CSE 531 Homework 1 Solution

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1. (a) True. Since $f(n) \in O(f(n))$, clearly we have $O(f(n)) \subset O(O(f(n)))$. For the opposite, suppose $g(n) \in O(O(f(n)))$, that is $\exists\ c_1 > 0$, $h(n) \in O(f(n))$, s.t $g(n) \leq c_1 h(n)$ as $n \to \infty$. Since $h(n) \in O(f(n))$, we have some $c_2 > 0$, s.t $h(n) \leq c_2 f(n)$ as $n \to \infty$, thus $g(n) \leq c_1 \cdot c_2 f(n)$ as $n \to \infty$, i.e $g(n) \in O(f(n))$.

   (b) True. Since $f(n) \in \Theta(f(n))$, clearly we have $O(f(n)) \subset O(\Theta(f(n)))$. For the opposite, suppose $g(n) \in O(\Theta(f(n)))$, according to definition, $\exists h(n) \in \Theta(f(n))$, c_1, c_2, $c_3 > 0$, s.t $g(n) \leq c_1 h(n)$, $c_2 f(n) \leq h(n) \leq c_3 f(n)$ as $n \to \infty$. It gives $g(n) \leq c_1 \cdot c_3 f(n)$ as $n \to \infty$, i.e. $g(n) \in O(f(n))$.

   (c) False. The relation should be $\Theta(O(f(n)) \supsetneq \Theta(f(n))$. Let $f(n) = n$, $g(n) = 1 \in O(f(n))$. clearly $1 \in \Theta(1) \subset \Theta(O(f(n)))$ but $1 \notin \Theta(n) = \Theta(f(n))$ (since there doesn't exist $c > 0$ s.t $1 > c \cdot n$ as $n \to \infty$).

   (d) True. $\forall\ g(n)$, let $h(n) = \max\{g(n), f(n)\}$, $l(n) = \min\{g(n), f(n)\}$. We know $g(n) \leq \max\{g(n), f(n)\} \in \Omega(f(n))$, so $g(n) \in O(\Omega(f(n)))$. $g(n) \geq \min\{g(n), f(n)\} \in \Omega(O(f(n)))$, so $g(n) \in \Omega(O(f(n))$. So $\Omega(O(f(n))) = O(\Omega(f(n))) = \{\text{all the functions } : \mathbb{N}^+ \to \mathbb{R}^+\}$

   (e) True. By definition, $\exists c > 0$ s.t. $c(g(n) + h(n)) \leq f(n)$, since both $g(n)$ and $h(n)$ are positive, we have $c \cdot g(n) \leq f(n)$, $c \cdot h(n) \leq f(n)$ respectively. Therefore $f(n) = \Omega(g(n))$,$f(n) = \Omega(h(n))$.

   (f) False. Take $f(n) = 1+\frac{1}{n}$, $g(n) = 2$. $f(n) = \Theta(1)$ but $\log f(n) = \Theta(\frac{1}{n})$

   (g) True. Since both $f(n)$ and $g(n)$ are positive, clearly we have $\max(f(n), g(n)) \leq f(n) + g(n)$. For the other direction, WLOG we assume $f(n) \geq g(n)$, then $f(n) + g(n) \leq 2f(n) = 2\max(f(n), g(n))$

2. (a) When $n = 1$, it's trivially true. Now suppose when the statement holds for $n = k$:

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$$

   then for $n = k + 1$, we have:

$$\begin{aligned}
&\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} \\
=&1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!} \\
=&1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} \\
=&1 - \frac{1}{(k+2)!}
\end{aligned}$$

   It completes our proof.

(b) *Proof.* First we use mathematical induction to prove the maximum number of regions generated by the intersections of n unit circles on a plane is $O(n^2)$:
Given arbitrary n circles, let $r(k)$ be the number of regions generated by the first k circles. Then it's sufficient to show $r(k) \leq r(k-1)+\Theta(k)$ for all $k \leq n$.
Consider the intersection of the k-th circle (denoted as R_k) and the first $k-1$ circles. Since any two circle intersects at at most 2 points, there are at most $2(k-1)$ intersection points on R_k, which divides R_k into at most $2(k-1)$ many arcs. Any newly generated region must include at least one arc of R_k as its boundary, otherwise it's an old region. But for any arc of R_k, there are at most two regions include it as their boundary. Therefore, the newly generated region is upper-bounded by $4(k-1)$. Namely, $r(k) \leq r(k-1) + 4(k-1)$, implies $r(n) = O(n^2)$.

