Graph Algorithms

- Many problems in CS can be modeled as graph problems.
- Algorithms for solving graph problems are fundamental to the field of algorithm design.

Definition

- A graph $G = (V, E)$ consists of a vertex set V and an edge set E. $|V| = n$ and $|E| = m$.
- Each edge $e = (x, y) \in E$ is an unordered pair of vertices.
- If $(u, v) \in E$, we say v is a neighbor of u.
- The degree $deg(u)$ of a vertex u is the number of edges incident to u.

Fact

$$\sum_{v \in V} deg(v) = 2m$$

This is because, for each $e = (u, v)$, e is counted twice in the sum, once for $deg(v)$ and once for $deg(u)$.

Directed Graphs

Definition

- If the two end vertices of e are ordered, the edge is directed, and we write $e = x \rightarrow y$.
- If all edges are directed, then G is a directed graph.
- The in-degree $deg_{in}(u)$ of a vertex u is the number of edges that are directed into u.
- The out-degree $deg_{out}(u)$ of a vertex u is the number of edges that are directed from u.

Directed Graphs

Fact

$$\sum_{v \in V} deg_{in}(v) = \sum_{v \in V} deg_{out}(v) = m$$

This is because, for each $e = (u \to v)$, e is counted once ($deg_{in}(v)$) in the sum of in-degrees, and once ($deg_{out}(u)$) in the sum of out-degrees.

- The numbers n $(= |V|)$ and m $(= |E|)$ are two important parameters to describe the size of a graph.
- It is easy to see $0 \leq m \leq n^2$.
- It is also easy to show: if G is a tree (namely undirected, connected graph with no cycles), them $m = n - 1$.
- If m is close to n, we say G is sparse. If m is close to n^2, we say G is dense.
- Because n and m are rather independent to each other, we usually use both parameters to describe the runtime of a graph algorithm. Such as $O(n + m)$ or $O(n^{1/2}m)$.

Graph Representations

We mainly use two graph representations.

Adjacency Matrix Representation

We use a 2D array $A[1..n, 1..n]$ to represent $G = (V, E)$:

$$A[i,j] = \begin{cases} 1 & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

- Sometimes, there are other information associated with the edges. For example, each edge $e = (v_i, v_j)$ may have a weight $w(e) = w(v_i, v_j)$ (for example, MST). In this case, we set $A[i,j] = w(v_i, v_j)$.
- For undirected graph, A is always symmetric.
- The Adjacency Matrix Representation for directed graph is similar. $A[i,j] = 1$ (or $w(v_i, v_j)$ if G has edge weights) iff $v_i \to v_j \in E$.
- For directed graphs, $A[*, *]$ is not necessarily symmetric.

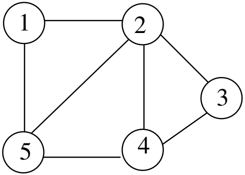
Graph Representations

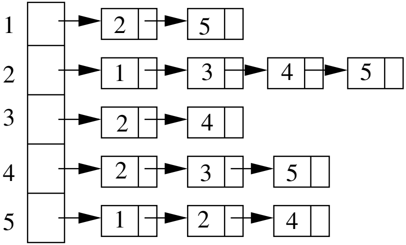
Adjacency List Representation

- For each vertex $v \in V$, there's a linked list $Adj[v]$. Each entry of $Adj[v]$ is a vertex w such that $(v, w) \in E$.

- If there are other information associated with the edges (such as edge weight), they can be stored in the entries of the adjacency list.
- For undirected graphs, each edge $e = (u, v)$ has two entries in this representation, one in $Adj[u]$ and one in $Adj[v]$.
- The Adjacency List Representation for directed graphs is similar. For each edge $e = u \to v$, there is an entry in $Adj[u]$.
- For directed graphs, each edge has only one entry in the representation.

Example





	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

Comparisons of Representations

Graph algorithms often need the representation to support two operations.

Neighbor Testing

Given two vertices u and v, is $(u, v) \in E$?

Neighbor Listing

Given a vertex u, list all neighbors of u.

When deciding which representation to use, we need to consider:

- The space needed for the representation.
- How well the representation supports the two basic operations.
- How easy to implement.

