Unit 2 : Methods of Proof

2.1 Chain of Reasoning

↳ Direct Proof

Kunkunwari: Use Chains of Reasoning to show that each argument is valid.

① If I study hard (p), then I will get a high grade (q).
My parents become happy (r) whenever I get a high grade.
I feel satisfied (s) when I get high grades.
I study hard
Therefore, my parents will be happy and I will feel satisfied.

Argument Form

$p \to q$
$q \to r$
$q \to s$
p
$\therefore r \wedge s$

①	$p \to q$	Premise
②	p	Premise
③	q	1,2 Modus Ponens
④	$q \to r$	Premise
⑤	r	4,3 Modus Ponens
⑥	$q \to s$	Premise
⑦	s	6,3 Modus Ponens
⑧	$r \wedge s$	5,7 Conjunction

QED

①	$p \to q$	Premise
②	$q \to r$	Premise
③	$p \to r$	1,2 Hypothetical Syllogism
④	$q \to s$	Premise
⑤	$p \to s$	1,4 Hypothetical Syllogism
⑥	p	Premise
⑦	r	3,6 Modus Ponens
⑧	s	5,6 Modus Ponens
⑨	$r \wedge s$	7,8 Conjunction

QED

If I come to class early (p), then either I woke up early (q) or I did not take a bath (r)
I take a bath whenever there is water (s)
I came to class early
I did not wake up early
Therefore, there was no water

$p \to (q \vee r)$
$s \to \sim r$
p
$\sim q$
$\therefore \sim s$

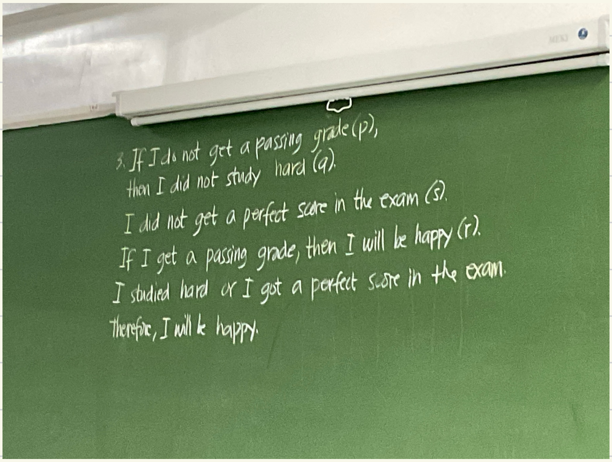
①	$p \to (q \vee r)$	Premise
②	p	Premise
③	$q \vee r$	1,2 Modus Ponens
④	$\sim q$	Premise
⑤	r	3,4 Disjunctive Syllogism
⑥	$s \to \sim r$	Premise
⑦	$\sim s$	6,5 Modus Tollens

If I come to class early (p), then either I woke up early (q) or I did not take a bath (r)
I take a bath whenever there is water (s)
I came to class early
I did not wake up early
Therefore, there was no water

$p \to (q \vee \sim r)$
$s \to r$
p
$\sim q$
$\therefore \sim s$

①	$p \to (q \vee \sim r)$	Premise
②	p	Premise
③	$q \vee \sim r$	1,2 Modus Ponens
④	$\sim q$	Premise
⑤	$\sim r$	3,4 Disjunctive Syllogism
⑥	$s \to r$	Premise
⑦	$\sim s$	6,5 Modus Tollens

QED



$\sim p \to \sim q$
$\sim s$
$p \to r$
$q \vee s$
$\therefore r$

5

1	$q \vee s$	Premise
2	$s \vee q$	1 Commutativity
3	$\sim s$	Premise
4	q	2,3 Disjunctive Syllogism
5	$\sim p \to \sim q$	Premise
6	$\sim(\sim p)$	4,3 Modus Tollens
7	p	6 Double Negation
8	$p \to r$	Premise
9	r	8,7 Modus Ponens

1	$\sim p \to \sim q$	Premise
2	$q \to p$	Contrapositive Law
3	$p \to r$	Premise
4	$q \to r$	2,3 Hypothetical Syllogism
5	$q \vee s$	Premise
6	$\sim s$	Premise
7	q	5,6 Disjunctive Syllo
8	r	4,7 Modus Ponens

