Math 35 25X — Homework 5

Kiran Jones
kiran.p.jones.27@dartmouth.edu

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Problem 4, Section 3.1

Use the definition of a limit to prove each of the following.

(a) $\lim_{x\to 2}(5x-11) = -1$

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $|x-2| < \delta$, then $|5x-11+1| < \epsilon$. We have:

$$|5x-11+1| = |5x-10| = 5|x-2|.$$

Thus, we need $5|x-2| < \epsilon$, which implies $|x-2| < \frac{\epsilon}{5}$. Therefore, we can choose $\delta = \frac{\epsilon}{5}$. Now, if $|x-2| < \delta$, then:

$$|5x-11+1| = 5|x-2| < 5\delta = 5\left(\frac{\epsilon}{5}\right) = \epsilon.$$

Hence, $\lim_{x\to 2}(5x-11) = -1$. □

(b) $\lim_{x\to 1}(x^2+x-1) = 1$

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $|x-1| < \delta$, then $|x^2+x-1-1| < \epsilon$. We have:

$$|x^2+x-1-1| = |x^2+x-2| = |(x-1)(x+2)|.$$

To ensure this is less than ϵ, we can bound $|x+2|$ for x close to 1. If we choose $\delta < 1$, then $|x-1| < 1$ implies $0 < x < 2$, so $|x+2| < 4$. Thus, we need:

$$|(x-1)(x+2)| < |x-1| \cdot |x+2| < \delta \cdot 4.$$

Therefore, we can choose $\delta = \frac{\epsilon}{4}$. Now, if $|x-1| < \delta$, then:

$$|(x-1)(x+2)| < \delta \cdot 4 = \frac{\epsilon}{4} \cdot 4 = \epsilon.$$

Hence, $\lim_{x\to 1}(x^2+x-1) = 1$. □

(c) $\lim_{x\to -2}(x-3x^2) = -14$

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $|x+2| < \delta$, then $|x-3x^2+14| < \epsilon$. This can be factored into

$$|x+2||3x-7| < \epsilon$$

If we choose $\delta < 1$, then

$$-3 < x < -1, -9 < 3x < -3 \text{ and } -16 < 3x - 7 < -10.$$

Thus, we need:

$$|x+2||3x-7| < |x+2| \cdot 16$$

Therefore, we can choose $\delta = \min\left(1, \frac{\epsilon}{16}\right)$. Now, if $|x+2| < \delta$, then:

$$|x - 3x^2 + 14| = |(x+2)(3x-7)| < |x+2| \cdot 16 < \delta \cdot 16 = \frac{\epsilon}{16} \cdot 16 = \epsilon.$$

□

(d) $\lim_{x\to 4}(\sqrt{x}) = 2$

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $|x-4| < \delta$, then $|\sqrt{x} - 2| < \epsilon$. We have:

$$|\sqrt{x} - 2| = \frac{|x-4|}{|\sqrt{x}+2|}.$$

For x close to 4, we can bound $|\sqrt{x}+2|$. If we choose $\delta < 1$, then $3 < \sqrt{x} < 5$, so $|\sqrt{x}+2| < 7$. Thus, we need:

$$\frac{|x-4|}{|\sqrt{x}+2|} < \epsilon \implies |x-4| < 7\epsilon.$$

Therefore, we can choose $\delta = \min(1, 7\epsilon)$. Now, if $|x-4| < \delta$, then:

$$|\sqrt{x} - 2| = \frac{|x-4|}{|\sqrt{x}+2|} < \frac{\delta}{7} < \epsilon.$$

Hence, $\lim_{x\to 4}(\sqrt{x}) = 2$. □

(e) $\lim_{x\to -2}(x^3) = -8$

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $|x+2| < \delta$, then $|x^3 + 8| < \epsilon$. We have:

$$|x^3 + 8| = |(x+2)(x^2 - 2x + 4)|.$$

For x close to -2, we can bound $|x^2 - 2x + 4|$. If we choose $\delta < 1$, then $-3 < x < -1$, so:

$$|x^2 - 2x + 4| < C$$

for some constant C. The maximum value for $x \in [-3, -1]$ occurs at $x = -3$, where:

$$|-3^2 - 2(-3) + 4| = |9 + 6 + 4| = 19.$$

Therefore, we can let $\delta = \min(1, \frac{\epsilon}{19})$. Now, if $|x+2| < \delta$, then:

$$|x^3 + 8| = |(x+2)(x^2 - 2x + 4)| < |x+2| \cdot 19 < \delta \cdot 19 = \frac{\epsilon}{19} \cdot 19 = \epsilon.$$

Therefore, $\lim_{x \to -2}(x^3) = -8$. □

(f) $\lim_{x \to 1}(\frac{4}{3x+2}) = \frac{4}{5}$

Proof. Let $\epsilon > 0$ be given. We want to find $\delta > 0$ such that if $|x-1| < \delta$, then $\left|\frac{4}{3x+2} - \frac{4}{5}\right| < \epsilon$. We have:

$$\left|\frac{4}{3x+2} - \frac{4}{5}\right| = \left|\frac{20 - 12x}{(3x+2)5}\right|.$$

To ensure this is less than ϵ, we can bound the denominator. If we choose $\delta < 1$, then $2 < 3x + 2 < 5$, so $|(3x+2)5| > 10$. Thus, we need:

$$\left|\frac{20 - 12x}{(3x+2)5}\right| < \frac{|20 - 12x|}{10} < \epsilon.$$

Therefore, we can choose $\delta = \min(1, \frac{10\epsilon}{12})$. Now, if $|x-1| < \delta$, then:

$$\left|\frac{4}{3x+2} - \frac{4}{5}\right| < \frac{\delta}{10} < \epsilon.$$

Hence, $\lim_{x \to 1}(\frac{4}{3x+2}) = \frac{4}{5}$. □

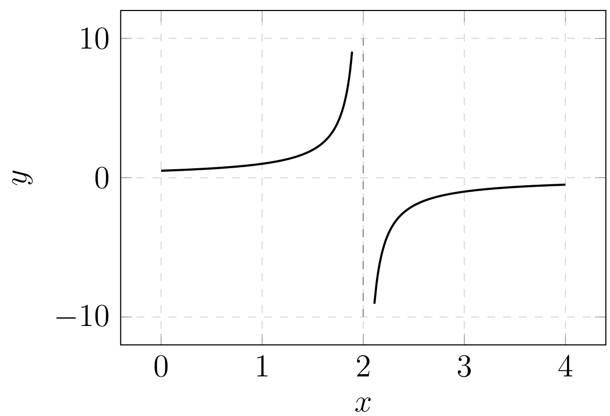
Problem 34, Section 3.1

Provide definitions for each of the following. Give an example, both numerical and graphical, for each limit.

(a) $\lim_{x \to c^+} f(x) = -\infty$

Definition: The limit $\lim_{x \to c^+} f(x) = -\infty$ means that as x approaches c from the right, the function $f(x)$ decreases without bound, i.e., for every negative number M, there exists a $\delta > 0$ such that if $0 < x - c < \delta$, then $f(x) < M$.
Example: Consider the function $f(x) = -\frac{1}{x-2}$ as $x \to 2^+$. As x approaches 2 from the right, $f(x)$ goes to $-\infty$.
Graphically, this can be observed in the following plot as the function approaches the vertical asymptote at $x = 2$ from the right:

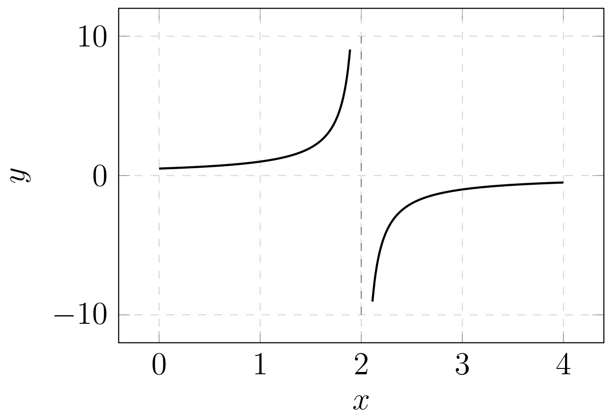
Plot of $y = -\frac{1}{x-2}$ with its asymptote



