta}}{r} + \frac{\tau_{\theta r}}{r} + \frac{\tau_{\theta\theta}}{r\tan\theta} - \frac{\tau_{\phi\phi}}{r\tan\theta}\right]\hat{e}_\theta \\ \left[\frac{\partial\tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\tau_{\phi\phi}}{\partial\phi} + 2\frac{\tau_{r\phi}}{r} + \frac{\tau_{\phi r}}{r} + \frac{\tau_{\theta\phi}}{r\tan\theta} + \frac{\tau_{\phi\theta}}{r\tan\theta}\right]\hat{e}_\phi \end{matrix}$$

$$\underline{\nabla}\cdot[\rho\,\underline{v}\,\underline{v}\,] = \begin{matrix} \left[\cancel{\frac{\partial[\rho v_r v_r]}{\partial r}} + \frac{1}{r}\cancel{\frac{\partial[\rho v_\theta v_r]}{\partial\theta}} + \frac{1}{r\sin\theta}\cancel{\frac{\partial[\rho v_\phi v_r]}{\partial\phi}} + 2\cancel{\frac{[\rho v_r v_r]}{r}} + \cancel{\frac{[\rho v_\theta v_r]}{r\tan\theta}} - \cancel{\frac{[\rho v_\theta v_\theta]}{r}} - \frac{[\rho v_\phi v_\phi]}{r}\right]\hat{e}_r \\ \left[\cancel{\frac{\partial[\rho v_r v_\theta]}{\partial r}} + \frac{1}{r}\cancel{\frac{\partial[\rho v_\theta v_\theta]}{\partial\theta}} + \frac{1}{r\sin\theta}\cancel{\frac{\partial[\rho v_\phi v_\theta]}{\partial\phi}} + 2\cancel{\frac{[\rho v_r v_\theta]}{r}} + \cancel{\frac{[\rho v_\theta v_r]}{r}} + \cancel{\frac{[\rho v_\theta v_\theta]}{r\tan\theta}} - \frac{[\rho v_\phi v_\phi]}{r\tan\theta}\right]\hat{e}_\theta \\ \left[\cancel{\frac{\partial[\rho v_r v_\phi]}{\partial r}} + \frac{1}{r}\cancel{\frac{\partial[\rho v_\theta v_\phi]}{\partial\theta}} + \frac{1}{r\sin\theta}\cancel{\frac{\partial[\rho v_\phi v_\phi]}{\partial\phi}} + 2\cancel{\frac{[\rho v_r v_\phi]}{r}} + \cancel{\frac{[\rho v_\phi v_r]}{r}} + \cancel{\frac{[\rho v_\theta v_\phi]}{r\tan\theta}} + \cancel{\frac{[\rho v_\phi v_\theta]}{r\tan\theta}}\right]\hat{e}_\phi \end{matrix}$$

$$\frac{\partial\,v_\phi}{\partial\phi} = 0$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left[\cancel{\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}} + r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right]$$

$$\tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right) + \cancel{\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}}\right]$$

$$\tau_{\theta\theta} = \mu\left[2\left[\cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}} + \cancel{\frac{v_r}{r}}\right]\right] - \left(\frac{2}{3}\mu - \kappa\right)\cancel{\underline{\nabla}\cdot\underline{v}}$$

$$\tau_{\phi\phi} = \mu\left[2\left[\cancel{\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}} + \cancel{\frac{v_r + v_\theta\cot\theta}{r}}\right]\right] - \left(\frac{2}{3}\mu - \kappa\right)\cancel{\underline{\nabla}\cdot\underline{v}}$$

$$\left[\underline{\nabla}\cdot\underline{\underline{\tau}}\right]_\phi = \left[\frac{\partial \tau_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + 2\frac{\tau_{r\phi}}{r} + \frac{\tau_{\theta\phi}}{r\tan\theta} + \frac{\tau_{\phi\theta}}{r\tan\theta}\right]\hat{e}_\phi$$

$$\tau_{r\phi} = \tau_{\phi r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \qquad \tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]$$

$$\left[\underline{\nabla}\cdot\underline{\underline{\tau}}\right]_\phi = \left[\frac{\partial \tau_{r\phi}}{\partial r} + 2\frac{\tau_{r\phi}}{r} + \frac{1}{r}\frac{\partial \tau_{\theta\phi}}{\partial \theta} + 2\frac{\tau_{\phi\theta}}{r\tan\theta}\right]\hat{e}_\phi$$

$$\frac{\tau_{r\phi}}{r} = \mu\left[\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)\right] \qquad \frac{\tau_{\phi\theta}}{r\tan\theta} = \mu\left[\frac{\cos\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]$$

$$\rho \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial\, v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial\, v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial\, v_\phi}{\partial \phi} + \frac{v_\phi\, v_r}{r} + \frac{v_\theta\, v_\phi \cot\theta}{r}$$
$$= -\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\, v_\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial\, v_\phi}{\partial\phi}\right) - \frac{v_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial\, v_\phi}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial\, v_\phi}{\partial\phi}\right] + \rho g_\phi$$

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\, v_\phi}{\partial\theta}\right) - \frac{v_\phi}{r^2\sin^2\theta}\right] = 0$$

$$v_\phi = r\,\omega\, f(\phi)$$

$$f'' + f'\cot\theta + f(1 - \cot^2\theta) = 0$$

$$\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left[\sin^3\theta\frac{d}{d\theta}\left(\frac{f}{\sin\theta}\right)\right] = 0$$

$$\frac{1}{\sin^2\theta}\frac{d}{d\theta}\left[\sin^3\theta\frac{d}{d\theta}\left(\frac{f}{\sin\theta}\right)\right] = 0$$

$$f = c_1\left[\cot\theta + \frac{1}{2}\left[\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right]\sin\theta\right] + c_2$$

$$v_\phi = r\omega\left[1 - \frac{\left[\cot\theta + \frac{1}{2}\left[\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right]\sin\theta\right]}{\left[\cot\beta + \frac{1}{2}\left[\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right)\right]\sin\beta\right]}\right]$$

$$\tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]$$

$$c_3 = \left[\cot\beta + \frac{1}{2}\left[\ln\left(\frac{1+\cos\beta}{1-\cos\beta}\right)\right]\sin\beta\right]$$

$$T = \frac{4\,\pi\,\mu\,R^3}{3\sin^3\beta\;\,c_3}$$

$$\tau_{\phi\theta} = \tau_{\theta\phi} = \mu\left[\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{v_\phi}{\sin\theta}\right)\right]$$

Application (This is Honor class level topic) Microscale fluid dynamics and Electro-osmotic flow)

$$dU = Tds - pdV + \sum_{j=1}^{t} \mu_j dN_j + \sum_{i=1}^{M} \psi dq_i$$

$$dG = -SdT + Vdp + \sum_{j=1}^{t} \mu_j dN_j + \sum_{i=1}^{M} \psi dq_i$$

$$dG = -SdT + Vdp + \sum_{i=1}^{M} \left(\mu_j + z_i e \psi\right) dN_i$$

$$\mu_j^* = \mu_j + z_i e \psi$$

$$\mu^*(x) = \mu(x) + ze\psi(x) = \mu^o + kT \ln[c(x)] + ze\psi(x)$$

Nernst Equation

$$kT \ln\left[\frac{c(x_2)}{c(x_1)}\right] = -ze[\psi(x_2) - \psi(x_1)]$$

$$\ln\left[\frac{a(x_2)}{a(x_1)}\right] = -\frac{ze[\psi(x_2)-\psi(x_1)]}{kT}$$

$$\psi = \psi_o - \frac{RT}{zF}\ln a$$

ε_o=permittivity of the vacuum
D=Dielectric constant
$F = e\,N$, Faraday Constant =96500 C/mol
$R = N\,k$ =8314.34 J/kmol-K

Electrostatic Potential

$$\delta w = -\underline{f}\cdot d\underline{\ell} = -q\underline{E}\cdot d\underline{\ell}$$

$$w_{AB} = -q\int_A^B \underline{E}\cdot d\underline{\ell}$$

$$\psi_B - \psi_A = \frac{w_{AB}}{q_{test}} = -\int_A^B \underline{E}\cdot d\underline{\ell}$$

$$\underline{E} = -\underline{\nabla}\psi$$

$$\underline{f} = \frac{q_1 q_2}{(4\pi\varepsilon_o D)r^2}\,\underline{e}_r$$

$$\underline{E} = \frac{q_1}{(4\pi\varepsilon_o D)r^2}\,\underline{e}_r$$

$$\int \underline{E}\cdot\underline{n}\,dA = \int 4\pi\frac{q_1\underline{e}_r}{(4\pi\varepsilon_o D)r^2}\,r dr = \frac{q_1}{\varepsilon_o D}$$

$$\int \underline{E}\cdot\underline{n}\,dA = \frac{q_1}{\varepsilon_o D} = \frac{1}{\varepsilon_o D}\int \rho_e dV = \int \underline{\nabla}\cdot\underline{E}\,dV$$

Poisson´s equation

$$\frac{\rho_e}{\varepsilon_o D} = \underline{\nabla}\cdot\underline{E} \qquad \underline{E} = -\underline{\nabla}\psi \qquad \frac{\rho_e}{\varepsilon_o D} = -\nabla^2\psi$$

Poisson-Boltzmann Model

z=valence (charge/ion)

e=charge pf proton

n_i = number of ions, per unit volume

$$n_+(x) = n_\infty e^{-\frac{ze\psi(x)}{kT}}$$

$$n_i = C_i N_A$$

Concentration of negative ions

$$n_-(x) = n_\infty e^{\frac{ze\psi(x)}{kT}}$$

Charge density

$$\rho_e(x) = \sum_i z_i e n_i(x) = ze\left[n_+(x) - n_-(x)\right] = F\sum_i z_i C_i$$

$$\nabla^2\psi = \frac{zen_\infty}{D\varepsilon_o}\left(e^{ze\psi/kT} - e^{-ze\psi/kT}\right) = \frac{2zen_\infty}{D\varepsilon_o}\sinh\left(\frac{ze\psi}{kT}\right)$$

Linearized Poisson-Boltzmann or Debye-Hückel equation

$$\sinh(\beta) \approx \beta + \frac{\beta^3}{3!} + \frac{\beta^5}{5!} + \frac{\beta^7}{7!} + \cdots$$

$$\nabla^2 \psi \approx \frac{2zen_\infty}{D\varepsilon_o}\left(\frac{ze\psi}{kT}\right) = \kappa^2 \psi$$

1/κ= Debye Length, screening or shielding distance

Bjerrum length

$$\psi \approx Be^{\kappa y} + Ae^{-\kappa y}$$

ζ= Zetha potential = ψ_o

$$\ell_B \approx \frac{e^2 N}{DRT}$$

$$\psi \approx \psi_o e^{-\kappa y}$$

z=valency

n_∞= bulk ion concentration

$$\frac{\rho_e}{\varepsilon_o D} = -\nabla^2 \psi = -\psi_o \kappa^2 e^{-\kappa y}$$

$$\rho_e = -\psi_o \varepsilon_o D \kappa^2 e^{-\kappa y}$$

$$\kappa^2 \approx \frac{2zen_\infty}{D\varepsilon_o}\left(\frac{ze}{kT}\right)$$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x + \rho_e E_x + \mu \frac{\partial^2 V_x}{\partial y^2}$$

$$0 = -\frac{\partial p}{\partial x} + \rho g_x - \psi_o \varepsilon_o D \kappa^2 e^{-\kappa y} E_x + \mu \frac{\partial^2 V_x}{\partial y^2}$$

$$\kappa^2 \approx \frac{2zen_\infty}{D\varepsilon_o}\left(\frac{ze}{kT}\right) = \frac{8\pi z^2 n_\infty}{D(NkT)}\left(\frac{e^2 N}{4\pi\varepsilon_o}\right) = \frac{8\pi z^2 n_\infty}{DRT}\left(\frac{e^2 N}{4\pi\varepsilon_o}\right) = \frac{8\,\pi\, z^2 NC_\infty}{DRT}\left(\frac{e^2 N}{4\pi\varepsilon_o}\right)$$

ε_o=permittivity of the vacuum=8.85x10^{-12} farad/m=8.85x10^{-12} C^2/(J-m)

D=Dielectric constant

F = e N , Faraday Constant =96500 C/mol

R = N k =8314.34 J/kmol-K

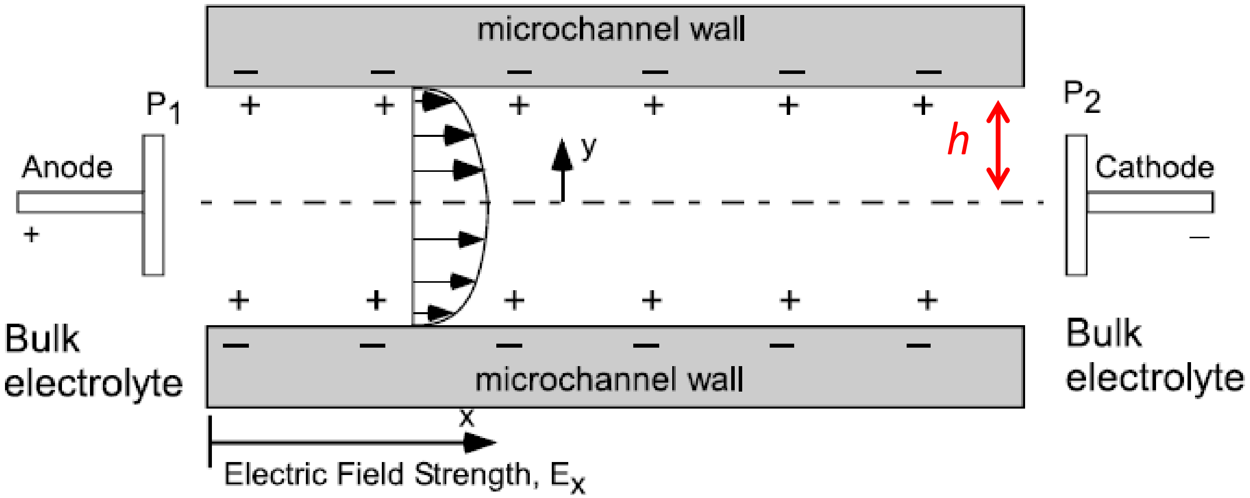
e=1.6x10^{-19} C

C (mol/dm^3)	$1/\kappa$ (A)
0.5	4.3
0.2	6.8
0.1	9.62
.05	13.6
0.02	21.5
0.01	30.4
0.005	43
0.002	68
0.001	96.2

Vacuum	1
Glass	5-10
Mica	3-6
Mylar	3.1
Neoprene	6.70
Plexiglas	3.40
Polyethylene	2.25
Polyvinyl chloride	3.18
Teflon	2.1
Germanium	16
Strontiun titanate	310
Water	80.4
Water(0C)	88
Water (25C)	78.54
Glycerin	42.5
Glycerol(25C)	37
Liquid ammonia(-78°C)	25
Benzene	2.284
Air(1 atm)	1.00059
Air(100 atm)	1.0548
Heptane(0C)	1.958
Heptane (30C)	1.916
Methanol	33

$$0 = -\frac{\partial p}{\partial x} + \rho g_x - \psi_o \varepsilon_o D \kappa^2 e^{-\kappa(h-y)} E_x + \mu \frac{\partial^2 V_x}{\partial y^2}$$

$$0 = \frac{\alpha}{\mu} - \frac{\psi_o \varepsilon_o D \kappa^2 e^{-\kappa h} e^{\kappa y} E_x}{\mu} + \frac{\partial^2 V_x}{\partial y^2}$$



Integrating and applying boundary conditions

$$0 = \frac{\alpha}{\mu} y - \frac{\psi_o \varepsilon_o D \kappa^2 e^{-\kappa h} e^{\kappa y} E_x}{\mu \kappa} + c_1 + \frac{dV_x}{dy}$$

$$0 = \frac{\alpha}{\mu}(0) - \frac{\psi_o \varepsilon_o D \kappa^2 e^{-\kappa h} E_x}{\mu \kappa} + c_1 + \left.\frac{dV_x}{dy}\right|_{y=0}$$

By symmetry, there is no gradient of velocity at the center

$$0 = \frac{\alpha}{\mu} y - \frac{\psi_o \varepsilon_o D \kappa^2 e^{-\kappa h}\left(e^{\kappa y} - 1\right) E_x}{\mu \kappa} + \frac{dV_x}{dy}$$

$$0 = \frac{\alpha}{2\mu} y^2 - \frac{\psi_o \varepsilon_o D \kappa^2 e^{-\kappa h}\left(\frac{e^{\kappa y}}{\kappa} - y\right) E_x}{\mu \kappa} + V_x + C_2$$

$$0 = \frac{\alpha}{2\mu} h^2 - \frac{\psi_o \varepsilon_o D \kappa^2 e^{-\kappa h}\left(\frac{e^{\kappa h}}{\kappa} - h\right) E_x}{\mu \kappa} + 0 + C_2$$

No slip boundary condition, velocity at the walls is zero

Collecting terms

$$0=\frac{\alpha}{2\mu}\left[y^2-h^2\right]-\frac{\psi_o\varepsilon_o D\kappa^2 e^{-\kappa h}\left(\frac{\left(e^{\kappa y}-e^{\kappa h}\right)}{\kappa}-(y-h)\right)E_x}{\mu\kappa}+V_x$$

$$-V_x=\frac{\alpha}{2\mu}\left[y^2-h^2\right]-\frac{\zeta\varepsilon_o D e^{-\kappa h}\left(e^{\kappa y}-e^{\kappa h}\right)E_x}{\mu}+\frac{\zeta\varepsilon_o D\kappa e^{-\kappa h}(y-h)E_x}{\mu}$$

Recasting in dimensionless form

$$-V_x = \frac{\alpha}{2\mu}\left[y^2 - h^2\right] - \frac{\zeta\varepsilon_o D e^{-\kappa h}\left(e^{\kappa y} - e^{\kappa h}\right)E_x}{\mu} + \frac{\zeta\varepsilon_o D\kappa e^{-\kappa h}\left(y - h\right)E_x}{\mu}$$

$$V_x = \frac{\alpha}{2\mu}\left[h^2 - y^2\right] + \frac{\zeta\varepsilon_o D e^{-\kappa h}\left(e^{\kappa y} - e^{\kappa h}\right)E_x}{\mu} - \frac{\zeta\varepsilon_o D\kappa e^{-\kappa h}\left(y - h\right)E_x}{\mu}$$

$$V_x = \frac{\alpha h^2}{2\mu}\left[1 - \eta^2\right] + \frac{\zeta\varepsilon_o D e^{-\kappa h} E_x}{\mu}\left[\left(e^{\kappa h\eta} - e^{\kappa h}\right) + \kappa h\left(1 - \eta\right)\right]$$

$$\frac{V_x\mu}{\alpha h^2} = \frac{1}{2}\left[1 - \eta^2\right] + \frac{\zeta\varepsilon_o D E_x}{\alpha h^2} e^{-\kappa h}\left[\left(e^{\kappa h\eta} - e^{\kappa h}\right) + \kappa h\left(1 - \eta\right)\right]$$

$$U^* = \frac{\mu V_x}{\left(-\nabla p + \rho g\right)h^2} = \frac{1}{Re\ Eu}\left(\frac{L}{h}\right) = \frac{1}{2}\left[1 - \eta^2\right] + \beta e^{-\gamma}\left[\left(e^{\gamma\eta} - e^{\gamma}\right) + \gamma\left(1 - \eta\right)\right]$$

$$\beta = \frac{\zeta\varepsilon_o D E_x}{\alpha h^2} \qquad \eta = \frac{y}{h} \qquad \gamma = \kappa\, h \qquad \mathrm{Re} = \frac{\rho V h}{\mu} \qquad Eu = \frac{\left(-\underline{\nabla} p + \rho \underline{g}\right)L}{\rho V^2}$$

$$U^* = \frac{\mu V_x}{\left(-\nabla p + \rho g\right)h^2} = \frac{1}{Re\ Eu}\left(\frac{L}{h}\right) = \frac{1}{2}\left[1-\eta^2\right] + \beta e^{-\gamma}\left[\left(e^{\gamma\eta} - e^{\gamma}\right) + \gamma\left(1-\eta\right)\right]$$

$$\beta = \frac{\zeta\varepsilon_o D E_x}{\alpha h^2} \qquad \eta = \frac{y}{h} \qquad \gamma = \kappa\, h \qquad \mathrm{Re} = \frac{\rho V\, h}{\mu} \qquad Eu = \frac{\left(-\underline{\nabla} p + \rho \underline{g}\right)L}{\rho V^2}$$

$$V^* = \frac{\mu V_x}{\zeta\varepsilon_o D E_x} = \frac{1}{Re\ Eu\ \beta}\left(\frac{L}{h}\right) = \frac{1}{2\beta}\left[1-\eta^2\right] + e^{-\gamma}\left[\left(e^{\gamma\eta} - e^{\gamma}\right) + \gamma\left(1-\eta\right)\right]$$