Comparisons of Representations

Adjacency List

- Space:
  - Each entry in the list needs $O(1)$ space.
  - Each edge has two entries in the representation. So there are totally $2m$ entries in the representation.
  - We also need $O(n)$ space for the headers of the lists.
  - Total Space: $\Theta(m+n)$.
- Neighbor Testing: $O(deg(v))$ time. (We need to go through $Adj(v)$ to see if another vertex u is in there.)
- Neighbor Listing: $O(deg(v))$. (We need to go through $Adj(v)$ to list all neighbors of v.)
- More complex.

Comparisons of Representations

Adjacency Matrix

- Space: $\Theta(n^2)$, independent from the number of edges.
- Neighbor Testing: $O(1)$ time. (Just look at $A[i,j]$.)
- Neighbor Listing: $\Theta(n)$. (We have to look the entire row i in A to list the neighbors of the vertex i.)
- Easy to implement.

- If an algorithm needs neighbor testing more often than the neighbor listing, we should use Adj Matrix.
- If an algorithm needs neighbor testing less often than the neighbor listing, we should use Adj List.
- If we use Adj Matrix, the algorithm takes at least $\Omega(n^2)$ time since even set up the representation data structure requires this much time.
- If we use Adj List, it is possible the algorithm can run in linear $\Theta(m+n)$ time.

Breadth First Search (BFS)

BFS is a simple algorithm that travels the vertices of a given graph in a systematic way. Roughly speaking, it works like this:

- It starts at a given starting vertex s.
- From s, we visits all neighbors of s.
- These neighbors are placed in a queue Q.
- Then the first vertex u in Q is considered. All neighbors of u that have not been visited yet are visited, and are placed in Q ...
- When finished, it builds a spanning tree (called BFS tree).

Before describing details, we need to pick a graph representation. Because we need to visit all neighbors of a vertex, it seems we need the neighbor listing operation. So we use Adj list representation.

Input: An undirected graph $G = (V, E)$ given by Adj List.
s: the starting vertex.

Basic Data Structures: For each vertex $u \in V$, we have

- Adj$[u]$: the Adj list for u.
- color$[u]$: It can be one of the following;
  - white, (u has not been visited yet.)
  - grey, (u has been visited, but some neighbors of u have not been visited yet.)
  - black, (u and all neighbors of u have been visited.)
- $\pi[u]$: the parent of u in the BFS tree.
- $d[u]$: the distance from u to the starting vertex s.

In addition, we also use a queue Q as mentioned earlier.

BFS: Algorithm

BFS(G, s)

$Q \leftarrow \emptyset$

for each $u \in V - \{s\}$ **do**

    $\pi[u] =$ NIL; $d[u] = \infty$; color[u]=white

$d[s] = 0$; color$[s]$=grey; $\pi[s] =$ NIL

Enqueue(Q, s)

while $Q \neq \emptyset$ **do**

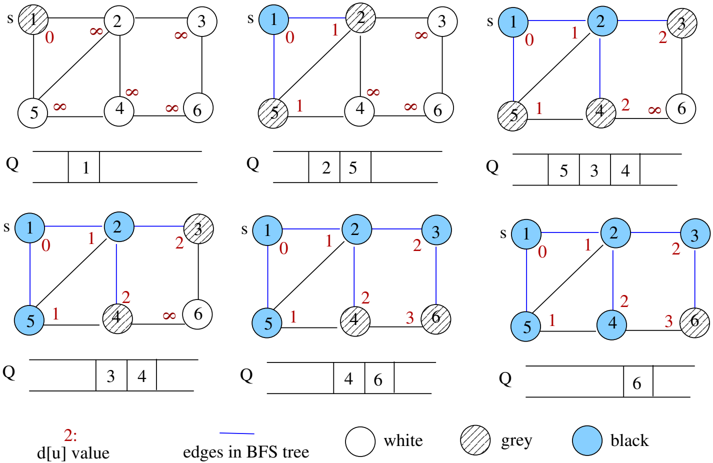
    $u \leftarrow$**Dequeue**(Q)

    for each $v \in Adj[u]$ **do**

        if color$[v]$ $=$ white

        then color$[v]$ $=$ grey; $d[v] \leftarrow d[u] + 1$; $\pi[v] \leftarrow u$; **Enqueue**(Q, v)

    color$[u]$ $=$ black


s
Q
2:
d[u] value
edges in BFS tree
white
grey
black

BFS: Algorithm

- BFS is not unique.
- The execution depends on the order in which the neighbors of a vertex i appear in Adj(i).
- In the example above, the neighbors of i appear in Adj(i) in increasing order.
- If the order is different, then the progress of the BFS algorithm would be different. And the BFS tree T constructed by the algorithm would be different.
- However, regardless of which order we use, the properties of the BFS algorithm and BFS tree are always true.

