SURFACE AREA

Summary

Essentially, we are adding together all of the squares on top of the surface. This can be done through a certain cross product formula that leads to the following equation:

$$SA = \iint\limits_R \sqrt{f_x^2 + f_y^2 + 1} dxdy$$

Prior formulas

These will be useful to deriving the above equation.

A parlellogram is equal to 2 times the triangles that make it up. Also, the area of a triangle can be found as:

$$\triangle ABD = \frac{1}{2}|AB||AD|\sin(\theta)$$

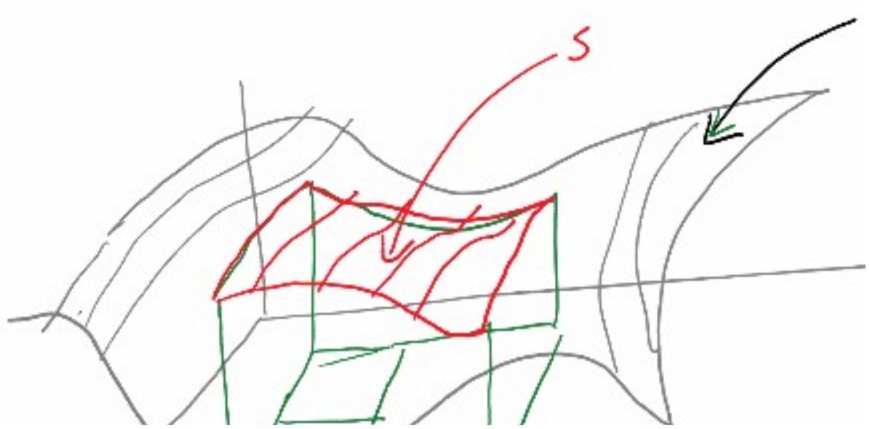
So the paralellogram $ABCD$ can be found as:

$$\square ABCD = 2\triangle ABD = |AB||AD|\sin(\theta)$$

If the angles are at 90 deg, then $\sin(\theta) = 1$, meaning:

$$\square ABCD = |AB||AD|$$

This time, we are going to find the surface area of a function in a given region:



Finding the area

Essentially, we are finding the sum of all the rectangles on top of the function.

$$Area = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} (Area\ T_{ij}) \begin{bmatrix} \Delta x_i = x_{i+1} - x_i \\ \Delta y_j = y_{j+1} - y_j \end{bmatrix}$$

The rectagles are simply parallelograms which have perpendicular sides. The areas of these rectangles can be defined as:

$$T_{ij} = |\vec{a} \times \vec{b}|$$

Given $\vec{a}$ and $\vec{b}$ are the sides of T_{ij}, defined as:

$$\vec{a} = \langle x_i, x_{i+1} \rangle$$

$$\vec{b} = \langle y_j, y_{j+1} \rangle$$

Then the rectangle R which actually rests on the surface is defined as

$$R_{ij} = |\vec{u} \times \vec{v}|$$

$$\vec{u} = \langle \Delta x_i, 0, 0 \rangle$$

$$\vec{v} = \langle 0, \Delta y_j, 0 \rangle$$

And we can get this by using the change in height of z

$$\vec{a} = \vec{u} + |\vec{u}| f_x(x_i, y_j)$$

$$\vec{b} = \vec{v} + |\vec{v}| f_y(x_i, y_j)$$

To get:

$$\vec{a} = \Delta x_i \hat{i} + f_x(x_i, y_j) \Delta x_i \hat{k}$$

$$\vec{b} = \Delta y_j \hat{j} + f_y(x_i, y_j) \Delta x_i \hat{k}$$

So the area of T_{ij} is:

$$|\vec{a} \times \vec{b}| = \begin{vmatrix} i & j & k \\ \Delta x_i & 0 & f_x(\Delta x_i) \\ 0 & \Delta y_j & f_y(\Delta y_j) \end{vmatrix}$$

$$= |\hat{i}(-f_x \Delta x_i \Delta y_j) - \hat{j}(f_y \Delta x_i \Delta y_j) + \hat{k}(\Delta x_i \Delta y_j)|$$

$$= \sqrt{(-f_x \Delta x_i \Delta y_j)^2 + (f_y \Delta x_i \Delta y_j)^2 + (\Delta x_i \Delta y_j)^2}$$

$$= \sqrt{f_x^2 + f_y^2 + 1} * \Delta x_i \Delta y_j$$

$$Area = \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \sum_{i=1}^{n} \sum_{j=1}^{m} \sqrt{f_x^2 + f_y^2 + 1} \Delta x_i \Delta y_j$$

