Line Integrals

Overview

This type of integrals is not related to triple integrals, but is defined as an integral of a function over a line.

This means we are computing the value at each point and summing them up over the length of a curve C. This means finding the value of the function $f(x_i, y_i)$ at each point, then multiplying by the length of the subarc between that point and the next Δs_i

$$\lim_{\Delta s_i \to 0} \sum f(x_i, y_i) \Delta s_i$$

This way we can split the line into many sections which can be summed up to find the integral, and we can show this as the following formula:

$$\int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

(We use the sqare root above because it represent the length of the curve as we integrate)

This is the line integral with respect to the length, or $\int_C f(x, y) ds$

. However, we can also find line integrals with respect to other variables:

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$$

This can also work for triple integrals and multiple types per function. For Example:

$$\int_C f(x,y,z)dx + g(x,y,z)dz =$$

$$\int_a^b f(x(t),y(t),z(t))x'(t) + g(x(t),y(t),z(t))z'(t)dt$$

▼ Click to show example

Question

Find the line integral $\int_C xy^2 ds$, where C is the upper half of the unit circle.

Solution

To find this, we need to first find a parameterization of C. Since C is a half circle above the x-axis, we can use a standard parameterization: $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le \pi$

Next, we can use the formula above to get:

$$\int_C xy^2 ds = \int_0^\pi \cos(t)(\sin(t))^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$\int_0^\pi \cos(t)(\sin(t))^2 \sqrt{(-\sin(t))^2 + (\cos(t))^2} dt$$

$$\int_0^{\pi} \cos(t)(\sin(t))^2 dt$$

What if C is in 3D space?

For the function $f(x, y, z)$, we have

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

where $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \leq t \leq b$ is a parameterization of C.

Even though there are an unlimited number of parameterizations of C, which one is used does not affect the answer. The right-hand side of the formulas do not depend on the parameterization of C.

Also, the length of the curve C is equal to:

$$\int_C 1\ ds = \int_a^b |r'(t)|dt$$

If we define the vector function $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \leq t \leq b$, then $\vec{r}(t)$ is a vector parameterization of C. Also, the formula can be rewritten as:

$$\int_C f(x,y,z)ds = \int_a^b f(\vec{r}(t))|r'(t)|dt$$

This is because, by definition, $|r'(t)|$ is

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

▸ Click to show example

These are useful becuase you can apply functions along a line, such as the weight of a varying density wire. However, the main use of these is computing the total force applied along a path.

Given a smooth curve through space asume that at each point in the 3D space a force $\vec{F}$ is acting. This means we can define $\vec{F}$ as a function of x, y, z that results in a force x_i, y_i, z_i, otherwise known as a vector function.

This is also called a '**vector field**'.

If, for example, F represents a gravitational force, it can be called a gravitational field, for example.

Question

What is the work that the force field $\vec{F}$ does to move a particle from point A to point B along curve C.

This is asking how the forces act along the path that the particle travels when acted on that force at each point.

For Example: if $\vec{F}$ is a constant and the curve C is straight line, then the work W done by $\vec{F}$ will be $W = \vec{F} \cdot \vec{AB}$. This is the general formula from high school.

In the general case, the work is equal to

$$W = \int_C \vec{F}(x, y, z) * \vec{T}(x, y, z) ds$$

Where $\vec{T}$ is the unit vector at the point (x,y,z), which is the tangent vector to C at that point with length equal to 1.

This is called the line integral of the vector field $\vec{F}$ along the curve C

$$\vec{T} = \frac{\vec{r}'(x_0, y_0, z_0)}{|\vec{r}'(x_0, y_0, z_0)|}$$

using the formula above,

$$\int_a^b \vec{F}(\vec{r}(t)) * \frac{\vec{r}'(t)}{|\vec{r}'(t)|} * |\vec{r}'(t)|dt$$

And the two $|\vec{r}'(t)|$ cancel, so it simplifies to:

$$\int_a^b \vec{F}(\vec{r}(t)) * \vec{r}'(t)dt$$

The line integral of $\vec{F}$ along C is denoted by

$$\int_C \vec{F} \cdot dr = \int \vec{F}(\vec{r}(t))\vec{r}'(t)dt$$

This integral will be discussed in more detail next.