Second, to show the maximum number of regions generated by the intersections of n unit circles on a plane is $\Omega(n^2)$, we just need to provide a certain arrangement.
For convenience let's consider $2n$ unit circles under Cartesian coordinate system. The coordinates of the center of a circle solely determines its position. Place the first n circles so that the i-th circle centered at $(1 + i/n^2, 0)$. Place the rest of n circles so that the $n + i$-th circle centered at $(0, 1 + i/n^2)$. For sufficient large n, there are $\Theta(n^2)$ many regions inside the rectangle bounded by $(0, 0)$ and $(1/n, 1/n)$. □

3. (a) Stirling formula:

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n = O(n^n)$$

(b)

$$\begin{aligned}\int_1^n x \log x dx &= \frac{1}{2}x^2 \log x \mid_1^n - \int_1^n \frac{1}{2}xdx \\ &= \frac{1}{2}n^2 \log n + o(n^2 \log n) = \Theta(n^2 \log n)\end{aligned}$$

(c)

$$\begin{aligned}\sum_{i=0}^{k} &= \log 2^k + \log 2^{k-1} + \ldots + \log 2^1 + \log 2^0 \\ &= \log(2^k \cdot 2^{k-1} \cdot \ldots \cdot 2 \cdot 1) \\ &= \log(2^{k+(k-1)+\ldots+2+1}) = \log(2^{\frac{k(k+1)}{2}}) \\ &= \frac{k^2 + k}{2} = \frac{1}{2}(\log^2 n + \log n) = \Theta(\log^2 n)\end{aligned}$$

(d) Wrong. By limit test,

$$\lim_{n\to\infty}\frac{n^n}{2^n}=\lim_{n\to\infty}\frac{2^{n\log n}}{2^n}=\lim_{n\to\infty}2^{n(\log n-1)}=\infty$$

4.

$$n^{\frac{1}{\lg n}}, \lg^*(\lg(n)), \sqrt{(\lg(n))}, (\lg n)^{\lg(\lg n)}, 2^{\sqrt{2\lg n}}, n^5, [\lg(\lg n)]^{\lg n}, 2^{n^{0.001}}, n!, 2^{2^n}$$

$$n^{\frac{1}{\lg n}}=(2^{\lg n})^{\frac{1}{\lg n}}=2$$

Denote $k=\lg^*\lg n$, $f(k)=\lg n$, $f(k-1)=\lg\lg(n)$, $f(k-2)=\lg\lg\lg(n)\dots$.
Then we have $f(i-1)=\lg(f(i))$, $f(i)=2^{f(i-1)}$, for $i\in(1,\dots,k)$.
First, it's easy to see by induction $k\le f(k)$:

(a) $k=1$, $f(1)=2^{2^x}\ge 2$.

(b) Suppose $k=n>1$, $n\le f(n)$.

(c) For $k=n+1$, $f(n+1)=2^{f(n)}>2^n>n+1$.

Second, according to the definition of $\lg^*\lg n$, we have $0<f(0)<2$ and $\lg n=2^{f(k-1)}$.
So $k<2^{(k-1)/2}<2^{f(k-1)/2}=\sqrt{\lg n}$ for $n\to\infty$.
The above proves $n^{\frac{1}{\lg n}}\le\lg^*(\lg(n))\le\sqrt{(\lg(n))}$.

Let $x=\lg(n)$

$$\sqrt{(\lg(n))}=x^{1/2}=2^{1/2\lg x}$$

$$(\lg n)^{\lg(\lg n)}=2^{(\lg(\lg n))\cdot\lg(\lg n)}=2^{(\lg(x))^2}$$

$$2^{\sqrt{2\lg n}}=2^{\sqrt{2x}}$$

$$n^5=2^{5\lg n}=2^{5x}$$

$$[\lg(\lg n)]^{\lg n}=2^{[\lg(\lg(\lg n))]\cdot\lg n}=2^{[\lg(\lg x)]\cdot x}$$

$$[\lg(\lg n)]^{\lg n}=2^{[\lg(\lg(\lg n))]\cdot\lg n}\ll 2^{n^{0.001}}$$

Remember $\lg x\ll x^{\varepsilon}$ for $\forall\varepsilon>0$, we can get

$$\sqrt{(\lg(n))}\ll(\lg n)^{\lg(\lg n)}\ll 2^{\sqrt{2\lg n}}\ll n^5\ll[\lg(\lg n)]^{\lg n}\ll 2^{n^{0.001}}$$

