Methods of Proof

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1.1 Terminologies

Definition 1.1.1 (Axiom)**.** Statement that is always true.

Definition 1.1.2 (Theorem)**.** A proposition that has been proved to be true.

Definition 1.1.3 (Lemma)**.** A theorem that is usually not too interesting in its own right but is useful in proving another theorem.

Definition 1.1.4 (Corollary)**.** A theorem that follows quickly from another theorem.

Definition 1.1.5 (Conjecture)**.** Conjecture is a statement that is believed to be true but not yet proved.

1.2 Direct method of proof

Consider an implication:$p \longrightarrow q$. If p is false, then the implication is always true. Thus, show that if p is true, then q is true. To perform a direct proof, assume that p is true, and show that q must therefore be true.

Example 1.2.1. *Use direct method of proof to show that if n is even then n^2 is also even.*

Solution. Assume n is even. Thus, $n = 2k$, for some integer k. Now, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since n^2 is 2 times an integer, n^2 is thus even. □

Example 1.2.2. *Use direct method of proof to show that every integer divisible by 6 is divisible by 3.*

Solution. Let m be an integer divisible by 6. There exists an integer y such that $m = 6y$. Therefore $m = 3(2y)$, implying m is divisible by 3.

□

1.3 Indirect method of proof (Contrapositive)

Consider an implication: $p \longrightarrow q$. It's contrapositive is $\neg q \longrightarrow \neg p$. This is logically equivalent to the original implication! If the antecedent $(\neg q)$ is false, then the contrapositive is always true. Thus, show that if $\neg q$ is true, then $\neg p$ is true. To perform an indirect proof, do a direct proof on the contrapositive.

Example 1.3.1. *Prove that for an integer n if $n^3 + 5$ is odd, then n is even.*

Solution. The contrapositive becomes: If n is odd, then $n^3 + 5$ is even. Now we need to prove this contrapositive using direct method. Since n is odd we can find an integer k such that $n = 2k+1$.Therefore $n^3+5 = (2k+1)^3+5 = 8k^3 + 12k^2 + 6k + 6 = 2(4k^3 + 6k^2 + 3k + 3)$. As $2(4k^3 + 6k^2 + 3k + 3)$ is 2 times an integer, it is even. □

Example 1.3.2. *Let n be an integer. Show that if n^2 is even, then n is also even.*

Solution. We want to prove that if n is odd, then n^2 is odd. If n is odd, then $n = 2t + 1$ for some integer t . Hence, $n^2 = 4t^2 + 4t + 1 = 2(2t^2 + 2t) + 1$ is odd. This completes the proof. $\square$

1.4 Proof by Contradiction

To prove by contradiction, we assume that what we want to prove is not true, and then show that the consequences of this are not possible. That is, the consequences contradicts either what we have just assumed, or something we already know to be true (or, indeed, both).

Example 1.4.1. *Prove by contradiction that for an integer n if $n^3 + 5$ is odd, then n is even.*

Solution. Assume that n^3+5 is odd, and n is odd. Now, $n = 2k+1$ for some integer k. It therefore follows that $n^3+5 = (2k+1)^3+5 = 8k^3+12k^2+6k+6 = 2(4k^3 + 6k^2 + 3k + 3)$ is even contradicting our assumption that $n^3 + 5$ is odd. Therefore n must be even. $\square$

Example 1.4.2. *Prove by contradiction that $\sqrt{2}$ is irrational*

Solution. Suppose $\sqrt{2}$ is rational. Then integers a and b exist so that $\sqrt{2} = \frac{a}{b}$. Without loss of generality we can assume that a and b have no factors incommon (i.e., the fraction is in simplest form). Multiplying both sides by b and squaring, we have $2b^2 = a^2$ so we see that a^2 is even. This means that a is even. So $a = 2m$ for some integer m. Then $2b = a^2 = (2m)^2 = 4m^2$ which, after dividing by 2, gives $b^2 = 2m^2$. So b^2 is even. This means $b = 2n$ for some integer n. So both a and b are even contradicting our assumption that they have no common factor. Therefore $\sqrt{2}$ must be irrational $\square$

1.5 Mathematical Induction

Principle of mathematical induction :
Let P be a proposition defined on the integers $n \geq 1$ such that:

1. $P(1)$ is true.
2. $P(k+1)$ is true whenever $P(k)$ is true.

Then P is true for every integer $n \geq 1$.

Example 1.5.1. *Use mathematical induction to prove that the sum of the first n odd numbers is n^2, that is,* $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

Solution. $P(n)$ is true for $n = 1$, that is, $P(1) : 1 = 1^2$. Now, assume $P(k)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k-1) = k^2$$

Adding $2k+1$ to both sides of the above equation we obtain

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = (k+1)^2$$

We have shown that $P(k+1)$ is true whenever $P(k)$ is true. By the principle of mathematical induction, P is true for all positive integers n. $\square$

Example 1.5.2. *Use mathematical induction to prove* $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$.

Solution. $P(1)$ is true since $1 = \frac{1}{2}(1)(1+1)$. Assuming $p(k)$ is true we have

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$$

Adding $k+1$ to both sides of the above equation we obtain

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{1}{2}[k(k+1) + 2(k+1)] \\ &= \frac{1}{2}[(k+1)(k+2)] \end{aligned}$$

$\square$

1.6 Disproof by Counterexample

To disprove a statement of the form "$\forall x \in D$, if $P(x)$ then $Q(x)$", find a value of x in D for which the hypothesis $P(x)$ is true and the conclusion $Q(x)$ is false. Such an x is called a counterexample.

Example 1.6.1. *Disprove following statement by finding a suitable counterexample: for all real numbers a and b, if $a^2 = b^2$ then $a = b$.*

Solution. For any real number x we see that $x^2 = (-x)^2$ and yet $x \neq -x$ $\square$