VISCOELASTICITY

Rheology and Fluid Mechanics

FLUID MECHANIC

Equations of motion for the system + Rheological properties of the fluid $\longrightarrow$ Predicted pressure drop vs. flow rate for the fluid in the system

RHEOLOGIST

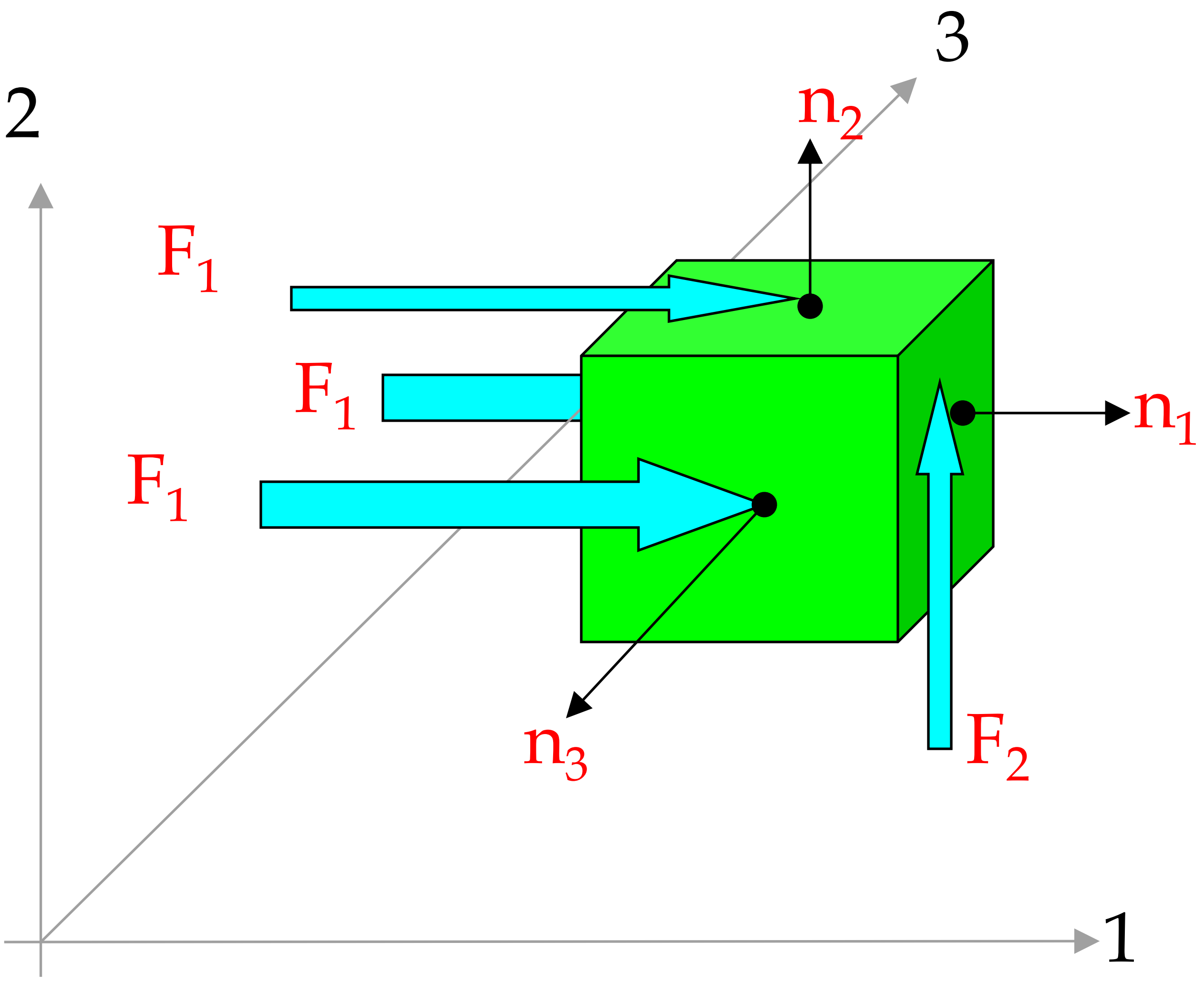
Equations of motion for the system + Measured pressure drop vs. flow rate for the fluid in the system $\longrightarrow$ Rheological properties of the fluid

