Towards an implementation in LambdaProlog of the two level Minimalist Foundation

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Outline

1. Introduction
2. Type checkers for the two levels of the Minimalist Foundation
3. Interpreting the extensional level in the intensional level
4. Conclusions and Future Works

Mathematical Proofs

SYLOW I. *If p is a prime and p^k, $k \geq 0$, divides $|G|$ (assumed finite), then G contains a subgroup of order p^k.*

Proof. We shall prove the result by induction on $|G|$. It is clear if $|G| = 1$, and we may assume it holds for every group of order $< |G|$. We first prove a special case of the theorem (which goes back to Cauchy): if G is finite abelian and p is a prime divisor of $|G|$ then G contains an element of order p. To prove this we take an element $a \neq 1$ in G. If the order r of a is divisible by p, say $r = pr'$, then $b = a^{r'}$ has order p. On the other hand, if the order r of a is prime to p, then the order $|G|/r$ of $G/\langle a \rangle$ is divisible by p and is less than $|G|$. Hence this factor group contains an element $b\langle a \rangle$ of order p. We claim that the order s of b is divisible by p, for we have $(b\langle a \rangle)^s = b^s\langle a \rangle = 1(=\langle a \rangle)$. Hence the order p of $b\langle a \rangle$ is a divisor of s. Now, since b has order divisible by p, we obtain an element of order p as before. After this preliminary result we can quickly give the proof. We consider the class equation (41′): $|G| = |C| + \sum [G:C(y_j)]$. If $p \nmid |C|$ then $p \nmid [G:C(y_j)]$ for some j. Then $p^k \mid |C(y_j)|$ and the subgroup $C(y_j)$ has order $< |G|$ since y_j is not in the center. Then, by the induction hypothesis, $C(y_j)$ contains a subgroup of order p^k. Next suppose $p \mid |C|$. Then, by Cauchy's result, C contains an element c of order p. Now $\langle c \rangle$ is a normal subgroup of G of order p, and the order $|G|/p$ of $G/\langle c \rangle$ is divisible by p^{k-1}. Hence, by induction, $G/\langle c \rangle$ contains a subgroup of order p^{k-1}. This subgroup has the form $H/\langle c \rangle$ where H is a subgroup of G containing $\langle c \rangle$. Then

$$|H| = [H:\langle c \rangle]|\langle c \rangle| = p^{k-1}p = p^k. \quad \square$$

Mathematical Proofs

```
Lemma galois_fixedField K E :
  reflect (fixedField 'Gal(E / K) = K) (galois K E).

Lemma mem_galTrace K E a : galois K E → a \in E → galTrace K E a \in K.

Lemma mem_galNorm K E a : galois K E → a \in E → galNorm K E a \in K.

Lemma gal_independent_contra E (P : pred (gal_of E)) (c_ : gal_of E → L) x
    P x → c_ x != 0 →
  exists2 a, a \in E & \sum_(y | P y) c_ y × y a != 0.

Lemma gal_independent E (P : pred (gal_of E)) (c_ : gal_of E → L) :
    (∀ a, a \in E → \sum_(x | P x) c_ x × x a = 0) →
   (∀ x, P x → c_ x = 0).

Lemma Hilbert's_theorem_90 K E x a :
   generator 'Gal(E / K) x → a \in E →
 reflect (exists2 b, b \in E ∧ b != 0 & a = b / x b) (galNorm K E a == 1).
```

On paper

- Set Theory (ZFC)
- Quotients, functions as graphs, extensionality of $\in$...
- Classical logic

On proof-assistants

- Type Theories
- Intensionality, computational semantic, code extraction...
- Constructive logic

Notation

$$\frac{P_1 \qquad P_2 \qquad \ldots \qquad P_n}{C}$$

The judgement C holds if all the judgements $P_1, P_2, \ldots, P_n$ hold.

The Minimalist Foundation

- Ideated by Maietti and Sambin in 2005
- Completed by Maietti in 2009
- Is compatible with the most influential constructive foundations
- Has an extensional level (with quotients and subsets) and an intensional level (decidable type-checking)
- Forget-restore principle

In 2009 Maietti succesfully interpreted the extensional level (emTT) in the intensional level (mTT)

Outline of Our Work

Work in Progress

- Type checkers for the two levels of the Minimalist Foundation (implemented in λProlog).
- Implementation (in λProlog) of the interpretation from the extensional level to the intensional level.

Future Works

- Formal validation of the interpretation (in Abella).
- Proof assistant over the extensional level (in *ELPI* = λProlog + Constraint Programming)
- Code extraction at the intensional level.

What Programming Language to Formalize a Theory?

