Lecture 1

1 Functions

To understand the word function, we consider the following scenario and definitions. For example, the growth of a sidling is an instance of a functional relation, since the growth may be affected by variations in temperature, moisture, sunlight, etc. If all these factors remain constant, then the *growth is a function of time.*

Definition 1.1 (Variables). *A variable is an object, event, time period, or any other type of category you are trying to measure.*

Consider the formula used for calculating the volume of a sphere of radius r.

$$V = \frac{4}{3}\pi r^3 \tag{1}$$

Then,

i) V and r vary with different spheres. Hence, they are called variables.

ii) π and $\frac{4}{3}$ are constants, irrespective of the size of the sphere.

There are two types of variables, i.e., independent and dependent variables.

Definition 1.2 (Independent and dependent variables). *Independent variable refers to the input value while dependent variable refers to the output value.*

For example from formula (1), the volume, V, depends on the value of the radius, r, of the sphere. In this case, r is called the independent variable while V is called the dependent variable since it is affected by the variation of r. Similarly, for the function $y = ax^2 + bx + c$, a, b and c are constants, x is the independent variable and y is the dependent variable.

Definition 1.3 (Function). *A function is a rule that assigns/associates each element in the independent set, say X, to a unique element in the dependent set, say Y.*

Examples of functions are

i) Linear functions e.g., $y = x + 5$

ii) Quadratic functions e.g., $y = x^2 - 2x + 5$

iii) Cubic functions e.g., $y = x^3 - 1$

iv) Quartic functions e.g., $y = 2x^4 + x^3 - 1$

v) Trigonometric functions e.g., $y = \sin(2x + 5)$

vi) Logarithmic functions (log to base 10) e.g., $y = \log(3x + 1)$

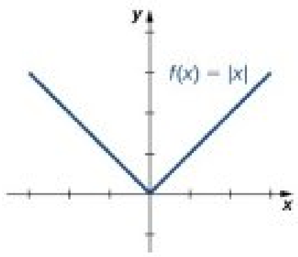
vii) Natural logarithmic functions (log to base $e \approx 2.71828$) e.g., $y = \ln(5x + 1)$

viii) Inverse of trigonometric functions e.g., $y = \tan^{-1}(2x + 1)$

ix) Exponential functions e.g., $y = e^{2x+1}$

x) Absolute value functions e.g., $y = |x|$. This function is defined as

$$y = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$



→ Note: in the above examples the variable y depends on the variable x. Thus, we say that the dependent variable y is a function of the independent variable x. Using function notation, we write $y = f(x)$, where f is a function. The function $f(x)$ is read as f of x, meaning that f depends on x.

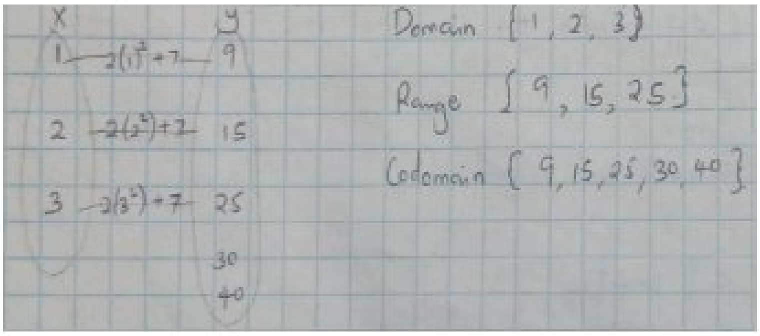
1.1 Domain, Range and Codomain

Definition 1.4 (Domain). *A domain consists of all the elements in the independent set (i.e., the set of inputs), X, for which the function is defined.*

Definition 1.5 (Range). *A range refers to a set of all the images of the elements in the domain.*

Definition 1.6 (Codomain). *A codomain consists of all the elements in the dependent set (i.e., the set of outputs), Y.*

For example, consider the diagram below



Example(s):

1. Find the domain and range of the following functions.

   (a) $f(x) = (x-4)^2 + 5$

   Solution

   ☐ Since $f(x)$ is defined (or is a real number) for any real number x, the domain of f is the interval $(-\infty, \infty)$.

   ☐ Let $y = (x-4)^2 + 5$. Making x the subject, we have $x = 4 \pm \sqrt{y-5}$. This function is defined if $y - 5 \geq 0$ or $y \geq 5$. Therefore, the range is the interval $[5, \infty)$.

   (b) $f(x) = 2x^2 - 5x + 1$

   Solution

   ☐ Since $f(x)$ is defined (or is a real number) for any real number x, the domain of f is the interval $(-\infty, \infty)$.