BFS: Analysis

- Lines 1 and 5: The queue operations take $O(1)$ time.
- Line 2-3: Loop takes $\Theta(n)$ time.
- Lines 4: $O(1)$ time.
- Lines 6-11:
  - Each vertex is enqueued and dequeued exactly once.
  - Since each queue operation takes $O(1)$ time, the total time needed for all queue operations is $\Theta(n)$.
  - Lines 8-10: Each item in $Adj[u]$ is processed once.
  - When an item is processed, $O(1)$ operations are needed.
  - So the total time needed is $\Theta(m) \cdot \Theta(1) = \Theta(m)$.
- Add everything together:

BFS algorithm takes $\Theta(n + m)$ time.

BFS: Main Property

Theorem

Let $G = (V, E)$ be a graph. Let $d[u]$ be the value computed by BFS algorithm. Then for any $(u, v) \in E$, $|d[u] - d[v]| \leq 1$.

Proof: First, we make the following observations:

- Each vertex $v \in V$ is enqueued and dequeued exactly once.
- Initially color$[v]$ =white. When it is enqueued, color$[v]$ becomes grey. When it is dequeued, color$[v]$ becomes black. The color remains black until the end.
- The $d[v]$ value is set when v is enqueued. It is never changed again.
- At any moment during the execution, the vertices in Q consist of two parts, Q_1 followed by Q_2 (either of them can be empty).
  - For all $w \in Q_1$, $d[w] = k$ for some k.
  - For all $x \in Q_2$, $d[x] = k + 1$.

BFS: Main Property

Without loss of generality, suppose that u is visited by the algorithm before v.
Consider the while loop in BFS algorithm, when u is at the front of Q.
There are two cases.
Case 1: color$[v]$ $=$ white at that moment.

- Since $v \in Adj[u]$, the algorithm set $d[v] = d[u] + 1$, and color$[v]$=grey.
- $d[v]$ is never changed again. Thus $d[v] - d[u] = 1$.

Case 2: color$[v]$ $=$ grey at that moment.

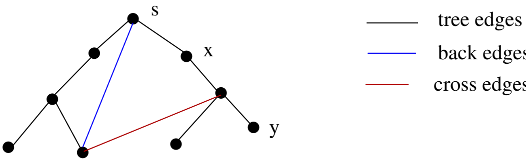
- Then v is in Q at that moment.
- By the previous observation, $d[u] = k$ for some k, and $d[v] = k$ or $k + 1$. Thus $d[v] - d[u] \leq 1$.

BFS: Main Property

Definition

Let $G = (V, E)$ be a graph and T a spanning tree of G rooted at the vertex s. Let x and y be two vertices. Let (u, v) be an edge of G.

- If x is on the path from y to s, we say x is an ancestor of y, and y is a descendent of x
- If $(u, v) \in T$, we say (u, v) is a tree edge.
- If $(u, v) \notin T$ and u is an ancestor of v, we say (u, v) is a back edge.
- If neither u is an ancestor of v, nor v is an ancestor of u, we say (u, v) is a cross edge.



BFS: Main Property

Theorem

Let T be the BFS tree constructed by the BFS algorithm. Then there are no back edges for T.

Proof: Suppose there is an back edge (u, v) for T. Then $|d[u] - d[v]| \geq 2$. This is impossible.

Shortest Path Problem

Let $G = (V, E)$ be a graph and s a vertex of G. For each $u \in V$, let $\delta(s, u)$ be the length of the shortest path between s and u.
Problem: For all $u \in V$, find $\delta(s, u)$ and the shortest path between s and u.

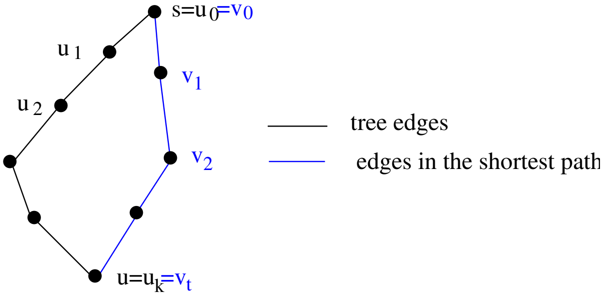
Theorem

Let $d[u]$ be the value computed by BFS algorithm and T the BFS tree constructed by BFS algorithm. Then for each vertex $u \in V$,

- $d[u] = \delta(s, u)$.
- The tree path in T from u to s is the shortest path.

