Chapter 9 Review Questions & Problems

Algebraic Operations for Vectors

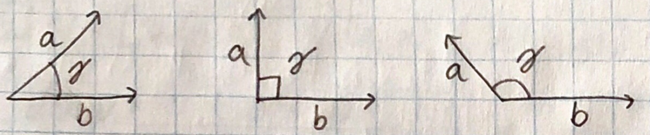
Let $a=[3,2,7]$, $b=[6,5,-4]$, $c=[1,8,0]$ and $d=[9,-2,0]$. Calculate the following expressions.

$\boxed{12}$ $a\cdot b$, $a\cdot c$ and $a\times c$

- The Inner Product (Dot Product) of two vectors a and b, is the product of their lengths times the cosine of their angle. That is:

$$a\cdot b = |a||b|\cos\gamma$$

or $a\cdot b = a_1b_1 + a_2b_2 + a_3b_3$ if $a=[a_1, a_2, a_3]$ & $b=[b_1, b_2, b_3]$



$a\cdot b>0$ $\quad$ $a\cdot b=0$ $\quad$ $a\cdot b<0$

- So let's calculate $a\cdot b$

$a=(a_1, a_2, a_3)=(3,2,7)$ & $b=(b_1, b_2, b_3)=(6,5,-4)$

$$\begin{aligned} a\cdot b &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= 3(6)+2(5)+7(-4) \\ &= 0 \end{aligned}$$

- Now, let's do $a\cdot c$

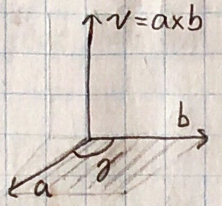
$a=[3,2,7]$ & $c=[1,8,0]$

$$\begin{aligned} a\cdot c &= 3(1)+2(8)+7(0) \\ &= 19 \end{aligned}$$

- The Vector Product (Cross Product) of two vectors a and b is the vector $v=a\times b$. If $a\times b=0$, then a and b have the same or opposite direction; in any other case, $a\times b$ has a length $|v|$

$$|v| = |a\times b| = |a||b|\sin\gamma$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = a \times b \qquad \text{If } a = [a_1, a_2, a_3] \text{ \& } b = [b_1, b_2, b_3]$$



The shaded area is $|v|$, γ is the angle between a & b, and the direction of v is perpendicular to a & b.

- A easy way to remember & compute $v = a \times b$

$$\begin{pmatrix} v_1 & v_2 & v_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -a_1 b_3 + a_3 b_1 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} = v$$

- So, let's calculate $a \times c$

$a = [3, 2, 7]$ & $c = [1, 8, 0]$

$$v = \begin{bmatrix} a_2 c_3 - a_3 c_2 \\ -a_1 c_3 + a_3 c_1 \\ a_1 c_2 - a_2 c_1 \end{bmatrix} = \begin{bmatrix} 2(0) - 7(8) \\ -(3)(0) + 7(1) \\ 3(8) - (2)(1) \end{bmatrix}$$

$$v = \begin{bmatrix} -56 \\ 7 \\ 22 \end{bmatrix}$$ //

$\boxed{14}$ $3a \cdot 4a$, $12 a \cdot a$, $12|a|^2$, $|b|^2$

- The Scalar Multiplication ca of any vector $a = [a_1, a_2, a_3]$ and any scalar c is given by:

$ca = [ca_1, ca_2, ca_3]$

- So, let's compute $3a \cdot 4a$

$a = [3, 2, 7]$

$3a = [3(3), 3(2), 3(7)]$
$= [9, 6, 21]$ & $4a = [4(3), 4(2), 4(7)]$
$= [12, 8, 28]$

$$3a \cdot 4a = 9(12) + 6(8) + 21(28)$$
$$= 744$$

- Now, Let's do $12a \cdot a$

$$a = [3, 2, 7]$$

$$12a = [12(3), 12(2), 12(7)]$$
$$= [36, 24, 84]$$

$$12a \cdot a = 36(3) + 24(2) + 84(7)$$
$$= 744$$

- The Length $|a|$ of a vector a is expressed in terms of componentes as:

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad \text{if } a = [a_1, a_2, a_3]$$

- So, let's calculate $12|a|^2$

$$a = [3, 2, 7]$$

$$|a| = \sqrt{3^2 + 2^2 + 7^2}$$
$$= \sqrt{64}$$

$$12|a|^2 = 12(\sqrt{64})^2$$
$$= 744$$

- Now, let's do $|b|^2$

$$b = [6, 5, -4]$$

$$|b| = \sqrt{6^2 + 5^2 + (-4)^2}$$
$$= \sqrt{77}$$

$$|b|^2 = (\sqrt{77})^2$$
$$= 77$$

$\boxed{16}$ $a \cdot (b \times c)$, $(a \times b) \cdot c$, $(a\ b\ c)$

~~(a = [3, 2, 7], b = [6, 5, -4] & c = [1, 8, 0])~~

- The Scalar Triple Product (Mixed Triple Product) of three vectors a, b, c is denoted by $(a\ b\ c)$ and is defined by:

$$(a\ b\ c) = a \cdot (b \times c)$$

If $a = [a_1, a_2, a_3]$, $b = [b_1, b_2, b_3]$ & $c = [c_1, c_2, c_3]$

$$a \cdot (b \times c) = a_1 \begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix} + a_2 \begin{bmatrix} b_3 & b_1 \\ c_3 & c_1 \end{bmatrix} + a_3 \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

Note that the sum above is the expansion of a 3rd order determinant by its 1st row.

$$a \cdot (b \times c) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

- So, Let's compute $a \cdot (b \times c)$

$a = [3, 2, 7]$, $b = [6, 5, -4]$ & $c = [1, 8, 0]$

$$(a\ b\ c) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= 3(5(0) - (-4)(8)) - 2(6(0) - (-4)(1)) + 7(6(8) - 5(1))$$

$$= 96 - 8 + 301$$

$$= 389 //$$

- Now, Let's do $(a \times b) \cdot c$

$$v = a \times b \Rightarrow \begin{pmatrix} v_1 & v_2 & v_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \Rightarrow \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -a_1b_3 + a_3b_1 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

$$v = \begin{pmatrix} 2(-4) - 7(5) \\ -(3)(-4) + 7(6) \\ 3(5) - 2(6) \end{pmatrix} = \begin{pmatrix} -43 \\ 54 \\ 3 \end{pmatrix}$$

$$(a \times b) \cdot c = \begin{pmatrix} -43 \\ 54 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix}$$

$$= (-43(1)) + 54(8) + 3(0)$$

$$= 389 //$$

- Since $(a\ b\ c) = a \cdot (b \times c)$, $(a\ b\ c) = 389 //$

• $(a\ b\ c)$ is the usual notation for the scalar triple product, and the other two expressions ($a\cdot(b\times c)$ & $(a\times b)\cdot c$) justify that "$(a\ b\ c)$" does not indicate where the dot and the cross is being used because it does not matter.

$\boxed{22}$ Find the angle between the planes $4x+3y-z=2$ and $x+y+z=1$.

• Let's take the functions' coefficients to define the vectos a & b.

$$4x+3y-z=2 \Rightarrow a=[4,3,-1]$$
$$x+y+z=1 \Rightarrow b=[1,1,1]$$

From the Dot Product definition, let's do:

$$a\cdot b = |a||b|\cos\gamma$$

$$a\cdot b = 4(1)+3(1)+(-1)(1) = 6 \quad \& \quad |a||b| = \sqrt{4^2+3^2+(-1)^2}\sqrt{1^2+1^2+1^2} = \sqrt{78}$$

$$6 = \sqrt{78}\cos\gamma$$
$$\gamma = \arccos\left(\frac{6}{\sqrt{78}}\right)$$
$$= 47.2^\circ //$$

$\boxed{24}$ Find u such that a, b, c, d, and u are in equilibrium.

