Portfolio Theory and Investments

Assignment 1

Q1) A1)

Cost of Asset 1 $(p_1) =$ Cost of Asset 1 $(p_2) = \$150$

Initial Wealth (Endowment) of the Individual $= \$150$

Utility function of the Individual $= v(c) = -e^{-c}$

Asset 1: $(z_{11}, z_{12}) = (100, 200)$

Asset 2: $(z_{21}, z_{22}) = (200, 100)$

Table 1.1 Yield Table for Asset 1 & 2 under different states:

	State 1	State 2
Asset 1	100	200
Asset 2	200	100

a) To show: State-contingent budget line: $c_1 + c_2 = \$300$

We know, the contingent income of a person in two asset world, can be represented by:

$$\binom{c_1}{c_2} = q_1 \binom{z_{11}}{z_{12}} + q_2 \binom{z_{21}}{z_{22}}$$

More generally,

$c_1 = z_{11}q_1 + z_{21}q_2$

$\boldsymbol{c_1 = 100q_1 + 200q_2}$ ---- (1)

$c_2 = z_{21}q_1 + z_{22}q_2$

$\boldsymbol{c_2 = 200q_1 + 100q_2}$ ---- (2)

Additionally, the individual's budget constraint is:

$150 = 150\text{q}_1 + 150\text{q}_2$

$Therefore,\ \mathbf{q_1 = 1 - q_2}$ ---- (3)

Using (1), (2) and (3), we get the following:

$\boxed{\boldsymbol{c_1 + c_2 = \$300}}$

--- Hence proved ---

b) *To show: Optimal Consumption in State 1:* $\boldsymbol{c_1^* = 150 + \frac{1}{2}\, ln\left(\frac{\pi}{1-\pi}\right)}$

We will use utility function v(c) and assume that the state 1 occurs with a probability $\boldsymbol{\pi}$

The optimal consumption is derived by maximizing the expected utility (E) over both the states:

$$max\, E = \pi(-e^{-c_1}) + (1-\pi)(-e^{-c_2})$$

To maximize E, we must find the First Order Conditions (Partial Derivatives) w.r.t. c_1 & c_2 and put them = 0

$$\frac{\partial E}{\partial c_1} = \pi(-e^{-c_1})(-1) + 0 = \pi(e^{-c_1})$$

$$\boldsymbol{\pi(e^{-c_1}) = 0} \qquad \text{---- (1)}$$

$$\frac{\partial E}{\partial c_2} = 0 + (1-\pi)(-e^{-c_2})(-1) = (1-\pi)(e^{-c_2})$$

$$\boldsymbol{(1-\pi)(e^{-c_2}) = 0} \qquad \text{---- (2)}$$

Equating (1) & (2),

$$\frac{\boldsymbol{\pi}}{\boldsymbol{(1-\pi)}} = \frac{(e^{-c_2})}{(e^{-c_1})} = (\boldsymbol{e^{-c_2+c_1}})$$

$$\boldsymbol{c_1 - c_2 = ln\left(\frac{\pi}{(1-\pi)}\right)} \qquad \text{---- (3)}$$

We also know that, $c_1 + c_2 = \$300$ ---- (4)

Substituting (3) in (4), we get:

$$\boxed{\boldsymbol{c_1^* = 150 + \frac{1}{2}\, ln\left(\frac{\pi}{1-\pi}\right)}}$$

--- Hence proved ---

c) *To show: Optimal units of Asset 1:* $\boldsymbol{c_1^* = 200 - 100q_1^*}$ ***& obtain expression of*** $\boldsymbol{q_1^*}$ ***in terms of*** $\boldsymbol{\pi}$

From a) (1), we know: $\boldsymbol{c_1 = 100q_1 + 200q_2}$ ---- (1)

And from a) (3), we know: $\boldsymbol{q_1 = 1 - q_2}$ ---- (2)

Upon substituting (2) in (1), we get the optimal units & consumption of Asset 1:

$$\boldsymbol{c_1 = 100q_1 + 200(1-q_1) = 200 - 200q_1 + 100q_1}$$

$\boxed{\boldsymbol{c_1^* = 200 - 100q_1^*}}$ ---- (3)