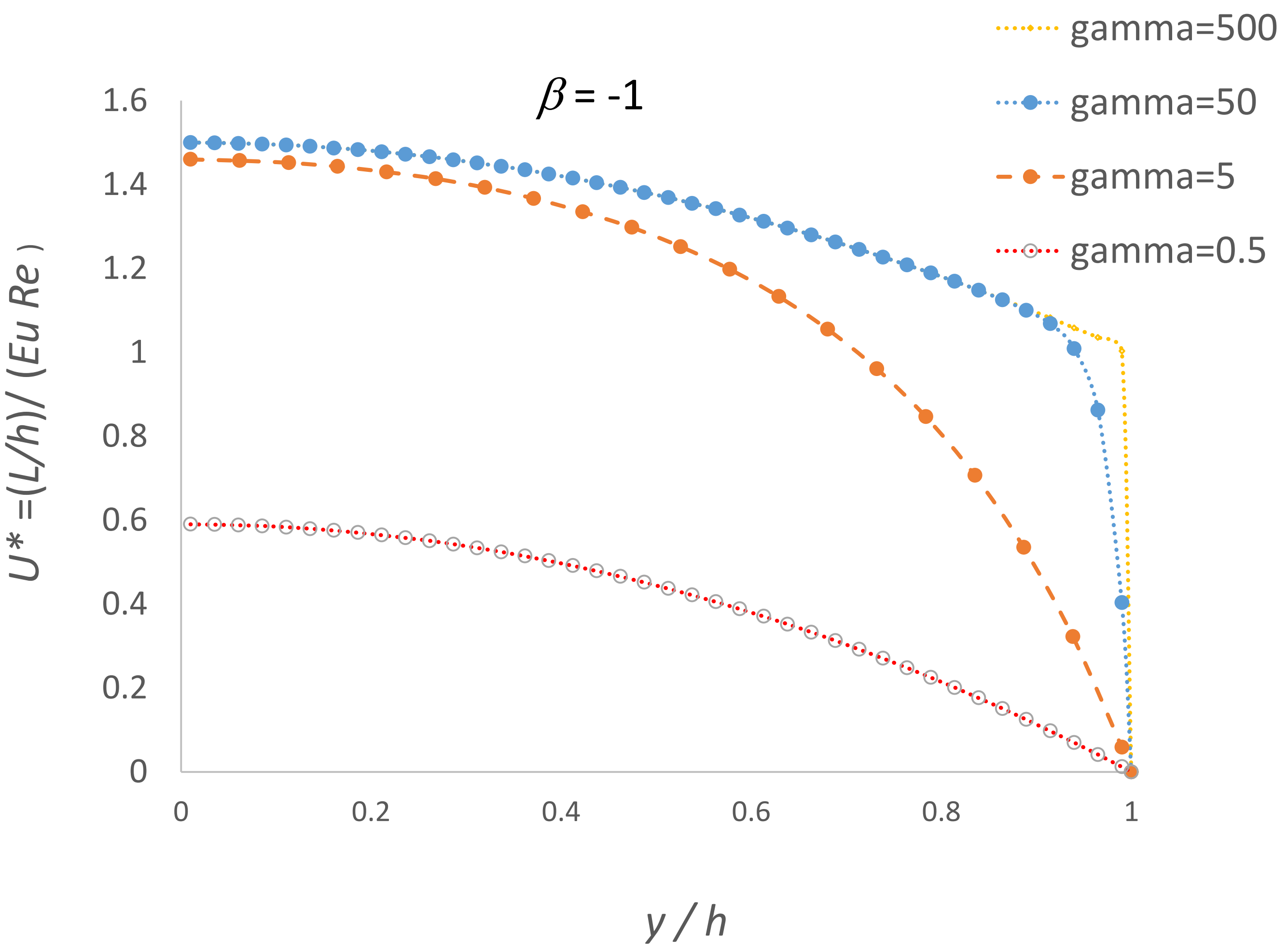
Integration of the velocity in height is used to calculate the average velocity

$$-\langle V_x \rangle h = \frac{\alpha}{2\mu}\left[\frac{y^3}{3} - h^2 y\right]_0^h - \left.\frac{\zeta\varepsilon_o D e^{-\kappa h}\left(\frac{e^{\kappa y}}{\kappa} - y e^{\kappa h}\right)E_x}{\mu}\right|_0^h + \left.\frac{\zeta\varepsilon_o D\kappa e^{-\kappa h}\left(\frac{y^2}{2} - hy\right)E_x}{\mu}\right|_0^h$$

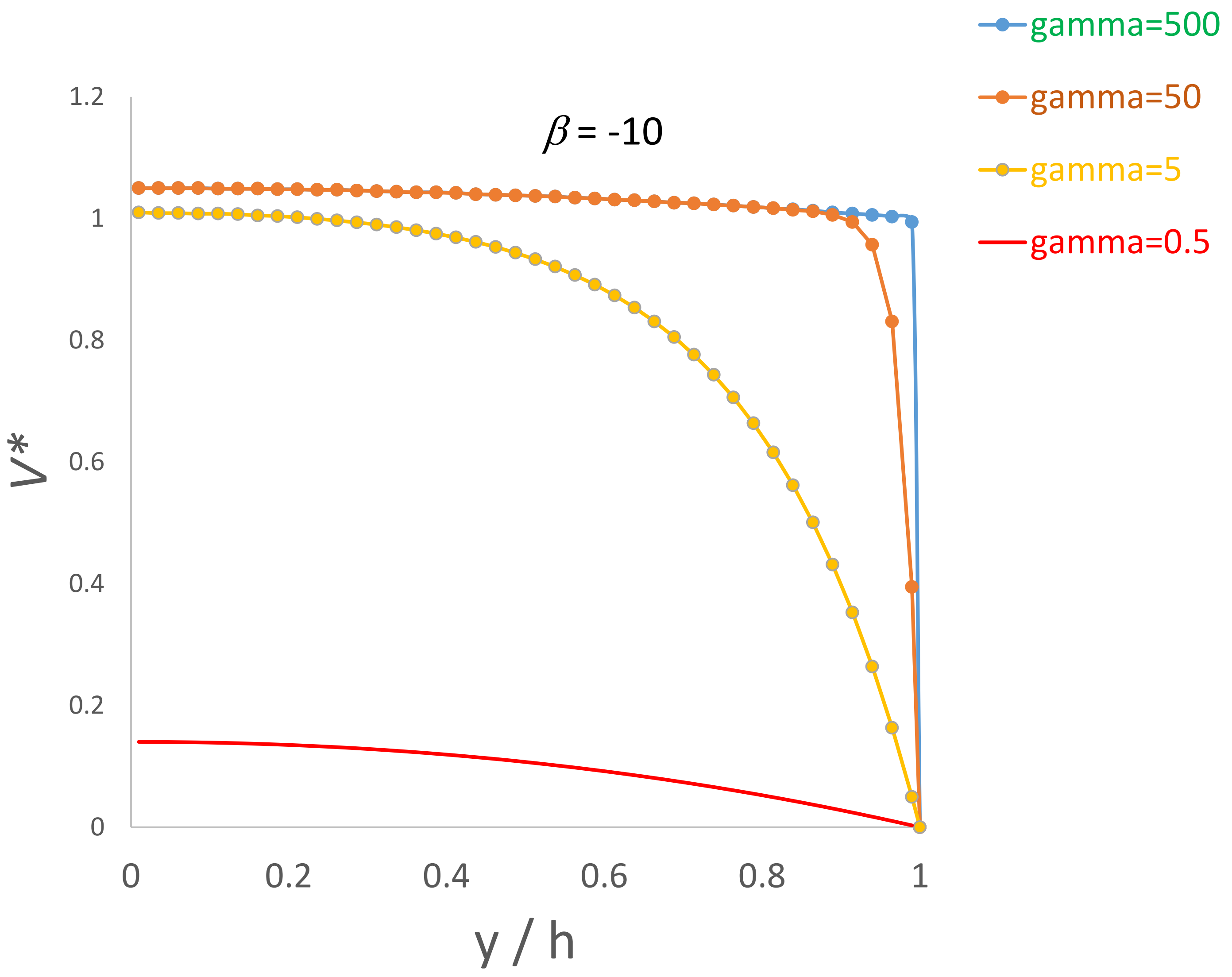
$$-\langle V_x \rangle h = -\frac{\alpha}{\mu}\frac{h^3}{3} - \frac{\zeta\varepsilon_o D\kappa e^{-\kappa h} h^2 E_x}{2\mu} - \frac{\zeta\varepsilon_o D e^{-\kappa h}\left(\frac{\left[e^{\kappa h} - 1\right]}{\kappa} - h e^{\kappa h}\right)E_x}{\mu}$$

$$-\langle V_x \rangle h = -\frac{\alpha}{\mu}\frac{h^3}{3} - \frac{\zeta\varepsilon_o D\kappa e^{-\kappa h} h^2 E_x}{2\mu} - \frac{\zeta\varepsilon_o D e^{-\kappa h}\left(\left[e^{\kappa h} - 1\right] - \kappa h e^{\kappa h}\right)E_x}{\mu\kappa}$$

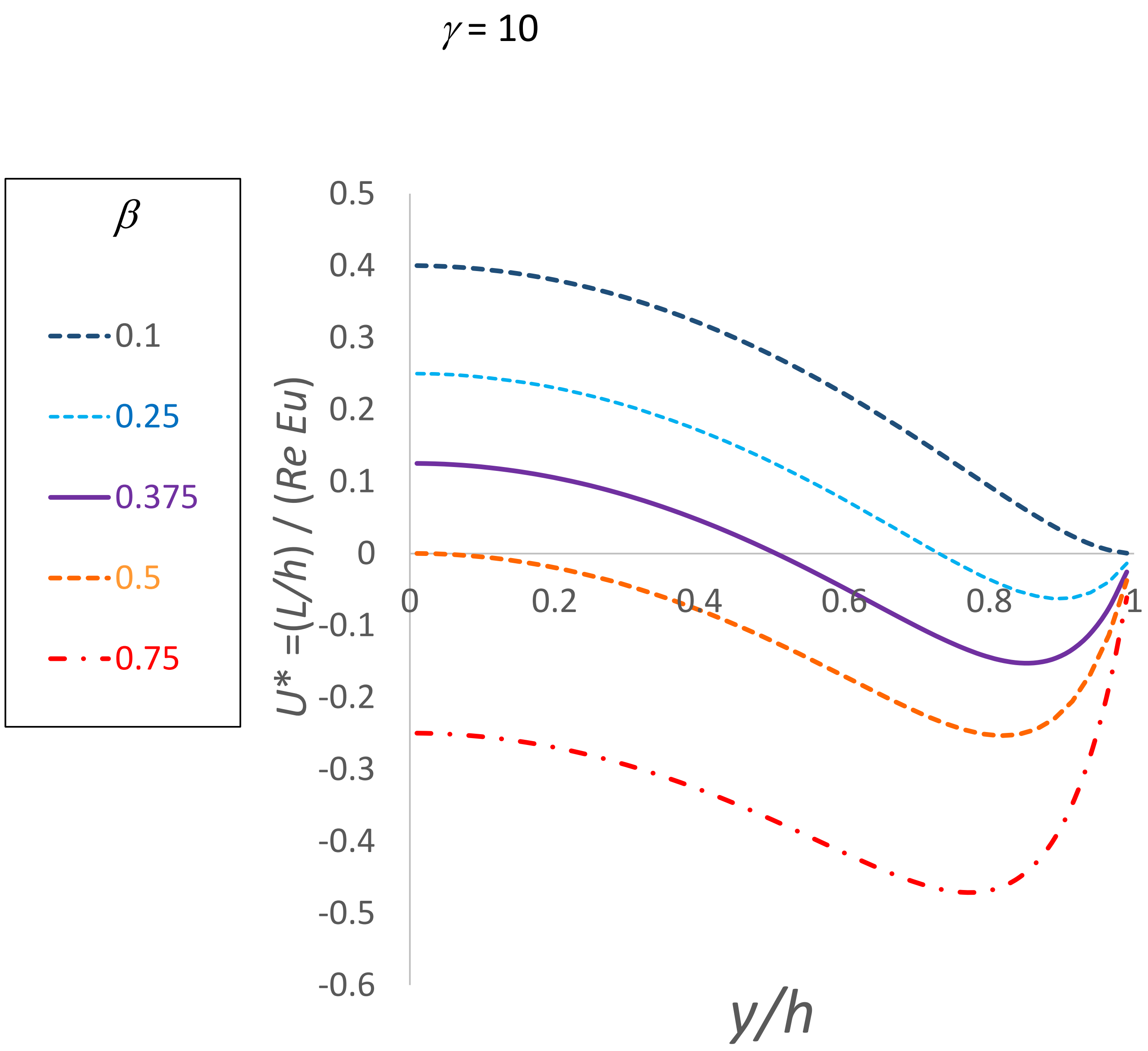
$$\langle V_x \rangle = \frac{\alpha}{\mu}\frac{h^2}{3} + \frac{\zeta\varepsilon_o D\kappa e^{-\kappa h} h E_x}{2\mu} + \frac{\zeta\varepsilon_o D\left(\frac{\left(1 - e^{-\kappa h}\right)}{kh} - 1\right)E_x}{\mu}$$



β = -1
gamma=500
gamma=50
gamma=5
gamma=0.5
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1
U* =(L/h)/ (Eu Re)
y / h



gamma=500
gamma=50
gamma=5
gamma=0.5
β = -10
V*
y / h
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1



γ = 10
β
0.1
0.25
0.375
0.5
0.75
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
0
0.2
0.4
0.6
0.8
1
U* =(L/h) / (Re Eu)
y/h

$$-V_x = \frac{\alpha}{2\mu}\left[y^2 - h^2\right] - \frac{\zeta\varepsilon_o D E_x e^{-\kappa h}\left(e^{\kappa y} - e^{\kappa h}\right)}{\mu} + \frac{\zeta\varepsilon_o D E_x \kappa e^{-\kappa h}\left(y - h\right)}{\mu}$$

$$-V_x = \frac{\alpha h^2}{2\mu}\left[\eta^2 - 1\right] - \frac{\zeta\varepsilon_o D E_x e^{-\kappa h}\left(e^{\kappa h\eta} - e^{\kappa h}\right)}{\mu} + \frac{\zeta\varepsilon_o D E_x h\kappa e^{-\kappa h}\left(\eta - 1\right)}{\mu}$$

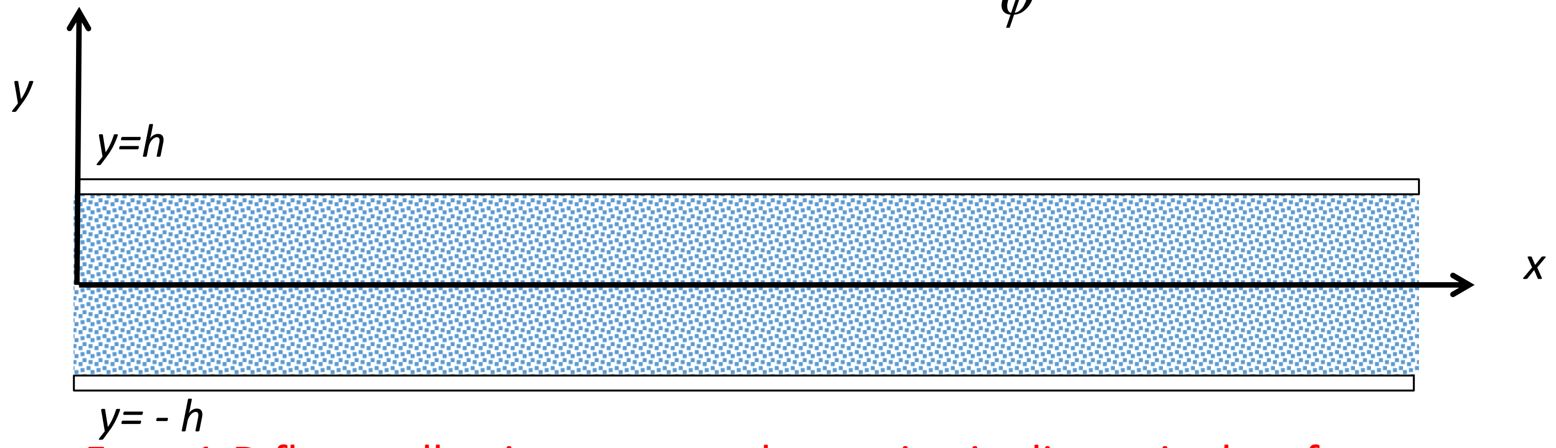
$$\alpha = -\frac{\partial p}{\partial x} + \rho g_x$$

$$-V_x = \frac{\alpha h^2}{2\mu}\left[\eta^2 - 1\right] - \frac{\zeta\varepsilon_o D E_x}{\mu}\left[e^{-\kappa h}\left(e^{\kappa h\eta} - e^{\kappa h}\right) - h\kappa e^{-\kappa h}\left(\eta - 1\right)\right]$$

$$V_x = \frac{\alpha h^2}{2\mu}\left[1 - \eta^2\right] + \frac{\zeta\varepsilon_o D E_x e^{-\kappa h}}{\mu}\left[\left(e^{\kappa h\eta} - e^{\kappa h}\right) + h\kappa\left(1 - \eta\right)\right]$$

Flow throw porous media **(This is Honor class level topic)**

$$-\underline{\nabla} p + \rho \underline{g} - \mu \underline{\underline{\kappa}}^{-1} \cdot \underline{u} - \rho \underline{\beta} \cdot |\underline{u}| \underline{u} + \frac{\mu}{\phi} \underline{\nabla} \cdot \left(\underline{\nabla} \underline{u} \right) = 0$$



For a 1-D flow, collecting terms and recasting in dimensionless form:

$$-\frac{\partial p}{\partial x} + \rho g_x - \frac{\mu}{\kappa} u - \rho \beta\, u^2 + \frac{\mu}{\phi} \frac{d^2 u}{dy^2} = 0$$

$$\kappa = \frac{\phi^3\, D_p^2}{150(1-\phi)^2}$$

$$\beta = \frac{1.75(1-\phi)}{\phi^3 D_p}$$

$$\left[-\frac{\partial p}{\partial x} + \rho g_x - \frac{\mu}{\kappa} u - \rho \beta\, u^2 + \frac{\mu}{\phi} \frac{d^2 u}{dy^2} = 0 \right] \left(\frac{\phi\, h^2}{\mu} \right) \left(\frac{\rho\, h}{\mu} \right)$$

Pseudo 1-D model for porous media in dimensionless form:

$$\left[-\frac{\partial p}{\partial x}+\rho g_x-\frac{\mu}{\kappa}u-\rho\beta\,u^2+\frac{\mu}{\phi}\frac{d^2u}{dy^2}=0\right]\left(\frac{\phi\,h^2}{\mu}\right)\left(\frac{\rho\,h}{\mu}\right)$$

$$\frac{\rho h^3\phi(-\nabla P)}{\mu^2}-\frac{\phi\,h^2}{\kappa}z-\phi\,h\,\beta\,z^2+\frac{d^2z}{d\eta^2}=0$$

$$\frac{d^2z}{d\eta^2}+c-b^2z-a\,z^2=0$$

$$c=\frac{\rho h^3\phi(-\nabla P)}{\mu^2}$$

$$b^2=\frac{\phi\,h^2}{\kappa}$$

$$a=\phi\,h\,\beta$$

Boundary conditions

$$u\big|_{y=h}=0 \qquad z\big|_{\eta=1}=0$$

$$\left.\frac{du}{dy}\right|_{y=0}=0 \qquad \left.\frac{dz}{d\eta}\right|_{\eta=0}=0$$

$$z=\frac{\rho\,u\,h}{\mu} \qquad \eta=\frac{y}{h}$$

$$\frac{d^2 z}{d\eta^2} + c - b^2 z - a\, z^2 = 0$$

Boundary conditions

$$z\Big|_{\eta=1} = 0$$

$$\left.\frac{dz}{d\eta}\right|_{\eta=0} = 0$$

Finite differences

$$\frac{z_{i-1} - 2z_i + z_{i+1}}{(\Delta\eta)^2} + c - b^2 z_i - a\, z_i^{\,2} = 0$$

In N is the number of points, including center (i=1), and excluding the point at the interface which will be specified as boundary condition

$$z\Big|_{\eta=1} = 0 \Rightarrow z_{N+1} = 0$$

$$\left.\frac{dz}{d\eta}\right|_{\eta=0} = 0$$

To symmetric boundary condition is satisfied with an artifact $z_0 = z_2$

Orthogonal collocation

$$\sum_{j=1}^{N+1} B_{i,j} z_j + c - b^2 z_i - a\, z_i^{\,2} = 0$$

In N is the number of internal points, excluding center (i=1), and excluding the point at the interface which will be specified as boundary condition (N+1)

$$z\Big|_{\eta=1} = 0 \Rightarrow z_{N+1} = 0$$

Shooting method

$$\frac{dY_1}{d\eta} = Y_2$$

$$\frac{dY_2}{d\eta} = -c + b^2 Y_1 + a\, Y_1^{\,2}$$

$$Y_1(1) = 0$$

$$Y_2(0) = 0$$

Solve the equation for the following cases

$$\frac{d^2 z}{d\eta^2} + c - b^2 z - a\, z^2 = 0$$

I. For a=0

II. Neglecting the second order derivative

III. For, a=0.4, b=4.45, c=8

Reflect about porous media analysis

Void volume in terms of total volume and void fraction:

Total volume in terms of solid volume and solid fraction :

$$D_H = \frac{4A_F}{W_p} = \frac{4A_F L}{W_p L} = \frac{4V_{void}}{A_{solid}} = \frac{4V\varphi}{A_{solid}} = \frac{4\left(\frac{V_{solid}}{1-\varphi}\right)\varphi}{A_{solid}}$$

Ratio of flowing area to wetted perimeter, if numerator and denominator are multiplied by length of porous media results in the ratio of porous volume to the surface area. ϕ is the porosity, then hydraulic diameter con be expressed in terms of the geometric properties of the solid. Assuming the solid is made up of spherical grains, the hydraulic diameter can be expressed in terms of void fraction and diameter of the grain.

$$D_H = \frac{4N_p\varphi\left(\frac{4}{3}\pi\, r_p^3\right)}{(1-\varphi)N_p\left(4\pi r_p^2\right)} = \frac{2D_p\varphi}{3(1-\varphi)}$$

The Reynolds number expressed in terms of hydraulic diameter takes the form:

$$Re^* = \frac{\rho\, uD_H}{\mu} = \frac{2\rho\, uD_p\varphi}{3\mu(1-\varphi)} = \frac{2\rho\, u_o D_p}{3\mu(1-\varphi)} = \frac{2}{3}\frac{Re_p}{(1-\varphi)}$$

$$Re_p = \frac{\rho\, u_o D_p}{\mu}$$

To predict energy loss

$$D_H = \frac{2D_p\varphi}{3(1-\varphi)}$$

$$\hat{E}_v = 4 f_F \frac{L_A}{D_H}\frac{u^2}{2} = 4 f_F \frac{L_A}{D_H}\frac{u_o^2}{2\varphi^2} = 4 f_F \frac{L_A}{\left[\frac{2D_p\varphi}{3(1-\varphi)}\right]}\frac{u_o^2}{2\varphi^2} = 4 f_F \frac{L_A}{D_p}\frac{3(1-\varphi)}{2\varphi^3}\frac{u_o^2}{2}$$

$$\hat{E}_v = 4 f_F \frac{L}{D_p}\frac{3\tau(1-\varphi)}{2\varphi^3}\frac{u_o^2}{2} = 4 f_p \frac{L}{D_p}\frac{u_o^2}{2} \qquad f_p = 3\tau\, f_F \frac{(1-\varphi)}{2\varphi^3}$$

The simplest form of friction factor based on previous knowledge has the form:

$$f_F = \alpha + \frac{\beta}{Re_H} = \alpha + \frac{3\beta(1-\varphi)}{2Re_p} \qquad Re^* = \frac{2}{3}\frac{Re_p}{(1-\varphi)}$$

$$f_p = 3\tau\left[\alpha_f + \frac{3\beta_f(1-\varphi)}{2Re_p}\right]\frac{(1-\varphi)}{2\varphi^3} = \frac{(1-\varphi)}{2\varphi^3}\left[3\tau\,\alpha_f + \frac{9\tau\,\beta_f(1-\varphi)}{2Re_p}\right] = \frac{(1-\varphi)}{2\varphi^3}\left[a_f + \frac{b_f(1-\varphi)}{Re_p}\right]$$

$$\tau = \frac{L_A}{L} = \sqrt{T}$$

τ, and T are two different forms of express tortuosity

$$f_p = \frac{(1-\varphi)}{2\varphi^3}\left[3\,\tau\,\alpha_f + \frac{9\,\tau\,\beta_f(1-\varphi)}{2Re_p}\right] = \frac{(1-\varphi)}{2\varphi^3}\left[\,a_f + \frac{b_f(1-\varphi)}{Re_p}\right]$$

$$f_p = \frac{(1-\varphi)}{2\varphi^3}\left[\,1.75 + \frac{150(1-\varphi)}{Re_p}\right]$$

Ergun proposed these values

Reflecting about the physical interpretation of the values, it is well known that for a channel, the contribution of laminar viscous dissipation is

$$b_f = \frac{9\,\tau\,(16\,\Psi)}{2}$$

Ψ is 1 for cylindrical channels and 1.5 for flat channel, and tortuosity which provides a measure of the directness of the path the fluid may take. By definition given tortuosity will always have a value greater than 1, and can be greater than 2 or even larger. Typical values of b_f found in literature rang from 150 up to 180.

In terms of pressure gradient

$$-\nabla p=\frac{\Delta p}{L}=\frac{\rho}{D_p}\frac{(1-\varphi)}{\varphi^3}\left[1.75+\frac{150\mu(1-\varphi)}{\rho D_p u_0}\right]u_0^2=\frac{1.75\rho}{D_p}\frac{(1-\varphi)u_0^2}{\varphi^3}+\frac{150\mu(1-\varphi)^2 u_o}{\varphi^3 D_p^2}$$

Forchheimer and Darcy terms

$$-\nabla p=\frac{1.75\rho}{D_p}\frac{(1-\varphi)u_0^2}{\varphi^3}+\frac{150\mu(1-\varphi)^2 u_o}{\varphi^3 D_p^2}=\rho\,\beta\,u_0^2+\frac{\mu\,u_o}{\kappa}$$

Recasting the equation and including the viscous dissipation close to the walls to satisfy no-slip boundary condition

$$-\underline{\nabla}p+\rho\underline{g}-\mu\underline{\underline{\kappa}}^{-1}\cdot\underline{u}-\rho\underline{\beta}\cdot|\underline{u}|\underline{u}+\frac{\mu}{\phi}\underline{\nabla}\cdot(\underline{\nabla}\underline{u})=0$$

This is called Darcy-Brinkman-Forchheimer equation

$$\kappa=\frac{\phi^3\,D_p^2}{150(1-\phi)^2}$$

$$\beta=\frac{1.75(1-\phi)}{\phi^3 D_p}$$

Method of Finite Fourier Transform to solve fluid dynamics problems ((This is Honor class level topic))

$$f(x, y) = \sum g_i(x) h_i(y)$$

$$F(y) = \int_a^b w(x) f(x, y) \Phi(x) \, dx$$

Self-Adjoint Eigenvalue problems and Sturm-Liouville Theory

$$\Im_x = \frac{1}{w}\left[\frac{d}{dx}\left(p\frac{d(\ )}{dx}\right)+q\right]$$

$$\frac{1}{w}\left[\frac{d}{dx}\left(p\frac{d\Phi}{dx}\right)+q\Phi\right] = -\lambda^2\Phi$$

$$\left\langle \Im_x u, v\right\rangle = \int_a^b w\left[\frac{1}{w}\frac{d}{dx}\left(p\frac{du}{dx}+qu\right)\right]vdx$$

$$\frac{d^2u}{dx^2}+f_1(x)\frac{du}{dx}+f_0(x)u=0$$

$$\frac{d}{dx}\left(p\frac{du}{dx}\right)+qu=0$$

FTT Problems and corresponding Eigenvalues Problems

		Coordinate	w	p	Eigenfunction
Rectangular (x,t)	$\frac{\partial \Theta}{\partial t}=\frac{\partial^2 \Theta}{\partial x^2}$	x	1	1	$\sin(\lambda x),\cos(\lambda x)$
Rectangular (x,y)	$\frac{\partial^2 \Theta}{\partial x^2}+\frac{\partial^2 \Theta}{\partial y^2}=0$	x	1	1	$\sin(\lambda x),\cos(\lambda x)$
		y	1	1	$\sin(\lambda y),\cos(\lambda y)$
Cylindrical (r,t)	$\frac{\partial \Theta}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right)$	r	r	r	$J_0(\lambda r),Y_0(\lambda r)$
Cylindrical (r,z)	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right)+\frac{\partial^2 \Theta}{\partial z^2}=0$	r	r	r	$J_0(\lambda r),Y_0(\lambda r)$
		z	1	1	$\sin(\lambda z),\cos(\lambda z)$
Cylindrical (r,θ)	$r\frac{\partial}{\partial r}\left(r\frac{\partial \Theta}{\partial r}\right)+\frac{\partial^2 \Theta}{\partial \theta^2}=0$	θ	1	1	$\sin(\lambda \theta),\cos(\lambda \theta)$
Spherical (r,t)	$\frac{\partial \Theta}{\partial t}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Theta}{\partial r}\right)$	r	r^2	r^2	$\frac{\sin(\lambda r)}{r},\frac{\cos(\lambda r)}{r}$
Spherical $(r,\eta=\cos\theta)$	$\frac{\partial}{\partial r}\left(r^2\frac{\partial \Theta}{\partial r}\right)+\frac{\partial}{\partial \eta}\left[\left(1-\eta^2\right)\frac{\partial \Theta}{\partial \eta}\right]=0$	η	1	$1-\eta^2$	$P(\eta)$

$P(\eta)$= Legendre polynomial

Analytical Techniques
FFT (Finite Fourier Transform)
Short version

L is the maximum length in the transformed coordinate, if problem is written in dimensionless form then *L*=1

Case	*Boundary conditions*	*Basis Functions*	
I	$\Phi_n(0)=0=\Phi_n(L)$	$\Phi_n(x)=\sqrt{\frac{2}{L}}\,sin\left(\frac{n\pi x}{L}\right)$	*n*=1,2,3,…
II	$\frac{d\Phi_n}{dx}(0)=0=\Phi_n(L)$	$\Phi_n(x)=\sqrt{\frac{2}{L}}\,cos\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{L}\right)$	*n*=0,1,2,3,…
III	$\Phi_n(0)=0=\frac{d\Phi_n}{dx}(L)$	$\Phi_n(x)=\sqrt{\frac{2}{L}}\,sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{L}\right)$	*n*=0,1,2,3,…
IV	$\frac{d\Phi_n}{dx}(0)=0=\frac{d\Phi_n}{dx}(L)$	$\Phi_n(x)=\sqrt{\frac{2}{L}}\,cos\left(\frac{n\pi x}{L}\right)$	*n*=1,2,3,…
		$\Phi_0(x)=\sqrt{\frac{1}{L}}$	

All functions satisfy $\frac{d^2\Phi_n}{dx^2}=-\lambda^2\,\Phi_n$

$w(x)=1$

Cylindrical coordinates

L= is the maximum radius, if written in dimensionless form then *L*=1

Case	*Boundary conditions*	*Characteristic equation*	*Basis Functions*
I	$\Phi_n(L)=0$	$J_0(\lambda_n L)=0$	$\Phi_n(r)=\frac{\sqrt{2}}{L}\frac{J_0(\lambda_n r)}{J_1(\lambda_n L)}$
II	$\frac{d\Phi_n}{dr}(L)=0$	$J_1(\lambda_n L)=0$	$\Phi_n(r)=\frac{\sqrt{2}}{L}\frac{J_0(\lambda_n r)}{J_0(\lambda_n L)}$ *n*=1,2,3,… $\Phi_0(r)=\frac{\sqrt{2}}{L}$
III	$A\Phi_n(L)+\frac{d\Phi_n}{dr}(L)=0$	$\frac{\lambda_n}{A}=\frac{J_0(\lambda_n L)}{J_1(\lambda_n L)}$	$\Phi_n(r)=\frac{\sqrt{2}}{L}\frac{\lambda_n}{\sqrt{A^2+\lambda_n^2}}\frac{J_0(\lambda_n r)}{J_0(\lambda_n L)}$

All functions satisfy $\frac{d\Phi_n}{dr}(0)=0$ $\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Phi_n}{dr}\right)=-\lambda^2\,\Phi_n$ $\quad w(r)=r$

Spherical coordinates

Case	*Boundary conditions*	*Characteristic equation*	*Basis Functions*
I	$\Phi_n(L)=0$	$\lambda_n L = n\pi$ *n*=1,2,3,…	$\Phi_n(r)=\sqrt{\frac{2}{L}}\left[\frac{sin(n\pi r/L)}{r}\right]$
II	$\frac{d\Phi_n}{dr}(L)=0$	$\lambda_n L = \tan(\lambda_n L)$	$\Phi_n(r)=\sqrt{\frac{2}{L}}\left[\frac{sin(\lambda_n r)}{r\ sin(\lambda_n L)}\right]$ *n*=1,2,3,… $\Phi_0(r)=\sqrt{\frac{3}{L^3}}$
III	$A\Phi_n(L)+\frac{d\Phi_n}{dr}(L)=0$	$\lambda_n L = (1-AL)\tan(\lambda_n L)$	$\Phi_n(r)=\sqrt{\frac{2}{L}\left[\frac{1-AL}{sin^2(\lambda_n L)-AL}\right]}\left[\frac{sin(\lambda_n r)}{r}\right]$

All functions satisfy $\frac{d\Phi_n}{dr}(0)=0$ $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_n}{dr}\right)=-\lambda^2\ \Phi_n$ $w(r)=r^2$

$$\lambda_n^2 = n(n+1)$$ n=0, 1,2,3,...

$$\Phi_n(\eta) = \sqrt{\frac{2n+1}{2}} P_n(\eta)$$

Legendre Polynomials

$$(m+1)P_{m+1}(\eta) - (2m+1)\eta P_m(\eta) + mP_{m-1}(\eta) = 0$$

$$P_0(\eta) = 1 \qquad P_1(\eta) = \eta \qquad P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m}\left[\left(x^2 - 1\right)^m\right]$$

All functions satisfy $\frac{d\Phi_n}{dr}(0) = 0 \quad \frac{d}{dr}\left(\left(1-\eta^2\right)\frac{d\Phi_n}{d\eta}\right) = -\lambda^2 \Phi_n \qquad w(\eta) = 1$

$$\int x\, J_0(x)\, dx \;=\; x J_1(x)$$

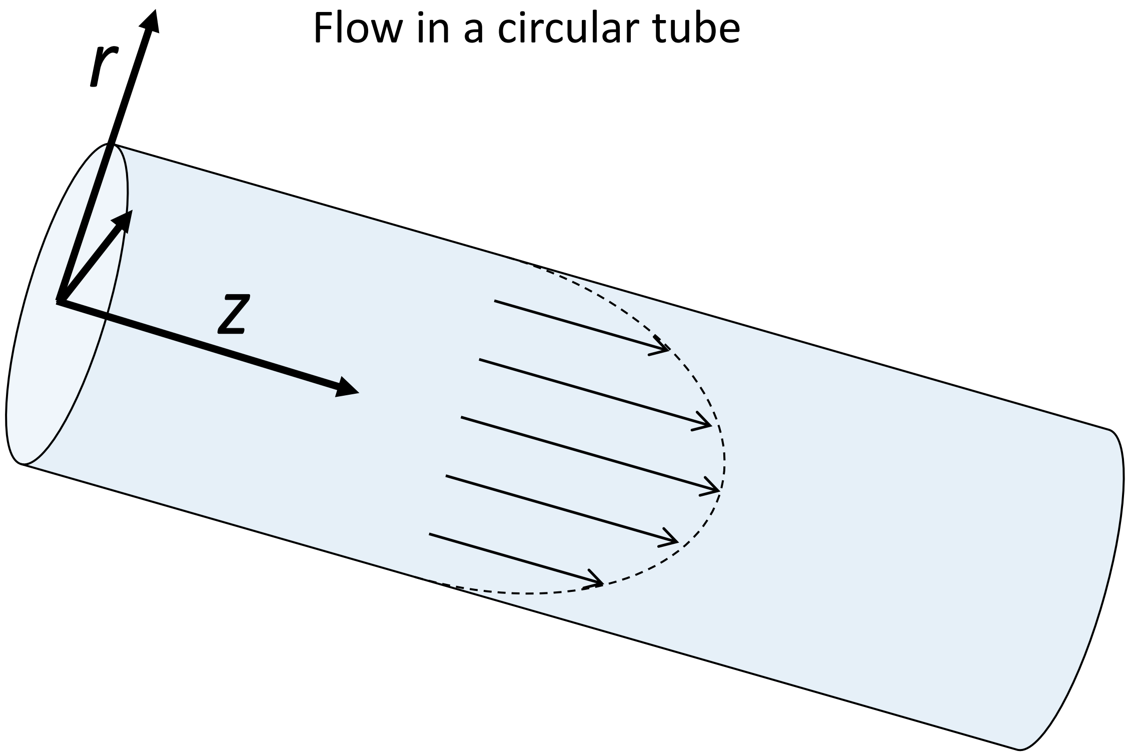
$$\int J_1(x)\, dx \;=\; -\, J_0(x)$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{71}$$

$$\int \sin^2(ax)dx = \frac{x}{2} - \frac{\sin(2ax)}{4a}$$

$$\int \sin(px)\sin(qx)dx = \frac{\sin[(p-q)x]}{2(p-q)} - \frac{\sin[(p+q)x]}{2(p+q)}$$

Flow in a circular tube



Analysis will be done for transient flow

Determine the velocity of a Newtonian fluid in a circular tube of radius, R when the fluid is subjected to a time-step pressure gradient. It is assumed that the fluid is initially at rest and that for time t>0 the axial, pressure gradient is given by

$$\rho g_z - \frac{\partial p}{\partial z} = \rho a = \rho\, a\, f(t) = \rho\, a\, U(t)$$

U(t) is unit step function.

From continuity and from linear momentum balance

$$\frac{\partial(\rho V_z)}{\partial t}+\left[\frac{\partial(\rho V_r V_z)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta V_z)}{\partial \theta}+\frac{\partial(\rho V_z V_z)}{\partial z}+\frac{\rho V_r V_z}{r}\right]=\rho g_z+\frac{\partial(T_{rz})}{\partial r}+\frac{1}{r}\frac{\partial(T_{\theta z})}{\partial \theta}+\frac{\partial(T_{zz})}{\partial z}+\frac{1}{r}T_{rz}-\frac{\partial p}{\partial z}$$

$$\frac{\partial(\rho V_z)}{\partial t}=\rho g_z+\frac{\partial(T_{rz})}{\partial r}+\frac{1}{r}T_{rz}-\frac{\partial p}{\partial z}$$

$$\frac{\partial(\rho V_z)}{\partial t}==\rho a+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial V_z}{dr}\right)$$

For an incompressible fluid, the momentum equation is multiplied by R^2 and divided by viscosity μ, and recasting this equation.

$$\frac{\partial(V_z)}{\partial \tau}=V_A+\frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial V_z}{d\eta}\right)$$

Two new dimensionless variables are set, and a dimensional perturbation velocity V_A.

$$\tau=\frac{t\mu}{\rho R^2}\qquad \eta=\frac{r}{R}\qquad V_A=\frac{aR^2\rho}{\mu}=V_o\gamma$$

γ is the dimensionless perturbation step signal, and V_o is the amplitude of the perturbation velocity

The linear momentum equation in terms of dimensionless velocity *u*.

$$\frac{\partial (V_z)}{\partial \tau} = V_A + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial V_z}{d\eta}\right)$$

$$u = \frac{V_z}{V_o}$$

The dimensionless form of the equation is: $$\frac{\partial u}{\partial \tau} = \gamma + \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial u}{d\eta}\right)$$

Using finite Fourier transform, and the orthonormal Bessel function for cylindrical coordinates, in this case we will use the Type I Eigenvalue problem, i.e. no slip boundary condition $u(\tau,\eta=1)=0$:

$$\Phi_n = \sqrt{2}\,\frac{J_0(\lambda_n \eta)}{J_1(\lambda_n)}$$

$$\int_0^1 \eta \Phi_n \frac{\partial u}{\partial \tau} d\eta = \int_0^1 \eta \Phi_n \gamma d\eta + \int_0^1 \frac{\eta \Phi_n}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial u}{d\eta}\right) d\eta$$

$$\int_0^1 \eta \Phi_n u(\eta,\tau) d\eta = \Psi_n(\tau)$$

Ψ Is the finite Fourier transform of the function *u, and in cylindrical coordinates the weight function is the dimensionless radius* η *.*

The Finite Fourier transform is applied to all the terms of the equation .

$$\int_0^1 \eta\Phi_n \frac{\partial u}{\partial\tau} d\eta = \frac{d\Psi_n}{d\tau} = \int_0^1 \eta\Phi_n \gamma d\eta + \int_0^1 \frac{\eta\Phi_n}{\eta} \frac{\partial}{\partial\eta}\left(\eta\frac{\partial u}{d\eta}\right) d\eta$$

The last term is integrated by parts twice, and stopped once you can identify the Eigenvalue problem operator.

$$\frac{d\Psi_n}{d\tau} = \int_0^1 \eta\Phi_n \gamma d\eta + \Phi_n\left(\eta\frac{du}{d\eta}\right)\Bigg|_0^1 - \eta u\left(\frac{d\Phi_n}{d\eta}\right)\Bigg|_0^1 + \int_0^1 \eta u \frac{d}{\eta d\eta}\left(\eta\frac{d\Phi_n}{d\eta}\right) d\eta$$

$$\frac{1}{\eta}\left[\frac{d}{d\eta}\left(\eta\frac{d\Phi_n}{d\eta}\right)\right] = -\lambda^2\Phi_n$$

$$\frac{d\Psi_n}{d\tau} = \int_0^1 \eta\Phi_n \gamma d\eta + \Phi_n\left(\eta\frac{du}{d\eta}\right)\Bigg|_0^1 - \eta u\left(\frac{d\Phi_n}{d\eta}\right)\Bigg|_0^1 + \int_0^1 \eta u \frac{d}{\eta d\eta}\left(\eta\frac{d\Phi_n}{d\eta}\right) d\eta$$

