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Contents

Abstract

In this paper we will present four approaches to investing in discrete time markets. In chapter 3 we focus on log optimum investment, presenting a detailed proof of the KKT conditions. We use this to prove the log optimum portfolio has the best asymptotic performance over any causal portfolio. In chapter 4 we consider a market with stock sequences belonging to a discrete finite set. This chapter presents a Bayesian sequential portfolio selection algorithm whose wealth grows at a rate asymptotically as large as the wealth of the optimal portfolio selected with future knowledge of the empirical stock distribution [1]. In chapter 5 we consider the μ weighted universal portfolio for two Dirichlet weightings. We prove these portfolios have, to first order, the same asymptotic wealth as the optimal portfolio chosen in hindsight. Chapter 6 deals with the addition of transaction costs and investment in interval policies. We present the universal policy and show this strategy achieves the same asymptotic wealth as the best hindsight policy, to first order. In this chapter we prove the original result that the difference in growth exponents between these polices converges to 0 and illustrate this via simulation.

1 Introduction

The problem of portfolio selection in discrete time markets is an important field in mathematical finance. A range of algorithms have been proposed, each with varying degrees of assumptions, which aim to perform optimally in a variety of different ways. In this paper we explore a number of algorithms which are information theoretic in flavour. We start with the algorithms which make the strongest and most unrealistic assumptions, progressively weakening these until we arrive at the concept of universal investment.

A significant part of this paper will revolve around the central question of: Is it possible for an investor to perform as well as a hindsight investor who has future knowledge of the stock market? In general the answer is, unsurprisingly, no. However if we are prepared to restrict the hindsight investor to only being able to select a single and constant investment strategy, as well as loosening our objective to require only performing as well asymptotically (to first order), then the answer to the question becomes yes. Investment strategies which achieve this objective are called universal investments. The field was first developed by Thomas Cover, in the early 1990's, who presented the μ weighted universal portfolio for investment in discrete time markets without transaction costs [3]. The wealth generated from such a universal portfolio is, to first order, asymptotically the same as the wealth of the best constant portfolio a hindsight investor can select. In 2005, Iyegngar extended this field to include markets with transaction costs. As discussed further in chapter 6, such markets drastically change the notion of what an optimal strategy would look like and thus what approach we should take to match the hindsight investors performance. This paper will give a rigorous introduction to universal investment strategies, both in markets with and without transaction costs, as well as prove new material regarding their optimality. We also include original demonstrations which illustrate these concepts in practice.

Before presenting universal investment, this paper also looks at two important variants of the market model, the first of which illustrates the log optimum portfolio for markets following a known distribution. This is central concept in discrete time markets and appears at various points throughout this paper. It is shown in chapter 3 that the log optimum portfolio performs asymptotically better than any causal (non future relying) portfolio. Thus throughout this paper, the log optimum portfolio serves as an implicit benchmark for measuring the performance of an investment strategy.

2 Preliminaries and Notation

This paper focuses on discrete time markets where an investor invests a fraction of their wealth in $m \geq 1$ stocks at discrete time points $i = 1, 2, 3....$ We do not allow the investor to short or long a stock by taking negative positions in an asset.

Definition 2.0.1 (Price Relatives, Portfolios and Wealth). *Let $m \geq 1$ be the number of stocks of interest. A vector of price relatives is denoted by $X = (X_1, ..., X_m)$, where $X_k \geq 0$ is the fractional change in stock price of the k^{th} asset over the trading period. Throughout this paper we will use the convention that X_i is random and x_i is a realisation. This should be clear from the context and varies between chapters.*
A portfolio $b = (b_1, ..., b_m) \in B$ is a fractional allocation of wealth amongst the m stocks where $B = \{b \in [0,1]^m : \sum_{i=1}^{m} b_i = 1\}$ is the set of valid portfolios. The wealth relative from following $b \in B$ with stock vector X is defined as $S := b^T X$ which is the fractional change in wealth from investment in that period.

Example 2.0.1. *Let $X = (1.07, 0.98)$ and $b = (0.6, 0.4)$. Then stocks 1 and 2 increased and decreased by 7% and 2% from the previous period respectively. b places 60% of the investors wealth in stock 1 and 40% in stock 2. Investment in b increases the wealth by a factor of $S = 1.034$ (3.4%).*

Remark 2.0.1. *Note that the set B of valid portfolios is a convex set.*

Definition 2.0.2 (Time n Wealth). *Let $\{X_i\}_{i=1}^{n}$ denote a sequence of price relatives and $b = \{b_i\}_{i=1}^{n}, b_i \in B$, a sequence of portfolios. Define*

$$S_n \equiv S_n(b) := \prod_{i=1}^{n} b_i^T X_i \ , \quad S_0 = 1,$$

as the time n wealth from following b.

Note that at the end of the i^{th} period, the fraction of wealth invested in the k^{th} stock is

$$b_{ik}^{'} = \frac{b_{ik} X_{ik}}{b_i^T X_i}.$$

For an investor to follow the portfolio $b = \{b_i\}_{i=1}^{n}$ therefore requires a lot of effort as at each step i they are required to buy and sell assets to obtain their desired distribution of wealth, b_i, for the i^{th} trading period. In this paper we frequently look at the case where the investor's portfolio remains constant at each step i.e. $b_i = b \ \forall i$. The corresponding time n wealth is $S_n = \prod_{i=1}^{n} b^T X_i$. We refer to b as a constantly rebalanced portfolio with its name owing to the fact that after each step we are required to rebalance our assets to maintain a constant fraction of our wealth b_k invested in the k^{th} stock.

3 The Log Optimal Portfolio

In this section we deal with the most unrealistic scenario where we assume $X_1, ..., X_n \overset{iid}{\sim} F(\cdot)$ with $F(\cdot)$ known. We also make the assumption that their are no transaction costs. Obviously this model has limitations as the distribution of stock sequences changes in response to a variety of complex events. However the analysis of log optimal investment yields interesting results and is referenced throughout this paper. To motivate the theory suppose $X_1, ..., X_n \overset{iid}{\sim} F(\cdot)$ and the investor is restricted to selecting a single constantly rebalanced portfolio b. Which portfolio should they select to maximise the asymptotic growth exponent, $\log(S_n)/n$, of their wealth? By the law of large numbers, we have:

$$\frac{log(S_n)}{n} = \frac{\sum_{i=1}^{n} log(b^T X_i)}{n} \xrightarrow{P} \mathbb{E}(log(b^T X)) := W(b, F). \tag{1}$$

where $W(b, F)$ is the growth rate of our wealth. Thus for large n, $S_n \approx 2^{nW(b,F)}$. This suggests that to maximise the asymptotic return we should seek the portfolio which maximises the exponent $W(b, F)$, the expected log return for one time step. In this section we develop this theory and show that this approach is optimal for a number of notions of "optimality", other than maximal asymptotic growth rate. The theory behind this will closely follow from [6].

3.1 Log Optimality and the KKT conditions

Definition 3.1.1 (Growth Rate and Log Optimal Portfolio)**.** *For a stock distribution $X \sim F(\cdot)$, define the growth rate of a portfolio b as*

$$W(b, F) = \int \log(b^T X) dF(x) = \mathbb{E}(log(b^T X)).$$

Furthermore define the log optimal portfolio and optimal growth rate as $b^* = argmax_{b \in B} W(b, F)$ and $W^*(F) \equiv W(b^*, F)$ respectively.

We now prove some properties of the growth rate and log optimal portfolio that will be used in subsequent proofs.

Lemma 3.1.1 (Concavity of growth rate)**.** *The growth rate $W(b, F)$ is concave in the portfolio b.*

Proof. For $\lambda \in (0, 1)$ and portfolios $b_1, b_2 \in B$, by the concavity of the logarithm,

$$\begin{aligned} W(\lambda b_1 + (1-\lambda)b_2, F) &= \mathbb{E}(log((\lambda b_1 + (1-\lambda)b_2)^T X)) \geq \mathbb{E}(\lambda \log(b_1^T X) + (1-\lambda)\log(b_2^T X)) \\ &= \lambda W(b_1, F) + (1-\lambda)W(b_2, F). \end{aligned}$$

□

Lemma 3.1.1 and Remark 2.0.1 imply that to find the optimal growth rate $W^*(F)$, we have to maximise the concave function $W(b, F)$ over the convex set of portfolios B.

Lemma 3.1.2 (Convexity of Optimal Growth Rate)**.** *The optimal growth rate $W^*(F)$ is convex in the stock distribution F.*

Proof. In this proof we use the notation $W^*(F) \equiv W(b^*(F), F)$ to distinguish which stock distribution the portfolio b^* is log optimal for. Let $\lambda \in (0, 1)$ and F_1, F_2 be two stock distributions. By linearity of the integral in the measure, we have

$$\begin{aligned} W^*(\lambda F_1 + (1-\lambda)F_2) &= W(b^*(\lambda F_1 + (1-\lambda)F_2), \lambda F_1 + (1-\lambda)F_2) \\ &= \lambda W(b^*(\lambda F_1 + (1-\lambda)F_2), F_1) + (1-\lambda)W(b^*(\lambda F_1 + (1-\lambda)F_2), F_2). \end{aligned}$$

By definition of the log optimal portfolios $b^*(F_1)$ and $b^*(F_2)$, for distributions F_1 and F_2 respectively, we have

$$W^*(\lambda F_1 + (1-\lambda)F_2) \leq \lambda W(b^*(F_1), F_1) + (1-\lambda)W(b^*(F_2), F_2) = \lambda W^*(F_1) + (1-\lambda)W^*(F_2).$$

□

Lemma 3.1.3 (Convexity of Log Optimal portfolios). *The set of log optimal portfolios $B^* := \{b^* \in B : W(b^*, F) = W^*(F)\}$ is a convex set.*

Proof. Let $b_1^*, b_2^* \in B^*$ and $\lambda \in (0, 1)$. By lemma 3.1.1 we have

$$W(\lambda b_2^* + (1-\lambda)b_2^*, F) \geq \lambda W(b_2^*, F) + (1-\lambda)W(b_2^*, F) = \lambda W^*(F) + (1-\lambda)W^*(F) = W^*(F). \quad (2)$$

By Remark 2.0.1, $\lambda b_2^* + (1-\lambda)b_2^* \in B$. By definition $W^*(F) = \max_{b\in B} W(b, F)$ is the maximum growth rate over B and therefore we must have equality in (2) implying $\lambda b_2^* + (1-\lambda)b_2^* \in B^*$. $\square$

These three lemmas help to characterise our objective and maximisation problem. We now look at the notion of directional derivatives. These allow us to compute the rate of change of a multivariate function in a particular direction and will be useful for characterising stationary points of the multivariate objective $W(b, F)$. We now state a few results that will be used in the proof of the KKT conditions.

Definition 3.1.2 (Directional Derivative). *The directional derivative of a scalar function $f(\vec{x})$ along a direction $\vec{v}$ is*

$$D_{\vec{v}} f(\vec{x}) = \lim_{h\to 0} \frac{f(\vec{x} + h\vec{v}) - f(\vec{x})}{h}. \quad (3)$$

Remark 3.1.1 (Zero Directional Derivatives). *A concave function $f(\vec{x})$ attains its maximum at $\vec{x}$ iff the directional derivative at $\vec{x}$ in any direction $\vec{v}$ is less than or equal to 0 [4], [7].*

The following proposition will also be useful for proving the KKT conditions.

Proposition 3.1.1 (Extreme Portfolios). *Let b^* be the log optimum portfolio. Then*

$$\mathbb{E}\left(\frac{b^T X}{{b^*}^T X}\right) \leq 1 \ \forall b \in B \iff \mathbb{E}\left(\frac{X_i}{{b^*}^T X}\right) \leq 1 \ \forall i = 1, ..., m \iff \mathbb{E}\left(\frac{e_i^T X}{{b^*}^T X}\right) \leq 1 \ \ \forall e_i \in E$$

where $E := \{e_i, i = 1, ..., m\} \subset B$ is the set of extreme standard basis portfolios.

Proof. The second equivalence follows trivially from the definition of e_i. We therefore focus on the first equivalence. The forward direction follows by taking the standard basis portfolios $e_i \in B$. For the reverse direction let $b \in B$ and suppose

$$\mathbb{E}\left(\frac{X_i}{{b^*}^T X}\right) \leq 1 \ \forall i = 1, ..., m.$$

Then

$$\mathbb{E}\left(\frac{b^T X}{{b^*}^T X}\right) = \sum_{i=1}^{m} b_i \mathbb{E}\left(\frac{X_i}{{b^*}^T X}\right) \leq \sum_{i=1}^{m} b_i = 1.$$

$\square$

We are now in a position to derive the KKT conditions for the log optimum portfolio. These conditions have some important consequences and are insightful for understanding why we maximise the expected log return.

Theorem 3.1.1 (KKT Conditions for Log Optimum Portfolio). *Let $X \sim F(\cdot)$. The log optimum portfolio b^* satisfies the following necessary and sufficient conditions:*

$$\mathbb{E}\left(\frac{X_i}{{b^*}^T X}\right) \begin{cases} = 1, & b_i^* > 0 \\ \leq 1, & b_i^* = 0 \end{cases}$$

Proof. The proof of this follows a similar idea to [page 617, [6]]. As stated in Lemma 3.1.1, the objective $W(b,F)$ is concave in b. Thus by Remark 3.1.1, b^* is log optimum iff $\forall b \in B$, the directional derivative, in the direction b^* to b, is less than or equal to 0. Letting $\lambda \in (0,1)$ and $b \in B$, define the line segment from b to b^* by $b_\lambda := (1-\lambda)b^* + \lambda b$. Then

$$\lim_{h\to 0} \frac{W(b^* + hb_\lambda, F) - W(b^*, F)}{h} \leq 0 \iff \lim_{\lambda\to 0^+} \frac{W(b_\lambda, F) - W(b^*, F)}{\lambda} \leq 0 \iff \frac{d}{d\lambda} W(b_\lambda, F)\Big|_{\lambda=0^+} \leq 0$$

where the derivative is one sided at 0 to ensure b_λ remains in B. By the dominated convergence theorem we have:

$$\frac{d}{d\lambda} W(b_\lambda, F)\Big|_{\lambda=0^+} = \lim_{\lambda\to 0^+} \frac{\mathbb{E}(\log(b_\lambda^T X)) - \mathbb{E}(\log({b^*}^T X))}{\lambda} = \lim_{\lambda\to 0^+} \frac{1}{\lambda} \mathbb{E}\left(log\left(\frac{(1-\lambda){b^*}^T X + \lambda b^T X}{{b^*}^T X} \right)\right)$$

$$= \mathbb{E}\left(\lim_{\lambda\to 0^+} \frac{1}{\lambda} log\left(\frac{(1-\lambda){b^*}^T X + \lambda b^T X}{{b^*}^T X} \right)\right) = \mathbb{E}\left(\frac{b^T X}{{b^*}^T X} \right) - 1 \leq 0$$

where the last equality follows from L'Hopital's rule. Thus $\forall b \in B$

$$\mathbb{E}\left(\frac{b^T X}{{b^*}^T X} \right) \leq 1 \tag{4}$$

We now consider 2 cases, one where (4) holds with equality and one with inequality.

1. If $\exists \lambda < 0$ such that $b_\lambda \in B$, then the line segment b_λ can be extended beyond the maximiser b^* in the simplex B. In this case the global maximiser of $W(b_\lambda, F)$ occurs at $b^* \equiv b_0$ and the two sided derivative is equal to 0. Thus
$$\frac{d}{d\lambda} W(b_{\lambda,F)}\Big|_{\lambda=0} = 0 \iff \mathbb{E}\left(\frac{b^T X}{{b^*}^T X} \right) = 1 \tag{5}$$

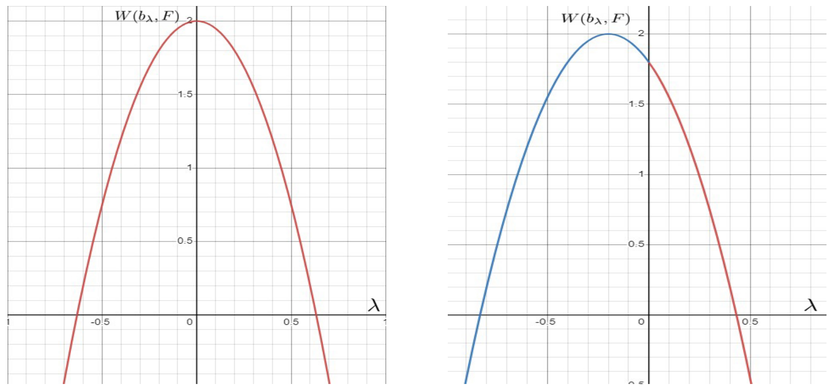
2. If $b_\lambda \notin B \ \forall \lambda < 0$, then the line segment b_λ can not be extended beyond b^* in the simplex due to it violating the constraints in B. In this case $b^* \equiv b_0$ may not globally maximise $W(b_\lambda, F)$ over $\lambda \in \mathbb{R}$, but (by definition) does subject to $b_\lambda \in B$. In this case $W(b_\lambda, F)$ is decreasing or stationary at $\lambda = 0$. In either case we have
$$\frac{d}{d\lambda} W(b_\lambda, F)\Big|_{\lambda=0^+} \leq 0 \iff \mathbb{E}\left(\frac{b^T X}{{b^*}^T X} \right) \leq 1. \tag{6}$$

The 2 cases are visualised in Figure 1. This argument holds for all $b \in B$. By Proposition 3.1.1 and (4) we have

$$\mathbb{E}\left(\frac{b^T X}{{b^*}^T X} \right) \leq 1 \ \forall b \in B \iff \mathbb{E}\left(\frac{e_i^T X}{{b^*}^T X} \right) \leq 1 \ \forall e_i \in E \iff \mathbb{E}\left(\frac{X_i}{{b^*}^T X} \right) \leq 1 \ \forall i = 1, ..., m$$

Applying the same argument as 1 and 2, with $e_i \in B$, we conclude

$$\mathbb{E}\left(\frac{X_i}{{b^*}^T X} \right) \begin{cases} = 1, & \text{line segment from } e_i \text{ to } b^* \text{ can be extended beyond } b^* \text{ in B} \\ \leq 1, & \text{line segment from } e_i \text{ to } b^* \text{ can not be extended beyond } b^* \text{ in B} \end{cases}.$$



(a) Maximum achieved globally at $\lambda = 0$

(b) Maximum achieved within simplex at $\lambda = 0$

Figure 1: Plot of $W(b_\lambda, F)$ against λ

In 1b, the blue line illustrates possible values of $W(b_\lambda, F)$ for invalid portfolio's b_λ, attained outside the simplex. In this case the maximum is not globally achieved at $\lambda = 0$. In 1a, the global maximum is achieved at $\lambda = 0$.

