Assignment 3: The Glacier Thickness Problem

The gravity profile across a glacier

Parker (Geophysical Inverse Theory, Princeton 1994) gives the following problem: A gravity profile is measured across a glacier with the intent of estimating the thickness of the ice. In this setting it is permissible to approximate the glacier and the valley that it partly fills by an infinitely long system, thus allowing us to work in a cross-sectional plane. To be useful the original observations must be carefully corrected in several ways, for example, remove the gravitational attraction of the surrounding mountains. We shall assume this has been done properly. The data shown in Figure 1 represent the gravitational deficit from the replacement of dense rock (density, ρ= 2700 $kg\ m^{-3}$) by ice (ρ = 1000 $kg\ m^{-3}$) measured at twelve equally spaced stations across the glacier. It is therefore convenient to think of the disturbance to the gravity as being caused by a body of negative density contrast $\Delta\rho$ = -1700 $kg\ m^{-3}$ which naturally produces a negative gravity anomaly.

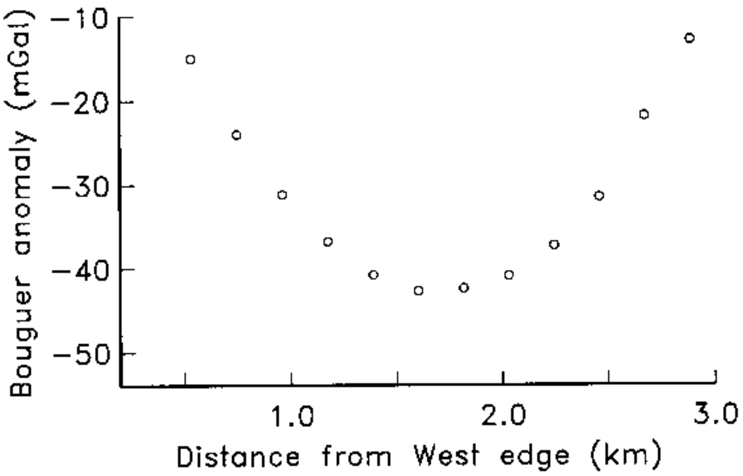

Figure 1: Terrain-corrected gravity anomaly values at twelve equally spaced stations across a glacier.

Distance from edge(m)	**Anomaly (mgal)**	**Distance from edge(m)**	**Anomaly (mgal)**
535	−15.0	1819	−42.4
749	−24.0	2033	−40.9
963	−31.2	2247	−37.3
1177	−36.8	2461	−31.5
1391	−40.8	2675	−21.8
1605	−42.7	2889	−12.8

Table 1: Gravity Anomaly Values (1 mgal is 10^{-5}m/s^2)

The solution to the forward problem is given by the following formula for the gravity anomaly at x_j:

$$\Delta g_j = \Delta g\left(x_j\right) = G\Delta\rho \int_0^a \ln\left[\frac{\left(x-x_j\right)^2 + h(x)^2}{\left(x-x_j\right)^2}\right] dx, \tag{1}$$

where x_j is the horizontal coordinate of the j'th observation, $h(x)$ is the thickness of the glacier at position x, G is the gravitational constant, and we specify that the valley floor outcrops at $x = 0\ km$ and $x = a = 3.42\ km$.

The inverse problem

We wish to estimate $h(x)$ from observations of the gravity along the x-axis. The first step in the analysis is a discretization of the problem.

1. Perform such a discretization by representing the model as a series of M regular elements with rectangular cross sections, each of which are Δx wide, and has height $h(x_l)$, $l = 1 \dots M$. The model vector $\mathbf{m}$ will then consist of the M height values. The discretized version of (1) will inherit the denominator $(x - x_j)^2$ in the log. This introduces a singularity that may cause numerical problems. To remove the singularity, you can introduce a small, additive term δ in the denominator:
$$(x - x_j)^2 \rightarrow (x - x_j)^2 + \delta \tag{2}$$

2. Formulate the discrete inverse problem. This includes choosing a reasonable value for the number of model parameters, M.

