CSE431/531, Problem Set 1 Due Fri 9/21, in class

Note: There are 6 problems with a total of 130 points. You are required to do all the problems. In the problems, the base of $\log n$ is 2.

1. ($7 \times 5 = 35$ points) Let $f(n)$, $g(n)$, and $h(n)$ be functions from N^+ to R^+. Prove or disprove the following assertions. To disprove, you only need to give a counter example for the functions which make the assertion false.
   (a) $O(O(f(n))) = O(f(n))$
   (b) $O(\Theta(f(n))) = O(f(n))$
   (c) $\Theta(O(f(n))) = \Theta(f(n))$
   (d) $\Omega(O(f(n))) = O(\Omega(f(n)))$
   (e) If $f(n) = \Theta(g(n) + h(n))$, then $f(n) = \Omega(g(n))$ and $f(n) = \Omega(h(n))$.
   (f) If $f(n) = \Theta(g(n))$, then $\log f(n) = \Theta(\log g(n))$
   (g) $f(n) + g(n) = \Theta(\max(f(n), g(n)))$

2. ($2 \times 5 = 10$ points) Use mathematical induction to prove the following.
   (a) $\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$.
   (b) The maximum number of regions generated by the intersections of n unit circles (i.e., all with radius 1) on a plane is $\Theta(n^2)$, where each region is bounded by some arcs of these circles with no other arc in its interior.

3. ($4 \times 5 = 20$ points) Prove or disprove the following assertions.
   (a) $n! = O(n^n)$
   (b) $\sum_{i=1}^{n} i \log i = \Theta(n^2 \log n)$
   (c) If $n = 2^k$, then $\sum_{i=0}^{k} \log(n/2^i) = \Theta(\log^2 n)$
   (d) $n^n = O(2^n)$

4. (15 points) Rank the following functions in asymptotically increasing order based on O-notation and justify your ordering: $n!$, $(lgn)^{lg(lgn)}$, $[lg(lgn)]^{lgn}$, $2^{n^{0.001}}$, $n^{1/lgn}$, $lg^*(lgn)$, $2^{\sqrt{2lgn}}$, 2^{2^n}, n^5, $\sqrt{lgn}$.

5. ($8 \times 5 = 40$ points) Find a closed form for each $T(n)$. You may assume that $T(1) = 1$.
   (a) $T(n) = T(n-1) + 3^n$
   (b) $T(n) = 4T(n/3) + n^{1.5}$
   (c) $T(n) = 5T(n/8) + n$
   (d) $T(n) = T(\sqrt{n}) + logn$
   (e) $T(n) = 1 + 2\sum_{i=1}^{n-1} T(i)$
   (f) $T(n) = 3T(n/2) + n\log^2 n$
   (g) $T(n) = 2T(n/2) + n/\log\log n$
   (h) $T(n) = \sqrt{n}T(\sqrt{n}) + n^2$

6. (10 points) The sequence $< a_n >$ is defined for $n \geq 0$ by $a_0 = 3$, $a_1 = 5$, and $a_{n+2} = 8a_{n+1} - 15a_n$ for $n \geq 0$. Find a closed form for a_n.