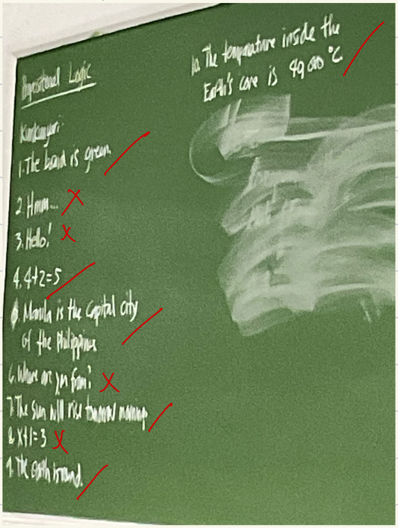
Logic

→ has many ty pes

① <u>Propositional logic</u>

* Proposition → <u>declarative</u> sentence that is either true or false, but not both
↳ relays a general idea

ex. Proposition or not



<u>Propositional Variables</u>

∴ p, q, r, s, t

ex. p = 1+1 = 2
q = It is rainy now.
r = Today is May 7.
s = Today is monday.

simple/atomic proposition

* can combine propositions using logical operators / connectives

Logical Operators (Connectives)

A. Negation

negation operator

target

Written: $\sim p$ ($\neg p$)

Read: "not p"

O.K.: "It is not the case that p."

"It is not true that p."

Kunkunyari: ETFSISF express the following sentences in symbolic form

1. $1+1 \neq 2$. $\sim p$
2. It is not true that it is raining now. $\sim q$ ($\neg q$)
3. It is not the case that today is not May 7. $\sim(\sim r)$

Truth Table

↳ summarizes the truth value of a compound proposition for all possible combinations of truth values of the component propositions

not symbol/atomic proposition

ex.

p	$\sim p$
T	F
F	T

B. Conjunction

Written: $p \wedge q$

Read: "p and q"

O.K.: "p but q"

Kunkunwari: ETFSISF

1. It is raining now and $1+1=2$. $q \wedge p$
2. $1+1 \neq 2$ but today is not May 7. $\sim p \wedge \sim r$
3. It is not true that $1+1=2$, and it is raining now. $\sim p \wedge q$
4. It is not true that $1+1=2$ and it is raining now. $\sim(p \wedge q)$
5. Today is not May 7, but it is not the case that $1+1 \neq 2$ and it is raining now. $\sim r \wedge \sim(\sim p) \wedge q$

$\sim r \wedge \sim(\sim p \wedge q)$

Paradox

ex. This sentence is false.

T - F

F - T

Truth Table for Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

* Conjunction is true when both conjuncts are true.

C. Disjunction

Written: $p \vee q$

Read: "p or q"

OK.: "either p or q"

Kunkunwari: ETFSISF

1. $1+1 \neq 2$ or it is raining now. $\sim p \vee q$ — $\sim T \vee F$ / $F \vee F$ / $\therefore$ False
2. Either it is not raining now or today is not May 7. $\sim q \vee \sim r$ — $\sim F \vee \sim F$ / $T \vee T$ / $\therefore$ TRUE
3. It is not the case that either $1+1 \neq 2$ and it is raining now or $1+1=2$. $\sim[(\sim p \wedge q) \vee p]$ — $\sim[(\sim T \wedge F) \vee T]$ / $\sim[(F \wedge F) \vee T$ / $\sim[F \vee T]$ / $\sim T$ / F / $\therefore$ False
4. Either it is not true that $1+1=2$ and today is May 7, or it is raining now. $\sim(p \wedge r) \vee q$
5. It is raining now or $1+1 \neq 2$, and either today is not May 7 or it is not raining now. $(q \vee \sim p) \wedge (\sim r \vee \sim q)$
6. $1+1=2$ or it is raining now and today is May 7. $p \vee q \wedge r$

Truth Table for Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

* Disjunction is true when at least one of the disjuncts are true

D. Implication (Conditional)

Written: $p \rightarrow q$

- p → antecedent / hypothesis / premise
- q → consequence / consequent / conclusion
- $\rightarrow$ → implication operator