Comment

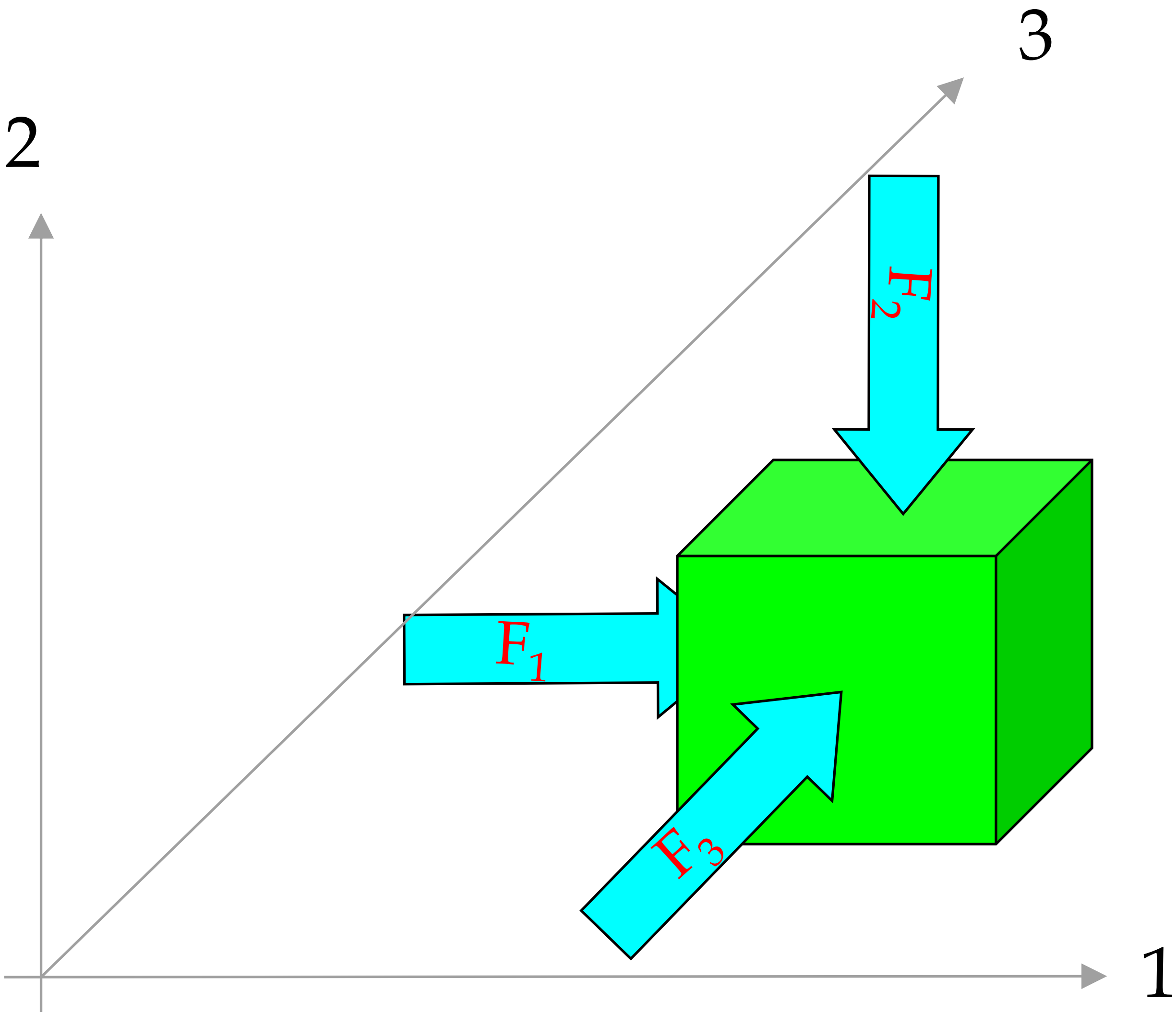
Rehological phenomena do not deal with motion *per se,* but with *the deformation* of a fluid element (volume) and *the surface forces* acting on such element.

The deformation on the fluid element can occur in all directions at once and the surface forces can be acting in one or more surfaces of the fluid element.

Tangential surface forces

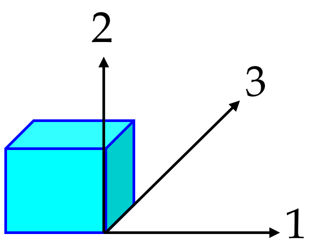


Perpendicular surface forces



Stress Tensor

$$\begin{bmatrix} \frac{F_1}{A_1} & \frac{F_2}{A_1} & \frac{F_3}{A_1} \\ \frac{F_1}{A_2} & \frac{F_2}{A_2} & \frac{F_3}{A_2} \\ \frac{F_1}{A_3} & \frac{F_2}{A_3} & \frac{F_3}{A_3} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



$$\sigma_{12}$$

Direction of the normal surface vector

Direction of the force

Stress Tensor

- Features:
  - The stress tensor is symmetric ($\sigma_{ij} = \sigma_{ji}$)
  - Therefore, there are six independent components

- Then,
  - how could the stress tensor, with its six independent components, be represented?
  - can this be represented by a combination of other tensors?

