Summary, sections 14.1 - 14.8

Chapter 14 - Partial Derivatives and their Uses

As a quick note before we get into the rest: this chapter focuses on differentiating multivariable functions and using those derivatives. Multivariable functions are defined as a function that goes from a pair or set of real numbers to a single real number.

$$y = f(x)$$

Becomes

$$z = f(x, y)$$

Or even

$$s = f(x, y, z)$$

Partial Derivatives

The partial derivative of a multivariable function depends on the direction it is taken in. The most common are the derivatives in the directions of x and y:

$$f_x = \frac{d}{dx} f(x, y)$$

$$f_y = \frac{d}{dy} f(x, y)$$

This also works with higher numbers of variables:

$$f_z = \frac{d}{dz} f(x, y, z)$$

These derivatives can be found by simply differentiating with respect to one of the variables and treating the other as a constant

▼ Click to show examples

Question

Find partial derivatives for the following functions:

$$f(x, y) = 2x + 6y$$

$$f(x,y) = 2xy + 4x^2 - 5y^2x$$ $$f(x,y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = y \ln x$$

Solution

$$f_x = 2, f_y = 6$$

$$f_x = 2y + 8x - 5y^2$$

$$f_y = 2x - 10xy$$

$$f_x = \frac{x}{\sqrt{(x^2 + y^2)}}$$

$$f_y = \frac{y}{\sqrt{(x^2 + y^2)}}$$

$$f_x = \frac{y}{x}$$

$$f_y = \ln x$$

Second derivatives require differentiating each part into more parts:

$$f_{xx} = \frac{d}{dx} f_x$$

$$f_{xy} = \frac{d}{dy} f_x$$

$$f_{yx} = \frac{d}{dx} f_y$$

$$f_{yy} = \frac{d}{dy} f_y$$

▼ Click to show lecture example

$$z = f(x, y) = x^2 y + e^{xy}$$

$$\frac{df}{dx} = f_x = 2xy + ye^{xy}$$

$$f_{xx} = 2y + y^2 e^{xy}$$

Chain Rule One

Given a two-variable function $z = f(x, y)$, use the two portions as functions themselves. $x = g(t), y = h(t)$

$$z(t) = f(g(t), h(t))$$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$$

▼ Click to show Example

Question Let $z = x^2 y + 3xy^4$, where $x = \sin(2t), y = \cos(t)$. Find $\frac{dz}{dt}$ at $t = 0$

Solution

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}\frac{dx}{dt} + \frac{\delta f}{\delta y}\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}(2\cos(2t)) + \frac{\delta f}{\delta y}(-\sin(t))$$

$$\frac{dz}{dt} = \frac{\delta f}{\delta x}(2) + \frac{\delta f}{\delta y}(0)$$

$$\frac{dz}{dt} = (2yx + 3y^4)(2)$$

$$\frac{dz}{dt} = (2y(t)x(t) + 3(y(t))^4)(2)$$

$$\frac{dz}{dt} = (2y(0)x(0) + 3(y(0))^4)(2)$$

$$\frac{dz}{dt} = (2\cos(0)\sin(0) + 3(\cos(0))^4)(2)$$

$$\frac{dz}{dt} = (3)(2) = 6$$

Chain Rule Two

Given a two-variable function $z = f(x, y)$, use the two portions as multivariable functions themselves. $x = g(s, t)$, $y = h(s, t)$

$$\frac{\delta z}{\delta s} = \frac{\delta z}{\delta x} * \frac{\delta x}{\delta s} + \frac{\delta z}{\delta y} * \frac{\delta y}{\delta s}$$

$$\frac{\delta z}{\delta t} = \frac{\delta z}{\delta x} * \frac{\delta x}{\delta t} + \frac{\delta z}{\delta y} * \frac{\delta y}{\delta t}$$

▼ Click to show Example 3

Question

$z = e^x \sin(y)$, where $x = st^2$, $y = s^2t$. Find $\frac{\delta z}{\delta s}$

