SUMMARY, CHAPTER 16 - VECTOR CALCULUS

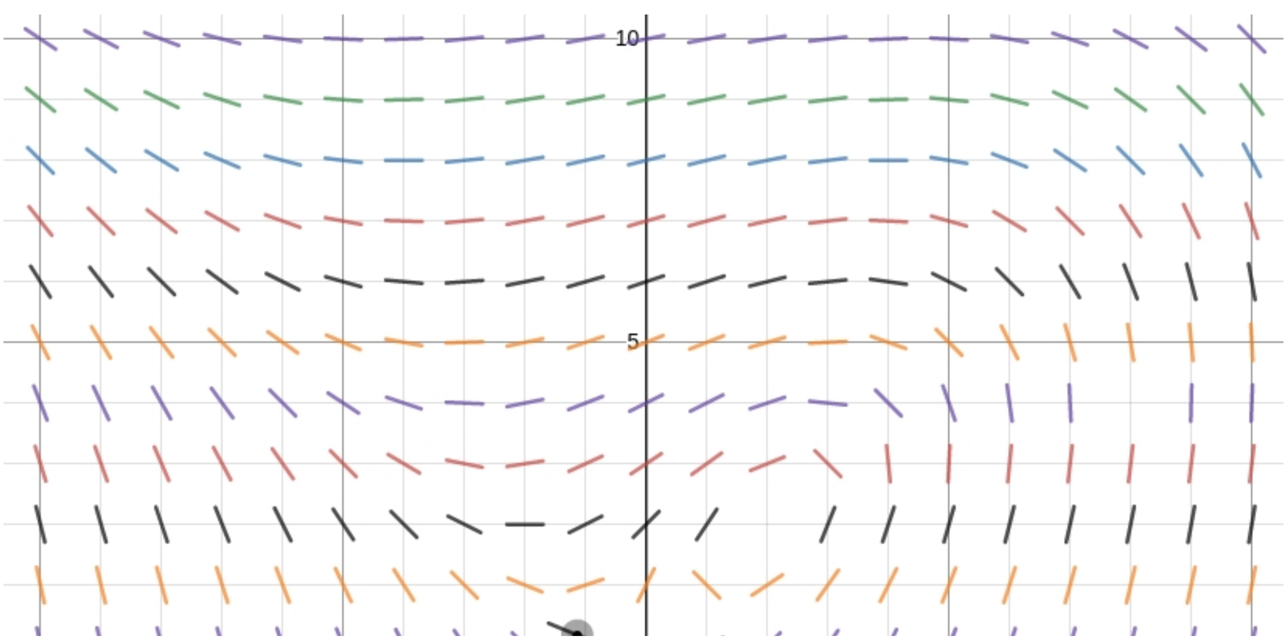
Vector Fields

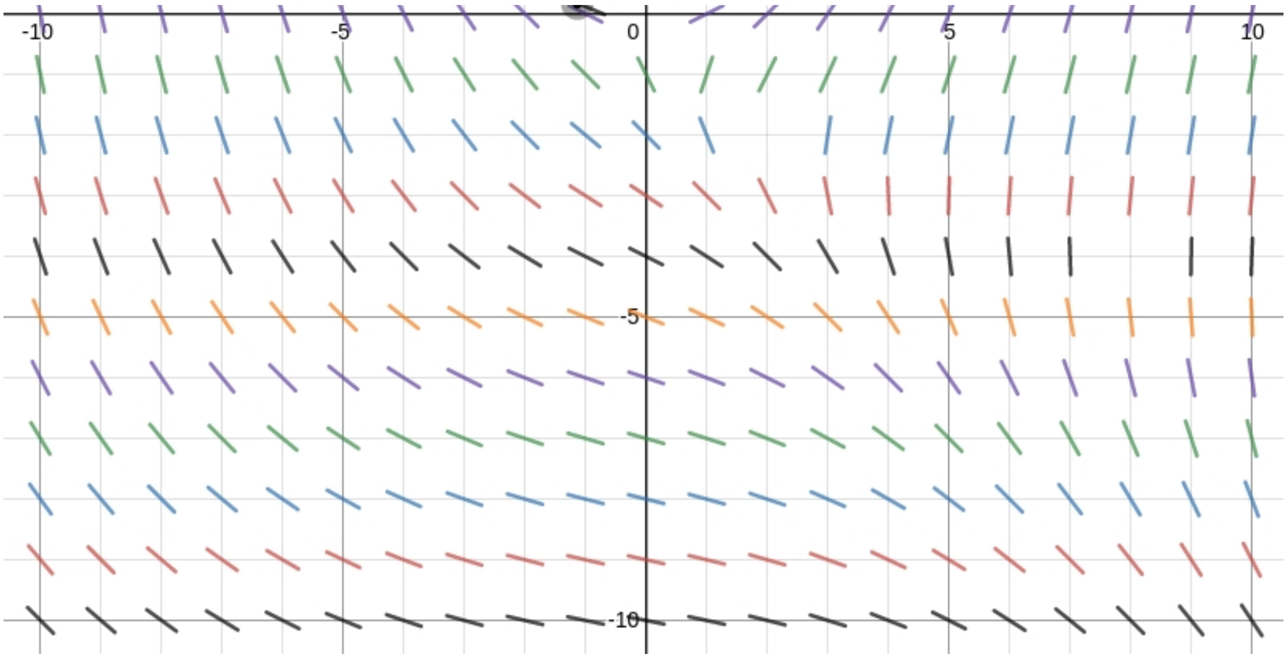
Vector fields are functions that represent vectors for each point in space. They go from a location (vector or multiple variables) to another vector that can represent speed, force, or another vector quantity.