Zero order tensors: SCALARS

- A scalar is a quantity which can be described by a single component.
- Therefore, it is independent of the specific coordinate system (CS) used as a reference frame for describing the system.
- It can be specified in terms of its magnitude
- The magnitude does not depend on the coordinate system
- Examples: temperature, pressure, density

First order tensors: VECTORS

- It can be specified in terms of its components
  - v_1, v_2 and v_3
    (in the direction x_1, x_2 and x_3)
- Its components in one coordinate system (CS) can be easily transformed to another CS or they can be rotated in any given CS with no problem (there are rules to do so)
- It can be specified in terms of its magnitude and direction
- The magnitude does not depend on the coordinate system

First order tensors: VECTORS

- Examples: force, velocity, gradient of a scalar

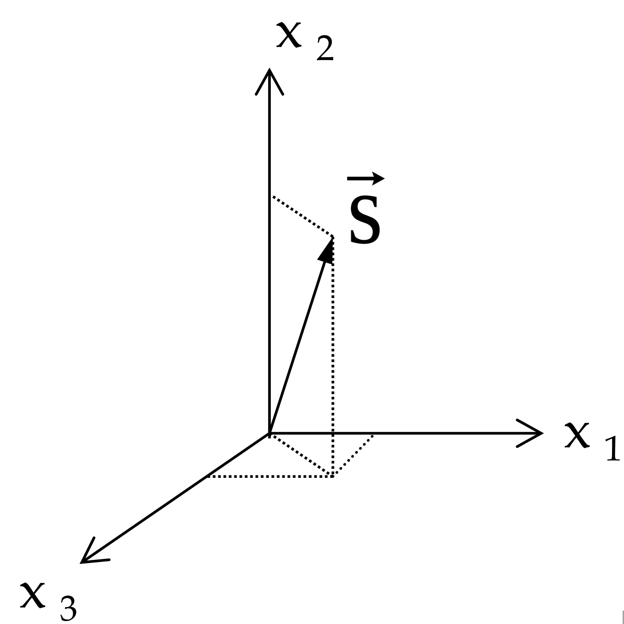
Velocity

$$\vec{v} = v_1\vec{i}_1 + v_2\vec{i}_2 + v_3\vec{i}_3$$

Gradient of Temperature

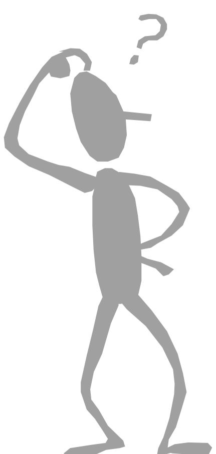
$$\vec{\nabla} T = \frac{dT}{d\vec{s}} =$$

$$\vec{i}_1 \frac{\partial T}{\partial x_1} + \vec{i}_2 \frac{\partial T}{\partial x_2} + \vec{i}_3 \frac{\partial T}{\partial x_3}$$



Second order tensor: Dyads or, simply, Tensor

- "Getting the feeling for it":
- If we consider the gradient of a vector quantity, we can notice that there are three components of the vector and each of them may change in any or all of the three possible directions. Therefore, there are 9 possible components of the gradient of a vector. Each component is associated with a magnitude and two directions (the direction of the component of the vector and the direction in which it varies.



$$\frac{d\vec{v}}{d\vec{S}} = \nabla\vec{v} = \frac{\partial v_i}{\partial x_j} =$$

Gradient of a vector = Tensor

$$\frac{d\vec{v}}{d\vec{S}} = \frac{\partial v_i}{\partial x_j} = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}$$

TENSOR

- It can be specified in terms of its components
- Its components in one coordinate system (CS) can be easily transformed to another CS or they can be rotated in any given CS with no problem (there are rules to do so).
- It can not be specified in terms of its magnitude and direction.
- Examples: strain tensor and stress tensor.