Solution

$$\frac{\delta z}{\delta s} = \frac{\delta z}{\delta x} * \frac{\delta x}{\delta s} + \frac{\delta z}{\delta y} * \frac{\delta y}{\delta s}$$

$$\frac{\delta z}{\delta s} = e^x \sin(y) * t^2 + e^x \cos(y) * 2st$$

This is not desirable because it depends on 4 variables, so plug st^2 and s^2t for x and y

$$\frac{\delta z}{\delta s} = e^{st^2} \sin(s^2t) * t^2 + e^{st^2} \cos(s^2t) * 2st$$

Directional Derivative

The derivative can also be taken in a certain direction. So, we just add together the parts of the vector multiplied by the gradient. The formula for this is:

$$D_{\vec{u}} f(x, y) = f_x(x, y) * a + f_y(x, y) * b$$

Which means that the derivative in the direction of $\vec{u} = \langle a, b \rangle$ is equal to the partial derivatives multiplied by the components of $\vec{u}$

Gradient Vector

The gradient is the vector made of the two partial derivatives.

$$\nabla f = \langle f_x f_y \rangle$$

This gradient vector defines the direction of the largest slope.

Local Maximum and Minimum

The local maximum and minimum can be found by finding critical points, then checking to see if they are max, min, saddle, or unknown

Critical Points

Critical points are all the places where $f'(x, y) = 0$. Meaning:

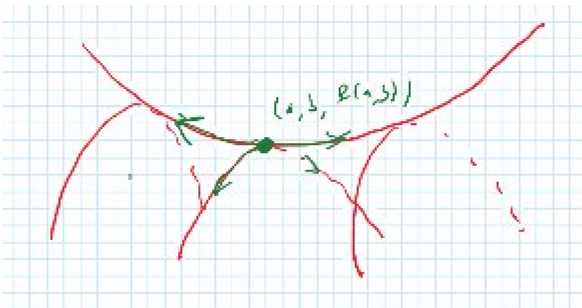
$$f_x = f_y = 0$$

Second Derivatives Test

We can use the second derivative to see if they are maximums or minimums.

$$D(x, y) = \begin{vmatrix} f_{xx} f_{xy} \\ f_{yx} f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy} f_{yx}$$

1. If $D(a, b) > 0$, we have (a, b) is an extremum point.
    1. If $f_{xx}(a, b) > 0$, then $f(a, b)$ is a *local minimum* (concave up)
    2. If $f_{xx}(a, b) < 0$, then $f(a, b)$ is a *local maximum* (concave down)
    3. If $f_{xx}(a, b) = 0$, then the test is inconclusive
2. If $D(a, b) < 0$, then $f(a, b)$ is a *saddle point*
3. If $D(a, b) = 0$, then the results are inconclusive

Saddle points are like this: 

▼ Click to show example for local max and min

Question

Find the local max and min of the function $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 1$

Solution

First, we find the critical points of the function, then use the second derivative test to check wheter they are maximum or minimum.

- find where $f_x = f_y = 0$
- of those, find which ones $D > 0$ (if $D < 0$, it is a saddle point)
- of those, if $f_{xx} < 0$, it is a **maximum**, if $f_{xx} < 0$, it is a **minimum**

$$f_x = 6xy - 6x = 0$$

$$f_y = 3x^2 + 3y^2 - 6y = 0$$

The first equation gives that $x = 0$ or $y = 1$. By plugging those into the second equation:

- if $x = 0, y = 0, 2$
- if $y = 1, x = 1, -1$

These are the critical points. $(0, 0), (0, 2), (1, 1), (-1, 1)$

So we test these points in the second derivative. (keep in mind that $f_{xy} = f_{yx}$)

$$D = f_{xx}f_{yy} - f_{xy}f_{yx}$$

$$D = (6y - 6)^2 - (6x)^2$$

- $D(0,0) = 36 > 0$
- $D(0,2) = 36 > 0$
- $D(1,1) = -36 < 0$ - it is a saddle point
- $D(-1,1) = -36 < 0$ - it is a saddle point

So now we test in f_{xx}

- $f_{xx}(0,0) = -6 < 0$, so $(0,0)$ is a maximum
- $f_{xx}(0,2) = 6 > 0$, so $(0,2)$ is a minimum

Absolute Maximum and Minimum

Finding Max and Min on the Boundary of the Domain

The absolute max and min can be found by checking all local max and min point in the closed domain D along with all the critical points at the edge of the domain

▼ Click to show example 7 (14.7)

Question

Find the absolute max and min values of the function $(x, y) = x^2 - 2xy + 2y$ On the rectangle $D = (x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2$

Solution

Since D is closed, an absolute max and min are obtainable.

