Partial derivatives

Functions of multiple varibles

A function is a map from one set of numbers to another set. In a function, there is only one result per source. We have also demonstrated functions that go from a set of real numbers to a set of vectors.

In a multivariable function, set two is a set of real numbers, and set one is a set of *pairs* of real numbers.

Examples:

$$f(x, y) = x + y$$

$$f(1, -1) = 0$$

$$f(1, \sqrt{2}) = 1 + \sqrt{2}$$

$$f(x, y) = x = x + 0y$$

In this, y has no effect, and is called a fictive variable.

$$f(x, y) = \frac{1}{x - y}$$

Domain: $(x_0, y_0) \in D$ means $f(x, y)$ is defined at (x_0, y_0)

The domain of this is $D = \{(x, y) \mid x = y\}$

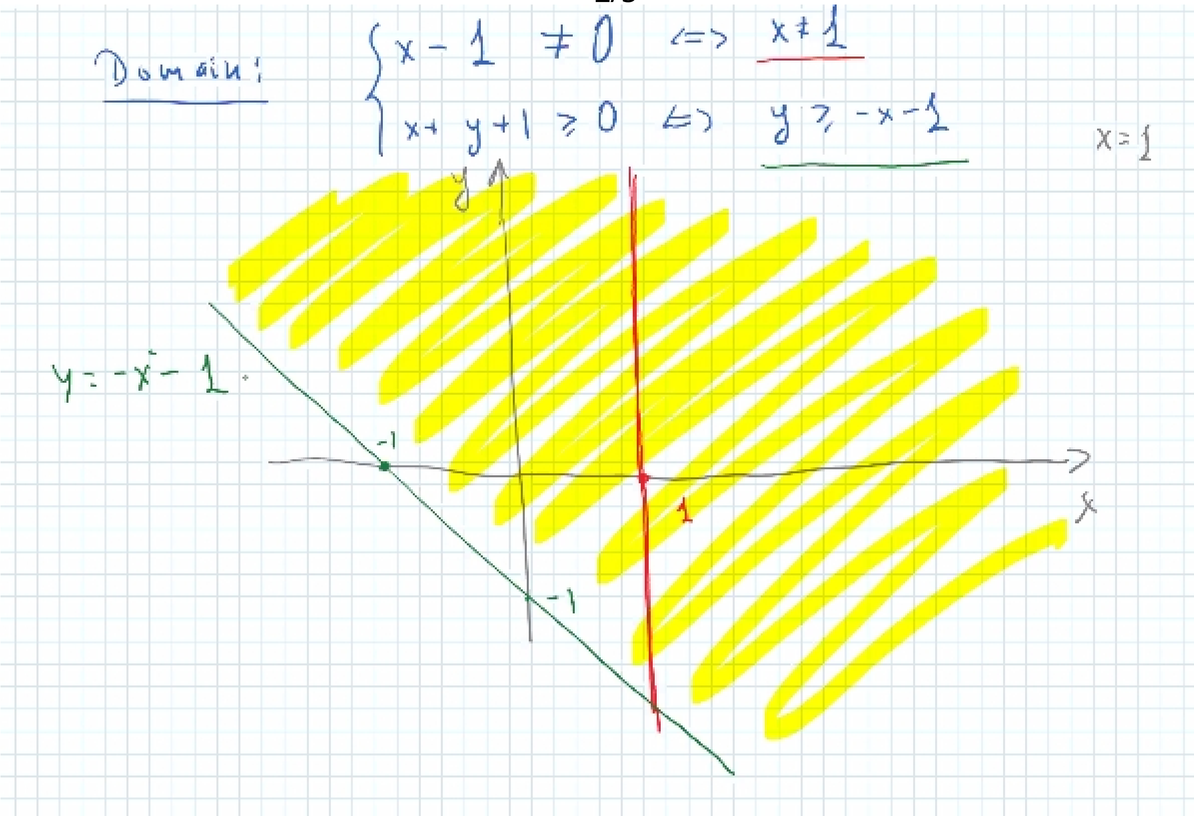
▼ Click to show Example 1

Question 1:

For each of the functions, evaluate f(3,2) and find and sketch domain:

1. $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$
    - Solution:

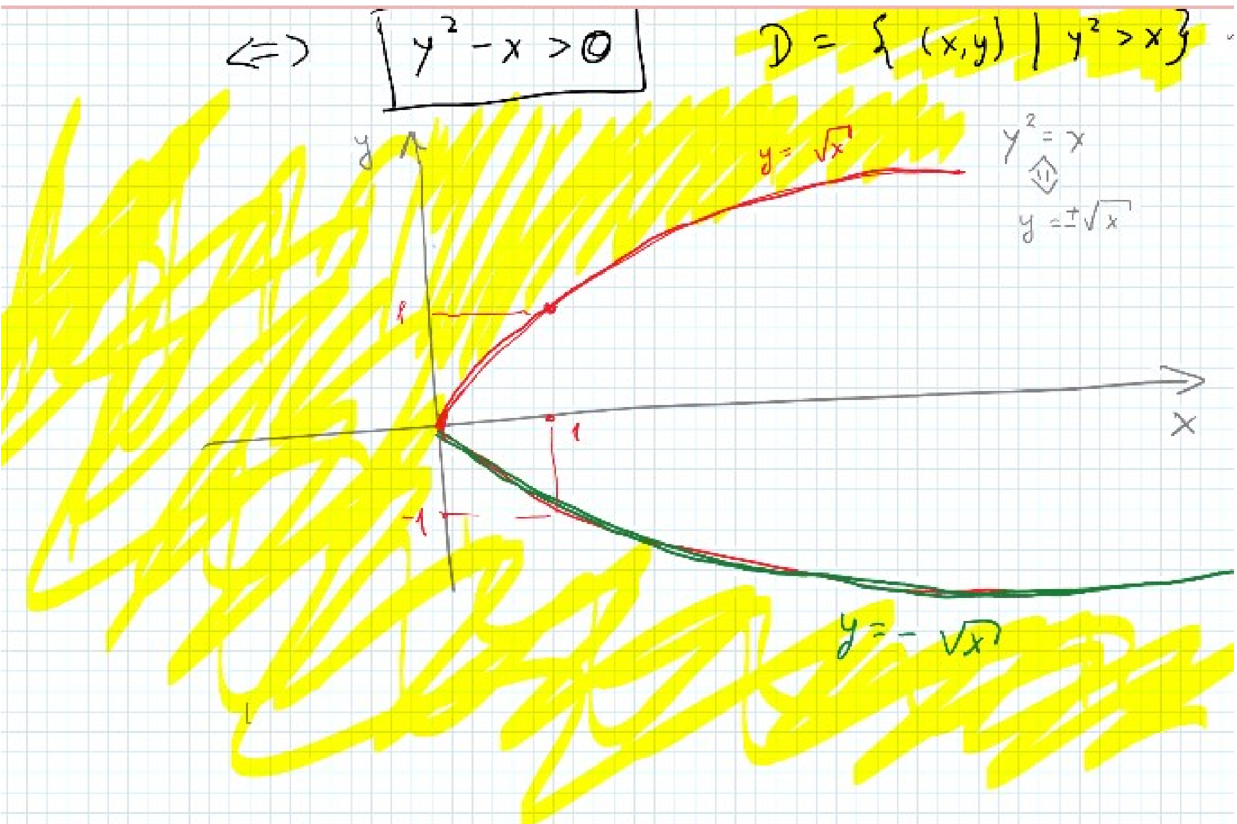
      Domain: $\{\begin{matrix} x - 1 = 0 \iff x = 1 \\ x + y + 1 \geq 0 \iff y \geq -x - 1 \end{matrix}$



1. $f(x, y) = x \ln(y^2 - x)$
   - Solution:

$$f(3, 2) = 3\ln(2^2 - 3) = 3\ln(1) = 0$$

$$D = \{(x, y) \mid y^2 > x\}$$



▼ Click to show Example 4

Question 4

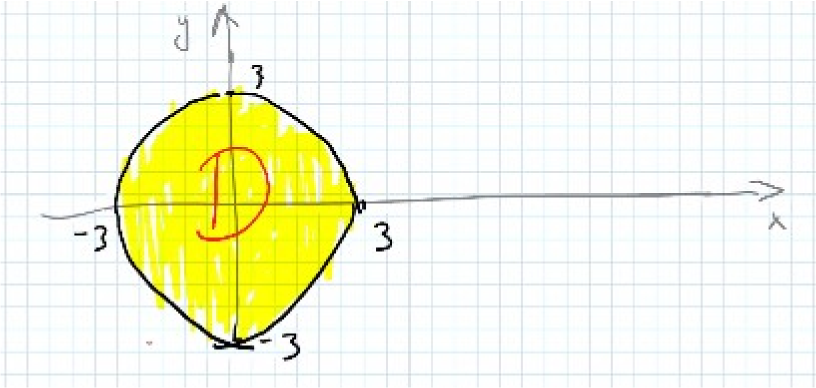
Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Solution