Read: "if p, then q"

O.K.: "p implies q" — "p only if q"

"if p, q" — "q when p"

"q if p" — "q whenever p"

"p is sufficient for q" — "q follows from p"

"q is necessary for p" — "q unless $\sim p$"

Additional:

Definition: Converse, Contrapositive, Inverse

Given a conditional statement $p \rightarrow q$,

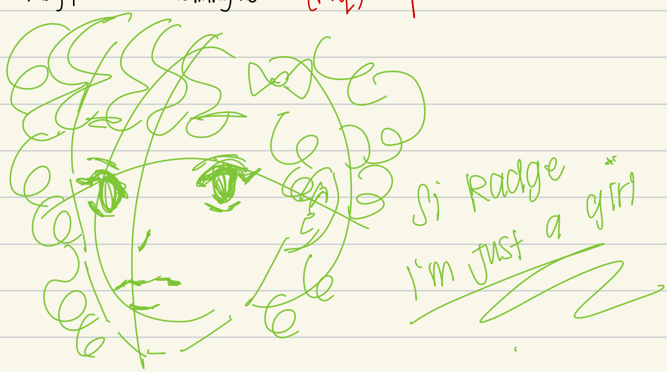
- its converse is $q \rightarrow p$
- its contrapositive is $\sim q \rightarrow \sim p$; and
- its inverse is $\sim p \rightarrow \sim q$

Kunkunwari: ETFSISF

1. If $1+1=2$, then it is raining now. $p \rightarrow q$
2. Today being May 7 is a necessary condition for $1+1$ to not be equal to 2. $\sim p \rightarrow r$
3. Today is not May 7 only if $1+1 \neq 2$. $\sim r \rightarrow \sim p$
4. It is not true that if it is raining now, then today is May 7 and $1+1=2$. $\sim[q \rightarrow (r \wedge p)]$
5. If it is not true that $1+1=2$, then today is May 7 or it is raining now. $\sim p \rightarrow (r \vee q)$
6. $1+1=2$ whenever it is not true that either today is May 7 or it is raining now. $\sim(r \vee q) \rightarrow p$

Truth Table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T



Consider

p: You get a perfect score in the exam

q: You will pass

"If you get perfect on the exam, then you will pass" $p \rightarrow q$

"You will pass if you get a perfect score in the exam" $p \rightarrow q$

"You will pass only if you get a perfect in the exam." $q \rightarrow p$

E. Bi-implication / Bi-Conditional

Form: $p \leftrightarrow q$

Read: "p if and only if q"

O.K.: "p iff q"

"p is necessary and sufficient condition for q"

"p implies q, and conversely."

Kunkunwari: ETFSISF

1. 1+1=2 and it is not raining now if and only if 1+1≠2. $p \wedge \sim q \leftrightarrow \sim p$
2. It is not the case that it is not raining now or today is May 7, if and only if 1+1=2. $\sim(\sim q \vee r) \leftrightarrow p$
3. Either today is not May 7 iff it is raining now, or it is not true that 1+1=2 and today is May 7. $(\sim r \leftrightarrow q) \vee \sim(p \wedge r)$
4. If it is not the case that 1+1=2 iff today is May 7 and it is raining now, then 1+1≠2. $\sim(p \leftrightarrow r \wedge q) \rightarrow \sim p$
5. 1+1=2 or it is raining now, iff today is May 7. $(p \vee q) \leftrightarrow r$

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Dagdag Kaalaman: Exclusive Disjunction

Form: $p \oplus q$

Read: "p or q, but not both"

O.K.: "p XOR q"

