Triple Integrals

These integrals are just a generalization of double integrals.

Notation

$$\iiint\limits_D f(x,y,z)dxdydz = \iiint\limits_D f(x,y,z)dV$$

Where $D \in \mathbb{R}^3$ is a 3D region, and dV stands for change in volume.

Assuming D is a rectangular box given by:

$$D = \{(x,y,z)|a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$$

D can also be defined as $[a,b],[c,d],[e,f]$

Computation

To find this integral, we break the volume up into small cubes, instead of squares or lines like a two- or one-dimensional integral. So we can define the integral as the sum of all of these cubes, or the sum of their multiplied sides and the function:

$$\iiint\limits_D f(x,y,z)dV = \lim_{\begin{cases}\Delta x_i \to 0\\ \Delta y_j \to 0\\ \Delta z_l \to 0\end{cases}} \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{l=1}^{k} f(x_i,y_j,z_l)\Delta x_i \Delta y_j \Delta z_l$$

Where

$$\Delta x_i = x_i - x_{i-1}$$

$$\Delta y_j = y_j - y_{j-1}$$

$$\Delta z_l = z_l - z_{l-1}$$

Since $\Delta x_i \Delta y_j \Delta z_l$ represents the volume of each individual box, you can see why we shorten it to dV.

To explain this relationship more intuitively, the function $f(x,y,z)$ can be visualized as a density function, where is gives the value of the density in some point in space. For example, $f(x, y, z) = \frac{1}{x+y+z}$ would be very dense in the center (near the origin), and gradually decrease as the distance from the center increases.

Fubini's Theorem for Triple Integrals

If $D = [a, b], [c, d], [e, f]$, then:

$$\iiint\limits_D f(x, y, z)dV = \int\limits_a^b \int\limits_c^d \int\limits_e^f f(x, y, z)dzdydx = \left(\int\limits_a^b \left(\int\limits_c^d \left(\int\limits_e^f f(x, y, z)dz \right) dy \right) dx \right)$$

The order of these integrals can be rearranged as necessary, but the boundaries of each variable must be paired with it in the integration.

$$\int\limits_e^f \int\limits_c^d \int\limits_a^b f(x, y, z)dxdydz$$

▼ Click to show Example 1

Question 1 (sec 15.6)

Find the triple integral $\iiint\limits_B xyz^2 dV$, where B is the rectangular box $[0, 1], [-1, 2], [0, 3]$

Solution

$$\iiint xyz^2 dV = \int\limits_0^1 \int\limits_{-1}^2 \int\limits_0^3 xyz^2 dzdydx$$

$$\int\limits_0^3 xyz^2 dz = \frac{1}{3} xyz^3|_0^3$$

$$\int\limits_{-1}^2 9xydy = \frac{9}{2} xy^2|_{-1}^2$$

$$\int\limits_0^1 \left(18x - \frac{9}{2} x \right) dx$$

$$9x^2 - \frac{9}{4} x^2|_0^1 = 9 - \frac{9}{4}$$

$$\frac{27}{4}$$

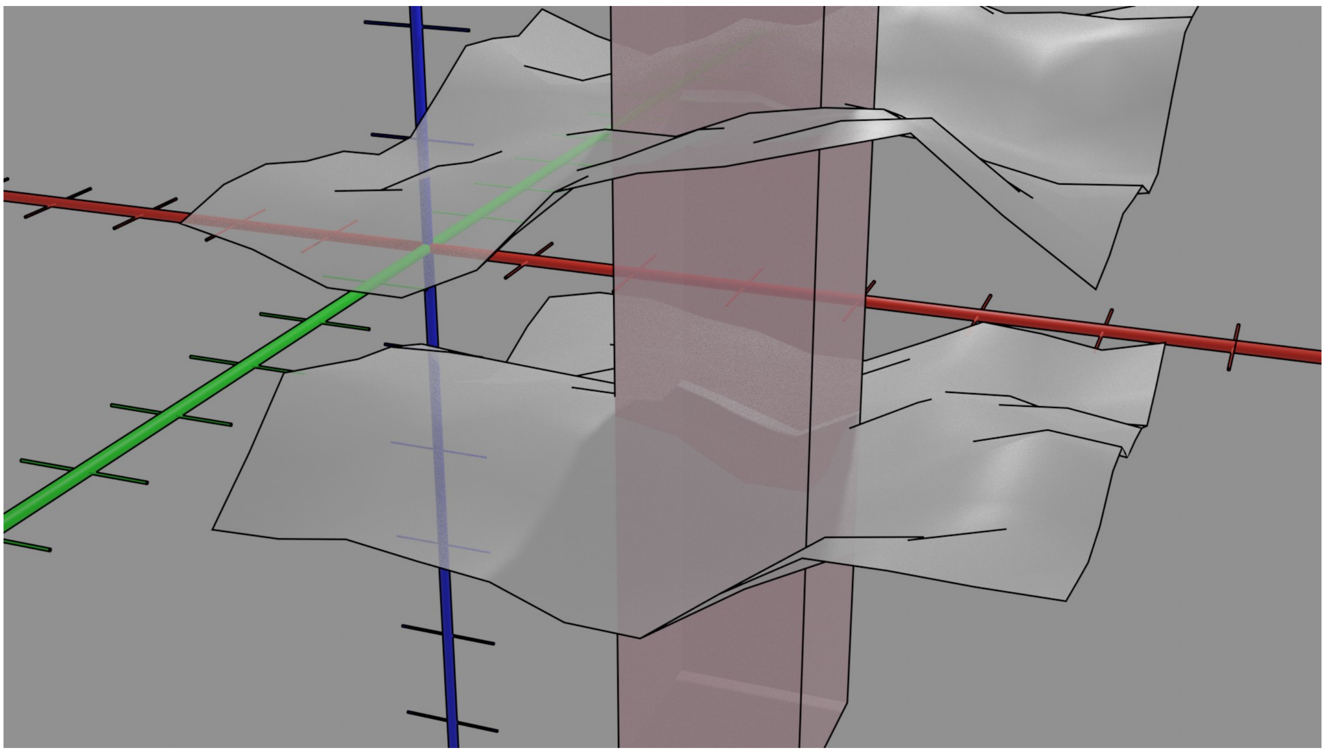
Type 1,2, and 3 Domains

Type 1

The domain $D \in \mathbb{R}^3$ is said to be type 1 if it can be given as

$$D = \{(x, y, z)|(x, y) \in \mathbb{R}, h_1(x, y) \leq z \leq h_2(x, y)\}$$

This means that z is not a constant value, but a function.



So, we can insert these functions into the normal formula:

$$\iiint\limits_D f(x, y, z)dV = \iint\limits_R \int\limits_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)dzdydx = \int\limits_a^b \int\limits_c^d \int\limits_{h_1(x,y)}^{h_2(x,y)} f(x, y, z)dzdydx$$

So we first integrate the inner integral, which leaves us with a double integral.

Type 2 and 3

This is just where instead of z being defined by a function, it is x or y

$$D = \{(x, y, z)|(y, z) \in \mathbb{R}, h_1(x, y) \leq x \leq h_2(x, y)\}$$

$$\iiint\limits_D f(x,y,z)dV = \iint\limits_R \int\limits_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dxdydz$$

$$D = \{(x,y,z)|(x,z) \in \mathbb{R}, h_1(x,y) \leq y \leq h_2(x,y)\}$$

$$\iiint\limits_D f(x,y,z)dV = \iint\limits_R \int\limits_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)dydxdz$$