   ☐ Let $y = 2x^2 - 5x + 1$ or $2x^2 - 5x + (1-y) = 0$. Making x the subject (use quadratic formula), we have $x = \dfrac{5 \pm \sqrt{25 - 8(1-y)}}{4}$. This function is defined if $25 - 8(1-y) \geq 0$ or $y \geq -\dfrac{17}{8}$. Therefore, the range is the interval $\left[-\frac{17}{8}, \infty\right)$.

   (c) $f(x) = \dfrac{4}{x^2 - 5x + 6}$

   Solution

   → Note: $4/0 = \infty$ (infinity), vvvv large value, undefined, indeterminate.

   ☐ The function $f(x)$ is defined when the denominator is nonzero, i.e., if $x^2 - 5x + 6 \neq 0$. Solving yields $x \neq 2$ and $x \neq 3$. Therefore, the domain of f includes all the real numbers of x except $x = 2$ and $x = 3$, i.e., the set $(-\infty, \infty) \setminus \{2, 3\}$ or $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

   ☐ Let $y = \dfrac{4}{x^2 - 5x + 6}$ or $x^2 - 5x + \left(6 - \frac{4}{y}\right) = 0$. Making x the subject (use quadratic formula), we have

   $$x = \frac{5 \pm \sqrt{25 - 4\left(6 - \frac{4}{y}\right)}}{2}$$

   This function is defined if $25 - 4\left(6 - \frac{4}{y}\right) \geq 0$ or $y \geq -16$. Therefore, the range is the interval $[-16, \infty)$.

(d) $f(x) = \sqrt{x-1}$

Solution

- ☐ Since $f(x)$ is defined (or is a real number) if $x - 1 \geq 0$ or $x \geq 1$, the domain of f is the interval $[1, \infty)$.
- ☐ Let $y = \sqrt{x-1}$. Making x the subject, we have $x = y^2 + 1$. This function is defined for any real number y. Therefore, the range is the interval $(-\infty, \infty)$.

(e) $f(x) = 2|x-3| + 4$

Solution

- ☐ Since $f(x)$ is defined for all real numbers, the domain of f is the interval $(-\infty, \infty)$.
- ☐ Since for all $|x-3| \geq 0$, the function $f(x) = 2|x-3| + 4 \geq 4$. Therefore, the range is all the values of y for which $y \geq 4$ or the interval $[4, \infty)$.

Exercise:

1. Find the domain and range of the following functions.

   (a) $f(x) = 6 - x^2$. [ans: domain $(-\infty, \infty)$, range $(-\infty, 6]$]

   (b) $f(x) = \dfrac{6+3x}{1-2x}$. [ans: domain $(-\infty, 0.5) \cup (0.5, \infty)$, range $(-\infty, 1.5) \cup (1.5, \infty)$]

   (c) $f(x) = \dfrac{x+5}{x-2}$. [ans: domain $(-\infty, 2) \cup (2, \infty)$, range $(-\infty, 1) \cup (1, \infty)$]

   (d) $f(x) = \sqrt{4-2x} + 5$. [ans: domain $(-\infty, 2]$, range $(-\infty, \infty)$]

   (e) $f(x) = \sqrt{\dfrac{x^2-16}{x^2-2x-24}}$.[ans: domain $(-\infty, -4) \cup [4, 6) \cup (6, \infty)$, range $\left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right] \backslash \{-1, 1\}$]

1.2 Evaluation of functions

This involves replacing x in the function by the suggested value and retaining the rule of the function.

Example(s):

1. Given $f(x) = 2x + 1$. Find: (i) $f(0)$, (ii) $f(1)$, (iii) $f(x+2)$, and (iv) $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$.

   Solution

   i) $f(0) = 2(0) + 1 = 0 + 1 = 1$

   ii) $f(1) = 2(1) + 1 = 2 + 1 = 3$

   iii) $f(x+2) = 2(x+2) + 1 = 2x + 4 + 1 = 2x + 5$

   iv) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{[2(x+h)+1] - [2x+1]}{h} = \dfrac{2x + 2h + 1 - 2x - 1}{h} = \dfrac{2h}{h} = 2.$

2. Given $f(x) = 3x^2 - 2x + 4$. Find: (i) $f(0)$, (ii) $f(-1)$, (iii) $f(x+2)$, and (iv) $\dfrac{f(x+h) - f(x)}{h}$ for $h \neq 0$.

