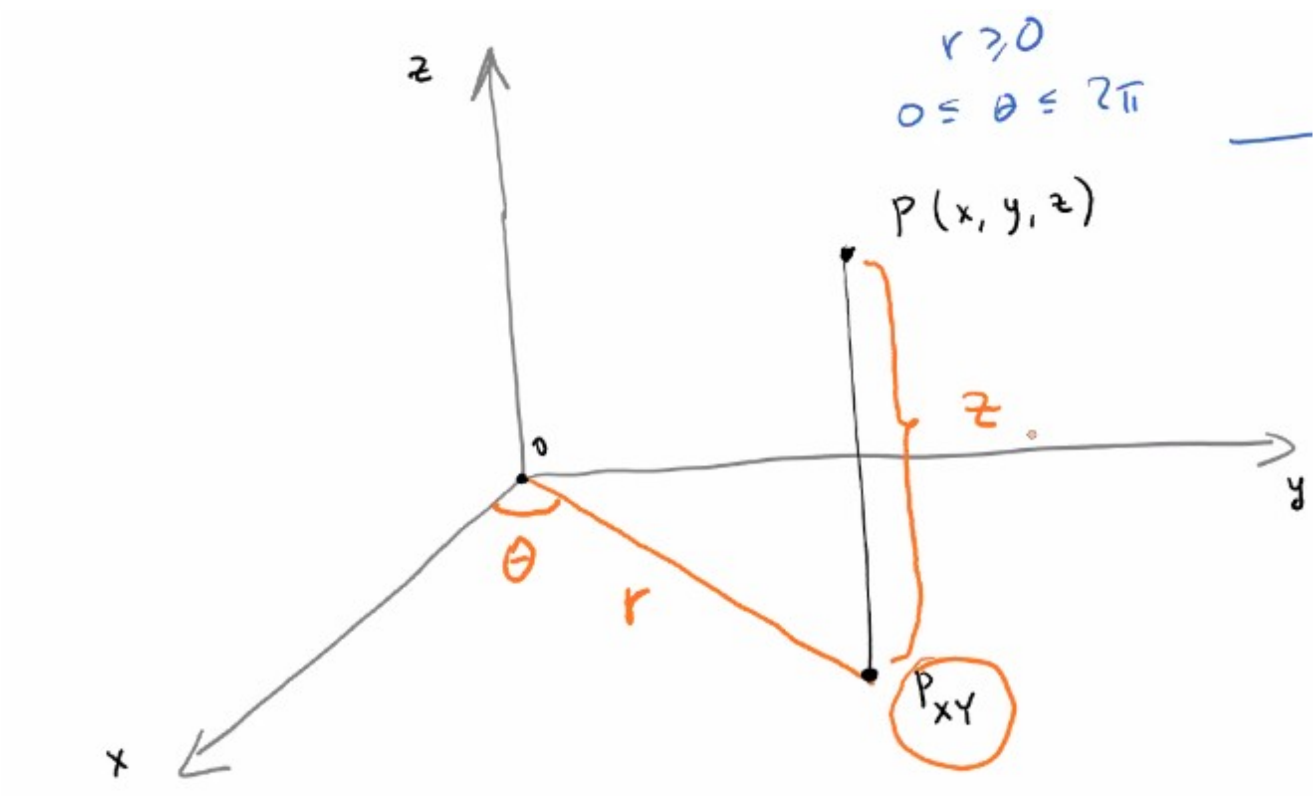
Triple Integrals in Cylindrical Coordinates

Cylindrical Coordinates

Recall that polar coordinates use a radtius and angle (r, θ) to represent a coordinate in space. Each pair of numbers corresponds to only one point.

We can extend this similarly to cylindrical coordinates, where instead of using (x, y, z), we use a radius, angle, and height. In this case, P_{xy}, the point on the xy-plane corresponding to (x, y, z), can be represented in polar coordinates as (r, θ). Therefore, we can show the 3-D point by adding a z-value as height: (r, θ, z).

This grouping also uniquely describes point (x, y, z), so it is a valid representation.



To convert from one to the other, we can use the trigonometric equations for normal polar coordinates and leave z the same.