Proof: Let $P = \{s = u_0, u_1, \ldots, u_k = u\}$ be the path from s to u in the BFS tree T. Then: $d[u] = d[u_k] = k$, $d[u_{k-1}] = k - 1$,$d[u_{k-2}] = k - 2 \ldots$
Suppose $P' = \{s = v_0, v_1, v_2, \ldots, v_t\}$ is the shortest path from s to u in G. We need to show $k = t$.
Toward a contradiction, suppose $t < k$. Then there must exist $(v_i, v_{i+1}) \in P'$ such that $|d[v_i] - d[v_{i+1}]| \geq 2$. This is impossible.

Shortest Path Problem



BFS algorithm solves the Single Source Shortest Path problem in $\Theta(n+m)$ time.

Connectivity Problem

Definition

- A graph $G = (V, E)$ is connected if for any two vertices u and v in G, there exists a path in G between u and v.
- A connected component of G is a maximal subgraph of G that is connected.
- G is connected if and only if it has exactly one connected component.

Connectivity Problem

Given $G = (V, E)$, is G a connected graph?
If not, find the connected components of G.

We can use BFS algorithm to solve the connectivity problem.

Connectivity Problem

In the **BFS** algorithm, delete the lines 2-3 (initialization of vertex variables).

Connectivity$(G = (V, E))$

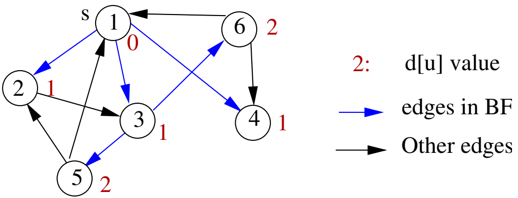
1. **for** each $i \in V$ **do**
2.   color$[i]$ $=$white; $d[i] = \infty$; $\pi[i] = nil$;
3. count = 0;   (count will be the number of connected components)
4. **for** $i = 1$ **to** n **do**
5.   **if** color$[i]$ $=$white **then**
6.   **call BFS**(G, i);   count = count+1
7. **output** count;
8. **end**

Connectivity Problem

- This algorithm outputs count, the number of connected components.
- If count$= 1$, G is connected. The algorithm also constructs a BFS tree.
- If count> 1, G is not connected. The algorithm also constructs a BFS spanning forest F of G. F is a collection of trees.
- Each tree corresponds to a connected component of G.

BFS for Directed Graphs

BFS algorithm can be applied to directed graphs without any change.



Definition

Let $G = (V, E)$ be a directed graph, T a spanning tree rooted at s. An edge $e = u \to v$ is called:

- tree edge if $e = u \to v \in T$.
- backward edge if u is a decedent of v.
- forward edge if u is an ancestor of v.
- cross edge if u and v are unrelated.

BFS for Directed Graphs: Property

Theorem

Let $G = (V, E)$ be a directed graph. Let T be the BFS tree constructed by BFS algorithm. Then there are no forward edges with respect to T.

Theorem

Let $d[u]$ be the value computed by BFS algorithm and T the BFS tree constructed by BFS algorithm. Then for each vertex $u \in V$,

- The tree path in T from s to u is the shortest path.
- $d[u] =$ the length of the shortest path from s to u.

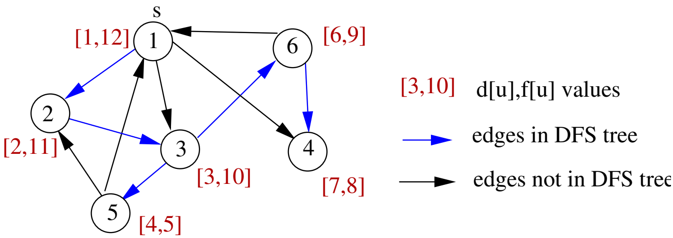
Depth First Search (DFS)

- Similar to BFS, Depth First Search (DFS) is another systematic way for visiting the vertices of a graph.
- It can be used on directed or undirected graphs. We discuss DFS for directed graph first.
- DFS has special properties, making it very useful in several applications.
- As a high level description, the only difference between BFS and DFS: replace the queue Q in BFS algorithm by a stack S. So it works like this:

