SMA 2100:DISCRETE MATHEMATICS

November 30, 2024

4 Propositional Logic

Definition 1: Logic

•

Any 'formal system' can be considered a logic if it has:

✓ A well-defined syntax.

✓ A well-defined semantics.

✓ A well-defined proof-theory.

Propositional logic (PL) is the simplest form of logic where all the statements are made by propositions.

4.1 Propositions

Definition 2: Proposition

✓ A proposition (also known as a statement) is a declarative (states a fact) sentence that is either true or false, but not both.

Example 1

The following sentences are propositions:

(i) 7 is odd.

(ii) $1 + 1 = 3$.

(iii) There are seven days in a week.

Propositions (i) and (iii) are true while proposition 2 is false. Now consider the following sentences.

(i) Do your homework.

(ii) Where are you going?

(iii) $x + 1 = 4$.

Sentences (i) and (ii) are not propositions because they are not declarative sentences. Sentence (iii) is not a proposition because it is neither true nor false (depending on the value of x.).

Exercise 1

✓ ***Determine whether or not the following sentences are propositions.***

i. $2 + 3 = 7$.

ii. Are you going out?

iii. $x - y$.

iv. December is the last month of the year.

v. $0 - 7$.

vi. Evaluate the sum $2 + 2$.

vii. This sentence is false.

Definition 3: Truth values

✓ The truth value of a proposition is true, denoted by T, if it is a true proposition and false, denoted by F, if it is a false proposition.

Definition 4: Atomic propositions

✓ An atomic (or simple or primitive) proposition is one whose truth or falsity does not depend on the truth or falsity of any other proposition.

✓ Such propositions cannot be broken down into two or more complete sentences.

Definition 5: Propositional variables

✓ Variables that are used to denote propositions are called propositional variables.

✓ It is a standard practice to use the lower-case letters $p, q, r, s, \ldots$ as propositional variables.

4.2 Logical Operators

✓ Any two or more atomic propositions can be combined to form a compound proposition using binary logical operators (or connectives).

✓ Binary logical operators are those that operate on two propositions. They include conjunction, disjunction, implication etc.

✓ An operator that operates on one proposition is called a unary operator. A negation is an example of a unary logical operator.

Remark 1

✓ The truth or falsity of a compound proposition depends on the truth or falsity of each of the simple propositions together with the way in which they are connected to form the compound propositions.

Definition 6: Truth table

✓ A truth table in logic is a table that shows how the truth or falsity of a compound statement depends on the truth or falsity of the simple statements from which it's constructed.

✓ If there are n atomic propositions in a compound statement, then there are 2^n possible combinations of truth values, since each atomic proposition can be either true or false. So, we'll need 2^n rows in the truth table (in addition to the header row).

✓ We can use the following three steps to construct truth tables.

(i) A truth table should be such that each component of the compound statement is represented, as well as the entire statement (compound) itself. This will constitute the header row of the truth table.

(ii) Write out all the possible combinations of truth values for each individual proposition.

(iii) Complete the rest of the table using the basic properties of the the logical connective(s) involved.

4.2.1 Negation

Definition 7: Negation

✓ Let p be a proposition. The sentence "It is not the case that p" is a proposition called the negation of p. The negation of p is denoted by $\neg p$ or $\sim p$. If p is true then $\neg p$ should be false, and vice versa.

p	$\neg p$
T	F
F	T

Table 1: Truth table for the negation operator

Example 2

✓ ***Write the negation of each of the following propositions. Express your answers in simple English.***

(i) Today is Friday.

(ii) Michael's PC runs Linux.

(iii) My smartphone has at least 32 GB of memory.

✓ ***Solution*:**

(i) "It is not the case that today is Friday" or "Today is not Friday".

(ii) "It is not the case that Michael's PC runs Linux" or "Michael's PC does not run Linux".

(iii) "It is not the case that my smartphone has at least 32 GB of memory" or "My smartphone has less than 32 GB of memory".