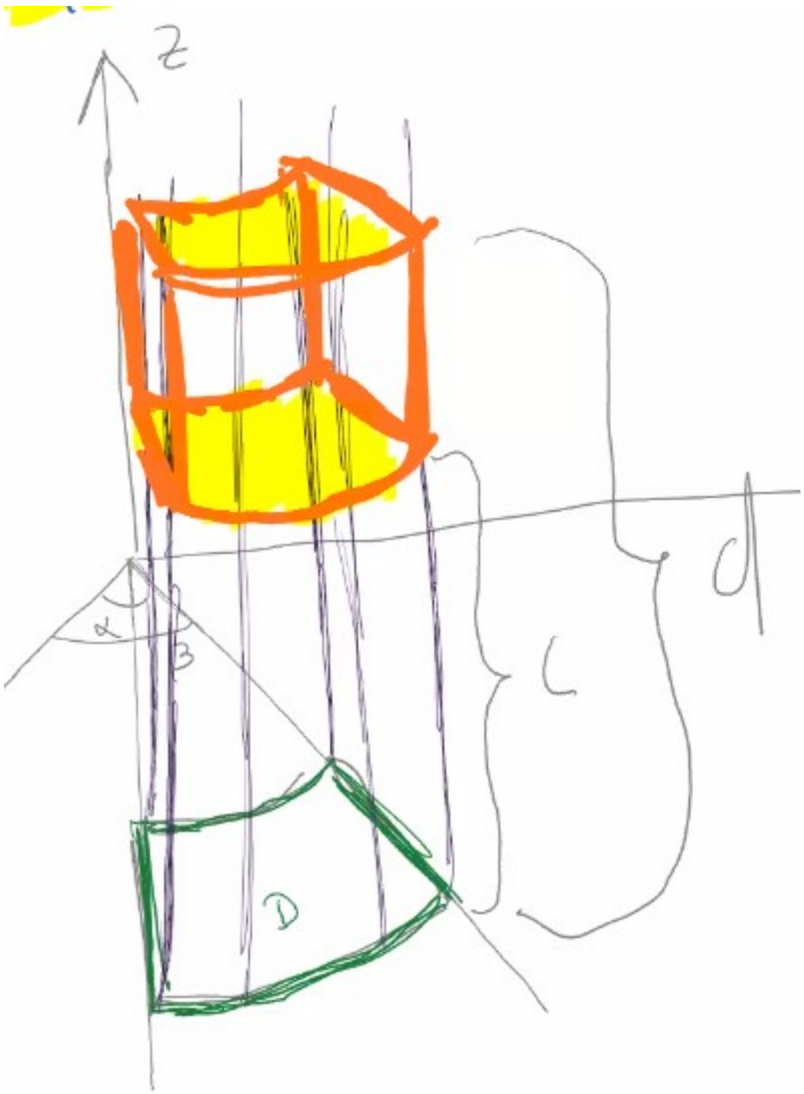
$$P = (x, y, z) \cong (\sqrt{x^2 + y^2}, \arctan(y/x), z) = (r, \theta, z)$$

$$P = (r, \theta, z) \cong (r\cos(\theta), r\sin(\theta), z) = (x, y, z)$$

So, what if we used cylindrical coordinates to do a triple integral?

To answer this, we need to know what the analog of polar rectangles is for cylindrical coordinates (a polar rectangular prism):

$$R = \{(r, \theta, z) | \alpha \leq \theta \leq \beta, a \leq r \leq b, c \leq z \leq d\} = [a, b] \times [\alpha, \beta] \times [c, d]$$



This simply extends the polar rectangle into cylindrical coordinates by adding a range in z

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So we are taking an integral over this volume:

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left(\int_c^d f(x,y,z)dz\right)dxdy$$

Which we can represent in cylindrical coordinates (note that f is multiplied by r, as with polar integrals):

$$\int\limits_\alpha^\beta \int\limits_a^b \int\limits_c^d f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$$

These limits can also be represented by a function, as with other types of multiple integrals. This allows the domain to be distorted in two of the 3 coordinates:

$$E = \{(r,\theta,z)|\alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta), g_1(x,y) \leq z \leq g_2(x,y)\}$$

$$\int\limits_\alpha^\beta \int\limits_{h_1(r)}^{h_2(r)} \int\limits_{g_1(x,y)}^{g_2(x,y)} f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$$

or

$$\int\limits_\alpha^\beta \int\limits_{h_1(r)}^{h_2(r)} \int\limits_{g_1(r\cos(\theta), r\sin(\theta))}^{g_2(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$$

