Math 35 25X — Homework 1

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Problem 6

Prove that the reciprocal of an irrational number is an irrational number.

Proof. Let x be some irrational number. Assume that its reciprocal, $\frac{1}{x} = \frac{p}{q}$ is a rational number for some integers p, q with $p, q \neq 0$. If p or $q = 0$, then x cannot be irrational: either $x = 0$ or x is undefined. Solving for x gives us:

$$x = \frac{q}{p}.$$

This contradicts the assumption that x is irrational. Therefore, the reciprocal of an irrational number must also be irrational. □

Problem 12

Prove that $\sqrt{2}+\sqrt{3}$ is irrational.

Proof. Let $x=\sqrt{2}+\sqrt{3}$. Assume that x is rational, i.e. $x \in \mathbb{Q}$. Then we can rearrange the equation to isolate one of the square roots: $\sqrt{3}=x-\sqrt{2}$. Squaring both sides gives us:

$$3=(x-\sqrt{2})^2=x^2-2x\sqrt{2}+2.$$

Rearranging this gives us:

$$2x\sqrt{2}=x^2-1.$$

Dividing both sides by $2x$ (as $x \neq 0$) gives us:

$$\sqrt{2}=\frac{x^2-1}{2x}.$$

Since x is rational, both x^2-1 and $2x$ are rational numbers, and thus their quotient $\frac{x^2-1}{2x}$ is also rational. This implies that $\sqrt{2}$ is rational, which contradicts the fact that $\sqrt{2}$ is irrational. Therefore, our assumption that $x=\sqrt{2}+\sqrt{3}$ is rational must be false, and $\sqrt{2}+\sqrt{3}$ is irrational. $\square$

Problem 20

Use the properties of the ordered field to prove the following: if $x < y$ and $z > 0$, then $xz < yz$.

Ordered Field Properties

1. If $x > 0$ and $y > 0$, then $x + y > 0$.
2. If $x > 0$ and $y > 0$, then $xy > 0$.
3. $x < y$ if and only if $y - x > 0$.

Proof. Let $x, y, z \in F$, where F is an ordered field. Using OF-P3, as $x < y$, then $y - x > 0$. Let a equal $y - x$. Then, we have $a > 0$. Using OF-P2, as $a > 0$ and $z > 0$, we can combine these two inequalities to get $az > 0$. Expanding az gives us $az = (y - x)z = yz - xz$, or $yz - xz > 0$. Applying OF-P3, we can rearrange this inequality to be $xz < yz$. Thus, if $x < y$ and $z > 0$, then $xz < yz$. □