Symmetric and Anti-symmetric tensors

- A symmetric tensor is defined as:

$$S_{ij}=S_{ji}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} = S_{ij}$$

- An anty-simmetric is defined as:

$$A_{ij} = -A_{ji} \quad \text{for i} \neq \text{j and}$$
$$A_{ij} = 0 \quad \text{for i} = \text{j}$$

$$\begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix} = A_{ij}$$

Symmetric and Anti-symmetric tensors

It is said that any tensor Tij can be represented as:

$$T_{ij} = S_{ij} + A_{\mathrm{ij}}$$

where:

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$$

and:

$$A_{ij} = \frac{1}{2}(T_{ij} - T_{ji})$$

then:

$$\mathrm{T_{ij}} = \frac{1}{2}\left[\mathrm{T_{ij}} + \mathrm{T_{ji}}\right] + \frac{1}{2}\left[\mathrm{T_{ij}} - \mathrm{T_{ji}}\right]$$

Symmetric and Anti-symmetric tensors

Let us prove that: $T_{ij} = \frac{1}{2}\left[T_{ij} + T_{ji}\right] + \frac{1}{2}\left[T_{ij} - T_{ji}\right]$

$$T_{ij} + T_{ji} = \begin{bmatrix} T_{11} + T_{11} & T_{12} + T_{21} & T_{13} + T_{13} \\ T_{21} + T_{12} & T_{22} + T_{22} & T_{23} + T_{32} \\ T_{31} + T_{13} & T_{32} + T_{23} & T_{33} + T_{33} \end{bmatrix}$$

$$T_{ij} - T_{ji} = \begin{bmatrix} T_{11} - T_{11} & T_{12} - T_{21} & T_{13} - T_{31} \\ T_{21} - T_{12} & T_{22} - T_{22} & T_{23} - T_{32} \\ T_{31} - T_{13} & T_{32} - T_{23} & T_{33} - T_{33} \end{bmatrix}$$

$$\frac{1}{2}(T_{ij} + T_{ji}) = \frac{1}{2}\begin{bmatrix} 2T_{11} & T_{12} + T_{21} & T_{13} + T_{13} \\ T_{21} + T_{12} & 2T_{22} & T_{23} + T_{32} \\ T_{31} + T_{13} & T_{32} + T_{23} & 2T_{33} \end{bmatrix}$$

$$\frac{1}{2}(T_{ij} - T_{ji}) = \frac{1}{2}\begin{bmatrix} 0 & T_{12} - T_{21} & T_{13} - T_{31} \\ T_{21} - T_{12} & 0 & T_{23} - T_{32} \\ T_{31} - T_{13} & T_{32} - T_{23} & 0 \end{bmatrix}$$

$\text{Observe that}: r1c2 = r2c1$

$\text{that is}: (T_{12} + T_{21}) = (T_{21} + T_{12})$

$$\frac{1}{2}(T_{ij} + T_{ji}) = S_{ij}$$

$\text{Observe that}: r1c2 = r2c1$

$\text{that is}: (T_{12} - T_{21}) = -(T_{21} - T_{12})$

$$\frac{1}{2}(T_{ij} - T_{ji}) = A_{ij}$$

INVARIANTS

If T_{ij} represents any symmetric second-order tensor, an orthogonal coordinate system (defined by the three unit vectors n_i) can be found, such that the only non-zero components of T_{ij} in this system are the three normal (diagonal) components. These coordinates are referred as the principle axes.

The condition for this statement to be true is that the vector formed by the inner product (dot product) of T_{ij} and n_j, must be parallel to n_j.
That is:

$$T^i{}_j \bullet n^j = \lambda n^i$$

Where λ is a scalar constant, or scale factor. The previous equation can rearranged as:

$$\left(\mathbf{T}^{\mathbf{i}}{}_{\mathbf{j}} - \lambda \delta^{\mathbf{i}}{}_{\mathbf{j}}\right)\mathbf{n}^{\mathbf{j}} = \mathbf{0}$$

Invariants

The non trivial solution to this equation is:

$$\left|\mathbf{T}^{\mathbf{i}}{}_{\mathbf{j}} - \boldsymbol{\lambda}\boldsymbol{\delta}^{\mathbf{i}}{}_{\mathbf{j}}\right| = \mathbf{0}$$

$$\begin{vmatrix} \left(T^1{}_1 - \lambda\right) & T^1{}_2 & T^1{}_3 \\ T^2{}_1 & \left(T^2{}_2 - \lambda\right) & T^2{}_3 \\ T^3{}_1 & T^3{}_2 & \left(T^3{}_3 - \lambda\right) \end{vmatrix} = 0$$

$$\boldsymbol{\lambda}^3 - \mathbf{I}\boldsymbol{\lambda}^2 + \mathbf{II}\boldsymbol{\lambda} - \mathbf{III} = \mathbf{0}$$

