PARAMETRIC SURFACES - 16.6

Remember that we defined parametric equations for curves earlier:

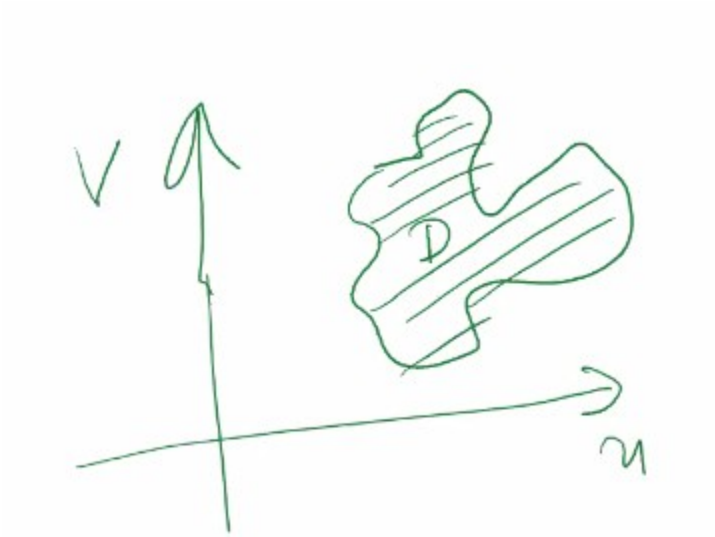
$$\vec{r}(t) = \begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

This shows that parametric just means that each coordinate has it's own function, taking in two separate *parameters*.

For parametric surfaces, the vector function $\vec{r}$ is dependent on two variables.

$$\vec{r}(u, v) = \begin{cases} x = x(u, v) \\ y = y(u, v) \\ z = z(u, v) \end{cases}$$

Where $(u, v) \in D$ is the domain in the (u, v)-plane.



One point to note is that an equation can be parameterized in many different ways; often infinitely many.

▾ Click to show example 1

Question 1

Identify the surface with the given vector equation:

$$r(u,v)(u+v)\hat{\imath}+(3-u)\hat{\jmath}+(1+4u+5v)\hat{k}$$

Solution

We can find z in terms of x and y by finding a pair of numbers which can be multiplied or added to x and y to get the equation for z.

$$z = ax + by + c$$

$$1+4u+5v = a(u+v)+b(3-u)+c$$

We can also define u and v in terms of x and y:

$$y = 3-u \rightarrow u = 3-y$$

$$x = u+v \rightarrow v = x-u \rightarrow v = x-(3-y)$$

$$z = 1+4u+5v = 1+4(3-y)+5(x-3+y)$$

So we can find that $z = 5x+y-2$

$$5x+y-z=2$$

So, we can describe S as the plane that passes through (0,0,-2) and has the normal vector $\vec{u} = \langle 5, 1, -1\rangle$

▾ Click to show example 2

Question 2

$$\vec{r}(s,t) = \langle s\cos t, s\sin t, s\rangle$$

Solution

From the above equation, we can see that if we add the squares of x and y:

$$x^2 + y^2 = (s\cos t)^2 + (s\sin t)^2 = s^2(\sin^2 + \cos^2) = s^2 = z^2$$

So the surface is a paraboloid.

▼ Click to show example 3

Question 3

$$\vec{r}(s,t) = \langle 3\cos t, s, \sin t\rangle$$

$$-1 \leq s \leq 1$$

Solution

We can see that in this equation, y is only determined by s, and the others are only of t. This means we can ignore y for now.

Again, we can recognize that there is a similarity to the $\sin^2 + \cos^2$ property, so we can try to define an equation that changes x and z to this property:

$$x^2 = 9\cos^2 t$$

$$9\cos^2 t + 9\sin^2 t = 9$$

$$z^2 = \sin^2 t \to (3z)^2 = 9\sin^2 t$$

$$x^2 + (3z)^2 = 9$$

This shows that any cross-section of the surface will be an ellipse, and since y increases with s, the surface is a cylinder.

Since the equation goes from $[-1, 1]$, the cylinder goes from -1 to 1 in height.

▼ Click to show example 4

Question 4

Find parametric representation of the surface S:

$$x^2 + y^2 + z^2 = a^2$$

Solution

In spherical coordinates:

$$\begin{cases} x = \rho \cos\theta \sin\phi \\ y = \rho \sin\theta \sin\phi \\ z = \rho \cos\phi \end{cases}$$

In spherical coordinates, we can say that a sphere is $\rho = a$. So, we can define the sphere as:

$$\begin{cases} x = a \cos\theta \sin\phi \\ y = a \sin\theta \sin\phi \\ z = a \cos\phi \end{cases}$$

We can simply define θ and ϕ as the other two parametric variables:

$$\vec{r}(\theta, \phi) = a\cos\theta\sin\phi\hat{\imath} + a\sin\theta\sin\phi\hat{\jmath} + a\cos\phi\hat{k}$$

$$\begin{vmatrix} 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi \end{vmatrix}$$

▼ Click to show example 5

Question 5

S: is the plane thrrough the origin that contains the vectors $\hat{\imath} - \hat{\jmath}$ and $\hat{\jmath} - \hat{k}$.

Solution

First, let us define the equation of the plane by finding the normal:

$$\vec{n} = (\hat{i} - \hat{j}) \times (\hat{j} - \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

So we can use that in the vector equation:

$$S : x + y + z = 0$$

$$\vec{r}(u, v) = u\hat{i} + v\hat{j} + (-u - v)\hat{k}$$

This is found by simply setting x and y equal to u and v, then solving for z.