Arc Length and Curvature

Arc Length

For a curve, if we want to find the length from P_1 to P_2, we use the length of straight lines at many points along the line and add them all up.

So, we want the integral of the length from $\vec{r}(t)$ to $\vec{r}(t+h)$ as $h \to 0$. So:

$$\int_a^b |\vec{r}(u)|' du$$

$$\int_a^b \left(\frac{d}{du}\sqrt{x^2+y^2+z^2}\right) du$$

$$\int_a^b \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

$$Or$$

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, du$$

Formal Definition:

So, given $\vec{r}_i$ at $r(t_i)$, and $\vec{r}_{i+1}$ at $r(t_{i+1})$,

$$\sum_{i=0}^{N} |\vec{r}_{i+1} - \vec{r}_i| = \sum_{i=0}^{N} \frac{|\vec{r}(t_i + \Delta t_i) - \vec{r}(t_i)|}{\Delta t_i} \Delta t_i$$

You may notice that the inner portion of this equation is the same as the derivative of a function ($f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$)

So as $\Delta t_i \to 0$, we can rewrite it as:

$$\sum_{i=0}^{N} |\vec{r}(t_i)|' \Delta t_i \to \int_{t_i}^{t_{N+1}} |\vec{r}(u)|' du$$

Then, by plugging in the equation for the length of a function (and taking the derivative with respect to u) we get:

$$\int_a^b \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Equation for the arc length

If the curve C is parameterized according to $r(t) = \langle f(t), g(t), h(t)\rangle$ then for $t = a$ & $t = b, a > b$ the length of the arc on C that is in between the points is

$$\int_a^b |\vec{r}'(u)|\, du = \int_a^b \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du$$

Relation to 2D

Let $y = f(x)$ for the curve C. If:

$$\begin{cases} x = t \\ y = f(t) \\ z = 0 \end{cases}$$

$$L = \int_a^b \sqrt{(t')^2 + (f')^2 + (0)^2}$$

$$\int_a^b \sqrt{1 + (f'(t))^2}$$

Section 13.3

▼ Click to show Example 1

Question 1

Curve C = $\vec{r}(t) = \cos(t)i + \sin(t)j + tk$. Find the length of arc on C from $(1, 0, 0)$ to $(1, 0, 2\pi)$

Solution

The start point is at $\vec{r}(0)$ and the end is $\vec{r}(2\pi)$

$$L = \int_0^{2\pi} \sqrt{(\cos'(u))^2 + (\sin'(u))^2 + (u')^2}\, du$$

$$L = \int_0^{2\pi} \sqrt{(-\sin(u))^2 + (\cos(u))^2 + 1}\, du$$

$$L = \int_0^{2\pi} \sqrt{1+1}\, du = \sqrt{2}\ \Big|_0^{2\pi} = \sqrt{2} * 0 + \sqrt{2} * 2\pi$$

Arc Length Function

A function that gives the length from $t = 0$ to t, denoted by $L = s(t)$. If we use these values in the above function, we get:

$$s(t) = \int_0^t |\vec{r}'(u)|\, du$$

Property

Similarly, we can take the derivative to get:

$$\frac{ds}{dt} = |\vec{r}'(t)|$$

Creating a Natural Equation

This is a more detailed paraphrasing of both the lecture and the book. I am still not sure how all of it works together, but what is below seems to make sense. If you see a mistake, please tell me.

The equation $\vec{r}(t)$, because t is a somewhat arbitrary value, has certain properties that have no relation to the curve itself. For example, the point moves with a certain "velocity" (given by the length of the derivative $\mid \vec{r}'(t) \mid$, as in the speed that the point changes as t moves). This is entirely unrelated to the properties of the curve itself. Therefore, to get a more "natural" representation of the curve, we want the create a function that depends on s, not a particular coordinate system.

This means that instead of $\vec{r}(t)$ we are using $\vec{r}_0(s)$, where s is the length of the arc.

To get that function, we need a way to get from $\vec{r}(t)$ to $\vec{r}_0(s)$. The best way to do this is to define t in terms of s and plug that into $\vec{r}(t)$. So, we are looking for $t(s)$. Since we already have a function $s(t)$ (the length function), we can take the inverse to get $t(s)$.

$$t(s) = s^{-1}(t)$$

To get this, it is useful to use the property above and integrate:

$$\frac{ds}{dt} = \mid \vec{r}'(t) \mid$$

$$s(t) = \int_0^t \mid \vec{r}'(u) \mid du = \int_0^t \frac{ds}{dt} du$$

$$t(s) = s^{-1}(t) = inverse\,(\int_0^t \frac{ds}{dt} du)$$

(See example 2 for more applicable clarification)

Then, we plug that into $\vec{r}(t)$ to get the new function.

$$\vec{r}_0(s) = \vec{r}(t(s))$$

This function is useful because it is only dependent on the curve itself, and has no arbitrary reliance on a specific coordinate system

How he explained it:

We want to find a parametrization $\vec{r}_0(s)$ so that the "travel speed" with respect to $\vec{r}_0(s)$vis constant & is = to 1

$$\int_a^b \mid \vec{r}'(u) \mid du = b - a$$

$$\Downarrow$$

$$\mid \vec{r}'(u) = 1 \mid$$

then $\int_0^s \mid \vec{r}_0'(u) \mid du = s$ One can define $\vec{r}_0(s) = \vec{r}(t(s))$

$$s(t) = f(t) = \int_0^t \mid u'(t) \mid dt$$

$$t(s) = f^{-1}(s)$$

▼ Click to show Example 2

Question 2

$\vec{r}(t) = \cos(t)i + \sin(t)j + tk$ Reparameterize it with respect to arc length measured from (1,0,0) in the direction ofincreasing t.

Solution

The start point is (1,0,0), and the direction is +t

Find $s(t)$:

$$s(t) = \int_0^t \mid \vec{r}'(u) \mid du = \sqrt{2}t$$

find the inverse (t(s))

$$t(s) = \frac{s}{\sqrt{2}} \rightarrow \vec{r}_0(s) = \vec{r}(t(s))$$

$$= \cos(\frac{s}{\sqrt{2}})\hat{i} + \sin(\frac{s}{\sqrt{2}})\hat{j} + s/\sqrt{2}\hat{k}$$