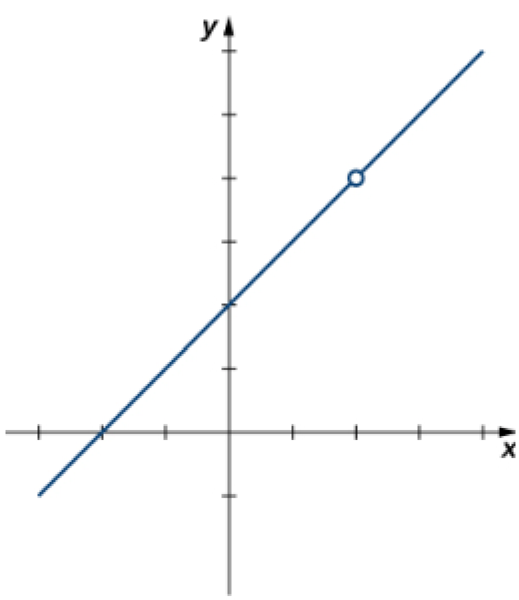
Introduction to Limits and Estimation using Tables and Graphs

Learning Objectives:

- Using correct notation, describe the limit of a function.
- Use a table of values to estimate the limit of a function or to identify when the limit does not exist.
- Use a graph to estimate the limit of a function or to identify when the limit does not exist.

The concept of a limit or limiting process, essential to the understanding of calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet, the formal definition of a limit—as we know and understand it today—did not appear until the late 19th century. We, therefore, begin our quest to understand limits, as our mathematical ancestors did, by using an intuitive approach.

We begin our exploration of limits by taking a look at the graph. In particular, let's focus our attention on the behavior of the graph at and around $x=2$.



$$f(x) = \frac{x^2 - 4}{x - 2}$$

Intuitive Definition of a Limit

Let’s first take a closer look at how the function $f(x) = (x^2 - 4)/(x - 2)$ behaves around $x = 2$ in Figure [redacted]. As the values of *x* approach 2 from either side of 2, the values of $y = f(x)$ approach 4. Mathematically, we say that the limit of $f(x)$ as *x* approaches 2 is 4. Symbolically, we express this limit as

$$\lim_{x \to 2} f(x) = 4.$$

From this very brief informal look at one limit, let’s start to develop an **intuitive definition of the limit**. We can think of the limit of a function at a number *a* as being the one real number *L* that the functional values approach as the *x*-values approach *a*, provided such a real number *L* exists. Stated more carefully, we have the following definition:

DEFINITION

Let $f(x)$ be a function defined at all values in an open interval containing *a*, with the possible exception of *a* itself, and let *L* be a real number. If *all* values of the function $f(x)$ approach the real number *L* as the values of $x \, (\neq a)$ approach the number *a*, then we say that the limit of $f(x)$ as *x* approaches *a* is *L*. (More succinct, as *x* gets closer to *a*, $f(x)$ gets closer and stays close to *L*.) Symbolically, we express this idea as

$$\lim_{x \to a} f(x) = L.$$

(2.3)

We can estimate limits by constructing tables of functional values and by looking at their graphs. This process is described in the following Problem-Solving Strategy.

The problem-solving strategy in simpler words:

Evaluating a Limit Using a Table of Functional Values

Step 1: Begin by completing a table of functional values f(x) for two sets of x values. One set of x values approaching a and less than a, and another set of x values approaching a and greater than a. Your table might look as below:

x	$f(x)$		*x*	$f(x)$
$a - 0.1$	$f(a - 0.1)$		$a + 0.1$	$f(a + 0.1)$
$a - 0.01$	$f(a - 0.01)$		$a + 0.01$	$f(a + 0.01)$
$a - 0.001$	$f(a - 0.001)$		$a + 0.001$	$f(a + 0.001)$
$a - 0.0001$	$f(a - 0.0001)$		$a + 0.0001$	$f(a + 0.0001)$
Use additional values as necessary.			Use additional values as necessary.	

Step 2: Look at the values in each of the f(x) columns and determine whether the values seem to be approaching a single value as we move down each column. Make note

these chosen values work nearly every time, on very rare occasions we may need to modify our choices.
Step 3: If both columns approach a common f(x) value, “L”, we conclude that

$$\lim_{x \to a} f(x) = L.$$

If both columns approach different f(x) values, conclude that the limit does not exist.
Step 4: Confirm your answers using a graphing calculator or computer software. Use the trace feature to move along the graph of the function and watch the f(x) value as x-values approach ‘a’ from both directions.

We apply this Problem-Solving Strategy to compute a limit in Example 2.4.