Now using result from b) $\boldsymbol{c_1^* = 150 + \frac{1}{2}\, ln\left(\frac{\pi}{1-\pi}\right)}$ and equating against (3), we get:

$$\boldsymbol{200 - 100q_1 = 150 + \frac{1}{2}\, ln\left(\frac{\pi}{1-\pi}\right)}$$

$$\boldsymbol{100q_1 = 50 - \frac{1}{2}\, ln\left(\frac{\pi}{1-\pi}\right)}$$

$$\boxed{\boldsymbol{q_1^* = \frac{1}{2} - \frac{1}{200} ln\left(\frac{\pi}{1-\pi}\right)}}$$

--- Hence proved ---

d) To find: Values of q_1^*, c_1^* as $\pi \to 0$

Value of $\boldsymbol{c_1^*}$ as the probability of State 1 becomes very small:

$$\lim_{\pi \to 0} \boldsymbol{c_1^*} = 150 + \frac{1}{2}\, \boldsymbol{ln}\left(\frac{0}{1-0}\right)$$

$$\lim_{\pi \to 0} \boldsymbol{c_1^*} = 150 + \frac{1}{2} \boldsymbol{ln}(\boldsymbol{0})$$

$$\lim_{\pi \to 0} \boldsymbol{c_1^*} = 150 + (-\infty)$$

$$\lim_{\pi \to 0} \boldsymbol{c_1^*} = (-\infty)$$

Therefore, as $\boldsymbol{\pi}$ becomes very small, the value of $\boldsymbol{c_1^*}$ approaches negative infinity $(-\infty)$

Value of $\boldsymbol{q_1^*}$ as the probability of State 1 becomes very small:

$$\lim_{\pi \to 0} \boldsymbol{q_1^*} = \frac{1}{2} - \frac{1}{200} \boldsymbol{ln}\left(\frac{\pi}{1-\pi}\right)$$

$$\lim_{\pi \to 0} \boldsymbol{q_1^*} = \frac{1}{2} - \frac{1}{200} \boldsymbol{ln}\left(\frac{0}{1-0}\right)$$

$$\lim_{\pi \to 0} \boldsymbol{q_1^*} = \frac{1}{2} - \frac{1}{200} \boldsymbol{ln}(\boldsymbol{0})$$

$$\lim_{\pi \to 0} \boldsymbol{q_1^*} = \frac{1}{2} - (-\infty)$$

$$\lim_{\pi \to 0} \boldsymbol{q_1^*} = +\infty$$

Therefore, as $\boldsymbol{\pi}$ becomes very small, the value of $\boldsymbol{q_1^*}$ approaches positive infinity $(+\infty)$

Q2) A2)

2) Given the competitive economy with I investors, all have the same utility function $U = \mu^{10}e^{-\sigma}$ and each individual is endowed with exactly one unit each of assets 1,2, and 3. Also given all payoff distributions are unrelated (σ_{ab}= 0 for all a ≠b)

Given each individual has one of each asset, The initial budget of the investor becomes 1 + 0.46 + 0.04 = 1.5

$$\overline{w} = 1.5$$

a) The single asset portfolio for asset 1: $q_1 = \overline{w}/P_1 = 1.5/1 = 1.5$ units, Asset 2 $q_2 = \overline{w}/P_2 = 1.5/0.46 = 3.26$ units and $q_3 = \overline{w}/P_3 = 1.5/0.04 = 37.5$ units.

   The points corresponding to the assets on $\mu\sigma$ the graph with μ on Y-axis and σ on X-axis will be Points be $(q_i * \sigma_i, q_i * \mu_i)$

   Asset 1 (a1): (0*1.5, 1*1.5) = (0, 1.5)

