Math 35 25X — Homework 3

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Problem 28, Section 1.3

Let A and B be nonempty bounded sets. Prove that

$$\inf(A \cup B) = \min\{\inf A, \inf B\} \quad \text{and} \quad \sup(A \cup B) = \max\{\sup A, \sup B\}.$$

Is there an analogous result for $A \cap B$, assuming that this set is nonempty? Provide either a proof or a counterexample.

Proof. As A and B are nonempty bounded sets, we can define $a = \inf A$ and $b = \inf B$.

- If $a < b$, then for all $x \in A$, we have $x \geq a$ and for all $y \in B$, we have $y \geq b > a$. Thus, for all $z \in A \cup B$, we have $z \geq a$. Since a is the infimum of A, the infimum of B is greater than a (i.e. $b > a$), and the two sets are bounded, $\inf(A \cup B) = a = \min\{a, b\}$.
- If $b < a$, then by similar reasoning, we can show that $\inf(A \cup B) = b = \min\{a, b\}$.
- If $a = b$, then $\inf(A \cup B) = a = b = \min\{a, b\}$.

To prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$ let $c = \sup A$ and $d = \sup B$.

- If $c < d$, then for all $x \in A$, we have $x \leq c < d$ and for all $y \in B$, we have $y \leq d$. Thus, for all $z \in A \cup B$, we have $z \leq d$. Since d is the supremum of B, the supremum of A is less than d (i.e. $c < d$), and the two sets are bounded, $\sup(A \cup B) = d = \max\{c, d\}$.
- If $d < c$, then by similar reasoning, we can show that $\sup(A \cup B) = c = \max\{c, d\}$.
- If $c = d$, then $\sup(A \cup B) = c = d = \max\{c, d\}$.

Now, we will consider the intersection $A \cap B$. If $A \cap B$ is nonempty, then we can define $e = \inf(A \cap B)$ and $f = \sup(A \cap B)$.

- Letting $a = \inf A$ and $b = \inf B$, we have $e \geq a$ and $e \geq b$. We can guarentee that $e \geq \max\{a, b\}$. In the case of equality $a = b = e$, which means that A and B share an infimum. If this is the case, then $\inf(A \cap B) = a = b = e$. In all other cases, the greatest lower bound of the intersection can only occur once both lowest bounds have been satisfied of both sets, i.e. $e \geq \max\{a, b\}$.
- Similarly, letting $c = \sup A$ and $d = \sup B$, we have $f \leq c$ and $f \leq d$. We can guarentee that $f \leq \min\{c, d\}$. In the case of equality $c = d = f$, which means that A and B share a supremum. If this is the case, then $\sup(A \cap B) = c = d = f$. In all other cases, the least upper bound of the intersection can only occur once both highest bounds have been satisfied for both sets, i.e. $f \leq \min\{c, d\}$.

$\square$

Problem 4, Section 1.4

Let S be a nonempty set of real numbers that is bounded above and let $\beta = \sup S$. Suppose that $\beta \notin S$. Prove that for each $\epsilon > 0$, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite.

Proof. Assume that the set $T = \{x \in S : x > \beta - \epsilon\}$ is finite. Then, there exists some $x_0 \in T$ such that $x_0 = \max T$. As $x_0 \in T$ and $T \subset S$, we have $x_0 \in S$. As β is the supremum of S and $\beta \notin S$, we have $\beta > x_0$. Let $\epsilon = \beta - x_0 > 0$. Then, we have

$$T = \{x \in S : x > \beta - (\beta - x_0)\} = \{x \in S : x > x_0\}.$$

However, as x_0 is the maximum of T, there are no elements in T that are greater than x_0. This contradiction implies that the set T cannot be finite, and thus must be infinite. Therefore, for any $\epsilon > 0$, the set $\{x \in S : x > \beta - \epsilon\}$ is infinite. $\square$

Problem 7, Section 1.4

Prove that the union of two disjoint countably infinite sets is countably infinite by finding a one-to-one correspondence between the union of the sets and the set $\mathbb{Z}^+$