EXAMPLE 2.4

Evaluating a Limit Using a Table of Functional Values 1

Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ using a table of functional values.

Example 2.4:
Step 1: Let us calculate the given function as x values approach 0 from both directions. We will use x values as -0.1, -0.01, -0.001, -0.0001, 0.1, 0.01, 0.001, 0.0001.

When $x = -0.1$,

$$f(-0.1) = \frac{\sin(-0.1)}{-0.1} = \frac{-0.0998334166468}{-0.1} = 0.998334166468$$

When $x = -0.01$,

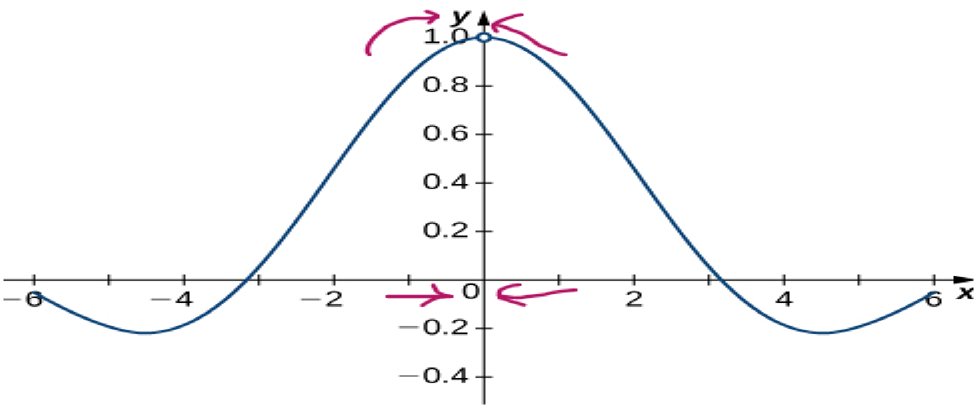
$$f(-0.01) = \frac{\sin(-0.01)}{-0.01} = \frac{-0.0099983333417}{-0.01} = 0.999983333417$$

Proceed further for the remaining x values to get the below table.

x	$\frac{\sin x}{x}$		x	$\frac{\sin x}{x}$
−0.1	0.998334166468		0.1	0.998334166468
−0.01	0.999983333417		0.01	0.999983333417
−0.001	0.999999833333		0.001	0.999999833333
−0.0001	0.999999998333		0.0001	0.999999998333

Step 2: The left set of values approaches one. The right set of values approaches one.
Step 3: As both columns approach a common f(x) value, one, it is fairly reasonable to conclude that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Step 4: A calculator or computer-generated graph of $f(x) = \frac{(\sin x)}{x}$ would be similar to that shown below and it confirms our estimate.



EXAMPLE 2.5

Evaluating a Limit Using a Table of Functional Values 2

Evaluate $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$ using a table of functional values.

Example 2.5:

Step 1: Let us calculate the given function as x values approach 4 from both directions. We will use x values as 3.9, 3.99, 3.999, 3.9999, 3.99999, 4.1, 4.01, 4.001, 4.0001, and 4.00001.

When $x = 3.9$,

$$f(3.9) = \frac{\sqrt{3.9} - 2}{3.9 - 4} = 0.251582341869$$

When $x = 3.99$,

$$f(3.9) = \frac{\sqrt{3.99} - 2}{3.99 - 4} = 0.25015644562$$

When $x = 4.1$,

$$f(4.1) = \frac{\sqrt{4.1} - 2}{4.1 - 4} = 0.248456731317$$

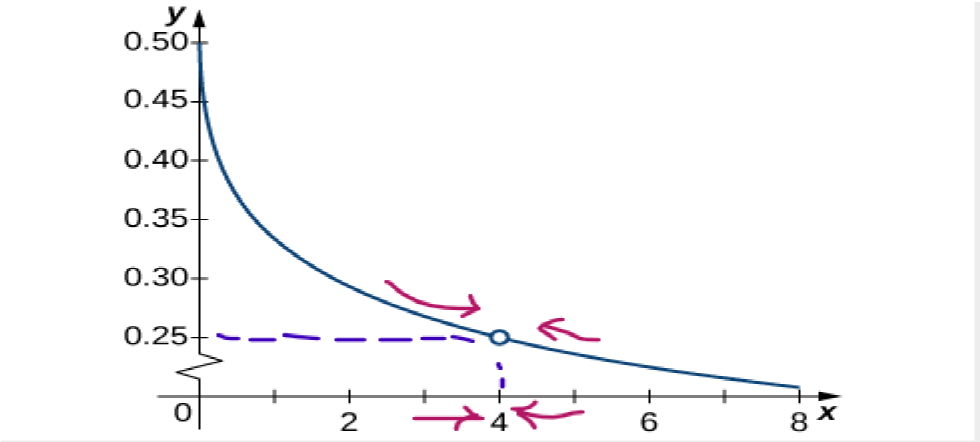
Proceed further for the remaining x values to get the below table.