High Level Description of DFS

- Start at the starting vertex s.
- Visit a neighbor u of s; visit a neighbor v of u ...
- Go as far as you can go, until reaching a dead end.
- Backtrack to a vertex that still has unvisited neighbors, and continue

DFS: Example



s
[1,12]
1
6
[6,9]
2
[2,11]
3
4
[3,10]
[7,8]
5
[4,5]
[3,10] d[u],f[u] values
edges in DFS tree
edges not in DFS tree

DFS: Recursive algorithm

- It is easier to describe the DFS by using a recursive algorithm.
- DFS also computes two variables for each vertex $u \in V$:
  - $d[u]$: The time when u is "discovered", i.e. pushed on the stack.
  - $f[u]$: the time when u is "finished", i.e. popped from the stack.
- These variables will be used in applications.

DFS: Recursive algorithm

DFS(G)

1. **for** each vertex $u \in V$ **do**
2. color$[u] \leftarrow$ white; $\pi[u] =$ NIL
3. $time \leftarrow 0$
4. **for** each vertex $u \in V$ **do**
5. **if** color$[u] =$ white **then DFS-Visit**(u)

DFS-Visit(u)

1. color$[u] \leftarrow$ grey; $time \leftarrow time + 1$; $d[u] \leftarrow time$
2. **for** each vertex $v \in Adj[u]$ **do**
3. **if** color$[v] =$ white
4. **then** $\pi[v] \leftarrow u$; **DFS-Visit**(v)
5. color$[u] \leftarrow$ black
6. $f[u] \leftarrow time \leftarrow time + 1$

DFS: Properties

Let T be the DFS tree of G by DFS algorithm. Let $[d[u], f[u]]$ be the time interval computed by DFS algorithm. Let $u \neq v$ be any two vertices of G.

- The intervals of $[d[u], f[u]]$ and $[d[v], f[v]]$ are either disjoint or one is contained in another.
- $[d[u], f[u]]$ is contained in $[d[v], f[v]]$ if and only if u is a descendent of v with respect to T.

DFS: Properties

Classification of Edges

Let $G = (V, E)$ be a directed graph and T a spanning tree of G. The edge $e = u \rightarrow v$ of G can be classified as:

- tree edge if $e = u \rightarrow v \in T$.
- back-edge if $e \notin T$ and v is an ancestor of u.
- forward-edge if $e \notin T$ and u is an ancestor of v.
- cross-edge if $e \notin T$, v and u are unrelated with respect to T.

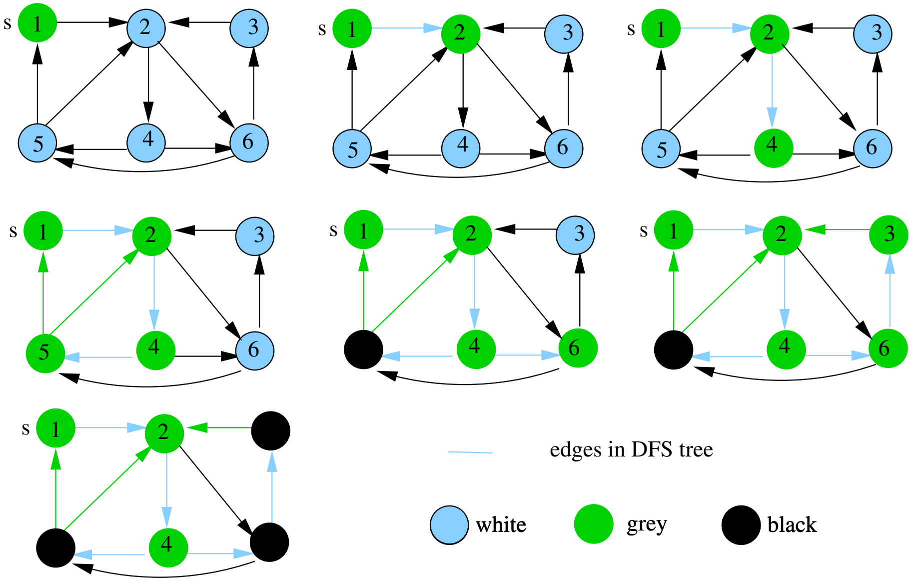
DFS: Properties

Classification of Edges

Let $G = (V, E)$ be a directed graph and T the spanning tree of G constructed by DFS algorithm. The classification of the edges can be done as follows.