$p \to (\sim r \to q)$
$\sim(s \wedge \sim p)$
$\sim r$
$\therefore s \to q$

1	$\sim(s \wedge \sim p)$	Premise
2	$\sim s \vee \sim(\sim p)$	1 De Morgan's Law
3	$\sim s \vee p$	2 Double Negation
4	$s \to p$	3 Material Implication
5	$p \to (\sim r \to q)$	Premise
6	$\sim r$	Premise
7	$p \to (q)$	5,6 Modus Ponens
8	$p \to q$	7 Associativity
9	$s \to q$	4,8 Hypothetical Syllogism

2.2 Proof by Contradiction

> Indirect proof

$T \to F \equiv F$

Inverse: $p \wedge \sim p \equiv F$

Dominance: $p \wedge F \equiv F$

Ex. 1. $p \to q$

$q \to r$

$r \to s$

$\underline{p}$

$\therefore r \wedge s$

Negated Conclusion:

$\sim(r \wedge s)$

Proof

1.	$p \to q$	Premise
2.	p	Premise
3.	q	1,2 Modus Ponens
4.	$q \to r$	Premise
5.	r	4,3 Modus Ponens
6.	$\sim(r \wedge s)$	Premise
7.	$\sim r \vee \sim s$	6 De Morgan's Law
8.	$\sim s$	7,5 Disjunctive Syllogism
9.	$r \to s$	Premise
10.	s	9,5 Modus Ponens
11.	$s \wedge \sim s$	10,8 Conjunction
12.	FALSE	11 Inverse

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$p \to (q \vee \neg r)$

$s \to r$

p

$\neg q$

$\therefore \neg s$

$\sim(\neg s)$

1.	$p \to (q \vee \neg r)$	Premise
2.	p	Premise
3.	$q \vee \neg r$	1,2 Modus Ponens
4.	$\neg q$	Premise
5.	$\neg r$	3,4 Disjunctive Syllogism
6.	$s \to r$	Premise
7.	$\neg s$	6,5 Modus Tollens
8.	$\neg(\neg s)$	Premise
9.	s	8 Double Negation
10	$s \wedge \neg s$	9,7 Conjunction
11.	False	11 Inverse

QED

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2.2.3

$q \to p$

$\sim r \to \sim s$

$q \vee \sim r$

s

$\therefore p$

Negated Conclusion

$\sim p$

1	$\sim r \to \sim s$	Premise
2	s	Premise
3	$\sim(\sim r)$	1,2 Modus Tollens
4	r	3 Double Negation
5	$\sim p$	Premise
6	$q \to p$	Premise
7	$\sim q$	6,5 Modus Tollens
8	$q \vee \sim r$	Premise
9	$\sim r$	8,7 Disjunctive Syllogism
10	$r \wedge \sim r$	4,9 Conjunction
11	FALSE	10 Inverse

QED

2.3 Proof of Resolution

Consider:

$$\begin{array}{l} \sim p \to q \\ \underline{q \to r} \\ \therefore \sim p \to r \end{array} \quad \Bigg| \quad \begin{array}{l} \sim(\sim p) \vee q \\ \underline{\sim q \vee r} \\ \therefore \sim(\sim p) \vee r \end{array}$$

Resolution:
- only works in expressions that are clause
- how pcs think, operate
  - can solve by itself

$$\begin{array}{l} p \vee \cancel{q} \\ \cancel{\sim q} \vee r \\ \therefore p \vee r \end{array}$$

- clauses
- resolvent
- cancelled

Clauses and Clausal Form

- proposition in clausal form
- disjunction/s of simple proposition or their negations
- simple proposition / negation of these simple proposition
  - no logical operations
- only negation & disjunction
- conjunctive normal form (CNF)

Converting propositions to clauses

(Guidelines / Principles)