Truth Table

p	q	$p \oplus q$	$p \vee q$
T	T	F	T
T	F	T	T
F	T	T	T
F	F	F	F

Precedence Rules

A. Different Operators

Consider: $p \rightarrow \sim q \leftrightarrow r \vee s \wedge t$

① Negation

$p \rightarrow (\sim q) \leftrightarrow r \vee s \wedge t$

② Conjunction

$p \rightarrow (\sim q) \leftrightarrow r \vee (s \wedge t)$

③ Disjunction

$p \rightarrow (\sim q) \leftrightarrow [r \vee (s \wedge t)]$

④ Implication

$[p \rightarrow (\sim q)] \leftrightarrow [r \vee (s \wedge t)]$

⑤ Bi-implication

$[p \rightarrow (\sim q)] \leftrightarrow [r \vee (s \wedge t)]$

B. Same Operators

1. for $\wedge$, $\vee$, and $\leftrightarrow$: evaluate left to right
   a. $p \wedge q \wedge r \vee s \wedge t$ $\rightarrow [(p \wedge q) \wedge r] \vee (s \wedge t)$
   b. $p \vee q \wedge r \vee s \vee t$ $\rightarrow \{[p \vee (q \wedge r)] \vee s\} \vee t$
   c. $p \leftrightarrow q \leftrightarrow r$ $\rightarrow (p \leftrightarrow q) \leftrightarrow r$
2. for $\rightarrow$: evaluate right to left

   $p \rightarrow q \rightarrow r \rightarrow s$

   $p \rightarrow [q \rightarrow (r \rightarrow s)]$

1.4 - Evaluating Compound Propositions

Evaluate the following proposition if p, q, r, and s are true:

$p \vee \sim(r \to \sim s) \leftrightarrow (q \oplus \sim r) \wedge \sim p$

$T \vee \sim(T \to \sim T) \leftrightarrow (T \oplus \sim T) \wedge \sim T$

$T \vee \sim(T \to F) \leftrightarrow (T \oplus F) \wedge F$

$(T \vee \sim F) \leftrightarrow (T \wedge F)$

$T \vee T \leftrightarrow F$

$T \leftrightarrow F$

FALSE

What not to do

a. $T \to F \vee T$

TRUE

$(T \to F) \vee T$

$F \vee T$

True

$T \to (F \vee T)$

$T \to T$

True

EXAMPLE

1. $r \vee s \wedge \sim p$

$F \vee F \wedge \sim T$

$F \vee F \wedge F$

$F \vee F$

False

2. $p \to s \vee q \wedge r$

$T \to F \vee F \wedge T$

$T \to F \vee F$

$T \to F$

FALSE

3. $q \vee r \wedge \neg p \leftrightarrow s \to \neg r$

$F \vee T \wedge \neg F \leftrightarrow T \to \neg T$

$F \vee T \wedge T \leftrightarrow T \to F$

$F \vee T \leftrightarrow T \to F$

$T \leftrightarrow T \to F$

$T \leftrightarrow F$

FALSE

Truth tables of Compound Propositions

① $\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$

Number of rows: 2^n, where n is the total number of propositional variables

$\therefore$ p, q ; $n=2$; $2^2=4$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

2. $p \rightarrow q \wedge r \leftrightarrow p \wedge (\neg q \vee \neg r)$

$2^3 = 8$

* All false = contradiction (Absurdity)
* All true = Tautology (Valid statement / Proposition)

p	q	r	$\neg q$	$\neg r$	$\neg q \vee \neg r$	$p \wedge (\neg q \cup \neg r)$	$q \wedge r$	$p \rightarrow q \wedge r$	$p \rightarrow q \wedge r \leftrightarrow p \wedge (\neg q \vee \neg r)$
T	T	T	F	F	F	F	T	T	F
T	T	F	F	T	T	T	F	F	F
T	F	T	T	F	T	T	F	F	F
T	F	F	T	T	T	T	F	F	F
F	T	T	F	F	F	F	T	T	F
F	T	F	F	T	T	F	F	T	F
F	F	T	T	F	T	F	F	T	F
F	F	F	F	T	T	F	F	T	F

③ $p \vee \neg q \to p \wedge q$

$2^n = 2^2 = 4$

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$p \vee \neg q \to p \wedge q$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

