Absolute Maximum and Minimum

Extremum and Critical Points

Critical points are where $\frac{\delta f}{\delta x}(x, y) = \frac{\delta f}{\delta y}(x, y) = 0$

Theorem:

If (x, y) is an extremum point, then (x, y) is a critical point

Theorem (second deriv test)

If (x_0, y_0) is a critical point, then

1. $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is local max
2. $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is local max
3. $D > 0$, then the point is a saddle point (nether max nor min)

$$D = \begin{vmatrix} f_{xx} f_{xy} \\ f_{yx} f_{yy} \end{vmatrix} = \begin{vmatrix} f_{xx} f_{xy} \\ f_{xy} f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

$$f_{xy} = f_{yx}$$

Absolute minimum and maximum values

Definition

We say that if $(x_0, y_0) \in$ Domain of $f(x, y)$, then (x_0, y_0) is an absolute min/max if for any point $(x, y) \in D$, we have $f(x, y) \geq f(x_0, y_0)$. (for max it is $f(x, y) \leq f(x_0, y_0)$)

Functions do not always have a max or min value. For Ex: $f(x) = 1/x, x > 0$. It does not have an absolute maximum in the domain from $(0, +\infty)$, because $\lim_{x \to 0} 1/x = \infty$

On the other hand, $f(x) = 1/x, x \in [1, 3]$ has a max. Since for any a we have $f(x) \leq 1$, and since $f(1) = 1$, 1 is the absolute max value.

Theorem (Extreme value theorem for 2-var functions)

If f is continuous and the domain D of $f(x, y)$ is ***closed***, then the function f reaches its absolute maximum and minimum values on D.

This means there are points (x_0, y_0) and (x_1, y_x) both $\in D$ such that for any $(x, y) \in D$, we have

$$f(x_0, y_0) \leq f(x, y) \leq f(x_1, y_1)$$

Definition

The set $D \in R^2$ is said to be a closed set if any point on D can be approximated by points on D, and vice versa if $(x, y) \in R^2$ can be approximated by points on D, then $(x, y) \in D$

So this means that if a value is on the edge, but not in the set, it can be approximated by D but is not in the set, so the set is not closed. (The boundaries are either open or closed. Open does not include the boundary, but closed does)

Ex: the domain of $\sqrt{x}$ is closed if limited to an area of [0,3] because the edge point (0) is in the domain (as is 3), but $\frac{1}{\sqrt{x}}$ is not, because the point at zero does not exist (the set is (0,3]), but it can be approximated by the $\lim\limits_{x \to 0}$.

Let f be continuous and its domain D, is closed. Then to find absolute max and min values of f, one can find the max and min values on the boundary of D and the critical points in D, then choose the max and min values of f at that point.

▼ Click to show example 7 (14.7)

Question 7

Find the absolute max and min values of the function $(x, y) = x^2 - 2xy + 2y$ On the rectangle $D = (x, y) \mid 0 \le x \le 3, 0 \le y \le 2$

Solution

Since D is closed, an absolute max and min are obtainable.

1. Find the critical points (x,y):

$$\frac{\delta f}{\delta x}(x, y) = 2x - 2y = 0 \ \rightarrow \ x = y$$

$$\frac{\delta f}{\delta y}(x, y) = -2x + 2 = 0 \ \rightarrow \ x = 1$$

   Therefore, $(1, 1)$ is the only critical point. $f(1, 1) = 1^2 + 2 \ast 1 \ast 12 \ast 1 = 1$
2. Find the points at the boundary of D We split the edges into four parts: $D =$

$$(0, y) \mid 0 \le y \le 2U$$

$$(x, 2) \mid 0 \le x \le 3U$$

$$(3, y) \mid 0 \le y \le 2U$$

$$(x, 0) \mid 0 \le x \le 3$$

$$0 \le f(0, y) = 2y \le 4$$

$$0 \le f(x, 2) = x^2 - 4x + 4 = (x - 2)^2 \le 4$$

$$1 \le f(3, y) = 9 - 6y + 2y = 9 - 4y \le 9$$

$$0 \le f(x, 0) = x^2 \le 9$$

   These points are the edges, and their max is 9 and min is 0. These are at the points $(3, 0)$ and $(0, 0)$. These are absolute (and local) maximum and minimum respectively