Applying boundary conditions for the second and third term , they both vanished Φ_n (η=1)=0 and $u(\eta=0,\tau)$=0 .

$$\frac{d\Psi_n}{d\tau} = \int_0^1 \eta\Phi_n \gamma d\eta + \Phi_n\left(\eta\frac{du}{d\eta}\right)\Bigg|_0^1 - \eta u\left(\frac{d\Phi_n}{d\eta}\right)\Bigg|_0^1 + -\lambda_n^2\int_0^1 \eta u\Phi_n d\eta$$

$$\frac{d\Psi_n}{d\tau}+\lambda_n^2\Psi_n=\int_0^1\eta\Phi_n\gamma d\eta+\Phi_n\left(\eta\frac{du}{d\eta}\right)\Bigg|_0^1-\eta u\left(\frac{d\Phi_n}{d\eta}\right)\Bigg|_0^1$$

$$\frac{d\Psi_n}{d\tau}+\lambda_n^2\Psi_n=\gamma\int_0^1\eta\frac{\sqrt{2}J_0(\lambda_n\eta)}{J_1(\lambda_n)}d\eta+\Phi_n\left(\eta\frac{du}{d\eta}\right)\Bigg|_0^1-\eta u\left(\frac{d\Phi_n}{d\eta}\right)\Bigg|_0^1$$

Integrating the remaining term in the left hand side of the equation.

$$\frac{d\Psi_n}{d\tau}+\lambda_n^2\Psi_n=\gamma\int_0^1(\eta\lambda_n)\frac{\sqrt{2}J_0(\lambda_n\eta)}{\lambda_n{}^2J_1(\lambda_n)}d(\lambda_n\eta)=\gamma(\eta\lambda_n)\frac{\sqrt{2}J_1(\lambda_n\eta)}{\lambda_n{}^2J_1(\lambda_n)}\Bigg|_0^1=\frac{\sqrt{2}}{\lambda_n}\gamma$$

Recasting the equation in a simplified form.

$$\frac{d\Psi_n}{d\tau}+\lambda_n^2\left(\Psi_n-\frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right)=0 \qquad \frac{d\left(\Psi_n-\frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right)}{d\tau}+\lambda_n^2\left(\Psi_n-\frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right)=0$$

Integration the first order differential equation.

$$\frac{d\left(\Psi_n - \frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right)}{\left(\Psi_n - \frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right)} = -\lambda_n^2 d\tau \qquad \ln\left(\Psi_n - \frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right) = -\lambda_n^2\tau + C_1$$

Applying the initial condition (remember the fluid is at rest, then Ψ_n (τ=0)=0 .

$$\left(\Psi_n - \frac{\sqrt{2}}{\lambda_n{}^3}\gamma\right) = C_2 e^{-\lambda_n^2\tau}$$

At τ=0 Ψ_n=0

$$\Psi_n = \frac{\sqrt{2}}{\lambda_n{}^3}\gamma\left[1 - e^{-\lambda_n^2\tau}\right]$$

Final expression of dimensionless velocity , in terms of dimensionless variables.

$$u = \sum \Phi_n(\eta)\Psi_n(\tau) = \sum \frac{2}{\lambda_n{}^3}\gamma\left[1 - e^{-\lambda_n^2\tau}\right]\frac{J_0(\lambda_n\eta)}{J_1(\lambda_n)}$$

The Eigenvalues λ_n are the roots of the zeroth order Bessel function.

$J_0(\lambda_n)=0$

For a pulsatile flow in a tube, determine the velocity resulting of a Newtonian fluid in a tube with time-periodic pressure gradient.

Use the finite Fourier transform of differential equation as starting point.

$$\frac{d\Psi_n}{d\tau} + \lambda_n^2 \Psi_n = \frac{\sqrt{2}}{\lambda_n}\gamma\left[1 + \beta \sin(\omega\tau)\right]$$

a) Under what conditions the decaying term is negligible, and what is the physical meaning of this finding, use properties of water and some other fluids to explain your findings.
b) Explain under what circumstances the pressure and velocity will be on phase.

Use the following hints to solve your problem:

$$\frac{dy}{dx} + f(x)y = g(x) \qquad F(x) = \int f(x)dx \qquad y = e^{-F(x)}\left[c_1 + \int e^{F(x)} g(x)dx\right]$$

$$\int e^{bx}\sin(ax)dx = \frac{1}{b^2 + a^2} e^{bx}\left[b\sin ax - a\cos ax\right]$$

$$\Psi_n = \left(\frac{\sqrt{2}}{\lambda_n}\gamma\right)\left(\left[\frac{\beta\omega}{\lambda_n^4+\omega^2}-\frac{1}{\lambda_n^2}\right]e^{-\lambda_n^2\tau}+\frac{1}{\lambda_n^2}+\left[\frac{\beta\lambda_n^2}{\lambda_n^4+\omega^2}\right]\left[\sin(\omega\tau)-\frac{\omega}{\lambda_n^2}\cos(\omega\tau)\right]\right)$$

$$\Phi_n = \sqrt{2}\,\frac{J_0(\lambda_n\eta)}{J_1(\lambda_n)}$$

$$\Psi_n = \left(\frac{\sqrt{2}}{\lambda_n^3}\gamma\right)\left(\left[\frac{\beta\omega\lambda_n^2}{\lambda_n^4+\omega^2}-1\right]e^{-\lambda_n^2\tau}+1+\left[\frac{\beta\lambda_n^4}{\lambda_n^4+\omega^2}\right]\left[\sin(\omega\tau)-\frac{\omega}{\lambda_n^2}\cos(\omega\tau)\right]\right)$$

$$u = \sum \Phi_n(\eta)\Psi_n(\tau)$$

$$u = \sum\left(\frac{2}{\lambda_n^3}\gamma\right)\left(\left[\frac{\beta\omega\lambda_n^2}{\lambda_n^4+\omega^2}-1\right]e^{-\lambda_n^2\tau}+1+\left[\frac{\beta\lambda_n^4}{\lambda_n^4+\omega^2}\right]\left[\sin(\omega\tau)-\frac{\omega}{\lambda_n^2}\cos(\omega\tau)\right]\right)\frac{J_0(\lambda_n\eta)}{J_1(\lambda_n)}$$

$$\Psi_n = \left(\frac{\sqrt{2}}{\lambda_n^3}\gamma\right)\left(\left[\frac{\beta\omega\lambda_n^2}{\lambda_n^4+\omega^2}-1\right]e^{-\lambda_n^2\tau}+1+\left[\frac{\beta\lambda_n^4}{\lambda_n^4+\omega^2}\right]\left[\sin(\omega\tau)-\frac{\omega}{\lambda_n^2}\cos(\omega\tau)\right]\right)$$

For values of $\tau > \dfrac{4}{\lambda_n^2}$ Velocity is not function of time anymore

$\dfrac{\omega}{\lambda_n^2} < 0.01$ Velocity is in phase with pressure

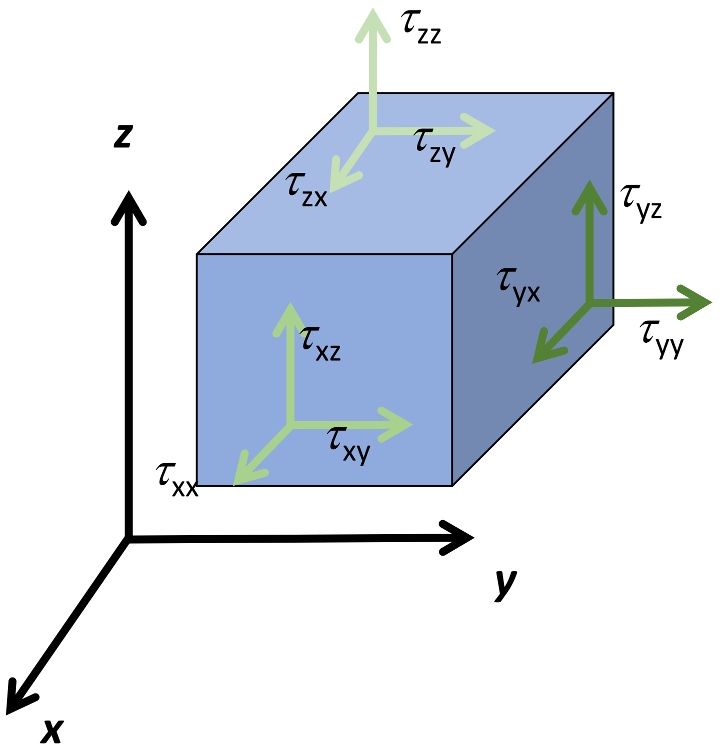
We've arranged a global civilization in which most crucial elements - transportation, communications, and all other industries; agriculture, medicine, education, entertainment, protecting the environment; and even the key democratic institution of voting - profoundly depend on science and technology. We have also arranged things so that almost no one understands science and technology. This is a prescription for disaster. We might get away with it for a while, but sooner or later this combustible mixture of ignorance and power is going to blow up in our faces.

THE DEMON-HAUNTED WORLD

Carl Sagan

Energy balance

Stress tensor is another construct to simplify the analysis



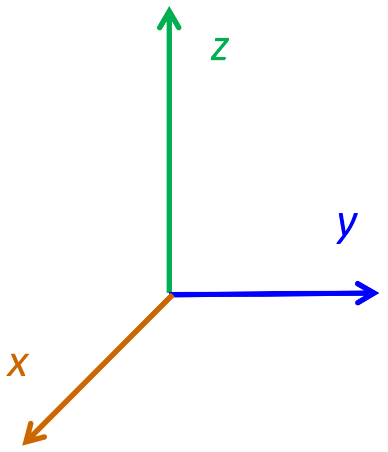
$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

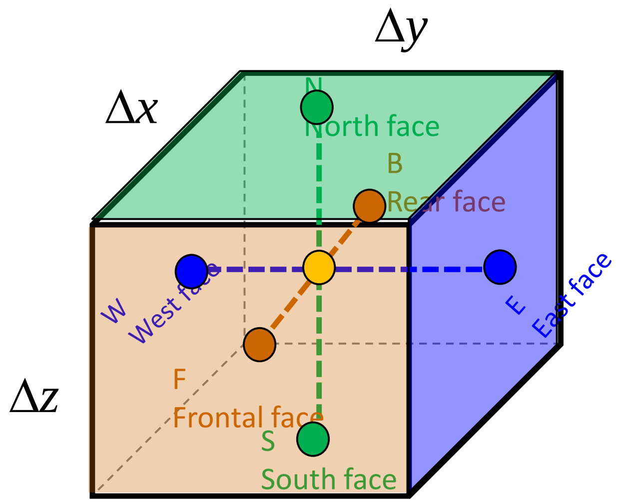
$$\underset{\rightarrow}{f_S} = \underset{\rightarrow}{n} \cdot \underline{\underline{\tau}}$$

$$\underset{\rightarrow}{f_S} = -p\underset{\rightarrow}{n} \cdot \underline{\underline{I}}$$

τ_{yx}=Is seen to be the force per unit area on a plane perpendicular to the *y* axis, acting in the ***x*** direction and exerted by the fluid at greater ***y***.

$$\underline{q}(x,y,z)=\hat{\mathrm{i}}\,q_x(x,y,z)+\hat{\mathrm{j}}\,q_y(x,y,z)+\hat{\mathrm{k}}\,q_z(x,y,z)$$

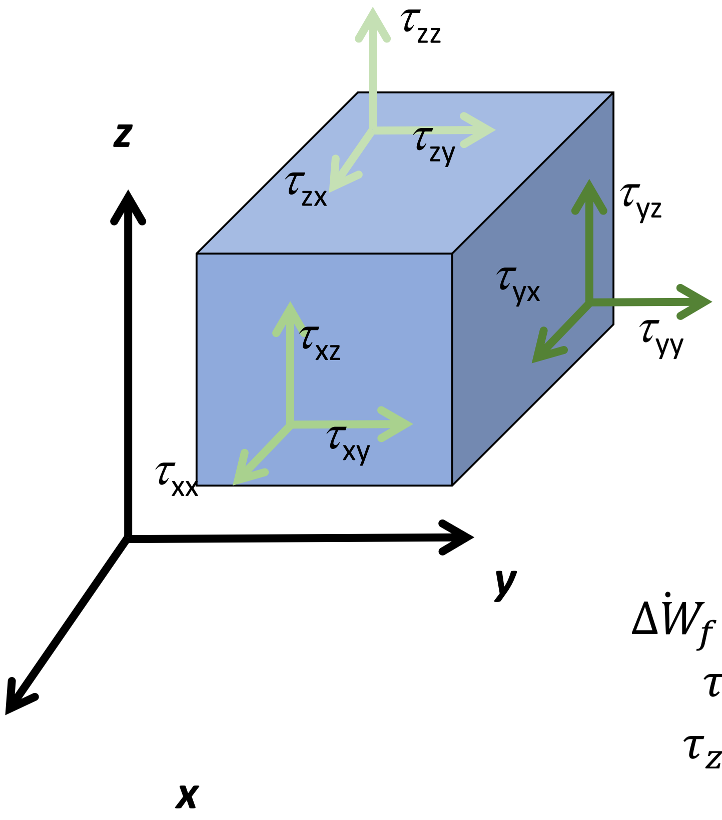




Coordinates of the points

Point	x	y	z	Normal
C	x	y	z	
F	$x+\Delta x/2$	y	z	$\underline{n}_F = \hat{\mathrm{i}}$
B	$x-\Delta x/2$	y	z	$\underline{n}_B = -\hat{\mathrm{i}}$
E	x	$y+\Delta y/2$	z	$\underline{n}_E = \hat{\mathrm{j}}$
W	x	$y-\Delta y/2$	z	$\underline{n}_W = -\hat{\mathrm{j}}$
N	x	y	$z+\Delta z/2$	$\underline{n}_N = \hat{\mathrm{k}}$
S	x	y	$z-\Delta z/2$	$\underline{n}_S = -\hat{\mathrm{k}}$

To calculate the energy rate loss by viscous dissipation



$$\underline{\underline{\tau}} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

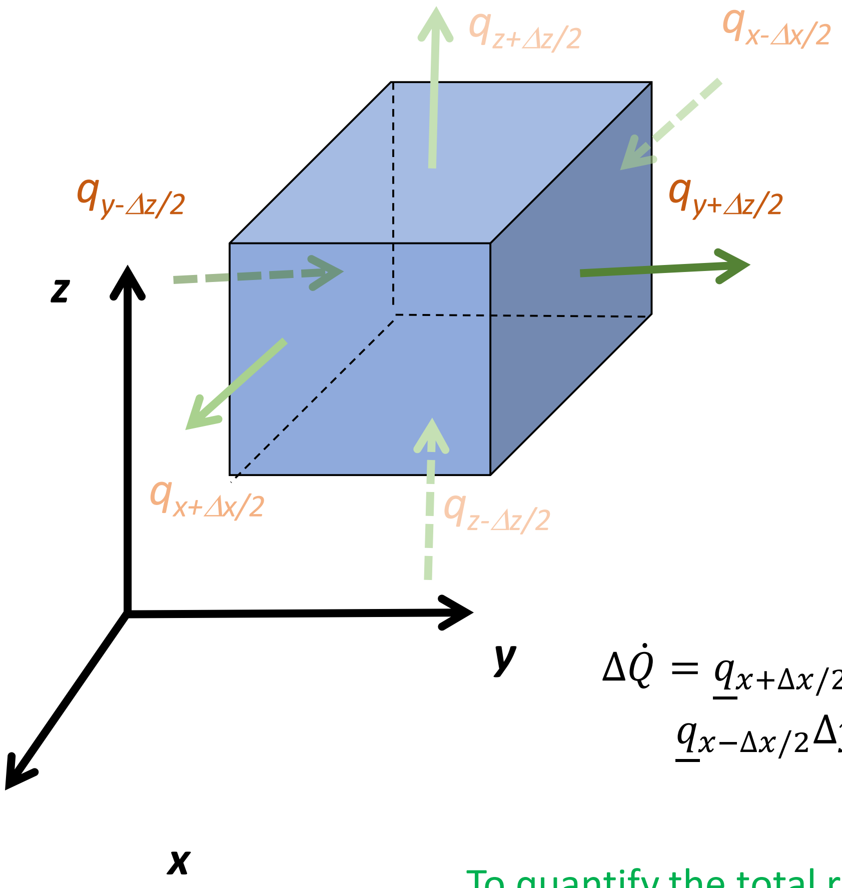
$$\dot{W}_f = \underline{F_f} \cdot \underline{V}$$

$$\Delta\dot{W}_f = \tau_{xx}\Delta y\Delta z\, V_x n_x + \tau_{xy}\Delta y\Delta z\, V_y n_x + \tau_{xz}\Delta y\Delta z\, V_z n_x + \tau_{yx}\Delta x\Delta z\, V_x n_y + \tau_{yy}\Delta x\Delta z\, V_y n_y + \tau_{yz}\Delta x\Delta z\, V_z n_y + \tau_{zx}\Delta y\Delta x\, V_x n_z + \tau_{yz}\Delta y\Delta x\, V_y n_z + \tau_{zz}\Delta y\Delta x\, V_z n_z + \cdots$$

$$\dot{W}_f = \oiint \underline{\underline{\tau}} \cdot \underline{V} \cdot \underline{n} dA = \oiint \underline{n} \cdot \underline{\underline{\tau}} \cdot \underline{V} \cdot dA = \iiint \underline{\nabla} \cdot \left[\underline{\underline{\tau}} \cdot \underline{V}\right] dV$$

Only 3 faces are shown, but remember we have six faces

To calculate the energy rate by heat



$$\underline{q} = -k\underline{\nabla T}$$

$$\underline{q}_x = -k\frac{\partial T}{\partial x}\hat{\mathrm{i}}$$

$$\underline{q}_y = -k\frac{\partial T}{\partial y}\hat{\mathrm{j}}$$

$$\underline{q}_z = -k\frac{\partial T}{\partial z}\hat{\mathrm{k}}$$

$$\Delta\dot{Q} = \underline{q}_{x+\Delta x/2}\Delta y\Delta z \cdot \underline{n}_F + \underline{q}_{y+\Delta y/2}\Delta x\Delta z \cdot \underline{n}_E + \underline{q}_{z+\Delta z/2}\Delta y\Delta x \cdot \underline{n}_N + \underline{q}_{x-\Delta x/2}\Delta y\Delta z \cdot \underline{n}_B + \underline{q}_{y-\Delta y/2}\Delta x\Delta z \cdot \underline{n}_W + \underline{q}_{z-\Delta z/2}\Delta y\Delta x \cdot \underline{n}_S$$

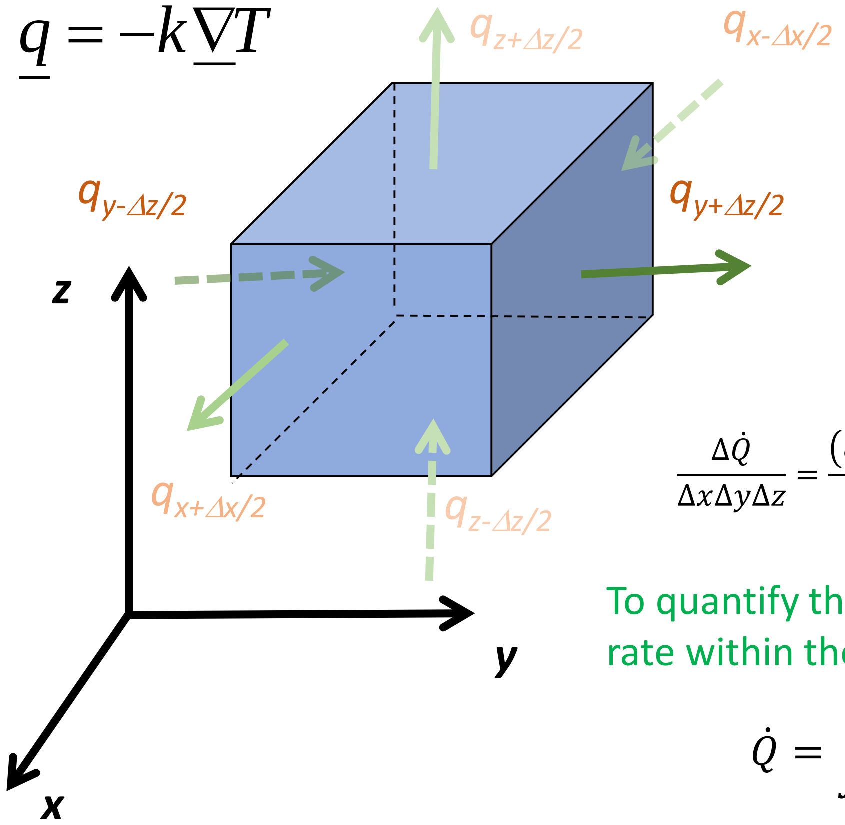
To quantify the total rate of energy by heat crossing the entire system:

$$\dot{Q} = \oiint \underline{q} \cdot \underline{n} dA = \oiint \underline{n} \cdot \underline{q}\, dA$$