In Proposition 3.1.2 we show that the line segment from e_i to b^* can be extended beyond b^* in the simplex if and only if $b_i^* > 0$. Thus b^* must satisfy the following necessary and sufficient conditions:

$$\mathbb{E}\left(\frac{X_i}{{b^*}^T X}\right) \begin{cases} = 1, & b_i^* > 0 \\ \leq 1, & b_i^* = 0 \end{cases}.$$

□

Proposition 3.1.2 (Extending Line Segments in the Simplex)**.** *Let $e_i \in E$ be an extreme portfolio. The line segment b_λ, from e_i to b^*, can be extended beyond b^* in the simplex iff $b_i^* > 0$.*

Proof. Let $e_i \in E$. For the forward implication suppose the line segment b_λ, from e_i to b^*, can be extended beyond b^* in the simplex. Then $\exists \lambda < 0$ such that $b_\lambda \in B$.

$$0 \leq (b_\lambda)_i \leq 1 \iff 0 \leq (1-\lambda)b_i^* + \lambda \leq 1 \implies \frac{-\lambda}{1-\lambda} \leq b_i^* \leq 1.$$

Since $\lambda < 0$, we conclude

$$0 < \frac{-\lambda}{1-\lambda} \leq b_i^*.$$

This proves the forward direction. For the reverse direction suppose that $b_i^* > 0$. We need to find $\lambda < 0$ such that $b_\lambda = (1-\lambda)b^* + \lambda e_i \in B$. First note that $\forall \lambda < 0$,

$$\sum_{j=1}^{m} (b_\lambda)_j = \sum_{j=1}^{m} (1-\lambda)b_j^* + \lambda = 1.$$

Thus we only require a $\lambda < 0$ such that $0 \leq (b_\lambda)_j \leq 1$ $\forall j \in \{1, ..., m\}$. We claim that $\lambda^* = -b_i^* < 0$ works. Clearly

$$(b_{\lambda^*})_i = (1-\lambda^*)b_i^* + \lambda^* = (1+b_i^*)b_i^* - b_i^* = b_i^{*2} \in [0,1].$$

For $j \neq i$, $(b_{\lambda^*})_j = (1-\lambda^*)b_j^* \geq 0$. Since $(b_{\lambda^*})_j \geq 0 \, \forall j$ and

$$\sum_{j=1}^{m} (b_\lambda)_j = 1,$$

this implies that $(b_{\lambda^*})_j \in [0,1] \; \forall j$. Thus $b_{\lambda^*} \in B$, proving the reverse implication. □

This concludes the proof of the KKT conditions for the log optimal portfolio.

3.2 Consequences of the KKT Conditions

Theorem 3.2.1 (Wealth Ratio)**.** *Let b^* be the log optimum portfolio and $S^* = {b^*}^T X$. Then $\forall b \in B$, with corresponding wealth $S = b^T X$,*

$$\mathbb{E}\left(\frac{S}{S^*}\right) \leq 1$$

and

$$\mathbb{E}\left(\log \frac{S}{S^*}\right) \leq 0$$

Proof. Let $b \in B$ and $S = b^T X$. Then by Theorem 3.1.1 we have

$$\mathbb{E}\left(\frac{S}{S^*}\right) = \sum_{i=1}^{m} b_i \mathbb{E}\left(\frac{X_i}{{b^*}^T X}\right) \leq \sum_{i=1}^{m} b_i = 1.$$

By Jensen's inequality we have

$$\mathbb{E}\left(\log\left(\frac{S}{S^*}\right)\right) \leq \log\left(\mathbb{E}\left(\frac{S}{S^*}\right)\right) \leq \log(1) = 0$$

□

This shows the log optimum portfolio not only maximises the asymptotic growth rate, but also the expected wealth ratio. In this sense the log optimum portfolio performs on average, and asymptotically, better than any other portfolio $b \in B$.

Definition 3.2.1 (Causal Portfolio Strategy)**.** *A casual portfolio strategy $(b_i)_{i=1}^n$ is a sequence of portfolios with portfolio $b_i = b_i(X_1^{i-1})$ being used on day i and dependent only on past values of the stock sequence.*

Causal Portfolios are the only type of portfolio an investor can use in practice without knowledge of the future.

Proposition 3.2.1 (Maximising Wealth Over Causal Investment)**.** *Let S_n^* and S_n be the time n wealth's from following b^* and some causal strategy $(b_i)_{i=1}^n$ respectively. Then*

$$\mathbb{E}(\log(S_n)) \leq \mathbb{E}(\log(S_n^*)) = nW^*(F)$$

Thus the log optimum portfolio maximises the logarithm of the time n wealth over all causal portfolio strategies.

Proof. Let $(b_i)_{i=1}^n$ be a causal portfolio strategy. Then

$$\begin{aligned}\mathbb{E}(\log(S_n^*)) &= \sum_{i=1}^n \mathbb{E}\left(\log\left({b^*}^T X_i\right)\right) = nW^*(F) \geq \sum_{i=1}^n \mathbb{E}\left(\log\left(b_i^T X_i\right)\right) \\ &= \mathbb{E}\left(\log\left(\prod_{i=1}^m b_i^T X_i\right)\right) = \mathbb{E}(\log(S_n))\end{aligned}$$

where the inequality follows by definition of $W^*(F)$ being the maximizer of $\mathbb{E}(\log(b^T X))$ and X_i being IID. □

Up to this point our motivation for using the log optimum portfolio has been mostly in terms of expected performance relative to other portfolio's. At the beginning of Chapter 3 we motivated the log optimum portfolio as maximising the asymptotic growth exponent of S_n. We now formally show that the log optimum wealth S_n^* asymptotically has growth exponent larger than any other wealth S_n (to first order in the exponent). First we define what the growth exponent is.

Definition 3.2.2 (Growth Exponent)**.** *Let $(b_i)_{i=1}^n$ be a portfolio sequence and S_n the time n wealth. The growth exponent is defined as*

$$W_n \equiv W_n(b) := \frac{1}{n}\log(S_n).$$

The growth exponent is not to be confused with the growth rate.

Theorem 3.2.2 (Asymptotic Optimality)**.** *Let $(b_i)_{i=1}^n$ be any causal portfolio strategy, with time n wealth S_n, and S_n^* the time n wealth of the log optimum portfolio strategy. Then,*

$$\limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{S_n}{S_n^*}\right) \equiv \limsup_{n\to\infty} W_n(b) - W_n(b^*) \leq 0 \ \ a.s. \tag{7}$$

Proof. By Markov's inequality and Theorem 3.1.1,

$$\mathbb{P}\left(\frac{S_n}{S_n^*} > n^2\right) \leq n^{-2}\mathbb{E}\left(\frac{S_n}{S_n^*}\right) \leq n^{-2} \iff \mathbb{P}\left(\frac{1}{n}\log\left(\frac{S_n}{S_n^*}\right) \geq \frac{2\log(n)}{n}\right) \leq n^{-2}.$$

Summing both sides over all n gives

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{1}{n}\log\left(\frac{S_n}{S_n^*}\right) \geq \frac{2\log(n)}{n}\right) \leq \frac{\pi^2}{6} < \infty.$$

Define the event

$$E_n = \left\{\frac{1}{n}\log\left(\frac{S_n}{S_n^*}\right) \geq \frac{2\log(n)}{n}\right\}.$$

Since

$$\sum_{n=1}^{\infty} \mathbb{P}(E_n) < \infty$$

, by the Borel Cantelli lemma applied to the events E_n, we have

$$\mathbb{P}\left(\limsup_{n\to\infty} E_n\right) = \mathbb{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} E_k\right) = \mathbb{P}(\{E_n \text{ occurs infinitely often}\}) = 0.$$

Hence, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\frac{1}{n}\log\left(\frac{S_n}{S_n^*}\right) \leq \frac{2\log(n)}{n}$$

or otherwise E_n occurs infinitely often. Thus, almost surely,

$$\limsup_{n\to\infty} \frac{1}{n}\log\left(\frac{S_n}{S_n^*}\right) \leq \limsup_{n\to\infty} \frac{2\log(n)}{n} = 0.$$

□

Comparing the growth exponent of a candidate portfolio (in this case b^*) to other portfolios, in a similar manner to (7), will be a common metric for measuring performance throughout this paper.

Theorem 3.2.3 (Robustness of Log optimal Portfolio)**.** *Consider the usual set up $X_1, ..., X_n \overset{iid}{\sim} F$ and denote the log optimum portfolio as b_f. Let G be another distribution and let b_g be the corresponding log optimum portfolio. Then*

$$\Delta W := W(b_f, F) - W(b_g, F) \leq D(f||g)$$

where $D(\cdot||\cdot)$ is the KL divergence.

This result shows that log optimum investment is in some regards robust against misspecification of the stock distribution.

Proof. The true stock distribution is F with expectation denoted $\mathbb{E}$. When expectation is taken over the stock distribution G we denote this $\mathbb{E}_G$.

$$\begin{aligned}
W(b_f, F) - W(b_g, F) = \mathbb{E}(\log(b_f^T X)) - \mathbb{E}(\log(b_g^T X)) = \int_x f(x)\log\left(\frac{b_f^T x}{b_g^T x}\right) dx \qquad (8)\\
= \int_x f(x)\log\left(\frac{f(x)}{g(x)}\right) dx + \int_x f(x)\log\left(\frac{g(x)b_f^T x}{f(x)b_g^T x}\right) dx \qquad (9)\\
= D(f||g) + \int_x f(x)\log\left(\frac{g(x)b_f^T x}{f(x)b_g^T x}\right) dx \qquad (10)\\
\leq D(f||g) + \log\left(\int_x \cancel{f(x)}\frac{g(x)b_f^T x}{\cancel{f(x)}b_g^T x} dx\right) \qquad (11)\\
= D(f||g) + \log\left(\mathbb{E}_G\left(\frac{b_f^T X}{b_g^T X}\right)\right) \leq D(f||g) + log(1) \qquad (12)\\
= D(f||g) \qquad (13)
\end{aligned}$$

where the inequality in (11) follows from the concavity of the logarithm and Jensen's inequality, and the inequality in (12) follows from Theorem 3.2.1. □

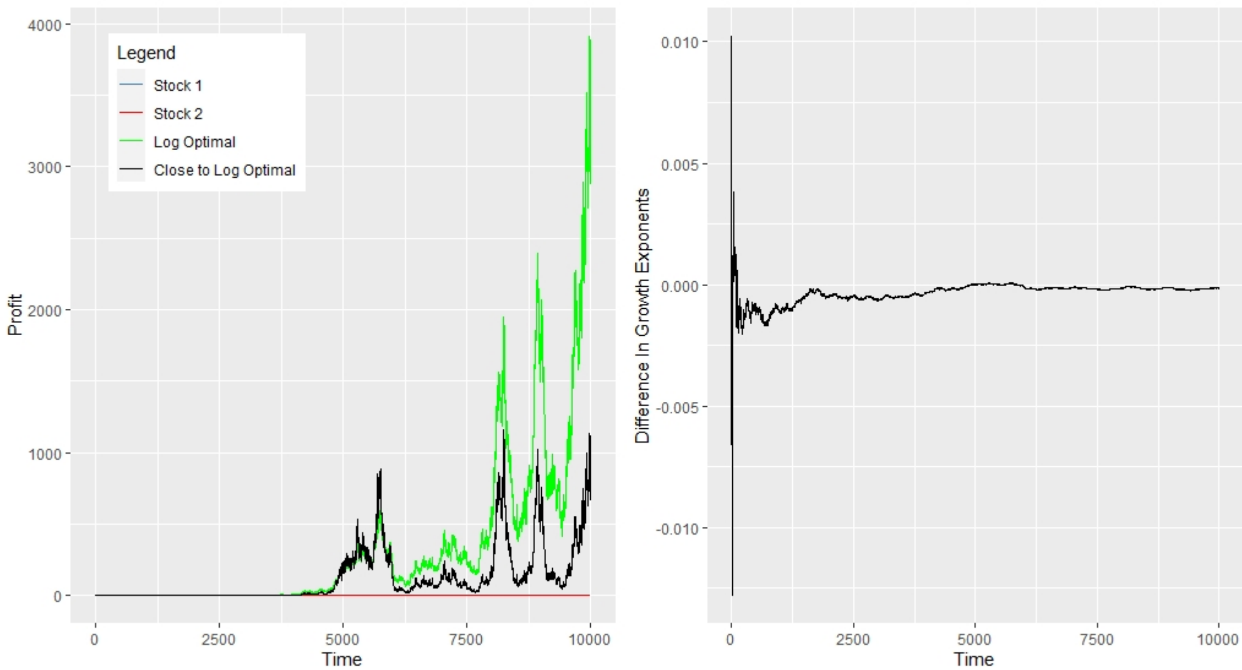


Figure 2: Left hand figure shows the comparison between the profit of different investment strategies over n=10000 time steps. The right hand figure plots the difference in growth exponents between the log optimal portfolio and the portfolio b_3.

3.3 Demonstration of Log Optimal Investment

All R code used in this paper has been written by the author and can be found at Github . In the following demonstration we consider the case of $m = 2$ stocks and $n = 10000$ trading periods. We suppose that our stock distribution is known to follow a shifted uniform Dirichlet distribution i.e. $(X_{i1}, X_{i2}) = (D_{i1}+0.51, D_{i2}+0.49)$ where $(D_{i1,}, D_{i2}) \overset{iid}{\sim} Dir(1, 1)$. In this case both stocks are centered at approximately 1, with stock 1 slightly biased towards returning a profit and stock 2 slightly biased towards returning a loss. Investment in stock 1 should therefore perform marginally better. We will show that investing all our wealth in stock 1 performs considerably worse than investing in the log optimum portfolio. Computation using R yields a log optimum portfolio of $b^* = (0.56, 0.44)$. As expected it places more wealth in stock 1 than stock 2, however the difference of 12% is surprisingly high given the marginal performance difference between the two stocks. We compare the performance of the log optimum portfolio to $b_1 = (1, 0)$ and $b_2 = (0, 1)$ (investment solely in stock 1 and 2 respectively) as well as the portfolio $b_3 = (0.6, 0.4)$, which is close to log optimum. The results are shown on the left hand side of Figure 2. As expected the log optimum portfolio out performs the other 3 investment strategies, yielding, in this simulation, a return of over $4000\times$ the initial wealth. This performs considerably better than investment solely in stocks 1 and 2, whose corresponding profits are almost 0! Looking at figure 2 we see that portfolio b_3 and the log optimum portfolio have similar growth exponents. Theorem 3.2.2 states that the log optimum portfolio should have the best asymptotic growth exponent over any causal portfolio. The right hand side of Figure 2 demonstrates this result, plotting the difference in growth exponents of S_n

and S_n^*, generated from b_3 and b^* respectively. As expected from the theorem, for large n we have

$$\frac{1}{n}log\left(\frac{S_n}{S_n^*}\right) \leq 0$$

indicating that to first order in the exponent, S_n^* asymptotically exceeds S_n. Surprisingly the convergence is very close to 0, even though the portfolio b_3 is not that close to b^*.

As motivated by Theorem 3.2.3, we may also like to know how robust the log optimal portfolio is to misspecification of the distribution. Suppose the stock market follows the aforementioned shifted distribution, with $(D_{i1}, D_{i2}) \overset{iid}{\sim} f \overset{D}{=} Dir(1,1)$, but instead we slightly misspecify the distribution to be $(D_{i1}, D_{i2}) \overset{iid}{\sim} g \overset{D}{=} Dir(0.9999, 0.9999)$. We assume the shifting parameters are known to be 0.51 and 0.49. Through R we find $D(f||g) = 8.8898 \times 10^{-8}$ and $\Delta W = 2.8866 \times 10^{-11}$. Thus, even when the misspecified stock distribution g is extremely close to the true distribution f, the difference in growth rate ΔW does not exceed the KL divergence.

3.4 Game Theory and Log Optimal Investment

To conclude the discussion on log optimality we present a scenario in which log optimal investment is the solution to an optimisation problem. As presented in [2], consider a two person 0 sum game in which player i start with £1, selects a randomisation random variable $W_i \in W := \{W : \mathbb{E}(W) \leq 1, W \geq 0\}$, with corresponding distribution G_i, and a portfolio $b_i \in B$. The set W is the set of fair randomisations. Player i then gambles their £1 according to W_i and distributes the resulting wealth according to their selected portfolio b_i. Player 1 is then paid an amount dictated by the following payoff function:

$$P_\phi = P_\phi(W_1, W_2, b_1, b_2) := \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2 S_2}\right)\right) = \mathbb{E}\left(\phi\left(\frac{W_1 b_1^T X}{W_2 b_2^T X}\right)\right)$$

where ϕ is any non decreasing function. Player 1's gain is player 2's loss, as characterised by 0 sum games. This is called the Stock Market ϕ Game.

Definition 3.4.1 (Value of the ϕ Game)**.** *If*

$$\sup_{b_1,W_1} \inf_{b_2,W_2} P_\phi = \inf_{b_2,W_2} \sup_{b_1,W_1} P_\phi := v_\phi$$

, then v_ϕ is the value of the game corresponding to the minimax payoff. Let W_1^*, W_2^* and b_1^*, b_2^* denote the randomisation random variables and portfolios for players 1 and 2 respectively, which achieve v_ϕ.

