Derivatives and Integrals of Vector Functions

TL;DR: To take the limit, derivative, or integral of a vector function, do the same to each component function.

Derivatives

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

$$\vec{r}^{'}(t) = \langle f^{'}(t), g^{'}(t), h^{'}(t) \rangle$$

$$=$$

$$\lim_{h \to 0} \frac{r(t+h) - r(t)}{h} =$$

$$\langle \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \to 0} \frac{h(t+h) - h(t)}{h} \rangle$$

Given C as the curve given by $\vec{r}(t)$, the point P on the line denoted by $\vec{r}(t)$ has a tangent line with the following properties:

- The tangent line is parallel to the vector $\vec{r}^{'}(t)$.

We call $\vec{r}^{'}(t)$ a **tangent vector** to the curve C.

$T(t) = \frac{\vec{r}^{'}(t)}{|\vec{r}^{'}(t)|}$ is the **unit tangent** vector.

Rules for dot and cross products

$$\frac{d}{dt}[\vec{u}(t) \cdot \vec{v}(t)] = \vec{u}^{'}(t) \cdot \vec{v}^{'}(t) + \vec{u}(t) \cdot \vec{v}(t)$$

$$\frac{d}{dt}[\vec{u}(t) \times \vec{v}(t)] = \vec{u}^{'}(t) \times \vec{v}^{'}(t) + \vec{u}(t) \times \vec{v}(t)$$

Sec 13.2

▼ Click to show Example 1

Question 1:

Find $\vec{r}^{'}(t)$ where $\vec{r}(t) = (1 + t^3)\hat{i} + te^{-t}\hat{j} + sin(2t)\hat{k}$

Solution:

$$r^{'}(t) = (1 + t^3)^{'}\hat{i} + (te^{-t})^{'}\hat{j} + sin(2t)^{'}\hat{k}$$

$$= 3t^2\hat{i} + (e^{-t} - te^{-t})\hat{j} + (2\cos(2t))\hat{k}$$

$$= \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle$$

▼ Click to show Example 2

Question 2:

Find the unit tangent vector at the point P, where $t = 0$

Solution:

since $t = 0, P = (1, 0, 0)$

$$T(t) = \frac{\vec{r}'(t)}{|\ \vec{r}'(t)\ |} = \frac{\langle 0, 1, 2\rangle}{\sqrt{5}}$$

$$= \langle 0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\rangle$$

▼ Click to show Example 3

Question 3:

Find the parametric equations for the tangent line to the helix with parametric equations at the point $(0, 1, \pi/2)$

$$x = 2\cos(t), y = \sin(t), z = t$$

Notes:

The equation is an ellipse on the x-y plane When $t = 0$, $(x, y, z) = (2, 0, 0)$. From there, the point follows a spiral path, creating a helix.

Solution:

We need to find which value of t results in this point. Since $z = \frac{\pi}{2}$, $t = \frac{\pi}{2}$, so

$$\vec{r}(\frac{\pi}{2}) = \langle 0, 1, \frac{\pi}{2}\rangle$$

We know that $\vec{r}'(\frac{\pi}{2})$ is parallel to the tangent. Therefore, the vector equation of the tangent is: (using $\vec{r}(\frac{\pi}{2})$ as the initial point and $\vec{r}'(\frac{\pi}{2})$ as the direction)

$$\vec{r}(\frac{\pi}{2}) + t * \vec{r}'(\frac{\pi}{2})$$

$$\vec{r}'(t) = \langle -2\sin(t), \cos(t), 1\rangle$$

$$\downarrow$$

$$\vec{r}'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle$$

So the equation is

$$\langle 0, 1, \frac{\pi}{2}\rangle + t\langle -2, 0, 1\rangle$$

and the parametric equations are

$$\begin{cases} x = 0 + (-2)t = -2t \\ y = 1 + 0t = 1 \\ z = \frac{\pi}{2} + 1t = \frac{\pi}{2} + 1t \end{cases}$$

Integrals

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$, then the **definite integral**

$$\int_a^b \vec{r}(t)dt = \langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt\rangle$$

If $\vec{R}(t)$ is an anti-derivative of $\vec{r}(t)$ (i.e. $\vec{R}'(t) = \vec{r}(t)$) then

$$\int_a^b \vec{r}(t)dt = \vec{R}(t) \mid_a^b = \vec{R}(b) - \vec{R}(a)$$

▼ Click to show Example 5

Question 5:

$\vec{r}(t) = 2\cos(t)\hat{i} + \sin(t)\hat{j} + 2t\hat{k}$. Find $\int_0^{\pi/2} r(t)dt$

Solution:

First, find $\vec{R}(t)$

$$\vec{R}(t) = \int \vec{r}(t) = \langle \int 2\cos(t)dt, \int \sin(t)dt, \int 2tdt$$

$$= \langle 2sin(t), -\cos(t), t^2 \rangle (+C)$$

Therefore

$$\int_0^{\pi/2} \vec{r}(t)dt = \vec{R}(t) \mid_0^{\pi/2} = \langle 2sin(t), -\cos(t), t^2 \rangle \Big|_0^{\pi/2}$$

$$\langle 2, 1, \frac{\pi^2}{4} \rangle$$