Surface Integrals (16.7)

The main integral

As discussed in the previous lecture, a surface can be described as:

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

The surface integral of a function $f(x,y,z)$ over the surface S is:

$$I = \iint_S f(x,y,z)ds = \iint_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dudv$$

Where

$$\vec{r}_u = \left\langle \frac{\delta x}{\delta u}, \frac{\delta y}{\delta u}, \frac{\delta z}{\delta u} \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\delta x}{\delta v}, \frac{\delta y}{\delta v}, \frac{\delta z}{\delta v} \right\rangle$$

To find this, let's visualize how this integral would work. It sums up very small rectangles over the top of the surface. This means we are summing up the areas of all of those small rectangles.

$$I = \sum_i \sum_j f(P_{ij})\Delta_{ij}$$

Similar to previous lectures, we can find the area of the Surface by using 1 as the function we are integrating:

$$I = \iint_S 1 ds = \iint_D |\vec{r}_u \times \vec{r}_v| dudv$$

If $z = f(x, y)$, and S is the graph of f, then

$$S = \vec{r}(u, v) = u\hat{i} + v\hat{j} + f(u, v)\hat{k}$$

$$S = \vec{r}(x, y) = x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

This is simply a way of converting a normal surface to a parameterized version for use in this equation.

$$\vec{r}_x \times \vec{r}_y = \sqrt{\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1}$$

So we can find the area like this:

$$Area\ of\ S = \iint_S 1 ds = \iint_D \sqrt{\left(\frac{\delta f}{\delta x}\right)^2 + \left(\frac{\delta f}{\delta y}\right)^2 + 1} dydx$$

▾ Click to show Example 1

Question 1

Find:

$$\iint_S (x + y + z) ds$$

Given:

$$\begin{cases} x = u + v \\ y = u - v \\ z = 1 + 2u + v \end{cases}$$

$$\vec{r}(u,v) = (u+v)\hat{i} + (u-v)\hat{j} + (1+2u+v)\hat{k}$$

Where $0 \leq x \leq 1$ and $0 \leq y \leq 2$. So we can use the formula:

$$\iint_D [(u+v) + (u-v) + (1+2u+v)]|\vec{r}_u \times \vec{r}_v| dudv$$

And find the magnitude as follows:

$$\vec{r}_u = \langle 1, -1, 2 \rangle$$

$$\vec{r}_v = \langle 1, -1, 1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & -1 & 2 \\ 1 & -1 & 1 \end{vmatrix} = \sqrt{14}$$

$$\iint_D (4u + v + 1)\sqrt{14} dudv$$

▾ Click to show Example 2

Question 2

$$\iint_S xyzds$$

$$S = \begin{cases} x = u\cos v \\ y = u\sin v \\ z = u \end{cases}$$

$$0 \leq u \leq 1, 0 \leq v \leq \frac{\pi}{2}$$

$$\iint_D u\cos v + u\sin v + u|\vec{r}_u \times \vec{r}_v| dudv$$

$$\vec{r}_u = \langle \cos v, \sin v, 1 \rangle$$

$$\vec{r}_v = \langle -u \sin v, u \cos v, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = -u \cos vi - u \sin vj + uk$$

$$\iint\limits_D u \cos v + u \sin v + u\sqrt{u^2(\cos^2 + \sin^2 + 1)}$$

▼ Click to show Example 3

Example 3

$$I = \iint\limits_S (x^2 + y^2) ds$$

$$\vec{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$$

$$u^2 + v^2 = 1$$

$$\iint (u^2 + v^2)^2 |\vec{r}_u \times \vec{r}_v| dudv$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{32(u^2 + v^2)}$$

$$\iint (u^2 + v^2)^{5/2} \sqrt{32} dudv$$

This would be easiest to compute as a polar integral:

$$\sqrt{32} \iint r^5 r d\theta dr$$

Unit normal vector

The unit normal vector of S at point P is the vector that is perpendicular to the surface at that point.

This vector can be found as:

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

So, we are finding the normal vector of the plane that contains the vectors $\vec{r}_u$ and $\vec{r}_v$, which give a plane that is tangent to the surface at that point.

Surface Integrals of Vector Fields

We can integrate a vector field $\vec{F}(x, y, z)$ over S like this:

$$\iint \vec{F} \cdot ds = \iint_S \vec{F} \cdot \vec{n} ds$$

To help make this equation a bit easier to compute, we can use:

$$\iint_D \vec{F} \cdot (\vec{r}_u \times r_v) dudv$$

This equation comes from converting the previous one to a normal integral.

$$\iint \vec{F} \cdot ds = \iint_S \vec{F} \cdot \vec{n} ds = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| dudv$$

So, it cancels to:

$$\iint_D \vec{F} \cdot (\vec{r}_u \times r_v) du dv$$

Note that this uses both dot product and cross product. Also, this is called flux, sometimes flux of $\vec{F}$ across S, related to fluid dynamics.