An easier way:

$$\begin{array}{ccccc} & & \quad - & \quad - & \quad - \end{array}$$

$$\left|\begin{array}{ccc} (T_1^1-\lambda) & T_2^1 & T_3^1 \\ T_1^2 & (T_2^2-\lambda) & T_3^2 \\ T_1^3 & T_2^3 & (T_3^3-\lambda)\end{array}\right|\left|\begin{array}{cc} (T_1^1-\lambda) & T_2^1 \\ T_1^2 & (T_2^2-\lambda) \\ T_1^3 & T_2^3 \end{array}\right|$$

$$\begin{array}{ccccc} & & \quad + & \quad + & \quad + \end{array}$$

$$=(T_1^1-\lambda)(T_2^2-\lambda)(T_3^3-\lambda)+T_2^1T_3^2T_1^3+T_3^1T_1^2T_2^3$$

$$=-T_1^3(T_2^2-\lambda)T_3^1-T_2^3(T_1^1-\lambda)T_3^2-(T_3^3-\lambda)T_1^2T_2^1$$

$$= -\lambda^3 + (T_1^1 + T_2^2 + T_3^3)\lambda^2$$

$$- (T_2^2 T_3^3 - T_2^3 T_3^2 + T_1^1 T_3^3 - T_1^3 T_3^1 + T_1^1 T_2^2 - T_1^2 T_2^1)\lambda$$

$$- (+T_1^3 T_2^2 T_3^1 + T_2^3 T_3^2 T_1^1 + T_3^3 T_1^2 T_2^1$$

$$- T_1^1 T_2^2 T_3^3 - T_2^1 T_3^2 T_1^3 - T_3^1 T_2^1 T_2^3) = 0$$

$$= \lambda^3 - \boxed{(T_1^1 + T_2^2 + T_3^3)}\lambda^2$$

$$+ \boxed{(T_2^2 T_3^3 - T_2^3 T_3^2 + T_1^1 T_3^3 - T_1^3 T_3^1 + T_1^1 T_2^2 - T_1^2 T_2^1)}\lambda$$

$$+ \boxed{\begin{aligned} &(+T_1^3 T_2^2 T_3^1 + T_2^3 T_3^2 T_1^1 + T_3^3 T_1^2 T_2^1 \\ &\quad - T_1^1 T_2^2 T_3^3 - T_2^1 T_3^2 T_1^3 - T_3^1 T_2^1 T_2^3) \end{aligned}} = 0$$

$$\boxed{= \lambda^3 - I\lambda^2 + II\lambda + III = 0}$$

Isotropic and Anisotropic Stress and Strain

- The total stress and strain tensors (indeed, any symmetric second-order tensor) can be expressed as the sum of an isotropic tensor and an anisotropic tensor.
- An isotropic stress, such as static pressure (which is independent of direction), acting on an isotropic material element will result in, at most, a change in volume but no change in shape of the element. This volume change is an isotropic strain, which is also known as volumetric strain.
- Since isotropic stress and strain are independent of direction, they are scalar quantities.
- Conversely, when a purely anisotropic stress acts on an isotropic material element, the result is a change in shape but not in volume. This is referred to as a shear strain.
- The anisotropic shear stress on the other hand , may have components which act normal as well as tangential to various surfaces of the element. Thus a shear strain may result from anisotropic normal and/or shear(tangential) stresses.
- The total tensor may be resolved into a isotropic strain (volumetric) $ev=3/2e_{av}$ and a shear strain (anisotropic) $\gamma_{ij.}$
- This shear strain is also called the deviatoric stress, since it represents the deviation from isotropy.

“Summarizing”

TENSORS

- The tensor has nine components
- These nine components contain all the information necessary to **transform one vector into another one** that has certain prescribed relationship with the first
- In the mathematical language it is say that:
  - the tensor operates on one vector to yield a second vector,
  - this second vector contains information from both the original vector and the tensor.

TENSORS

- Examples:
  - the nine components of the strain tensor, can be used to **operate on** the components of the vector describing the relative position of fluid particles within an undeformed fluid element **to yield** the corresponding position vectors after deformation.
  - the nine components of the stress tensor, can be used to **operate on** the components of the unit normal vector (which defines the orientation of a surface of a fluid element) **to yield** the surface force vector acting on that element.

TENSORS

- Therefore:
  - the strain tensor contains a complete description of the deformation that a fluid element undergoes during **some** flow process
  - the stress tensor contains a complete description of the state of the stress acting at a point in the fluid element at a particular time.

Stress Tensor

- The quantitaive specification of the forces acting on a solid body due to a contact with another body is straightforward, it requires to give the components of the force vector acting at the interface.