Proof. Let A, B, be two disjoint countably infinite sets. As they are countably infinite, there exists some pair of functions, f_1 and f_2 such that $f_1 : A \mapsto \mathbb{Z}^+$ and $f_2 : B \mapsto \mathbb{Z}^+$ are both one to one and onto. Let $f_3 : A \cup B \mapsto \mathbb{Z}^+$ where

$$f_3 : \begin{cases} 2f_1(n) & \text{if } n \in A \\ 2f_2(n) - 1 & \text{if } n \in B \end{cases}$$

As $A \cap B = \emptyset$, f_3 is well defined. As all elements in A are mapped to even numbers and all elements in B are mapped to odd numbers, the union $A \cup B$ is a countably infinite set. $\square$

Problem 7, Section 2.1

Find an example of a sequence with the given property.

(a) The sequence is monotone but not bounded.

$\{a_n\}_{n=1}^{\infty}$ with $a_n = n \quad (n \in \mathbb{N})$.

(b) The sequence is bounded but not monotone.

$\{a_n\}_{n=1}^{\infty}$ with $a_n = (-1)^n \quad (n \in \mathbb{N})$.

(c) The sequence is bounded and strictly increasing.

$\{a_n\}_{n=1}^{\infty}$ with $a_n = 1 - \frac{1}{n} \quad (n \in \mathbb{N})$.

(d) The sequence is convergent but not monotone.

$\{a_n\}_{n=1}^{\infty}$ with $a_n = \frac{(-1)^n}{n} \quad (n \in \mathbb{N})$.

(e) The sequence is strictly decreasing but not convergent.

$\{a_n\}_{n=1}^{\infty}$ with $a_n = -n \quad (n \in \mathbb{N})$.

(f) The sequence is neither bounded nor monotone.

$\{a_n\}_{n=1}^{\infty}$ with $a_n = (-n)^n \quad (n \in \mathbb{N})$.

(g) The sequence is bounded below, not bounded above, and contains an infinite number of negative terms.

$$\{a_n\}_{n=1}^{\infty} \quad \text{with} \quad a_n = \begin{cases} -1 & \text{if } n \mod 2 = 1 \\ n & \text{if } n \mod 2 = 0. \end{cases}$$

Problem 12, Section 2.1

Let $\{a_n\}$ and $\{b_n\}$ be two sequences and suppose that the set $\{n : a_n \neq b_n\}$ is finite. Prove that the sequences either both converge to the same limit or both diverge.

Proof. Let $S = \{n : a_n \neq b_n\}$. As S is finite, $\exists x \in \mathbb{R}$ such that $x = \max S$. Hence, for all $n \in \mathbb{N} : n > x$, we have $a_n = b_n$. If a_n converges to some $L \in \mathbb{R}$, this implies that b_n also converges to this same L. By the same logic, if a_n does not converge then b_n cannot either as $a_n = b_n$ where $n > x$. Thus, if a_n converges, then b_n converges to the same limit, and if a_n diverges, then b_n diverges as well. $\square$

Problem 36, Section 2.1

Let a_n and b_n be two convergent sequences with limits a and b, respectively, and suppose that $a_n \leq b_n$ for all n. Prove that $a \leq b$.

Proof. Assume that $a > b$. Let $\epsilon = \frac{a-b}{2}$, which is always positive as $a - b > 0$. From the definition of a convergent sequence, we have $\exists N > 0$, $\forall n > N$, $|a_n - a| < \epsilon = \frac{a-b}{2}$. This can be expressed as $a - \frac{a-b}{2} < a_n < a + \frac{a-b}{2}$, or partially as $\frac{a+b}{2} < a_n$. Similarly, we can write $|b_n - b| < \epsilon = \frac{a-b}{2}$, leading to $b - \frac{a-b}{2} < b_n < b + \frac{a-b}{2}$, with the right side reducing to $b_n < \frac{a+b}{2}$. Rearanging these inequalities yields

$$\frac{a+b}{2} < a_n \leq b_n < \frac{a+b}{2}$$

which is a contradiction. Therefore, our assumption that $a > b$ must be false, and we conclude that $a \leq b$.

□