Directional Derivatives

Formal Definition

Given a unit vector $\vec{u} = \langle a, b \rangle$, $\sqrt{a^2 + b^2} = 1$, then the directional derivative of $f(x, y)$ at (x, y) is :

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

This represents the slope along the tangent line of the function along the unit vector's direction.

The partial derivatives $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$ are particular instances of directional derivatives. Specifically,

$$\frac{\delta f}{\delta x} = D_{\hat{\imath}} f, \hat{\imath} = \langle 1, 0 \rangle$$

$$\frac{\delta f}{\delta y} = D_{\hat{\jmath}} f, \hat{\jmath} = \langle 0, 1 \rangle$$

Theorem

If $\vec{u} = \langle a.b \rangle$, then

$$D_{\vec{u}} f(x, y) = f_x(x, y) \ast a + f_y(x, y) \ast b$$

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Question

Find $D_{\vec{u}} f(x, y)$, where $f(x, y) = x^3 - 3xy + 4y^2$ and $\vec{u} = \langle 1/2, \sqrt{3}/2 \rangle$

Solution

First, find $\frac{\delta f}{\delta x}$ and $\frac{\delta f}{\delta y}$

$$\frac{\delta f}{\delta x} = 3x^2 - 3y$$

$$\frac{\delta f}{\delta y} = -3x + 8y$$

$$D_{\vec{u}} f(x, y) = \frac{1}{2}(3x^2 - 3y) + \frac{\sqrt{3}}{2}(-3x + 8y)$$

Gradient vector

$$\langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y} \rangle$$

Notation for gradient is ∇f. Namely, $\nabla f = \langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y} \rangle$

By the theorem above, we have that

$$D_{\vec{u}} f(x, y) = \frac{\delta f}{\delta x} \ast a + \frac{\delta f}{\delta y} \ast b$$

$$D_{\vec{u}}f(x, y) = \langle a.b \rangle \cdot \langle \frac{\delta f}{\delta x}, \frac{\delta f}{\delta y} \rangle$$

Therefore,

$$D_{\vec{u}}f(x, y) = \vec{u} \cdot \nabla f$$

Gradient vector defines the direction of the largest slope

Why is this true?

$$\mid D_{\vec{u}}f(x, y) \mid \ = \nabla f \cdot \vec{u}$$

$$= \ \mid \nabla f \mid \ * \ \mid \vec{u} \mid \ * \cos\theta$$

$$= \ \mid \nabla f \mid \ * \cos\theta \leq \ \mid \nabla f \mid$$

Moreover, $D_{\vec{u}}f(x, y) = \ \mid \nabla f \mid$ iff $\cos\theta = 1$ ($\theta = 0$), in which case u has the direction of ∇f and $D_u f = \ \mid \nabla f \mid$