Only 3 faces are shown, but remember we have six faces

To calculate the energy rate by heat

$$\underline{q} = -k\underline{\nabla}T$$



$$\Delta\dot{Q} = q_{x+\Delta x/2}\Delta y\Delta z + q_{y+\Delta y/2}\Delta x\Delta z + q_{z+\Delta z/2}\Delta y\Delta x + -q_{x-\Delta x/2}\Delta y\Delta z - q_{y-\Delta y/2}\Delta x\Delta z - q_{z-\Delta z/2}\Delta y\Delta x$$

It can also expressed by unit volume and quantify within the entire volume as:

$$\frac{\Delta\dot{Q}}{\Delta x\Delta y\Delta z} = \frac{\left(q_{x+\Delta x/2} - q_{x-\Delta x/2}\right)}{\Delta x} + \frac{\left(q_{y+\Delta y/2} - q_{y-\Delta y/2}\right)}{\Delta y} + \frac{\left(q_{z+\Delta z/2} - q_{z-\Delta z/2}\right)}{\Delta z}$$

To quantify the total amount of energy rate within the entire system:

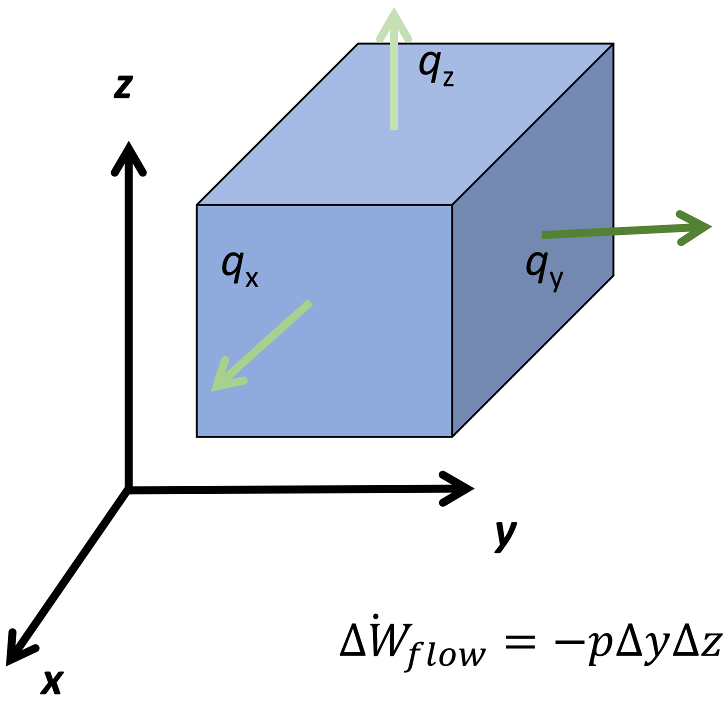
$$\dot{Q} = \iiint \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right] dV = \iiint \underline{\nabla}\cdot\underline{q}\,dV$$

Comparing with the previous equation the Gauss' Law can be inferred as:

$$\dot{Q} = \oiint \underline{q}\cdot\underline{n}\,dA = \oiint \underline{n}\cdot\underline{q}\,dA = \iiint \underline{\nabla}\cdot\underline{q}\,dV$$

Only 3 faces are shown, but remember we have six faces

To calculate the energy rate by flow



$$\Delta\dot{W}_{flow} = -p\Delta y\Delta z\, V_x n_x - p\Delta x\Delta z\, V_y n_y - p\,\Delta y\Delta x\, V_z n_z - \cdots$$

$$\dot{W}_{flow} = -\oiint p\,\underline{n}\cdot\underline{V}\,dA = -\iiint \underline{\nabla}\cdot(p\,\underline{V})dV$$

Only 3 faces are shown, but remember we have six faces

$$\frac{dB}{dt} = \iiint \left[\frac{\partial\left(\rho \hat{B}\right)}{\partial t} + \underline{\nabla} \cdot \left(\rho \; \hat{B} \; \underline{v}\right) \right] dV$$

$$\frac{\partial\left(\rho \hat{E}\right)}{\partial t} + \underline{\nabla} \cdot \left(\rho \; \hat{E} \; \underline{v}\right) = \underline{\nabla} \cdot \left(\underline{\underline{\tau}} \cdot \underline{v}\right) - \underline{\nabla} \cdot \underline{q} - \underline{\nabla} \cdot \left(p \; \underline{v}\right)$$

$$\underline{\underline{\underline{\varepsilon}}} = \sum \varepsilon_{ijk} \; \hat{\mathrm{i}} \; \hat{\mathrm{j}} \; \hat{\mathrm{k}} \qquad \varepsilon_{ijk} = \begin{cases} 0 & i = j,\; j = k, or\; i = k \\ 1 & \mathrm{ijk} = 123, 231, \text{or } 312 \\ -1 & \mathrm{ijk} = 132, 213, \text{or } 321 \end{cases}$$

Incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9–61a}$$

x-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{9–61b}$$

y-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{9–61c}$$

z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial P}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{9–61d}$$

Incompressible continuity equation: $$\frac{1}{r}\frac{\partial(ru_r)}{\partial r}+\frac{1}{r}\frac{\partial(u_\theta)}{\partial\theta}+\frac{\partial(u_z)}{\partial z}=0 \tag{9–62a}$$

r-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)-\frac{u_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial z^2}\right] \tag{9–62b}$$

θ-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u_\theta}{\partial t}+u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}+u_z\frac{\partial u_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial P}{\partial\theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\theta}{\partial r}\right)-\frac{u_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{\partial^2 u_\theta}{\partial z^2}\right] \tag{9–62c}$$

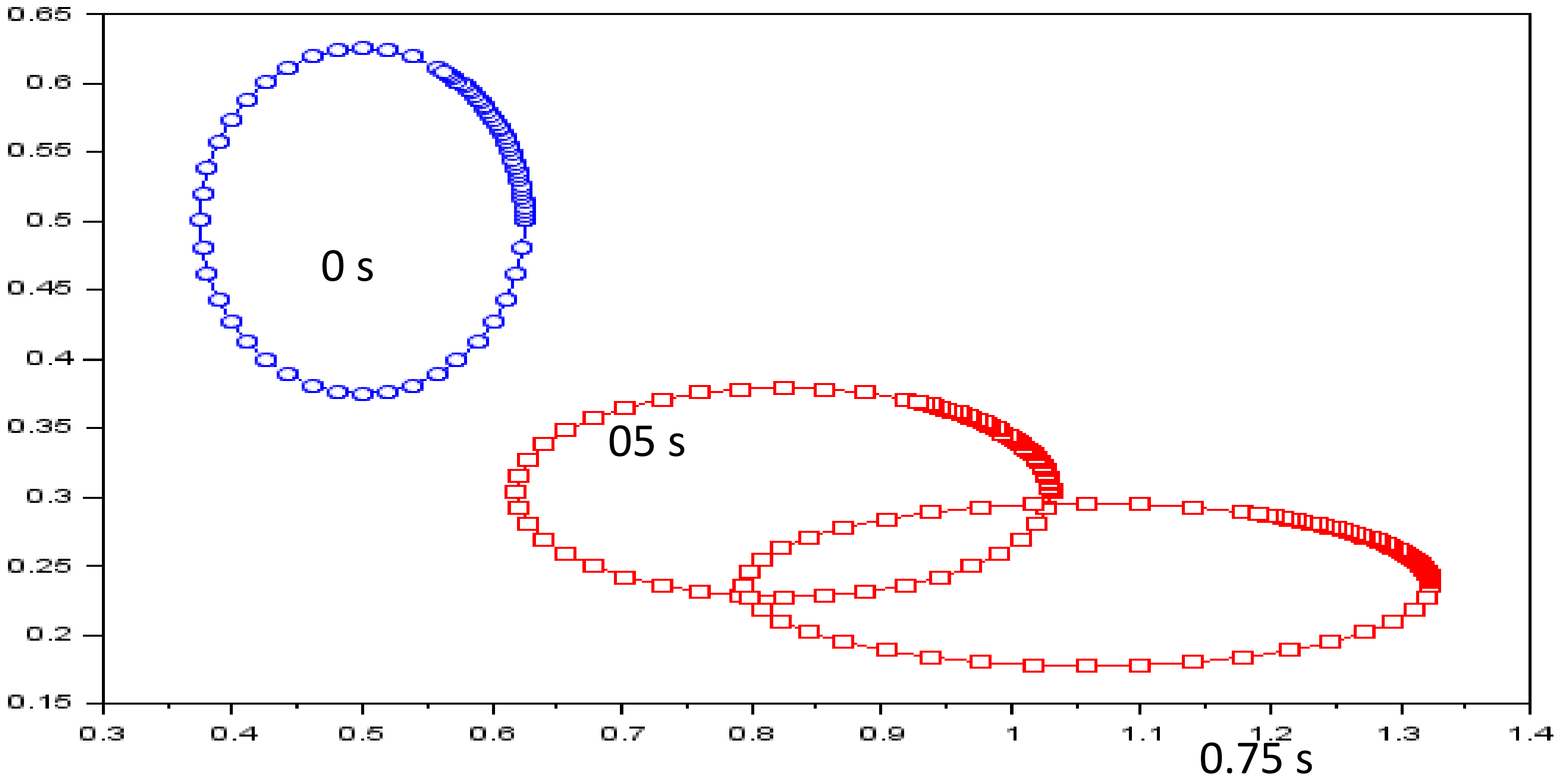
z-component of the incompressible Navier–Stokes equation:

$$\rho\left(\frac{\partial u_z}{\partial t}+u_r\frac{\partial u_z}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta}+u_z\frac{\partial u_z}{\partial z}\right)$$

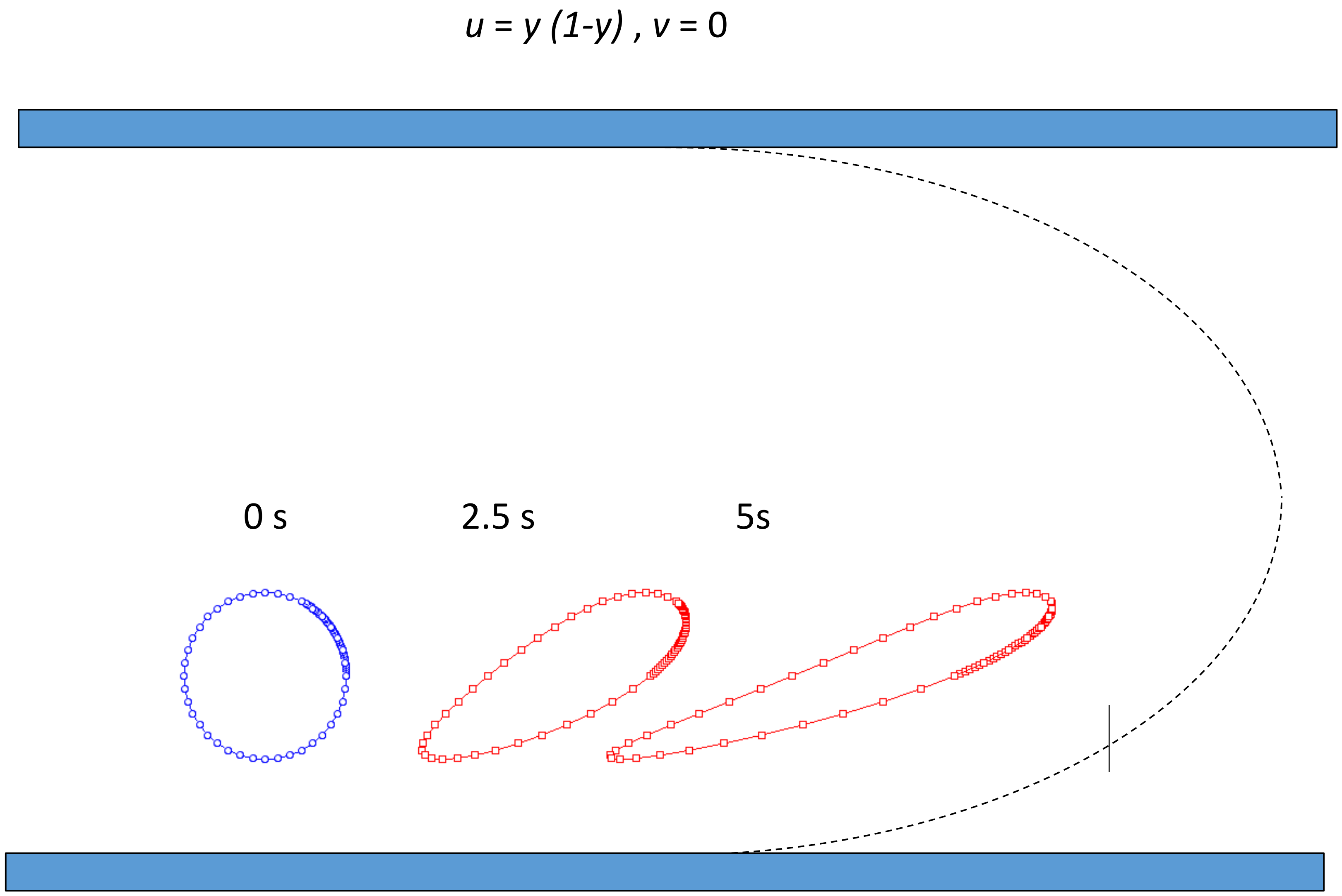
$$=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial\theta^2}+\frac{\partial^2 u_z}{\partial z^2}\right] \tag{9–62d}$$

$$\nabla(\boldsymbol{A}+\boldsymbol{B}) = \nabla\boldsymbol{A}+\nabla\boldsymbol{B}$$
$$\nabla\cdot(\boldsymbol{A}+\boldsymbol{B}) = \nabla\cdot\boldsymbol{A}+\nabla\cdot\boldsymbol{B}$$
$$\nabla\times(\boldsymbol{A}+\boldsymbol{B}) = \nabla\times\boldsymbol{A}+\nabla\times\boldsymbol{B}$$
$$\nabla\cdot(\alpha\boldsymbol{A}) = (\nabla\alpha)\cdot\boldsymbol{A}+\alpha(\nabla\cdot\boldsymbol{A})$$
$$\nabla\times(\alpha\boldsymbol{A}) = (\nabla\alpha)\times\boldsymbol{A}+\alpha(\nabla\times\boldsymbol{A})$$
$$\nabla\times(\nabla\times\boldsymbol{A}) = \nabla(\nabla\cdot\boldsymbol{A})-\nabla^2\boldsymbol{A}$$
$$\nabla\cdot(\boldsymbol{v}\times\boldsymbol{w}) = (\nabla\times\boldsymbol{v})\cdot\boldsymbol{w}+\boldsymbol{v}\cdot(\nabla\times\boldsymbol{w})$$
$$\nabla\times(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}-(\nabla\cdot\boldsymbol{v})\boldsymbol{w}-(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}+(\nabla\cdot\boldsymbol{w})\boldsymbol{v}$$
$$\nabla(\boldsymbol{v}\cdot\boldsymbol{w}) = (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}+\boldsymbol{w}\times(\nabla\times\boldsymbol{v})+\boldsymbol{v}\times(\nabla\times\boldsymbol{w})$$

$u = x$, $v = -y$



0.65
0.6
0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.1
1.2
1.3
1.4
0 s
05 s
0.75 s



$u = y\,(1-y)\ ,\ v = 0$
0 s
2.5 s
5s

Turbulent Flow (For CFD analysis)

$$\frac{\partial(\rho)}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

Continuity equation

$$\frac{\partial(\rho \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}\, \underline{v}) = \rho \underline{g} - \underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}}$$

Linear momentum equation

Velocity

Average Velocity

Fluctuating Component of Velocity

$$\underline{v} = \underline{V} + \underline{V}'$$

$$p = P + P'$$

Pressure

Average Pressure

Fluctuating Component of Pressure

Average Velocity

Average Fluctuating Velocity

$$\langle \underline{v} \rangle = \underline{V} \qquad \langle \underline{V}' \rangle = \underline{0}$$

$$\langle p \rangle = P \qquad \langle P' \rangle = 0$$

Plugging the velocity equation in terms of average velocity and fluctuating velocity into the linear momentum equation and continuity respectively for incompressible fluid, and averaging all of them, the resulting equations are:

$$\frac{\partial(\underline{V})}{\partial t} + \underline{\nabla} \cdot (\underline{V}\, \underline{V}) = -\frac{1}{\rho} \underline{\nabla} P + \frac{\mu}{\rho} \nabla^2 \underline{V} + \underline{\nabla} \cdot \langle \underline{V}' \underline{V}' \rangle$$

$$\underline{\nabla} \cdot (\underline{V}) = 0$$

Fluctuating components of velocity

$$\underline{V}' = \hat{\imath}\, u' + \hat{\jmath}\, v' + \hat{k}\, w'$$

Some properties of the fluctuating velocity field are organized in terms, to track its behavior

Reynolds Stress Tensor

$$\underline{\underline{\tau}}_{RST} = \langle \underline{V}'\underline{V}' \rangle = \begin{bmatrix} \langle u'u' \rangle & \langle u'v' \rangle & \langle u'w' \rangle \\ \langle u'v' \rangle & \langle v'v' \rangle & \langle v'w' \rangle \\ \langle u'w' \rangle & \langle v'w' \rangle & \langle w'w' \rangle \end{bmatrix}$$

Turbulence Kinetic Energy

$$k = \frac{1}{2}\left(\langle u'u' \rangle + \langle v'v' \rangle + \langle w'w' \rangle\right)$$

Rate at which turbulent kinetic energy is dissipated

$$\varepsilon = \frac{\mu}{\rho}\left(\underline{\nabla V'} : \left(\underline{\nabla V'}\right)^T\right)$$

The linear average momentum equation takes a new form after approximating the Reynolds stress tensor, to a form similar to the viscous stress tensor:

$$\frac{\partial(\underline{V})}{\partial t} + \underline{\nabla} \cdot (\underline{V}\,\underline{V}) = -\frac{1}{\rho}\underline{\nabla} P + \left(\frac{\mu}{\rho} + \nu_T\right)\nabla^2 \underline{V}$$

$$\nu_T = C_\mu \frac{k^2}{\varepsilon}$$

Turbulent kinematic eddy viscosity

$$k = \frac{1}{2}(\langle u'u'\rangle + \langle v'v'\rangle + \langle w'w'\rangle)$$

$$\varepsilon = \frac{\mu}{\rho}\left(\underline{\nabla V'} : \left(\underline{\nabla V'}\right)^T\right)$$

$$\underline{\underline{\tau}}_{RST} = \langle \underline{V'}\,\underline{V'}\rangle = \nu_T \left(\underline{\nabla}\,\underline{V} + \left(\underline{\nabla}\,\underline{V}\right)^T\right)$$

$$\frac{\partial k}{\partial t} + \underline{V} \cdot \underline{\nabla} k = \underline{\nabla} \cdot (\nu + C_2 \nu_T) \underline{\nabla} k + \underline{\langle V'V' \rangle} : \underline{\nabla V} - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + \underline{V} \cdot \underline{\nabla} \varepsilon = \underline{\nabla} \cdot (\nu + C_3 \nu_T) \underline{\nabla} \varepsilon + \frac{C_4}{k} \varepsilon \underline{\langle V'V' \rangle} : \underline{\nabla V} - \frac{C_5}{k} \varepsilon^2$$

Turbulent transport

Production

Dissipation

$$C_1 = 0.09; \quad C_2 = 1.0; \quad C_3 = 0.769; \quad C_4 = 1.44; \quad C_5 = 1.92$$

Continuity Equation

$$\frac{\partial(\rho)}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

Cauchy Equation of Motion

$$\frac{\partial(\rho \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}\, \underline{v}) = \rho \underline{g} - \underline{\nabla} p + \underline{\nabla} \cdot \underline{\underline{\tau}} + \sigma\, \kappa\, \delta\, \underline{\hat{n}}$$

Navier-Stokes

$$\underline{\nabla} \cdot (\underline{v}) = 0$$

$$\rho \frac{\partial(\underline{v})}{\partial t} + \rho \underline{v} \cdot \underline{\nabla}(\underline{v}) = \underline{F} - \underline{\nabla} p + \underline{\nabla} \cdot \left[\mu\left(\underline{\nabla}\, \underline{v} + (\underline{\nabla}\, \underline{v})^T\right)\right]$$

Brinkman Equation

$$\underline{\nabla} \cdot (\underline{v}) = 0$$

$$\rho \frac{\partial(\underline{v})}{\partial t} = -\frac{\mu}{\kappa} \underline{v} + \underline{F} - \underline{\nabla} p + \underline{\nabla} \cdot \left[\mu\left(\underline{\nabla}\, \underline{v} + (\underline{\nabla}\, \underline{v})^T\right)\right]$$