x	$\frac{\sqrt{x}-2}{x-4}$		x	$\frac{\sqrt{x}-2}{x-4}$
3.9	0.251582341869		4.1	0.248456731317
3.99	0.25015644562		4.01	0.24984394501
3.999	0.250015627		4.001	0.249984377
3.9999	0.250001563		4.0001	0.249998438
3.99999	0.25000016		4.00001	0.24999984

Step 2: After inspecting this table, we see that the functional values less than 4 appear to be decreasing toward 0.25 whereas the functional values greater than 4 appear to be increasing toward 0.25.

Step 3: As both columns approach a common f(x) value, 0.25, we conclude that $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = 0.25$.

Step 4: A calculator or computer-generated graph of $f(x) = \frac{\sqrt{x}-2}{x-4}$ would be similar to that shown below and it confirms our estimate.



At this point, we see from Example 2.4 and Example 2.5 that it may be just as easy, if not easier, to estimate a limit of a function by inspecting its graph as it is to estimate the limit by using a table of functional values. In Example 2.6, we evaluate a limit exclusively by looking at a graph rather than by using a table of functional values.

EXAMPLE 2.6

Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 2.15, evaluate $\lim\limits_{x \to -1} g(x)$.

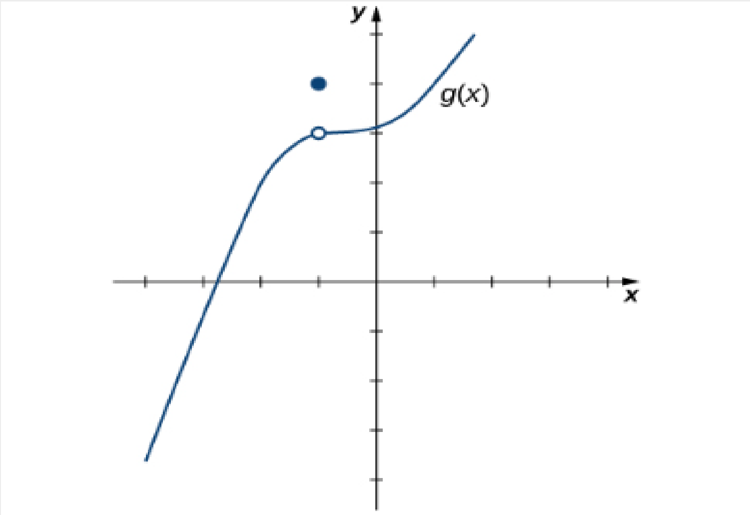


Figure 2.15 The graph of $g(x)$ includes one value not on a smooth curve.

[Show/Hide Solution]

Solution

Despite the fact that $g(-1) = 4,$ as the x-values approach −1 from either side, the $g(x)$ values approach 3. Therefore, $\lim_{x \to -1} g(x) = 3.$ Note that we can determine this limit without even knowing the algebraic expression of the function.

Based on Example 2.6, we make the following observation: It is possible for the limit of a function to exist at a point, and for the function to be defined at this point, but the limit of the function and the value of the function at the point may be different.

The Existence of a Limit

As we consider the limit in the next example, keep in mind that for the limit of a function to exist at a point, the functional values must approach a single real-number value at that point. If the functional values do not approach a single value, then the limit does not exist.

EXAMPLE 2.7

Evaluating a Limit That Fails to Exist

Evaluate $\lim_{x \to 0} \sin(1/x)$ using a table of values.

Example 2.7:

Step 1: Let us calculate the given function as x values approach 0 from both directions. We will use x values as -0.1, -0.01, -0.001, -0.0001, -0.00001, -0.000001, 0.1, 0.01, 0.001, 0.0001, 0.00001, and 0.000001.