4.2.2 Conjunction

Definition 8

✓ Any two propositions can be combined by the word "and" to form a compound proposition called the conjunction of the original propositions.

✓ Let p and q be propositions. Symbolically, $p \wedge q$, read as " p and q" denotes the conjunction of propositions p and q. The conjunction $p \wedge q$ is true when both p and q are true and false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 2: Truth table for the conjunction

Example 3

✓ ***Consider the following propositions:***

(i) Ice floats in water and $2+2=4$.

(ii) Ice floats in water and $2+2=5$.

(iii) China is in Europe and $2+1=3$

✓ In this example, only the first compound proposition is true.

Remark 2

✓ The conjunction $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ of n distinct propositions $p_1, p_2, \ldots, p_n$ is true when each of the n propositions is true and false otherwise.

Example 4

Construct the truth table for $\neg(p \wedge \neg q)$

***Solution*:**

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Table 3: Truth table for $\neg(p \wedge \neg q)$

Alternatively:

p	q	$\neg(p \wedge \neg q)$				
T	T	T	T	F	F	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	F	F	T	F

Table 4: Truth table for $\neg(p \wedge \neg q)$

4.2.3 Disjunction

Definition 9

Any two propositions can be combined using the word "or" to form a compound proposition called the disjunction of the original propositions. Symbolicaly, $p \vee q$ read as "p or q" is the disjunction of p and q. The truth value of $p \vee q$ is false when p and q are false, and true otherwise.

Example 5

Consider the following propositions:

1. Ice floats in water or $2 + 2 = 4$.
2. Ice floats in water or $2 + 2 = 5$.
3. China is in Europe or $2 + 5 = 3$.

In this example, only the last compound proposition is false.

Remark 3

The disjunction $p_1 \vee p_2 \vee \cdots \vee p_n$ of n distinct simple propositions $p_1, p_2, \ldots, p_n$ is false when each of the n propositions is false and true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 5: Truth table for disjunction

Exercise 2

✓ ***Let p="It is cold" and q="It is raining". Give a simple verbal sentence which describes each of the following statements:***

i. $\neg p$.

ii. $p \bigwedge q$

iii. $p \bigvee q$

iv. $p \bigvee \neg q$

4.2.4 Conditional Statements

Definition 10

✓ Let p and q be propositions. The conditional statement $p \longrightarrow q$ is the proposition "if p, then q" or "p implies q" or "p is sufficient for q" or "a sufficient condition for q is p" or "q whenever p" or "q is necessary for p" etc.

✓ The conditional statement $p \longrightarrow q$ is is false when p is true and q is false, and true otherwise. In the conditional statement $p \longrightarrow q$, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

✓ The statement $p \longrightarrow q$ is called a conditional statement because $p \longrightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an implication.

p	q	$p \longrightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 6: Truth table for $p \longrightarrow q$

Example 6

- ✓ Consider the conditional statetement "If you get 100% on the final, then you will get an A"
- ✓ This conditional statement means that if you manage to get 100% on the final, then you would expect to receive an A. If you do not get 100%, you may or may not receive an A depending on other factors. However, if you do get 100%, but you are not given an A, you will feel cheated.

Example 7

- ✓ Let p be the statement "Maria learns discrete mathematics" and q be the statement "Maria will find a good job". Express $p \longrightarrow q$ as a statement in English.
- ✓ ***Solution***: Any of the following statements is acceptable.

  i. If Maria learns discrete mathematics, then she will find a good job.

  ii. Maria will find a good job when she learns discrete mathematics.

  iii. For Maria to get a good job, it is sufficient for her to learn discrete mathematics.

  iv. Maria will find a good job unless she does not learn discrete mathematics.