Euler Equation

$$\frac{\partial(\rho)}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0$$

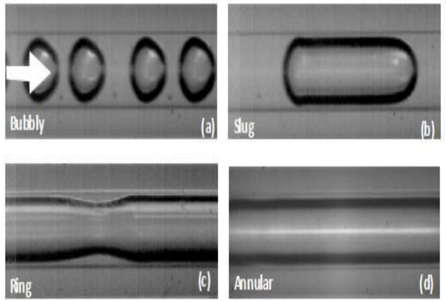
$$\rho \frac{\partial(\underline{v})}{\partial t} + \rho \underline{v} \cdot \underline{\nabla}(\underline{v}) = \underline{F} - \underline{\nabla} p$$

$$\frac{\partial p}{\partial t} + \underline{v} \cdot \underline{\nabla}\, p + \gamma\, p\, \underline{\nabla} \cdot \underline{v} = (\gamma - 1)(Q - \underline{F} \cdot \underline{u})$$

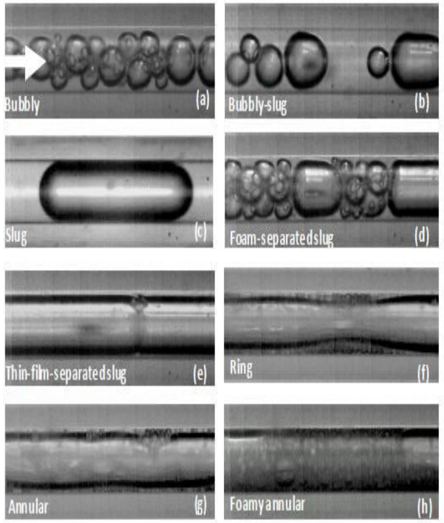
Darcy's Law

$$\alpha \frac{\partial(p)}{\partial t} + \underline{\nabla} \cdot \left(\frac{\kappa}{\mu} \underline{\nabla} p\right) = \underline{F}$$

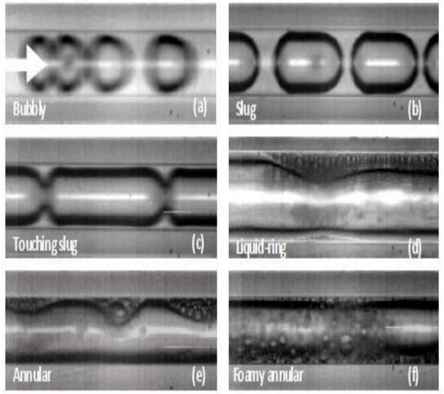
Two phase flow Flow



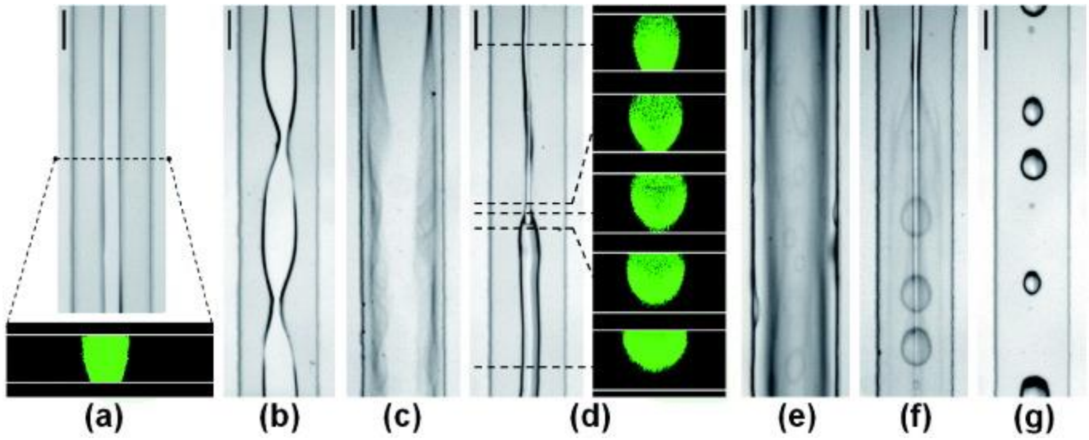
Two phase flow patterns with pure water



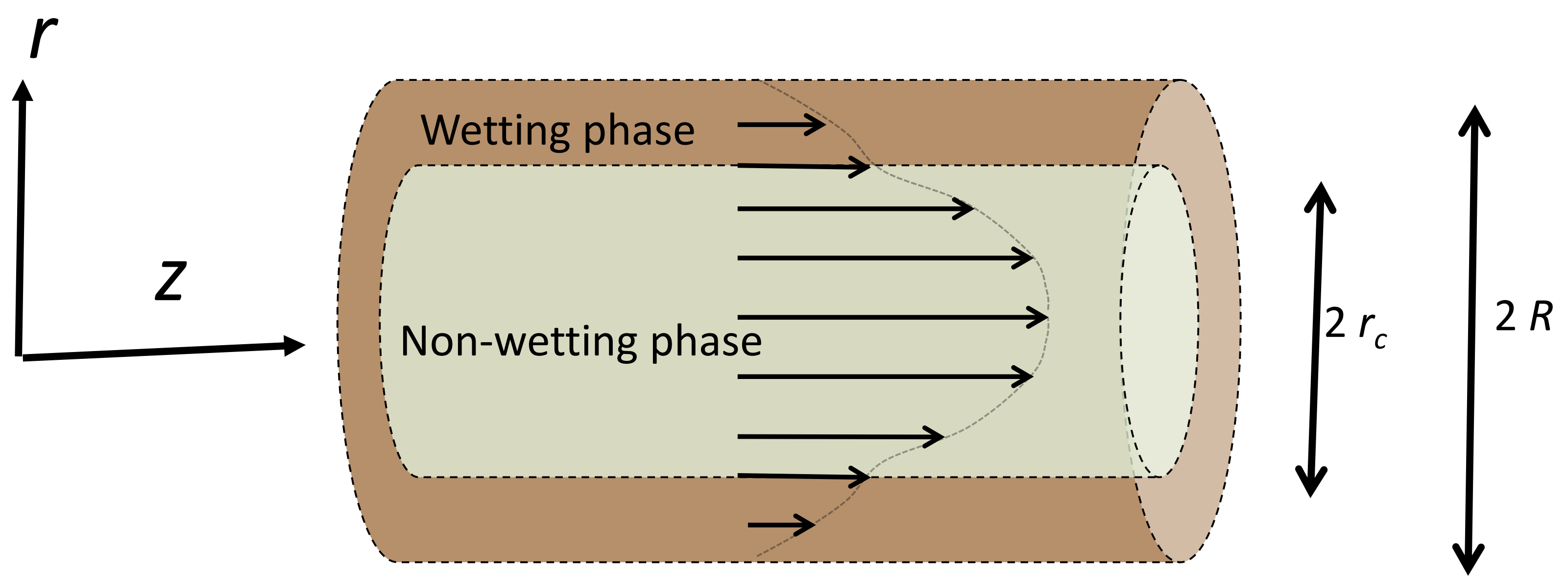
Two phase flow patterns with 1000 wppm



Two phase flow patterns with 400 wppm



Two phase laminar flow under steady-flow conditions



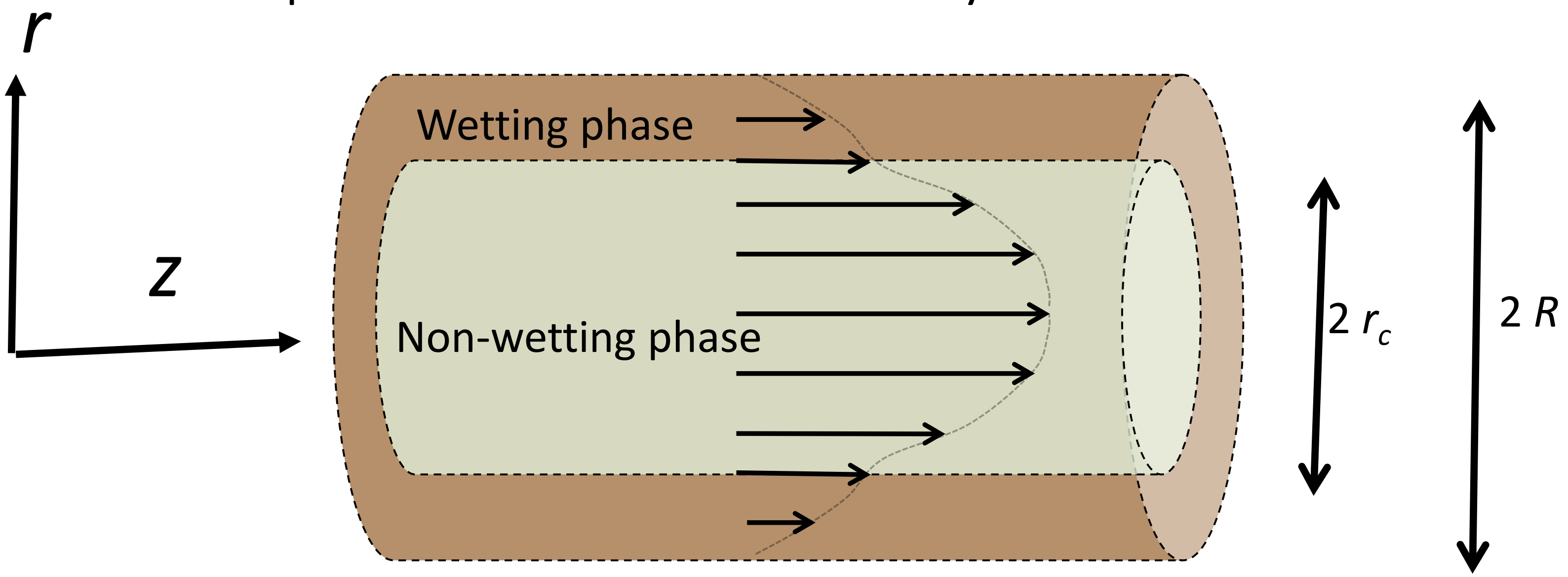
$$0 = \rho g_z + \frac{1}{r}\frac{\partial\left(rT_{rz}\right)}{\partial r} - \frac{\partial p}{\partial z}$$

$$\alpha = \rho g_z - \frac{\partial p}{\partial z}$$

$$-\alpha_{nw} = \frac{1}{r}\frac{\partial(r\tau_{nw})}{\partial r}$$

$$-\alpha_{w} = \frac{1}{r}\frac{\partial(r\tau_{w})}{\partial r}$$

Two phase laminar flow under steady-flow conditions



$$-\alpha_{nw} = \frac{1}{r}\frac{\partial\left(r\mu_{nw}\frac{\partial v_{nw}}{\partial r}\right)}{\partial r}$$

$$-\alpha_{w} = \frac{1}{r}\frac{\partial\left(r\mu_{w}\frac{\partial v_{w}}{\partial r}\right)}{\partial r}$$

Boundary conditions

$$v_{nw}\Big|_{r=0} = v_{max}$$ Finite velocity at the center

$$v_{nw}\Big|_{r=r_c} = v_w\Big|_{r=r_c}$$ No-slip boundary at the interface

$$\mu_{nw}\frac{dv_{nw}}{dr}\Big|_{r=r_c} = \mu_w\frac{dv_w}{dr}\Big|_{r=r_c}$$ Tangent stress balance at the interface

$$v_w\Big|_{r=R} = 0$$ No-slip boundary at the walls

$$-\alpha_{nw} = \frac{1}{r}\frac{d\left(r\mu_{nw}\frac{dv_{nw}}{dr}\right)}{dr}$$

Integrating the linear momentum equation, the results has the form of the strain rate

$$-\alpha_{nw}\frac{r^2}{2\mu_{nw}} + C_{1,nw} = r\frac{dv_{nw}}{dr}$$

Integrating the strain rate equation to obtain the expression for velocity, gives the form:

$$-\alpha_{nw}\frac{r^2}{4\mu_{nw}} + C_{1,nw}\ln r + C_{2,nw} = v_{nw}$$

Velocity field in radial direction for the non wetting phase

The same methodology is applied for the wetting phase, resulting the following equations for strain rate and velocity field

$$-\alpha_{w}\frac{r^2}{2\mu_{w}} + C_{1,w} = r\frac{dv_{w}}{dr}$$

$$-\alpha_{w}\frac{r^2}{4\mu_{w}} + C_{1,w}\ln r + C_{2,w} = v_{w}$$

Applying boundary conditions to obtain the integration constants, results in the following equations.

At the venter the velocity is finite, then $C_{1,nw}$ =0

$$-\alpha_{nw}\frac{r^2}{4\mu_{nw}} + C_{1,nw}\ln r + C_{2,nw} = v_{nw}$$

$$-\alpha_{nw}\frac{r^2}{4\mu_{nw}} + C_{2,nw} = v_{nw}$$

$$-\alpha_{nw}\frac{r^2}{2\mu_{nw}} = r\frac{dv_{nw}}{dr}$$

For the wetting phase at the walls the velocity is set to zero, then:

$$-\alpha_{w}\frac{R^2}{4\mu_{w}} + C_{1,w}\ln R + C_{2,w} = 0$$

At the interface (i.e. at $r=r_c$) the velocity of both phases are equal, as well as the tangential shear stress, then:

$$-\alpha_{nw}\frac{r_c^2}{4\mu_{nw}} + C_{1,nw}\ln r_c + C_{2,nw} = -\alpha_{w}\frac{r_c^2}{4\mu_{w}} + C_{1,w}\ln r_c + C_{2,w}$$

$$\mu_{nw}\left(-\alpha_{nw}\frac{r_c^2}{2\mu_{nw}} + C_{1,nw}\right) = \mu_{w}\left(-\alpha_{w}\frac{r_c^2}{2\mu_{w}} + C_{1,w}\right)$$

If we neglect the effect of gravity, $\alpha_w = \alpha_{nw}$:

$$\mu_{nw}\left(-\alpha_w \frac{r_c^2}{2\mu_{nw}} + C_{1,nw}\right) = \mu_w\left(-\alpha_w \frac{r_c^2}{2\mu_w} + C_{1,w}\right)$$

Then $C_{1,nw} = C_{1,w} = 0$

$$-\alpha_{nw}\frac{r_c^2}{4\mu_{nw}} + C_{2,nw} = -\alpha_w \frac{r_c^2}{4\mu_w} + C_{2,w}$$

$$\boxed{\alpha_w \frac{R^2}{4\mu_w} = C_{2,w}}$$

$$\boxed{C_{2,nw} = \alpha_w \frac{R^2 - r_c^2}{4\mu_w} + \alpha_{nw}\frac{r_c^2}{4\mu_{nw}}}$$

$$-\alpha_w \frac{r^2}{4\mu_w} + C_{1,w}\ln r + C_{2,w} = v_w$$

$$\boxed{v_w = \alpha_w \frac{R^2 - r^2}{4\mu_w}}$$

$$\boxed{v_{nw} = \alpha_w \frac{R^2 - r_c^2}{4\mu_w} + \alpha_{nw}\frac{(r_c^2 - r^2)}{4\mu_{nw}}}$$

$$v_w = \frac{\alpha\, R^2}{4\mu_w}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$v_{nw} = \frac{\alpha\, R^2}{4\mu_w}\left(1 - \left(\frac{r_c}{R}\right)^2\right) + \frac{\alpha\, R^2}{4\mu_{nw}}\left(\left(\frac{r_c}{R}\right)^2 - \left(\frac{r}{R}\right)^2\right)$$

Integration across their respective domains to obtain the average velocity, ends up with the equations:

$$\langle\, v_w \rangle = \frac{2\pi \int_{r_c}^{R} r\, v_w dr}{\pi R^2} = \frac{\alpha\, R^2}{2\mu_w} \int_{r_c/R}^{1} \zeta[1 - \zeta^2] d\zeta$$

$$\langle\, v_{nw} \rangle = \frac{2\pi \int_{0}^{r_c} r\, v_{nw} dr}{\pi R^2} = \frac{\alpha\, R^2}{2\mu_{nw}} \int_{0}^{r_c/R} \zeta[\zeta_c^2 - \zeta^2] d\zeta + \frac{\alpha\, R^2}{2\mu_w}[1 - \zeta_c^2] \int_{0}^{r_c/R} \zeta d\zeta$$

Before integration let's define saturation (s) as the volume fraction of the non-wetting phase, then the ratio between the radii is:

$$s = \frac{\pi r_c^2}{\pi R^2} = \left(\frac{r_c}{R}\right)^2 = \zeta_c^2$$

The dimensionless radius of the interface between flowing phases:

$$\zeta_c = \sqrt{s}$$

$$\langle v_w \rangle = \frac{\alpha R^2}{2\mu_w} \int_{\sqrt{s}}^{1} \zeta[1-\zeta^2]d\zeta = \frac{\alpha R^2}{8\mu_w}(1-s)^2$$

$$\langle v_{nw} \rangle = \frac{\alpha R^2}{2\mu_{nw}} \int_{0}^{\sqrt{s}} \zeta(s-\zeta^2)d\zeta + \frac{\alpha R^2}{2\mu_w}[1-s] \int_{0}^{\sqrt{s}} \zeta d\zeta = \frac{\alpha R^2 s^2}{8\mu_{nw}} + \frac{\alpha R^2}{8\mu_w}(2s)[1-s]$$

Another parameter characteristic of the two phase flow is the viscosity ratio ($M=\mu_{nw}/\mu_w$, i.e. the viscosity of the non-wetting fluid divided by the viscosity of the wetting fluid), sometimes confused with mobility ratio:

$$\langle v_w \rangle = \frac{\alpha R^2}{8\mu_w}(1-s)^2$$

$$\langle v_{nw} \rangle = \frac{\alpha R^2}{8\mu_{nw}}[s^2 + 2Ms(1-s)]$$

Volumetric flow rates

$$\frac{\dot{V}_w}{\pi R^2} = \frac{\alpha R^2}{8\mu_w}(1-s)^3$$

$$\frac{\dot{V}_{nw}}{\pi R^2} = \frac{\alpha R^2}{8\mu_{nw}}[s^3 + 2Ms^2(1-s)]$$

α is the pressure gradient

$$\alpha = -\nabla p$$

If the non-wetting fluid is the only fluid transported the equation has the form:

$$\frac{\dot{V}_{nw}}{\pi R^2} = \frac{\alpha\, R^2}{8\mu_{nw}} = \frac{-\nabla p\, R^2}{8\mu_{nw}} = \langle v_{nw} \rangle = -\frac{k}{\mu_{nw}} \nabla p$$

A new term pops up , called permeability (k):

$$k = \frac{R^2}{8}$$

This new equation is called Darcy law, and in vector form is:

$$\langle \underline{v}_{nw} \rangle = -\frac{k}{\mu_{nw}} \underline{\nabla} p$$

In the case of two-phase flow, the superficial velocities of each phase have the form:

$$\frac{\dot{V}_w}{\pi R^2} = \frac{-k \nabla p}{\mu_w} (1-s)^3 = \frac{-k\, k_{rw} \nabla p}{\mu_w}$$

$$\frac{\dot{V}_{nw}}{\pi R^2} = \frac{-k \nabla p}{\mu_{nw}} [s^3 + 2Ms^2(1-s)] = \frac{-k\, k_{rnw} \nabla p}{\mu_{nw}}$$

These are called relative permeabilities

$$k_{rw} = (1-s)^3$$

$$k_{rnw} = [s^3 + 2Ms^2(1-s)]$$

This flux is called superficial velocity, and for flow in porous media the equations have the form:

$$u_w = \frac{-k\, k_{rw} \nabla p}{\mu_w}$$

$$u_{nw} = \frac{-k\, k_{rnw} \nabla p}{\mu_{nw}}$$

Modified Darcy Law

$$k = \frac{\phi\, R^2}{8}$$

Permeability

$$k_{rw} = (1-s)^3$$

$$k_{rnw} = [s^3 + 2Ms^2(1-s)]$$

Relative permeability

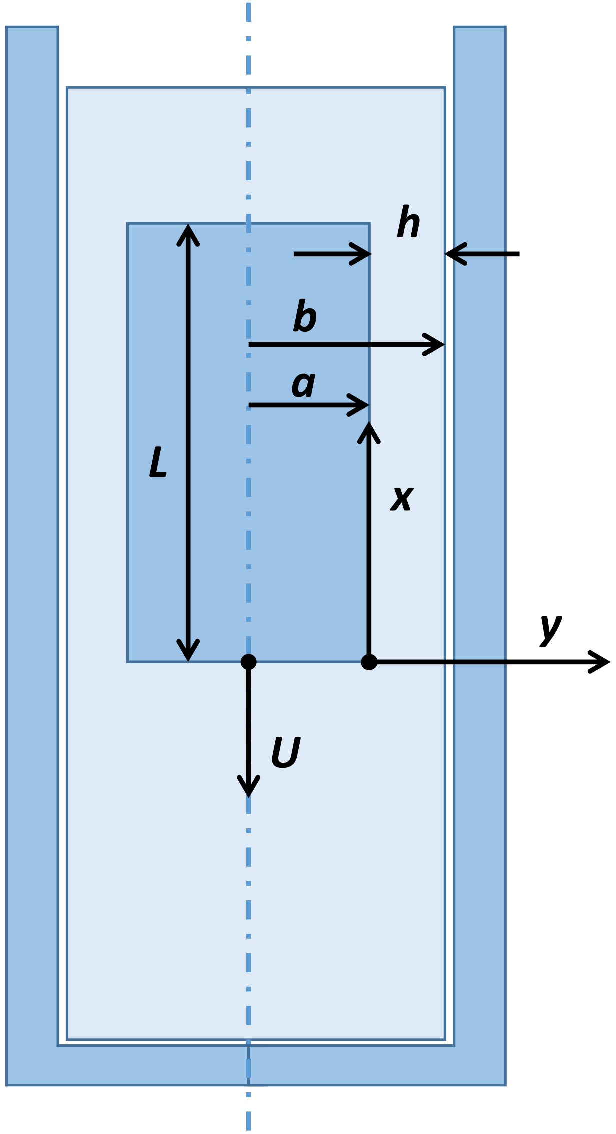
This set of equations are used to predict the two-phase flow in porous media, of course the expressions of permeability and relative permeability are for a porous media of cylindrical and homogeneous pore space with a pore radius “*R*”, and void fraction ϕ.