When $x = -0.1,$

$$f(-0.1) = \sin\left(\frac{1}{-0.1}\right) = -\sin\left(\frac{1}{0.1}\right) = -\sin(10) = 0.544021110889$$

When $x = -0.01,$

$$f(-0.01) = \sin\left(\frac{1}{-0.01}\right) = -\sin\left(\frac{1}{0.01}\right) = -\sin(100) = 0.50636564111$$

When $x = 0.1,$

$$f(0.1) = \sin\left(\frac{1}{0.1}\right) = \sin(10) = -0.544021110889$$

When $x = -0.01,$

$$f(0.01) = \sin\left(\frac{1}{0.01}\right) = \sin(100) = -0.50636564111$$

Proceed further for the remaining x values to get the below table.

x	$\sin\left(\frac{1}{x}\right)$		x	$\sin\left(\frac{1}{x}\right)$
−0.1	0.544021110889		0.1	−0.544021110889
−0.01	0.50636564111		0.01	−0.50636564111
−0.001	−0.8268795405312		0.001	0.826879540532
−0.0001	0.305614388888		0.0001	−0.305614388888
−0.00001	−0.035748797987		0.00001	0.035748797987
−0.000001	0.349993504187		0.000001	−0.349993504187

Step 2: After examining the table of functional values, we can see that f(x) values do not seem to approach any one single value.

Step 3:It appears the limit does not exist. Before drawing this conclusion, let's take a more systematic approach. Take the following sequence of x-values approaching 0:

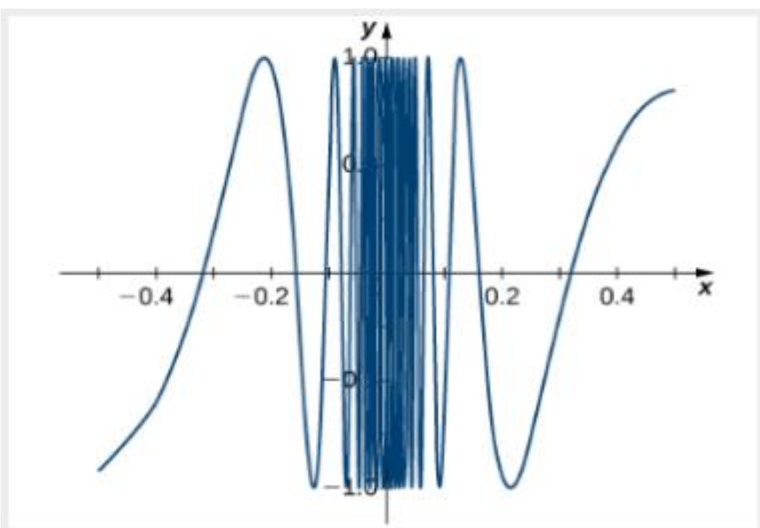
$$\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \frac{2}{9\pi}, \frac{2}{11\pi}, \ldots.$$

The corresponding y-values are

1, -1, 1, -1, 1, -1,.....

At this point we can indeed conclude that $\lim\limits_{x\to 0}\sin(1/x)$ does not exist. Mathematicians frequently abbreviate "does not exist" as DNE.

Step 4: A calculator or computer-generated graph of $f(x) = \sin(1/x)$ would be similar to that shown below and it confirms our estimate.



We can see that sin(1/x) oscillates ever more widely between -1 and 1 as x approaches 0.

Summary:

In this subsection, you learned how to describe limits using correct notation. You also learned how to evaluate a limit using a table of functional values and using graphs.

MCQ 1:

a. Which of the following table is the correct output while estimating $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1}$ rounded to six decimal places?

b. Estimate $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1}$.

Option A:

a.

x	f(x)	x	f(x)
0.9	-1.111111	1.1	-0.909091
0.99	-1.010101	1.01	-0.990099
0.999	-1.001001	1.001	-0.999001
0.9999	-1.000100	1.0001	-0.999900
0.99999	-1.000010	1.00001	-0.999990

b. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = -1$

Option B:

a.

x	f(x)	x	f(x)
0.9	1.111111	1.1	0.909091
0.99	1.010101	1.01	0.990099
0.999	1.001001	1.001	0.999001
0.9999	1.000100	1.0001	0.999900
0.99999	1.000010	1.00001	0.999990

b. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = 1$

Option C:

a.

x	f(x)	x	f(x)
-0.9	1.111111	-1.1	0.909091
-0.99	1.010101	-1.01	0.990099
-0.999	1.001001	-1.001	0.999001
-0.9999	1.000100	-1.0001	0.999900
-0.99999	1.000010	-1.00001	0.999990

b. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = 1$

Option D:

a.

x	f(x)	x	f(x)
-0.9	-1.111111	-1.1	-0.909091
-0.99	-1.010101	-1.01	-0.990099
-0.999	-1.001001	-1.001	-0.999001
-0.9999	-1.000100	-1.0001	-0.999900
-0.99999	-1.000010	-1.00001	-0.999990

b. $\lim_{x \to 1} \frac{\frac{1}{x}-1}{x-1} = -1$

Correct Option: Option A

General Feedback:

a. To estimate the table values of f(x) for the given x values, substitute x values in the given function and simplify with the help of a calculator. Round these values to six decimal places.