Domain:

$$D = \{(x, y) \mid 9 - x^2 - y^2 \geq 0\}$$

$$D = \{(x, y) \mid x^2 + y^2 \leq 9\}$$



Range: The point z is in the range of $g(x, y)$ if there exists x, y such that $z = \sqrt{9 - x^2 - y^2}$

$$z \geq 0, z \leq 3$$

For any value $z \in (0, 3)$ there exists x such that $z = \sqrt{9 - x^2 - y^2}$
Therefore, the range of $g(x, y)$ is the interval $[0, 3]$

Graphing Multivariable functions

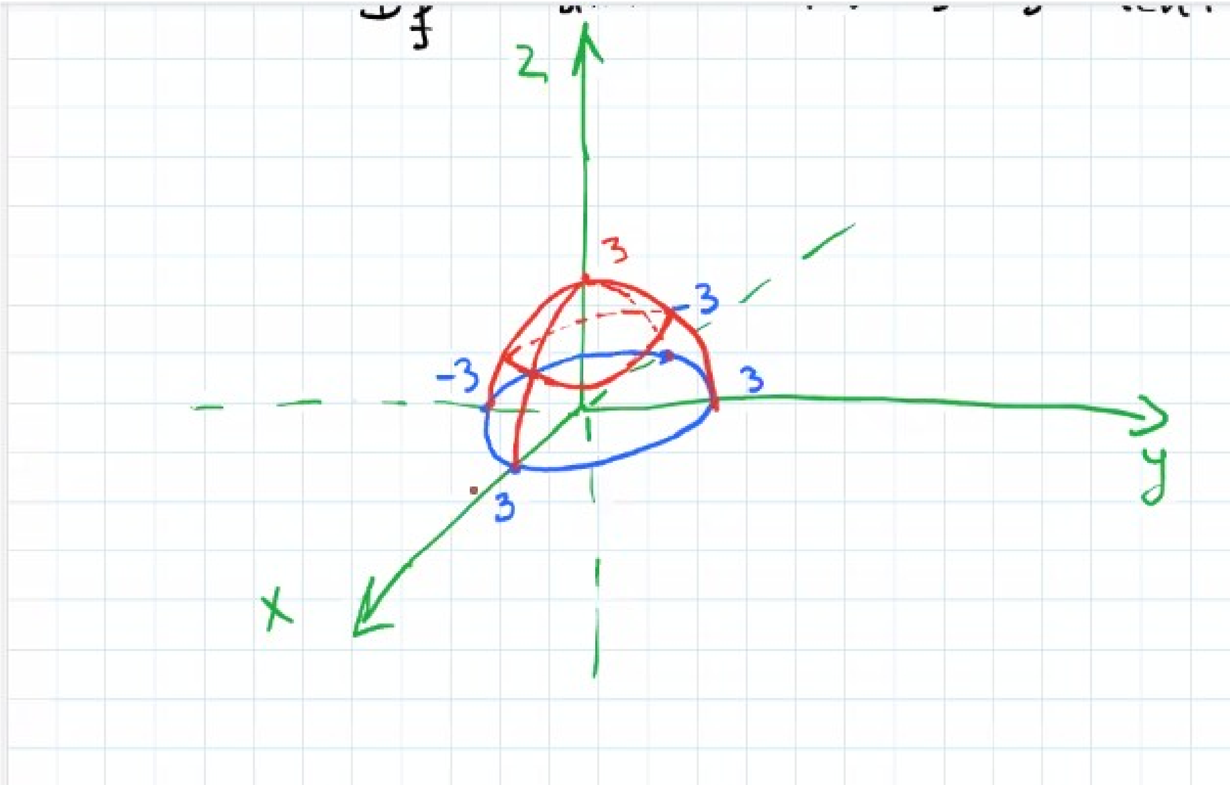
A two-dimensional function $y = f(x)$, if it is contiuous, represents a curve in 2-D space. In three dimensions, $z = f(x, y)$, if the function is continuous, will represent a surface in 3-D space.

