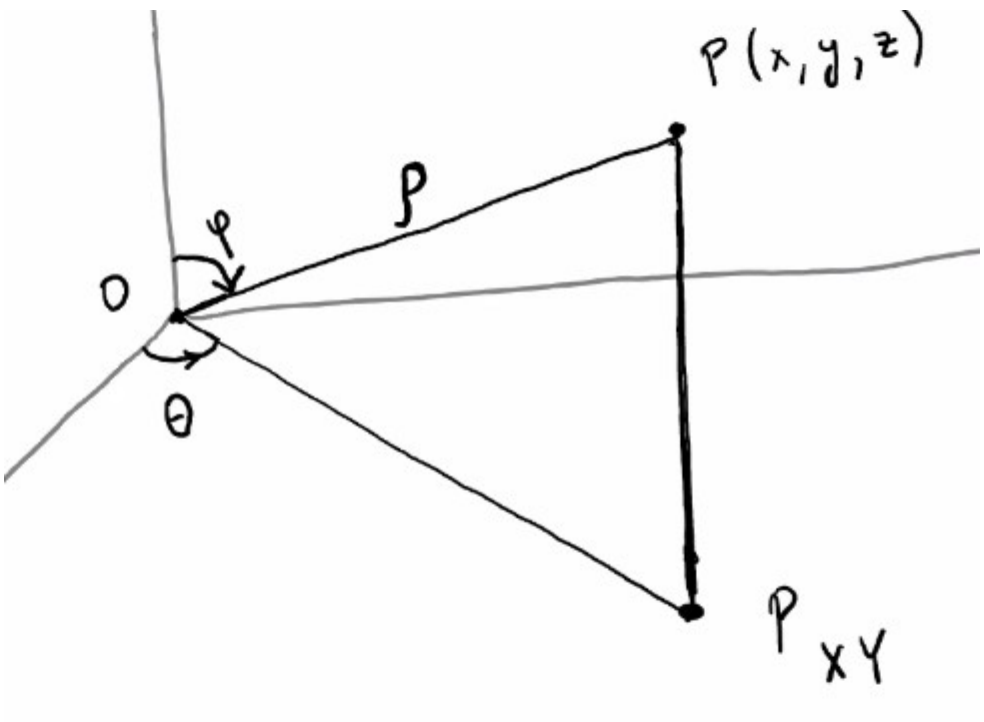
Triple Integrals In Spherical Coordinates

Explaining Spherical Coordinates

Similar to cylindrical coordinates, spherical coordinates represent part of the cartisian coordinate system as polar representations. In this case, however, the entire set is polar, instead of just xy. This means a point in polar coordinates is represented by:

$$P = (\rho, \theta \, \phi)$$



Equations in polar coordinates

Similar to normal polar coordinates, spherical coordinates can be used in functions. For example, the equation $\rho = C$ is the equation representing a sphere. Similarly, $\rho < C$ is a solid sphere.

This shows that polar coordinates are very useful for showing round objects with simplicity.

Another example: $\rho \cos \phi = 1$ This exuation can be interpreted as $z = 1$, because when converting from polar to cartesian, $z = \rho \cos \phi$

In this case, the equation should have been left as cartesian, because it becomes more complex in polar coordinates.

