Planes and Vector Applications

Equations of Lines

Vector Equation

The vector equation describes the line as a point and direction. For each value of t, it produces a point on the line

Given starting point $\vec{r}_0$ and vector paralell to the line $\vec{v}$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

Parametric Equation

The parametric equation defines x,y,and z separately as functions of t

Given starting point $\langle x_0, y_0, z_0 \rangle$ and vector $\langle a, b, c \rangle$

$$\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$$x(t) = x_0 + at$$

$$y(t) = y_0 + bt$$

$$z(t) = z_0 + ct$$

Symmetric Equation(s)

The symmetric equation is found by solving for t, and therefore each one can be set equal

$$t = \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

If one of the pieces a, b, or c is 0, it can still be written as $x = x_0$ for example, if a = 0

Line segments

If only the portion of a line from r_0 to r_1 is given by

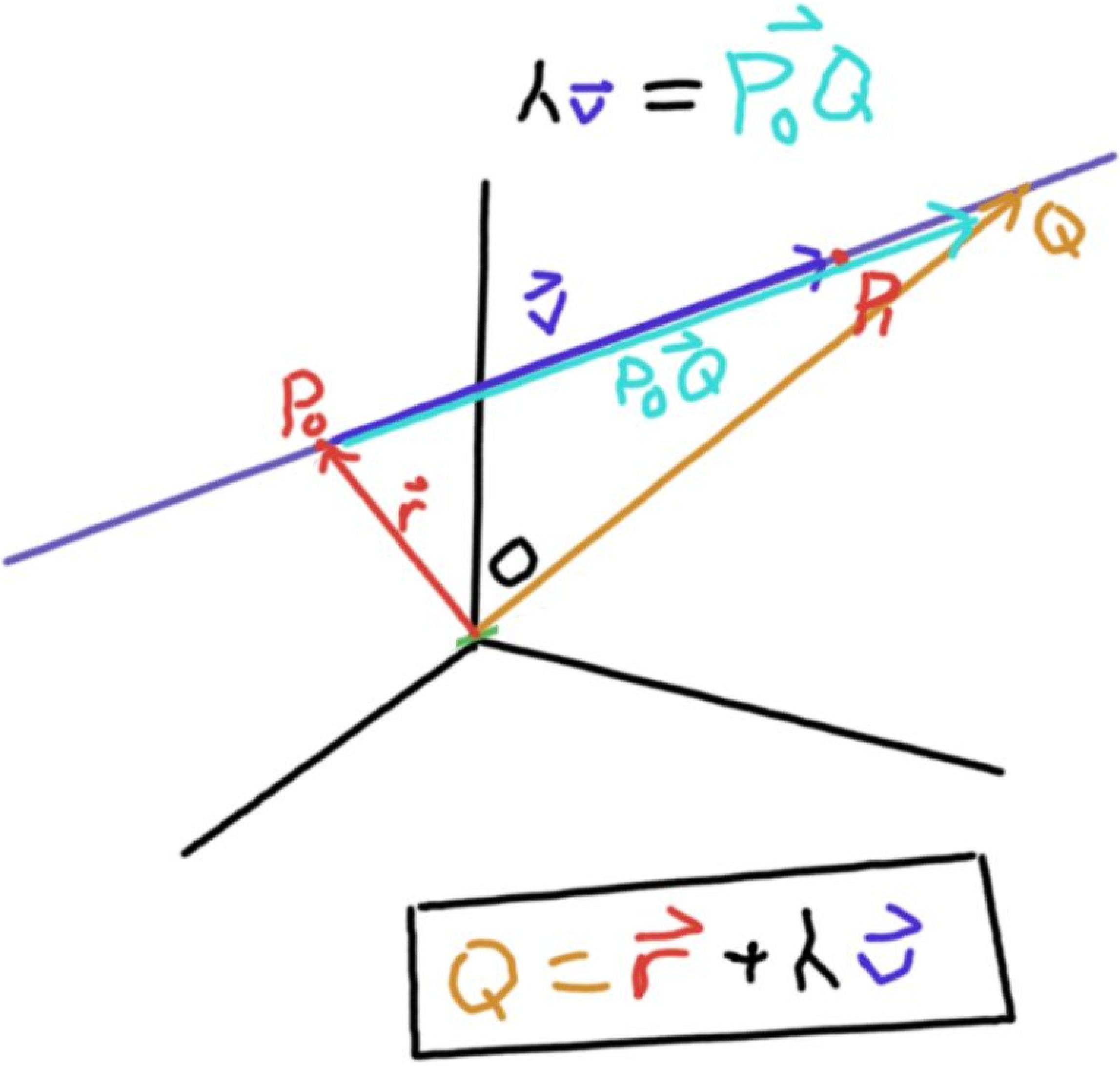
$$r(t) = (1 - t)\vec{r}_0 + t\vec{r}_1$$