Definition 3.4.2 (Primitive ϕ Game)**.** *The primitive ϕ game is the game where player i selects a randomisation distribution $W_i \in W$ and gambles their £1 according to this. Player 1 is then paid according to the payoff function*

$$P_\phi^{prim} = \mathbb{E}\left(\phi\left(\frac{W_1}{W_2}\right)\right).$$

Define the value of the primitive game as

$$\sup_{W_1 \in W} \inf_{W_2 \in W} P_\phi^{prim} = \inf_{W_2 \in W} \sup_{W_1 \in W} P_\phi^{prim} := v_\phi^{prim}$$

and let $W_1^{*'} \sim H_1^*$ and $W_2^{*'} \sim H_2^*$ be the minimax distributions which achieve v_ϕ^{prim}.

The primitive game is the same as the Stock Market ϕ Game, only the players don't invest the proceeds of their randomisation in the stock market.

Theorem 3.4.1 (Log Optimality and the Stock Market ϕ Game)**.** *Let b^*, S^* be the log optimum portfolio and wealth for stock distribution F. Consider the setup of the Stock Market ϕ Game with payoff function P_ϕ and let W_1^*, W_2^* and b_1^*, b_2^* be the minimax randomisations and portfolios for players 1 and 2 respectively. Let $W_1^{*'} \sim H_1^*$ and $W_2^{*'} \sim H_2^*$ be the minimax distributions in the primitive game. Then the value of the Stock Market ϕ Game and its corresponding minimax distributions coincide with those of the primitive game i.e. $v_\phi = v_\phi^{prim}$ and $W_1^* \sim H_1^*, W_2^* \sim H_2^*$. Furthermore the minimax portfolio for either player is the log optimum portfolio i.e. $b_1^* = b_2^* = b^*$.*

Proof. We first reduce the problem in the ϕ game to the primitive game. Note that for all $W_2 \in W$ and $b_2 \in B$, with $S_2 = b_2^T X$, we have by independence and Theorem 3.2.1 that

$$\mathbb{E}\left(\frac{W_2 S_2}{S^*}\right) = \mathbb{E}(W_2)\mathbb{E}\left(\frac{S_2}{S^*}\right) \leq \mathbb{E}(W_2) \leq 1.$$

In addition

$$\frac{W_2 S_2}{S^*} \geq 0.$$

Hence $W_2 S_2 / S^* \in W$ is a valid randomisation. Thus

$$\mathbb{E}\left(\phi\left(\frac{W_1^{*'} S^*}{W_2 S_2}\right)\right) = \mathbb{E}\left(\phi\left(\frac{W_1^{*'}}{W_2 S_2/S^*}\right)\right) \geq \mathbb{E}\left(\phi\left(\frac{W_1^{*'}}{W_2^{*'}}\right)\right) = v_\phi^{\text{prim}}$$

where the inequality follows by definition of $W_2^{*'}$ as the infimum. Similarly we find for all $W_1 \in W$ and $b_1 \in B, S_1 = b_1^T X$, that $W_1 S_1 / S^* \in W$ and hence

$$\mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2^{*'} S^*}\right)\right) = \mathbb{E}\left(\phi\left(\frac{W_1 S_1/S^*}{W_2^{*'}}\right)\right) \leq E\left(\phi\left(\frac{W_1^{*'}}{W_2^{*'}}\right)\right) = v_\phi^{\text{prim}}.$$

Hence

$$v_\phi = \sup_{W_1, b_1} \inf_{W_2, b_2} \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2 S_2}\right)\right) \geq \inf_{W_2, b_2} \mathbb{E}\left(\phi\left(\frac{W_1^{*'} S^*}{W_2 S_2}\right)\right) \geq v_\phi^{\text{prim}} \tag{14}$$

$$\geq \sup_{b_1, W_1} \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2^{*'} S^*}\right)\right) \geq \inf_{W_2, b_2} \sup_{W_1, b_1} \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2 S_2}\right)\right) = v_\phi. \tag{15}$$

Hence all inequalities must be equalities. Thus $v_\phi = v_\phi^*$ and

1. $$\sup_{W_1, b_1} \inf_{W_2, b_2} \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2 S_2}\right)\right) = \inf_{W_2, b_2} \mathbb{E}\left(\phi\left(\frac{W_1^* S^*}{W_2 S_2}\right)\right)$$

2. $$\sup_{b_1, W_1} \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2^* S^*}\right)\right) = \inf_{W_2, b_2} \sup_{W_1, b_1} \mathbb{E}\left(\phi\left(\frac{W_1 S_1}{W_2 S_2}\right)\right).$$

$W_1 = W_1^{*'}$ and $b_1 = b^*$ achieves the desired equality in 1 and similarly $W_2 = W_2^{*'}$ and $b_2 = b^*$ achieves it in 2. Hence $W_i^* \sim H_i^*$ and $b_i^* = b^*$ are solutions to the stock market ϕ game. □

To summarise this chapter we have shown that the log optimum porfolio is optimal in the following three ways:

- Asymptotically outperforming the growth exponent of any other causal portfolio to first order in the exponent.
- Maximises the expected wealth ratio and expected log wealth.
- It is the optimal strategy for either player in the Stock Market ϕ Game.

4 Empirical Bayes

The main issue with log optimum investment was the reliance on the fact that the price relatives, $(X_i)_{i=1}^n$, followed a known distribution $F(\cdot)$. In this chapter we somewhat weaken this by assuming that the stock distribution is over a discrete finite set, i.e at each period i, $X_i \in \mathbf{Z} := \{a_1, ..., a_M\}$, but with no assumptions about what the distribution, if any, may look like. We still make the assumption of no transaction costs. The goal of this chapter is to present a sequential portfolio selection algorithm which performs as well as the best constantly rebalanced portfolio selected with advanced knowledge of the empirical distribution of the stock sequence (or in other words, the number of realisations of each element in $\mathbf{Z}$ at time n). Results from this section follow closely from [1].

4.1 Motivation and Preliminaries

In this subsection we define the necessary notation needed throughout this chapter. We also discuss which constantly rebalanced portfolio a hindsight investor would select if they had knowledge of the empirical distribution.

Definition 4.1.1 (Probability Simplex)**.** *Let $\mathbf{Z} = \{a_1, ..., a_M\}$ denote the stock set. The probability simplex over $\mathbf{Z}$ is defined as*

$$\Delta(M) := \left\{ p \in \mathbb{R}^M : p_i \geq 0, \sum_{i=1}^{M} p_i = 1 \right\}$$

Note that $\Delta(m) = B$ since the set of valid portfolios B, over m stocks, is the probability simplex.

Definition 4.1.2 (Empirical Distribution and Density)**.** *Let $\mathbf{Z} := \{a_1, ..., a_M\}$. Given a sequence x_1^n, define it's Empirical Density function as*

$$P_n(a) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(x_i = a).$$

In this chapter it is also useful to consider the vector form:

$$P_n := \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(x_i)$$

where $\mathbb{I}(x_i) = e_k \in \Delta(M)$ for $x_i = a_k$. Let $F_n(x)$ be the corresponding discrete distribution function with $F_n(dx)$ being the measure which assigns mass $1/n$ to each x_i.

We use P_n to describe the empirical stock distribution e.g. for $\mathbf{Z} = \{a_1, ..., a_5\}$, $P_{10} = (1/10, 2/10, 2/10, 3/10, 2/10)$ would be a valid empirical density defining a distribution over the stock sequences $\mathbf{Z}$. Let $x_1^n \in \mathbf{Z}^n$ be a non random stock sequence and P_n the corresponding empirical density. The growth rate of a portfolio b under this empirical stock distribution is:

$$W(b, P_n) = \mathbb{E}_{X\sim P_n}(\log(b^T X)) = \int \log(b^T x) F_n(dx) = \frac{1}{n}\sum_{i=1}^{n} \log(b^T x_i) = \sum_{i=1}^{M} \log(b^T a_i) P_n(a_i).$$

Suppose that our investor is restricted to selecting a constantly rebalanced portfolio. Then on the stock sequence x_1^n,

$$\begin{aligned} S_n &= 2^{\sum_{i=1}^n \log(b^T x_i)} = 2^{n\left(\frac{1}{n}\sum_{i=1}^n \log(b^T x_i)\right)} \\ &= 2^{nW(b,P_n)} \leq 2^{nW^*(P_n)} \end{aligned}$$

where $W^*(P_n) = W(b^*, P_n)$ with b^* the log optimal portfolio corresponding to the empirical distribution P_n. This holds for any stock sequence x_1^n so a.s holds in the case of random sequences X_1^n. Thus regardless of which constantly rebalanced portfolio we select, we can never do better than the investor who knows the empirical distribution of the stock sequence in advance. As seen above, such an investor would select the empirical log optimum portfolio. Hence if we wish to perform as well as the empirical investor, we must consider sequential portfolios. Our notion of optimality of a portfolio is measured with respect to how well it performs against the investor who knows P_n in advance and invests log optimally i.e. we compare S_n against $2^{nW^*(P_n)}$ or equivalently the growth exponent W_n against the empirical growth rate $W^*(P_n)$. The aim of the rest of this chapter is to present a portfolio selection algorithm for which the growth exponent W_n ϵ achieves $W^*(P_n)$.

4.2 Empirical Bayes Achieving Portfolio

The rest of this chapter is dedicated to finding such an ϵ achieving portfolio. A few preliminaries are required. We now switch back to considering random stock sequences.

Theorem 4.2.1 (Bounded Optimal Growth Rate). *Let $X \in [0,1]^m$ be a vector of price relatives for one period and suppose each stock is logarithmically bounded in the following sense:*

$$|\log(X_i)| \leq L \;\; a.e \;\; \forall i = 1, ..., m$$

Let p and q be two stock measures. Then

$$|W^*(p) - W^*(q)| \leq L\|p - q\|_1.$$

The condition that our stocks X are logarithmically bounded prevents stock prices from exploding to ∞ or reducing to 0 in a single period.

Proof.

$$\begin{aligned} W^*(p) - W^*(q) &= W(b^*(p), p) - W(b^*(q), q) = W(b^*(p), p) - W(b^*(p), q) + W(b^*(p), q) - W(b^*(q), q) \\ &\leq W(b^*(p), p) - W(b^*(p), q) \leq |W(b^*(p), p) - W(b^*(p), q)| = \left|\int \log(b^*(p)^T x)(p - q)(dx)\right| \\ &\leq \int \left|\log(b^*(p)^T x)\right| |(p - q)(dx)|, \end{aligned}$$

where the first inequality follows from the definition of $b^*(q)$ maximising the growth rate $W(b, q)$. Define $\bar{x} = \max(x_1, ..., x_n), \underline{x} = \min(x_1, ..., x_n)$. Since $\log(\cdot)$ is increasing,

$$\log(\underline{x}) = \log\left(\underline{x}\sum_{i=1}^{m} b_i\right) \leq \log(b^T x) \leq \log\left(\bar{x}\sum_{i=1}^{m} b_i\right) = log(\bar{x}).$$

Hence $\left|\log(b^T x)\right| \leq \max(|\log(\underline{x})|, |\log(\bar{x})|) \leq L$. Thus

$$W^*(p) - W^*(q) \leq L \int |p - q|(dx) = L\|p - q\|_1.$$

The same argument shows $W^*(q) - W^*(p) \leq L\|p - q\|_1$. □

We now look at a specific version of Blackwell's approachability game. The general set up of such a game can be found in [Def 1.1, [8]]. It consists of a 4-tuple (X, Y, u, A), where X and Y are closed convex sets, and $A \subseteq \mathbb{R}^n$ is a target convex set that we wish to iteratively converge to. At each turn, player 1 selects a strategy $x_n \in X$ and player 2 selects $y_n \in Y$. Player 1 is then payed according to the vector valued payoff function $u(x_n, y_n)$. The goal of player 1 is to select $x_1, ...x_n \in X$ to ensure that regardless of the strategies played by player 2,

$$\lim_{n\to\infty} \min_{a\in A} \left\|a - \frac{1}{n}\sum_{i=1}^{n} u(x_n, y_n)\right\| = 0.$$

In other words, player 1 wishes to select strategies from their convex set to ensure that regardless of player 2's strategy, the average payoff converges to the target set A at the closest point. We study a specific version of this game, with payoff $u(x_n, y_n) = y_n$, in which at time n player 1 selects a strategy $x_n \in X$ which ensures that for any $y_n \in Y$, $u(x_n, y_n) = y_n \in \mathcal{H}$ for some hyperplane $\mathcal{H} = \{z \in \mathbb{R}^n : a^T z = b\}$, $a \in \mathbb{R}^n, b \in \mathbb{R}$. $\mathcal{H}$ is referred to as a forcing hyperplane, a specialisation of a forcible half space $\{z \in \mathbb{R}^n : a^T z \leq b\}$ (see Def 1.2, [8]), and $x_n \in X$ a forcing strategy.

Definition 4.2.1 (Diameter of Set)**.** *The diameter of a set U is the supremum of the distances between pairs in U i.e.*

$$D = \sup_{x,y\in U} \|x - y\|_2.$$

Theorem 4.2.2 (Blackwell's Approachability)**.** *Let $U \subset \mathbb{R}^d$ be a bounded set of diameter D and let A be a closed convex target subset of U. Define the sequence of moving averages $(q_i)_{i=1}^n$ as, $q_1 = y_1$, for any arbitrary point $y_1 \in U$, and given q_n, $n \geq 1$, define q_{n+1} inductively as follows:*

1. *If $q_n \notin A$ then let t_n be the closest point to q_n in the set A i.e. $t_n = argmin_{t\in A}\|t - q_n\|_2$. Let $\mathcal{H}_n$ be the supporting hyperplane to A passing through the point $t_n \in A$ and orthogonal to $q_n - t_n$. This is illustrated in Figure 3.*

2. *If $q_n \in A$, then let $\mathcal{H}_n$ be any supporting hyperplane of A.*

3. *Let $y_{n+1} \in \mathcal{H}_n$ be any point on $\mathcal{H}_n$ and inductively define the moving average as:*

$$q_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} y_i = \frac{n}{n+1}q_n + \frac{y_{n+1}}{n+1}.$$

Repeat procedure.

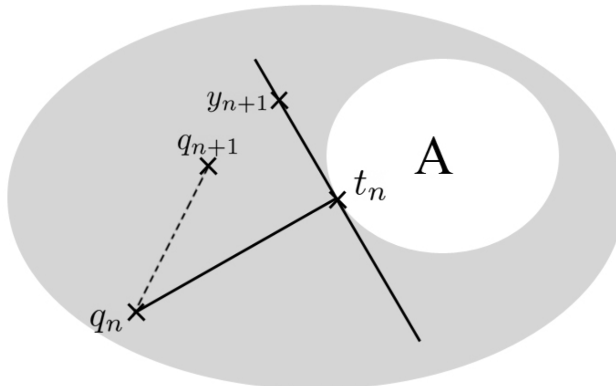


Figure 3: Blackwell's inductive selection for $q_n \in A$

For q_n defined above,

$$d_n := \|q_n - t_n\|_2 \leq \frac{D}{\sqrt{n}} \;\; \forall n \geq 1 \implies q_n \to t_n \in A$$

Remark 4.2.1.
- *Theorem 4.2.2 is a special case of the Blackwell game (X, Y, u, A) with $u(x, y) = y$. The space $\mathcal{H}_n$ defined is a forcible hyperplane and implicitly, selecting $y_{n+1} \in \mathcal{H}_n$ is short hand for player 1 playing a forcing strategy $x \in X$ which forces player 2 to select $y_{n+1} = u(x, y_{n+1}) \in \mathcal{H}_n$. For our needs we do not require the general theory, or even the notion of "player 1" nor sets X and Y, but it is insightful to understand that the theory can be applied much more generally.*
- *$t_n \in A$ is unique by the convexity of the set A and the strict convexity of the L2 norm.*

Proof. We prove that $d_n^2 \leq \frac{D^2}{n}$ by induction. First note that $\forall n \geq 1$, if $q_n \in A$, then $t_n = q_n$ and thus $d_n = 0$. Moreover we have

$$\begin{aligned} d_{n+1}^2 = \|q_{n+1} - t_{n+1}\|^2 \leq \|q_{n+1} - t_n\|^2 &= \left\| \frac{n}{n+1} q_n + \frac{1}{n+1} y_{n+1} - t_n \right\|^2 \\ &= \left\| \frac{n}{n+1}(q_n - t_n) + \frac{1}{n+1}(y_{n+1} - t_n) \right\|^2 \end{aligned}$$

where the first inequality follows from the definition of t_{n+1} being the closest point to q_{n+1} in A. For $q_n \notin A$, $q_n - t_n$ is orthogonal to $y_{n+1} - t_n \in \mathcal{H}_n$ by definition of how $\mathcal{H}_n$ was selected. If $q_n \in A$, then $q_n - t_n = 0$ is trivially orthogonal to $y_{n+1} - t_n$ with $d_n = 0$. In either case,

$$d_{n+1}^2 \leq \left\| \frac{n}{n+1}(q_n - t_n) \right\|^2 + \left\| \frac{1}{n+1}(y_{n+1} - t_n) \right\|^2 \leq \left(\frac{n}{n+1} \right)^2 \|q_n - t_n\|^2 + \frac{1}{(n+1)^2} \|y_{n+1} - t_n\|^2 \quad (16)$$

$$\leq \left(\frac{n}{n+1} \right)^2 d_n^2 + \frac{1}{(n+1)^2} D^2 \quad (17)$$

where the last inequality follows from the fact $y_{n+1}, t_n \in U$ and U has diameter D. We now begin the induction. For the base case note that

$$d_1^2 = \|q_1 - t_1\|^2 \le D^2$$

since $q_1 = y_1$ is chosen arbitrarily in U which has diameter D. Assume $d_n^2 \le \frac{D^2}{n}$. From (17),

$$d_{n+1}^2 \le \frac{nD^2}{(n+1)^2} + \frac{D^2}{(n+1)^2} = \frac{D^2}{n+1}.$$

Hence by induction the result follows. □

We now have an iterative method for constructing an average of terms q_n which converges to any convex set A. Since the empirical density P_n, defined in Definition 4.1.2, is the average of indicator terms, it is clear how this theorem might be used to construct a portfolio sequence whose growth exponent W_n converges to $W^*(P_n)$, the target exponent. The following theorem applies Blackwell's approachability argument to prove this.

