Integrals of Multiple Variables

Indefinite integrals represent the set of all antiderivatives of the function.

For most definite integrals, it represent the sum of the areas of many infinitesmal rectangles under the function.

$$\int_a^b f(x)dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_{i+1})\Delta x_i$$

Question:

Can we define integrals that involve functions of two variables? If so, what meaning does it carry.

Multivariable Integrals

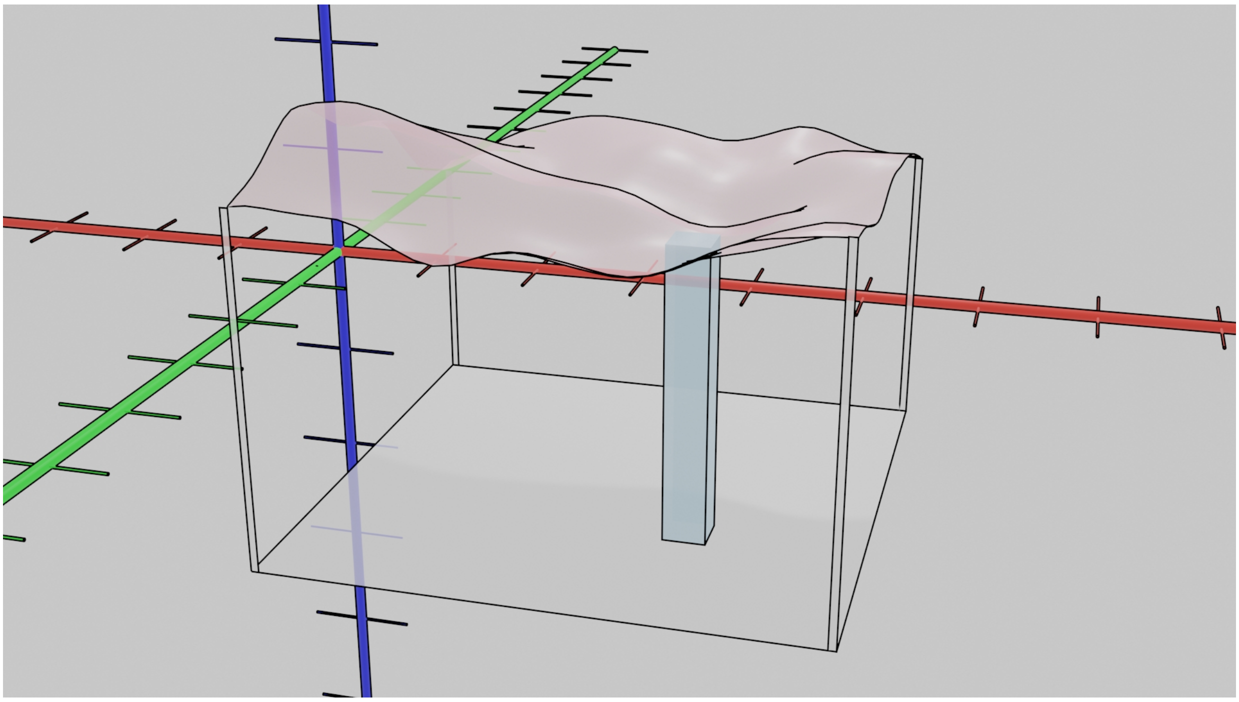
Let $z = f(x, y)$ and D be a bounded domain of this function. The domain is a closed shape on the xy axis, and the graph is a surface who's projection coincides with this shape. The volume under the function can be represented by length by width by height.

Since the graph is often curved, we will need to split it up into many rectangular boxes of different heights.

Each box will have a width and depth of Δx and Δy, and a height of $f(x, y)$. So, we add up all of them from a to b:

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy = \lim_{(\Delta x,\Delta y)\to(0,0)} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_{i+1}, y_{j+1})\Delta x_i \Delta y_j$$

$$\iint_D f(x,y)dxdy = \iint_{[a,b]\times[c,d]} f(x,y)dxdy$$



Example

$$z = f(x,y) = x + 0y = x$$

$$D = [0,1] \times [0,1]$$

The graph will create a prism with sides of 1. The double integral is:

$$\iint_D f(x,y)dxdy$$

$$\iint_{[0,1]\times[0,1]} xdxdy$$

$$\int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_a^b g(x)dx$$

$$\iint_{[0,1]\times[0,1]} xdxdy = \int_0^1 \left(\int_0^1 xdy \right) dx$$

$$g(x) = x\left(\int_0^1 1dy \right) = x * 1 = x$$

$$g(x) = \int_0^1 xdy = xy \mid_0^1 = 0 * x + 1 * x = x$$

$$\int_0^1 \left(\int_0^1 xdy \right) dx = \int_0^1 xdx = \frac{1}{2}x^2 \mid_0^1 = 0 + \frac{1}{2}(1)^2 = \frac{1}{2}$$

More Integrals (Oct 7 Lecture)

$$\iint f(x,y)dxdy, D \subseteq \mathrm{R} \times \mathrm{R}$$

when

$$D = [a,b] \times [c,d]$$

Fubini's Theorem:

$$\iint_D f(x,y)dxdy = \int_a^b \left(\int_c^d f(x,y)dy \right) dx = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

▼ Click to show Example 5 (Section 15.1)

Question 5

Evaluate the double integral $\iint_R (x-3y^2)dxdy$ where $R = (x, y) \mid 0 \le x \le 2, 1 \le y \le 2 = [0,2] \times [1,2]$

Solution

By Fubini, $\iint_R (x-3y^2)dxdy = \int_0^2 (\int_1^2 (x-3y^2)dy)\, dx$

$$(xy - y^3) \mid_1^2 = (2x - 2^3) - (1x - 1^3) = x - 7$$

$$\int_0^2 (x-7)dx = (\frac{1}{2}x^2 - 7x) \mid_0^2 = (2-14) - 0 = -12$$

Average Value (Sec 15.1)

By definition, the average value of g=f(x) that has domain [a,b] is given by $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$.

Average value of a two-variable function is

$$f_{avg} = \frac{1}{Area\ of\ D} \iint_D f(x,y)dxdy$$

Recall if you have numbers $n_1, n_2, ..., n_k$, the average of them = $\frac{n_1+n_2+...+n_k}{k}$

If we take the function from the previous example:

$$f_{avg} = \frac{1}{Area\ of\ R} \iint_R (x-3y^2)dxdy = \frac{1}{2 * 1}(-12) = -6$$