- But in a fluid element, one can
  - specify the stress vectors acting on each of the three mutually perpendicular planes going through a point in a fluid in order to be able to transform such specification to any other choice of planes by simple tranformation rules.
  - it is convenient to let these planes to be perpendicular to the coordinate axes, which allows the unit normal vector for a surface to become equal to one of the unit normal vectors for the coordinate system.

Some typical tensors

- **Stress tensor for simple shear**

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

- **Stress tensor for simple extension**

$$\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

- **Strain tensor for simple shear**

$$\begin{bmatrix} 0 & \dot{\gamma}_{12} & 0 \\ \dot{\gamma}_{21} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- **Strain tensor for simple extension**

$$\begin{bmatrix} 2\varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{bmatrix}$$

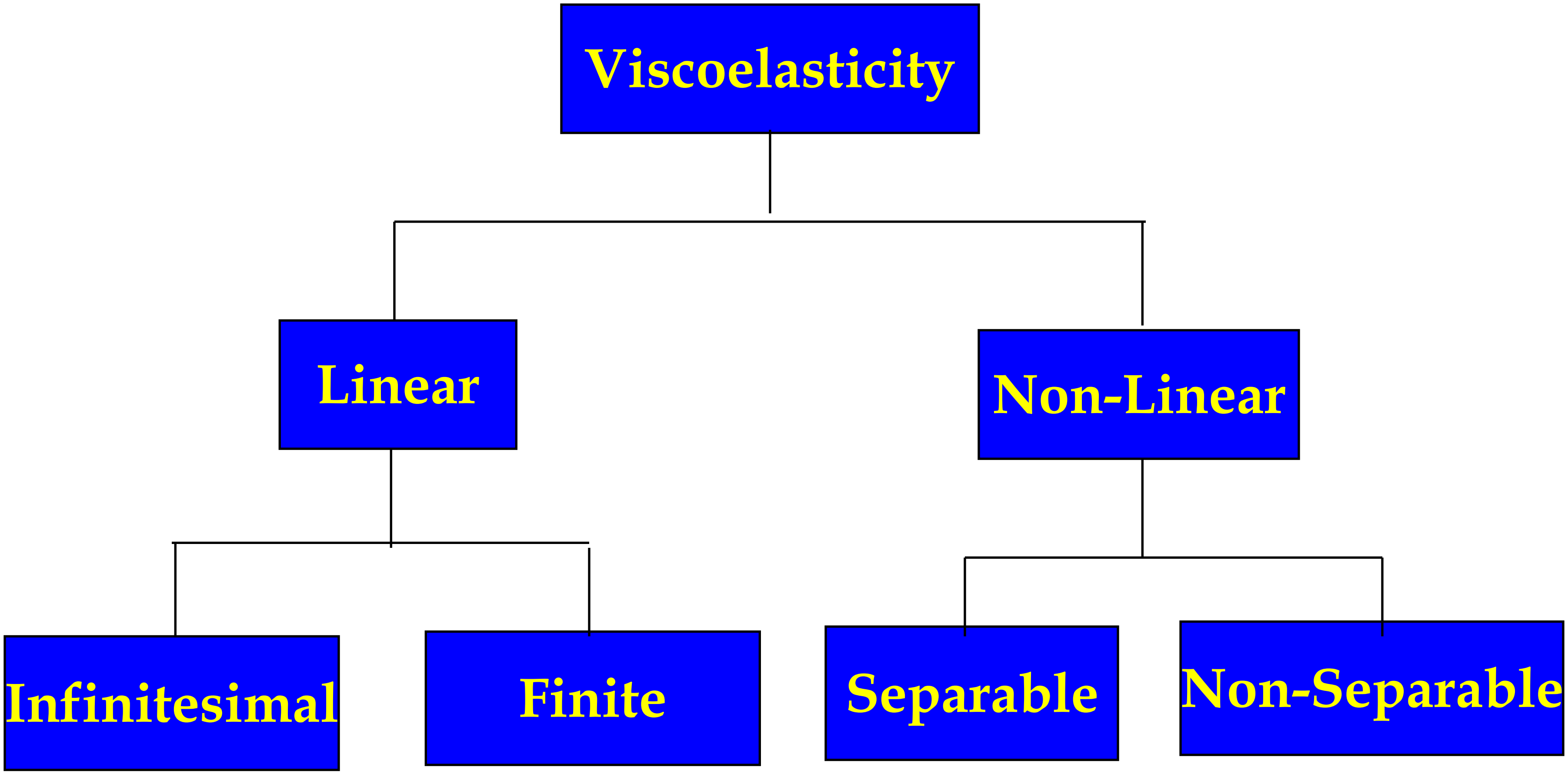
$$\sigma_E = \sigma_{11} - \sigma_{22} = \sigma_{11} - \sigma_{33}$$