Definition 11: Converse, inverse and contrapositive

- ✓ We can form some new conditional statements from the conditional statement $p \longrightarrow q$.

i. The proposition $q \longrightarrow p$ is called the **converse** of $p \longrightarrow q$.

ii. The **contrapositive** of $p \longrightarrow q$ is the proposition $\neg q \longrightarrow \neg p$.

iii. The **inverse** of $p \longrightarrow q$ is the proposition $\neg p \longrightarrow \neg q$.

Example 8

✓ Find the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining."

✓ ***Solution*:** Because "q whenever p" is one of the ways to express the conditional statement $p \longrightarrow q$, the original statement can be rewritten as "If it is raining, then the home team wins." Here, p="If it is raining" and q="The home team wins".

i. **Contrapositive**: If the home team does not win, then it is not raining.

ii. **Converse:** If the home team wins, then it is raining.

iii. **Inverse:** If it is not raining, then the home team does not win.

Example 9

✓ Construct the truth table of the proposition $(p \vee \neg q) \longrightarrow (p \wedge q)$

✓ ***Solution:***

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \longrightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Table 7: Truth table for $(p \vee \neg q) \longrightarrow (p \wedge q)$

4.2.5 Biconditional Statements

Definition 12

✓ Let p and q be propositions. The biconditional statement $p \longleftrightarrow q$ is the proposition " p if and only if q". The biconditional $p \longleftrightarrow q$ is true when p and q have the same truth values, and false otherwise. Bicondtional satatements are also called bi-implications. Some common ways of expressing the biconditional $p \longleftrightarrow q$ include:

i. say p is necessary and sufficient for q

ii. " p if then q, and conversely"

iii. " p iff q"

iv. " p exactly when q"

p	q	$p \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 8: Truth table for $p \longleftrightarrow q$

4.3 Precedence of logical operators

We generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. The negation operator is applied before all other logical operators.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\longrightarrow$	4
$\longleftrightarrow$	5

Table 9: Precedence of logical operators

4.4 Tautologies, contradictions and contingencies

Definition 13

✓ A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a ***tautology***. A compound proposition that is always false is called a ***contradiction***. A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

Example 10

✓ Use truth tables to show that $p \vee \neg p$ is a tautology and $p \wedge \neg p$ is a contradiction.

✓ ***Solution***:

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Table 10: Tautology and contradiction

4.5 Logical Equivalence

Definition 14: Logical equivalence

✓ When two compound propositions always have the same truth values, regardless of the truth values of its propositional variables, we call them equivalent (or logically equivalent). We use the symbol $\equiv$ to denote logical equivalence.

✓ Alternatively, $p \equiv q$ if and only if $p \longleftrightarrow q$ is a tautology.

Remark 4

It can be easily shown that:

i. The converse and the inverse of a conditional statement are logically equivalent.

ii. A conditional statement and its contrapositive are logically equivalent.

Example 11

Use the truth table to show that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

Solution:

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Table 11: Logical equivalence

Example 12

Use the truth table to show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Solution:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 12: Logical equivalence for $p \vee (p \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

4.5.1 Laws of the algebra of propositions

Idempotent laws	$p \vee p \equiv p$	$p \wedge p \equiv p$
Commutative laws	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Associative laws	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
Distributive laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation laws	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
Double negation law	$\neg(\neg p) \equiv p$	
Domination laws	$p \vee T \equiv T$	$p \wedge F \equiv F$
Identity laws	$p \wedge T \equiv p$	$p \vee F \equiv p$