① Remove $\leftrightarrow$ and $\to$

Ex.1: $q \leftrightarrow \sim r \equiv (q \to \sim r) \wedge (\sim r \to q)$

Ex.2: $r \to (s \wedge q) \equiv \sim r \vee (s \wedge q)$

② Reduce the scope of $\sim$

Ex 3: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

③ Distribute $\vee$ over $\wedge$

Ex 4: $\sim r \vee (s \wedge q) \equiv (\sim r \vee s) \wedge (\sim r \vee q)$

④ seperate clauses joined by $\wedge$

Ex. 5: $(\sim r \vee s) \wedge (\sim r \vee q)$ becomes $\sim r \vee s$ and $\sim r \vee q$

Propositional Resolution

Steps

1. Convert premises to clausal form.
2. Negate the conclusion and convert to clausal form.
3. Apply resolution until a contradiction is derived, or until resolution can no longer be applied.

Kunkunyari: Use resolution to show that the arguments are true.

1. Argument Form

Argument Form	Premise	CNF		
$p \to q$	$p \to q$	$\sim p \vee q$	(Material Implication)	1
$q \to r$	$q \to r$	$\sim q \vee r$	(Material Implication)	2
$q \to s$	$q \to s$	$\sim q \vee s$	(Material Implication)	3
p	p	p		4
$\therefore r \wedge s$				

Negated conclusion	CNF		
$\sim(r \wedge s)$	$\sim r \vee \sim s$	(De Morgan's Law)	5

Resolution Table

1.	$\sim p \vee q$	Clause 1
2.	$\sim q \vee r$	Clause 2
3.	$\sim q \vee s$	Clause 3
4.	p	Clause 4
5.	$\sim r \vee \sim s$	Clause 5
6.	q	1, 4
7.	r	6, 2
8.	s	3, 6
9.	$\sim s$	5, 7
10.	FALSE	8, 9

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① $p \to (q \vee \neg r)$
$s \to r$
p
$\neg q$
$\therefore \neg s$

	Premise	CNF		
	$p \to (q \vee \neg r)$	$\neg p \vee (q \vee \neg r)$	Material Implication	
		$\neg p \vee q \vee \neg r$	Associativity	1
	$s \to r$	$\neg s \vee r$	Material Implication	2
	p	p		3
	$\neg q$	$\neg q$		4
Negated Conclusion: $\neg(\neg s)$	$\neg(\neg s)$	s	Double Negation	5

Resolution Table

1	$\neg p \vee q \vee \neg r$	Clause 1
2	$\neg s \vee r$	Clause 2
3	p	Clause 3
4	$\neg q$	Clause 4
5	s	Clause 5
6	$q \vee \neg r$	1, 3
7	$\neg r$	6, 4
8	r	2, 5
9	FALSE	7, 8

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PBR

3. If I spend time on the internet (p), then I will get a failing grade (q) if I ignore the course site (r), If it is not true that I cheat (s) and I do not spend time on the internet I ignore the course site. Therefore, I will get a failing grad if I cheat.

$p \to (r \to q)$
$\sim(s \land \sim p)$
r
$\therefore s \to q$

Negated conc:
$\sim(s \to q)$

Premise		
$p \to (r \to q)$	$p \to (\sim r \lor q)$	Material implication
	$\sim p \lor (\sim r \lor q)$	Material impl
	$\sim p \lor \sim r \lor q$	Association
	$\sim r \lor \sim p \lor q$	Commutativity (1)
$\sim(s \land \sim p)$	$\sim s \lor \sim(\sim p)$	De M
	$\sim s \lor p$	DN (2)
r	r	(3)
$\sim(s \to q)$	$\sim(\sim s \lor q)$	MI
	$\sim(\sim s) \land \sim q$	DM
	$s \land \sim q$	DN
	s	simplification (4)
	$\sim q$	Simplification (5)

1	$\sim r \lor \sim p \lor q$	Clause 1
2	$\sim s \lor p$	clause 2
3	r	Clause 3
4	s	Clause 4
3	$\sim q$	clause 3
6.	$\sim p \lor q$	1,3
7.	p	2,4
8.	q	6,7
9.	FALSE	5,

