Examples of Polar Integrals

Review

Remember that we learned the integral of $f(x, y)$ over a polar rectangle R is:

$$\iint\limits_R f(x,y)dxdy = \int\limits_\alpha^\beta \int\limits_a^b f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

Where r, θ can represent polar coordinates as an angle and distance.

▼ Click to show example 1

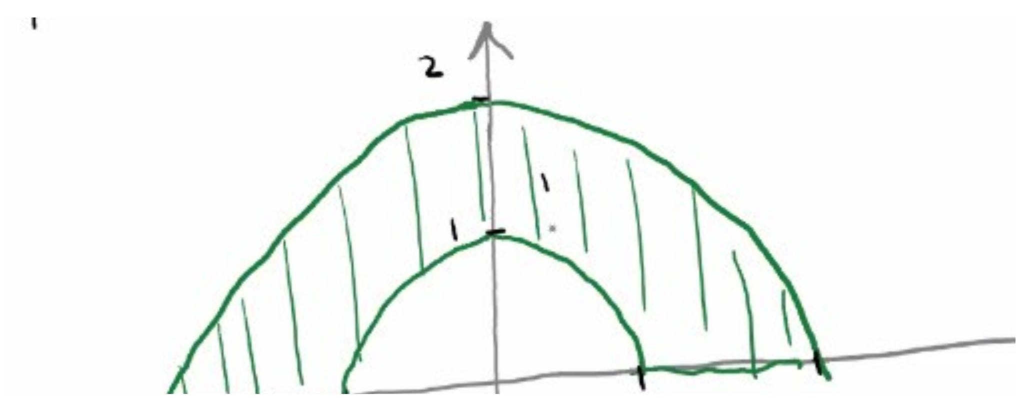
Question 1 (Sec. 15.3)

Find $\iint\limits_R (3x + 4y^2)dxdy$, where R is the region in the upper-half plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 2^2$

Solution

First we determine the domain. In this case, the area is a polar rectangle bounded by $r = 1, 2$ and $\theta = 0, \pi$

$$R = \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi\} = [1, 2] \times [0, \pi]$$



So we can use the formula:

$$\iint_R (3x+4y^2)dxdy = \int_0^{\pi}\int_1^{2}(3r\cos\theta + 4(r\sin\theta)^2)rdrd\theta$$

$$= \int 3r^2\cos\theta + 4r^3\sin^2\theta$$

$$= [r^3\cos\theta + r^4\sin^2\theta]|_1^2$$

$$= \int_0^{\pi}[7\cos\theta + 15\sin^2\theta]d\theta$$

$$\int sin^2\theta = \int \frac{1-\cos 2\theta}{2}$$

$$= [7\sin\theta + \frac{15}{2}(\theta - \frac{1}{2}\sin 2\theta)]|_0^{\pi}$$

$$= \frac{15}{2}\pi$$

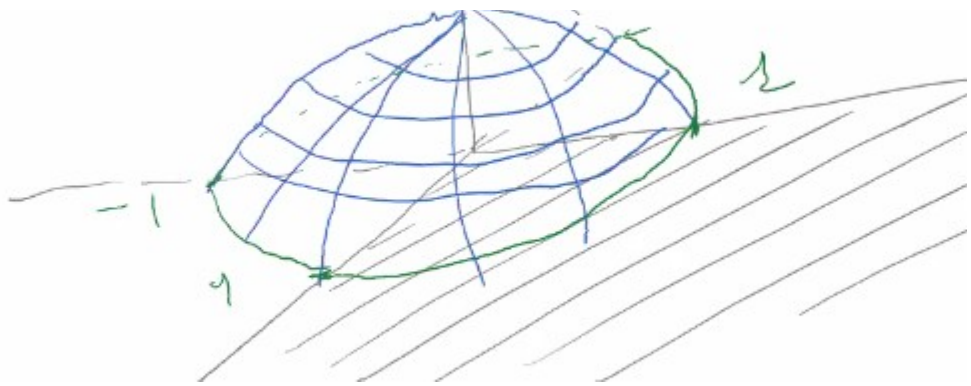
▼ Click to show example 2

Question 2

Find the volume of the solid bounded by the plane $z=0$ and the pararboloid $z=1-x^2-y^2$

Solution

First, we determine the bounds. Since the volume is a dome, the domain R is a circle with a radius of where the paraboloid intersects $z=0$, which is 1:



$$R = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le 2\pi\} = [0,1] \times [0,2\pi]$$

$$\iint\limits_D (1-x^2-y^2)dxdy = \int\limits_0^{2\pi}\int\limits_0^1 (1-(r\cos\theta)^2-(r\sin\theta)^2)rdrd\theta$$

$$= (1-r^2(\cos^2\theta+\sin^2\theta))$$

$$= \int_0^1 (1-r^2)rdr$$

$$\left(\frac{1}{2}r^2 - \frac{1}{4r^4}\right)|_0^1$$

$$= \int\limits_0^{2\pi} \frac{1}{4}d\theta$$

$$= \frac{\pi}{2}$$

Preview of next lecture

Next lecture we will be covering domains that are polar and have a variable radius or angle bounds.