Characteristics of λ-Prolog

1. very high level language, usable by a logician/mathematician
2. easy definition of structures with binders
3. α-equality and capture-avoiding substitution for free
4. simple encoding of inference rules
5. automatic management of non-determinism/backtracking
6. simple reasoning on the programs (simple semantics)

λProlog is the smallest extension to Prolog able to treat syntaxes with binders

Higher Order Logic Programming (HOLP)

λProlog = Prolog $\cup$ $\{\Rightarrow, \forall\}$ in queries

$$\frac{\begin{array}{c}[\mathrm{c}]\\ \vdots \\ \mathrm{q}\end{array}}{\mathrm{c} \Rightarrow \mathrm{q}}$$

Locally scoped,
hypothetical reasoning

$$\frac{\mathrm{c}\{y/x\} \qquad y \text{ fresh}}{\mathrm{pi}\ \mathrm{x} \backslash\ \mathrm{c}}$$

Generation of
fresh names

HOAS + $\{\Rightarrow, \forall\}$ for entering binders in recursive definition

The Hello-World of λProlog

Type-Checking for Simply Typed λ-calculus

$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma, x : A \vdash F\ x : B}{\Gamma \vdash \lambda x.F\ x : A \to B} \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}$$

Representation of Simply Typed λ-calculus

```
type app    term -> term -> term.
type lam    (term -> term) -> term.
```

Type-Checking/Inference in λProlog

```
of (app M N) B :- of M (arr A B), of N A.
of (lam F) (arr A B) :- pi x\ of x A => of (F x) B.
```

Outline

Preliminary Work: Minor Changes to the Calculus

Syntax directed version of the rules

From

$$\frac{x \in A \qquad A = B}{x \in B} \qquad \frac{f \in \Pi_{x \in B} C(x) \qquad t \in B}{f\ t \in C(t)}$$

to

$$\frac{f \in \Pi_{x \in B} C(x) \qquad t \in= B}{f\ t \in C(t)}$$

Deterministic equality check

From $(\lambda_{x \in B} C(x))\ t = C(t)$

to $(\lambda_{x \in B} C(x))\ t \rhd C(t)$ and $(s = t) := s \rhd^{*} {}^{*}\lhd t$

Preliminary Work: Major Changes to the Calculus

Problem: proofs are not recorded at the extensional level

$$\frac{true \in Eq(C, c, d)}{c = d \in C} \qquad \frac{true \in B \quad true \in C \quad B\ props \quad C\ props}{true \in B \wedge C}$$

Discarded solution

The typechecker takes the whole derivation in input.
The datatype for the derivation is yet another typed λ-calculus.

Partial solution

Keep proof terms as in the intensional level.
To a user we can still show *true* because of proof irrelevance.
It does not solve the problem of the *conv* rule.

Preliminary Work: Major Changes to the Calculus

Full solution: deterministic equality check in the ext. level

From arbitrary conversion proofs

$$\frac{\mathit{true} \in \mathit{Eq}(C, c, d)}{c = d \in C}$$

to contextual closure + context lookup rule

$$\frac{(x \in \mathit{Eq}(C, c, d)) \in \Gamma}{c = d \in C}$$

and new LetIn term constructor

$$\frac{p \in P \qquad t \in T\,[x \in P]}{\mathit{let}\ x := p \in P\ \mathit{in}\ t \in T}$$

Preliminary Work: Changes for Code Reuse

Π Introduction rule

$$\frac{B\ set \qquad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B]}{\lambda x^B.c(x) \in \Pi_{x\in B} C(x)}$$

```
of (lambda B F) (setPi B C) IE :-
  isType B _ IE,
  (pi x\ locDecl x B => isType (C x) _ IE)
  pi x\ locDecl x B => of (F x) (C x) IE.
```

Π Formation rule

$$\frac{B\ set \qquad C(x)\ set\ [x \in B]}{\Pi_{x\in B} C(x)\ set} \qquad \frac{B\ col \qquad C(x)\ col\ [x \in B]}{\Pi_{x\in B} C(x)\ col}$$

```
isType (setPi B C) KIND3 IE :-
  isType B KIND1 IE,
  pi x locDecl x B  => isType (C x) KIND2 IE,
  pts_pi KIND1 KIND2 KIND3.
```

Typechecking and future works

Typechecking

- Code reuse between levels.
- Code reduction via PTS-style.
- Extremely modular code.

Future works

- Complete and debug all the rules.
- The changes to the calculi have to be validated
- The ξ-rule at the intentional level must be removed. Requires a syntax directed version of explicit substitutions.

