Towards an implementation in LambdaProlog of the two level Minimalist Foundation

A. Fiori C. Sacerdoti Coen

University of Padova University of Bologna

Padova, 27/09/2017

Outline

Mathematical Proofs

SYLOW I. *If p is a prime and p^k, $k \geq 0$, divides $|G|$ (assumed finite), then G contains a subgroup of order p^k.*

Proof. We shall prove the result by induction on $|G|$. It is clear if $|G| = 1$, and we may assume it holds for every group of order $< |G|$. We first prove a special case of the theorem (which goes back to Cauchy): if G is finite abelian and p is a prime divisor of $|G|$ then G contains an element of order p. To prove this we take an element $a \neq 1$ in G. If the order r of a is divisible by p, say $r = pr'$, then $b = a^{r'}$ has order p. On the other hand, if the order r of a is prime to p, then the order $|G|/r$ of $G/\langle a \rangle$ is divisible by p and is less than $|G|$. Hence this factor group contains an element $b\langle a \rangle$ of order p. We claim that the order s of b is divisible by p, for we have $(b\langle a \rangle)^s = b^s\langle a \rangle = 1(=\langle a \rangle)$. Hence the order p of $b\langle a \rangle$ is a divisor of s. Now, since b has order divisible by p, we obtain an element of order p as before. After this preliminary result we can quickly give the proof. We consider the class equation (41′): $|G| = |C| + \sum [G:C(y_j)]$. If $p \nmid |C|$ then $p \nmid [G:C(y_j)]$ for some j. Then $p^k \mid |C(y_j)|$ and the subgroup $C(y_j)$ has order $< |G|$ since y_j is not in the center. Then, by the induction hypothesis, $C(y_j)$ contains a subgroup of order p^k. Next suppose $p \mid |C|$. Then, by Cauchy's result, C contains an element c of order p. Now $\langle c \rangle$ is a normal subgroup of G of order p, and the order $|G|/p$ of $G/\langle c \rangle$ is divisible by p^{k-1}. Hence, by induction, $G/\langle c \rangle$ contains a subgroup of order p^{k-1}. This subgroup has the form $H/\langle c \rangle$ where H is a subgroup of G containing $\langle c \rangle$. Then

$$|H| = [H:\langle c \rangle]|\langle c \rangle| = p^{k-1}p = p^k. \quad \square$$

Mathematical Proofs

```
Lemma galois_fixedField K E :
  reflect (fixedField 'Gal(E / K) = K) (galois K E).

Lemma mem_galTrace K E a : galois K E → a \in E → galTrace K E a \in K.

Lemma mem_galNorm K E a : galois K E → a \in E → galNorm K E a \in K.

Lemma gal_independent_contra E (P : pred (gal_of E)) (c_ : gal_of E → L) x
    P x → c_ x != 0 →
  exists2 a, a \in E & \sum_(y | P y) c_ y × y a != 0.

Lemma gal_independent E (P : pred (gal_of E)) (c_ : gal_of E → L) :
    (∀ a, a \in E → \sum_(x | P x) c_ x × x a = 0) →
  (∀ x, P x → c_ x = 0).

Lemma Hilbert's_theorem_90 K E x a :
   generator 'Gal(E / K) x → a \in E →
 reflect (exists2 b, b \in E ∧ b != 0 & a = b / x b) (galNorm K E a == 1).
```

On paper

- Set Theory (ZFC)
- Quotients, unctions as graphs, extensionality of $\in$, ...
- Classical logic

On proof-assistants

- Type Theories
- Intensionality
- Constructive and computational

The Minimalist Foundation

- Ideated by Maietti and Sambin in 2005
- Completed by Maietti in 2009
- Is compatible with the most influential constructive foundations
- Has an extensional level (with quotients and subsets) and an intensional level (decidable type-checking)
- Forget-restore principle

In 2009 Maietti succesfully interpreted the extensional level (emTT) in the intensional level (mTT)

Outline of Our Work

Work in Progress

- Type checkers for the two levels of the Minimalist Foundation (implemented in λProlog).
- Implementation (in λProlog) of the interpretation from the extensional level to the intensional level.

Future Works

- Formal validation of the interpretation (in Abella).
- Proof assistant over the extensional level (in *ELPI* = λProlog + Constraint Programming)
- Code extraction at the intensional level.

What Programming Language to Formalize a Theory?

