Vectors

Properties of a vector

The zero vector is non-directional

$$-\vec{0} = \vec{0}$$

Making a vector negative just flips it's direction

$$-\vec{V} = -1 * \vec{V} \rightarrow |\vec{V}| = |-\vec{V}|$$

$$|\vec{V}| = \sqrt{x^2 + y^2 + z^2}$$

You can multiply a vector by a scalar (λ)

$$\lambda\vec{a} = \langle \lambda a_x, \lambda a_y, \lambda a_z \rangle$$

$$\vec{x} + \vec{z} = \langle x_1 + x_2, y_1 + y_2, z_1 + z_2 \rangle$$

Dot Product

The dot product produces a scalar output

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

Properties

1. $$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$
2. $$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$
3. $$\vec{a} \cdot (\vec{b}\vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$
4. $$(\lambda\vec{a}) \cdot \vec{b} = \lambda * (\vec{a} \cdot \vec{b})$$
5. $$\vec{0} \cdot \vec{a} = 0$$

Proof for 2

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

by definition:

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = b_1 a_1 + b_2 a_2 + b_3 a_3$$

by associativity of multiplication:

$$a_1 b_1 + a_2 b_2 + a_3 b_3 = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Theorem for dot product

$$\vec{a} \cdot \vec{b} = |\vec{a}| * |\vec{b}| * \cos(\theta)$$

Proof:

- use the law of cosines with the triangle made by the vectors $\vec{a}, \vec{b}$, and $\vec{AB}$ (vector from A to B).
- show $\vec{AB} = \vec{b} - \vec{a}$
- simplify using above properties

Uses of the dot product

Checking for perpendicular vectors:

$\vec{a} \perp \vec{b}$ if $\vec{a} \cdot \vec{b} = 0$

If $|\vec{a}|$ and $|\vec{b}| = 1$ ($\vec{a}$ and $\vec{b}$ are unit vectors), then $\vec{a} \cdot \vec{b} = \cos(\theta)$

Projections

Projections are the "shadow" of a vector onto another vector

Scalar Projections

- scalar projections are the length of the "shadow" you get from drawing a line from the end of $\vec{b}$ that is perpendicular to $\vec{a}$
- scalar projection of $\vec{b}$ onto $\vec{a}$ is

$$|\vec{b}| \cos(\theta)$$

So using

$$\vec{a} \cdot \vec{b} = |\vec{a}| * (|\vec{b}| \cos(\theta))$$

$$|\vec{b}| \cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

So to find the scalar projection of $\vec{b}$ onto $\vec{a}$, use:

$$comp_a \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$$

Vector Projections

- vector projections are the actual vector that spans the "shadow"
- They are found by multiplying the scalar projection onto the direction (unit vector) of a, or $\frac{\vec{a}}{|\vec{a}|}$

$$proj_a \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} a$$

Cross Product

- Stewart 12.4
- the cross product is a vector that is perpendicular to both vectors
- results in a vector perpendicular to the plane that a and b are in

$$\vec{c} = \vec{a} \times \vec{b}$$

- note: if u and v are perpendicular, $\vec{u} \cdot \vec{v} = 0$
- so, to find $\vec{a} \times \vec{b}$, find:

$$\vec{c} \perp \vec{a} \rightarrow a_1c_1 + a_2c_2 + a_3c_3 = 0$$

$$\vec{c} \perp \vec{b} \rightarrow b_1c_1 + b_2c_2 + b_3c_3 = 0$$

Process:

- mult by b3 and a3

$$a_1b_3c_1 + a_2b_3c_2 + a_3b_3c_3 = 0$$

$$a_3b_1c_1 + a_3b_2c_2 + a_3b_3c_3 = 0$$

- Subtract $a_3b_3c_3$ and use it to set the rest equal

$$a_1b_3c_1 + a_2b_3c_2 = a_3b_1c_1 + a_3b_2c_2$$

$$(a_1b_3 - a_3b_1) * c_1 = (a_3b_2 - a_2b_3) * c_2$$

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \quad \begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}$$

- if we say c_1 and c_2 are the determinants of the above matrices, (as opposites) the equality is true
- Similarly, $c_3 = (a_1b_2 - a_2b_1)$
- so,

$$\vec{c} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Unit Vectors

$$\hat{i} = \langle 1, 0, 0 \rangle$$

$$\hat{j} = \langle 0, 1, 0 \rangle$$

$$\hat{k} = \langle 0, 0, 1 \rangle$$

$$\vec{a} = \langle a_1, a_2, a_3 \rangle = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

Properties of unit vectors

1. $i \times j = k$
2. $j \times k = i$
3. $i \times k = j$
4. $j \times i = -k$
5. $k \times j = -i$
6. $k \times i = -j$

With Cross Products:

- $$\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(The determinant of the matrix)

This is because of the properties of determinants:

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

Cross Product Theorem

If theta is the angle between $\vec{a}$ and $\vec{b}$, then

$$\mid \vec{c} \mid \ = \ \mid \vec{a} \times \vec{b} \mid$$

$$is$$

$$\mid \vec{a} \times \vec{b} \mid \ = \ \mid \vec{a} \mid \ \ast \ \mid \vec{b} \mid \ \ast \sin(\theta)$$

Corrolary

Using the fact that $\sin(0\,°, 180\,°) = 0$:

$$\vec{a} \ /\!/ \ \vec{b} \ \ iff \ \ \vec{a} \times \vec{b} = 0$$

Properties of Cross Product

λ is a scalar

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(\lambda\vec{a}) \times \vec{b} = (\lambda\vec{a} \times \vec{b}) = \vec{a} \times (\lambda\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$
5. $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$
6. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Triple Products

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$