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Design of the Interpretation

The Interpretation in a Nutshell

- In the Minimalist Foundation types are interpreted in dependent setoids.
- The interpretation on types is defined by structural recursion.
- For simple types (the singleton, the empty set, naturals) the setoid equality is the intensional propositional equality
- The equality of functions imposes the ξ rule
- Proof irrelevance is imposed by the interpretation
- Lack of impredicative quantifications avoids user-defined type equalities: this is NOT homotopy type theory

Design of the Interpretation

The Interpretation is Rich and Complex

- Requires lots of (proof) terms to be defined by meta-level recursion on types, terms and derivations of equalities
  - proofs of reflexivity, symmetry, transitivity
  - proofs that equivalences behave as congruences for every user defined function
  - canonical isomorphisms between interpretation of extensionally equal types
  - proofs that they are indeed isomorphisms
  - ...
- We are unable to directly use the proof of the paper as they are often given in categorical terms.

The Main Issue

Subsumption becomes coercions

- Equality used to fix mismatching (extensionally convertible) types must become term translation.
$$\frac{x \in A \qquad A = B}{x \in B} \text{ becomes } \frac{x \in A \qquad A \mathcal{R} B}{\sigma x \in B}$$
- σ is defined by recursion also over the proof of $A = B$ (compring the missing derivations of $Eq(T, c, d)$) Luckily we made proof search deterministic via let-ins and restricting to congruence rules and context lookup
- An example of an extensionally well typed term with mismatching types
$\forall_{x \in \mathbb{1}} \forall_{f \in (x =_{\mathbb{1}} \bigstar) \to \mathbb{1}} (\bigstar =_{\mathbb{1}} x) \Rightarrow f(rfl(\bigstar)) =_{\mathbb{1}} f(rfl(\bigstar))$

Interpretation of Types

```
forall singleton x0 \
  forall (colSigma (fun (propId singleton x0 star) singl
  forall (propId singleton x0 star) x2 \
  forall (propId singleton x0 star) x3 \
  forall (propId singleton star star) x4 \
  propId singleton (fun_app x1 x2) (fun_app x1 x3)) x1 \
forall (propId singleton star x0) x2 \ propId singleton
(fun_app (elim_colSigma x1 (x3 \
    fun (propId singleton x0 star) singleton) x3 \ x4 \
    (impl_app (impl_app (forall_app (forall_app (impl_ap
    (forall_app (forall_app (k_propId singleton) star) x
    x2) star) star) (id singleton star)) (id singleton s
(fun_app (elim_colSigma x1 (x3 \
    fun (propId singleton x0 star) singleton) x3 \ x4 \
    (impl_app (impl_app (forall_app (forall_app (impl_ap
    (forall_app (forall_app (k_propId singleton) star) x
    x2) star) star) (id singleton star)) (id singleton s
```

Auxiliary Predicates for the Interpretation

```
pippo (propEq T T1 T2) (propEq T T1' T2') (SIGMA) :-
    (pippoequ T1 T1' F1),
    (pippoequ T2 T2' F2),
    (trad T1 T1i),
    (trad T2 T2i),
    (trad T1' T1i'),
    (trad T2' T2i'),
    (trad T Ti),
    SIGMA = x\ impl_app (
      impl_app ( forall_app ( forall_app ( impl_app ( for
      forall_app (k_propId Ti) T1i) T1i') F1) T2i) T2i')

pippoequ (fun_app F X1) (fun_app F X2) H :-
  pippoequ X1 X2 G,
  trad F F',
  P2F' = elim_colSigma F' _ (x\ y\ y),
  trad X1 X1',
  trad X2 X2',
  H = forall_app (forall_app (forall_app P2F' X1') X2')
```

Outline

Conclusions

Implementing the Minimalist Foundation is non trivial

- Many different type constructors and rules.
- Many terms need to be provided during the interpretation.
- Extensional type theories pose issues to the implementors.
- Implementation choices impact the calculus.
- The good properties must be preserved.

But the constrained nature of the theory helps

- Structural recursion on types is facilitated by their very rigid structure.
- The propositional equality (int./ext.) is the only type constructor that directly takes terms as arguments.

Conclusions and Future Works

λProlog was a great choice

- Takes away the pain due to binders, α-conversion, capture avoiding substitution, etc.
- The code is in 1-1 relation with the new syntax oriented version of the formal inference rules.
- Joint Bologna/INRIA effort to combine λProlog with Constraint Programming to smoothly transition to proof assistant implementation.

In the future we wish to extend our work

- Complete and validate (in Abella) the type checkers and interpretation.
- Implement code extraction for the intensional level.
- Implement a proof assistant for the extensional level.
- Validate the proof assistant formalizing Sambin's Basic Picture book (porting proofs from Matita).