Realizzabilità e altre cose interessanti

CHAPTER 1

The Interpretation

As I understand it the speciality of a realizzability interpretation is in being looser, for example in this theory the term $natrec(x, 0, 0)$, where x is a free variable, belongs to every type despite not even being well formed in a type theory context [1].

1. Copycatting: Let's get vaguely serious

1.1. Starting definitions. We are going to give a different interpretation of the same language used in the Minimalist Foundation. Una delle differenza sarà che definiremo un'untyped relazione di riducibilità invece del giudizio di uguanza tipata; then using the distinction in **canonical** and **non-canonical** expression we will define the set of **normal** expressions (le espressioni normali in mtt sono normali anche qui) Let's give a simple grammar for our language (in this interpretation we don't actually diffenrentiate between types and elements): let $\mathcal{V}$ be an infinite set of variables (generally denoted with x, y, z, w and with [2]

[1]Ho risolto il problema!! tutte le *astrazioni-funzioni* sono delle λ nella realizzabilità tanto solanto i termini di arietà 0 possono essere tipati in **MTT**

[2]pensando di nuovo a come trattare il concetto delle *astrazioni* in *proglöf* credo che per ora la cosa migliore sia dimenticarsene e non usarle

DEFINITION 1.1. The set of expression $\mathcal{E}$ will be inductively generated from the set of variables $\mathcal{V}$ as closed under the following construct:

$$\lambda x.e \tag{1}$$
$$apply(e_1, e_2) \tag{2}$$
$$0 \tag{3}$$
$$succ(e) \tag{4}$$
$$natrec(e_1, e_2, e_3) \tag{5}$$
$$<e_1,\ e_2> \tag{6}$$
$$El_\Sigma(e_1, e_2) \tag{7}$$
$$N \tag{8}$$
$$\Sigma(e_1, e_2) \tag{9}$$
$$\Pi(e_1, e_2) \tag{10}$$

where x is a variable and each e_i is a previously constructed expression.

DEFINITION 1.2. An expression is said to be **canonical** if it is in the form $\lambda x.e$, 0, $succ(e)$, $<e_1, e_2>$

DEFINITION 1.3. the **contraction** relation ($\triangleright$) is defined by:

$$\begin{aligned} natrec(0, b, c) &\triangleright b \\ natrec(succ(a), b, c) &\triangleright apply(apply(c, a), natrec(a, b, c)) \\ apply(\lambda x.e, d) &\triangleright b\{a/c\} \\ p_0(<a, b>) &\triangleright a \\ p_1(<a, b>) &\triangleright b \end{aligned} \tag{11}$$

DEFINITION 1.4. Here we give the definition of ψ and Ψ as inference rules:

$$\frac{}{\Psi(N)}\ \mathcal{R}\text{-N}$$
$$\frac{\Psi(A) \qquad \Psi(B(a))\,[\psi(A, h, a)]}{\Sigma(A, B)}\ \mathcal{R}\text{-}\Sigma$$
$$\frac{\Psi(A) \qquad \Psi(B(a))\,[\psi(A, h, a)]}{\Pi(A, B)}\ \mathcal{R}\text{-}\Pi$$
$$\frac{\mathcal{F}(A)}{\Psi(A)}\ \mathcal{R}\text{-F} \qquad \frac{A \to B \qquad \Psi(B)}{\Psi(A)}\ \mathcal{R}\text{-}\!\to$$

$$\frac{a \to b \quad \mathcal{N}(b)}{\psi(N, a)} \tag{12}$$

$$\frac{c_{1b}\colon \Psi(\Pi(A, B)) \quad \psi(B(a), apply(b, a))[\psi(A, a)]}{\psi(\Pi(A, B), c_?, b)} \tag{13}$$

$$\frac{c_{1c}\colon \Psi(A) \quad a \to b \quad \mathcal{F}(b)}{\psi(A, a)} \tag{14}$$

$$\frac{c_{1d}(A, B, h)\colon \Psi(A) \quad \psi(B, a)}{\psi(A, a)} \tag{15}$$

$$\frac{}{\Phi(U)} \quad \frac{\Psi(A)}{\phi(U, A)} \tag{16}$$

The normalization theorem will be proved in two steps. In the first step we will prove basic properties of this *Realizzability Interpretation* (such as being normalizing and having other computational properties). In the second step we will give a proof of compatibility between the Minimalist Foundation and the

LEMMA 1.1 (qwe).