* Proposition that is not a tautology nor a contradiction is called a contingency (satisfiable)

1.6 - Logical Equivalence Laws

$\left[\begin{array}{ll} x^2+y^2=1 & \sin^2\theta+\cos^2\theta=1 \\ (2+5)=(8-1) & \end{array} \right]$ ↗ Quantitative

Equivalent / Logically Equivalent

T / F

Definition: Logical Equivalence

The compound propositions p and q are said to be logically equivalent, written $p \equiv q$, iff $p \leftrightarrow q$ is a tautology.

$p \equiv q \leftrightarrow$ "$p \leftrightarrow q$ is a tautology"

$p \leftrightarrow q$ is a <u>tautology</u> - T

p	q
T	T
F	F

Kunkunwari: Determine whether the two compound proposition are logically equivalent.

① $\sim(p \wedge q)$ and $\sim p \wedge \sim q$

$\sim(p \wedge q) \leftrightarrow (\sim p \wedge \sim q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$\sim(p \wedge q) \leftrightarrow (\sim p \wedge \sim q)$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	F	F
F	T	F	T	T	F	F	F
F	F	F	T	T	T	T	T

$\sim(p \wedge q) \not\equiv \sim p \wedge \sim q$

② $\sim(p \wedge q)$ and $\sim p \vee \sim q$

$\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(p \wedge q)$ and $\sim p \vee \sim q$
T	T	T	F	F	F	F	T
T	F	F	T	F	T	T	T
F	T	F	T	T	F	T	T
F	F	F	T	T	T	T	T

$\sim(p \wedge q) \equiv \sim p \vee \sim q$

③ $p \to q$
- inverse
- converse
- contrapositive

p	q	$p \to q$	converse $q \to p$	$\sim p$	$\sim q$	inverse $\sim p \to \sim q$	contrapositive $\sim q \to \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

$p \to q \not\equiv q \to p$

$p \to q \not\equiv \sim p \to \sim q$

$p \to q \equiv \sim q \to \sim p$

LOGICAL EQUIVALENCE LAWS

① Reflexivity (Reflexive Law)

$$p \equiv p$$

② Double Negation

$$\neg(\neg p) \equiv p$$

③ Commutativity (Commutative Laws)

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$
$$p \leftrightarrow q \equiv q \leftrightarrow p$$

④ Associativity (Associative Laws)

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$
$$(p \leftrightarrow q) \leftrightarrow r \equiv p \leftrightarrow (q \leftrightarrow r)$$

⑤ Distributivity (Distributive Laws)

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

⑥ Idempotency (Idempotent Laws)

$$p \wedge p \equiv p$$
$$p \vee p \equiv p$$

⑦ Identity (Identity Laws)

$$p \wedge T \equiv p$$
$$p \vee F \equiv p$$

⑧ Inverse (Negation Laws)

$$p \wedge \neg p \equiv F$$
$$p \vee \neg p \equiv T$$

⑨ Dominance (Domination Laws)

$$p \wedge F \equiv F$$
$$p \vee T \equiv T$$

⑩ Absorption (Absorption Laws)

$$p \wedge (p \vee q) \equiv p$$
$$p \vee (p \wedge q) \equiv p$$

⑪ De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

⑫ Contrapositive Law

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

⑬ Material Implication

$$p \rightarrow q \equiv \neg p \vee q$$

⑭ Material Equivalence

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

EXAMPLES

1. $\neg(p \to q) \equiv p \wedge \neg q$

$p \wedge \neg q \equiv \neg(p \to q)$
$\equiv \neg(\neg p \vee q)$ Material Implication
$\equiv \neg(\neg p) \wedge \neg q$ De-Morgan's Law
$p \wedge \neg q \equiv p \wedge \neg q$ Double Negation

