Double Integrals

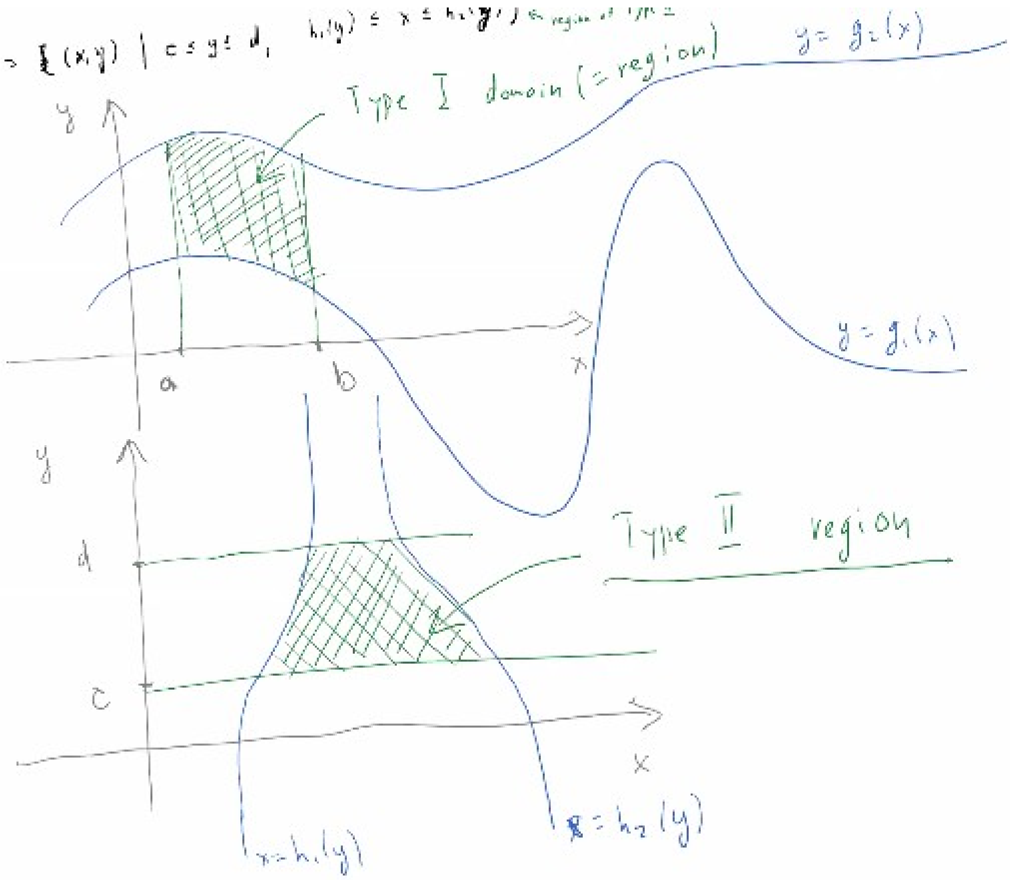
Another method of computing double integrals

This type of strategy works for the following:

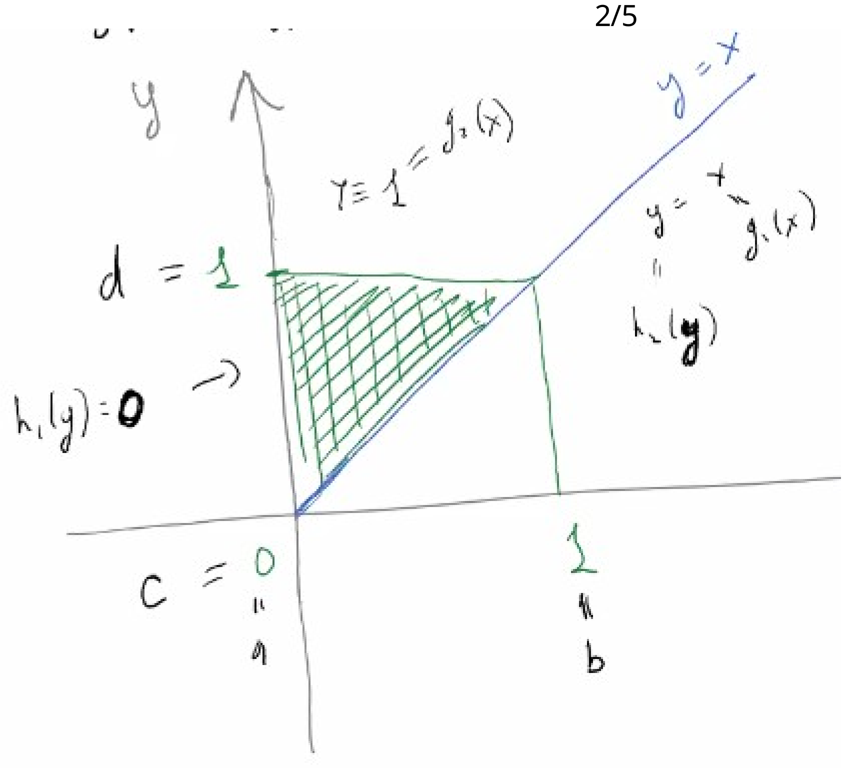
$$\iint_D f(x,y)dxdy$$

D is one of the following:

1. Region of type one: $D = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$
2. Region of type two: $D = \{(x,y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$