Theorem 4.2.3 (Empirical Bayes Portfolio Selection)**.** *For $n \ge 1$ let $x_1^n \in \mathbf{Z} = \{a_1, ..., a_M\}$ be a stock sequence. Then there exists a sequence of causal portfolios $(b_i)_{i=1}^n$, with b_i depending only $x_1, ..., x_{i-1}$ and M, for which the corresponding growth exponent W_n ϵ achieves $W^*(P_n)$ with*

$$W_n \ge W^*(P_n) - \frac{K}{\sqrt{n}}$$

for some constant K.

Proof. Let $W_n = \frac{1}{n}\log(S_n)$ be the growth exponent and let $L := \max_{i\in[M], j\in[m]} |\log(a_{ij})|$. Define the bounded set

$$U = \{(p, W) : p \in \Delta(M), |W| \le L\} \subset \mathbb{R}^{M+1}.$$

The diameter of U is

$$D^2 = \sup_{(p,W),(p',W')\in U} \|(p - p', W - W')\|_2^2 \tag{18}$$

$$= \sup_{-L\le W,W'\le L} (W - W')^2 + \sup_{p,p'} \|p - p'\|_2^2 = 4L^2 + 2. \tag{19}$$

Further define

$$A = \{(p, W) \in U : W \ge W^*(p)\}.$$

Note that $\forall\lambda \in [0, 1], (p, W), (p', W') \in A$, we have $\lambda p + (1 - \lambda)p' \in \Delta(M)$, $|\lambda W + (1 - \lambda)W^{'}| \le L$ by the triangle inequality, and by Lemma 3.1.2,

$$\lambda W + (1 - \lambda)W' \ge \lambda W^*(p) + (1 - \lambda)W^*(p') \ge W^*(\lambda p + (1 - \lambda)p').$$

Thus $\lambda(p, W) + (1 - \lambda)(p', W') \in A$ and so A is a closed convex subset of U. This will be our target set in Blackwell's. Let $q_1 = (\mathbb{I}(x_1), \log(S_1)) \in U$ be our arbitrary initial choice. For $n \ge 1$, suppose $q_n = (P_n, W_n) \in U$ and inductively define q_{n+1}, as follows:

1. Let $t_n = (\hat{P}_n, \hat{W}_n) = argmin_{t\in A}\|t - q_n\|_2$ denote the closest point to q_n in A. If $q_n \notin A$, then, t_n is on the boundary of A and thus $\hat{W}_n = W^*(\hat{P}_n) = W(b^*(\hat{P}_n), \hat{P}_n)$, where $b^*(\hat{P}_n)$ is the log optimum portfolio for distribution $\hat{P}_n$.

2. Set $b_{n+1} = b^*(\hat{P}_n)$ and denote the supporting hyperplane to A, passing through t_n and orthogonal to $q_n - t_n$, by
$$\mathcal{H}_n = \{\ (p, W(b^*(\hat{P}_n), p)) : p \in \Delta(M)\}$$

3. Consider the point
$$y_{n+1} = \left(\mathbb{I}(x_{n+1}), \log\left(b_{n+1}^T x_{n+1}\right)\right).$$
Note that $\mathbb{I}(x_{n+1}) = e_k \in \Delta(M)$ for some k such that $x_{n+1} = a_k \in \mathbf{Z}$. By our choice of b_{n+1} and our deterministic distribution $\mathbb{I}(x_{n+1})$ over $\mathbf{Z}$,
$$W(b^*(\hat{P}_n), \mathbb{I}(x_{n+1})) = \mathbb{E}_{\mathbb{I}(x_{n+1})}\left(\log\left(b^*(\hat{P}_n)^T X\right)\right) = \log\left(b_{n+1}^T x_{n+1}\right).$$
Thus $y_{n+1} \in \mathcal{H}_n$.

4. Let
$$q_{n+1} = \frac{n}{n+1} q_n + \frac{y_{n+1}}{n+1} = (P_{n+1}, W_{n+1}).$$

Note that if $q_n \in A$ in step 1, then $q_n = (P_n, W_n) = (\hat{P}_n, \hat{W}_n) = t_n$ and we proceed the same, defining $b_{n+1} = b^*(\hat{P}_n) = b^*(P_n)$ and using the same supporting hyperplane. Our definitions of $q_n = (P_n, W_n)$ and y_n imply that
$$q_n = \frac{1}{n}\sum_{i=1}^{n} y_i.$$
Thus
$$P_n = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(x_i)$$
is the empirical density, defined in Definition 4.1.2, and
$$W_n = \frac{1}{n}\sum_{i=1}^{n} \log(b_i^T x_i) = \frac{1}{n}\log(S_n)$$
is the growth exponent. By Blackwell's theorem, we therefore have $\|q_n - t_n\| \le \frac{D}{\sqrt{n}}$.
We claim that that the portfolio inductively defined as $b_{n+1} = b^*(\hat{P}_n)$ ϵ achieves $W^*(P_n)$. To see this consider the setup at time n: $q_n = (P_n, W_n), t_n = (\hat{P}_n, W^*(\hat{P}_n))$. Then
$$\max(\|P_n - \hat{P}_n\|_2, |W_n - W^*(\hat{P}_n)|) \le \|(P_n, W_n) - (\hat{P}_n, W^*(\hat{P}_n))\|_2 = \|q_n - t_n\|_2 \le \frac{D}{\sqrt{n}}. \tag{20}$$

Let $\delta(\vec{v})$ denotes the vector of signs. By the Cauchy Schwarz inequality
$$\|P_n - \hat{P}_n\|_1 = \delta(P_n - \hat{P}_n)^T (P_n - \hat{P}_n) \le \sqrt{\delta(P_n - \hat{P}_n)^T \delta(P_n - \hat{P}_n))}\|P_n - \hat{P}_n\|_2 \le \frac{\sqrt{M}D}{\sqrt{n}}. \tag{21}$$

Thus by Theorem 4.2.1 and (21) we have
$$|W^*(\hat{P}_n) - W^*(P_n)| \le L\|P_n - \hat{P}_n\|_1 \le \frac{LD\sqrt{M}}{\sqrt{n}}.$$

Thus by the triangle inequality, (20) and (19),

$$|W_n - W^*(P_n)| \leq |W_n - W^*(\hat{P}_n)| + |W^*(\hat{P}_n) - W^*(P_n)| \leq \|q_n - t_n\|_2 + \frac{LD\sqrt{M}}{\sqrt{n}}$$
$$\leq \frac{D}{\sqrt{n}}(1 + L\sqrt{M}) = \frac{\sqrt{4L^2+2}(L\sqrt{M}+1)}{\sqrt{n}}.$$

In particular we have
$$W_n \geq W^*(P_n) - \frac{K}{\sqrt{n}}$$
where $K = \sqrt{4L^2+2}(L\sqrt{M}+1)$. Thus the inductively defined portfolio $b_{n+1} = b^*(\hat{P}_n)$ depends only on the past and obtains a growth exponent W_n, which for large n, is arbitrarily close to the growth rate $W^*(P_n)$ of the empirical log optimum investor. □

Quite unbelievably, following b_n asymptotically obtains the same wealth as the investor who knows the empirical stock distribution in advance and log optimally invests accordingly.

5 The Universal Portfolio

In this section we move towards a more realistic scenario in which the stock sequence follows an unknown distribution in a set $\mathcal{P}$. For now we still make the assumption of no transaction costs. As stated in Chapter 3, if the stock sequence follows a distribution $F \in \mathcal{P}$, the log optimum portfolio is asymptotically the best one can aim for over any causal portfolio strategy (Theorem 3.2.2). However since F is unknown, we can not compute the log optimum portfolio nor use its growth rate $W^*(F)$ as a benchmark for this chapter. Similar to Chapter 4, any portfolio we propose will be measured relative to the best constantly rebalanced portfolio of a hindsight investor. Unlike Chapter 4, this wont be the empirical log optimum portfolio as we no longer assume a distribution over a finite set.

The main issue with investing purely based on past stock data, is that regardless of how intelligently we select a casual portfolio $b_i(x_1^{i-1})$, we can always select an adverse stock sequence x_1^i which tricks our causal portfolio into investing into the stocks which perform the worst at the i^{th} step [3]. Thus no such causal portfolio exists which can perform well at every step i on all possible stock sequences x_1^i. A natural approach to dealing with this would be to construct $b_i(x_1^{i-1})$ by taking a performance weighted average of portfolios over the known past, so as to stand the best chance of making a profit at the present. Taking such an average in some sense protects the investor against extreme stock sequences. In this chapter we develop such a portfolio, again comparing its performance against a hindsight investor. The theory follows closely from the works of Thomas Cover and can be found in [3].

5.1 The Universal Investment Strategy

Definition 5.1.1 (Best Constantly Rebalanced Portfolio)**.** *Let $b^* \equiv b^*(x_1^n) = argmax_{b\in B} S_n(b)$ denote the best constantly rebalanced portfolio in hindsight and $S_n^* = \max_{b\in B} S_n(b)$ the corresponding maximum wealth. Let*
$$W_n(b^*) \equiv W_n^*(x_1^n) = \frac{1}{n}\log(S_n^*)$$
be the growth exponent of this portfolio.

In this chapter the growth exponent $W_n^*(x_1^n)$ will be used as the bench mark for measuring the performance of candidate portfolios. By contrast chapter 3 used the empirical log optimum growth rate $W^*(P_n)$ as the benchmark. We now study a family of portfolios which have comparable performance to $b^*(x_1^n)$.

Definition 5.1.2 (Universal Portfolio's). *Let S_n^* be the wealth generated from the best constantly rebalanced portfolio. A causal portfolio $(b_i)_{i=1}^n$, with time n wealth $S_n \equiv S_n(b)$, is said to be a universal portfolio if*

$$\lim_{n\to\infty} \sup_{x_1^n} \frac{1}{n} \log\left(\frac{S_n^*}{S_n}\right) = \lim_{n\to\infty} \sup_{x_1^n} W_n^*(x_1^n) - W_n(x_1^n) = 0.$$

A universal portfolio is therefore a causal portfolio which, to first order in the exponent, has the same asymptotic growth exponent as the best constantly rebalanced portfolio in hindsight. In this chapter we seek a universal portfolio.

Remark 5.1.1. *Universal portfolios do not necessarily make a profit. If all the stocks decrease over the n periods, then even the best constantly rebalanced portfolio in hindsight will not make a profit. Whilst universal portfolios are also not profitable in this instance, they never loose money with exponent worse, to first order, than the hindsight investor. In such instances these portfolios yield the best worst case growth exponent.*

Remark 5.1.2 (IID Underlying Stock Distribution). *Since* log *is increasing, note that*

$$b^*(x_1^n) = argmax_{b\in B} \frac{1}{n} \log(S_n(b)).$$

If $X_i \overset{iid}{\sim} F$ for some unknown $F \in \mathcal{P}$, then by the law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} \log(S_n(b)) = \mathbb{E}_F(\log(b^T X)) = W(b, F) \implies b^*(x_1^n) \to argmax_{b\in B} W(b, F).$$

Thus for large n, the best constantly rebalanced portfolio $b^*(x_1^n)$ tends to the underlying log optimum portfolio. Therefore seeking a universal portfolio which asymptotically achieves $W_n^*(x_1^n)$ amounts to selecting a portfolio whose growth exponent is asymptotically the same (to first order) as the growth rate of the underlying log optimum portfolio.

We now define the μ weighted universal portfolio.

Definition 5.1.3 (μ Weighted Universal Portfolio (page 352, [3])). *For any distribution μ over the simplex B on m stocks, define the μ weighted universal portfolio by*

$$\hat{b}_i \equiv \hat{b}_i(x_1^{i-1}) = \frac{\int_{b\in B} b S_{i-1}(b, x_1^{i-1}) d\mu(b)}{\int_{b\in B} S_{i-1}(b, x_1^{i-1}) d\mu(b)}.$$

We will refer to this as the universal portfolio from now on.

In this paper we will focus on μ belonging to the family of Dirichlet distributions. Specifically we look at the uniform weighting when $\mu \sim Dir(1, ..., 1)$ and also $\mu \sim Dir(1/2, ..., 1/2)$. We will show that the universal portfolio proposed in Definition 5.1.3 is indeed universal for these two weightings.

Remark 5.1.3 (Uniform Weighted Universal Portfolio). *For $\mu \sim Dir(1, ..., 1)$, $d\mu(b) = \Gamma(m) db$. Thus*

$$\hat{b}_i = \frac{\int_{b\in B} b S_{i-1}(b, x_1^{i-1}) \cancel{\Gamma(m)} db}{\int_{b\in B} S_{i-1}(b, x_1^{i-1}) \cancel{\Gamma(m)} db} = \int_{b\in B} b p(b|x_1^{i-1}) db = \mathbb{E}_p(b)$$

where

$$p(b|x_1^{i-1}) = \frac{S_{i-1}(b, x_1^{i-1})}{\int_{b\in B} S_{i-1}(b, x_1^{i-1})db}$$

is a conditional distribution over $b \in B$. $p(b|x_1^{i-1})$ assigns greater mass to portfolio's which perform best on the known sequence x_1^{i-1}. Such portfolios have a greater contribution to the next portfolio $\hat{b}_i$. Therefore the universal portfolio is said to be performance weighted.

Lemma 5.1.1 (Wealth Generated By Universal Portfolio)**.** *The time n wealth, $\hat{S}_n$, generated by the universal portfolio is*

$$\hat{S}_n = \int_{b\in B} S_n(b, x_1^n)d\mu(b)$$

Proof.

$$\begin{aligned}\hat{S}_n &= \prod_{i=1}^n \hat{b}_i^T x_i = \prod_{i=1}^n \frac{\int_{b\in B} b^T x_i S_{i-1}(b, x_1^{i-1})d\mu(b)}{\int_{b\in B} S_{i-1}(b, x_1^{i-1})d\mu(b)} \\ &= \prod_{i=1}^n \frac{\int_{b\in B} S_i(b, x_1^i)d\mu(b)}{\int_{b\in B} S_{i-1}(b, x_1^{i-1})d\mu(b)} = \int_{b\in B} S_n(b, x_1^n)d\mu(b)\end{aligned}$$

where the final equality follows from the cancellation of cross terms and the assumption that $S_0 = 1$ in Definition 2.0.2. □

5.2 Universality For Dirichlet Weighting

Before proving the universality of our proposed portfolio, we first require some prerequisites.

Lemma 5.2.1 (Upper Bound On Fraction of Sums)**.** *For $a_i \geq 0, b_i \geq 0, i = 1, ..., n$ we have*

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \leq \max_j \frac{a_j}{b_j} \tag{22}$$

Proof. The proof presented here is a slightly altered version of the one found in [page 632, [6]]. Let

$$j^* = arg\max_j \frac{a_j}{b_j}.$$

If $a_{j^*} = 0$, then $\forall i$

$$\frac{a_i}{b_i} \leq \frac{a_{j^*}}{b_{j^*}} = 0 \implies a_i = 0.$$

Hence both sides of (22) are 0 and so it trivially holds. Similarly if $b_{j^*} = 0$, then the upper bound in (22) is ∞ and the inequality also holds trivially. Hence suppose $a_{j^*} \neq 0$ and $b_{j^*} \neq 0$. Note that $\forall i$,

$$\frac{a_i}{b_i} \leq \frac{a_{j^*}}{b_{j^*}} \implies \frac{a_i}{a_{j^*}} \leq \frac{b_i}{b_{j^*}}.$$

Thus

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} = \frac{a_{j^*}\left(1 + \sum_{i\neq j^*} \frac{a_i}{a_{j^*}}\right)}{b_{j^*}\left(1 + \sum_{i\neq j^*} \frac{b_i}{b_{j^*}}\right)} \leq \frac{a_{j^*}\left(1 + \sum_{i\neq j^*} \frac{b_i}{b_{j^*}}\right)}{b_{j^*}\left(1 + \sum_{i\neq j^*} \frac{b_i}{b_{j^*}}\right)} = \frac{a_{j^*}}{b_{j^*}} = \max_j \frac{a_j}{b_j}$$

□

Lemma 5.2.2 (Upper Bound On Fraction of Wealth)**.**

$$\frac{S_n^*}{\hat{S}_n} \leq \max_{j^n \in \{1,\dots,m\}^n} \frac{\prod_{i=1}^n b_{j_i}^*}{\int_{b \in B} \prod_{i=1}^n b_{j_i} d\mu(b)}$$

Proof. Note that for any portfolio b,

$$\begin{aligned} S_n = \prod_{i=1}^n b_i^T x_i = \prod_{i=1}^n \sum_{j=1}^m b_{ij} x_{ij} &= \sum_{j^n \in \{1,\dots,m\}^n} \prod_{i=1}^n b_{ij_i} x_{ij_i} \\ &= \sum_{j^n \in \{1,\dots,m\}^n} w(j^n) x(j^n) \end{aligned}$$

where

$$w(j^n) \equiv w(j^n, b) = \prod_{i=1}^n b_{ij_i}, \quad x(j^n) = \prod_{i=1}^n x_{ij_i}.$$

Because $\sum_{j^n} w(j^n) = 1$ and $w(j^n) \geq 0$, this alternative form of S_n can be viewed as the wealth generated in a single period with an investment of $w(j^n)$ in sequence j^n, with price relative $x(j^n)$. Hence

$$S_n^* = \sum_{j^n} \prod_{i=1}^n b_{j_i}^* x_{ij_i} = \sum_{j^n} w^*(j^n) x(j^n)$$

and similarly by Lemma 5.1.1,

$$\begin{aligned} \hat{S}_n = \int_{b \in B} S_n(b, x_1^n) d\mu(b) &= \sum_{j^n} \int_{b \in B} \prod_{i=1}^n b_{j_i} x_{ij_i} d\mu(b) \\ &= \sum_{j^n} \int_{b \in B} w(j^n) x(j^n) d\mu(b). \end{aligned}$$

By Lemma 5.2.2

$$\frac{S_n^*}{\hat{S}_n} = \frac{\sum_{j^n} w^*(j^n) x(j^n)}{\sum_{j^n} \int_{b \in B} w(j^n) x(j^n) d\mu(b)} = \frac{\sum_{j^n : x(j^n) > 0} w^*(j^n) x(j^n)}{\sum_{j^n : x(j^n) > 0} \int_{b \in B} w(j^n) x(j^n) d\mu(b)} \tag{23}$$

$$\leq \max_{j^n : x(j^n) > 0} \frac{\cancel{x(j^n)} w^*(j^n)}{\int_{b \in B} w(j^n) \cancel{x(j^n)} d\mu(b)} \leq \max_{j^n} \frac{w^*(j^n)}{\int_{b \in B} w(j^n) d\mu(b)} \tag{24}$$

$$= \max_{j^n} \frac{\prod_{i=1}^n b_{j_i}^*}{\int_{b \in B} \prod_{i=1}^n b_{j_i} d\mu(b)}, \tag{25}$$

where in (23) we removed sequences j^n, with $x(j^n) = 0$, which allowed us to cancel $x(j^n)$ in (24) without the possibility of having 0/0. □

Finally we require some additional lemmas from information theory. The next few lemmas and remarks are well established results and can be found in the part 3 information theory course.