So the final eqution is

$$= \iint\limits_R \sqrt{f_x^2 + f_y^2 + 1} dx dy$$

The easiest way to remember this is to elate it to the 2-D example, where the length of a curve is given as:

$$\int_a^b \sqrt{1 + f'(x)^2} dx$$

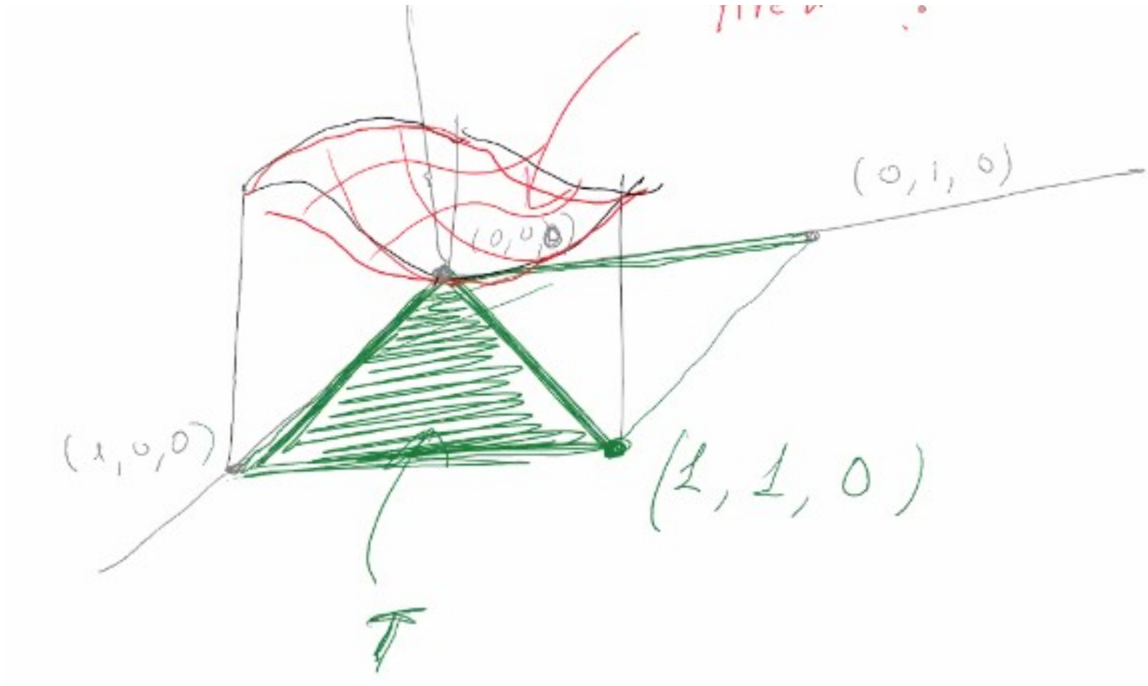
▼ Click to show Example 1

Question 1 (Sec 15.5)

Find the surface area of the part of the surface given by $z = x^2 + 2y$ that lies above the plane with vertices $(0,0), (1,0), (1,1)$

Solution

Let us first draw the area that we are trying to find to visualize it:



So we apply the formula above:

$$Area = \iint\limits_T \sqrt{f_x^2 + f_y^2 + 1} dxdy$$

$$Area = \iint\limits_T \sqrt{4x^2 + 4 + 1} dxdy$$

$$Area = \iint\limits_T \sqrt{4x^2 + 5} dxdy$$

Then we can find the region of T (both type1 and 2):

$$T = \{(x, y) | 0 \le x \le 1, 0 \le y \le x\}$$

And integrate:

$$Area = \int_0^1 \int_0^x \sqrt{4x^2 + 5} dydx$$

$$= y\sqrt{4x^2 + 5}|_0^x$$

$$= x\sqrt{4x^2 + 5}$$

$$\int_0^1 x\sqrt{4x^2+5}\,dx$$