- When $e = u \to v$ is first explored by DFS, color e by the color$[v]$.
- If color$[v]$ is white, then e is white and is a tree edge.
- If color$[v]$ is grey, then e is grey and is a back-edge.
- If color$[v]$ is black, then e is black and is either a forward- or a cross-edge.

For DFS tree of directed graphs, all four types of edges are possible.


edges in DFS tree
white
grey
black

DFS: Applications

Definition

A directed graph $G = (V, E)$ is called a directed acyclic graph (DAG for short) if it contains no directed cycles.

DAG Testing

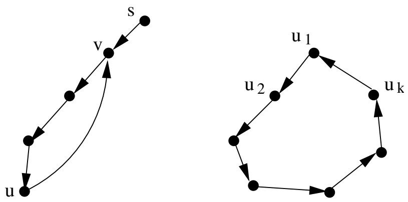
Given a directed graph $G = (V, E)$, test if G is a DAG or not.

Theorem

Let G be a directed graph, and T the DFS tree of G. Then G is DAG $\Longleftrightarrow$ there are no back edges.

DFS: Applications

Proof: $\Longrightarrow$ Suppose $e = u \to v$ is a back edge. Let P be the path in T from v to u. Then the directed path P followed by $e = u \to v$ is a directed cycle.



$\Longleftarrow$ Suppose $C = u_1 \to u_2 \to \cdots u_k \to u_1$ is a directed cycle. Without loss of generality, assume u_1 is the first vertex visited by DFS. Then, the algorithm visits $u_2, u_3, \ldots u_k$, before it backtrack to u_1. So $u_k \to u_1$ is a back edge.

DAG Testing in $\Theta(n+m)$ time

1. Run DFS on G. Mark the edges “white”, “grey” or “black”,
2. If there is a grey edge, report “G is not a DAG”. If not “G is a DAG”.

Topological Sort

Let $G = (V, E)$ be a DAG. A topological sort of G assigns each vertex $v \in V$ a distinct number $L(v) \in [1..n]$ such that if $u \to v$ then $L(u) < L(v)$.

Note: If G is not a DAG, topological sort cannot exist.

Application

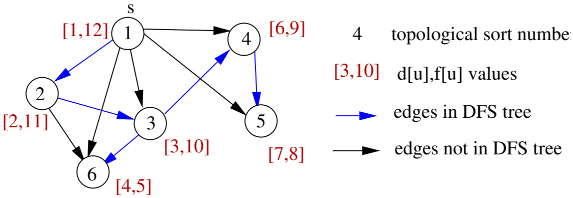
- The directed graph $G = (V, E)$ specifies a job flow chart.
- Each $v \in V$ is a job.
- If $u \to v$, then the job u must be done before the job v.
- A topological sort specifies the order to complete jobs.

Topological Sort

We can use DFS to find topological sort.

Topological-Sort-by-DFS(G)

1. Run DFS on G.
2. Number the vertices by decreasing order of $f[v]$ value. (This can be done as follows: During DFS, when a vertex v is finished, insert v in the front of a linked list.)



Clearly, this algorithm takes $\Theta(m+n)$ time.

Strong Connectivity

Definition

- A directed graph $G = (V, E)$ is strongly connected if for any two vertices u and v in V, there exists a directed path from u to v.
- A strongly connected component of G is a maximal subgraph of G that is strongly connected.

Strong Connectivity Problem

Given a directed graph G, find the strongly connected components of G.

Note: G is strongly connected if and only if it has exactly one strongly connected component.

Strong Connectivity

Application: Traffic Flow Map

- $G = (V, E)$ represents a street map.
- Each $v \in V$ is an intersection.
- Each edge $u \to v$ is a 1-way street from the intersection u to the intersection v.
- Can you reach from any intersection to any other intersection?
- This is so $\Longleftrightarrow$ G is strongly connected.
- All intersections within each connected component can reach each other.

This problem can be solved by using DFS. Without it, it would be hard to solve efficiently.