Table 13: Algebra of propositions

4.5.2 Logical equivalences involving conditional and biconditional statements

1. $p \longrightarrow q \equiv \neg p \vee q$
2. $p \longrightarrow q \equiv \neg q \longrightarrow \neg p$
3. $p \vee q \equiv \neg p \longrightarrow q$
4. $p \wedge q \equiv \neg(p \longrightarrow \neg q)$
5. $\neg(p \longrightarrow q) \equiv p \wedge \neg q$
6. $(p \longrightarrow q) \wedge (p \longrightarrow r) \equiv p \longrightarrow (q \wedge r)$
7. $(p \longrightarrow r) \wedge (q \longrightarrow r) \equiv (p \vee q) \longrightarrow r$
8. $(p \longrightarrow q) \vee (p \longrightarrow r) \equiv p \longrightarrow (q \vee r)$
9. $(p \longrightarrow r) \vee (q \longrightarrow r) \equiv (p \wedge q) \longrightarrow r$
10. $p \longleftrightarrow q \equiv (p \longrightarrow q) \wedge (q \longrightarrow p)$
11. $p \longleftrightarrow q \equiv \neg p \longleftrightarrow \neg q$
12. $p \longleftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
13. $\neg(p \longleftrightarrow q) \equiv \ p \longleftrightarrow \neg q$

Example 13

Use the above laws to show that $\neg(p \longrightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

***Solution*:**

$$\begin{aligned} \neg(p \longrightarrow q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg(\neg p) \wedge \neg q \\ &\equiv p \wedge \neg q \end{aligned} \tag{1}$$

Example 14

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Solution:

$$\begin{aligned} \neg(p \vee (\neg p \wedge q) &\equiv \neg p \wedge \neg(\neg p \wedge q) \\ &\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] \\ &\equiv \neg p \wedge (p \vee \neg q) \\ &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \\ &\equiv F \vee (\neg p \wedge \neg q) \\ &\equiv (\neg p \wedge \neg q) \vee F \\ &\equiv \neg p \wedge \neg q \end{aligned} \tag{2}$$

Example 15

Show that $(p \wedge q) \longrightarrow (p \vee q)$ is tautology.

Solution:

$$\begin{aligned} (p \wedge q) \longrightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\ &\equiv (\neg p \vee \neg q) \vee (p \vee q) \\ &\equiv (\neg p \vee p) \vee (\neg q \vee q) \\ &\equiv T \vee T \\ &\equiv T \end{aligned} \tag{3}$$

Exercise 3

✓ ***By translate the following statements into symbolic language. Hence, find the negation of each of the statement***

i. If John testifies and tells the truth, he will be found guilty and if he does not testify, he will be found guilty.

ii. Either the fire was intentionally produced or it was produced by internal combustion.

iii. If there are more cats than dogs, then there are more horses than dogs and there are fewer snakes than cats.

✓ ***Rewrite the following statements without using the conditional:***

i. If it is cold, he wears a hat.

ii. If productivity increases, then wages rise.

✓ ***Write the converse, inverse and contrapositive of following statements:***

i. If John is a poet, then he is poor.

ii. Only if Marc studies will he pass the test.

✓ ***Assuming that the proposition q is false, prove that:***

i. $p \wedge q$ is a contradiction.

ii. $p \longrightarrow q$ is logically equivalent to $\neg p$

✓ ***Determine which of the following is a tautology, contradiction or indeterminate:***

i. $(p \vee r) \wedge (p \longrightarrow q)$.

ii. $(p \vee q) \longleftrightarrow (q \vee r)$.

4.6 Arguments and rules of inference

Definition 15

An argument is formed when we try to connect bits of evidence (premises) in a way that will help the audience to draw a desired conclusion. More formally, an argument is an assertion that a given set of propositions $p_1, p_2 \ldots, p_n$, called premises yield(or has a consequence) q called the conclusion.

Example 16

Consider the following arguments a prosecutor might present to a jury:

i. The person who robbed the Mini-Mart drives a 1989 Toyota Tercel.
Gomer drives a 1989 Toyota Tercel.
Therefore, Gomer robbed the Mini-Mart.

ii. The person who drank my coffee left this fingerprint on the cup.
Gomer is the only person in the world who has this fingerprint.
Therefore, Gomer is the person who drank my coffee.

One of these arguments is convincing, and the other is not. Why? In the first argument we have the following:
Evidence (premises):

- The person who robbed the Mini-Mart drives a 1989 Toyota Tercel.
- Gomer drives a 1989 Toyota Tercel.