1. Find the critical points (x,y):

$$\frac{\delta f}{\delta x}(x, y) = 2x - 2y = 0 \rightarrow x = y$$

$$\frac{\delta f}{\delta y}(x, y) = -2x + 2 = 0 \rightarrow x = 1$$

   Therefore, $(1,1)$ is the only critical point. $f(1,1) = 1^2 + 2 * 1 * 12 * 1 = 1$
2. Find the points at the boundary of D We split the edges into four parts: $D =$

$$(0, y) \mid 0 \leq y \leq 2U$$

$$(x, 2) \mid 0 \leq x \leq 3U$$

$$(3, y) \mid 0 \leq y \leq 2U$$

$$(x, 0) \mid 0 \leq x \leq 3$$

$$0 \leq f(0, y) = 2y \leq 4$$

$$0 \leq f(x, 2) = x^2 - 4x + 4 = (x - 2)^2 \leq 4$$

$$1 \leq f(3, y) = 9 - 6y + 2y = 9 - 4y \leq 9$$

$$0 \leq f(x, 0) = x^2 \leq 9$$

   These points are the edges, and their max is 9 and min is 0. These are at the points $(3,0)$ and $(0,0)$. These are absolute (and local) maximum and minimum respectively

Lagrange Multipliers

Lagrange multipliers are a way to find the absolute max and min of a function $f(x, y)$ given a constraint $g(x, y) = k$

It states:

$$f(x, y) = \lambda g(x, y)$$

To solve these problems, we take the derivatives of each function and set them into these equations, and use the system of equations to solve for x, y

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g(x, y) = k \end{cases}$$

▼ Click to show Example for Lagrange

Question

Find max and min values for $f(x, y) = 8x^2 - 2y$ provided that $x^2 + y^2 = 1$ (= $g(x, y)$)

Solution

First we find all the points on (x,y), with some scalar(constant) λ that the following is true:

$$\{ \begin{matrix} \nabla f = \lambda \nabla g \\ g(x, y) = 1 \end{matrix}$$

Then, we choose among the pairs (x,y) that we found that are the largest/smallest

So, we comute the partial derivatives of each function, and set them equal with a scalar λ

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$\ldots$$

So, for this we have:

$$\begin{cases} 16x = 2\lambda x \\ -2 = 2\lambda y \\ x^2 + y^2 = 1 \end{cases}$$

Step 1: For the first equation, we have $x = 0$ or $\lambda = 8$. Using these in the third equation shows $y = \pm 1$. Therefore, $(0, -1)$ and $(0, 1)$ are candidates

If $\lambda = 8$, then according to the second equation, $y = -1/8$. Using this in the third equation, we get $x = \pm\sqrt{1 - \frac{1}{64}}$

Step 2: Finding the candidate points for which $f(x, y)$ takes smallest and largest values.

This just means we evaluate the function at each point and take the largest and smallest values.

- $f(0, -1) = 8 \ast 0^2 - 2(-1) = 2$
- $f(0, 1) = 8 \ast 0^2 - 2(1) = -2$

- $f(\pm\sqrt{1-\frac{1}{64}}, -1/8) = 8 * (1-\frac{1}{64}) - 2(-1/8) = 8 - 1/8 + 1/4 = 8 + 1/8$

Answer: The absolute max of $f(x, y)$ when $x^2 + y^2 = 1$ is equal to max$(2, -2, 8 + 1/8) = 8 + 1/8$
Absolute minimum is min$(2, -2, 8 + 1/8) = -2$