Characteristics of λ-Prolog

1. very high level language, usable by a logician/mathematician
2. easy definition of structures with binders
3. α-equality and capture-avoiding substitution for free
4. simple encoding of inference rules
5. automatic management of non-determinism/backtracking
6. simple reasoning on the programs (simple semantics)

λProlog is the smallest extension to Prolog able to treat syntaxes with binders

Higher Order Logic Programming (HOLP)

λProlog = Prolog $\cup$ $\{\Rightarrow, \forall\}$ in queries

$$\dfrac{\begin{array}{c}[\mathrm{c}]\\ \vdots \\ \mathrm{q}\end{array}}{\mathrm{c} => \mathrm{q}}$$

Locally scoped,
hypothetical reasoning

$$\dfrac{\mathrm{c}\{y/x\} \qquad y \text{ fresh}}{\mathrm{pi}\ \mathrm{x} \backslash\ \mathrm{c}}$$

Generation of
fresh names

HOAS + $\{\Rightarrow, \forall\}$ for entering binders in recursive definition

The Hello-World of λProlog

Type-Checking for Simply Typed λ-calculus

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \frac{\Gamma, x : A \vdash F\ x : B}{\Gamma \vdash \lambda x.F\ x : A \rightarrow B} \qquad \frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}$$

Representation of Simply Typed λ-calculus

```
type app    term -> term -> term.
type lam    (term -> term) -> term.
```

Type-Checking/Inference in λProlog

```
of (app M N) B :- of M (arr A B), of N A.
of (lam F) (arr A B) :- pi x\ of x A => of (F x) B.
```

Outline

Preliminary Work: Minor Changes to the Calculus

Syntax directed version of the rules

From

$$\frac{x \in A \qquad A = B}{x \in B} \quad \frac{f \in \Pi_{x \in B} C(x) \qquad t \in B}{f\ t \in C(t)}$$

to

$$\frac{f \in \Pi_{x \in B} C(x) \qquad t \in = B}{f\ t \in C(t)}$$

Deterministic equality check

From $(\lambda_{x \in B} C(x))\ t = C(t)$

to $(\lambda_{x \in B} C(x))\ t \triangleright C(t)$ and $(s = t) := s \triangleright^{*} {}^{*}\triangleleft\ t$

Preliminary Work: Major Changes to the Calculus

Problem: proofs are not recorded at the extensional level

$$\frac{true \in Eq(C, c, d)}{c = d \in C} \qquad \frac{true \in B \quad true \in C \quad B\ props \quad C\ props}{true \in B \wedge C}$$

Discarded solution

The typechecker takes the whole derivation in input.
The datatype for the derivation is yet another typed λ-calculus.

Partial solution

Keep proof terms as in the intensional level.
To a user we can still show *true* because of proof irrelevance.
It does not solve the problem of the *conv* rule.

Preliminary Work: Major Changes to the Calculus

Full solution: deterministic equality check in the ext. level

From arbitrary conversion proofs

$$\frac{true \in Eq(C, c, d)}{c = d \in C}$$

to contextual closure + context lookup rule

$$\frac{(x \in Eq(C, c, d)) \in \Gamma}{c = d \in C}$$

and new LetIn term constructor

$$\frac{p \in P \qquad t \in T\,[x \in P]}{let\ x := p \in P\ in\ t \in T}$$

Preliminary Work: Changes for Code Reuse

Π Introduction rule

$$\frac{B\ set \qquad c(x) \in C(x)\ [x \in B] \qquad C(x)\ set\ [x \in B]}{\lambda x^B.c(x) \in \Pi_{x \in B} C(x)}$$

```
of (lambda B F) (setPi B C) IE :-
  isType B _ IE,
  (pi x\ locDecl x B => isType (C x) _ IE)
  pi x\ locDecl x B => of (F x) (C x) IE.
```

Π Formation rule

$$\frac{B\ set \qquad C(x)\ set\ [x \in B]}{\Pi_{x \in B} C(x)\ set} \qquad \frac{B\ col \qquad C(x)\ col\ [x \in B]}{\Pi_{x \in B} C(x)\ col}$$

```
isType (setPi B C) KIND3 IE :-
  isType B KIND1 IE,
  pi x locDecl x B  => isType (C x) KIND2 IE,
  pts_pi KIND1 KIND2 KIND3.
```

Typechecking and future works

Typechecking

- Code reuse between levels.
- Code reduction via PTS-style.
- Extremely modular code.

Future works

- Complete and debug all the rules.
- The changes to the calculi have to be validated
- The ξ-rule at the intentional level must be removed.
  Requires a syntax directed version of explicit substitutions.

