Vector Functions and Space Curves

Vector functions

- a function is a link between one set and another
  - to be a function, each number from set 1 must have only one result in the second set
  - Set 1 is Domain, set 2 is Codomain (Range)

- In a vector function, Set 1 $\in$ R and Set 2 is a set of vectors

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

- $f(t), g(t)$, and $h(t)$ are component functions
- Given two different points, there is exactly one line that goes through each point

▼ Click to show example 1

Question 1:

If $r(t) = \langle t^3, \ln(3-t), \sqrt{t} \rangle$

then component vectors are

$$f(t) = t^3, g(t) = \ln(3-t), h(t) - \sqrt{t}$$

$$0 \leq t < 3$$

Limits and Continuity

Limits

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \rangle$$

▼ Click to show example 2

Question 2:

Find $\lim_{t \to 0} r(t)$ where $\vec{r}(t) = (1+t^3)\hat{i} + te^{-t}\hat{j} + \frac{sin(t)}{t}\hat{k}$

$$\lim_{t \to 0}(1+t^3) = 1$$

$$\lim_{t \to 0} te^{-t} = 0$$

$$\lim_{t \to 0} \frac{sin(t)}{t} = 1$$

$$\lim t \to 0 \vec{r}(t) = \langle 1, 0, 1 \rangle$$

Continuity

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is **continuous at point a** if the component functions are continuous. That is:

$$\lim_{t \to a} f(t) = f(a)$$

$$\lim_{t \to a} g(t) = g(a)$$

$$\lim_{t \to a} h(t) = h(a)$$

Space Curves

Space curves are defined as a vector function in 3d space, where the input is *t*, or time, and the output is a vector $\vec{v}$ which represents a point P

As time progresses, the point described by P will follow a line, or curve, through space

t is called a parameter for the curve C

and $\begin{cases} x = f(t) \\ y = g(t) \\ z = h(t) \end{cases}$ are called parametric functions

▼ Click to show example 3

Question 3:

Given $r(t) = \langle 1 + t, 2 + 5t, -1 + 6t \rangle$, find the vector equation.

The line equation: $(1 + t)\hat{\imath} + (2 + 5t)\hat{\jmath} + (-1 + 6t)\hat{k}$

The equation can be defined with:

1. the point $(1, 2, 1)$ is on the line and
2. the blue line is parallel to the vector $\vec{v} = \langle 1, 5, 6 \rangle$

Therefore the blue line can be defined by the vector equation $\vec{r} = \vec{r}_0 + t\vec{v}$, where $\vec{r}_0 = \langle 1, 2, -1 \rangle$ and $\vec{v} = \langle 1, 5, 6 \rangle$.

▼ Click to show example 4

Question 4:

Sketch the curve whose vector equation is

$$\vec{r}(t) = \cos(t)\hat{\imath} + \sin(t)\hat{\imath} + t\hat{k}$$

Solution:

Depict the points (x, y, z) such that

$$\begin{cases} x = \cos(t) \\ y = \sin(t) \\ z = t \\ for\ t \in \mathrm{R} \end{cases}$$

If we assume that *t* corresponds to time and we are at point $\langle \cos(t), \sin(t), t \rangle$ at time t, than our trajectory will describe the curve.

So, we simply need to set the result to this set of parametric eqations.

▼ Click to show example 5

Question 5:

Find the vector equation and the parametric equation for the line that joins $P(1,3,-2)$ and $Q(2,-1,3)$.

Solution:

First find the vector between the two points, and use that to define a line along that vector through one of these points.

According to the notes on Lines, the vector equation of a line is:

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

In this case, $\vec{r}_0 = \langle 1,3,-2\rangle$ and $\vec{v} = \langle 2-1, -1-3, 3-(-2)\rangle$

So the equation of the line passing through these points is:

$$\vec{r} = \langle 1+t, 3-4t, -2+5t\rangle$$

But this line is infinite, and we want the line to go only from P to Q, so we need to restrict t.

So, find the value of t at each point and set t to the interval between them.

In this case, since we used P as the initial point and $\vec{PQ}$ as the vector (meaning Q is the endpoint),

$$0 \le t \le 1$$

Additionallly, the parametric equation is the following:

$$\begin{cases} x = 1+t \\ y = 3-4t \\ z = -2+5t \end{cases}, \text{ where } 0 \le t \le 1$$

Defining a line

This section explains more explicitly how the above example is able to define a line.

Given P_0, P_1, $\vec{v}$ connecting them, a point Q on the line defined by P_0, P_1, and a vector $\vec{OQ}$ from the orogin, and $\vec{r}$ from the origin to P_0

For the vec $\vec{P_0Q}$, there exists a scalar λ such that $\vec{P_0Q} = \lambda \times \vec{v}$

Note that $\vec{OQ} = -\vec{r}_0 + \vec{P_0Q} = -\vec{r}_0 + \lambda\vec{v}$ (Vector equations of the line)

As we will see soon, it is better to denote λ by t (because t has the meaning of time)

$\vec{OQ} = \langle x,y,z\rangle$, so the above equation can be rewritten as:

$$\langle x,y,z\rangle = \langle x_0,y_0,z_0\rangle + t\langle x_1-x_0, y_1-y_0, x_1-z_0\rangle$$

$$\langle x,y,z\rangle = \langle x_0+t(x_1-x_0), y_0+t(y_1-y_0), z_0+t(z_1-z_0)\rangle$$

$$x = x_0 + t(x_1-x_0)$$

$$y = y_0 + t(y_1-y_0)$$

$$z = z_0 + t(z_1-z_0)$$

(parametric equation of the line) - or more generally:

$$x = x_0 + ta$$

$$y = y_0 + tb$$

$$z = z_0 + tc$$

These can be used as $f(t), g(t), h(t)$ from earlier.

Observation

every continuous vector function $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a curve in 3-D space

So as t increases, the point $\langle f(t), g(t), h(t) \rangle$ travels along a curve, which in this case is a straight line

The vector $\vec{v}$ from the origin to a point on the line can be given as $\vec{v} = \vec{r}(t)$

From Stewart 13.1

▼ Click to show example 6

Find a vector function that the curve of the intersection of the cylinder $x^2 + y^2 = 1$ and the plane y + z = 2

$$x^2 + y^2 = 1 \rightarrow y = \sqrt{1 - x^2}$$

$$y + z = 2 \rightarrow z = 2 - y \rightarrow 2 - \sqrt{1 - x^2}$$

$$\langle x, y, z \rangle = \langle x, \sqrt{1 - x^2}, 2 - \sqrt{1 - x^2} \rangle$$

So we can write:

$$\begin{cases} x = f(t) = t \\ y = g(t) = \sqrt{1 - t^2} \\ z = 2 - \sqrt{1 - t^2} \end{cases}$$

$$\vec{r}(t) = \langle t, \sqrt{1 - t^2}, 2 - \sqrt{1 - t^2} \rangle$$

But with this, t is constrained to $0 \leq t \leq 1$

Alternate solution

If we denote $x = \cos(t)$ then we can substitute it into the other equations, getting $y = \sqrt{1 - \cos^2(t)}$, which simplifies to $y = \sin(t)$. Similarly, $z = 2 - \sin(t)$. Then we get:

$$\vec{r}(t) = \langle \cos(t), \sin(t), 2 - \sin(t) \rangle$$