Inverse Problems
Tips Exercise 2: The Vertical Fault - Part 2

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1 Problem formulation

The observed gravity anomalies can be described in a discretized form as the sum over the contribution of the 5 individual layers.

$$d_j = \sum_{i=1}^{5} G\Delta\rho_i \log\left(\frac{x_j^2 + z_{base,i}^2}{x_j^2 + z_{top,i}^2}\right) \tag{1}$$

Because z_{base} and z_{top} are unknown, the relation between model and data space is no longer linear but has the more general form:

$$\mathbf{d} = g(\mathbf{m}) \tag{2}$$

This weakly non-linear problem can be linearized by a Taylor expansion around an initial model guess $\mathbf{m}_0$ and solved with the Tikhonov regularization. The optimal solution is found with an iterative approach, where we calculate the misfit between the modeled and observed gravity anomaly and adjust our model step-wise, until a predefined threshold is reached. An initial model space is defined with assumed values of $\Delta\rho$ and Δz, describing the change in density and layer thickness of the subsurface:

$$\boldsymbol{m_0} = \begin{pmatrix} \Delta\rho_{1,0} \\ \Delta\rho_{2,0} \\ \dots \\ \Delta\rho_{5,0} \\ \Delta z_{1,0} \\ \Delta z_{2,0} \\ \dots \\ \Delta z_{5,0} \end{pmatrix} \tag{3}$$

- use your solution from the last exercise to find a reasonable initial guess for the density variations.

2 Linearization with Taylor's theorem and Tikhonov regularisation

The Taylor's theorem gives an approximation of a k-times differentiable function around a given point by a polynomial of degree k and is defined as

$$f(x) = f(a) + f'(a)(x-a) + ... + \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{4}$$

- expand the function g($\mathbf{m}$) around a starting point $\mathbf{m}_0$ using the Taylor's theorem.
- ignore terms of second order and higher.
- use $\Delta\mathbf{m} = \mathbf{m}_0 - \mathbf{m}$ and $\Delta\mathbf{d} = \mathbf{d} - \mathbf{d}_0$ to rewrite your expanded function g($\mathbf{m}$) in a linear form.

$$\Delta\boldsymbol{d} = \left(\frac{\partial g_i}{\partial \boldsymbol{m_j}}\right)_{m=m_0} \Delta\boldsymbol{m} \tag{5}$$

This equation is now similar to $\boldsymbol{d} = \boldsymbol{G}\boldsymbol{m}$ we used in part 1 and can be solved using the Tikhonov Regularization:

$$\Delta\boldsymbol{m} = \left[\left(\frac{\delta\boldsymbol{g_i}}{\delta\boldsymbol{m_j}}\right)^T \left(\frac{\delta\boldsymbol{g_i}}{\delta\boldsymbol{m_j}}\right) + \varepsilon^2\boldsymbol{I}\right]^{-1} \left(\frac{\delta\boldsymbol{g_i}}{\delta\boldsymbol{m_j}}\right)^T \Delta\boldsymbol{d} \tag{6}$$

- $\mathbf{\Delta m}$ is the misfit between the true model and our assumed model. We can use this misfit to adjust our model for the next iteration. However, if we add the full $\mathbf{\Delta m}$ we risk to miss the optimal model as our steps might be too large. Therefore, we introduce the parameter α to only add part of $\mathbf{\Delta m}$. α takes a value between 0 and 1.

$$\mathbf{m}_{new} = \mathbf{m}_0 + \alpha \mathbf{\Delta m} \tag{7}$$

- we adjust our model until a certain criteria is reached, e.g. that the misfit is below a certain value or a pre-defined maximum number of iterations is reached. During each iteration, Eq. 6 is solved and Eq. 7 can be re-written as:

$$\mathbf{m}_{n+1} = \mathbf{m}_n + \alpha \mathbf{\Delta m}_{n+1}, \tag{8}$$

  where n is the number of iteration.

3 Tips for your code

1. **Load data**
2. **Define initial model**: The choice of the initial model has quite a big effect on the final solution. Therefore, $\boldsymbol{m_0}$ should be as close to the 'real' model. In our case, we can use the solution from part 1 of this exercise to make a reasonable choice of layer thicknesses and density variations.
3. **Define inversion parameters:** Assume some values for α , ε and the maximum number of iterations n_{max}
4. **Iteration loop**:
   - Design an iteration loop, e.g. **while** n < n_{max}, **for** i = 1: n_{max}...
   - Calculate the data we would observe from our model $\mathbf{m}_0$: $\boldsymbol{d_0} = g(\boldsymbol{m_0})$.
   - Define the difference between $\boldsymbol{d_0}$ and the real observations: $\Delta \boldsymbol{d} = \boldsymbol{d_{obs}} - \boldsymbol{d_0}$.
   - Find your 18x10 matrix $\frac{\partial g}{\partial m}$. To do that you have to find the derivatives $\frac{\partial g(m)}{\partial \Delta \rho_i}$ and $\frac{\partial g(m)}{\partial \Delta z_i}$
   - solve for $\mathbf{\Delta m}$ using Eq. 6
   - update your model using Eq. 8.

4 Significance of damping factor α

The implication of α can be evaluated by keeping the ε value constant and perform simulations with different values for α. The parameter regulates the amount of $\Delta\boldsymbol{m}$ we're adding to the new solution. The higher α, the faster the solution converges, as the model parameters are adjusted with larger steps. However, with large steps, we can also miss the optimal solution. The correct value of α has to be found by trial and error.