Strong Connectivity

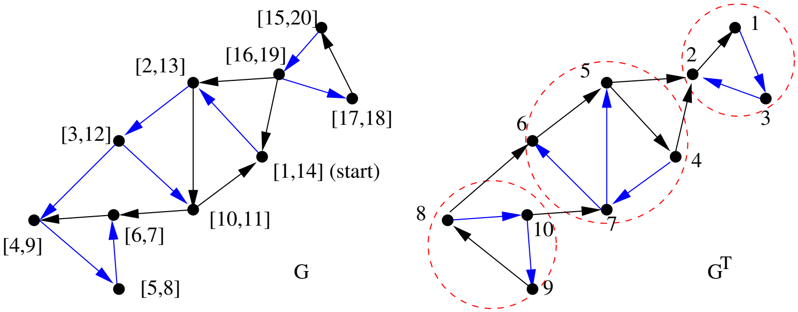
Strong-Connectivity-by-DFS(G)

1. Run DFS on G, compute $f[u]$ for all $u \in V$,
2. Order the vertices by decreasing $f[v]$ values.
3. Construct the transpose graph G^T, which is obtained from G by reversing the direction of all edges.
4. Run DFS on G^T, the vertices are considered in the order of decreasing $f[v]$ values.
5. The vertices in each tree in the DFS forest correspond to a strongly connected component of G.

Analysis:

- Steps 1 and 2: $\Theta(n+m)$ (step 2 is a part of step 1.)
- Step 3: $\Theta(n+m)$ (how?)
- Step 4 and 5: $\Theta(n+m)$ (step 5 is part of step 4.)

Strong Connectivity: Example



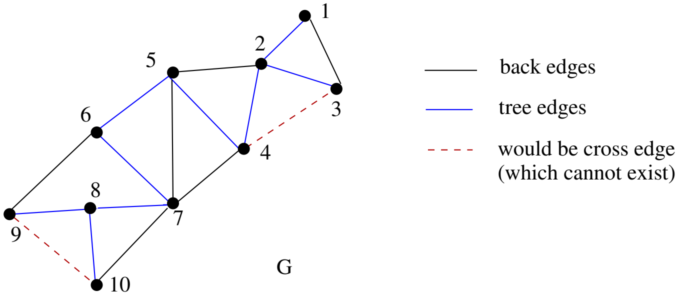
DFS for Undirected Graphs

- DFS algorithm can be used on an undirected graph $G = (V, E)$ without any change.
- It construct a DFS tree T of G.
- Recall that: for an undirected graph $G = (V, E)$ and a spanning tree T of G, the edges of G can be classified as:
  - tree edges
  - back edges
  - cross edges

Theorem

Let G be an undirected graph, and T the DFS tree of G constructed by DFS algorithm. Then there are no cross edges.

DFS for Undirected Graphs



Summary: Edge Types

For Directed Graphs

	Tree	Forward	Backward	Cross
BFS	yes	no	yes	yes
DFS	yes	yes	yes	yes

For Undirected Graphs

	Tree	Back-edge	Cross
BFS	yes	no	yes
DFS	yes	yes	no

Biconnectivity Problem

Definition

Let $G = (V, E)$ be an undirected connected graph.

- A vertex $v \in V$ is a cut vertex (also called articulation point) if deleting v and its incident edges disconnects G.
- G is biconnected if it is connected and has no cut vertices.
- A biconnected component of G is a maximal subgraph of G that is biconnected.
- G is biconnected if and only if it has exactly one biconnected component.

Biconnectivity Problem

Given an undirected graph $G = (V, E)$, is G biconnected?
If not, find the cut vertices and the biconnected components of G.

Biconnectivity Problem

Application

- G represents a computer network.
- Each vertex is a computer site.
- Each edge is a communication link.
- If v is a cut vertex, then the failure of v will disconnect the whole network.
- The network can survive any single site failure if and only if G is biconnected.

Simple-Biconnectivity(G)

1. **for** each vertex $v \in V$ **do**
2. delete v and its incident edges from G
3. test if $G - \{v\}$ is connected

This algorithm takes $\Theta(n) \times \Theta(n+m) = \Theta(n(n+m))$ time.

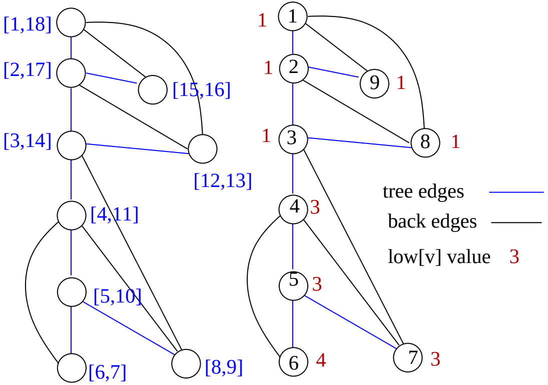
By using DFS, the problem can be solved in $O(n+m)$ time.