Conclusion:
Therefore, Gomer robbed the Mini-Mart.

In the second argument we have:
Evidence (premises):

- The person who drank my coffee left this fingerprint on the cup.
- Gomer is the only person in the world who has this fingerprint.

Conclusion:
Therefore, Gomer is the person who drank my coffee.

Definition 16

In a well-formulated argument, it should be logically impossible to reject the conclusion if we accept all of the evidence ("the truth of the premises forces the conclusion to be true;" or "the conclusion is a certain consequence of the premises"). Such an argument is said to be **valid**. In other words, an argument is valid if and only if in every case where all the premises are true, the conclusion is true. Otherwise, the argument is **invalid**.

Definition 17

An argument is poorly-formed if it is logically possible for the audience to believe all of the evidence and yet reject the conclusion. More formally: An argument is said to be invalid if it is logically possible for the conclusion to be false even though every premise is assumed to be true.

Remark 5

In the first the argument given in the example above, even if the Jury believes all of the evidence, they don't necessarily have to believe the conclusion (because there are many people besides Gomer who drive 1989 Toyota Tercels). That is what makes the first argument invalid. In the second argument, however, if the ury believes all of the evidence, then they must accept the conclusion. That is what makes the second argument valid.

We now have the following procedure that can be used to analyze arguments whose statements can be symbolized with logical connectives:

1. Symbolize (consistently) all of the premises and the conclusion.
2. Make a truth table having a column for each premise and for the conclusion.
3. If there is a row in the truth table where every premise column is true but the conclusion column is false then the argument is invalid. If there are no such rows, then the argument is valid. Rows where all the premises are true are called the critical rows.

Example 17

Use a truth table to test the validity of the following argument:
If the apartment is damaged, then the deposit won't be refunded.
The apartment isn't damaged.
Therefore, the deposit will be refunded.

Solution: *Step 1: Symbolize the argument.*
Let p be the statement "The apartment is damaged."
Let q be the statement "The deposit won't be refunded."
Then the argument has this form:

$$\begin{array}{r} p \longrightarrow q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

Step 2: Make a truth table having a column for each premise and for the conclusion.

		premise	premise	conclusion
p	q	$p \longrightarrow q$	$\neg p$	$\neg q$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	F
F	F	T	T	T

Notice that in the third row, both premises are true while the conclusion is false; this "bad row" tells us that the argument is INVALID.

Example 18

Use a truth table to test the validity of this argument:
If I had a hammer, I would hammer in the morning.
I don't hammer in the morning.
Therefore, I don't have a hammer.

Solution
Step 1: Symbolize the argument.

Let p be the statement "I have a hammer."
Let q be the statement "I hammer in the morning."
Then the argument has this form:

$$\begin{array}{r} p \longrightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Step 2: Make a truth table having a column for each premise and for the conclusion.

		premise	premise	conclusion
p	q	$p \longrightarrow q$	$\neg q$	$\neg p$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Notice that there is no row where the conclusion column is false while both premise columns are true; the absence of a "bad row" tells us that the argument is VALID.

It is enough to prove that the conjunction of all premises implies the conclusion. We do this in the following examples:

Let q be the statement "I hammer in the morning."
Then the argument has this form:

$$\begin{array}{r} p \longrightarrow q \\ q \longrightarrow r \\ \neg r \\ \hline \therefore \neg p \end{array}$$

The corresponding conditional is
$[(p \longrightarrow q) \wedge (q \longrightarrow r) \wedge \neg r] \longrightarrow \neg p$ and the respective truth table is

p	q	r	$p \longrightarrow q$	$q \longrightarrow r$	$\neg r$	$(p \longrightarrow q) \wedge (q \longrightarrow r) \wedge \neg r$	$\neg p$	$[(p \longrightarrow q) \wedge (q \longrightarrow r) \wedge \neg r] \longrightarrow \neg p$
T	T	T	T	T	F	F	F	T
T	T	F	T	F	T	F	F	T
T	F	T	F	T	F	F	F	T
F	T	T	T	T	F	F	T	T
T	F	F	F	T	T	F	F	T
F	T	F	T	F	T	F	T	T
F	F	T	T	T	F	F	T	T
F	F	F	T	T	T	T	T	T

Table 14: Truth table for $[(p \longrightarrow q) \wedge (q \longrightarrow r) \wedge \neg r] \longrightarrow \neg p$

Therefore, the above argument is valid since the corresponding conditional is a tautology.

