Math 35 25X — Homework 4

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Problem 12, Section 2.2

Use the definition to prove that $\{\frac{n}{n+3}\}$ is a Cauchy sequence.

Proof. Let $\epsilon > 0$ be given, and assume that $\{\frac{n}{n+3}\}$ is Cauchy. We want to find N such that for all $m, n \geq N$, we have

$$\left|\frac{n}{n+3} - \frac{m}{m+3}\right| < \epsilon.$$

We can rewrite this as

$$\left|\frac{n(m+3) - m(n+3)}{(n+3)(m+3)}\right| = \left|\frac{3(n-m)}{(n+3)(m+3)}\right|.$$

Since $(n+3)(m+3) > 0$ for all $n, m > 0$, Without a loss of generality, assume $m \geq n$. Letting $\epsilon = \frac{3(m+n)}{(m+3)(n+3)} > 0$ yields

$$\frac{3(n-m)}{(m+3)(n+3)} < \frac{3(m+n)}{(m+3)(n+3)} = \frac{3}{n+3}$$

As $n > N$, $n+3 > N+3$, and $\frac{3}{n+3} < \frac{3}{N+3}$. Using any $\epsilon > \frac{3}{N+3}$ the sequence is therefore Cauchy. □

Problem 20, Section 2.2

Let x_n be a sequence and suppose that the sequence $\{x_{n+1} - x_n\}$ converges to 0. Give an example to show that the sequence $\{x_n\}$ may not converge. Hence, the condition that $|x_n - x_m| < \epsilon$ for all $m, n \geq N$ is crucial in the definition of a Cauchy sequence.

Proof. Let $\{x_n\} = \log n$. The sequence $\{x_{n+1} - x_n\} = \log(n+1) - \log(n)$ converges to 0 as $n \to \infty$. However, the sequence $\{x_n\} = \log n$ is unbounded, as $\forall M \in \mathbb{R}$, $\exists n$ such that $\log n > M$, where $n = 10^M + 1$. Hence, the sequence $\{x_n\}$ does not converge. □

Problem 3, Section 2.3

Provide an example of a sequence with the given property.

(a) a sequence that has subsequence that converge to 1, 2, and 3

$x_n = (n \mod 3) + 1$

(b) a sequence that has subsequence that converge to ∞ and $-\infty$

$x_n = (-1)^n n$

(c) a sequence that has a strictly increasing subsequence, a strictly decreasing subsequence, and a constant subsequence

$$x_n = \begin{cases} 0 & \text{if } n \mod 3 = 0 \\ n & \text{if } n \mod 3 = 1 \\ -n & \text{if } n \mod 3 = 2 \end{cases}$$

(d) an unbounded sequence which has a convergent subsequence

$$x_n = \begin{cases} 0 & \text{if } n \mod 2 = 0 \\ n & \text{if } n \mod 2 = 1 \end{cases}$$

(e) a sequence that has no convergent subsequence

$x_n = n$

Problem 9, Section 2.3

Let $\{a_n\}$ be an unbounded sequence. Prove that there exists a subsequnce $\{a_{p_n}\}$ of $\{a_n\}$ such that $\{\frac{1}{a_{p_n}}\}$ converges to 0.

Proof. As $\{a_n\}$ is unbounded, we can choose an arbitrary subsequence a_{p_n} that is strictly increasing. This gives us the reciprocal subsequence $\{\frac{1}{a_{p_n}}\}$, which must be strictly decreasing. As $a_{p_n} \to \infty$, we have $\frac{1}{a_{p_n}} \to 0$. Thus, the subsequence $\{\frac{1}{a_{p_n}}\}$ converges to 0. $\square$

Problem 17, Section 2.3

Find the limit inferior and limit superior of the given sequence.

(a) $\{(\frac{n}{3}) - \lfloor\frac{n}{3}\rfloor\}$
   lim sup $= \frac{2}{3}$, lim inf $= 0$

(b) $\{(-1)^n(1 + \frac{1}{n})\}$
   lim sup $= 1$, lim inf $= -1$

(c) $\{n\sin(\frac{\pi n}{3})\}$
   lim sup $= \infty$, lim inf $= -\infty$

Problem 18, Section 2.3

Find the limit inferior and limit superior of the sequence $\{\lfloor 5 \sin n \rfloor\}$.

Proof. The sequence $\{\lfloor \sin n \rfloor\}$ is bounded between -5 and 4. The limit inferior is the greatest lower bound of the set of subsequential limits, which is -5, and the limit superior is the least upper bound of the set of subsequential limits, which is 4. Hence, $\limsup = 4$ and $\liminf = -5$. $\square$