$$\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$$
$$= \langle P(x,y), Q(x,y)\rangle$$

$$\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$
$$= \langle P(x,y), Q(x,y), R(x,y)\rangle$$

Example of a two-dimensional vector field:





Line Integrals

Line integrals represent taking the integral of all the values in a field along a certain path.

To explain what this means, we are basically taking the value of the function f at each point, represented by some parametric equation $\vec{r}(t) = \langle x(t), y(t) \rangle$, and multiplying it by the length of the change in t (this is essentially multiplying the force at that point by the portion of the line it influences.)

$$\lim_{\Delta t \to 0} \sum f(x(t), y(t))|\Delta t|$$

Which leads to the equation

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t))|\vec{r}'(t)| dt$$

This is the line integral with respect to the length (as shown by $|\vec{r}'(t)|$, normally denoted ds). However, the integral can be taken with respect to other variables, or even to multiple.

$$\int_C f(x,y)dx = \int_a^b f(x(t), y(t))\ x'(t)dt$$

$$\int_C f(x,y)dx + g(x,y)dy$$

$$= \int_a^b f(x(t), y(t))\ x'(t)dt + g(x(t), y(t))\ y'(t)dt$$

These can also be represented in 3D:

$$\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))|r'(t)|dt$$

To simplify this for integrals over vector fields, they are shown as the integral over a curve of a vector field:

$$\int_C \vec{F} \cdot dr$$

This means:

$$\int \vec{F}(\vec{r}(t))\vec{r}'(t)dt$$

Note: the length is cancelled out due to the derivation of this formula. See the notes for more.

Fundamental Theorem for Line Integrals

Given a conservative vector field $\vec{F}$, and a function f such that $\nabla f = \vec{F}$ the integral can be found as:

$$\int_C \vec{F} dr = f(B) - f(A)$$

Essentially, this means we can find the anti-derivative of a vector field as a function whose gradient is the vector field and evaluate it similar to a normal integral.

This only works for conservative vector fields, which can be found by comparing:

$$\frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x}$$

This only works for 2D fields; the full definition will be given below.

To actually find f, we just need to take the normal integral of one of the parts and then find what needs to be added to ensure the rest of the parts work.

▾ Click to show Example

Question

Find f such that $\nabla f = \vec{F}$, then find the integral.

$$\vec{F} = 3yz + 2x, 3xz, 3xy - 1\langle$$

$$\vec{r}(t) = \langle 2t, t^2, 3\rangle$$

$$0 \leq t \leq 1$$

Solution

First, we take the integral of P.

$$\int 3yz + 2x = 3xyz + x^2 + h(y, z)$$

In this case, we use $h(y, z)$ instead of a constant C because a constant with respect to x can be a function of y and z. So, now we find what $h(y, z)$ is. One way to do this is to take the derivative with respect to y and z and determine what else is needed, the other is to simply reason it out.

$$\frac{d}{dy}(3xyz + x^2 + h(y, z))$$

$$3xz + \frac{dh}{dy} = 3xz$$

Here we can see that the derivative already results in Q, so the function with respect to y is constant, meaning only z remains.