• By definition, forces are in equilibrium if their resultant is the zero vector.

$$a=[3,2,7],\ b=[6,5,-4],\ c=[1,8,0],\ d=[9,-2,0]$$

$$u = -(a+b+c+d)$$
$$= -\begin{pmatrix} 3+6+1+9 \\ 2+5+8+(-2) \\ 7+(-4)+0+0 \end{pmatrix}$$
$$= \begin{pmatrix} -19 \\ -13 \\ -3 \end{pmatrix} //$$

• Note that $u+(a+b+c+d) =$ zero vector.

$\boxed{26}$ Find the work done by $q = [5, 1, 0]$ in the displacement from $(4, 4, 0)$ to $(6, -1, 0)$.

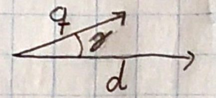
- The work done by q in the displacement is defined as

  $w = q \cdot d$, where w is the work, q is a constant force & d is the displacement.

  $$d = (6, -1, 0) - (4, 4, 0) = (2, -5, 0)$$

  $$q \cdot d = 5(2) + 1(-5) + 0(0) = \underline{5} \;//$$

- Given $W = q \cdot d = |q||d| \cos \gamma$, we can sketch the work done by q as:



$\boxed{30}$ Find the moment vector m of $p = [4, 2, 0]$ about $P: (5, 1, 0)$ if p acts on a line through $(1, 4, 0)$.

- The moment m of a force p about a point P is defined as the product:

  $m = |p| d$, where d is the perpendicular distance between P and the line of action. If r is the vector for P to any point to the line of action, then

  $d = |r| \sin \gamma \Rightarrow m = |r||p| \sin \gamma$, and since γ is the angle between r & p, then:

  $$m = r \times p$$

- So:

  r = the distance between P and the line of action
  $$= (1, 4, 0) - (5, 1, 0) = (-4, 3, 0)$$

  $$m = (-4, 3, 0) \times (4, 2, 0)$$
  $$= \begin{vmatrix} m_1 & m_2 & m_3 \\ -4 & 3 & 0 \\ 4 & 2 & 0 \end{vmatrix}$$
  $$= [3(0) - 0(2), -((-4)(0) - 0(4)), (-4)(2) - 3(4)]$$
  $$= [0, 0, -20]$$

32 Find the velocity, speed and acceleration of the motion given by:
$r(t) = [5\cos t, \sin t, 2t]$ at point $P = [5/\sqrt{2}, 1/\sqrt{2}, \pi/2]$.

- The tangent vector is called the velocity vector v, as it points to the instantaneous direction of motion and its length gives the speed $|v| = |r'|$
- The second vervative of $r(t)$ is the acceleration vector and it's denoted by a.
- So, let's compute the velocity of $r(t)$

$$v = r'(t)$$
$$= \underline{[-5\sin t, \cos t, 2]}_{//}$$

Now find a t, so that $r(t) = P$

$$5\cos t = \frac{5}{\sqrt{2}} \quad ; \sin t = \sin\left(\frac{1}{4}\pi\right) \quad ; 2t = 2\left(\frac{1}{4}\pi\right.$$
$$t = \arccos\left(\frac{1}{\sqrt{2}}\right) \qquad = \frac{1}{\sqrt{2}} \qquad = \frac{\pi}{2}$$
$$t = \frac{1}{4}\pi$$

- At $t = \pi/4$, $r(t) = P$ therefore

$$v(P) = \left[-5\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{\pi}{4}\right), 2\right]$$
$$= \underline{\left[-\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2\right]}_{//}$$

- Let's compute the speed at P.

$$\text{speed} = |v(P)|$$
$$= \sqrt{\left(-\frac{5}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + 2^2}$$
$$= \underline{\sqrt{17}}_{//}$$

- Let's calculate the acceleration at P.

$$a(t) = v'(t)$$
$$= [-5\cos t, -\sin t, 0]$$

$$a(P) = \left[-5\cos\left(\frac{\pi}{4}\right), -\sin\left(\frac{\pi}{4}\right), 0\right]$$
$$= \left[-\frac{5}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right]$$ //

34 Find an equation of the plane through (1, 0, 2), (2, 3, 5), (3, 5, 7)

- So, the general equation for any plane is given by:

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$ where: (x_0, y_0, z_0) is a point on plane. and (a, b, c) is a vector perpendicular to the plane.