By Stirling's formula, $n!\sim\sqrt{2\pi n}(\frac{n}{e})^n$.

$$\lg 2^{n^{0.001}}=n^{0.001}$$

$$\lg n! = \frac{1}{2}\lg(2\pi n) + n\lg(\frac{n}{e})$$
$$\lg 2^{2^n} = 2^n$$

Since $n^{0.001} \le n\lg(\frac{n}{e})$, $\frac{1}{2}\lg(2\pi n) + n\lg(\frac{n}{e}) \le 2^n$ when $n \to \infty$, we have $2^{n^{0.001}} \le n! \le 2^{2^n}$.

5. (a) $T(n) = T(n-1) + 3^n$

$$T(n) = T(n-1) + 3^n = T(n-2) + 3^{n-1} + 3^n = \cdots = T(1) + \sum_{i=2}^{n} 3^i$$
$$= 1 + \frac{3^{n+1}-3}{2} = \frac{3^{n+1}-1}{2}$$

(b) Assume $n = 3^k$, from $T(n) = 4T(n/3) + n^{1.5}$ we have:

$$\begin{aligned}
T(n) = T(3^k) &= 4T(3^{k-1}) + 3^{2k} = 4(4T(3^{k-2}) + 3^{1.5(k-1)}) + 3^{1.5k} \\
&= 4^2T(3^{k-2}) + 4\cdot 3^{1.5(k-1)} + 3^{1.5k} \\
&= \cdots \\
&= 4^kT(1) + \sum_{i=1}^{k} 4^{k-i}\cdot 3^{1.5i} \\
&= 4^k(1 + \sum_{i=1}^{k}\left(\frac{3\sqrt{3}}{4}\right)^i) \\
&= 4^k\frac{\left(\frac{3\sqrt{3}}{4}\right)^{k+1} - 1}{\frac{3\sqrt{3}}{4} - 1} \\
&= \Theta(3^{1.5k}) \\
&= \Theta(n^{1.5})
\end{aligned}$$

or using master theorem.

(c) Assume $n = 8^k$, from $T(n) = 5T(n/8) + n$ we have:

$$\begin{aligned}
T(n) = T(8^k) &= 5T(8^{k-1}) + 8^k \\
&= 5(5T(8^{k-2}) + 8^{k-1}) + 8^k \\
&= 5^2T(8^{k-2}) + 5 \cdot 8^{k-1} + 8^k \\
&= \cdots \\
&= 5^kT(1) + \sum_{i=1}^{k} 5^{k-i} \cdot 8^i \\
&= 5^k(1 + \sum_{i=1}^{k} (\frac{8}{5})^i) \\
&= 5^k \frac{\left(\frac{8}{5}\right)^{k+1} - 1}{\frac{8}{5} - 1} \\
&= \Theta(8^k) \\
&= \Theta(n)
\end{aligned}$$

or using master theorem.

(d) Assume $n = a^{2^k}$, then $\log n = 2^k \log a$

$$\begin{aligned}
T(n) &= T(\sqrt{n}) + \log n = T(n^{\frac{1}{4}}) + \log \sqrt{n} + \log n \\
&= \cdots = T(n^{\frac{1}{2^k}}) + \log n^{\frac{1}{2^{k-1}}} + \log n^{\frac{1}{2^{k-2}}} + \cdots + \log \sqrt{n} + \log n \\
&= T(a) + \log n^{\sum_{i=0}^{k-1}(\frac{1}{2})^i} = T(a) + \log n^{\frac{1-(\frac{1}{2})^k}{1-\frac{1}{2}}} = T(a) + (2 - (\frac{1}{2})^{k-1}) \log n \\
&= T(a) + 2\log n - 2\log a
\end{aligned}$$

(e) $T(n) - T(n-1) = 2T(n-1)$, so $T(n) = 3T(n-1) = 3^2T(n-2) = \cdots = 3^nT(1) = 3^n$.