TRUE

2. $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg p \wedge \neg q \equiv \neg(p \vee (\neg p \wedge q))$
$\equiv \neg p \wedge \neg(\neg p \wedge q)$ De-Morgan's Law
$\equiv \neg p \wedge (\neg(\neg p) \vee \neg q)$ De-Morgan's Law
$\equiv \neg p \wedge (p \vee \neg q)$ Double Negation
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ Distributivity
$\equiv F \vee (\neg p \wedge \neg q)$ Inverse
$\neg p \wedge \neg q \equiv \neg p \wedge \neg q$ Identity

3. $(p \wedge q) \to (p \vee q)$ is a tautology

$(p \wedge q) \to (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ Material Implication
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ De-Morgan's Law
$\equiv \neg p \vee \neg q \vee p \vee q$ Associativity
$\equiv \neg p \vee p \vee \neg q \vee q$ Commutativity
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ Associativity
$\equiv T \vee T$ Dominance
$(p \wedge q) \to (p \vee q) \equiv \text{TRUE}$ Definition of disjunction
$\therefore$ Tautology QED

Practice

① $p \to q \equiv \neg q \to \neg p$

$p \to q \equiv \neg q \to \neg p$

$\equiv \neg(\neg q) \vee \neg p$ Material Implication

$\equiv q \vee \neg p$ Double Negation

$\equiv \neg p \vee q$ commutativity

$\equiv p \to q$ Material Implication

② $p \to (p \vee q)$ is a tautology

$p \to (p \vee q) \equiv \neg p \vee (p \vee q)$ Material Implication

$\equiv (\neg p \vee p) \vee q$ Associativity

$\equiv (p \vee \neg p) \vee q$ Commutativity

$\equiv T \vee q$ Inverse

$\equiv q \vee T$ Commutativity

$\equiv T$ Dominance

$p \to p \vee q \equiv$ TRUE QED

③ $(p \wedge q) \to p$ is a tautology

$(p \wedge q) \to p \equiv \neg(p \wedge q) \vee p$ Material Implication

$\equiv (\neg p \vee \neg q) \vee p$ De-Morgan's Law

$\equiv \neg p \vee \neg q \vee p$ Associativity

$\equiv \neg q \vee p \vee \neg p$ Commutativity

$\equiv \neg q \vee (p \vee \neg p)$ Associativity

$\equiv \neg q \vee T$ Inverse

$(p \wedge q) \to p \equiv$ TRUE Dominance

QED

1.7 Logical Interference Rules

Arguments and Argument Forms

"If I study hard, then I will pass.
I study hard.
Therefore, I will pass."

Let: p: I study hard
q: I will pass

$$\left.\begin{array}{l} p \to q \\ \underline{p} \\ \quad q \end{array}\right\} \text{Argument Form}$$

to expressed symbolically

Truth Table

$[(p\to q)\wedge p] \to q$ → Tautology Form

p	q	$p\to q$	$(p\to q)\wedge p$	$([p\to q)\wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Logical Inference Laws

① Addition

$$\frac{p}{\therefore p \vee q}$$

$p \to (p \vee q)$

② Simplification

$$\frac{p \wedge q}{\therefore p}$$

$(p \wedge q) \to p$

③ Conjunction

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore p \wedge q}$$

$(p \wedge q) \to (p \wedge q)$

④ Modus Ponens

$$\frac{\begin{array}{c} p \to q \\ p \end{array}}{\therefore q}$$

$[(p\to q)\wedge p] \to q$

⑤ Modus Tollens

$$\frac{\begin{array}{c} p \to q \\ \sim q \end{array}}{\therefore \sim p}$$

$[(p\to q)\wedge \sim q] \to \sim p$

⑥ Hypothetical Syllogism

$$\frac{\begin{array}{c} p \to q \\ q \to r \end{array}}{\therefore p \to r}$$

$[(p\to q)\wedge(q\to r)] \to (p \to r)$

⑦ Disjunctive Syllogism

$p \vee q$ $\quad [(p \vee q) \wedge \sim p] \to q$

$\sim p$

$\therefore q$