Definition 5.2.1 (Type and Type Classes). *Let $x_1^n \in A^n$ for some finite alphabet $A = \{1, ..., m\}$. The type of the string x_1^n is the empirical density defined in Definition 4.1.2:*

$$P_n \equiv P_n(x_1^n) = \Big(\frac{n_1}{n}, ..., \frac{n_m}{n}\Big)$$

where $n_i \equiv n_i(x_1^n)$ are the number of elements $i \in A$ in x_1^n. Given a type P on A, the type class is defined as

$$T(P) = \{x_1^n \in A^n : P_n(x_1^n) = P\}.$$

Definition 5.2.2 (n-Type). *On an alphabet $A = \{1, ..., m\}$, the n-type is defined as*

$$\mathcal{P}_n = \{P \in \Delta(m) : nP(a) \in \mathbb{Z} \ \forall a \in A\}$$

where $\Delta(m) \equiv B$ is the probability simplex over A.

The n-type is the set of distributions in the simplex which could be generated as the type of a sequence $x_1^n \in A^n$.

Remark 5.2.1 (Size of Type Class). *For a type $P \in \mathcal{P}_n$,*

$$|T(P)| = \binom{n}{nP(1)}\binom{n - nP(1)}{nP(2)} \times \cdots \times \binom{n - nP(1) - ... - nP(m-1)}{nP(m)} = \frac{n!}{\prod_{i=1}^{m}(nP(i))!}$$

Lemma 5.2.3 (Number of n-types).

$$|\mathcal{P}_n| = \binom{n+m-1}{m-1} \leq (n+1)^{m-1}$$

Proof. For all $P \in \mathcal{P}_n$, we require $P(i) = K_i/n$ for some $K_i \in \{0, ..., n\}$ and

$$\sum_{i=1}^{m} K_i = n.$$

Thus the number of n-types corresponds to the number of ways we can split n elements up into m bins, each of size K_i. This ensures that the sum of all elements in all bins is n. This can be reformulated as the number of ways we can insert $m - 1$ separating bars into n elements (Stars and Bars Theorem). There are

$$\binom{n+m-1}{m-1}$$

ways of doing this. The upper bound follows from noting that the number of elements in the last bin K_m is fixed at $K_m = 1 - K_1 - ... - K_{m-1}$ and then dropping the requirement that the sum of all K_i has to equal n . There are $n + 1$ possible values for K_i for each $i = 1, ..., m - 1$. Hence $|T(P)| \leq (n+1)^{m-1}$. □

The following lemma can be found in the part 3 information theory lecture notes along with its proof.

Lemma 5.2.4. *Let $Q \in \Delta(m)$ be a distribution over the alphabet $\{1, ..., m\}$. The probability of a string x_1^n under the distribution Q^n satisfies:*

$$Q^n(x_1^n) = 2^{-n(H(P_n)+D(P_n||Q))} \leq 2^{-nH(P_n)}$$

where P_n denotes the type of x_1^n.

Proposition 5.2.1 (Property of a Dirichlet)**.** *For any $\alpha_i > 0, i = 1, ..., m,$*

$$\int_B \prod_{r=1}^{m} x_r^{\alpha_r - 1} dx = \frac{\prod_{r=1}^{m} \Gamma(\alpha_r)}{\Gamma(\alpha_1 + ... + \alpha_m)}$$

Proof. The density function of the $Dir(\alpha_1, ..., \alpha_m)$ integrates to 1. Hence

$$\int_B \frac{\Gamma(\alpha_1 + ... + \alpha_m)}{\prod_{r=1}^{m} \Gamma(\alpha_r)} \prod_{r=1}^{m} x_r^{\alpha_r - 1} dx = 1 \Longleftrightarrow \int_B \prod_{r=1}^{m} x_r^{\alpha_r - 1} dx = \frac{\prod_{r=1}^{m} \Gamma(\alpha_r)}{\Gamma(\alpha_1 + ... + \alpha_m)}.$$

□

Recall that we are trying to show

$$\lim_{n \to \infty} \sup_{x_1^n} \frac{1}{n} \log \left(\frac{S_n^*}{\hat{S}_n} \right) = 0$$

for the $Dir(1, ..., 1)$ and $Dir(1/2, ..., 1/2)$ weighted universal portfolios. The upper bound in Lemma 5.2.2 is independent of the stock sequence x_1^n. Hence if we find an upper bound on

$$\frac{\prod_{i=1}^{n} b_{j_i}^*}{\int_{b \in B} \prod_{i=1}^{n} b_{j_i} d\mu(b)}$$

which holds $\forall j^n \in \{1, ..., m\}^n$ and whose growth exponent tends to 0, then we have proved the result.

Theorem 5.2.1 (Universality Of Uniform Weighted Portfolio)**.** *Let $\hat{b}$ be the $Dir(1, ..., 1)$ weighted universal portfolio, with time n wealth $\hat{S}_n$, and let S_n^* be the time n wealth from the best constantly rebalanced portfolio in hindsight. Then*

$$\frac{S_n^*}{\hat{S}_n} \leq (n+1)^{m-1}$$

which implies universality.

Proof. For any portfolio $b \in B$, define the distribution $Q(i) = b_i$ over $\{1, ..., m\}$. This is valid since

$$\sum_{i=1}^{m} Q(i) = 1.$$

By Lemma 5.2.4, for $j^n \in \{1, ..., m\}^n$

$$Q^n(j^n) = \prod_{i=1}^{n} Q(j_i) = \prod_{i=1}^{n} b_{j_i} = 2^{-n(H(P_n) + D(P_n || Q))} \leq 2^{-nH(P_n)} \tag{26}$$

where P_n is the type of the sequence j^n. By Lemma 5.2.2

$$\frac{S_n^*}{\hat{S}_n} \leq \max_{j^n} \frac{\prod_{i=1}^{n} b_{j_i}^*}{\int_{b \in B} \prod_{i=1}^{n} b_{j_i} d\mu(b)} = \max_{j^n} \frac{Q^{*n}(j^n)}{\int_{b \in B} \prod_{i=1}^{n} b_{j_i} d\mu(b)}$$

where $Q^*(i) = b_i^*$. We know for $\mu \sim Dir(1,...,1)$ that $d\mu(b) = \Gamma(m)db$. For $r = 1,...,m$ and $j^n \in \{1,...,m\}^n$, let $n_r(j^n) := nP_n(r)$ denote the number of symbols r in j^n. For any sequence j^n, by Proposition 5.2.1,

$$\int_{b\in B} \prod_{i=1}^{n} b_{j_i} d\mu(b) = \Gamma(m) \int_{b\in B} \prod_{r=1}^{m} b_r^{n_r(j^n)} db = \Gamma(m) \int_{b\in B} \prod_{r=1}^{m} b_r^{(n_r(j^n)+1)-1} db = \frac{\Gamma(m)\prod_{r=1}^{m}\Gamma(n_r(j^n)+1)}{\Gamma(n+m)} \tag{27}$$

where we take $\alpha_r = n_r(j^n) + 1$ in the proposition and note the sum over $r = 1,...,m$ of $n_r(j^n)$ is n. Thus

$$\int_{b\in B} \prod_{i=1}^{n} b_{j_i} d\mu(b) = \frac{1}{\frac{(n+m-1)!n!}{(m-1)!n!}} \prod_{r=1}^{m} \Gamma(n_r(j^n)+1) = \frac{1}{\binom{n+m-1}{m-1}} \times \frac{1}{\frac{n!}{\prod_{r=1}^{m}(n_r(j^n))!}} \tag{28}$$

$$= \frac{1}{\binom{n+m-1}{m-1}} \times \frac{1}{|T(P_n)|} \geq \frac{1}{\binom{n+m-1}{m-1}} 2^{-nH(P_n)} \tag{29}$$

where the last equality follows from remark 5.2.1 and the inequality follows from the well established result that for any distribution P on A,

$$|T(P)| \leq 2^{nH(P)}.$$

Combining with (26) and using the bound in Lemma 5.2.3, we deduce

$$\frac{S_n^*}{\hat{S}_n} \leq \max_{j^n} \frac{\cancel{2^{-nH(P_n)}}}{\frac{1}{\binom{n+m-1}{m-1}}\cancel{2^{-nH(P_n)}}} = \binom{n+m-1}{m-1} \leq (n+1)^{m-1}. \tag{30}$$

This bound holds $\forall x_1^n$ and all sequences j^n. Thus

$$\lim_{n\to\infty} \sup_{x_1^n} \frac{1}{n} \log\left(\frac{S_n^*}{\hat{S}_n}\right) \leq \lim_{n\to\infty} \frac{m-1}{n} \log(n+1) = 0.$$

Consider the stock sequence $\tilde{x}_i = (1,...,1)\ \forall i = 1,2....$ Then $S_n^* = \hat{S}_n = 1$ and thus

$$0 = \lim_{n\to\infty} \frac{1}{n} \log\left(\frac{1}{1}\right) \leq \lim_{n\to\infty} \sup_{x_1^n} \frac{1}{n} \log\left(\frac{S_n^*}{\hat{S}_n}\right) \leq \lim_{n\to\infty} \frac{m-1}{n} \log(n+1) = 0.$$

Hence

$$\lim_{n\to\infty} \sup_{x_1^n} \frac{1}{n} \log\left(\frac{S_n^*}{\hat{S}_n}\right) = 0.$$

$\square$

This establishes the universality of the $Dir(1,...,1)$ weighted universal portfolio!

Theorem 5.2.2 (Universality of $Dir(1/2,...,1/2)$ Weighted Portfolio)**.** *For $\mu \sim Dir(1/2,...,1/2)$ we have*

$$\frac{S_n^*}{\hat{S}_n} \leq 2(n+1)^{\frac{m-1}{2}}$$

which implies universality.

Proof. The proof of this can be found on page 355 in [3]. It follows a very similar argument to Theorem 5.2.1. Again considering the sequence $\tilde{x}_i = (1, ..., 1)\ \forall i = 1, 2, ...,$ we have

$$0 \le \lim_{n\to\infty} \sup_{x_1^n} \frac{1}{n} \log\left(\frac{S_n^*}{\hat{S}_n}\right) \le \lim_{n\to\infty} \frac{m-1}{2n}\log(n+1) + \frac{1}{n} = 0$$

which proves the result. □

A slight extension to the above proofs of universality yields the following stronger and original result.

Lemma 5.2.5 (Steward - Convergence Of Difference in Growth Exponent)**.** *Let $\hat{S}_n$ be the time n wealth of the $Dir(1, ..., 1)$ or $Dir(1/2, ..., 1/2)$ weighted universal portfolio. Then for any stock sequence x_1^∞,*

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{S_n^*}{\hat{S}_n}\right) = 0$$

Proof. Note that from Lemma 5.1.1, $\hat{S}_n = \mathbb{E}_\mu(S_n(b, x_1^n)) \le S_n^*$ where $b \sim \mu$. Thus for all $n \ge 1$ and stock sequences x_1^n,

$$\frac{1}{n}\log\left(\frac{S_n^*}{\hat{S}_n}\right) = \frac{1}{n}\log\left(\frac{S_n^*}{\mathbb{E}_\mu(S_n(b, x_1^n))}\right) \ge \frac{1}{n}\log\left(\frac{S_n^*}{S_n^*}\right) = 0.$$

Thus by Theorems 5.2.1 and 5.2.2, $\forall n \ge 1$ and x_1^n,

$$0 \le \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{S_n^*}{\hat{S}_n}\right) \le \lim_{n\to\infty} \sup_{x_1^n} \frac{1}{n}\log\left(\frac{S_n^*}{\hat{S}_n}\right) \le 0$$

which concludes the proof. □

This result implies that both μ weighted universal portfolios have the same asymptotic growth exponent on any stock sequence x_1^n. The existing literature showed this only over the supremum of stock sequences.

5.3 Demonstrating the Universal Portfolio

Having seen the theory underlying the μ weighted universal portfolio, we now ask if this works in practice. In this brief subsection we demonstrate how the uniform weighted universal portfolio performs on both fake and real stock data. Again all code was written by the author and can be found at Github. For simplicity we consider the case of 2 stocks ($m = 2$) which allows for exact computation of the universal portfolio. For any portfolio $b = (b_1, b_2) \in B$, we can compute the multivariate integral in Definition 5.1.3 by integrating b_1 over $[0, 1]$ and rewriting b_2 as $1 - b_1$. Thus the universal portfolio on the i^{th} step is $(\hat{b}_{i1}, 1 - \hat{b}_{i1})$ where

$$\hat{b}_{i1} = \frac{\int_0^1 b \prod_{j=1}^{i-1} (bx_{j1} + (1-b)x_{j2})\, db}{\int_0^1 \prod_{j=1}^{i-1} (bx_{j1} + (1-b)x_{j2})\, db}.$$

This has an exact solution and in this paper is computed using the *polynom* package in R. For $m > 2$ stocks, the above multivariate integrals are computationally intractable. Recall from Remark 5.1.3 that $\hat{b}_i = \mathbb{E}_p(b)$ can be expressed as the expectation over the distribution $p \equiv p(b|x_1^{i-1})$. For $m > 2$ we can use Markov chain Monte Carlo methods to sample from the distribution $p(b|x_1^{i-1})$ and then approximate the expectation [10].

5.3.1 Random Stock Data

Similar to the demonstration in 3.3 we generate $n = 300$ data points from a shifted uniform Dirichlet distribution as follows: $X_i = (X_{i1}, X_{i2}) = (D_{i1} + 0.55, D_{i2} + 0.5)$ where $(D_{i1}, D_{i2}) \overset{iid}{\sim} Dir(1, 1)$. Figure 4 illustrates the results from universal investment. The top left figure shows the comparison between the performance of the universal portfolio, the underlying log optimal portfolio (unknown to the investor), investment solely in stocks 1 and 2 and the best constantly rebalanced portfolio in hindsight. We also plot log profit vs time and log profit vs log time which are shown in the top right and bottom left figures respectively. These make it easier to see the wealth from investment in stocks 1 and 2 as well as emphasise key features of interest. As expected from Remark 5.1.2 the wealth from log optimum investment and the best constantly rebalanced portfolio are similar for large n due to the price relatives being iid. By construction, the universal wealth appears to have the same asymptotic growth exponent as the hindsight portfolio; particularly evident in the bottom left figure. To this end, the bottom right figure shows that for large n

$$\frac{1}{n} \log \left(\frac{S_n^*}{\hat{S}_n} \right) \approx 0$$

demonstrating universality of our universal portfolio and confirming the result in Lemma 5.2.5. To summarise, the universal portfolio performs significantly better than investment in either stock, obtaining over $1000\times$ the initial investment in only 300 trading days; quite remarkable for a causal portfolio!

5.3.2 Real Stock Data

We now run the same code on real data using Google and Apple stock from 2023 to 2024. This data was scraped from Yahoo finance. We first had to normalise the stock prices into price relatives. The universal portfolio algorithm assumes the stock market follows an unknown distribution in a set $\mathcal{P}$. Inspecting the price relatives $X_i = (X_{i1}, X_{i2})$, it seems that any distribution (if at all) it follows will be incredibly complex. Unlike the random stock example, whatever underlying distribution it may follow, the price relatives are going to be correlated. Figure 5 shows the results of this. The left hand figure shows the universal portfolio obtains around a 50% increase in wealth, in between the performance of the 2 stocks. The best constantly rebalanced portfolio closely follows the stock with the best performance, in this case Google. The right hand side of figure 5 again demonstrates the universality of the universal portfolio. The performance of the universal portfolio is less impressive compared to its performance on the random iid stock data. In general the portfolio seems to perform worse on correlated stock data. This is likely due to the fact that the universal portfolio weights candidate portfolios, $b \in B$, with their performance $S_n(b, x_1^{i-1})$ over the entire history x_1^{i-1}. For correlated stock data this may be sub optimal as the portfolio the investor selects for the next step should not be so heavily influenced by the performance of a portfolio on stock data from e.g. 8 months ago. Adapting the universal portfolio's weighting in definition 5.1.3, whilst retaining the universality property, has potential for new research.

