Finding Maximum and Minimum Values of a Function

Finding Max and Min in single variable

From one-variable calculus:

$$y = f(x)$$

The local minimum is the minimum of a portion of the graphp, and the same is true for max. Also:

$$f^{'}(x_{max}) = f(x_{min}) = 0$$

$f^{''}(x) > 0$ - concave up
$f^{''}(x) < 0$ - concave down

Definitions for multivariable

Definition

We say that (a, b) is a local etremum (max or min), if in a small neighborhood (the area around the point) at (a, b), $f(a, b)$ is either maximum or minimum (it is greater than all its neighbors).

Theorem

If (a, b) is an extremum point for f(x,y), then we have:

$$\frac{\delta f}{\delta x}(a, b) = \frac{\delta f}{\delta y}(a, b) = 0$$

Definition

We say that (a, b) is a critical point if $\frac{\delta f}{\delta x}(a, b) = \frac{\delta f}{\delta y}(a, b) = 0$.

The Theorem above says that if (a, b) is an extremum point, then it must be a critical point.

The inverse is not true

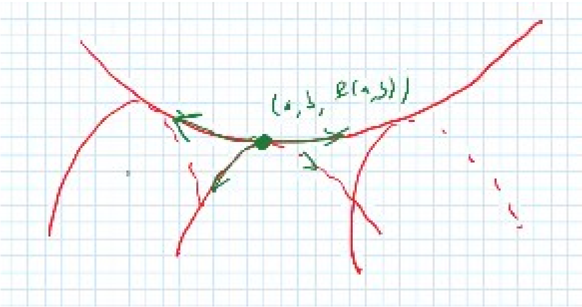
Second Derivative Test

$$D(x, y) = \begin{vmatrix} f_{xx}f_{xy} \\ f_{yx}f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx}$$

Let (a, b) be a critical point for the function $z = f(x, y)$ Then, if

1. If $D(a, b) > 0$, we have (a, b) is an extremum point.
    1. If $f_{xx}(a, b) > 0$, then $f(a, b)$ is a *local minimum* (concave up)
    2. If $f_{xx}(a, b) < 0$, then $f(a, b)$ is a *local maximum* (concave down)
    3. If $f_{xx}(a, b) = 0$, then the test is inconclusive
2. If $D(a, b) < 0$, then $f(a, b)$ is *netiher local maximum nor minimum*
3. If $D(a, b) = 0$, then the results are inconclusive

If $D(a, b) > 0$, then (a, b) is called *saddle point*



▼ Click to show Example 3

Question 3 (Sec 14.7)

Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$

Solution

First, find critical point (a, b): $\frac{\delta f}{\delta x}(a, b) = \frac{\delta f}{\delta y}(a, b) = 0$

$$\left| \frac{\delta f}{\delta x} = 4x^3 - 4y = 0 \frac{\delta f}{\delta y} = 4y^3 - 4x = 0 \right|$$

$$y = x^3$$

$$x^9 - x = 0$$

$$x(x^8 - 1) = 0$$

$$x(x^4 - 1)(x^4 + 1) = 0$$

$$(x^4 + 1) > 0$$

$$x(x^2 - 1)(x^2 + 1) = 0$$

$$(x^2 + 1) > 0$$

$$x(x - 1)(x + 1) = 0$$

$$x = -1, 0, 1$$

$$x^3 = y = -1, 0, 1$$

Therefore, the critical points are $(-1, -1), (0, 0), (1, 1)$

After finding the critical points, let us compute D at that point

$$D(x, y) = f_{xx} f_{yy} - f_{xy} f_{yx}$$

$$D(x, y) = 144x^2y^2 - 16$$

$$f_{xx} = 12x^2 \geq 0$$

$$D(-1, -1) = 144 - 16 = 128 > 0$$

$$f_{xx} = 12(-1)^2 > 0$$

$(-1, -1)$ is a *local min* by the second derivative test

$$D(0, 0) = -16$$

$(-1,-1)$ is a saddle point by the second derivative test

$$D(1,1) = 144 - 16 = 128 > 0$$

$$f_{xx} = 12(1)^2 > 0$$

$(1,1)$ is a *local min* by the second derivative test