Definition

The graph of the function $f(x, y)$ is the set $\Gamma = \{(x, y, z)\}$, where $(x, y) \in D$ (the domain of f) and $z = f(x, y)$

Example

Consider $z = f(x, y) = \sqrt{9 = x^2 - y^2}$
D_f = Disk of radius 3 centered at $(0, 0)$

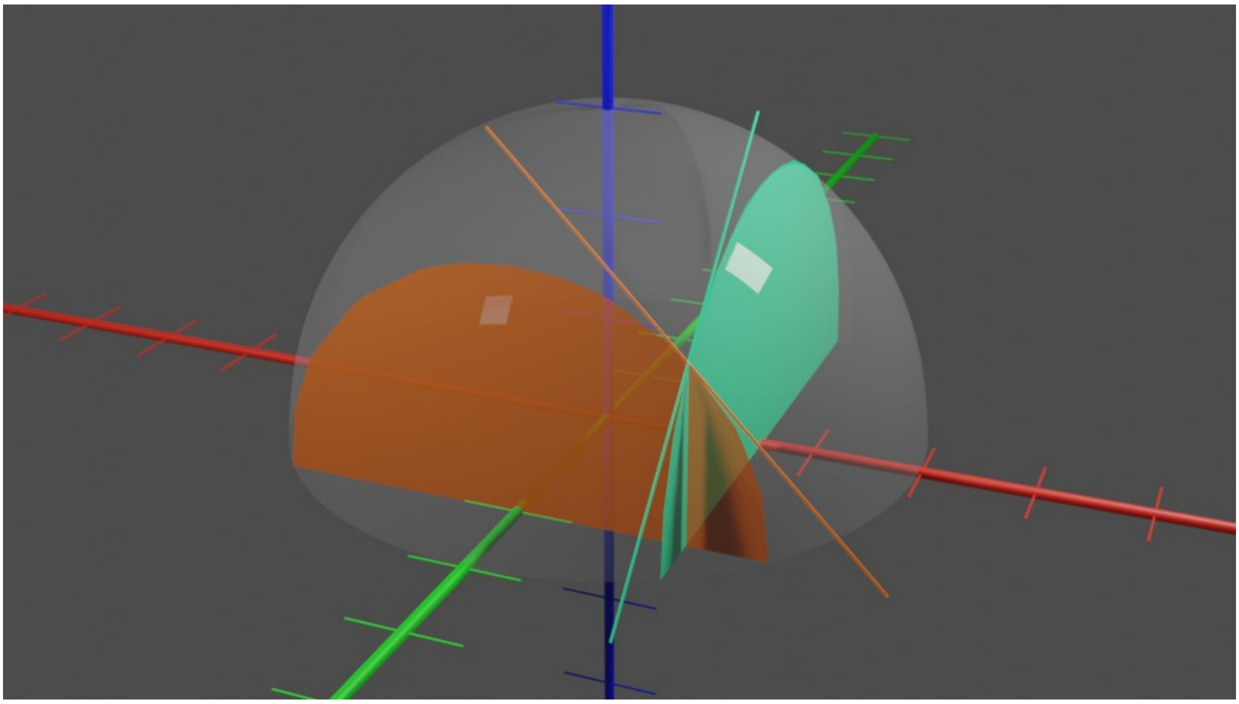


Partial Derivatives

Video Link

Recall: With $y = f(x)$ the derivative $f'(x)$ represents the slope of the tangent line. However, this does not work verbatim for graphs of three dimensions.

In three dimensions, the rate of change changes based on which direction you are traveling. In three dimensions, the natural directions are in the zy and zx planes. So, we can take the derivative in the y direction to find the slope of the y direction. This also works for the x direction.



The partial derivative of the function $f(x, y)$ are $f_x(a, b)$ (the derivative of the x direction) and $f_y(a, b)$ (the derivative of the y direction)

Formal definition

Given the function $z = f(x, y)$, Let $(a, b) \in D_f$. Then, the partial derivative of f at (a, b) with respect to x is defined as:

$$\frac{\delta f}{\delta x} = f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Similarly, the partial derivative with respect to y is:

$$\frac{\delta f}{\delta y} = f_y(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

Section 14.3

▼ Click to show Example 1

Question 1

Given $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$

Solution

To compute the partial derivative f_x, leave y constant and have x be the only variable.

For simplicity, replace y with constant b. Then:

$$g(x) = f(x, b) = x^3 + x^2b^3 - 2b^2$$

Then, plugging a into x to comute the derivative at (a, b):

$$f_x(a, b) = g^{'}(b) = 3a^2 + 2ab^3$$

Next, replace x with constant a. Then:

$$h(x) = f(a, y) = a^3 + a^2y^3 - 2y^2$$

Then, plugging b into y to compute the derivative at (a, b):

$$f_y(a, b) = g^{'}(b) = 3a^2b^2 - 4b$$

So now we can plug in the values:

$$f_x(a, b) = 3a^2 + 2ab^3$$

$$f_x(2, 1) = 3(2)^2 + 2(2)(1)^3 = 16$$

$$f_y(a, b) = 3a^2b^2 - 4b$$

$$f_y(2, 1) = 3(2)^2(1)^2 - 4(1) = 8$$