6 Universal Policies

In this final chapter we consider the most general scenario in which the market follows an unknown distribution and has transaction costs. The addition of these costs drastically changes the problem of finding an optimal portfolio. The best constantly rebalanced portfolio was used as our benchmark in Chapters 4 and 5, however retaining a constant portfolio b requires rebalancing at each step which incurs large transaction costs. Therefore an optimal investor should only switch portfolio's under certain conditions when it is optimal

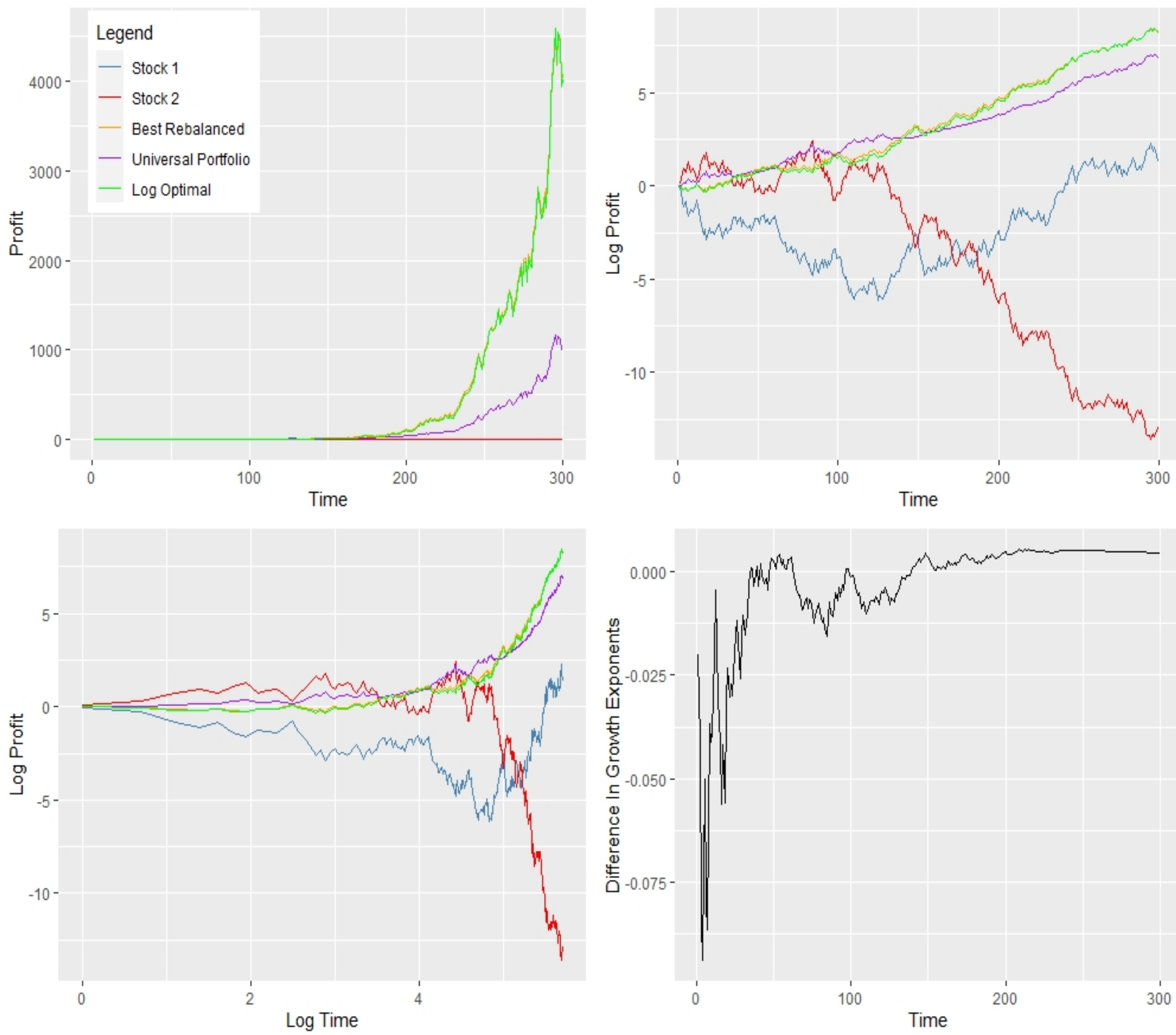


Figure 4: Performance of the uniform weighted universal portfolio on random stock data. Top left and right figures show profit and log profit vs time respectively for various strategies. The bottom left figure shows log profit vs log time and the bottom right figure shows the difference in growth exponent between the universal portfolio and the best hindsight portfolio.

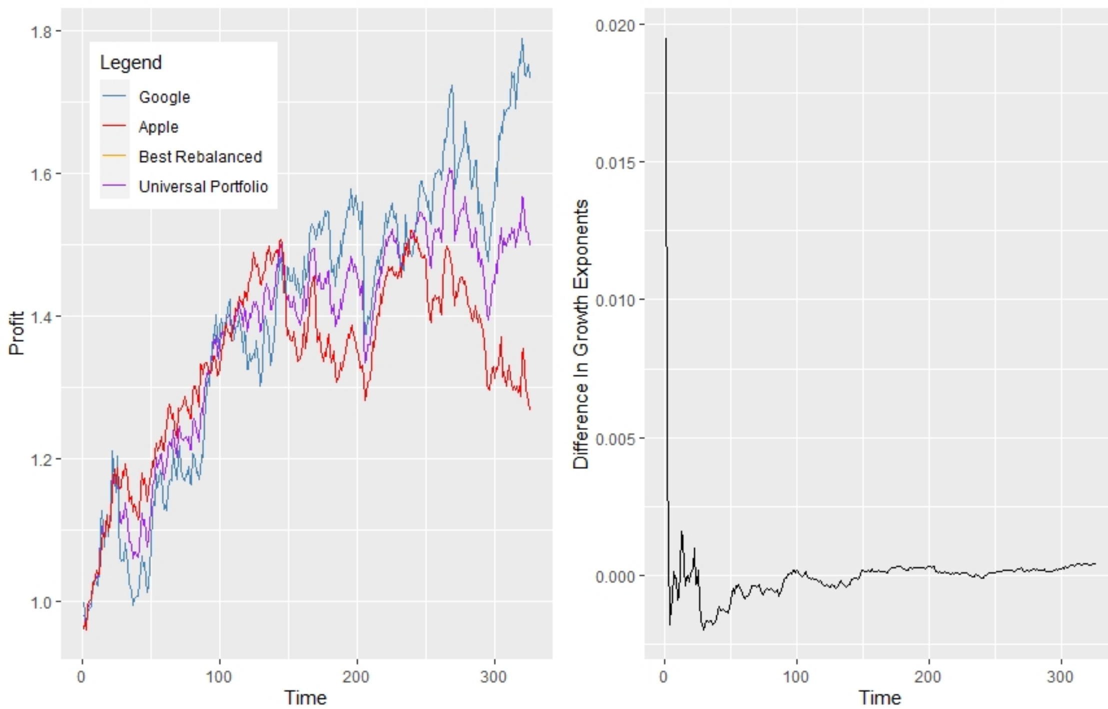


Figure 5: Performance of universal investment on Google and Apple stock collected from 2023 to 2024. This is analogous to figure 4

to incur the transaction fee. In this chapter we look at one such family of conditions called Interval Policies. These will form the basis of our benchmark and construction of our optimal portfolio as developed in [5].

6.1 Markets With Proportional Transaction Costs

In this chapter, we restrict ourselves to markets with $m = 2$ stocks. Further, we suppose that all stock sequences, x_1^∞, are regular whereby they satisfy the condition that $\forall n \geq 1$ there exists $0 < a \leq b$ such that

$$a \leq \frac{x_{n1}}{x_{n2}} \leq b.$$

This ensures neither stock grows arbitrarily larger than the other at any point. For each asset i we define a corresponding transaction rate λ_i. We assume transactions are symmetric. Therefore selling £1 of asset i yields £$1-\lambda_i$ and buying £1 of asset i costs £$1+\lambda_i$. Our market is assumed to have proportional transaction costs (transaction fee's are proportional to the amount traded). Hence selling £1 of asset i and using this money to buy asset j yields the investor

$$£\frac{1-\lambda_i}{1+\lambda_j}$$

of asset j. Throughout this chapter we assume all transactions, unless stated otherwise, are done optimally to minimise transaction fees.

Definition 6.1.1 (Policies). *A policy π is any allowable trading strategy where at each step n the investor can switch from the current portfolio b_n to some z_n. We will be primarily interested in the set of causal self financing polices Π. An investor following such a policy has no knowledge of the future and can not add or withdraw money from outside the system.*

Definition 6.1.2 (Net Realised Wealth and Cumulative Wealth). *Let π be a self financing policy, with transitions from portfolio b_n to z_n at time n. The Net Realised Wealth $w(b_n, z_n)$ is the remaining wealth from optimally trading £1 of portfolio b_n for z_n. Let x_n denote the n^{th} stock vector. Then £1 of portfolio z_n is worth $W(z_n) := £x_{n1}z_n + x_{n2}(1 - z_n)$ at the beginning of the $n + 1^{th}$ period, and is invested in the portfolio*

$$b_{n+1} = \frac{x_{n1}z_n}{W(z_n)}.$$

Under policy π we then switch from b_{n+1} to z_{n+1} and repeat. The wealth relative of π, at the beginning of the n^{th} period, is

$$S_1 = 1 \ , S_n = S_{n-1}w(b_{n-1}, z_{n-1})W(z_{n-1}) = \prod_{i=1}^{n-1} w(b_i, z_i)W(z_i)$$

which is invested in the portfolio b_n. We assume $b_1 = 1/2$.

Note that in this chapter we let S_n denote the wealth at the beginning of the n^{th} period, whereas in previous chapters S_n was the wealth at the end.

Proposition 6.1.1 (Computing the Net Realised Wealth). *Let $w := w(b_n, z_n)$ denote the net realised wealth from trading £1 of portfolio b_n for z_n. Suppose λ_1 and λ_2 are the transaction rates for assets 1 and 2. Then*

$$\lambda_1|wz_n - b_n| + \lambda_2|w(1 - z_n) - (1 - b_n)| = 1 - w. \tag{31}$$

Proof. w is the amount of money we have invested in portfolio z_n after trading £1 of b_n for z_n. $1 - w$ is the amount we lost in transaction fees. After the switch we have $£wz_n$ in asset 1 and $£w(1 - z_n)$ in asset 2. Thus we have a difference of $£|wz_n - b_n|$ and $£|w(1 - z_n) - (1 - b_n)|$ invested in assets 1 and 2 respectively. Regardless of which way the transactions went (sell some of asset 1 and buy asset 2 or vice versa), the wealth traded into or out of each asset is 'taxed' at a rate of λ_1 and λ_2 respectively. Thus

$$\lambda_1|wz_n - b_n| + \lambda_2|w(1 - z_n) - (1 - b_n)| = 1 - w$$

with both sides representing the wealth lost in transaction fees. □

Definition 6.1.3 (Interval Policies [5]). *Let $\mathcal{IP} = \{\alpha = [\alpha(1), \alpha(2)] : 0 \leq \alpha(1) \leq \alpha(2) \leq 1\}$. For $\alpha \in \mathcal{IP}$, a policy π^α is called a Interval Policy if it has the form*

$$\pi^\alpha(b_k) = \begin{cases} \alpha(1), & \text{if } b_k < \alpha(1) \\ b_k, & \text{if } b_k \in [\alpha(1), \alpha(2)] \\ \alpha(2), & \text{if } b_k > \alpha(2) \end{cases}$$

and only depends on the current portfolio b_k. At the beginning of the k^{th} period, the policy π^α switches from portfolio b_k to $z_k := \pi^\alpha(b_k)$.

An interval policy $\pi^\alpha, \alpha \in \mathcal{IP}$, is markovian and regulates the investors portfolio b_k to ensure it remains in the no trade interval $[\alpha(1), \alpha(2)]$. A detailed discussion of why we consider interval policies can be found on page 362 in [5]. In general they are asymptotically and arbitarly close to being growth optimal for *iid* markets, as considered in chapter 3. For an interval policy corresponding to $\alpha \in \mathcal{IP}$, let $(b_n^\alpha)_{n=1}$ and $(z_n^\alpha)_{n=1}$ denote the sequence of unregulated and regulated portfolios respectively. Additionally let S_n^α denote cumulative wealth obtained by the policy π^α.

Example 6.1.1. *Let $\alpha = [0.4, 0.6] \in \mathcal{IP}$, $b_1^\alpha = 1/2$. Consider the market $X = \{(1.1, 0.5), (1.05, 1.02)\} = \{x_1, x_2\}$, $\lambda_1 = 0.05$, $\lambda_2 = 0.1$ and suppose the investor follows the interval policy π^α. Then*

$$S_2^\alpha = W(0.5, 0.5)(0.5 \times 1.1 + 0.5 \times 0.5) = 0.8$$

with unregulated portfolio

$$b_2^\alpha = \frac{0.5 \times 1.1}{0.8} = 0.6875 \notin \alpha.$$

The policy π^α truncates b_2^α to $z_2^\alpha = 0.6$. Solving equation (31) gives a net realised wealth of $w\left(b_2^\alpha, z_2^\alpha\right) = 0.987$. Hence

$$S_3^\alpha = S_2^\alpha w\left(b_2^\alpha, z_2^\alpha\right) W\left(Z_2^\alpha\right) = 0.8 \times 0.987 \times (0.6 \times 1.05 + 0.4 \times 1.02) = 0.812.$$

The unregulated and regulated portfolios are $(b_1^\alpha, b_2^\alpha) = (0.5, 0.6875)$ and $(z_1^\alpha, z_2^\alpha) = (0.5, 0.6)$ respectively. $(S_1^\alpha, S_2^\alpha, S_3^\alpha) = (1, 0.8, 0.812)$ are the cumulative wealth's generated by π^α.

As is becoming a recurring theme in this paper, we now define a benchmark policy, chosen in hindsight, which we aim to match or beat using a causal approach.

Definition 6.1.4 (Universal Policies)**.** *Let $\alpha_n^* = arg\max_{\alpha \in \mathcal{IP}} S_n^\alpha(x_1^{n-1})$ be the optimal interval policy chosen in hindsight with corresponding wealth S_n^*. A policy $\pi \in \Pi$, with wealth S_n^π, is universal if for all regular market sequences x_1^∞,*

$$\liminf_{n \to \infty} \frac{1}{n} \log\left(\frac{S_n^\pi}{S_n^*}\right) = 0.$$

We aim to find such a policy. Note the similarity of universal policies to universal portfolios defined in 5.1.2.

6.2 The Uniform Weighted Universal Policy

Definition 6.2.1 (Uniform Weighted Universal Policy)**.** *Define*

$$F(d\alpha) = \frac{d\alpha}{\int_{\alpha \in \mathcal{IP}} d\alpha}$$

as the uniform measure on $\mathcal{IP}$ which places mass

$$f(\alpha) = \frac{1}{\int_{\alpha \in \mathcal{IP}} d\alpha}$$

on the interval policy α. The uniform weighted universal policy $\hat{\pi}$, referred to as the universal policy from now on, invests £$F(d\alpha)$ of the initial $S_1 =$ £1 in each interval policy π^α and manages this pool separately. Let $\hat{S}_n$ denote the wealth from this policy. Then

$$\hat{S}_n = \int_{\mathcal{IP}} S_n^\alpha F(d\alpha) = \int_{\mathcal{IP}} S_n^\alpha f(\alpha) d\alpha = \mathbb{E}_f(S_n^\alpha),$$

where the expectation is over the uniform distribution f on $\mathcal{IP}$.

As defined in Definition 6.1.1, to each policy π, exists portfolios b_n and z_n where b_n is the portfolio your wealth is in at the beginning of the n^{th} step and z_n the portfolio you wish to use for the n^{th} step. We now look at the portfolios generated by the universal policy.

Lemma 6.2.1 (Portfolios Generated by the Universal Policy)**.** *The sequence of portfolios $\hat{b}_n$ and $\hat{z}_n$ generated by the universal policy $\hat{\pi}$ are*

$$\hat{b}_n = \frac{\int_{\mathcal{IP}} S_n^\alpha b_n^\alpha d\alpha}{\int_{\mathcal{IP}} S_n^\alpha d\alpha} \tag{32}$$

and

$$\hat{z}_n = \frac{\int_{\mathcal{IP}} w(b_n^\alpha, z_n^\alpha) S_n^\alpha z_n^\alpha d\alpha}{\int_{\mathcal{IP}} w(b_n^\alpha, z_n^\alpha) S_n^\alpha d\alpha} = \mathbb{E}_g(z_n^\alpha) \tag{33}$$

where g is a distribution over $\mathcal{IP}$ with

$$\alpha \sim g(\alpha) = \frac{w(b_n^\alpha, z_n^\alpha) S_n^\alpha}{\int_{\mathcal{IP}} w(b_n^{\alpha'}, z_n^{\alpha'}) S_n^{\alpha'} d\alpha'}.$$

$\hat{z}_n$ is the portfolio sequence of interest.

Proof. Each policy π^α produces a wealth S_n^α invested in the unregulated portfolio b_n^α. We invest a fraction $F(d\alpha)$ of our initial wealth in this portfolio. Thus at the beginning of the n^{th} period we have £$S_n^\alpha F(d\alpha) b_n^\alpha$ invested in stock 1 in each alpha policy. Thus the fraction of the total wealth invested in stock 1, across all policies, is:

$$\hat{b}_n = \frac{\int_{\mathcal{IP}} S_n^\alpha b_n^\alpha F(d\alpha)}{\int_{\mathcal{IP}} S_n^\alpha F(d\alpha)} = \frac{\int_{\mathcal{IP}} S_n^\alpha b_n^\alpha d\alpha}{\int_{\mathcal{IP}} S_n^\alpha d\alpha}$$

This gives (32). Each alpha policy mandates a move from b_n^α to z_n^α. After this transition, there is £$w(b_n^\alpha, z_n^\alpha) S_n^\alpha F(d\alpha) z_n^\alpha$ invested in stock 1, in each policy, with a total wealth of $w(b_n^\alpha, z_n^\alpha) S_n^\alpha F(d\alpha)$ over both stocks. Hence the fraction of the total wealth, over all α policies, invested in stock 1 is:

$$\hat{z}_n = \frac{\int_{\mathcal{IP}} w(b_n^\alpha, z_n^\alpha) S_n^\alpha z_n^\alpha F(d\alpha)}{\int_{\mathcal{IP}} w(b_n^\alpha, z_n^\alpha) S_n^\alpha F(d\alpha)} = \frac{\int_{\mathcal{IP}} w(b_n^\alpha, z_n^\alpha) S_n^\alpha z_n^\alpha}{\int_{\mathcal{IP}} w(b_n^\alpha, z_n^\alpha) S_n^\alpha}$$

which gives (33). $\square$

The portfolio $\hat{z}_n$ is the complement to the uniform weighted universal portfolio defined in Chapter 5.