- Let's give names the the points... and compute (a, b, c)

$A = (1, 0, 2)$
$B = (2, 3, 5)$
$C = (3, 5, 7)$

- To create a perpendicular vector, we compute the cross product of to vector, but we don't have vectors, but points. So let's create some vectors.

$\vec{AB} = (2-1, 3-0, 5-2)$
$= (1, 3, 3)$

$\vec{AC} = (3-1, 5-0, 7-2)$
$= (2, 5, 5)$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} a & b & c \\ 1 & 3 & 3 \\ 2 & 5 & 5 \end{vmatrix}$$
$$= [3(5) - 3(5), -(1(5) - 3(2)), 1(5) - 3(2)]$$
$$= [0, 1, -1]$$

so; $a = 0$, $b = 1$ & $c = -1$ and substituing (x_0, y_0, z_0) with A, we have:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$
$$0(x-1) + 1(y-0) + (-1)(z-2) = 0$$
$$y - z + 2 = 0$$

$$z = y + 2$$ //

Grad, Div, Curl, ∇^2, $D_v f$

Let $f = zy + yx$, $v = [y, z, 4z - x]$, $w = [y^2, z^2, x^2]$

36 Find grad f & f grad f at (3, 4, 0)

- The gradient of a given function $f(x, y, z)$ is denoted by grad f or ∇f (nabla) and is defined by:

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$$

- So, let's compute ∇f at $(3, 4, 0)$

$f = zy + yx$

$\frac{\partial f}{\partial x} = y \; , \; \frac{\partial f}{\partial y} = z + x \; , \; \frac{\partial f}{\partial z} = y$

$$\begin{aligned}\nabla f &= [y, x+z, y] \\ &= [4, 3+0, 4] \\ &= [4, 3, 4] \;\#\end{aligned}$$

- Let's calculate $\cancel{f}\, f\nabla f$ at the same point $(3, 4, 0)$

$$\begin{aligned}f(3,4,0) &= 0(4) + 4(3) \\ &= 12\end{aligned}$$

$$\begin{aligned}f\nabla f &= 12[4, 3, 4] \\ &= [48, 36, 48] \;\#\end{aligned}$$

$\boxed{38}$ Find div v, div w

- $\text{div}\, v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$ is called the divergence of v or the divergence of the vector field defined by v.

i) if the divergence is greater than 0, then the vector field represents a source (outflow)
ii) if the divergence is less than 0, then the vector field represents a sink (inflow)
iii) if the divergence is equal to 0, then no source or sink is present within the field.

- Let's calculate div v

$v = [y, z, 4z - x]$

$\frac{\partial v_1}{\partial x} = 0 \; , \; \frac{\partial v_2}{\partial y} = 0 \; , \; \frac{\partial v_3}{\partial z} = 4$

$$\begin{aligned}\text{div}\, v &= 0 + 0 + 4 \\ &= 4 \;\#\end{aligned}$$

- Now let's do div w

$w = [y^2, z^2, x^2]$

$\frac{\partial w_1}{\partial x} = 0, \quad \frac{\partial w_2}{\partial y} = 0, \quad \frac{\partial w_3}{\partial z} = 0$

$\text{div}\, w = 0 + 0 + 0$
$= 0$ //

$\boxed{39}$ Find curl v, curl w

- The curl of the vector function v or of the vector field given by v is defined by the symbolic determinant.

$$\text{curl}\, v = \nabla \times v = \begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{pmatrix}$$
$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k$$

- Let's compute curl v

$v = [y, z, 4z - x]$

$$\text{curl}\, v = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}, \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$
$$= (0 - 1, \; 0 - (-1), \; 0 - 1)$$
$$= (-1, 1, -1) \; //$$

- Now, let's do curl w

$w = [y^2, z^2, x^2]$

$$\text{curl}\, w = \left(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}, \frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}, \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y}\right)$$
$$= (0 - 2z, \; 0 - 2x, \; 0 - 2y)$$
$$= [-2z, -2x, -2y] \; //$$