$$0 \leq t \leq 1$$

Explanation

Defining a line

This section explains more explicitly how the above example is able to define a line.



Given $P_0, P_1, \vec{v}$ connecting them, a point Q on the line defined by P_0, P_1, and a vector $\vec{OQ}$ from the orogin, and $\vec{r}$ from the origin to P_0

Since $\vec{v}$ and $\vec{P_0Q}$ are parallel, one can be scaled to be the other. In other words, for the vec $\vec{P_0Q}$, there exists a scalar λ such that $\vec{P_0Q} = \lambda \times \vec{v}$.

Note that $\vec{OQ} = \vec{r} + \vec{P_0Q} = \vec{r} + \lambda\vec{v}$. This gives the vector equation of the line:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$

As we will see soon, it is better to denote λ by t (because t has the meaning of time)

$\vec{OQ} = \langle x, y, z \rangle$, so the above equation can be rewritten as:

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t\langle x_1 - x_0, y_1 - y_0, x_1 - z_0 \rangle$$

$$\langle x, y, z \rangle = \langle x_0 + t(x_1 - x_0), y_0 + t(y_1 - y_0), z_0 + t(z_1 - z_0) \rangle$$

$$x = x_0 + t(x_1 - x_0)$$

$$y = y_0 + t(y_1 - y_0)$$

$$z = z_0 + t(z_1 - z_0)$$

(parametric equation of the line) - or more generally:

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

Equations of Planes

- planes are infinite
- Every plane has an equation that describes it
- The solutions to such an equation are all in the same plane - they represent it
- A plane can be defines using as few as three (non-colinear) points
- Thes epoints can allow you to find an equation Using a point P on the plane, a vector $\vec{n}$ (called a normal vector) perpendicular ($\perp$) to the plane with endpoint Q, and another vector in the plane ending in point A

Observations

1. point A is on the plane if and only if (iff) the angle between $\vec{PA}$ and $\vec{n}$ is 90°
2. the point A is on the plane iff $\vec{n} \cdot \vec{PA} = 0$

Assume $\vec{n} = \langle a, b, c \rangle$, $P = \langle x_0, y_0, z_0 \rangle$ Then from observation 2, point $A = \langle x, y, z \rangle$ is on the plane iff

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

(meaning that the vector created by $\vec{PA}$ is perpendicular to $\vec{n}$)

Scalar form: (scalar because it can be multiplied by a constant and describe the same plane)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

Linear form:

$$ax + by + cz = d$$

$$where$$

$$d = ax_0 + by_0 + cz_0$$

Proposition

Any of these two equations describes a plane for which $\vec{n} = \langle a, b, c \rangle$ is a normal vector to the plane

▼ Expand to show example 4

Find an equation of the plane through the point (2,4,-1) with the normal vector $\vec{n} = \langle 2, 3, 4 \rangle$.

Solution

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

(General scalar form of plane equation) In this case, we have $\vec{n} = \langle 2, 3, 4 \rangle = \langle a, b, c \rangle$

$$\langle x_0, y_0, z_0 \rangle = \langle 2, 4, -1 \rangle$$

(can be any point on the plane) So:

$$2(x-2) + 3(y-4) + 4(z+1) = 0$$

$$\downarrow$$

$$2x + 3y + 4z = 12$$

▼ Expand to show example 5

Find an equation of the plane through the points P = (1,3,2), Q = (3,-1,6), and R = (5,2,0).

