FUNDAMENTAL THEOREM FOR LINE INTEGRALS

Integrals for vector fields

For integrating a points movement in a vector field, we have the following equation (from the previos notes)

$$\int \vec{F} \cdot dr = \int \vec{F} \cdot \vec{T} dr$$

Today we will discuss an integration technque for these vector fields.

Def: Conservative Vector fields

A vector field is conservative if there is a function f such that $\nabla f = \vec{F}$

Examples

1. If $\vec{F}(x, y) = x^2 \hat{i} + y^2 \hat{j} = \langle x^2, y^2 \rangle$ then $\vec{F}$ is conservative

This is because we can define $f(x, y) = 1/3x^3 + 1/3y^3$ Then ∇f is $\langle x^2, y^2$

2. $\vec{F}(x, y) = x^2 \hat{i} + (x^2 + y^2) \hat{j}$ This is not a conservative vector field

The fundamental theorem of Line Integrals

Let C be a curve with initial point A and end point B. Let also $\vec{F}$ be a conservative vector field such that for some $f, \nabla f = \vec{F}$

Then,

$$\int\limits_C \vec{F} dr = f(B) - f(A)$$

In other words,

$$\int\limits_C \nabla f \cdot dr = f(B) - f(A)$$

This is very similar to the normal romanian integral

$$\int\limits_a^b F'(x)dx = F(x)|_a^b = F(b) - F(a)$$

Uses

This tells us that the force needed to move a particle from A to B is not dependent on the path, since only $f(A)$ and $f(B)$ matter

If C is a closed curve, meaning $(A = B)$, then the total force is equal to 0. This is true because the direction going one way is the opposite of the force needed to move the other way.

The opposite is also correct, if the function is 'nice', then if the force along a closed curve is negative, than the field is conservative

▸ Click to show Example

Determining if a field is conservative

If $\vec{F}(x, y) = P(x, y)\hat{\imath} + Q(x, y)\hat{\jmath} = \langle P(x, y), Q(x, y)\rangle$ Then $\vec{F}(x, y)$ is conservative if and only if

$$\frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x}$$

If this is not true, it is not conservative.

Now we can show why $\vec{F}(x, y) = x^2\hat{\imath} + (x^2 + y^2)\hat{\jmath}$ is is not a conservative vector field.

Because

$$\frac{\delta x^2}{\delta y} = 0 \neq \frac{\delta(x^2 + y^2)}{\delta x} = 2x$$

The previous function $\vec{F}(x,y) = x^2\hat{\imath} + y^2\hat{\jmath} = \langle x^2, y^2 \rangle$ is conservative because

$$\frac{\delta x^2}{\delta y} = 0 = \frac{\delta y^2}{\delta x}$$

What if the vector field depends on 3 variables?

$$\vec{F}(x,y,z) = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$

The above definition does not work here. Three dimensions will be given later, and will be more complex.

Examples

Is $\vec{F}$ conservative? If so, find f such that $\nabla f = \vec{F}$.

1. $\vec{F} = (x^y + y^2)\hat{\imath} + (x^2 + 2xy)\hat{\jmath}$

$$2y \neq 2x + 2y$$

Not conservative.

2. $\vec{F}(x,y) = (y^2 - 1x)\hat{\imath} + 2xy\hat{\jmath}$

$$2y = 2y$$

It is conservative.

Finding $f(x,y)$

To find $f(x,y)$ s.t. $\nabla f = \vec{F}$ is the same as finding f such that

$$\begin{cases} \frac{\delta f}{\delta x} = P(x,y) \to f(x,y) = \int P(x,y)dx \\ \frac{\delta f}{\delta y} = Q(x,y) \to f(x,y) = \int Q(x,y)dy \end{cases}$$

So for this problem, we can use:

$$\begin{cases} \int (y^2 - 1x)dx = y^2x - x^2 + h(y) \\ \int (2xy)dy = y^2x + g(x) \end{cases}$$

Where $h(y)$ is a function not depending on x, but possibly depending on y and $g(x)$ is a similar function for x.

So

$$f(x,y) = y^2x - x^2 + h(y) = y^2x + g(x)$$

$$h(y) = 0$$

$$g(y) = -x^2$$

$$f(x,y) = y^2x - x^2(+C)$$