3. Show that the problem is nonlinear.

We shall now use the MCMC method to generate acceptable solutions to the problem. The distribution of solutions is the *a posteriori* probability density

$$\sigma(\mathbf{m}) = \frac{\rho(\mathbf{m})L(\mathbf{m})}{\mu(\mathbf{m})},$$

where $\rho(\mathbf{m})$, $L(\mathbf{m})$ and $\mu(\mathbf{m})$ are, respectively, the *a priori* probability density, the *Likelihood function* and the *homogeneous* probability density.

We assume here that $\rho(\mathbf{m})$ is a normal distribution:

$$\rho(\mathbf{m}) = const \cdot \exp\left(-\frac{1}{2}\left(\mathbf{m} - \mathbf{m}_0\right)^T \mathbf{C}_m^{-1}\left(\mathbf{m} - \mathbf{m}_0\right)\right)$$

where $\mathbf{m}_0$ is a "preferred model", and $\mathbf{C}_m$ is the *a priori* covariance matrix. The vector $\mathbf{m}_0$ can, e.g., be found by a simple calculation where a measurement $\Delta g(l)$ above the l'th rectangular model element is approximated with the gravity response from a uniform, horizontal, infinite plate with density contrast $\Delta\rho = -1700\ kg\ m^{-3}$. If the thickness at x_l is equal to $h(x_l)$, the l'th component of $\mathbf{m}_0$ becomes:
$$\Delta g(l) = 2\pi G \Delta\rho h(x_l)\ . \tag{3}$$
where G is the gravitational constant. With this rough calculation we assume that the gravity anomaly in a point is mainly influenced by the glacier thickness immediately below the point, and is unaffected by the variation of thickness over the glacier profile. This is of course inaccurate, but good enough to provide a first guess about the thickness profile.

$\mathbf{C}_m$ is a matrix whose diagonal elements are the *a priori* variances. The square root of the l'th of these (the l'th a priori standard deviation) is an *a priori* measure of how much the l'th model parameter is expected to deviate from the l'te component of $\mathbf{m}_0$. A possible choice for the standard deviation in this case could be 300 m. We shall assume that the off-diagonal elements of $\mathbf{C}_m$ are zero.

The Likelihood function is given by

$$L(\mathbf{m}) = \text{constant } \cdot \exp\left(-\frac{1}{2}(\mathbf{d}_{obs} - g(\mathbf{m}))^T \mathbf{C}_d^{-1}(\mathbf{d}_{obs} - g(\mathbf{m}))\right),$$

where $g(\mathbf{m})$ is the forward function, and $\mathbf{C}_d$ is the data (noise) covariance matrix. We assume that $\mathbf{C}_d$ is diagonal, and that the standard deviation of the noise is 1.0 mgal. We further assume that the homogeneous probability density $\mu(\mathbf{m})$ is constant.

4. Set up a Metropolis Algorithm to find a solution to this problem, assuming a reasonable step length.
5. Run the algorithm for several iterations (for instance 10000). Compute the "acceptance ratio" which is the number of accepted models divided by the number of iterations. Adjust the steplength to obtain an acceptance ratio between 30% and 70%. This gives the most efficient algorithm.
6. Plot the value of $\log(L(\mathbf{m}^{(k)}))$ as a function of k, where $\mathbf{m}^{(k)}$ is the model parameter vector sampled in the k'th iteration. From inspection of this plot, do you see a burn-in phase in the beginning, where the algorithm has not yet reached its correct equilibrium sampling?
7. Find a large number of models $\mathbf{m}^{(k)}$, $k = 1 \ldots K$, distributed according to the *a posteriori* distribution. Remember, if necessary, to discard the models obtained during the burn-in phase.
8. Plot histograms of the 1D marginal distributions for all the model parameters $h(x_j)$.
9. Create a scatterplot of samples from the 2D marginal distribution of two neighbouring glacier thicknesses. Are the two parameters correlated?