Outline

Design of the Interpretation

The Interpretation in a Nutshell

- In the Minimalist Foundation types are interpreted in dependent setoids.
- The interpretation on types is defined by structural recursion.
- For simple types (the singleton, the empty set, naturals) the setoid equality is the intensional propositional equality
- The equality of functions imposes the ξ rule
- Proof irrelevance is imposed by the interpretation
- Lack of impredicative quantifications avoids user-defined type equalities: this is NOT homotopy type theory

Design of the Interpretation

The Interpretation is Rich and Complex

- Requires lots of (proof) terms to be defined by meta-level recursion on types, terms and derivations of equalities
  - proofs of reflexivity, symmetry, transitivity
  - proofs that equivalences behave as congruences for every user defined function
  - canonical isomorphisms between interpretation of extensionally equal types
  - proofs that they are indeed isomorphisms
  - ...
- We are unable to directly use the proof of the paper as they are often given in categorical terms.

The Main Issue

Subsumption becomes coercions

- Equality used to fix mismatching (extensionally convertible) types must become term translation.
$$\frac{x \in A \qquad A = B}{x \in B} \text{ becomes } \frac{x \in A \qquad A \mathcal{R} B}{\sigma x \in B}$$
- σ is defined by recursion also over the proof of $A = B$ (compring the missing derivations of $Eq(T, c, d)$)
Luckily we made proof search deterministic via let-ins and restricting to congruence rules and context lookup
- An example of an extensionally well typed term with mismatching types
$\forall_{x \in \mathbb{1}} \forall_{\color{red}{f \in (x =_{\mathbb{1}} \bigstar) \to \mathbb{1}}} (\bigstar =_{\mathbb{1}} x) \Rightarrow \color{blue}{f(rfl(\bigstar))} =_{\mathbb{1}} \color{green}{f(rfl(\bigstar))}$

Interpretation of Types

```
forall singleton x0 \
  forall (colSigma (fun (propId singleton x0 star) singl
  forall (propId singleton x0 star) x2 \
  forall (propId singleton x0 star) x3 \
  forall (propId singleton star star) x4 \
  propId singleton (fun_app x1 x2) (fun_app x1 x3)) x1 \
forall (propId singleton star x0) x2 \ propId singleton
(fun_app (elim_colSigma x1 (x3 \
    fun (propId singleton x0 star) singleton) x3 \ x4 \
    (impl_app (impl_app (forall_app (forall_app (impl_ap
    (forall_app (forall_app (k_propId singleton) star) x
    x2) star) star) (id singleton star)) (id singleton s
(fun_app (elim_colSigma x1 (x3 \
    fun (propId singleton x0 star) singleton) x3 \ x4 \
    (impl_app (impl_app (forall_app (forall_app (impl_ap
    (forall_app (forall_app (k_propId singleton) star) x
    x2) star) star) (id singleton star)) (id singleton s
```

Auxiliary Predicates for the Interpretation

```
pippo (propEq T T1 T2) (propEq T T1' T2') (SIGMA) :-
    (pippoequ T1 T1' F1),
    (pippoequ T2 T2' F2),
    (trad T1 T1i),
    (trad T2 T2i),
    (trad T1' T1i'),
    (trad T2' T2i'),
    (trad T Ti),
    SIGMA = x\ impl_app (
      impl_app ( forall_app ( forall_app ( impl_app ( for
      forall_app (k_propId Ti) T1i) T1i') F1) T2i) T2i')

pippoequ (fun_app F X1) (fun_app F X2) H :-
  pippoequ X1 X2 G,
  trad F F',
  P2F' = elim_colSigma F' _ (x\ y\ y),
  trad X1 X1',
  trad X2 X2',
  H = forall_app (forall_app (forall_app P2F' X1') X2')
```

Outline

Conclusions

Implementing the Minimalist Foundation is non trivial

- Many different type constructors and rules.
- Many terms need to be provided during the interpretation.
- Extensional type theories pose issues to the implementors.
- Implementation choices impact the calculus.
- The good properties must be preserved.

But the constrained nature of the theory helps

- Structural recursion on types is facilitated by their very rigid structure.
- The propositional equality (int./ext.) is the only type constructor that directly takes terms as arguments.

Conclusions and Future Works

λProlog was a great choice

- Takes away the pain due to binders, α-conversion, capture avoiding substitution, etc.
- The code is in 1-1 relation with the new syntax oriented version of the formal inference rules.
- Joint Bologna/INRIA effort to combine λProlog with Constraint Programming to smoothly transition to proof assistant implementation.

In the future we wish to extend our work

- Complete and validate (in Abella) the type checkers and interpretation.
- Implement code extraction for the intensional level.
- Implement a proof assistant for the extensional level.
- Validate the proof assistant formalizing Sambin's Basic Picture book (porting proofs from Matita).