CHAPTER 2

Expression Language

The following is based on the ideas that I first read in the book "programming in Martin Löf type thery" [1]. Basically every expression we write in a mathematical expression has an arity (0,$\alpha \to \beta$, $\alpha_1 \otimes \alpha_2 \ldots \otimes \alpha_n$) which work similarly to how types in simply typed lambda calculus work. This simply is a way to ensure that expression are well-formed (even if it doesn't ensure that they are reasonable); we chose to add the $\otimes \ldots \otimes$ constructor to freely chose when to *curry* and *uncurry* function application; we could have added more, but there seem to be no advantage in doing so.

The rule for arity are the expected ones:

1. Everithhung has an Arity

123123123123

2. The previous section was a bad idea

As already pointed out in the footnote 1 on page 5 it was not a good idea to start from the wellformedness of formulas as to prove the normalization theorem we need just the concepts of **canonical** and **non-canonical** and the **tree-like** structure of expressions to define the reducibility relation ricursively.

In the whole article, I think, nothing else was used.

[1]Non sono più convinto di questa cosa; nel libro si parla escusivamente di *MLTT* dove c'è un enorme rigidità nelle costruzioni ammesse, al contrario quando siamo in realizzabilità abbiamo decisamente una maggior libertà: potremmo per esempio scrivere l'espressione $\mathrm{apply}(0,0)$ semplicemente questa non farebbe parte di alcun tipo. rimarrebbe ora da decidere come trattare il concetto di arietà funzionale, la Coquand definisce il costruttore di funzione come $\lambda x.e$ dando nomi espliciti alle variabili e richiamandosi all'operatore di sostituzione $B\{a/x\}$ per rappresentare i "conti" seguendo questa struttura, la consueta sintassi di funzione diventa obsoleta, il lambda calcolo funzionale (i.e. $(x).f$ o $\langle x \rangle .f$) diventa di fatto non strettamente necessario negli usi che servono nell'articolo.
Questo mi porta a propormi di non accanirmi sulle a rietà di non preoccuparsi troppo del $\lambda x.e$ dei Π-tipi (a cui n on sono troppo abituato: $\lambda x.e = \lambda((x).e)$)

3. Properties of the RI

LEMMA 3.1. *If $\Psi(A)$ holds the followings are true:*

(1) If $a \in \mathcal{F}$ then $\psi(A, c_{(1)_{\mathcal{F}}}, a)$ is derivable.
(2) If $\psi\,(A, a)$ and $a = b$ we have also $\psi\,(A, b)$
(3) If $\psi\,(A, a)$ then a is normalizzable
(4) If $A = B$ then $\Psi(B)$
(5) If $A = B$ and $\Psi(B)$ then $\psi\,(A, \cdot) \stackrel{\mathsf{ext}}{=} \psi\,(B, \cdot)$
(6) If A is normalizzable
(7) If $A = B$ and $a = b$ then $\psi\,(A, a)$ implies $\psi\,(B, b)$

LEMMA 3.2. *The lemma above is still true if we replace all Ψ with Φ and all ψ with ϕ. Moreover if $\Psi(A)$ holds then also $\Phi(A)$ holds and we have $\psi(A) \stackrel{\mathsf{ext}}{=} \phi(A)$*

PROOF 3.1. This proof will be by induction in the derivation of $\Psi(A)$.

$\square$