When $x = -0.9$,

$$f(-0.9) = \frac{\frac{1}{-0.9} - 1}{-0.9 - 1} = \frac{-1.1111111.... - 1}{-1.9} = \frac{-2.1111111....}{-1.9} = -1.11111$$

When $x = -0.99$, $f(-0.99) = \frac{\frac{1}{-0.99}-1}{-0.99-1} = \frac{-1.01010101....-1}{-1.99} = \frac{-2.01010101....}{-1.99} = -1.010101$

When $x = 1.1$,

$$f(1.1) = \frac{\frac{1}{1.1} - 1}{1.1 - 1} = \frac{0.90909090.... - 1}{0.1} = \frac{-0.09090909....}{0.1} = -0.909091$$

Proceeding in this way, the correct output is

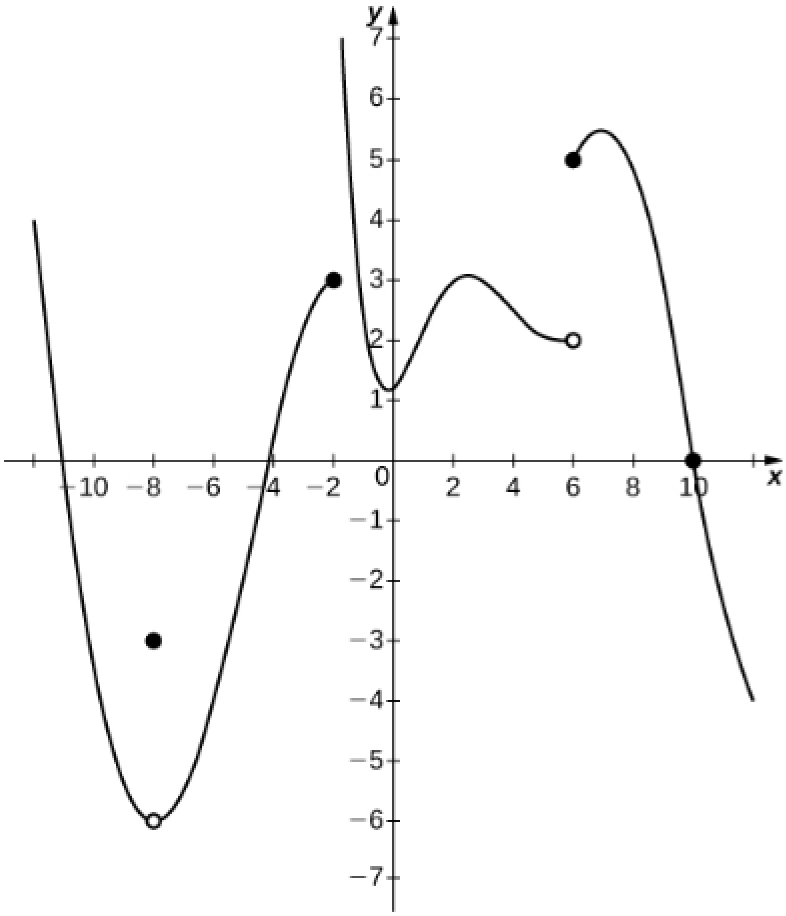
x	f(x)	x	f(x)
0.9	-1.111111	1.1	-0.909091
0.99	-1.010101	1.01	-0.990099
0.999	-1.001001	1.001	-0.999001
0.9999	-1.000100	1.0001	-0.999900
0.99999	-1.000010	1.00001	-0.999990

b. Values in each column in the table above approaching -1. Thus, it is fairly reasonable to conclude $\lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1} = -1$.

Specific Feedback:
Option A: Key
Option B: You used an incorrect function to evaluate the limit.
Option C: You used incorrect x values to evaluate the limit. X values should be close to 1.
Option D: Though the final limit is correct, you used incorrect x values and incorrect function to evaluate the limit.

MCQ 2:

Consider the following graph of the function $y = f(x)$ to estimate $\lim_{x \to -8} f(x)$



Option A: -8
Option B: -3
Option C: -6
Option D: Limit does not exist

Correct option: Option C

General Feedback:

Despite the fact that $f(-8) = -3$, as the x-values approach -8 from either side, the $f(x)$ values approach -6.
Therefore, $\lim_{x \to -8} f(x) = -6$.

Specific Feedback:

Option A: This is the value where you need to evaluate the limit.
Option B: This is the function's value at $x = -8$, that is $f(-8) = -3$

Option C: Key
Option D: It is possible for the limit of a function to exist at a point, though the limit of the function and the value of the function at the point is different.