(b) $\lim_{x \to c^-} f(x) = \infty$

Definition: The limit $\lim_{x \to c^-} f(x) = \infty$ means that as x approaches c from the left, the function $f(x)$ increases without bound, i.e., for every positive number M, there exists a $\delta > 0$ such that if $0 < c - x < \delta$, then $f(x) > M$.
Example: Consider the same function $f(x) = -\frac{1}{x-2}$ as $x \to 2^-$. As x approaches 2 from the left, $f(x)$ goes to ∞.
Graphically, this can be observed in the following plot as the function approaches the vertical asymptote at $x = 2$ from the left:

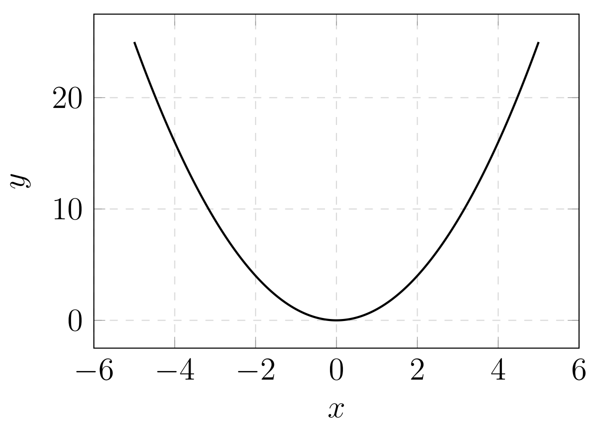
Plot of $y = -\frac{1}{x-2}$ with its asymptote



(c) $\lim_{x\to\infty} f(x) = \infty$

Definition: The limit $\lim_{x\to\infty} f(x) = \infty$ means that as x increases without bound, the function $f(x)$ also increases without bound, i.e., for every positive number M, there exists a number N such that if $x > N$, then $f(x) > M$.
Example: Consider the function $f(x) = x^2$. As x approaches ∞, $f(x)$ also approaches ∞.
Graphically, this can be observed in the following plot:

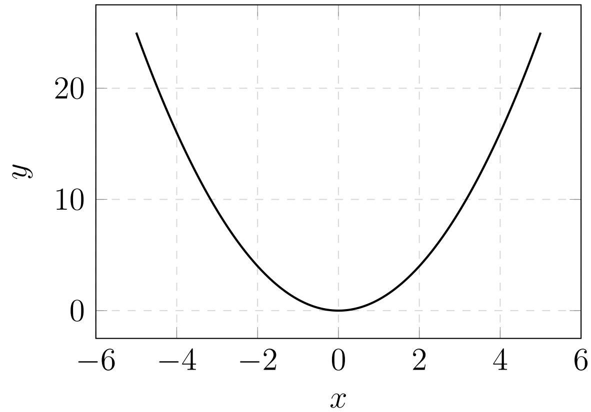
Plot of $y = x^2$



(d) $\lim_{x\to-\infty} f(x) = \infty$

Definition: The limit $\lim_{x\to-\infty} f(x) = \infty$ means that as x decreases without bound, the function $f(x)$ increases without bound, i.e., for every positive number M, there exists a number N such that if $x < N$, then $f(x) > M$.
Example: Consider the same function $f(x) = x^2$. As x approaches $-\infty$, $f(x)$ approaches ∞.
Graphically, this can be observed in the following plot:

Plot of $y = x^2$



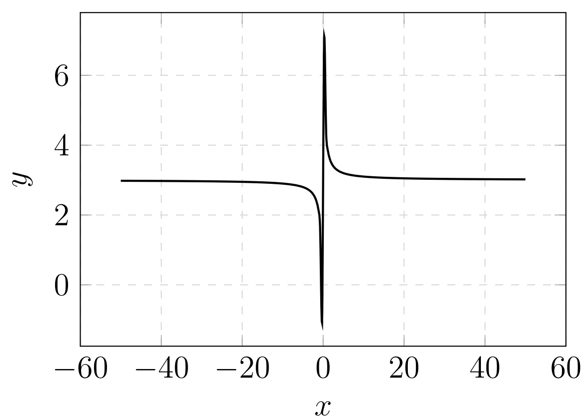
(e) $\lim_{x \to -\infty} f(x) = L$

Definition: The limit $\lim_{x \to -\infty} f(x) = L$ means that as x decreases without bound, the function $f(x)$ approaches the value L, i.e., for every positive number ϵ, there exists a number N such that if $x < N$, then $|f(x) - L| < \epsilon$.

Example: Let $L = 3$, and consider the function $f(x) = 3 + \frac{1}{x}$ as $x \to -\infty$. As x approaches $-\infty$, $f(x)$ approaches 3.

Graphically, this can be observed in the following plot:

Plot of $y = 3 + \frac{1}{x}$



Problem 27, Section 3.1

Give an example of a bounded function that does not have a limit at any point.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Problem 4, Section 3.2

Let $f : [a, b] \to \mathbb{R}$ be continuous at $c \in [a, b]$ and suppose that $f(c) > 0$. Prove that there exists a positive number m and an interval $[u, v] \subseteq [a, b]$ such that $c \in [u, v]$ and $f(x) \geq m$ for all $x \in [u, v]$.

Proof. Since f is continuous at c, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x-c| < \delta$, then $|f(x) - f(c)| < \epsilon$. Choose $\epsilon = \frac{f(c)}{2}$, which is positive since $f(c) > 0$. Then, there exists a $\delta > 0$ such that if $|x - c| < \delta$, then:

$$|f(x) - f(c)| < \frac{f(c)}{2}.$$

This implies:

$$f(x) > f(c) - \frac{f(c)}{2} = \frac{f(c)}{2} > 0.$$

Let $m = \frac{f(c)}{2}$. Now, consider the interval $[u, v] = [\max\{a, c - \delta\}, \min\{b, c + \delta\}]$. Then, for all $x \in [u, v]$, we have $f(x) \geq m$. Thus, there exists a positive number m and an interval $[u, v] \subseteq [a, b]$ such that $c \in [u, v]$ and $f(x) \geq m$ for all $x \in [u, v]$. □

Problem 7, Section 3.2

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and suppose that $f(x) = 0$ for each rational number $x \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. Let $x_0 \in [a, b]$ be arbitrary. Since the rational numbers are dense in the real numbers, there exists a sequence of rational numbers $\{q_n\}$ such that $q_n \to x_0$ as $n \to \infty$. Since f is continuous on $[a, b]$, we have:

$$f(x_0) = f\left(\lim_{n\to\infty} q_n\right) = \lim_{n\to\infty} f(q_n).$$

But since $f(q_n) = 0$ for all rational numbers q_n, we have:

$$\lim_{n\to\infty} f(q_n) = 0.$$

Therefore, $f(x_0) = 0$. Since x_0 was arbitrary, $f(x) = 0$ for all $x \in [a, b]$.

$\square$

Problem 10, Section 3.2

Use the definition of continuity to prove Theorem 3.13.

Proof. Let I be an interval in $\mathbb{R}$ and let $g : I \to \mathbb{R}$ be continuous on I. Let c be an arbitrary point in I, and let f be a function defined on the interval $g(I) \subseteq J$. If g is continuous at c andf f is continuous at $g(c)$, then $f \circ g$ is continuous at c. Hence, if g is continuous on I and f is continuous on J, then $f \circ g$ is continuous on I. □