Tangent Planes

Partial Derivatives

From last lesson, a multivariable function can be derived in two parts, one for f_x and one for f_y.

These partial derivatives can each be derived once more to obtain a double derivative:

$$f_{xx} = \frac{d}{dx} f_x(x, y)$$

$$f_{xy} = \frac{d}{dy} f_x(x, y)$$

$$f_{yx} = \frac{d}{dx} f_y(x, y)$$

$$f_{yy} = \frac{d}{dy} f_y(x, y)$$

▼ Click to show lecture example

$$z = f(x, y) = x^2 y + e^{xy}$$

$$\frac{df}{dx} = f_x = 2xy + ye^{xy}$$

$$f_{xx} = 2y + y^2 e^{xy}$$

Tangent planes

A function of two variables will not have a tangent line, but a tangent plane. At any single point, a surface (hat is continuous and differentiable) will have two tangent lines, one in the y and one in the x direction. We want the equation of the plane that contains those two lines, which will be tangent to that point.

The equation of the tangent plane at the point $P\,(x_0, y_0, z_0)$ is given by the equation

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(Essentially point-slope form for three dimensions)

$$z - z_0 = m_x(x - x_0) + m_y(y - y_0)$$

▼ Click to show Example 1

Question 1

Find the tangent plane to the ellliptic paraboloid $z = 2x^2 + y^2$ at the point $P\,(1, 1, 3)$

Solution:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f_x = 4x_0 = 4$$

$$f_y = 2y_0 = 2$$

$$z - 3 = 4(x - 1) + 2(y - 1)$$

Chain Rule with two-variable Functions

Chain Rule One

Given a two-variable function $z = f(x, y)$, use the two portions as functions themselves. $x = g(t), y = h(t)$

$$z(t) = f(g(t), h(t))$$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$$

▼ Click to show Example

Question 2

Let $z = x^2y + 3xy^4$, where $x = \sin(2t), y = \cos(t)$. Find $\frac{dz}{dt}$ at $t = 0$

Solution

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}(2\cos(2t)) + \frac{\delta f}{\delta y}(-\sin(t))$$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}(2) + \frac{\delta f}{\delta y}(0)$$

$$\frac{dz}{dt} = (2yx + 3y^4)(2)$$

$$\frac{dz}{dt} = (2y(t)x(t) + 3(y(t))^4)(2)$$

$$\frac{dz}{dt} = (2y(0)x(0) + 3(y(0))^4)(2)$$

$$\frac{dz}{dt} = (2\cos(0)\sin(0) + 3(\cos(0))^4)(2)$$

$$\frac{dz}{dt} = (3)(2) = 6$$