Some domains are of both types at the same time. This happens when you can either use g(x) or h(y) to describe the bounds



Theorem / Equation

If D is type one,

$$D = (x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)$$

then we have:

$$\iint_D f(x, y) dxdy = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y) dy \right) dx$$

The same can be done for type two:

$$D = (x, y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)$$

$$\iint_D f(x, y) dxdy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x, y) dx \right) dy$$

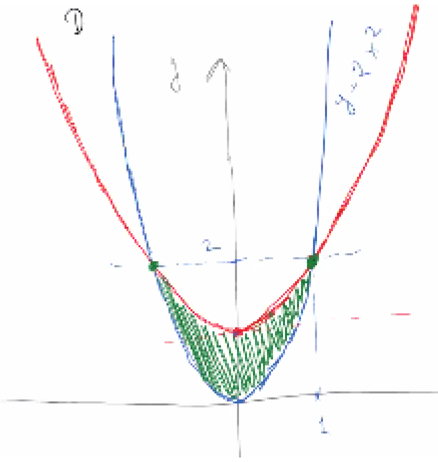
▼ Click to show Example 1 (15.2)

Question

Compute $\iint (x - 2y) dxdy$, where D is the region that is bounded by the functions $y = 2x^2$ and $y = 1 + x^2$

Solution

First, we figure out where the bounded region is.



So we have that the domain is bounded by these functions from -1 to 1:

$$D = (x, y) \mid -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2$$

So we use theorem one:

$$\iint_D (x + 2y)dxdy = \int_{-1}^{1} \left(\int_{2x^2}^{1+x^2} f(x, y)dy \right) dx$$

$$\int_{2x^2}^{1+x^2} f(x, y)dy = (xy + y^2) \mid_{2x^2}^{1+x^2} = [x(1 + x^2) + (1 + x^2)^2] - [x(2x^2) + 4x^4]$$

$$= -3x^4 - x^3 + 2x^2 + x + 1$$

$$\int_{-1}^{1} \left(\int_{2x^2}^{1+x^2} f(x, y)dy \right) dx = \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)dx$$

$$= (\frac{-3}{5}x^5 - \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x) \mid_{-1}^{1}$$

$$\frac{-6}{5} + \frac{4}{3} + 2$$

▼ Click to show Example 2 (15.2)

Question

Find the volume of the solid that is under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by $y = 2x$ and $y = x^2$

Solution

First, determine the volume that is being integrated. This is the volume above the bounded area in the xy-plane, so the domain is type one (defined by two functions of x: y = f(x)), and we just need to find the bounds of a and b on either side. This is the solution of the two functions:

$$2x = x^2$$

$$x^2 - 2x = 0$$

$$x = 0, x = 2$$

$$a = (0,0), b = (2,4)$$

$$D = \{(x,y) \mid 0 \le x \le 2, x^2 \le y \le 2x\}$$

$$Volume = \iint_D (x^2 + y^2)dxdy$$

$$\int_0^2 \int_{x^2}^{2x} (x^2 + y^2)dydx$$

$$\int_{x^2}^{2x} (x^2 + y^2)dy = (x^2y - 1/3y^3) \mid_{x^2}^{2x}$$

$$= \frac{1}{3}x^6 - x^4 + \frac{14}{3}x^3$$

$$\int_0^2 (\frac{1}{3}x^6 - x^4 + \frac{14}{3}x^3)dx = \frac{1}{21}x^7 - \frac{1}{5}x^5 + \frac{7}{6}x^4$$

$$= \frac{716}{6} - \frac{2^7}{21} - \frac{2^5}{5}$$

Solution 2

This can also be done with domain two logic:

$$h_1(y) = \frac{1}{2}y, h_2(y) = y^{1/2}$$

$$D = \{(x,y) \mid 0 \le x \le 4, \frac{1}{2}y \le y \le y^{1/2}\}$$

$$\int_0^4 \int_{\frac{1}{2}y}^{y^{1/2}} (x^2 + y^2)dxdy$$

So to summarize this method:

1. Detrmine which type of domain it is
2. Find the edges of the bounds by finding the solutions of the bounding equations
3. Compute the inner portion of the integral with the bounding functions
4. Compute the outer integral with the results of the inner

▼ Click to show example 5 (15.2)

Question

Evaluate the iterated integral $\int_0^1 \int_x^1 sin(y^2)dydx$

Solution

$$\int_0^1 \int_x^1 \sin(y^2)dydx$$

This integral cannot be nicely integrated, requiring conversion to a Talor polynomial.

$$f(y) = sin(y^2)$$

$$\int_x^1 f(y)dy = \int_x^1 \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} * y^n dy$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \int_x^1 y^n dy$$

Instead, we rewrite it as a couble integral:

$$\iint_D \sin(y^2)dydx$$

$$D = \{(x, y) \mid 0 \leq x \leq 1, x \leq y \leq 1\}$$

This is both a type one and type two domain:

$$D = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$$

$$\iint_D \sin(y^2)dydx = \int_0^1 \int_0^y \sin(y^2)dxdy$$

$$= x\sin(y^2) \mid_0^y = y\sin(y^2)dy$$

$$\int_0^1 y\sin(y^2)dy = -\frac{1}{2}\cos 1 - (1)\frac{1}{2}\cos 0$$

$$= \frac{1}{2}(1 - cos(1))$$