4.6.1 Common patterns of reasoning

Contrapositive Reasoning or Modus Tollens : Any argument that can be reduced to the form

$$\begin{array}{r} p \longrightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

will be a valid argument.This is a common form of valid reasoning known as Contrapositive Reasoning or Modus Tollens.

Example 19

Without making a truth table, we know automatically that the following arguments are valid:
If it rains, then I won't go out.
I went out.
Therefore, it didn't rain.
Here is another example of Contrapositive Reasoning:
All cats have rodent breath.
Whiskers doesn't have rodent breath.
Thus, Whiskers isn't a cat.

Fallacy of the Inverse : Any argument that can be reduced to the form

$$\begin{array}{r} p \longrightarrow q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

will always be an invalid argument. This is a common form of invalid reasoning known as fallacy of the inverse.

Example 20

If you drink Pepsi, then you are happy.
You don't drink Pepsi.
Therefore, you aren't happy.

Here is another example:
All firefighters are courageous.
Gomer the Bold isn't a firefighter.
Thus, Gomer the Bold isn't courageous.

Example 21

Determine the validity or invalidity of the following arguments
Argument 1:
If I get a huge tax refund, then I'll buy a Yugo.
I didn't buy a Yugo.
Therefore, I didn't get a huge tax refund.
Argument 2:
If I get a huge tax refund, then I'll buy a Yugo.
I didn't get a huge tax refund.
Therefore, I didn't buy a Yugo.

Solution : Argument 1 is VALID (it is Contrapositive Reasoning), whereas Argument 2 is INVALID (it is Fallacy of the Inverse).

Fallacy of the Converse : Any argument that can be reduced to the form

$$\begin{array}{r} p \longrightarrow q \\ q \\ \hline \therefore p \end{array}$$

will be an invalid argument. This is a common form of invalid reasoning known as Fallacy of the Converse.

Example 22

Test the validity of the following arguments.

Argument 1:
If I eat Wheaties, then I am healthy.
I am healthy.
Therefore, I eat Wheaties.

Arguement 2:
All great writers are philosophical.
Thoreau was philosophical.
Thus, Thoreau was a great writer.

Solution: Both arguments are INVALID, because they are examples of Fallacy of the Converse. It is not necessary to make the truth tables, although the truth tables will verify the claims that these arguments are invalid.

Direct Reasoning or Modus Ponens : Any argument that can be reduced to the form

$$\begin{array}{r} p \longrightarrow q \\ p \\ \hline \therefore q \end{array}$$

is a valid argument.

Example 23

If I quit school, I'll sell apples on the corner.
I did quit school.
Therefore, I sell apples on the corner.

Here is another example:
No beggars are choosers.
Gomer is a beggar.
Hence, Gomer is not a chooser.

Disjunctive syllogism : Any argument that can be reduced to the form

$$\begin{array}{r} p \vee q \\ \neg q \\ \hline \therefore p \end{array}$$

will be a valid argument. This is a common form of valid reasoning known as disjunctive syllogism.

Remark 6

Because the "or" connective is symmetric, this pattern can also be written as

$$\begin{array}{r} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

Example 24

The following argument is valid:
Socrates is in Athens or Socrates is in Sparta.
Socrates isn't in Sparta.
Thus, Socrates is in Athens.