Remark 6.2.1 (Comparison to the Universal Portfolio)**.** *Recall from Remark 5.1.3 the form of the uniform universal portfolio:*

$$\hat{b}_n = \mathbb{E}_p(b)$$

where

$$p(b|x_1^{n-1}) = \frac{S_{n-1}(b, x_1^{n-1})}{\int_B S_{n-1}(b, x_1^{n-1}) db}.$$

By comparison $\hat{z}_n = \mathbb{E}_g(z_n^\alpha)$ where g is the counterpart to the distribution p with transaction costs. In both cases we take a performance weighted average ,over the past, of candidate portfolios. In Chapter 5 we invested fractionally over all portfolios $b \in B$, whereas in this chapter we invest fractionally over portfolios z_n^α generated by some interval policy $\alpha \in \mathcal{IP}$. Thus the universal portfolio invests in a broader range of portfolios.

We now prove the universality of the policy $\hat{\pi}$. The proof of this requires the notion of uniform equicontinuity which can be found in [page 234, [11]].

Definition 6.2.2 (Uniform Equicontinuity). *Let (X, d_X) and (Y, d_Y) be 2 metric spaces and $F = \{f_\beta : X \longrightarrow Y : \beta \in \Omega\}$ a family of functions from one space to the other. Then F is uniformly equicontinouse if*

$$\forall \epsilon > 0, \exists \delta > 0 : x, y \in X, d_X(x, y) < \delta \implies \sup_{\beta \in \Omega} d_Y(f_\beta(x), f_\beta(y)) < \epsilon.$$

In other words if 2 inputs x, y in the metric space (X, d_X) are 'close' under the metric d_X, then for any $f \in F$, $f(x)$ and $f(y)$ are close under the metric d_Y. We now consider the metric spaces $(\mathcal{IP}, d_X)$, $(\mathbb{R}^2, d_Y)$ where $d_X = d_Y = \|\cdot\|_\infty$ and $F = \{(b_n^{(\cdot)}, z_n^{(\cdot)})\}$ contains a single function $(b_n^{(\cdot)}, z_n^{(\cdot)}) : \mathcal{IP} \longrightarrow \mathbb{R}^2$. The following lemma establishes a form of uniform equicontinuity of the family F with respect to these metric spaces.

Lemma 6.2.2 (Uniform Equicontinuity in Interval Policy). *For a fixed $\alpha \in \mathcal{IP}$, $\alpha \subset (0,1)$, there exists a $\delta_0 > 0, m \geq 1$ such that $\forall \theta \in \mathcal{IP}$ with $\|\theta - \alpha\|_\infty \leq \delta \leq \delta_0$ and all regular market sequences x_1^∞, $n \geq 1$, we have*

$$\|(b_n^\alpha - b_n^\theta, z_n^\alpha - z_n^\theta)\|_\infty \leq m\delta.$$

Proof. The proof of this involves a lot of case by case analysis and can be found in [page 364 [5]]. In this proof we find an $m \geq m_j$ and define

$$\delta_0 = \min\left(\delta_1, \frac{(\alpha(2) - \alpha(1))}{2m}\right)$$

where $\delta_1 = \min(\alpha(1)/2, (1 - \alpha(2))/2)$. □

The following theorem presented by Iyengar in [5] will be used to prove the universality of our policy. The original proof had some minor issues which have been corrected.

Theorem 6.2.1 (Comparison of Asymptotic Growth Exponents). *Let $\alpha_n^* = [\alpha_n^*(1), \alpha_n^*(2)]$ be the optimal interval policy in hindsight and S_n^* the corresponding wealth. Suppose for large enough n, $\alpha_n^* \subset (0,1)$ and is non empty i.e.*

$$\liminf_{n \to \infty} (\alpha_n^*(2) - \alpha_n^*(1)) > 0.$$

Let $\hat{S}_n$ denote the universal policy wealth. Then for all regular stock sequences x_1^∞ we have

$$\liminf_{n \to \infty} \frac{1}{n} \log\left(\frac{\hat{S}_n}{S_n^*}\right) \geq 0. \tag{34}$$

Proof. It is a well known fact from analysis that for any sequence a_n $\exists$ a subsequence a_{n_j} which converges to the limit infimum of a_n (see [9]). Hence $\exists\{n_j : j \geq 1\}$ such that

$$\lim_{j \to \infty} \frac{1}{n_j} \log\left(\frac{\hat{S}_{n_j}}{S_{n_j}^*}\right) = \liminf_{n \to \infty} \frac{1}{n} \log\left(\frac{\hat{S}_n}{S_n^*}\right).$$

We assume α_n^* is eventually non empty and contained in $(0,1)$ thus eventually $\alpha_{n_j}^* \subset (0,1)$ and is non empty. Hence $\forall \epsilon > 0, \exists j_0$ such that $\forall j \geq j_0$,

1. $\alpha_{n_j}^*(1) > \epsilon$, $\alpha_{n_j}^*(2) < 1 - \epsilon$ ($\alpha_{n_j}^* \in (0,1)$)

2. $\alpha^*_{n_j}(2) - \alpha^*_{n_j}(1) > \epsilon$ ($\alpha^*_{n_j}$ is eventually non empty).

Applying lemma 6.2.2 to $\alpha^* := \alpha^*_{n_j}$, we deduce there exists $m_j \geq 1, \delta_{j,0}$ such that $\forall \theta = [\theta(1), \theta(2)] \in \mathcal{IP}$ with $\max_{i=1,2} |\theta(i) - \alpha^*(i)| \leq \delta \leq \delta_{j,0}$, we have

$$\max\{|b_n^\theta - b_n^{\alpha^*}|, |z_n^\theta - z_n^{\alpha^*}|\} \leq m_j \delta. \tag{35}$$

By Lemma 6.2.4 $\exists m, \delta_0 > 0$ such that $\delta_0 \leq \delta_{j,0}$ and $m_j \leq m$ for all $j \geq j_0$. We require δ_0 and m as these allow the upper bound in (35) to hold for all $j \geq j_0$. Fix such a $j \geq j_0$. Let $z_k^* := z_k^{\alpha^*_{n_j}}$, $b_k^* := b_k^{\alpha^*_{n_j}}$ and $S_k^* := S_k^{\alpha^*_{n_j}}$ denote the regulated, unregulated and wealth relatives for policy $\alpha^* := \alpha^*_{n_j}$. Fix $\delta \leq \delta_0$ and let

$$\mathcal{I}_\delta = \{\theta \in \mathcal{IP} : \max_{i=1,2} |\theta(i) - \alpha^*(i)| \leq \delta\} \subset \mathcal{IP}$$

be the set of interval policies for which the conditions of Lemma 6.2.2 hold with upper bound δ. Let $\theta \in \mathcal{I}_\delta$ be an arbitrary interval policy. Since $S_1^\theta = 1$,

$$S_{n_j}^\theta = \prod_{k=1}^{n_j} w(b_k^\theta, z_k^\theta) W(z_k^\theta) \geq \prod_{k=1}^{n_j} w(b_k^\theta, b_k^*) w(b_k^*, z_k^*) w(z_k^*, z_k^\theta) W(z_k^\theta) \tag{36}$$

$$= \prod_{k=1}^{n_j} w(b_k^*, z_k^*) W(z_k^*) \prod_{k=1}^{n_j} w(b_k^\theta, b_k^*) w(z_k^*, z_k^\theta) \prod_{k=1}^{n_j} \frac{W(z_k^\theta)}{W(z_k^*)} \tag{37}$$

$$= S_{n_j}^* \prod_{k=1}^{n_j} w(b_k^\theta, b_k^*) w(z_k^*, z_k^\theta) \prod_{k=1}^{n_j} \frac{W(z_k^\theta)}{W(z_k^*)} \tag{38}$$

with the inequality following from the fact that it is suboptimal to transition from b_k^θ to z_k^θ via b_k^* and z_k^*. We now bound the term

$$\prod_{k=1}^{n_j} \frac{W(z_k^\theta)}{W(z_k^*)}.$$

Consider the following 2 cases

- $0 \leq z_k^\theta \leq z_k^*$
- $z_k^* \leq z_k^\theta \leq 1$

If $0 \leq z_k^\theta \leq z_k^*$ then the θ policy invests less in stock 1 than the policy α^*. Hence $\exists \mu \in [0,1]$ such that $z_k^\theta = \mu z_k^*$. Note that $\theta \in \mathcal{I}_\delta$ so $\max\{|b_k^\theta - b_k^*|, |z_k^\theta - z_k^*|\} \leq m\delta$ where $\delta \leq \delta_0 \leq \delta_{j,0}$ and is independent of j, maintaining the generality of the argument. Since $j \geq j_0$ we also have $\alpha^*(1) \geq \epsilon$. Hence

$$\mu = \frac{z_k^\theta}{z_k^*} \geq \frac{z_k^* - m\delta}{z_k^*} = \left(1 - \frac{m\delta}{z_k^*}\right) \geq \left(1 - \frac{m\delta}{\alpha^*(1)}\right) \geq \left(1 - \frac{m\delta}{\epsilon}\right). \tag{39}$$

Hence

$$\begin{aligned} W(z_k^\theta) &= \mu z_k^* x_k(1) + (1 - \mu z_k^*) x_k(2) = \mu(z_k^* x_k(1) + (1 - z_k^*) x_k(2)) + (1-\mu) x_k(2) = \mu W(z_k^*) + (1-\mu) W(0) \\ &\geq \mu W(z_k^*) \geq \left(1 - \frac{m\delta}{\epsilon}\right) W(z_k^*) \end{aligned}$$

A similar proof shows that for $z_k^* \leq z_k^\theta \leq 1$:

$$W(z_k^\theta) \geq \left(1 - \frac{m\delta}{\epsilon}\right) W(z_k^*).$$

Combining this with (38), we deduce

$$\frac{S_{n_j}^\theta}{S_{n_j}^*} \geq \left(1 - \frac{m\delta}{\epsilon}\right)^{n_j} \prod_{k=1}^{n_j} w(b_k^\theta, b_k^*) w(z_k^*, z_k^\theta). \tag{40}$$

Now impose the constraint on δ that $\delta \to 0$ as $j \to \infty$. By lemma 6.2.5 we deduce that for portfolios u, v with $|u - v| \leq m\delta$, $\exists j_1$ such that $\forall j \geq j_1$,

$$w(u, v) \geq 1 - \frac{2m\delta\lambda_{max}}{1 - \lambda_{max}}$$

where $\lambda_{max} = \max(\lambda_1, \lambda_2)$. Since $|b_k^\theta - b_k^*| \leq m\delta$ and $|z_k^\theta - z_k^*| \leq m\delta$ we deduce

$$\frac{S_{n_j}^\theta}{S_{n_j}^*} \geq \left(1 - \frac{m\delta}{\epsilon}\right)^{n_j} \left(1 - \frac{2m\delta\lambda_{max}}{1 - \lambda_{max}}\right)^{2n_j} \geq (1 - c\delta)^{3n_j} \tag{41}$$

where $c := \max(m/\epsilon, 2m\lambda_{max}/(1 - \lambda_{max}))$. Recall from Definition 6.2.1,

$$\hat{S}_{n_j} = \int_{\mathcal{IP}} S_{n_j}^\alpha f(\alpha) d\alpha = \frac{\int_{\mathcal{IP}} S_{n_j}^\alpha d\alpha}{\int_{\mathcal{IP}} d\alpha} \geq \frac{\int_{\mathcal{I}_\delta} S_{n_j}^\alpha d\alpha}{\int_{\mathcal{IP}} d\alpha} = \frac{S_{n_j}^* \int_{\mathcal{I}_\delta} \frac{S_{n_j}^\alpha}{S_{n_j}^*} d\alpha}{\int_{\mathcal{IP}} d\alpha}. \tag{42}$$

In the numerator we integrate only over $\alpha \in \mathcal{I}_\delta$. Hence applying (41) we deduce

$$(42) \geq S_{n_j}^* (1 - c\delta)^{3n_j} \frac{\int_{\mathcal{I}_\delta} d\alpha}{\int_{\mathcal{IP}} d\alpha}. \tag{43}$$

We have

$$\int_{\mathcal{IP}} d\alpha = \int_0^1 \int_{\alpha(1)}^1 d\alpha(2) d\alpha(1) = 1/2.$$

For any $\alpha \in \mathcal{I}_\delta$, note that

1. $a := \max(0, \alpha^*(1) - \delta) \leq \alpha(1) \leq \min(1, \alpha^*(1) + \delta) := b$. This follows from both the constraints imposed by membership of $\mathcal{I}_\delta$ as well as $0 \leq \alpha(1) \leq 1$.

2. $c := \max(\alpha(1), \alpha^*(2) - \delta) \leq \alpha(2) \leq \min(1, \alpha^*(2) + \delta) := d$, again following from constraints imposed by membership of $\mathcal{I}_\delta$ as well as the fact $\alpha(1) \leq \alpha(2) \leq 1$.

Thus

$$\int_{\mathcal{I}_\delta} d\alpha = \int_a^b \int_c^d d\alpha(2) d\alpha(1) \geq \frac{k}{2}\delta^2$$

for some $k \geq 0$. This can be seen by working through all possible combinations of the maxima and minima. Thus

$$\hat{S}_{n_j} \geq S_{n_j}^* (1 - c\delta)^{3n_j} k\delta^2. \tag{44}$$

Since $\delta \leq \delta_0$ was chosen arbitrarily such that $\delta \to 0$ as $j \to \infty$, take $\delta := \delta_j = \frac{1}{n_j+1}$ and let $j_2 = \min(j : \delta_j \leq \delta_0)$. Then $\forall j \geq \max(j_0, j_1, j_2)$, $\delta = \delta_j \leq \delta_0$ and (44) holds. Thus

$$\frac{\hat{S}_{n_j}}{S^*_{n_j}} \geq \left(1 - \frac{c}{n_j + 1}\right)^{3n_j} k \left(\frac{1}{n_j + 1}\right)^2.$$

Therefore for any regular market sequence x_1^∞,

$$\liminf_{n\to\infty} \frac{1}{n} \log\left(\frac{\hat{S}_n}{S^*_n}\right) = \lim_{j\to\infty} \frac{1}{n_j} \log\left(\frac{\hat{S}_{n_j}}{S^*_{n_j}}\right) \geq \lim_{j\to\infty} 3\log\left(1 - \frac{c}{n_j+1}\right) + \frac{\log(k)}{n_j} - \frac{2\log(n_j+1)}{n_j} = 0.$$

□

The theorem itself falls short of proving the universality of our portfolio, as defined in Definition 6.1.4. The following original result completes this proof.

Lemma 6.2.3 (Steward - Universality Of Uniform Weighted Portfolio)**.** *For any regular portfolio sequence* x_1^∞*,*

$$\lim_{n\to\infty} \frac{1}{n} \log\left(\frac{\hat{S}_n}{S^*_n}\right) = 0.$$

Proof. From Definition 6.2.1, $\hat{S}_n = \mathbb{E}_f(S_n^\alpha)$ where $\alpha \sim f$ and f is the uniform distribution over $\mathcal{IP}$. Hence, for any $n \geq 1$ and any regular sequence x_1^n,

$$\frac{1}{n}\log\left(\frac{\hat{S}_n}{S^*_n}\right) = \frac{1}{n}\log\left(\frac{\mathbb{E}_f(S_n^\alpha)}{S_n^{\alpha_n^*}}\right) \leq \frac{1}{n}\log\left(\frac{\mathbb{E}_f(S_n^{\alpha_n^*})}{S_n^{\alpha_n^*}}\right) = 0.$$

Hence the ratio is less than or equal to 0 at the end point n. Thus by theorem 5.2.1,

$$0 \leq \liminf_{n\to\infty} \frac{1}{n}\log\left(\frac{\hat{S}_n}{S^*_n}\right) \leq \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{\hat{S}_n}{S^*_n}\right) \leq 0$$

which concludes the result. □

This proves the universality of the universal policy. In particular we actually showed a stronger result, proving that in the limit, the universal policy has the same growth exponent as the hindsight policy (as oppose to only in the limit infimum). Neither the convergence to 0 in the limit, nor in the limit infimum, has been shown before. We now prove the two results used in the proof of Theorem 6.2.1.

Lemma 6.2.4 (Existence of δ_0)**.** *Consider the setup in the proof of Theorem 6.2.1. There exists* $m, \delta_0 > 0$ *such that* $\delta_0 \leq \delta_{j,0}$ *and* $m_j \leq m$ *for all* $j \geq j_0$*.*

Proof. The existence of $m \geq m_j$ for all $j \geq j_0$ can be found in [5]. Recall from Lemma 6.2.2 that

$$\delta_{j,0} = \min\left(\delta_1, \frac{\alpha^*_{n_j}(2) - \alpha^*_{n_j}(1)}{2m_j}\right)$$

where $\delta_1 = \min(\alpha^*_{n_j}(1)/2, (1-\alpha^*_{n_j}(2))/2)$. For all $j \geq j_0$ we have $\delta_1 \geq \epsilon/2$. Since $m_j \geq 1$,

$$\delta_{j,0} \geq \min\left(\frac{\epsilon}{2}, \frac{\epsilon}{2m_j}\right) = \frac{\epsilon}{2m_j} \geq \frac{\epsilon}{2m} := \delta_0.$$

□

Lemma 6.2.5 (Bound on Net Realised Wealth)**.** *Suppose $u, v \in [0,1]$ are two portfolios such that $|u-v| \leq m\delta$. Further suppose that $\delta \to 0$ as $j \to \infty$. Then $\exists j_1 \geq 1$ such that for all $j \geq j_1$,*

$$w(u,v) \geq 1 - \frac{2m\delta\lambda_{max}}{1-\lambda_{max}}.$$

where $\lambda_{max} := \max(\lambda_1, \lambda_2)$.