$$\frac{d}{dz}(3xyz + x^2 + h(z))$$

$$3xy + \frac{dh}{dz} = 3xy - 1$$

So $\frac{dh}{dz} = -1$, meaning $h(z) = -z$. Now we have the full function:

$f = 3xyz + x^2 - z$

To evaluate the integral, we simply plug in the initial and final points:

$$\vec{r}(0) = \langle 0, 0, 3 \rangle$$

$$\vec{r}(1) = \langle 2, 1, 3 \rangle$$

So:

$$\int_C F \cdot dr = f(2, 1, 3) - f(0, 0, 3)$$

$$= 18 + 4 - 3 - (-3) = 22$$

Green's Theorem

Green's theorem allows us to convert a line integral into a double integral (for curves in the xy-plane):

$$\int\limits_C F \cdot dr = \iint\limits_D \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y} \right) dxdy$$

Green's Theorem only works when the curve is closed.

Curl and Divergence

Curl

Curl is defined as the cross product of ∇ and $\vec{F}$. It represents the rotation of the movement field (as a vector that represents the axis of rotation).

$$curl\ \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \end{vmatrix}$$

$$= \left(\frac{\delta R}{\delta y} \right) - \left(\frac{\delta Q}{\delta z} \right) \hat{i} + \left(\frac{\delta P}{\delta z} \right) - \left(\frac{\delta R}{\delta x} \right) \hat{j} + \left(\frac{\delta Q}{\delta x} \right) - \left(\frac{\delta P}{\delta y} \right) \hat{k}$$

Divergence

Divergence is defined as the dot product of ∇ and $\vec{F}$. It represents the movement of a particle away from that point.

$$div\ \vec{F} = \nabla \cdot \vec{F} = \frac{\delta P}{\delta x} + \frac{\delta Q}{\delta y} + \frac{\delta R}{\delta z}$$

Parametric Surfaces

Parametric surfaces are parametric functions of two variables.

$$\vec{r}(u,v) = \begin{cases} x = f(u,v) \\ y = g(u,v) \\ z = h(u,v) \end{cases}$$

This takes two variables (u, v) and outputs a point in space (x, y, z). This allows for a surface to be created, and is the 2D (suface) equivalent of a line (curve).

▼ Click to show Example

Question
Find the equation of the surface with the given parameterization:

$$\vec{r}(u,v) = \langle v\cos u, v\sin u, v\rangle$$

Solution
To make this easier, convert the parametric equations to a normal equation. To do this, try and convert it to (x,y,z).

This problem becomes much easier when you notice that we can use $x^2 + y^2$ to cancel out the trigonometric functions:

$$x^2 + y^2 = s^2(\cos^2 + \sin^2) = s^2 = z^2$$

$$z^2 = x^2 + y^2$$

This equation represents a paraboloid.

Surface Areas

We can find the surface area by summing up the rectangles on the surface of the function. Since the area of these paralellograms can be given as $|side1 \times side2|$, we can represent this as:

$$\iint_D |\vec{r}_u \times \vec{r}_v| dA$$

This means the double integral over the domain of S of the length of the cross product of the parial derivatives of $S\ (\vec{r}(u,v))$.

Integrals over Surfaces

It turns out the previous equation can be extended to general surface integrals, just replacing the function (which is 1 for area) with the function we are integrating:

$$\iint\limits_S f(x,y,z)ds = \iint\limits_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|dudv$$

As a side note, if S is the graph of $z = f(x,y)$, then $|\vec{r}_u \times \vec{r}_v|$ can be given more simply as:

$$\sqrt{\left(\frac{\delta f}{\delta x}\right) + \left(\frac{\delta f}{\delta y}\right) + 1}$$

Over a vector field, the equation becomes:

$$\iint\limits_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)dudv$$

This is also called **Flux**.

Stokes' Theorem

Stokes theorem allows us to change a curve integral to a surface integral.

$$\int\limits_C \vec{F} \cdot dr = \iint\limits_S curl\ \vec{F} \cdot dS$$

This only works for a closed curve (to make the surface S).

The Divergence Theorem

The divergence theorem allows us to convert a surface integral to a triple integral. It uses divergence in the formula, hence the name:

$$\iint\limits_S \vec{F} \cdot dS = \iiint\limits_E div\ \vec{F}\ dV$$

This only works for a closed surface (to make the volume E).

Note that $div\ \vec{F}$ is scalar, so we will not have a vector as an answer.