Solution

The points form a triangle, and because of the above properties of planes, it describes a plane

What we need

1. A point on the plane - any of the given points, ex. use P = (1,3,2)
2. A normal vector $\vec{n}$ of the plane - needs to be found

- to get this, use $\vec{PQ} \times \vec{PR}$ because it gives a vector perpendicular to both, and both vectors are in the plane

$$\vec{n} = \vec{PQ} \times \vec{PR}$$

$$\vec{PR} = \langle 2, -4, 4 \rangle$$

$$\vec{PR} = \langle 4, -1, -2 \rangle$$

$$\vec{n} = \langle 2, -4, 4 \rangle \times \langle 4, -1, -2 \rangle$$

$$\begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix} = i(8+4) - j(-4-16) + k(-2+16) = \langle 12, 20, 14 \rangle$$

So now we can plug in values to scalar form:

$$12(x-1) + 20(y-3) + 14(z+2) = 0$$

$$\downarrow$$

$$12x + 20y + 14z = 12 + 60 + 28 = 100$$

And simplifying:

$$6(x-1) + 10(y-3) + 7(z+2) = 0$$

$$\downarrow$$

$$6x + 10y + 7z = 50$$

▼ Expand to show example 6

Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$

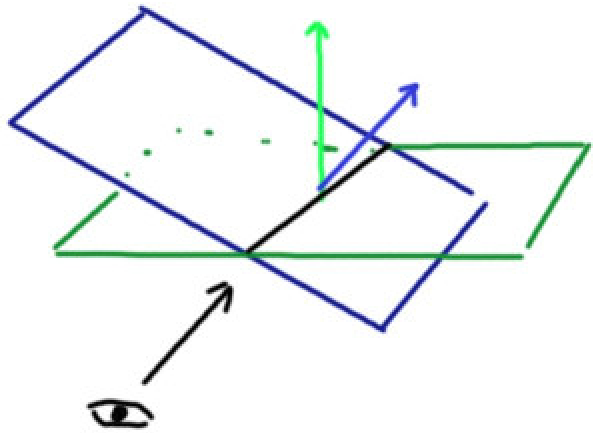
Solution

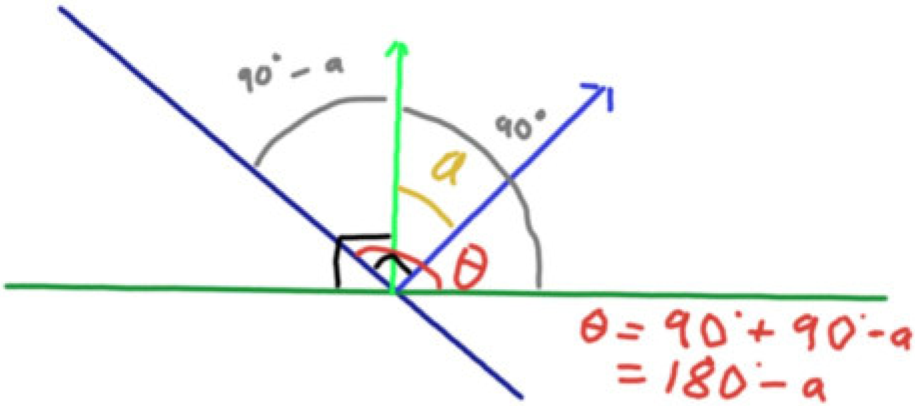
The two planes cross eachother, and we want the angle between them. We can take the normal vectors of these equations as:

$$\vec{n}_1 = \langle 1, 1, 1 \rangle$$

$$\vec{n}_2 = \langle 1, -2, 3 \rangle$$

We want to find the angle between the planes (α) using the angle between the vectors (θ). This is found using $\theta = 180 - \alpha$ (use the larger by convention), and θ can be found using a





dot product.

$$\vec{n}_1 \cdot \vec{n}_2 = |\vec{n}_1| * |\vec{n}_2| * \cos(\theta)$$

$$\cos(\theta) = \frac{(\vec{n}_1 \cdot \vec{n}_2)}{|\vec{n}_1| \; |\vec{n}_2|}$$

$$\vec{n}_1 \cdot \vec{n}_2 = 1 * 1 + 1 * -2 + 1 * 3 = 2$$

$$|\vec{n}_1| = \sqrt{1+1+1} = \sqrt{3}$$

$$|\vec{n}_2| = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{n}_1| \; |\vec{n}_2| = \sqrt{42}$$

$$\cos(\theta) = \frac{2}{\sqrt{42}}$$

$$\theta = \arccos\left(\frac{2}{\sqrt{42}}\right)$$