Proof. Wlog assume that $v \geq u$, or else switch the asset labels. Recall the net realised wealth $w(u,v)$ is the remaining wealth from trading £1 of portfolio u for portfolio v. As $j \to \infty$, $\delta \to 0$ and therefore $u \to v$ and $w(u,v) \to 1$. Hence by continuity $\exists j_1$ such that $\forall j \geq j_1$, $w(u,v)v \geq u$. Thus we have more money in asset 1 after the switch than before. Let ζ be the amount of stock 2 that we sell to fund the increase in stock 1. After transaction fees, the sale of £ζ of asset 2 nets

$$£\frac{1-\lambda_2}{1+\lambda_1}\zeta$$

of asset 1. Thus

$$\frac{1-\lambda_2}{1+\lambda_1}\zeta = w(u,v)v - u.$$

Of the initial £1, only £ζ of this is traded and incurs transaction fees. Hence $\forall j \geq j_1$

$$w(u,v) = (1-\zeta) + \zeta\frac{1-\lambda_2}{1+\lambda_1} = 1 - \frac{\lambda_2+\lambda_1}{1+\lambda_1}\zeta = 1 - \frac{\lambda_2+\lambda_1}{1-\lambda_2}\frac{1-\lambda_2}{1+\lambda_1}\zeta = 1 - \frac{\lambda_2+\lambda_1}{1-\lambda_2}(w(u,v)v-u) \quad (45)$$

$$\geq 1 - \frac{\lambda_2+\lambda_1}{1-\lambda_2}|v-u| \geq 1 - \frac{2\lambda_{max}}{1-\lambda_{max}}|v-u| \geq 1 - \frac{2m\delta\lambda_{max}}{1-\lambda_{max}}. \quad (46)$$

□

These lemmas complete the proof of Theorem 6.2.1. Lemma 6.2.5 is a more rigorous argument than the one presented by Iyengar, who neglects to mention this bound only holds for sufficiently large j.

6.3 Universal Interval Policy Demonstration

The universal policy $\hat{\pi}$ can not be used in practice as it involves placing £$F(d\alpha)$ in an infinite number of policies $\pi^\alpha, \alpha \in \mathcal{IP}$. As noted in Definition 6.2.1, the wealth generated by $\hat{\pi}$ can be written as $\hat{S}_n = \mathbb{E}_f(S^\alpha_n)$ where f is the uniform distribution over $\mathcal{IP}$. We can approximate the wealth $\hat{S}_n$ by taking a large sample $\alpha_1, ..., \alpha_m \overset{iid}{\sim} f(\cdot)$ and noting

$$\hat{S}_n \approx \tilde{S}^{(m)}_n := \frac{1}{m}\sum_{i=1}^{m} S^{\alpha_i}_n$$

where $\tilde{S}^{(m)}_n$ is generated by investing £$1/m$ in each sample policy. Thus investing ones initial £1 wealth uniformly over a sufficiently large number of policies $\alpha_1, ..., \alpha_m \overset{iid}{\sim} f$, produces the same asymptotic results.

Implementing this in practice involved a lot of optimisation. We set the transaction rates at $\lambda_1 = \lambda_2 = 0.01$ and generated $m = 300$ random uniform intervals $\alpha_1, ..., \alpha_{300} \overset{iid}{\sim} f$ which were used to approximate $\hat{S}_n$. For each interval α_j, computing $S_n^{\alpha_j}$ involved solving equation (31) for the net realised wealth $w(b_i^{\alpha_j}, z_i^{\alpha_j})$ at every time point $i = 1, ..., n$. Computation of $S_n^* = \max_{\alpha \in \mathcal{IP}} S_n^{\alpha}$ was done using the **constrOptim** function, imposing the constraint that $0 \leq \alpha(1) < \alpha(2) \leq 1$. We ran this on 5 random start points and returned the maximum. This was computationally expensive but ensured accuracy.

6.3.1 Random Stock Data

Analogous to the demonstration in 5.3.1, we generated $n = 750$ stock data points as follows: $X_i = (X_{i1}, X_{i2}) = (D_{i1} + 0.52, D_{i2} + 0.53)$ where $(D_{i1}, D_{i2}) \overset{iid}{\sim} Dir(1, 1)$. Computation yields an optimal interval policy in hindsight of $\alpha_n^* = [0.363, 0.631]$ with corresponding wealth $S_n^* = £1.77 \times 10^8$. We estimate the wealth $\hat{S}_n$ of the universal policy as described above. Figure 6 shows the results. The top left figure shows the comparison between the wealth of the universal policy compared to investment solely in stocks 1 and 2, and the best hindsight policy. Again because the wealth generated from stocks 1 and 2 is negligible in comparison to the other two strategies, we also plot log profit vs time and log profit vs log time. These are shown in the top right and bottom left figures respectively. The universal policy increases the investor's wealth by a factor of 1.21×10^7, considerably better than investment solely in either stock! As expected from universality, the universal wealth appears to grow with a similar exponent to the best hindsight policies wealth (up to logarithmic factors). This is clear from the two log scale plots. Mirroring 5.3.1, the bottom right figure shows the universality property from Lemma 6.2.3. For $i = 1, ..., n$, this figure plots

$$\frac{1}{i} \log \left(\frac{\hat{S}_i}{S_i^*} \right). \tag{47}$$

As predicted from the lemma, this quantity appears to converge to 0. This implies S_n^* and $\hat{S}_n$ have the same asymptotic growth exponent, verifying universality. It is worth noting that the convergence is extremely slow and can only be consistently seen for $n \geq 2000$. Increasing the transaction rates to $\lambda_1 = 0.1$ and $\lambda_2 = 0.2$ widens the optimal interval policy to $\alpha_n^* = [0.275, 0.713]$. Thus the optimal policy only regulates more extreme portfolios, as expected from higher transaction costs.

6.3.2 Real Stock Data

Similar to 5.3.2 we run the algorithm on Google and Apple stock from 2023 to 2024 with transaction rates of $\lambda_1 = \lambda_2 = 0.01$. The results are shown in Figure 7. The best hindsight policy was $\alpha_n^* = [0.999, 1]$. Thus the universal policy regulates our portfolio to maintain effectively 100% of our wealth in Google. The right hand figure demonstrates the universality property.

7 Conclusion

Depending on the assumptions we are prepared to make, we have seen a variety of different investment strategies an investor could use to yield optimal performance. In chapter 3 we made the strongest assumptions of a known stock distribution, arguing log optimal investment was the correct approach. In subsequent chapters we weakened these assumptions until arriving at the universal policy, the most realistic investment scenario. Under each set of assumptions we defined an oracle strategy, chosen in hindsight, and then constructed a causal investment strategy which asymptotically matched or outperformed this! We have therefore provided a comprehensive response to the question posed at the beginning of this paper under a range of different market assumptions. We also extended the existing literature from chapters 5 and 6, proving lemmas 5.2.5 and 6.2.3 on the convergence of the difference in growth exponents to 0. We then verified and demonstrated the theory via practical implementation of the algorithms. In addition this paper made rigorous a number of existing proofs, most notably the KKT conditions for the log optimum portfolio (theorem 3.1.1) and theorem 6.2.1. Future work on universal investment was suggested in 5.3.2 after the analysis of the performance of the universal portfolio on real stock data. Another area of research would be adapting the universal investment strategies seen in this paper to allow investors to take short positions in stocks (negative positions), moving the theory one step closer to a realistic market model. In this instance the set of valid portfolios would become $B = \{b \in [-1,1]^m : \sum_{i=1}^m b_i = 1\}$. This is no longer a probability simplex and thus many of the proofs presented in this paper will not extend trivially.

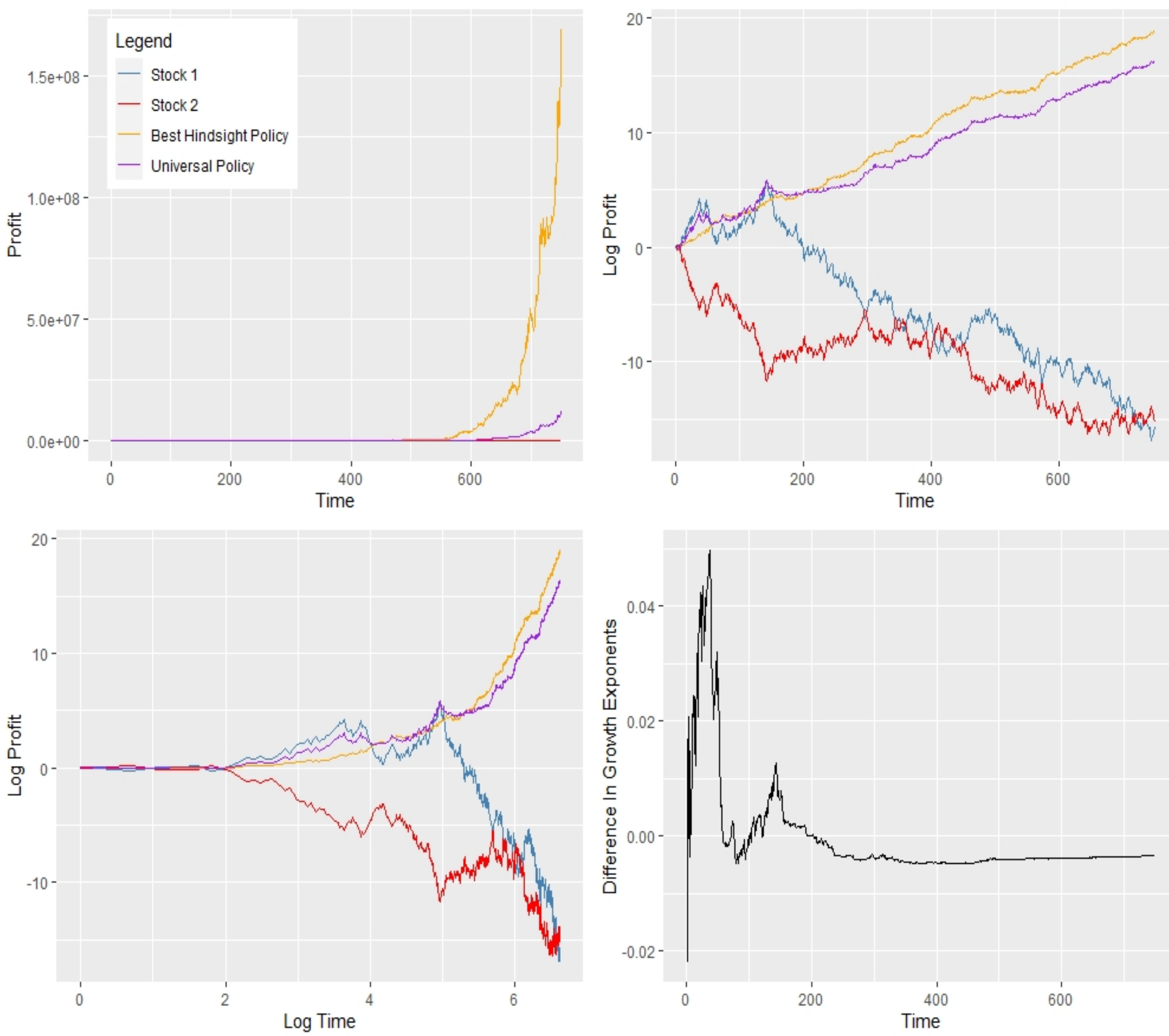


Figure 6: Universal Policy performance on random data. Top left and right figures show profit and log profit vs time respectively. The bottom left figure shows log profit vs log time and the bottom right figure shows the difference in growth exponent between the universal policy and the best hindsight policy.

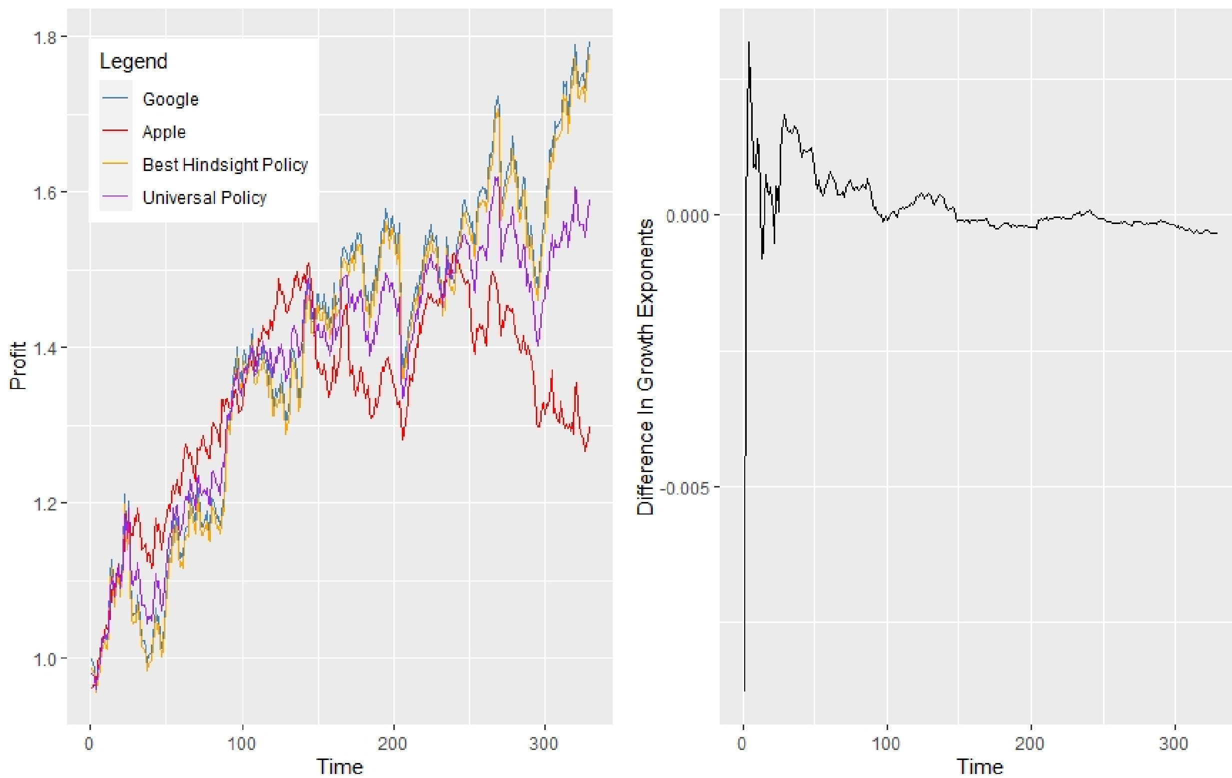


Figure 7: Universal Policy performance on real stock data

References

[1] Thomas M Cover and David H Gluss. "Empirical Bayes stock market portfolios". In: *Advances in Applied Mathematics* 7.2 (1986), pp. 170–181. ISSN: 0196-8858. DOI: https://doi.org/10.1016/0196-8858(86)90029-1. URL: https://www.sciencedirect.com/science/article/pii/0196885886900291.

[2] Robert Bell and Thomas M. Cover. "Game-Theoretic Optimal Portfolios". In: *Management Science* 34.6 (1988), pp. 724–733. ISSN: 00251909, 15265501. URL: http://www.jstor.org/stable/2632126 (visited on 02/23/2024).

[3] T.M. Cover and E. Ordentlich. "Universal portfolios with side information". In: *IEEE Transactions on Information Theory* 42.2 (1996), pp. 348–363. DOI: 10.1109/18.485708.

[4] Oliver Knill. *Directional derivative Math21a*. Harvard, 2004. URL: https://people.math.harvard.edu/~knill/teaching/summer2004/handouts/directional.pdf.

[5] Garud Iyengar. "Universal Investment In Markets With Transaction Costs". In: *Mathematical Finance* 15.2 (Apr. 2005), pp. 359–371. DOI: 10.1111/j.0960-1627.2005.. URL: https://ideas.repec.org/a/bla/mathfi/v15y2005i2p359-371.html.

[6] Thomas M. Cover and Joy A. Thomas. *Elements of Information Theory (Wiley Series in Telecommunications and Signal Processing)*. USA: Wiley-Interscience, 2006. ISBN: 0471241954.

[7] Carlos Fernandez-Granda. *Lecture notes 7: Convex optimization*. NYU, 2017. URL: https://cims.nyu.edu/~cfgranda/pages/OBDA_fall17/notes/convex_optimization.pdf.

[8] GabrieleFarina. "Blackwell approachability and regret minimizationon simplex domains". In: (2021). URL: chrome-extension://efaidnbmnnnibpcajpcglclefindmkaj/https://www.mit.edu/~gfarina/2021/15888f21_L04_blackwell_rm/L04_blackwell_rm.pdf.

[9] Lafferriere Nguyen. *3.6: Limit superior and limit inferior of functions*. Sept. 2021. URL: https://math.libretexts.org/Bookshelves/Analysis/Introduction_to_Mathematical_Analysis_I_(Lafferriere_Lafferriere_and_Nguyen)/03:_Limits_and_Continuity/3.06:_Limit_Superior_and_Limit_Inferior_of_Functions.

[10] Sudeep Raja. *Online Portfolio Selection - Cover's Universal Portfolio*. Nov. 2022.

[11] Marek Biskup. *Equicontinuity*. 2023. URL: https://www.math.ucla.edu/~biskup/131bh.1.23s/.