▼ Click to show explanation

We can do this by inputting each part:

$$\iiint\limits_E f(x,y,z)dV = \iint\limits_D \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\right)dxdy$$

We can then abstract the inner portion as another function,

$$G(x,y) = \int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz$$

Now we can insert it into a normal polar integral:

$$\iint_R G(x,y) = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)} G(r\cos(\theta), r\sin(\theta))r\ drd\theta$$

So we can evaluate the whole:

$$\int_{\alpha}^{\beta}\int_{h_1(r)}^{h_2(r)}\int_{g_1(r\cos(\theta),r\sin(\theta))}^{g_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)r\ dzdrd\theta$$

▼ Click to show Exercise 25 (Sec 15.8)

Question

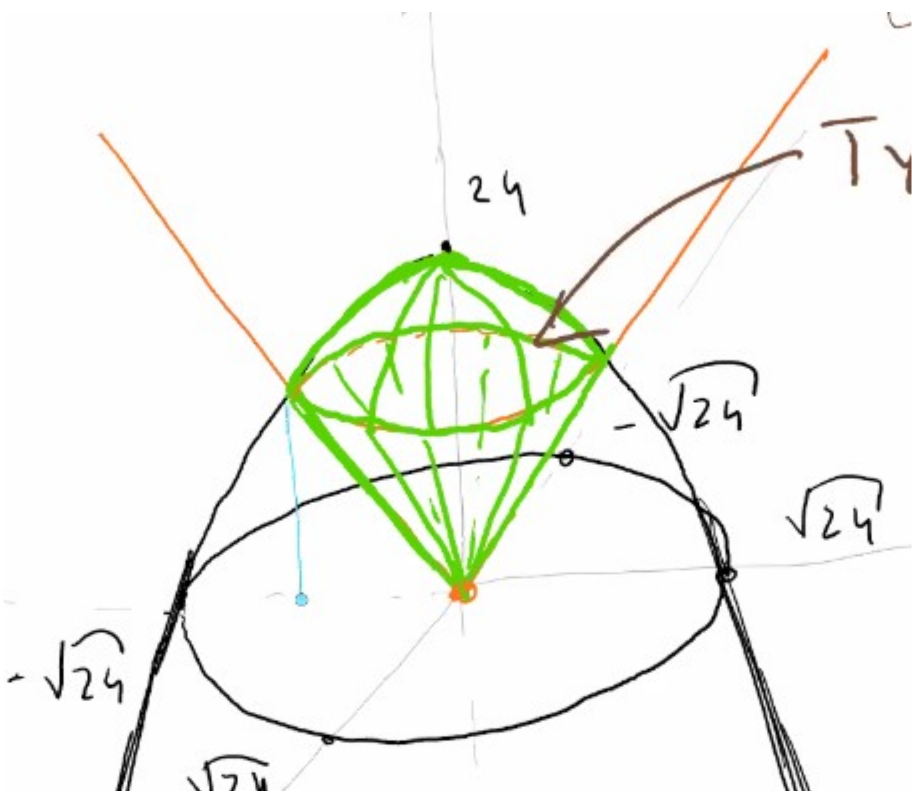
Find the voume of the solid E that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2+y^2}$

Solution

We can find the volume by simply integrating over 1

$$Volume\ of\ E = \iiint_E 1dV$$

These two equations will look like this:



We can find the location where the functions intersect by solving the system of z:

$$\begin{cases} z = 24 - x^2 - y^2 \\ z = 2\sqrt{x^2 + y^2} \end{cases} = x^2 + y^2 = 4^2$$

So since $r^2 = x^2 + y^2$, we can find that r goes from 0 to 4. We also know it is a full circle, so the angle is from 0 to 2π.

So let's set up the integral:

$$\iint\limits_D \int\limits_{2\sqrt{x^2+y^2}}^{24-x^2-y^2} 1\, dV = \int\limits_0^{2\pi} \int\limits_0^4 \int\limits_{2\sqrt{x^2+y^2}}^{24-x^2-y^2} r\, dz dr d\theta$$

Which we can simplify to (using $r^2 = x^2 + y^2$):

$$\int\limits_0^{2\pi} \int\limits_0^4 \int\limits_{2r}^{24-r^2} r\, dz dr d\theta$$

This can now be evaluated.