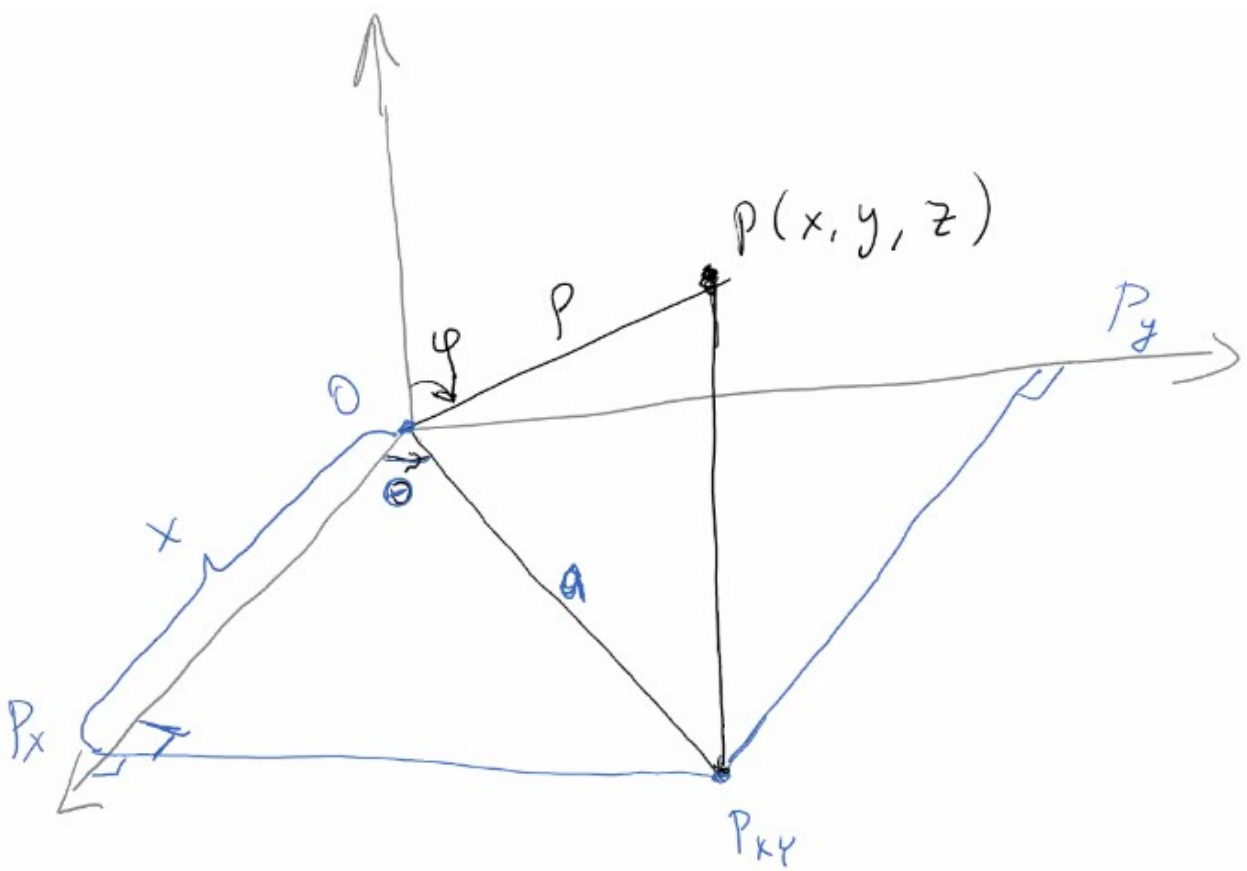
A final example: $\rho = \cos \phi$ In this case, the object ends up being sphereical object not centered at the origin, an equation that would be more complex in cartesian coordinates.

Conversion

Cartesian coordinates can be converted to spherical using the following relationships:

$$\begin{cases} x = \rho \cos \theta \sin \phi \\ y = \rho \sin \theta \sin \phi \\ z = \rho \cos \phi \end{cases} \cong \begin{cases} \rho = z^2 + y^2 + z^2 \\ \theta = \arctan(y/x) \\ \phi = \arccos(\rho/z) \end{cases}$$

These come from the fact that trigonometric identities can be used by representing this point as a pair of triangles: xy and θz



We can find a as

$$a = \frac{x}{\cos\theta} = \rho\sin\phi$$

So we can find that

$$x = \rho\cos\theta\sin\phi$$

Similarly,

$$y = \rho\sin\theta\sin\phi$$

z is simpler, as it can be found with just the first triangle:

$$z = \rho\cos\phi$$

Example

$z = x^2y^2$ in spherical coords is:

$$\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$$

$$\cos\phi = \rho^3\sin^4\phi\sin^2\theta\cos^2\theta$$

Spherical Integrals

Domains

First, let's represent a domain in spherical coords:

$$E = \{(\rho, \theta\phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

This can be calles a spherical rectangular box, or more accurately, a spherical wedge.

Integrals

These can be represented as:

$$\iiint\limits_E f(x, y, z)dV = \int\limits_a^b\int\limits_\alpha^\beta\int\limits_c^d f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi \; d\rho d\theta d\phi$$

Note that it is multiplied by $\rho^2\sin\phi$ in the second, similar to conversion to polar in double integrals.

Example

Compute $\iiint\limits_E y^2\ dV$ where E is the solid hemisphere given by $x^2 + y^2 + z^2 < 9$, $y \geq 0$

First, let's convert the domain to spherical coords. Since $\rho^2 = x^2 + y^2 + z^2$, we can see that $0 \leq \rho \leq 3$. By visualizing the domain, we can deduce that $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq \pi$.

By the above equation:

$$\iiint\limits_E y^2 dV = \int\limits_a^b \int\limits_\alpha^\beta \int\limits_c^d (\rho \sin\theta \sin\phi)^2\ \rho^2 \sin\phi\ d\rho d\theta d\phi$$

$$\int\limits_a^b \int\limits_\alpha^\beta \int\limits_c^d \rho^4 \sin^2\theta \sin^3\phi\ d\rho d\theta d\phi$$

This can now be integrated normally