QED

Mathematical induction 1

$n=1$:	$1=1$	$=1^2$
$n=2$:	$1+3=4$	$=2^2$
$n=3$:	$1+3+5=9$	$=3^2$
$n=4$:	$1+3+5+7=16$	$=4^2$
$n=5$:	$1+3+5+7+9=25$	$=5^2$
$n=10$:	$1+3+5+7+9+11+13+15+17+19=100$	$=10^2$

Conjecture:

$$\sum_{i=1}^{n}(2i-1)=n^2$$

$$P(n): \sum_{i=1}^{n}(2i-1)=n^2$$

element of set of integers

$\forall n P(n), n \in \mathbb{Z} \wedge n \geq 1$

restricting the domain to the set of integers that are greater than or equal to 1

Mathematical Induction

A statement $P(n)$ involving the positive integer n is true for all positive integer values of n if the following two conditions are satisfied:

(i) $P(1)$ is true.

(ii) If k is an arbitrary positive integer for which $P(k)$ is true, then $P(k+1)$ is also true; that is $P(k) \to P(k+1)$.

Basis Step

$$P(n): \sum_{i=1}^{n}(2i-1)=n^2$$

Proving

$p(1)$:

$$P(1): \sum_{i=1}^{1}(2i-1)=n^2$$

$2(1)-1=1$

$2-1=1$

$1=1$

P(1) holds

$$P(k): \sum_{i=1}^{k}(2i-1)=k^2$$

$$P(k+1): \sum_{i=1}^{k+1}(2i-1)=(k+1)^2$$

$(k+1)^2=$

Example 1: Prove that $P(n): \sum_{i=1}^{n}(2i-1)=n^2$

$P(k+1)$:

$$\sum_{i=1}^{k+1}(2i-1)=(k+1)^2$$

$(k+1)^2=[2(1)-1]+[2(2)-1]+[2(3)-1]+\cdots+$

Example 1: Prove that $P(n): \sum_{i=1}^{n}(2i - 1) = n^2$

$P(k+1)$:

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

$$(k+1)^2 = [2(1)-1] + [2(2)-1] + [2(3)-1] + \cdots + [2k-1] + [2(k+1)-1]$$

Example 1: Prove that $P(n): \sum_{i=1}^{n}(2i - 1) = n^2$

$P(k+1)$:

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

$$(k+1)^2 = [2(1)-1] + [2(2)-1] + [2(3)-1] + \cdots + [2k-1] + [2(k+1)-1]$$

$$= \sum_{i=1}^{k}(2i-1) + [2(k+1)-1]$$

$$= k^2 + [2(k+1)-1]$$

2(k+1)-1 = 2k + 2 -1

Example 1: Prove that $P(n): \sum_{i=1}^{n}(2i - 1) = n^2$

$P(k+1)$:

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

$$(k+1)^2 = [2(1)-1] + [2(2)-1] + [2(3)-1] + \cdots + [2k-1] + [2(k+1)-1]$$

$$= \sum_{i=1}^{k}(2i-1) + [2(k+1)-1]$$

$$= k^2 + (2k+1) \quad \text{// Multiplied 2 and (k + 1)}$$

Example 1: Prove that $P(n): \sum_{i=1}^{n}(2i - 1) = n^2$

$P(k+1)$:

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

$(k+1)^2 = [2(1)-1] + [2(2)-1] + [2(3)-1] + \cdots + [2k-1] + [2(k+1)-1]$

$= \sum_{i=1}^{k}(2i-1) + [2(k+1)-1]$

$= k^2 + (2k+1)$ // Multiplied 2 and (k + 1)

$= k^2 + 2k + 1$ // Associativity

Example 1: Prove that $P(n): \sum_{i=1}^{n}(2i - 1) = n^2$

$P(k+1)$:

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

$(k+1)^2 = [2(1)-1] + [2(2)-1] + [2(3)-1] + \cdots + [2k-1] + [2(k+1)-1]$

$= \sum_{i=1}^{k}(2i-1) + [2(k+1)-1]$

$= k^2 + (2k+1)$ // Multiplied 2 and (k + 1)