Disjunctive Fallacy : Any argument that can be reduced to one of these forms:

$$\begin{array}{r} p \vee q \\ p \\ \hline \therefore \neg q \end{array}$$

or

$$\begin{array}{r} p \vee q \\ q \\ \hline \therefore \neg p \end{array}$$

is automatically INVALID. This incorrect attempt to use the disjunctive syllogism is called DISJUNCTIVE FALLACY.

Example 25

The following arguments are INVALID, because they are examples of Disjunctive Fallacy:
Argument 1:
Today isn't Sunday or I can stay home.
I can stay home.
Therefore, today is Sunday.
Argument 2:
Fido is a poodle or has brown fur.
Fido is a poodle.
Therefore, Fido doesn't have brown fur.

Transitive Reasoning: Any argument that can be reduced to the form

$$\begin{array}{r} p \longrightarrow q \\ q \longrightarrow r \\ \hline \therefore p \longrightarrow r \end{array}$$

will be a valid argument. This is a common form of valid reasoning known as Transitive Reasoning.

Example 26

The following arguments are valid because they are examples of Transitive Reasoning:
Argument 1:
If today is Monday, then tomorrow is Tuesday.
If tomorrow is Tuesday, then the day after tomorrow is Wednesday.
Therefore, if today is Monday, then the day after tomorrow is Wednesday.

Argument 2:
All bulldogs are mean-looking dogs.
All mean-looking dogs are good watchdogs.
Therefore, all bulldogs are good watchdogs.

Argument 3:
If I eat my spinach, then I'll become muscular.
If I become muscular, then I'll become a professional wrestler.
If I become I professional wrestler, then I'll bleach my hair.
If I bleach my hair, then I'll wear sequined tights.
If I wear sequined tights, then I'll be ridiculous.
Therefore, if I eat my spinach, then I'll be ridiculous.

Exercise 4

✓ ***By using symbols write and analyze the validity of the following reasoning:***

i. If n is an integer divisible by 8, n is divisible by 4. If n is divisible by 4, then n is even. The integer n is not even. Therefore, n is not divisible by 8.

ii. If n is an integer divisible by 12, then n is divisible by 6. If n is divisible by 6, then n is divisible by 3. The integer n is not divisible by 12. Therefore, n is not divisible by 3.

iii. If there are more cats than dogs, then there are more horses than dogs and there are fewer snakes than cats.

✓ ***Test the validity of the following argument:***

If I study, then I will not fail mathematics.
If I do not play basketball, then I will study.
But I failed mathematics.
Therefore I must have played basketball.

✓ ***Show that the following argument is invalid.***

$$\begin{array}{r} p \vee q \\ \neg p \\ r \longrightarrow \neg q \\ \hline \therefore r \end{array}$$

5 Predicate Logic

5.1 Open sentences

✓ In an arithmetic sentence $x + 1 = 2$, the letter x is called a variable. The set of values for which the variable x may be subsituted is called its **domain** or **universe**.

✓ In this case, the most reasonable domain is some set of numbers. Here, the variable x is said to be **free**, in that any member of the domain is allowed to be substituted into the sentence.

✓ The sentence $x + 1 = 2$ could be represented by the symbol $p(x)$. The sentence $p(3)$ is the false proposition $3 + 1 = 2$, while the sentence $p(1)$ is the true proposition $1 + 1 = 2$. A sentence like $p(x)$ with free variable x is called a **predicate** or **open sentence**.

✓ The idea is that the truth of the proposition $p(x)$ depends on or is a function of the variable x. Thus, some authors refer to predicates as propositional functions. If you were asked to determine the truth value of $p(x)$, the question would be meaningless. The sentence is sometimes true (when x is replaced by 1) and sometimes false (when x is not replaced by 1).

✓ The truth of $p(x)$ is an open question until a value for x is specified. Similarly, let $P(z)$ be the predicate "z was president of the United States in 1955". Then P(Dwight David Eisenhower) is true, P(John F. Kennedy) is false, and $P(z)$ is open.

