FUNCTIONS

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Functions

1.1 Functions

Definition 1.1.1 (A function)**.** Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. If f is a function from A to B, wee write $f : A \longrightarrow B$.

The set A is called the domain of f and B the codomain. If $a \in A$ and $b \in B$ such that $f(a) = b$, then b is called the image of a under f and a the pre-image of b. The set of all images denoted $f(A)$ or $range(f)$ is called the range of f. Note that $f(A)$ is a subset of the codomain.

Example 1.1.1. *Functions on finite sets can be defined by listing all the assignments. If* $A = \{1, 2, 3, 4\}$ *and* $B = \{r, s, t, u, v\}$ *then* $f(1) = t, f(2) = s, f(3) = u, f(4) = t$ *defines a function from* A *to* B*. The assignment can be done quite arbitrarily, without recourse to any particular formula.*

Example 1.1.2. *The following are not functions from* $A = \{1, 2, 3, 4, 5\}$ *to* $B = \{r, s, t, u\}$*:* $f(1) = t, f(2) = s, f(3) = r, f(3) = u, f(4) = u, f(5) = r$

and
$g(1) = u, g(2) = r, g(4) = s, g(5) = t$

Example 1.1.3. *Determine the domain of the function* $f(x) = x^2 - 1$

Solution. The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers. In interval form, the domain of f is $(-\infty, \infty)$ □

Example 1.1.4. *Find the domain of the function* $f(x) = \dfrac{x+1}{2-x}$

Solution. When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x. That is, $2 - x = 0$ implying $x = 2$. Now, we will exclude 2 from the domain. The answers are all real numbers where $x < 2$ or $x > 2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution: $(-\infty, 2) \cup (2, \infty)$. □

Example 1.1.5. *Determine the domain of the function* $f(x) = \sqrt{7-x}$

Solution. Here, we exclude any real numbers that result in a squareroot of negative number. That is, $7 - x \geq 0$. I therefore follows that $x \leq 7$ and the domain becomes $(-\infty, 7]$ □

Example 1.1.6. *Determine the domain and range of the function* $f(x) = \dfrac{2}{x+1}$

Solution. We cannot evaluate the function at 1 because division by zero is undefined. The domain is $(-\infty, -1) \cup (-1, \infty)$. Because the function is never zero, we exclude 0 from the range. The range is $(-\infty, 0) \cup (0, \infty)$. □

1.2 Types of functions

Definition 1.2.1 (One-to-One or Injective)**.** A function $f : A \longrightarrow B$ is said to be one-to-one or injective if it assigns each element of the domain a unique element of the codomain, i.e, distinct elements in A are assigned disntinct elements in B.

If $f : X \longrightarrow Y$ is one-to-one, then $\forall x_1, x_2 \in X$ such that $f(x_1) \neq f(x_2)$ implies $x_1 \neq x_2$ or equivalently if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Example 1.2.1. *Show that $f(x) = x + 5$ from the set of real numbers $\mathbb{R}$ to the set of real $\mathbb{R}$ numbers is an injective function.*

Solution. Let $x_1, x_2 \in \mathbb{R}$ and assume that $f(x_1) = f(x_2)$. Therefore

$$x_1 + 5 = x_2 + 5$$

$$x_1 = x_2$$

$\square$

Example 1.2.2. *Determine whether $f(x) = x^2 - 2$ from the set of real numbers $\mathbb{R}$ to the set of real $\mathbb{R}$ numbers is an injective function.*

Solution. The function is not one to one since $\forall x, -x \in \mathbb{R}$ we have $f(x) = f(-x)$ and yet $x \neq -x$. $\square$

Definition 1.2.2 (Onto or Surjective)**.** A function $f : A \longrightarrow B$ is said to be onto or surjective if it assigns each element of the range has a pre-image.

If $f : X \longrightarrow Y$ is onto, then $\forall y \in Y\ \exists x \in X$ such that $y = f(x)$. Equivalently, f is onto if $f(X) = Y$.

Example 1.2.3. *Determine whether or not* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *defined by* $f(x) = 2x - 3$ *is onto.*

Solution. The domain of f is $\mathbb{R}$ and $range(f) = \mathbb{R}$ since for any y we can find a real number x such that $x = \frac{y+3}{2}$. The function is also one-to-one. □

Example 1.2.4. *Determine whether or not* $g : \mathbb{Z} \longrightarrow \mathbb{Z}$ *defined by* $g(x) = 2x + 1$ *is onto.*

Solution. Let $y = g(x) = 2x + 1$,it follows that $x = \frac{y-1}{2}$ is an integer only when y is odd. Therefore g is not onto since only odd integers have pre-images. □

Definition 1.2.3 (Bijective function)**.** A function $f : A \longrightarrow B$ is said to be bijective if it is both one-to-one and onto. For example the function given in example 1.2.3 is bijective.

Definition 1.2.4 (Identity function)**.** A function $I : A \longrightarrow A$ is said to be an identity function if $\forall a \in A$ $f(a) = a$.

Definition 1.2.5 (Identity function)**.** Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions. Then the composition of f followed by g, denoted by gof, is defined as the function $gof : A \longrightarrow C$ given by $gof(x) = g(f(x)), \forall x \in A$.

Example 1.2.5. *If* $f(x) = 2x$ *and* $g(x) = x + 1$*, then find* $(fog)(x)$ *if* $x = 1$.

Solution. $(fog)(x) = f(g(x)) = f(x + 1) = 2(x + 1) = 2x + 2$
$\therefore (fog)(1) = 2(1) + 2 = 4$ □

Example 1.2.6. *If* $f(x) = 3x^2$*, then find* $(fof)(x)$.

Solution. $(fof)(x) = f(f(x)) = f(3x^2) = 9x^4$ $\square$

Note that:

1. Composition of functions is not commutative in general.
2. Composition of functions is associative, i.e, $fo(goh) = (fog)oh$, for any functions f, g and h.

Definition 1.2.6 (Inverse functions)**.** Let $f : A \longrightarrow B$. If there exist a function $f^{-1} : B \longrightarrow A$, then f^{-1} is called the inverse function of f. This function exists if and only if f is bijective.

Example 1.2.7. *If* $f(x) = \frac{2x+1}{x-3}, x \neq 3$, *determine the formular for* $f^{-1}(x)$

Solution. We can find a formula for $f^{-1}(x)$ using the following method:

1. In the equation $y = f(x)$, if possible solve for x in terms of y to get a formula $x = f^{-1}(y)$.
2. Switch the roles of x and y to get a formula for f^{-1} of the form $y = f^{-1}(x)$.

$y = \frac{2x+1}{x-3}$
$xy - 3y = 2x + 1$
$x(y - 2) = 3y + 1$
$x = \frac{3y+1}{y-2}$
$y = \frac{3x+1}{x-2}$, i.e, $f^{-1}(x) = \frac{3x+1}{x-2}$ $\square$