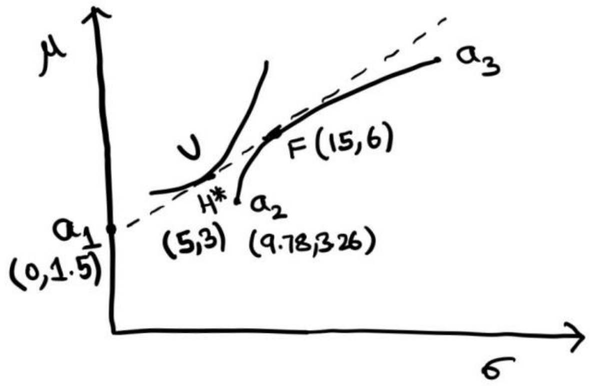
   Asset 2 (a2): (3*3.26, 1*3.26) = (9.78, 3.26)

   Asset 3 (a3): (37.5*4, 37.5*1) = (150, 37.5)

   The utility curve would be drawn assuming utility is constant so

   $\mu^{10}e^{-\sigma} = C$ which gives us $\mu = (Ce^{\sigma})^{\frac{1}{10}}$

   The indifference curve passes through a single asset portfolio of risky assets a_2,a_3 and F. The budget line from a_1 and H^* should touch indifference curve at F. The indifference curve needs to be convex as the individual will be willing to give less and less of asset 2 for asset 3 and the rate of change of slope will be negative.



b) Given if each individual's optimum portfolio $H^*$is the same as the endowed portfolio and we have one of each asset A, B, C. The mean of the portfolio H^* will be sum of individual means of asset A, asset B, and asset C. i.e.

$$\mu_{H^*} = 1 * \mu_1 + 1 * \mu_2 + 1 * \mu_3 = 1 + 1 + 1 = 3$$

and the variance of the portfolio H^* will be the summation of variances.

$$\sigma^2{}_{H^*} = 1^2 * \sigma_1 + 1^2 * \sigma_2 + 1^2 * \sigma_3 = (0)^2 + (3)^2 + (4)^2 = 25$$

$$\sigma_{H^*} = 5$$

The corresponding point for the portfolio H^* in $\mu\sigma$ the graph will be (5,3).

The optimal portfolio would be $H^* = (1/1.5) * a_1 + (0.46/1.5) * a_2 + (0.04/1.5) * a_3$

Mutual fund would not have asset 1 as it has zero variance. So, a mutual fund is a combination of Asset 2 and Asset 3 that minimizes variance for a given return.

Given the coefficients for asset 2 and asset 3 in optimal portfolio. We would get new weights for for asset 2 to be $\frac{(0.46/1.5)}{(0.46/1.5) + (0.04/1.5)} = 0.92$ and asset 3 $\frac{(0.04/1.5)}{(0.46/1.5) + (0.04/1.5)} = 0.08$.

The mutual fund equation becomes $F = 0.92a_2 + 0.08a_3$

This means in mutual fund we would have 92% of Asset 2 and 8% of Asset 3.

This means for every one dollar in mutual fund we have 0.92 dollar in Asset 2 and 0.08 dollar in Asset 3.

For one dollar in mutual fund, we have 0.92/0.46 = 2 units of Asset 2 and 0.08/0.04 = 2 units of Asset 3.

Using the units above for 1 dollar let’s find Asset Allocation in terms of units adjusted to budget 1.5.
For Asset 2 is (0.92/0.46) *(1.5/1) = 3 units.
For Asset 3 is (0.08/0.04) *(1.5/1) = 3 units.

Finding the mean for Mutual Fund portfolio for 3 units of Asset 2 and 3 units of Asset 3 would be $\mu_F = 3 * 1 + 3 * 1 = 6$

The variance for the Mutual Fund portfolio will be $\sigma^2{}_F = 3^2 * 3^2 + 3^2 * 4^2 = 225$ which give

$$\sigma_F = 15.$$

The corresponding point in $\mu\sigma$ the graph would be (15, 6).

Since the mutual fund comprises a_2 and a_3, and the individual spends (0.46 + 0.04)/ (1 + 0.46 + 0.04) = ⅓ of their income on the mutual fund, they effectively reduce their risk exposure while ensuring that the weighted returns are optimized.

c) The price of risk reduction will be the slope of the budget line which passes through Asset 1 (0, 1.5) and Optimal Portfolio (5,3). Which gives the slope $\theta = (3 - 1.5)/(5 - 0) = 3/10$.