Superficial velocity or Darcy velocity is different from interstitial velocity:

$$v \approx \frac{u}{\phi}$$

u = Superficial or Darcy velocity
v = interstitial or local velocity
ϕ = void fraction or porosity

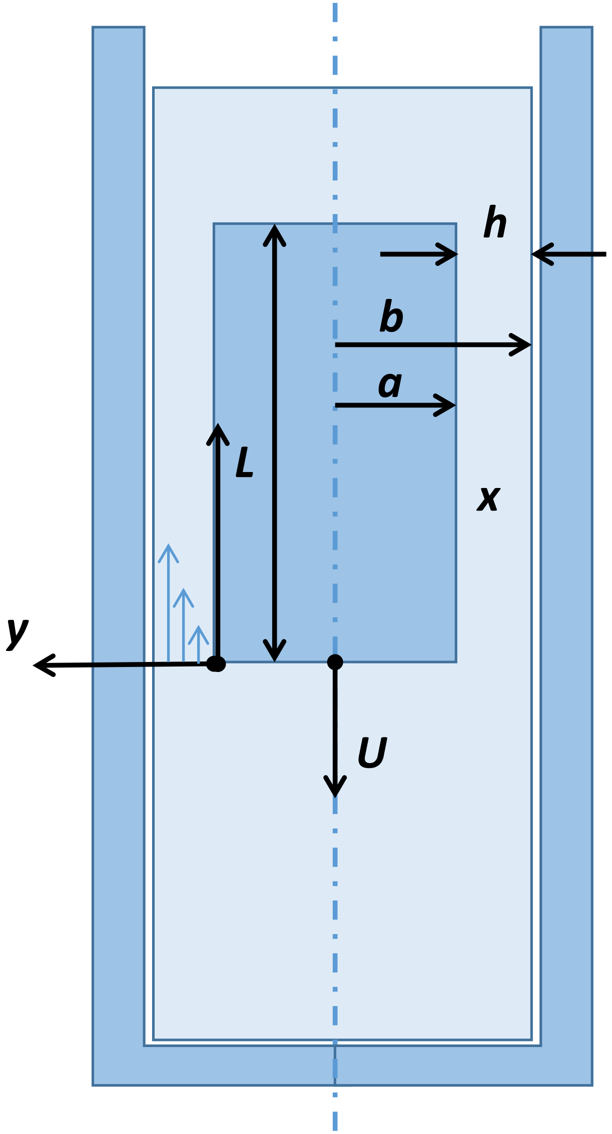
Falling-bar viscometer



Falling-bar viscometer. A method for determining the viscosity of liquid is to measure the terminal velocity of a bar settling in a container with a similar cross sectional geometry than the bar. As shown in figure, the bar half thickness measured from the center is ***a***, and the half width of the container ***b***, and the velocity relative to the container is ***U***. The bar length is ***L***, the gap between the bar and container is ***h***=***b***-***a***. Assume that ***h***/***L*** is small enough that the flow in the gap can be modeled as fully developed. Moreover, assume that the pressure variation above and below the cylinder is very nearly that in a static fluid. For simplicity, assume also that ***h***/***a***<< 1, allowing the gap to be approximated as space between parallel plates

a) Solve for $v_x(y)$ in the gap in terms of $-dp/dx+\rho g_x$. A convenient frame is one where the bar is stationary and the container wall moves upward at velocity ***U***.
b) By considering the rate at which the falling cylinder displaces liquid, evaluate the mean velocity in the gap and find $\alpha = -dp/dx+\rho g_x$
c) Use the overall force balance on the cylinder (which has a density ρ_s) to show how to calculate μ from ***U*** and the dimensions.
d) Demonstrate that when ***h*** → 0, the solution match for that of falling cylinder which has the form:

$$\frac{\mu U}{\Delta\rho\, g\, b^2} = \frac{\kappa^2}{2}\left[ln\left(\frac{1}{\kappa}\right) - \frac{(1-\kappa^2)}{(1+\kappa^2)}\right] \qquad \kappa = \frac{a}{b} \qquad \Delta\rho = \rho_s - \rho$$



$$-\rho\, g - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$a + \frac{\partial \tau_{yx}}{\partial y} = 0 \qquad -\rho\, g - \frac{\partial p}{\partial x} = a$$

$$\frac{\partial \tau_{yx}}{\partial y} = -a$$

$$\tau_{yx} = -a\, y + c_1$$

$$\mu \frac{du_x}{dy} = -a\, y + c_1$$

$$\frac{du_x}{dy} = -\frac{a}{\mu}\, y + \frac{1}{\mu} c_1$$

$$u_x = -\frac{a}{2\,\mu}\, y^2 + \frac{1}{\mu} c_1 y + c_2$$

$$u_x = -\frac{a}{2\,\mu}\, y^2 + \frac{1}{\mu} c_1 y + c_2$$

For y=0 u_x =0, then c_2 =0

For y=h u_x=u_o, then

$$u_o + \frac{a}{2\,\mu}\, h^2 = \frac{1}{\mu} c_1 h$$

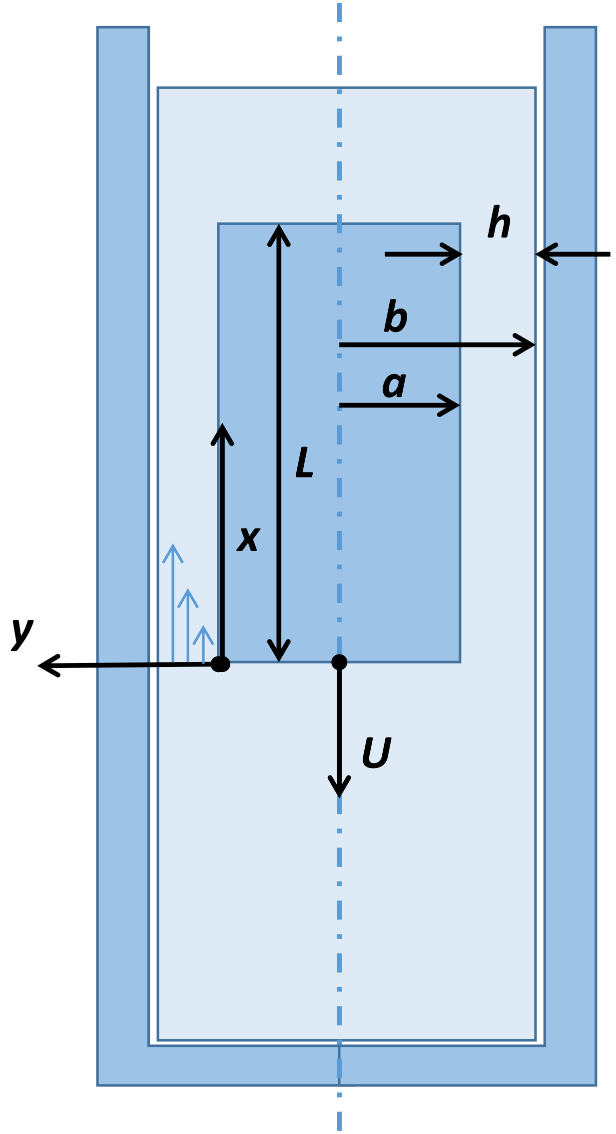
$$u_x = -\frac{ah^2}{2\mu}\left(\frac{y}{h}\right)^2 + \left(u_o + \frac{ah^2}{2\mu}\right)\frac{y}{h}$$

$$\frac{u_x}{u_0} = -\frac{ah^2}{2\mu u_o}\left(\frac{y}{h}\right)^2 + \left(1 + \frac{ah^2}{2\,\mu u_o}\right)\frac{y}{h}$$

$$\frac{u_x}{u_0} = -\alpha \left(\frac{y}{h}\right)^2 + (1+\alpha)\,\frac{y}{h} \qquad \alpha = \frac{ah^2}{2\mu u_o}$$

Using conservation of linear momentum

$$\rho\, \underline{g} + \underline{\nabla} p - \underline{\nabla} \cdot \underline{\underline{\tau}} = \frac{\partial(\rho \underline{v})}{\partial t} + \underline{\nabla} \cdot (\rho\, \underline{v}\, \underline{v})$$



For the x-axis, the equation is simplified to

$$-\rho\, g - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0$$

Pressure gradient and the forces are coupled in a variable f

$$f = -\rho\, g - \frac{\partial p}{\partial x}$$

$$f + \frac{\partial \tau_{yx}}{\partial y} = 0$$

$$\frac{\partial \tau_{yx}}{\partial y} = -f$$

Stress gradient is integrated to obtain the expression for the stress

$$\tau_{yx} = -f\, y + c_1$$

For a Newtonian fluid, stress is function of velocity gradient

$$\mu \frac{du_x}{dy} = -f\, y + c_1$$

$$\frac{du_x}{dy} = -\frac{f}{\mu}\, y + \frac{1}{\mu} c_1$$

Integrating the velocity gradient equation, gives the equation for the velocity profile within the liquid filling the gap between bar and walls of the container

$$u_x = -\frac{f}{2\,\mu}\, y^2 + \frac{1}{\mu} c_1 y + c_2$$

Integration constants are evaluated via boundary conditions. Using a frame of reference moving with the falling bar

For y=0 u_x =0, then c_2 =0
For y=h u_x=u_o, then

At the surface of the bar, velocity is zero, but at the walls of the container the velocity of the fluid is u_0 (*)

$$u_o + \frac{f}{2\,\mu}\, h^2 = \frac{1}{\mu} c_1 h$$

The final expression for velocity profile is

$$u_x = -\frac{f h^2}{2\mu} \left(\frac{y}{h}\right)^2 + \left(u_o + \frac{f h^2}{2\mu}\right) \frac{y}{h}$$

Collecting terms, to end up with a dimensionless equation, we have:

$$\frac{u_x}{u_0} = -\frac{f h^2}{2\mu u_o} \left(\frac{y}{h}\right)^2 + \left(1 + \frac{f h^2}{2\,\mu u_o}\right) \frac{y}{h}$$

The ratio between body and surface forces with viscous forces is coupled in the term α

$$\alpha = \frac{f h^2}{2\mu u_o}$$

$$\frac{u_x}{u_0} = -\alpha \left(\frac{y}{h}\right)^2 + (1+\alpha)\, \frac{y}{h}$$

$$\frac{u_x}{u_0} = -\alpha \left(\frac{y}{h}\right)^2 + (1+\alpha)\,\frac{y}{h} \qquad \zeta = \frac{y}{h}$$

The dimensionless distance between the surface of the bar and the surface of the container is called ζ

$$\frac{u_x}{u_0} = -\alpha\,\zeta^2 + (1+\alpha)\,\zeta$$

The dimensionless velocity as function of force ratio, and positon ζ

$$\tau_{yx} = \mu \frac{du_x}{dy} \qquad \tau_{yx} = \frac{u_0\,\mu}{h}\frac{d(u_x/u_0)}{d\zeta}$$

The velocity profile is derived to estimate the viscous stress within the fluid

For this system the viscous stress takes the form:

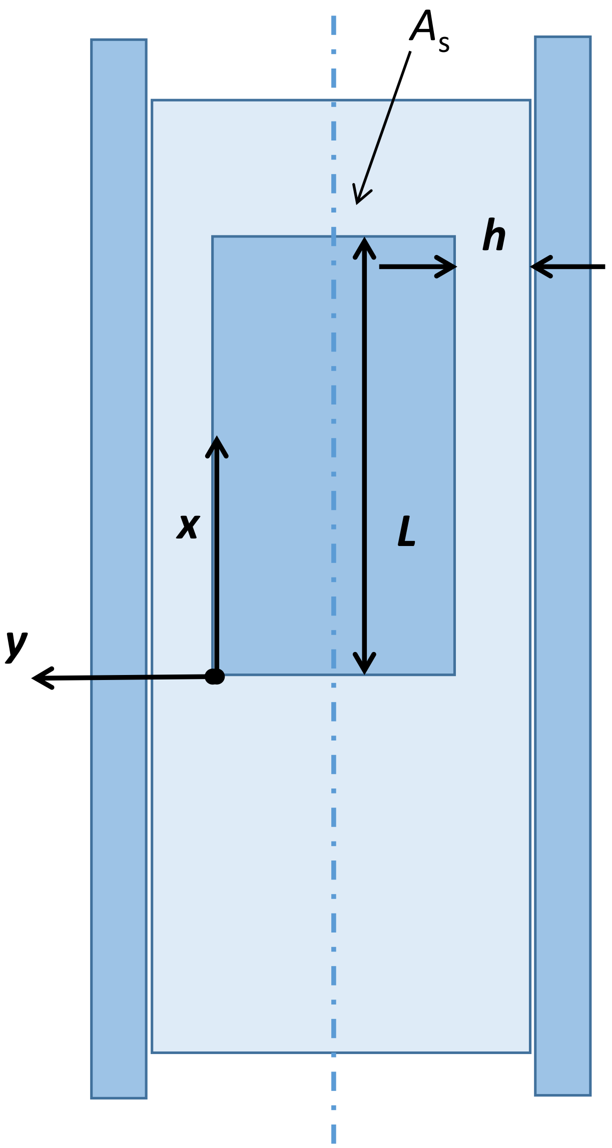
$$\tau_{yx} = \frac{u_0\,\mu}{h}[-2\,\alpha\,\zeta + (1+\alpha)]$$

The viscous stress at the surface of the bar:

$$\tau_{yx}\Big|_{y=0} = \tau_w = \frac{u_0\,\mu}{h}(1+\alpha)$$

The fluid displaced by the falling bar should coincide with fluid quizzed by the constrained side gap

$$u_0 A_s = \int_0^h w\,u_x dy \qquad u_0 A_s = u_0 h\,w \int_0^1 (u_x/u_0) d\zeta$$



$$u_x/u_0 = -\alpha\,\zeta^2 + (1+\alpha)\,\zeta$$

$$u_0 A_s = u_0 h\, w \int_0^1 (u_x/u_0)\, d\zeta$$

The fluid displaced by the falling bar should coincide with fluid quizzed by the constrained side gap

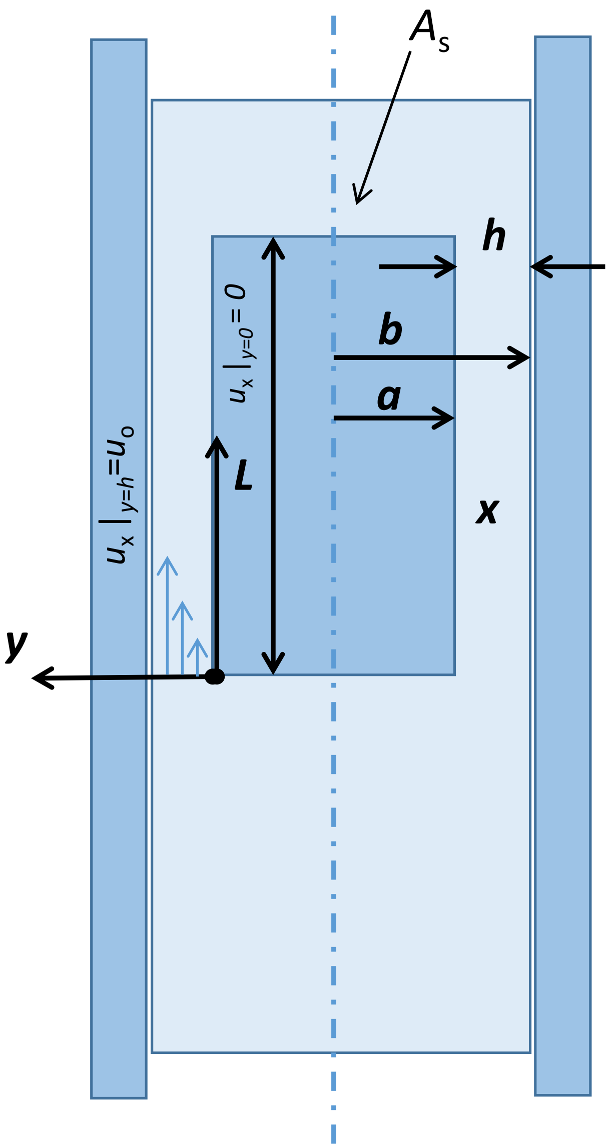
w is the perimeter or external perimeter of the bar

$$u_0 A_s = u_0 h\, w \int_0^1 \left[-\alpha\,\zeta^2 + (1+\alpha)\,\zeta\,\right] d\zeta$$

$$\frac{A_s}{h\,w} = \left[-\alpha\frac{\zeta^3}{3} + (1+\alpha)\,\frac{\zeta^2}{2}\,\right]\Bigg|_0^1$$

After integration, the area ratio has the form:

$$\frac{A_s}{h\,w} = 1/2 + \alpha/6 \qquad \alpha = \frac{f h^2}{2\mu u_o} \qquad f = -\rho\, g - \frac{\partial p}{\partial x}$$



The ratio between bar area and gap flow are related to pressure gradient, gap thickness, viscosity and terminal velocity

The next step is to relate the pressure gradient within the system with the forces acting over the bar, this is done with a force balance

A force balance can be used to relate the terminal velocity with the geometric, transport and physical properties of the materials involved

Viscous stress acting upwards, body forces of bar acting downwards, and pressure over the entire solid should be considered.

$$\tau_w\, w\, L - \rho_s L A_s g + pA_s\Big|_x - pA_s\Big|_{x+L} = 0$$

$$\tau_w\, w\ - \rho_s A_s g - A_s \frac{\Delta p}{L} = 0$$

Assuming the pressure gradient within the fluid filling the gap is uniform, allows to replace pressure difference per unit length by the pressure gradient

$$\frac{\tau_w\, w}{A_s} - \rho_s\, g - \frac{\partial p}{\partial x} = 0$$

Previously the fluid body forces and pressure surface forces were coupled in the parameter f

$$f = -\rho\, g - \frac{\partial p}{\partial x}$$

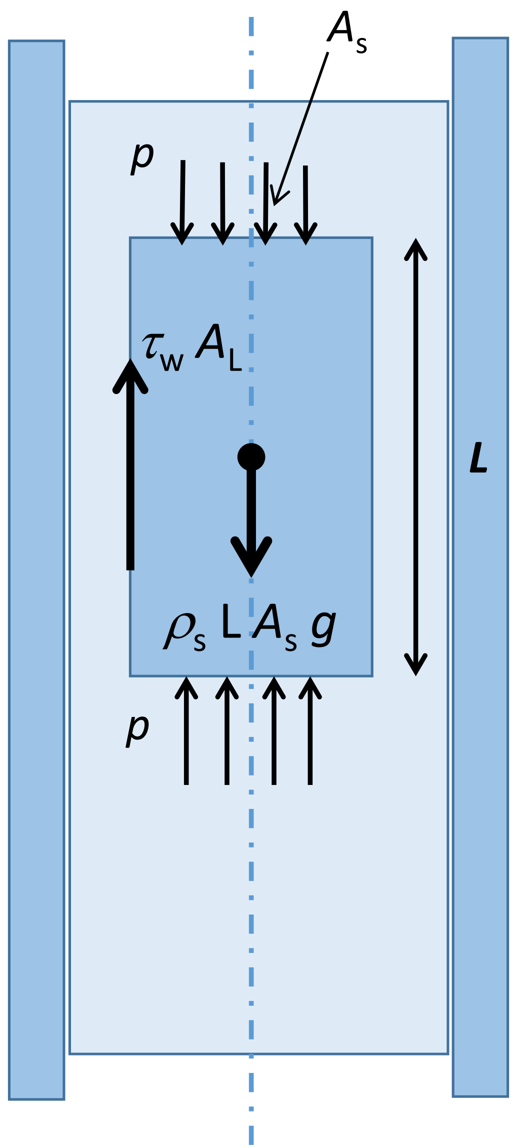
$$\frac{\tau_w\, w}{A_s} - \rho_s\, g + \ \rho\, g\ + f = 0$$

This parameter f was previously related with velocity, thickness and viscosity with a parameter called α. This a parameter was as well related with viscous stress at eh surface of the bar

$$\frac{\tau_w\, w}{A_s} + f = \Delta\rho\, g \qquad \alpha = \frac{f\, h^2}{2\mu u_o} \qquad \tau_w = \frac{u_0\, \mu}{h}(1+\alpha)$$