$= k^2 + 2k + 1$ // Associativity

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$P(n): \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$P(k+1): \sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$ // substitute k+1 as n

$= \frac{(k+1)(k+2)((2k+2)+1)}{6}$ // simplify

$= \frac{(k+1)(k+2)(2k+3)}{6}$ // simplify

$= \frac{(k^2+3k+2)(2k+3)}{6}$ // Multiply (k+1) and (k+2)

$= \frac{2k^3+9k^2+13k+6}{6}$ // Multiply (k^2+3k+2) and $(2k+3)$

$= \frac{2}{6}k^3 + \frac{9}{6}k^2 + \frac{13}{6}k + 1$ // Divide 6

$= \frac{1}{3}k^3 + \frac{3}{2}k^2 + \frac{13}{6}k + 1$ // simplify

2.7 - Mathematical Induction

Use mathematical Induction to show that each formula is true for all positive integers n.

2. $P(n): \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$\forall$ - For all

$P(k) \to P(k+1)$

$T \to T$

$T \to F$

False

$P(1): \sum_{i=1}^{1} i^2 = \frac{1(1+1)(2(1)+1)}{6}$

$1^2 = \frac{1(2)(3)}{6}$

$1 = \frac{6}{6}$

$1 = 1$ P(1) holds

$P(k): \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$ → inductive hypothesis

$P(k+1): \sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$\frac{(k+1)(k+2)(2k+3)}{6} = i^2 + 2^2 + 3^2 + \dots + k^2 + (k+1)^2$

$= \sum_{i=1}^{k} i^2 + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + \frac{(k+1)^2}{1}$

$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$ // Multiplied $(k+1)^2$ by $\frac{6}{6}$ to make the denominators the same

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$ // Added both fractions

$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$ // factored out k+1

$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$ // Multiplied k(2k+1) and 6(k+1)

$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$ // combined like terms

$\frac{(k+1)(2k+3)(k+2)}{6} = \frac{(k+1)(2k+3)(k+2)}{6}$

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3. $P(n): \sum_{i=1}^{n} 2^i = 2(2^n - 1)$

$P(1): \sum_{i=1}^{1} 2^i = 2(2^1 - 1)$

$2^1 = 2(2-1)$

$2 = 2(1)$

$2 = 2$ P(1) holds → * equal

$P(k): \sum_{i=1}^{k} 2^i = 2(2^k - 1)$ $P(k) \rightarrow P(k+1)$

$P(k+1): \sum_{i=1}^{k+1} 2^i = 2(2^{k+1} - 1)$

$2(2^{k+1} - 1) = 2^1 + 2^2 + 2^3 + \ldots 2^k + 2^{k+1}$

$= \sum_{i=1}^{k} 2^i + 2^{k+1}$

$= 2(2^k - 1) + 2^{k+1}$

$= [2(2^k) - 2] + 2^{k+1}$ // Multiplied 2 to $(2^k - 1)$

$= 2^1(2^k) - 2 + 2^{k+1}$ // Made 2 have an exponent

$= 2(2^{k+1}) - 2$ $= 2^{k+1} - 2 + 2^{k+1}$ // simplified $2^1(2^k)$ by adding both exponents

$= 2(2^{k+1} - 1)$ QED $= 4^{k+1} - 2$ // Combined like terms

$2(2^{k+1} - 1)$ $= 2(2^{k+1} - 1)$ // Factored out 2

QED

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4. $P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$P(1): \sum_{i=1}^{1} i = \frac{1(1+1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$$1 = \frac{2}{2}$$