- Let T be the DFS tree of G.
- Re-number the vertices by increasing $d[v]$ values.
- For each vertex v, define:

  low$[v]$ $=$ the smallest vertex that can be reached from v or a descendent of v through a back edge.

- If v is a leaf of T, then $\text{low}[v] = \min \left\{ \begin{array}{c} v \\ \{w \mid (v,w) \text{ is a back-edge}\} \end{array} \right\}$
- If v is not a leaf of T, then $\text{low}[v] = \min \left\{ \begin{array}{c} v \\ \{w \mid (v,w) \text{ is a back-edge}\} \\ \{\text{low}[t] \mid t \text{ is a son of } v\} \end{array} \right\}$

Biconnectivity Problem



The vertices have been renumbered by increasing d[v] values.

In this figure, low means closer to the root. So the root is the lowest vertex. low$[v]$ is the lowest vertex that can be reached from v or a descendent of v thru a single back edge.

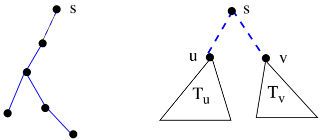
Biconnectivity Problem

Theorem

Let T be the DFS of $G = (V,E)$ rooted at the vertex s.

1. s is a cut vertex $\Longleftrightarrow$ s has at least two sons in T.
2. A vertex $a \neq s$ is a cut vertex $\Longleftrightarrow$ a has a son b such that $low[b] \geq a$.

Proof of (1): Suppose s has only one son. After deleting s, all other vertices are still connected by the remaining edges of T. So s is not a cut vertex.

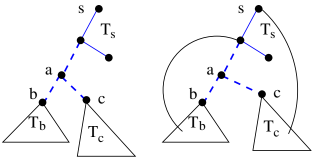


Suppose s has at least two sons u and v (there may be more). Let T_u be the subtree of T rooted at u and T_v be the subtree of T rooted at v. Because there are no cross edges, no edges connect T_u with T_v. So after s is deleted, T_u and T_v become disconnected. Hence s is a cut vertex.

Biconnectivity Problem

Proof of (2): Let T_s be the subtree of T above the vertex a. Let $b, c \dots$ be the sons of a. Let $T_b, T_c, \dots$ be the subtree of T rooted at $b, c \dots$

- Suppose a has a son b with low$[b] \geq a$. Because low$[b] \geq a$, no vertex in T_b is connected to T_s. Because there are no cross edges, no edges connect vertices in T_b and T_c. So after a is deleted, T_b is disconnected from the rest of G. So a is a cut vertex.



- Suppose for every son b of a we have low$[b] < a$. This means that there is a back edge connecting a vertex in T_b to a vertex in T_s. So after a is deleted, all subtrees $T_b, T_c, \dots$ are still connected to T_s, and G remains connected. So a is not a cut vertex.

We can now describe the algorithm. For conceptual clarity, the algorithm is divided into several steps. Actually, all steps can be and should be incorporated into a single DFS run.

Biconnectivity-by-DFS(G)

1. Run DFS on G
2. Renumber the vertices by increasing $d[*]$ values.
3. For all $u \in V$, compute low$[u]$ as described before.
4. Identify the cut vertices according to the conditions in the theorem.

Analysis

- Steps 1 and 2: takes $\Theta(m+n)$ time.
- Step 3: low$[u]$ is the minimum of k values:
  - the low$[*]$ values for all sons of u.
  - the values for each back-edge from u.
  - 1 for u itself, we charge this to the edge between u and its parent.
  - So $k = deg(u)$.
  - We compute low$[*]$ in post order. When computing low$[u]$, all values needed have been computed already. So it takes $\Theta(deg(u))$ time to compute low$[u]$.
  - So the total time needed to compute low$[u]$ for all vertices is Θ of the number of edges of G. This is $\Theta(m)$.
- Step 4: The total time needed for checking these conditions for all vertices is $\Theta(n)$.

The Biconnectivity problem can be solved in $\Theta(n+m)$ time

Notes

- The DFS based Biconnectivity algorithm was discovered by Tarjan and Hopcroft in 1972. (See Problem 22-2, Page 558).
- They advocated the use of adjacent list representation over the adjacent matrix representation for solving complex graph problems in linear (i.e. $O(n+m)$) time.
- This DFS algorithm is a good example. Without using adjacent list representation, the problem would take at least $\Theta(n^2)$ time to solve.