CURL AND DIVERGENCE

Curl

Curl is the measure of how much the field rotates, meaning that a particle will rotate as it travels through the field. Clockwise is negative adn counter-clockwise is positive. If curl is 0, the particle does not rotate, but may move.

Imporantly, if curl is 0, then the vector field is conservative, and it can be integrated, meaning that there is an f such that $\nabla f = \vec{F}$.

$$Curl = \left(\frac{\delta R}{\delta y} - \frac{\delta Q}{\delta z}\right)\hat{i} + \left(\frac{\delta P}{\delta z} - \frac{\delta R}{\delta x}\right)\hat{j} + \left(\frac{\delta Q}{\delta x} - \frac{\delta P}{\delta y}\right)\hat{k}$$

This is a bit of a pain to remember, so let's use a more convenient formula. Recall that

$$\nabla f = \langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z} \rangle$$

We can show ∇ as a special vector of derivatives and this as a scalar product.

$$\nabla = \frac{\delta}{\delta x}\hat{i} + \frac{\delta}{\delta y}\hat{j} + \frac{\delta}{\delta z}\hat{k}$$

Using this notation, curl can be defined as:

$$curl\ \vec{F} = \nabla \times \vec{F}$$

By computing the cross product of this we get back to the same formula, but through a much easier-to-remeber way.

Divergence

Divergence is the measure of how much the field pushes the particles away from their starting point. Ex: an explosion would be higly positive, a black hole would be higly negative. If the particle does not move, divergence is = 0.

Similarly to curl, divergence can be defined using ∇:

$$div\ \vec{F} = \nabla \cdot \vec{F}$$

This is a bit simpler to compute:

$$div\ \vec{F} = \frac{\delta P}{\delta x} + \frac{\delta P}{\delta y} + \frac{\delta P}{\delta z}$$

Theorem

If $\vec{F}$ is a vector field and P,Q,R have continuous second pertial derivatives, then:

$$div\ curl\ \vec{F} = 0$$

▼ Click to show Example 1

Question 1

Find curl and divergence for

$$\vec{F} = 0i + x^3yz^3j + yz^3k$$

Solution

$$Curl = \hat{\imath}(x^3yz^3dz + yz^3dy) - \hat{\jmath}(0dzyz^3dx) + \hat{k}(0dy + x^3yz^3dx)$$

$$= (z^3 - 3x^3yz^2)\hat{\imath} - 0\hat{\jmath} + (3x^2yz^3)\hat{k}$$

$$Div = \nabla \cdot \vec{F} = 0 + x^3z^3 + 3yz^2$$

Since neither of these is equal to 0, it will move away from it's location and rotate.

▼ Click to show Example 2

Question 2

$$\vec{F}(x, y, z) = ln(2y + 3z)\hat{\imath} + ln(x - z)\hat{\jmath} + ln(x + 2y)\hat{k}$$

Solution

$$curl\ \vec{F} = i(\frac{z}{(x+2y)} - (\frac{-1}{x-z}) - j\frac{1}{x+2y} - \frac{3}{(2y+3z)} + k\frac{1}{(x-z)} - \frac{2}{(2y+3z)} \neq 0$$

$$div\ F = 0 + 0 + 0$$

This means that particle does not move, but it does rotate (in a complex, chaotic way)

▼ Click to show Example 3

Question 3

$$\vec{F} = \frac{\sqrt{x}}{1+z}\hat{i} + \frac{\sqrt{y}}{1+x}\hat{j} + \frac{\sqrt{z}}{1+y}\hat{k}$$

Solution

This time, the curl is equal to 0, so F is conservative

The Divergence of F is:

$$\frac{1}{2(1+z)\sqrt{x}} + \dots$$

So, it is not = 0, meaning that the particles move away but do not rotate.

▼ Click to show final Example

Example

Show that $\vec{F}$ is conservative and find $f(x, y, z)$ such that $\nabla f = \vec{F}$

$$\vec{F} = y^2 z^3 i + 2xyz^3 j + 3xy^2 z^3 k$$

Solution

First, we check that $\vec{F}$ is conservative.

$$Curl\ \vec{F} = i(6xyz^2 - 6xyz^2) + j(3y^2z^2 - 3y^2z^2) + k(2yz^3 - 2yz^2)$$

To find the second part, we want to find $f(x, y, z)$ such that $\nabla f = \vec{F}$, meaning

$$\frac{\delta f}{\delta x} = P(x, y, z) = y^2 z^3$$

$$\frac{\delta f}{\delta y} = Q(x, y, z) = 2xyz^3$$

$$\frac{\delta f}{\delta z} = R(x, y, z) = 3xy^2 z^3$$

To do this, we simply need to integrate and ensure that the integral works for all three partial derivatives.

$$f = \int P = \int y^2 z^3 dx = xy^2 z^3 + h(y, z)$$

Then we see what h(y,z) needs to be to ensure that the derivatives are correct.

After differentiating it with respect to y and z, we can see that it already works, and that h(y,z) is actually = 0.

$$\frac{\delta}{\delta y} xy^2z^3 = 2xyz^3 = Q$$

$$\frac{\delta}{\delta z} xy^2z^3 = 3xy^2z^2 = R$$