$1 = 1$ P(1) holds

$$P(k) = \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

$$P(k+1) = \sum_{i=1}^{k+1} i = \frac{(k+1)(k+1+1)}{2}$$

$$\frac{(k+1)(k+2)}{2} = 1+2+3+\ldots+k+(k+1)$$

$$= \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$$

5. $P(n): \sum_{i=1}^{n} (3i-2) = \frac{n(3n-1)}{2}$

$$P(1): \sum_{i=1}^{1} (3i-2) = \frac{1(3(1)-1)}{2}$$

$$(3(1)-2) = \frac{1(2)}{2}$$

$$(3-2) = \frac{2}{2}$$

$$1 = 1$$

$$P(k): \sum_{i=1}^{k} (3i-2) = \frac{k(3k-1)}{2}$$

$$P(k+1): \sum_{i=1}^{k+1} (3i-2) = \frac{(k+1)(3(k+1)-1)}{2}$$

$$\frac{(k+1)(3k+2)}{2} = (3(1)-2) + (3(2)-2) + (3(3)-2) + \ldots + (3k-2) + (3(k+1)-2)$$

$$= \sum_{i=1}^{k} (3i-2) + (3(k+1)-2)$$

$$= \frac{k(3k-1)}{2} + (3k+3-2)$$

$$= \frac{k(3k-1)}{2} + (3k+1)$$

$$= \frac{k(3k-1)}{2} + \frac{2(3k+1)}{2}$$

$$= \frac{k(3k-1)+2(3k+1)}{2}$$

$$= \frac{3k^2-k+6k+2}{2}$$

$$= \frac{3k^2+5k+2}{2}$$

$$\frac{(k+1)(3k+2)}{2} = \frac{(k+1)(3k+2)}{2}$$

positive-integer values of n.

4. $P(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
5. $P(n): \sum_{i=1}^{n} (3i-2) = \frac{n(3n-1)}{2}$
6. $P(n): \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

6. $P(n): \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

$$P(1): \sum_{i=1}^{1} i^3 = \frac{1^2(1+1)^2}{4}$$

$$1^3 = \frac{1(2)^2}{4}$$

$$1 = \frac{4}{4}$$

$1 = 1$ P(1) holds

$$P(k): \sum_{i=1}^{k} i^3 = \frac{k(k+1)^2}{4}$$

$$P(k+1): \sum_{i=1}^{k+1} i^3 = \frac{(k+1)(k+1+1)^2}{4}$$

$$\frac{(k+1)(k+2)^2}{4} = 1^3+2^3+3^3+\ldots+k^3+(k+1)^3$$

$$= \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= \frac{k(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k(k+1)}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{k(k+1)+4(k+1)^3}{4}$$

$$= k^2+1k+$$

$(k+1)(k+1)$

$(k^2+2k+1)(k+1)$

$k^3+2k^2+k+k^2+2k+1$

k^3+3k^2+3k+1

$$P(n): 3 + \sum_{i=1}^{n} (3+5i) = \frac{(n+1)(5n+6)}{2}$$

$$P(1): 3 + \sum_{i=1}^{1} (3+5i) = \frac{(1+1)(5(1)+6)}{2}$$

$$3 + (3+5(1)) = \frac{(2)(5+6)}{2}$$

$$3 + 8 = \frac{22}{2}$$

$$11 = 11$$

$$P(n): 3 + \sum_{i=1}^{n} (3+5i) = \frac{(n+1)(5n+6)}{2}$$

$$P(k): 3 + \sum_{i=1}^{k} (3+5i) = \frac{(k+1)(5k+6)}{2}$$

$$P(k+1): 3 + \sum_{i=1}^{k+1} (3+5i) = \frac{(k+1+1)(5k+1+6)}{2}$$

$$\frac{(k+2)(5k+7)}{2} = (3 + (3+5(1))) + (3 + (3+5(2))) + (3+(3+5(3))) + \ldots + (3+(3+5k)) + (3+(3+5(k+1)))$$

$$= \left(3 + \sum_{i=1}^{k} (3+5i)\right) + (3+(3+5(k+1)))$$

$$= \left(\frac{(k+1)(5k+6)}{2}\right) + (3+(3+5k+5))$$

$$= \left(\frac{(k+1)(5k+6)}{2}\right) + \frac{2(5k+11)}{2}$$

$$= \frac{3k^2+5k+6k+6}{2} + \frac{10k+22}{2}$$

$$= \frac{5k^2+11k+6+10k+22}{2}$$

$$= \frac{5k^2+21k+28}{2}$$

$$= (5k+ \quad)(k+7$$