Comments

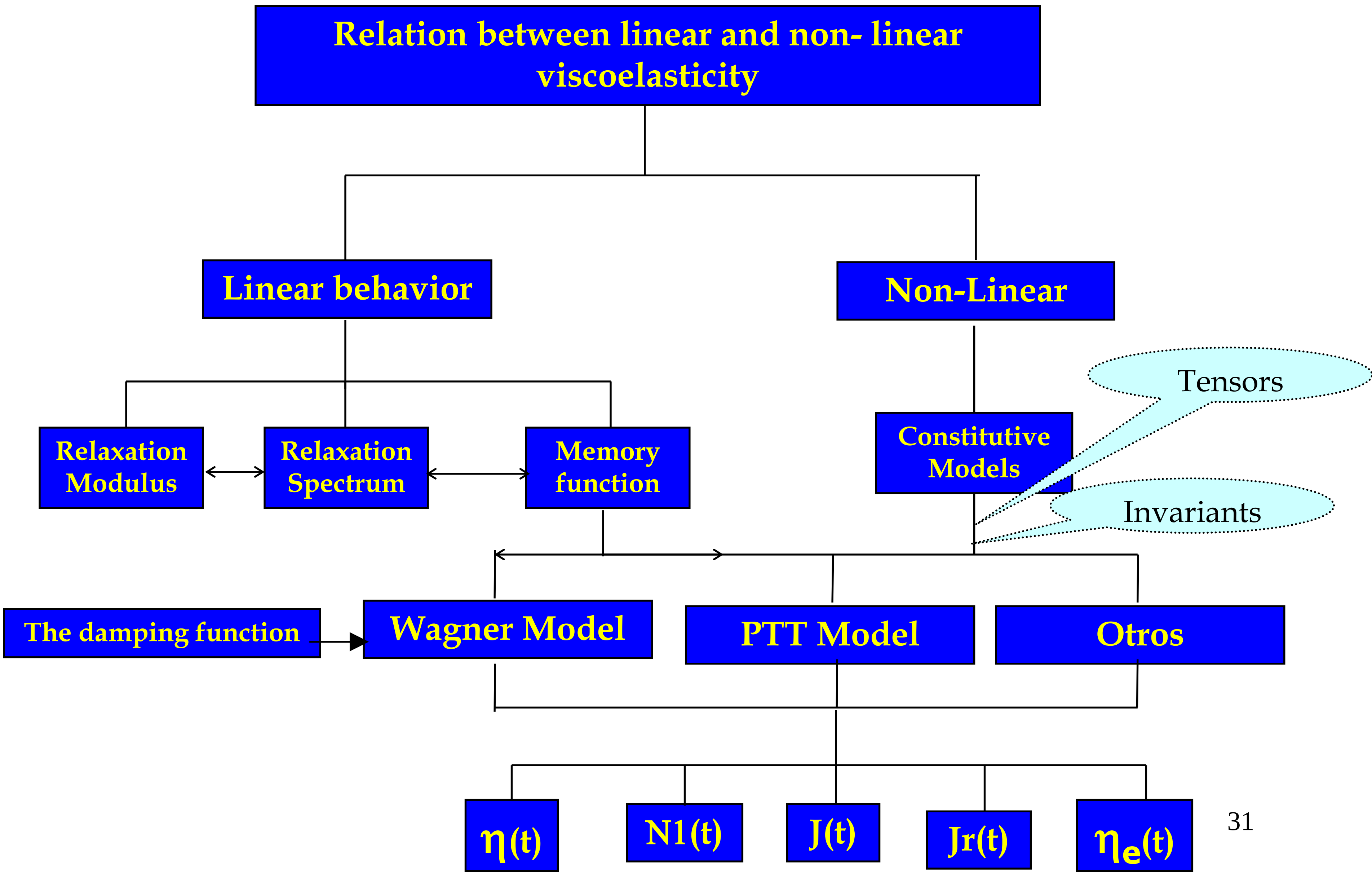
- The **stress** tensor provides an unambiguous description of the forces which may act at all points in a material. However, that is not the case for the **strain** tensor because it was defined in terms of displacement gradients with respect to a fixed reference frame and just for infinitesimal deformation. The strain tensor defined in this fashion is suitable only for purely viscous fluids with no memory.