Replacing parameter f and stress (τ_w) by α and properties the equation takes the from:

$$\left[\frac{2\, u_0\, \mu}{h^2}\frac{(1+\alpha)}{2}\right]\frac{w\, h}{A_s} + \alpha\frac{2\, \mu\, u_0}{h^2} = \Delta\rho\, g$$



Collecting terms of the geometry, and replace it with the relation of the parameter α, gives the relationship of density difference, gravity, terminal velocity, thickness with viscosity.

$$\left[\frac{2\,u_0\,\mu}{h^2}\frac{(1+\alpha)}{2}\right]\frac{w\,h}{A_s}+\alpha\frac{2\,\mu\,u_0}{h^2}=\Delta\rho\,g$$

You can use any dimensionless parameter as a measure of geometry to your convenience

$$\alpha=6\left[\frac{A_s}{h\,w}-1/2\right]$$

$$\frac{A_s}{h\,w}=\gamma=\frac{1}{\beta}=1/2+\alpha/6$$

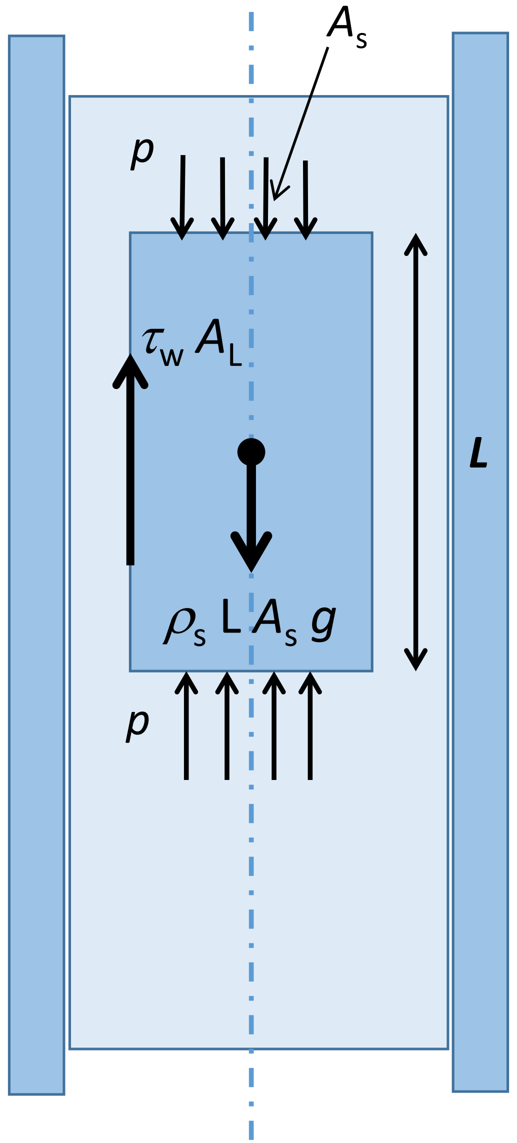
$$\beta=\frac{h\,w}{A_s}=1/\gamma=\frac{6}{3+\alpha}$$

$$\frac{2\,\mu\,u_0}{h^2}\left[\alpha+\frac{(1+\alpha)}{2\left[\frac{A_s}{h\,w}\right]}\right]=\Delta\rho\,g$$

$$\frac{2\,\mu\,u_0}{h^2}\left[\alpha+\frac{(1+\alpha)}{[1+\alpha/3]}\right]=\Delta\rho\,g$$

$$\frac{2\,\mu\,u_0}{h^2}\left[\frac{(\alpha^2+6\alpha+3)}{3+\alpha}\right]=\Delta\rho\,g$$

$$\boxed{\frac{2\,\mu\,u_0}{h^2}\left[\frac{6-\beta^2}{\beta}\right]=\Delta\rho\,g}$$



$$\frac{2\,\mu\,u_0}{h^2\Delta\rho\,g}=\left[\frac{\beta}{6-\beta^2}\right]$$

$$\beta=\frac{h\,w}{A_s}=1/\gamma=\frac{6}{3+\alpha}$$

$$\frac{2\,\mu\,u_0}{h^2\,\Delta\rho\,g}=\frac{b^2\,\kappa^2}{h^2}\left[ln\left(\frac{1}{\kappa}\right)-\frac{(1-\kappa^2)}{(1+\kappa^2)}\right]$$

$$\kappa=\frac{a}{b}\qquad h=b-a$$

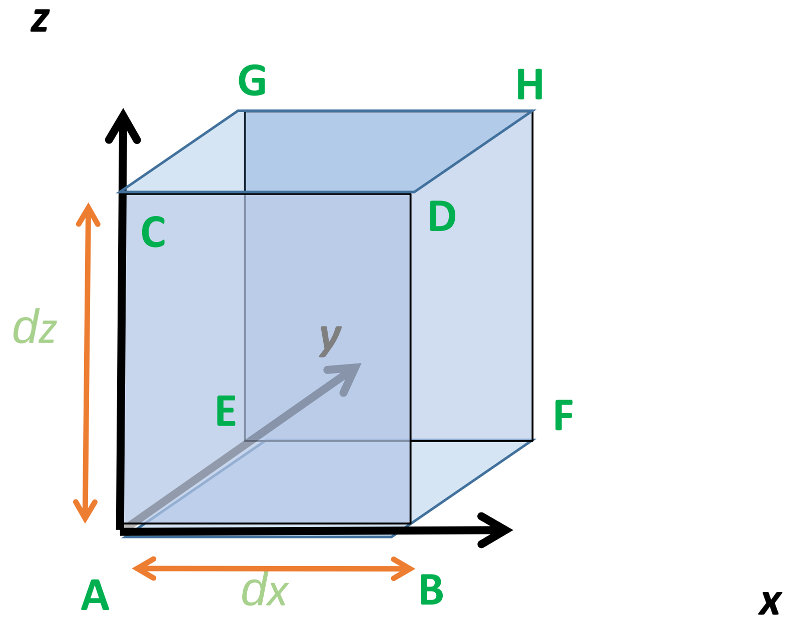
$$ln(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\frac{x^5}{5}+\cdots$$

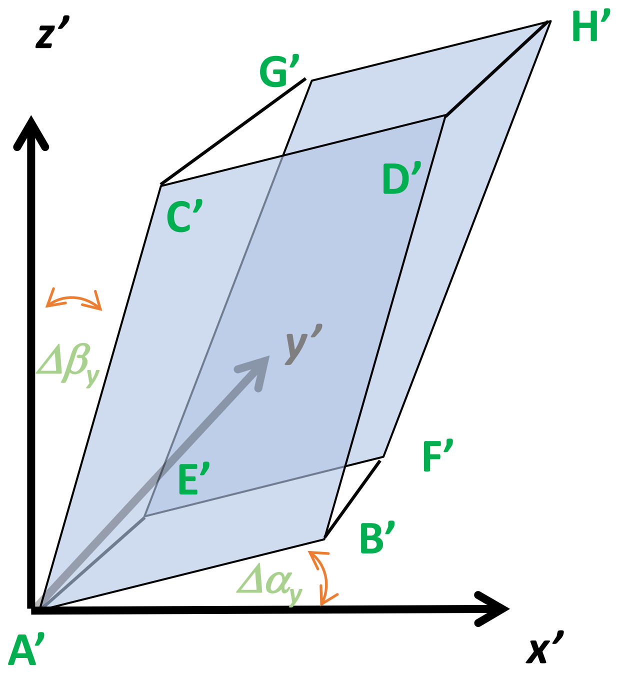
$$ln\left(\frac{1}{\kappa}\right)-\frac{(1-\kappa^2)}{(1+\kappa^2)}\approx ln\left(\frac{b}{a}\right)-\frac{(b^2-a^2)}{(b^2+a^2)}\approx -ln\left(1-\frac{h}{b}\right)-\frac{(2bh+h^2)}{2b^2}\approx (h/b)+\frac{(h/b)^2}{2}+\frac{(h/b)^3}{3}+\frac{(h/b)^4}{4}+\cdots$$

$$-ln\left(1-\frac{h}{b}\right)-\frac{(2ah+h^2)}{2a^2}\approx (h/b)+\frac{(h/b)^2}{2}+\frac{(h/b)^3}{3}+\frac{(h/b)^4}{4}+\frac{(h/b)^5}{5}+\cdots-(h/b)-\frac{(h/b)^2}{2}$$

$$-ln\left(1-\frac{h}{b}\right)-\frac{(2ah+h^2)}{2a^2}\approx\frac{(h/b)^3}{3}+\frac{(h/b)^4}{4}+\frac{(h/b)^5}{5}+$$

Quantification of: Translation, Deformation and Rotation (3D)





$$\underline{v} = \hat{\imath}\, u + \hat{\jmath}\, v + \hat{k}\, w$$ Velocity vector

$$\underline{r} = \hat{\imath}\, x + \hat{\jmath}\, y + \hat{k}\, z$$ position vector

In this cartoon points **A,B,C** and **D** are the corners of a rectangle of fluid element that is moving, after a time *t+Δt* these points have displaced and are labeled as **A',B',C'** and **D'**

Note: $\Delta\alpha$ is measured counter-clock wise form ***x***-axis, $\Delta\beta$ is measured clock wise sense respect to ***y***-axis.

The position of the points **r** and **r'** are:

	x-component	y-component	z-component
A	0	0	0
B	dx	0	0
C	0	0	dz
D	dx	0	dz
E	0	dy	0
F	dx	dy	0
G	0	dy	dz
H	dx	dy	dz

The position of the points **r** and **r'** are:

	x-component	y-component	z-component
A'	$u\,dt$	vdt	wdt
B'	$dx + [u + u_x dx]dt$	$[v + v_x dx\,]dt$	$[w + w_x dx\,]dt$
C'	$[u + u_z dz]dt$	$[v + v_z dz]dt$	$dz + [w + w_z dz]dt$
D'	$dx + [u + u_x dx + u_z dz]dt$	$[v + v_x dx + v_z dz]dt$	$dz + [w + w_x dx + w_z dz]dt$
E'	$[u + u_y dy]\Delta t$	$dy + [v + v_y dy]dt$	$[w + w_y dy]dt$
F'	$dx + [u + u_x dx + u_y dy]dt$	$dy + [v + v_x dx + v_y dy]dt$	$[w + w_x dx + w_y dy]dt$
G'	$[u + u_y dy + u_z dz]dt$	$dy + [v + v_y dy + v_z dz]dt$	$dz + [w + w_y dy + w_z dz]dt$
H'	$dx + [u + u_x dx + u_y dy + u_z dz]dt$	$dy + [v + v_x dx + v_y dy + v_z dz]dt$	$dz + [w + w_x dx + w_y dy + w_z dz]dt$

Dilatation or Extensional strain rate

$$\dot{\varepsilon}_{xx} = \frac{\left(\overline{\mathrm{A'B'}_x}\right) - \left(\overline{\mathrm{AB}_x}\right)}{\left(\overline{\mathrm{AB}_x}\right)dt}$$

$$\dot{\varepsilon}_{yy} = \frac{\left(\overline{\mathrm{A'E'}_y}\right) - \left(\overline{\mathrm{EC}_y}\right)}{\left(\overline{\mathrm{AE}_y}\right)dt}$$

$$\dot{\varepsilon}_{zz} = \frac{\left(\overline{\mathrm{A'C'}_z}\right) - \left(\overline{\mathrm{AC}_z}\right)}{\left(\overline{\mathrm{AC}_z}\right)dt}$$

$$\left(\overline{\mathrm{AB}_x}\right) = dx - 0$$

$$\left(\overline{\mathrm{A'B'}_x}\right) = dx + [u + u_x dx]dt - u\,dt$$

$$\dot{\varepsilon}_{xx} = \frac{dx + [u + u_x dx]dt - u\,dt\ - dx}{dxdt} = u_x$$

$$\dot{\varepsilon}_{xx} = u_x$$

$$\dot{\varepsilon}_{yy} = v_y$$

$$\dot{\varepsilon}_{zz} = w_z$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} u & v & w \end{bmatrix}$$

$$\underline{\underline{r}} + \left[\underline{l}\ \underline{v}^T + \underline{\underline{r}}\left(\underline{\nabla}\ \underline{v} \right) \right] \Delta t$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \Delta x & 0 & 0 \\ 0 & 0 & \Delta z \\ \Delta x & 0 & \Delta z \\ 0 & \Delta y & 0 \\ \Delta x & \Delta y & 0 \\ 0 & \Delta y & \Delta z \\ \Delta x & \Delta y & \Delta z \end{bmatrix}$$

$$\Delta t \begin{bmatrix} u & v & w \\ u & v & w \\ u & v & w \\ u & v & w \\ u & v & w \\ u & v & w \\ u & v & w \\ u & v & w \end{bmatrix}$$

$$\Delta t \begin{bmatrix} 0 & 0 & 0 \\ \Delta x & 0 & 0 \\ 0 & 0 & \Delta z \\ \Delta x & 0 & \Delta z \\ 0 & \Delta y & 0 \\ \Delta x & \Delta y & 0 \\ 0 & \Delta y & \Delta z \\ \Delta x & \Delta y & \Delta z \end{bmatrix} \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix}$$

$$\dot{\varepsilon}_{xx} = \frac{\left(\overline{\mathrm{A'B'}_x}\right) - \left(\overline{\mathrm{AB}_x}\right)}{\left(\overline{\mathrm{AB}_x}\right)\Delta t}$$

$$\dot{\varepsilon}_{yy} = \frac{\left(\overline{\mathrm{A'E'}_y}\right) - \left(\overline{\mathrm{EC}_y}\right)}{\left(\overline{\mathrm{AE}_y}\right)\Delta t}$$

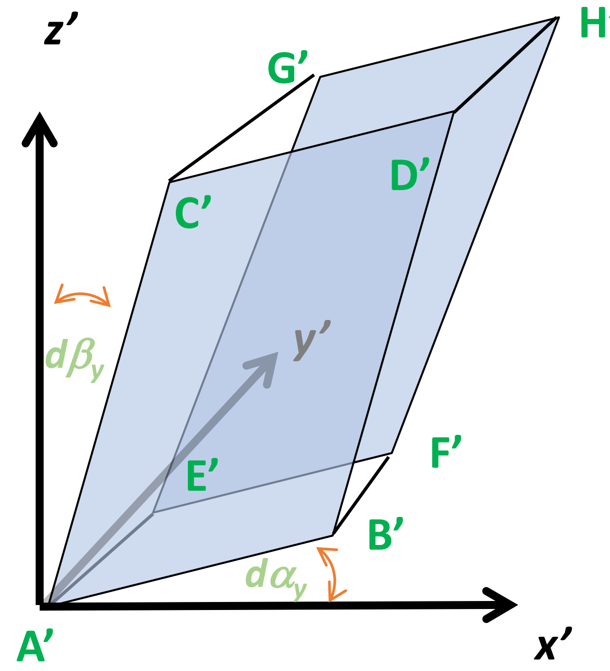
$$\dot{\varepsilon}_{zz} = \frac{\left(\overline{\mathrm{A'C'}_z}\right) - \left(\overline{\mathrm{AC}_z}\right)}{\left(\overline{\mathrm{AC}_z}\right)\Delta t}$$

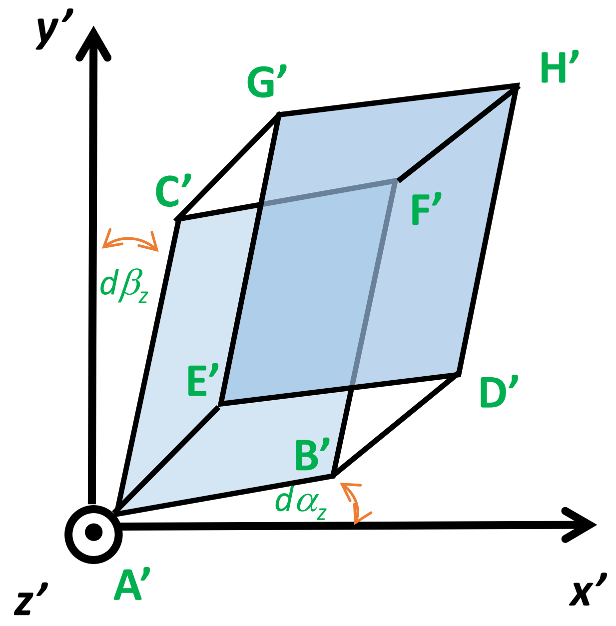
$$\dot{\varepsilon}_{xx} = u_x$$

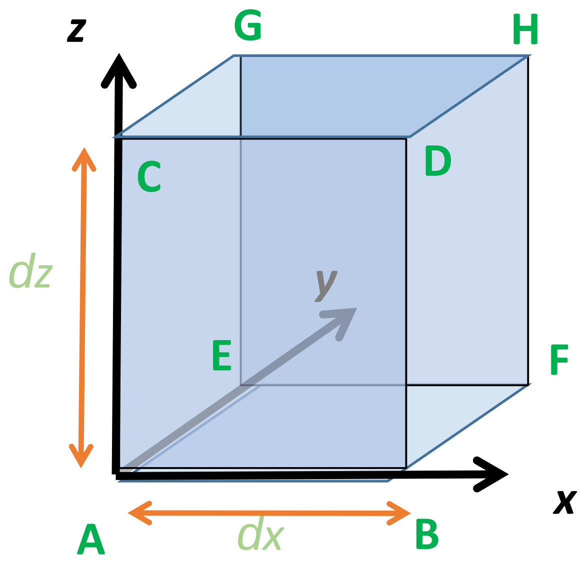
$$\dot{\varepsilon}_{yy} = v_y$$

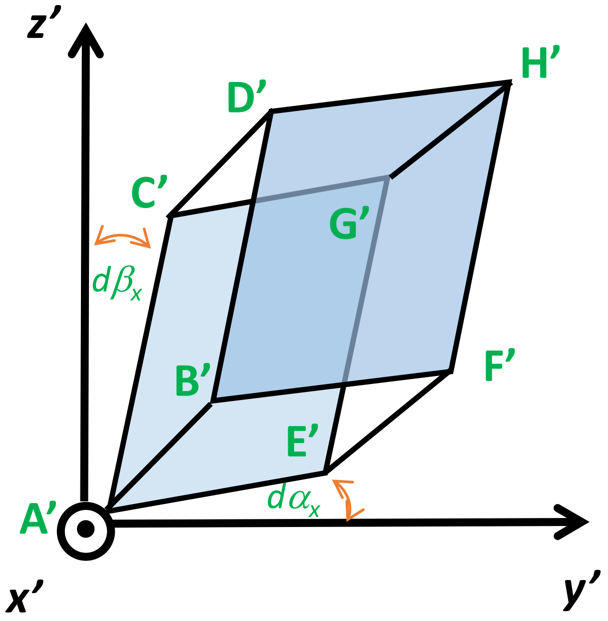
$$\dot{\varepsilon}_{zz} = w_z$$

Quantification of: Translation, Deformation and Rotation









$$d\theta_y = \frac{1}{2}\left(d\alpha_y - d\beta_y\right)$$

$$\dot{\varepsilon}_{xz} = \dot{\varepsilon}_{zx} = \frac{1}{2}\frac{\left(d\alpha_y + d\beta_y\right)}{dt}$$

$$z_{A'} = wdt$$

$$z_{B'} = [w + w_x dx\ ]dt$$

$$z_{C'} = dz + [w + w_z dz]dt$$

$$x_{A'} = u\ dt$$

$$x_{B'} = dx + [u + u_x dx]dt$$

$$x_{C'} = [u + u_z dz]dt$$

$$\tan\left(d\alpha_y\right) = \frac{z_{B'} - z_{A'}}{x_{B'} - x_{A'}} = \frac{[w + w_x dx\ ]dt - wdt}{dx + [u + u_x dx]dt - u\ dt}$$

$$\tan\left(d\alpha_y\right) = \frac{w_x dx\ dt}{dx\ [1 + u_x dt]} \approx d\alpha_y$$

$$\tan\left(d\beta_y\right) = \frac{x_{C'} - x_{A'}}{z_{C'} - z_{A'}} = \frac{[u + u_z dz]dt - u\ dt}{dz + [w + w_z dz]dt\ \ - wdt}$$

$$\tan\left(d\beta_y\right) = \frac{u_z dz\ dt}{dz[1 + w_z dt]} \approx d\beta_y$$

$$\frac{d\alpha_y}{dt} = \frac{w_x}{[1 + u_x dt]} = w_x$$

$$\frac{d\beta_y}{dt} = \frac{u_z}{[1 + w_z dt]} = u_z$$

$$\frac{d\theta_y}{dt} = \frac{1}{2}\left(\frac{d\alpha_y}{dt} - \frac{d\beta_y}{dt}\right)$$

$$\frac{d\theta_y}{dt} = \dot{\theta}_y = \frac{1}{2}(w_x - u_z)$$

$$\dot{\varepsilon}_{xz} = \dot{\varepsilon}_{zx} = \frac{1}{2}\frac{(w_x dt + u_z dt)}{dt}$$

$$\boxed{\dot{\varepsilon}_{xz} = \dot{\varepsilon}_{zx} = \frac{1}{2}(w_x + u_z) = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)}$$