With slope the equation of the budget line becomes

$$\mu - 3 = \frac{3}{10}(\sigma - 5)$$

$10\mu = 3\sigma + 15$ is the equation of the budget line.

The marginal rate of substitution (MRS) at optimal portfolio H^* would the slope of the budget line which is already proved as $3/10$.

Q3. There are 2 assets, where asset 1 is a riskless asset which give certain $50 income and asset 2 is risky asset which will give Z_{21} and Z_{22} income depends on which state is happen. Price of both assets are same with the wealth

A. from the table, we have to verify the red number

Category	Z_{21},Z_{22}	P(S_1) or π	C_1,C_2	E(c)	σ^2(c)
α	20,80	1/5	35,65	59	144
β	38,98	1/2	44,74	59	225
γ	30,90	1/3	40,70	60	200

First we have to use state claim formula for state 1 and 2

$$\binom{C1}{C2} = q1\binom{Z11}{Z12} + q2\binom{Z21}{Z22}$$

$C_1 = q1 \times Z_{11} + q2 \times Z_{21}$. .....(1)

$$C_2 = q1 \text{ x } Z_{12} + q2 \text{ x } Z_{22} \quad .....(2)$$

We know that $q1 = K1\, x \frac{W}{P1^A}$ and $q2 = K2\, x \frac{W}{P2^A}$

Then the expected value of income $E(c) = \boldsymbol{\pi} \text{ x } C_1 + (1-\boldsymbol{\pi}) \text{ x } C_2$ .....(3)

The variance of income $\sigma^2(c) = E[(c - E(c))^2] = E(c^2) - 2 \text{ x } [E(c)]^2 + [E(c)]^2$. ......(4)

where $E(c^2) = \boldsymbol{\pi} \text{ x } C_1^2 + (1-\boldsymbol{\pi}) \text{ x } C_2^2$

As price of asset 1 and asset 2 are the same with wealth (W) therefore q1 + q2 = 1. Furthermore, we know from the problem information that asset 1 and asset 2 are hold equally therefore q1 = 0.5 and q2 = 0.5 as W= $P1^A$

Then we calculate C_1 and C_2 with data from the table we plug in to the equation (1), (2), (3), and (4)

<u>for α</u> $C_1 = 0.5 \text{ x } 50 + 0{,}5 \text{ x } 20 = \$35$ $C_2 = 0.5 \text{ x } 50 + 0{,}5 \text{ x } 80 = \$65$ $E(c) = \frac{1}{5} \text{ x } 35 + \frac{4}{5} \text{ x } 65 = \$59$ $\sigma^2(c) = \frac{1}{5} \text{ x } 35^2 + \frac{4}{5} \text{ x } 65^2 - 2 \text{ x } [59]^2 + [59]^2 = \$144$	<u>for β</u> $C_1 = 0.5 \text{ x } 50 + 0{,}5 \text{ x } 38 = \$44$ $C_2 = 0.5 \text{ x } 50 + 0{,}5 \text{ x } 98 = \$74$ $E(c) = \frac{1}{2} \text{ x } 44 + \frac{1}{2} \text{x } 74 = \$59$ $\sigma^2(c) = \frac{1}{2} \text{ x } 44^2 + \frac{1}{2} \text{ x } 74^2 - 2 \text{ x } [59]^2 + [59]^2 = \$225$
<u>for γ</u> $C_1 = 0.5 \text{ x } 50 + 0{,}5 \text{ x } 30 = \$40$ $C_2 = 0.5 \text{ x } 50 + 0{,}5 \text{ x } 90 = \$70$ $E(c) = \frac{1}{3} \text{ x } 40 + \frac{2}{3} \text{x } 70 = \$60$ $\sigma^2(c) = \frac{1}{3} \text{ x } 40^2 + \frac{2}{3} \text{ x } 70^2 - 2 \text{ x } [60]^2 + [60]^2 = \$200$	