   Solution

   i) $f(0) = 3(0)^2 - 2(0) + 4 = 0 + 0 + 4 = 4$

   ii) $f(-1) = 3(-1)^2 - 2(-1) + 4 = 3 + 2 + 4 = 9$

   iii) $f(x+2) = 3(x+2)^2 - 2(x+2) + 4 = 3(x^2 + 4x + 4) - 2x - 4 + 4 = 3x^2 + 10x + 12$

iv)

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[3(x+h)^2-2(x+h)+4]-[3x^2-2x+4]}{h} \\ &= \frac{(3x^2+6hx+3h^2-2x-2h+4)-(3x^2-2x+4)}{h} = \frac{6hx+3h^2-2h}{h} \\ &= 6x+3h-2\end{aligned}$$

3. Given $f(x)=x^2-4x+3$. Find: (i) $f(1)$, (ii) $f(2)$, (iii) $f(a)$, and (iv) $f(a+h)$.

Solution

i) $f(x)=x^2-4x+3 \quad\Rightarrow\quad f(1)=1^2-4(1)+3=0$

ii) $f(x)=x^2-4x+3 \quad\Rightarrow\quad f(2)=2^2-4(2)+3=-1$

iii) $f(x)=x^2-4x+3 \quad\Rightarrow\quad f(a)=a^2-4a+3$

iv) $f(x)=x^2-4x+3 \quad\Rightarrow\quad f(a+h)=(a+h)^2-4(a+h)+3$

4. Given $\phi(\theta)=2\sin\theta$. Find: (i) $\phi(\frac{\pi}{2})$, (ii) $\phi(0)$, and (iii) $\phi(\frac{\pi}{3})$.

Solution

i) $\phi(\theta)=2\sin\theta \quad\Rightarrow\quad \phi(\frac{\pi}{2})=2\sin\left(\frac{\pi}{2}\right)=2$

ii) $\phi(\theta)=2\sin\theta \quad\Rightarrow\quad \phi(0)=2\sin(0)=0$

iii) $\phi(\theta)=2\sin\theta \quad\Rightarrow\quad \phi(\frac{\pi}{3})=2\sin\left(\frac{\pi}{3}\right)=2\times\dfrac{\sqrt{3}}{2}=\sqrt{3}$

Exercise:

(a) Given $f(x)=x^3+2x+1$, find: (i) $f(0)$, (ii) $f(-a)$, (iii) $f(x+2)$, and (iv) $\dfrac{f(x+h)-f(x)}{h}$ for $h\neq 0$.

(b) Given $g(x)=\dfrac{1}{\sqrt{x}+1}$, find: (i) $f(0)$, (ii) $f(1)$, (iii) $f(x+2)$, and (iv) $\dfrac{g(x+h)-g(x)}{h}$ for $h\neq 0$.

(c) Given $p(x)=\dfrac{6-2x}{1+3x}$, find: (i) $f(0)$, (ii) $f(-1)$, (iii) $f(2-x)$, and (iv) $\dfrac{p(x+h)-p(x)}{h}$ for $h\neq 0$.

(d) If $f(x)=2x^2-4x+1$, find (i) $f(1)$, (ii) $f(0)$, (iii) $f(2)$, (iv) $f(a)$, and $f(x+h)$.

(e) If $f(x)=(x-1)(x+5)$, find (i) $f(1)$, (ii) $f(0)$, (iii) $f(2)$, (iv) $f(a+1)$, and $f(\frac{1}{a})$.

(f) If $f(\theta)=\cos\theta$, find (i) $f(\frac{\pi}{2})$, (ii) $f(0)$, (iii) $f(\frac{\pi}{3})$, (iv) $f(\frac{\pi}{6})$, and (v) $f(\pi)$.

(g) If $f(x)=x^2$, find (i) $f(3)$, (ii) $f(3.1)$, (iii) $f(3.01)$, (iv) $f(3.001)$, and $\dfrac{f(3.001)-f(3)}{0.001}$.

(h) If $\phi(x)=2^x$, find (i) $\phi(0)$, (ii) $\phi(1)$, and (iii) $\phi(0.5)$.

1.3 Composite functions

The composition of functions is a function of another function. Consider the function f with domain A and range B, and the function g with domain D and range E. If B is a subset of D, then the composite function $(gof)(x)$ is the function with domain A and range E such that

$$(gof)(x)=g(f(x))$$

For example, given $f(x)=2x+1$ and $g(x)=5x-3$. Then,

$$(gof)(x)=g(f(x))=g(2x+1)=5(2x+1)-3=10x+2$$

Similarly,

$$(fog)(x) = f(g(x)) = f(5x-3) = 2(5x-3)+1 = 10x-5$$

$\rightarrow$ Note: $(fog)(x) \neq (gof)(x)$.

Exercise:

1. Given $f(x) = x^2 - 1,\ \ g(x) = x - 1$ and $h(x) = \sqrt{x}$. Find:
   (a) $(fog)(x)$
   (b) $(hog)(x)$
   (c) $(gog)(x)$
   (d) $(gohof)(x)$
2. Consider the functions $f(x) = x^2 + 1$ and $g(x) = 1/x$. Evaluate
   (a) $(fog)(4)$
   (b) $(gof)(-1/2)$
3. If $f(x) = \sqrt{x}$ and $g(x) = 4x + 2$, find the domain of $(fog)(x)$. [ans: $x \geq -0.5$ or $(-\infty, -0.5]$]