(f) Assume $n = 2^k$, from $T(n) = 3T(n/2) + n\log^2 n$ we have:

$$
\begin{aligned}
T(2^k) &= 3T(2^{k-1}) + 2^k \cdot k^2 \\
&= 3(3T(2^{k-2}) + 2^{k-1}(k-1)^2 + 2^k \cdot k^2 \\
&= 3^2T(2^{k-2}) + 3 \cdot 2^{k-1}(k-1)^2 + 2^k \cdot k^2 \\
&= \cdots \\
&= 3^kT(1) + \sum_{i=1}^{k} 3^{k-i}2^i \cdot i^2 \\
&= 3^k + 3^k \cdot \sum_{i=1}^{k} (\frac{2}{3})^i \cdot i^2 \\
&= 3^k(1 + \cdot o(3^k)) \\
&= \Theta(3^k) \\
&= \Theta(n^{\log_2 3})
\end{aligned}
$$

or using master theorem.

(g) Assume $n = 2^k$, from $T(n) = 2T(n/2) + \frac{n}{\log\log n}$ we have:

$$
\begin{aligned}
T(2^k) &= 2T(2^{k-1}) + \frac{2^k}{\log k} \\
&= 2(2T(2^{k-2}) + \frac{2^{k-1}}{\log(k-1)}) + \frac{2^k}{\log k} \\
&= 2^2T(2^{k-2}) + \frac{2^k}{\log(k-1)} + \frac{2^k}{\log k} \\
&= \cdots \\
&= 2^kT(1) + \sum_{i=1}^{k} \frac{2^k}{\log i} \\
&= 2^k + 2^k \sum_{i=1}^{k} \frac{1}{\log i}
\end{aligned}
$$

Using integral method, $\sum_{i=1}^{k} \frac{1}{\log i}$ is bounded by $\ln 2 \cdot \int_2^k \frac{1}{\ln x}dx$ which is called the Logarithmic integral function, or $li(x)$. It's asymptotic behavior is $li(x) = \Theta(x/\ln x)$. Therefore $\sum_{i=1}^{k} \frac{1}{\log i} = \Theta(k/\ln k)$. $T(n) = \Theta(n\log n/\log\log n)$.

(h) Assume $n = 2^{2^k}$, from $T(n) = \sqrt{n}T(\sqrt{n}) + n^2$ we have:

$$
\begin{aligned}
T(2^{2^k}) &= 2^{2^{k-1}}T(2^{2^{k-1}}) + 2^{2^k+2^k} \\
&= 2^{2^{k-1}}(2^{2^{k-2}}T(2^{2^{k-2}}) + 2^{2^k}) + 2^{2^k+2^k} \\
&= 2^{2^{k-1}+2^{k-2}}T(2^{2^{k-2}}) + 2^{2^k+2^{k-1}} + 2^{2^k+2^k} \\
&= 2^{2^k-2^{k-2}}T(2^{2^{k-2}}) + 2^{2^k+2^{k-1}} + 2^{2^k+2^k} \\
&= 2^{2^k-2^{k-2}}(2^{2^{k-3}}T(2^{2^{k-3}}) + 2^{2^{k-1}}) + 2^{2^k+2^{k-1}} + 2^{2^k+2^k} \\
&= 2^{2^k-2^{k-3}}T(2^{2^{k-3}}) + 2^{2^k+2^{k-2}} + 2^{2^k+2^{k-1}} + 2^{2^k+2^k} \\
&= \cdots \\
&= 2^{2^k-1}T(2) + \sum_{i=1}^{k} 2^{2^k+2^i} \\
&= 2^{2^k}/2 \cdot T(2) + 2^{2^k}\sum_{i=1}^{k} 2^{2^i} \\
&= n/2 \cdot T(2) + n \cdot \Theta(n) \\
&= \Theta(n^2)
\end{aligned}
$$

6. The characteristic equation is $x^2 = 8x - 15$. The roots are $x_1 = 3$ and $x_2 = 5$. So $a_n = A \cdot 3^n + B \cdot 5^n$. Plug in $a_0 = 3$ and $a_1 = 5$, we have

$$
\begin{aligned}
A + B &= 3 \\
3A + 5B &= 5
\end{aligned}
$$

So $A = 5$, $B = -2$. So $a_n = 5 \cdot 3^n - 2 \cdot 5^n$.