B. To confirm the preference of asset, we have to compute the expected utility for each asset composition category using expected utility function where,

$$E[U(c)] = \boldsymbol{\pi}\, U(c1) + (1-\boldsymbol{\pi})\, U(c2)$$
$$U(c) = -e^{-AC} \text{ with } A = \frac{\ln 4}{30} = 0.0462$$
$$U(c_1) = -e^{-AC1} \;\&\; U(c_2) = -e^{-AC2}$$

We input the formula using data from α,β,γ

<u>for α</u> $E[U(c)] = \frac{1}{5} x - e^{-0.0462 \times 35} + \frac{4}{5} x - e^{-0.0462 \times 65}$ $E[U(c)] = -0.07937$	**<u>for β</u>** $E[U(c)] = \frac{1}{2} x - e^{-0.0462 \times 44} + \frac{1}{2} x - e^{-0.0462 \times 74}$ $E[U(c)] = -0.0818$
<u>for γ</u> $E[U(c)] = \frac{1}{3} x - e^{-0.0462 \times 40} + \frac{2}{3} x - e^{-0.0462 \times 70}$ $E[U(c)] = -0.07875$	

From the computation, as there is minus sign, the bigger number in absolute number will be the smaller in minus sign. Therefore we can confirm that <u>asset γ will be the 1st rank</u>, <u>Asset α will be the 2nd rank</u>, and <u>asset β will be the last.</u>

C. We know the formula for asset position between q_1 and q_2 is:

$C_s(x) = (q_1 Z_1 + q_2 Z_{2s})$

$C_s(x) = (k_1 Z_1 + k_2 Z_{2s})*\frac{W}{Pa}$

Assume $k_2 = x$, $k_1 = 1-x$, where k2 is the fraction of risky asset

as wealth equal to price of every asset, $W = P^a$ therefore $q_2 = x$ and $q_1 = (1-x)$

$C_s(x) = ((1-x) Z_1 + x Z_{2s}x)$

$C_s(x) = ((1-x)50 + x Z_{2s}x)$

$C_s(x) = (50 - 50x + Z_{2s}x)$

$C_1(x) = (50 + x(z_{21} - 50))$ $\quad C_2(x) = (50 + x(z_{22} - 50))$

With expected value from utility function outcome

$E(u) = \pi U(C_1(x)) + (1 - \pi) U(C_2(x))$ $\qquad$ where $U(C_s(x)) = - e^{\{-A\, Cs(x)\}}$

$E(u) = \pi (-e^{\{-A\, C1(x)\}}) + (1 - \pi) (-e^{\{-A\, C2(x)\}})$

$E(U) = \pi (-e^{\{-A\, (50 + x(z21 - 50))\}}) + (1 - \pi) (-e^{\{-A\, (50 + x(z22 - 50))\}})$

To maximize the proportion, we need to get $\frac{dE(u)}{dX} = 0$ when inputting X variable

$\frac{dU}{dX}|_{x=1/2} = = A \pi (z_{21} - 50) (e^{\{-A\, (50 + x(z21 - 50))\}}) + A (1 - \pi) (z_{22} - 50) (e^{\{-A\, (50 + x\, (z22 - 50))\}})$

$\frac{dU}{dX}|_{x=1/2} = A \pi (z_{21} - 50) (e^{\{-A\, C1(x)\}}) + A (1 - \pi) (z_{22} - 50)) (e^{\{-A\, C2(x)\}})$

$\frac{dU}{dX}|_{x=1/2} = A (e^{\{-A\, C2(x)\}}) [\pi (z_{21} - 50) (e^{\{A\, (C2(x) - C1(x))\}}) + (1 - \pi) (z_{22} - 50)]$

for α, β, γ always $C_2(x) - C_1(x) = 30$ where $x = ½$ and $e^{\{30A\}} = 4$

$\frac{dU}{dX}$ (at $x = ½$) $= A (e^{-A\, C2(x)})(4 \pi (z_{21} - 50) + (1 - \pi) (z_{22} - 50))$

We can validate this with the data of α, β, γ

With	α	β	γ
π	1/5	1/2	1/3
z_{21}	20	38	30
z_{22}	80	98	90

We put this values in red bracket in above equation of $\frac{dU}{dX}|_{x=1/2}$, we always get zero

Therefore, α, β, γ are the same optimal choice at x = ½