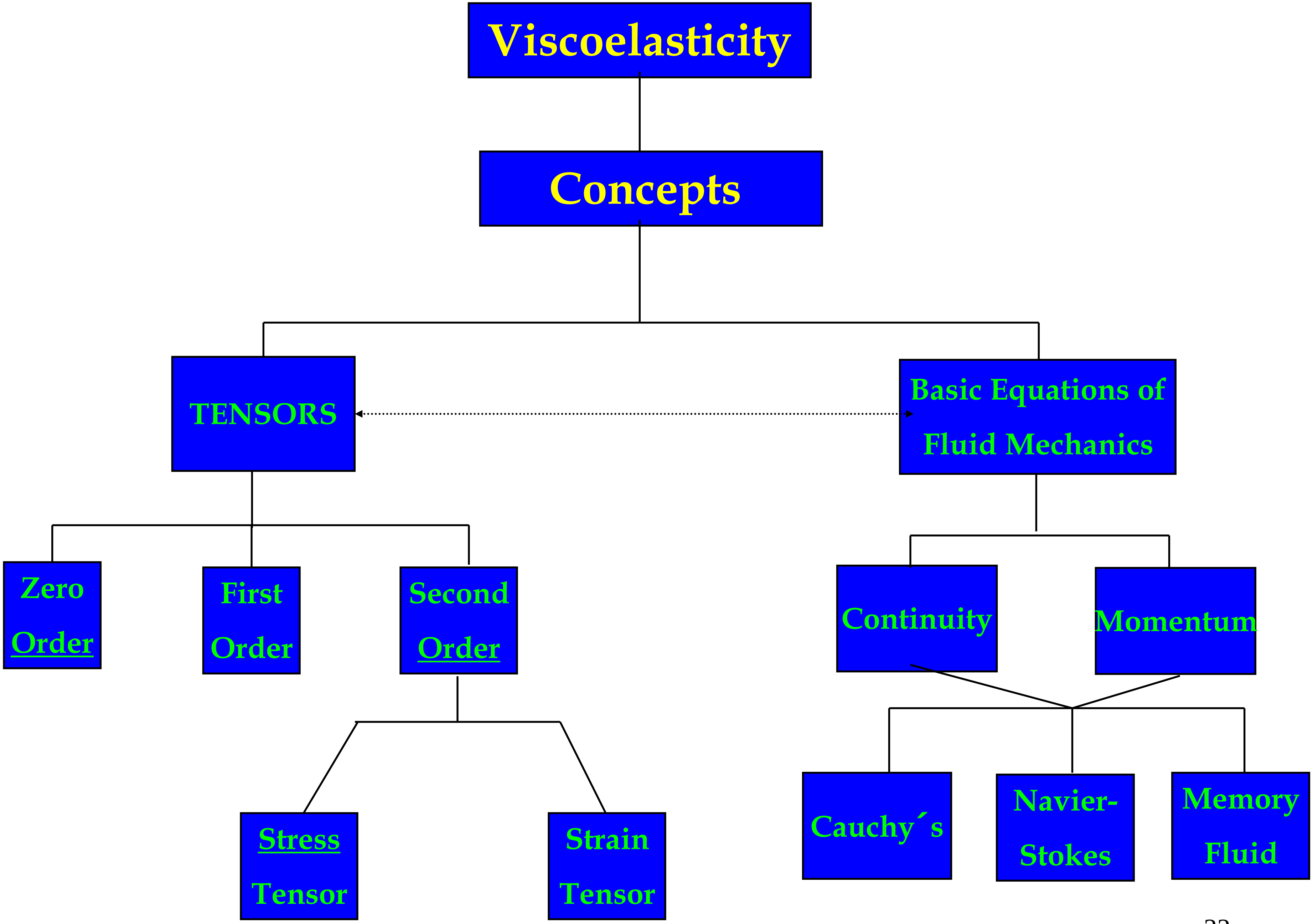
- For materials that present viscous and elastic properties, the instantaneous state of stress depends not only upon the instantaneous strain or rate of strain, but also upon the entire past history of deformation. Such memory property requires that the deformation be defined in a manner such that is always valid for a given material element regardless of the displacement history of the element with time. If all points of the material undergo only small (rather than infinitesimal) displacements, during this period, then the classical strain tensor is not enough for describing the real displacement and some other strain tensors have to be used.

Conceptual Map



Conceptual Map




Viscoelasticity
Concepts
TENSORS
Basic Equations of Fluid Mechanics
Zero Order
First Order
Second Order
Stress Tensor
Strain Tensor
Continuity
Momentum
Cauchy´s
Navier-Stokes
Memory Fluid