Definition 18: Quantifiers

✓ When we introduce quantifiers like all, every, some, there exist, etc., in front of a predicate, the variables in the sentence are bound by the quantifier.

✓ The two quantifiers we will use are the **Universal Quantifier**, $\forall$, commonly read as "for all" or "for every", and the **Existential Quantifier**, $\exists$, commonly read as "there exists" or "for some".

✓ For example, $\exists z P(z)$ is interpreted to mean the sentence "there exists z such that z was President of the United States in 1955". This sentence is considered closed, not open.

✓ The use of quantifiers and predicates is summarized as follows:

i. The closed sentence $\forall x Q(x)$ is true if and only if $Q(x)$ is true for every value in the domain of x.

ii. The closed sentence $\exists x Q(x)$ is true if and only if $Q(x)$ is true for at least one value in the domain of x.

✓ If $P(x)$ is a predicate and x has domain (or universal set) D, the truth set of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x. The truth set of $P(x)$ is denoted by

$$\{x : P(x)\}$$

Example 27

Let $A = \{1, 2, 3, 4, 5\}$. Determine the truth value of each of the following sentences.

i. $(\exists x \in A)(x + 3 = 10)$

ii. $(\forall x \in A)(x + 3 < 10)$

iii. $(\exists x \in A)(x + 3 < 5)$

Solution:

i. **False**. For no number in A is a solution to $x + 3 = 10$.

ii. **True**. For every number in A satisfies $x + 3 < 10$.

iii. **True**. For if $x = 1$, then $x + 3 < 5$.

Example 28

Determine the truth set of the propositional function $x + 3 > 7$ defined on the set of positive integers.

Solution: The truth set is $\{5, 6, 7, \ldots\}$

5.2 Negation of Quantified Predicates

✓ We first look at the negation of a sentence involving a universal quantifier. The general form for such a sentence can be written as $(\forall x \in U), P(x)$, where $P(x)$ is an open sentence and U is the universal set for the variable x . When we write

$$\neg(\forall x \in U)[P(x)]$$

we are asserting that the sentence $(\forall x \in U), P(x)$ is false. That is, there exists an element x in the universal set U such that $P(x)$ is false. This in turn means that there exists an element x in U such that $\neg P(x)$ is true, which is equivalent to saying that $(\exists x \in U)[\neg P(x)]$ is true. This explains why

$$\neg(\forall x \in U)[P(x)] \equiv (\exists x \in U)[\neg P(x)]$$

.

✓ Similarly when we write,

$$\neg(\exists x \in U)[P(x)]$$

we are asserting that $(\exists x \in U)[P(x)]$ is false. This is equivalent to saying that the truth set of the open sentence $P(x)$ is the empty set. That is, there is no element x in the universal set U such that $P(x)$ is true. This in turn means that for each element x in U , $\neg P(x)$ is true, and this is equivalent to saying that $(\forall x \in U)[\neg P(x)]$ is true. This explains why

$$\neg(\exists x \in U)[P(x)] \equiv (\forall x \in U)[\neg P(x)]$$

Example 29

✓ Rewrite each of the following sentences using quantifiers. Negate each of quantified sentence.

i. All dogs bark.

ii. Most cars are inexpensive.

Solution:

i. ∀d, d barks.

✓ Any of the following negation is acceptable:

i. ¬(∀ dogs d, d barks)

ii. $\exists$dogs d, $\neg$(d barks)
iii. $\exists$dogs d, d does not bark
iv. Some dogs do not bark.

ii. $\exists$c, c is inexpensive.

✓ Any of the following negation is acceptable:

i. $\neg$($\exists$ cars c, c is inexpensive)
ii. $\forall$cars c, $\neg$(c is inexpensive)
iii. $\forall$cars c, c is not expensive
iv. No car is inexpensive.

Exercise 5

Rewrite each of the following sentences using quantifiers. Negate each of quantified sentence.

i. Every day I bring lunch to work.
ii. Lazy people never prosper.
iii. Some people are born lucky.