GCSE Mathematics Higher

AQA Specification 8300

Part 2

Chapter Three: Measures and Accuracy

3.1 Exact Calculations

Surds

Sometimes the square root of a number will result in an irrational number. An **irrational number** is one which cannot be written as a fraction, a finite decimal or a recurring decimal. It has an infinite number or decimal places that don't appear to follow any pattern.
For example $\sqrt{2} = 1.4142135623095 \ldots$
When this is the case the numbers are left inside the root symbol and are called surds.
It is useful to leave a value in surd form when an exact answer is required.

Surds are simplified by finding factors of the number inside the root that have an integer square root.

Example:
Simplify $\sqrt{20}$

The first thing we need to do is look at factor pairs. For 20 we have:
1x20, 2x10 and 4x5
We will use 4x5 because 4 is a square number, this gives

$$\begin{aligned}\sqrt{20} &= \sqrt{4 \times 5} \\ &= \sqrt{4} \times \sqrt{5} \\ &= 2 \times \sqrt{5} \\ &= 2\sqrt{5}\end{aligned}$$

Example:
Simplify $2\sqrt{200}$

$2\sqrt{200}$ is just shorthand for $2 \times \sqrt{200}$ it is often easier to see what's going on if you write it out in full like this when you're working.

$$\begin{aligned}2 \times \sqrt{200} &= 2 \times \sqrt{100 \times 2} \\ &= 2 \times \sqrt{100} \times \sqrt{2} \\ &= 2 \times 10 \times \sqrt{2} \\ &= 20 \times \sqrt{2} \\ &= 20\sqrt{2}\end{aligned}$$

Example:
Simplify $\sqrt{75} + 3\sqrt{12}$

Each of these need to be simplified in turn before you deal with the addition

$$\begin{aligned}\sqrt{75} &= \sqrt{3 \times 25} \\ &= \sqrt{3} \times \sqrt{25} \\ &= \sqrt{3} \times 5 \\ &= 5\sqrt{3}\end{aligned}$$

$$\begin{aligned}3 \times \sqrt{12} &= 3 \times \sqrt{3 \times 4} \\ &= 3 \times \sqrt{3} \times \sqrt{4} \\ &= 3 \times \sqrt{3} \times 2 \\ &= 6\sqrt{3}\end{aligned}$$

So we have $\sqrt{75} + 3\sqrt{12} \;=\; 5\sqrt{3} + 6\sqrt{3} \;=\; 11\sqrt{3}$

If you have an expression with a surd as the denominator you should *rationalise the denominator*. This means multiplying it by a fraction equivalent to 1 (where the numerator and denominator are the same) such that the denominator becomes a rational number.

To change a surd to a rational number you should multiply it by itself

$\sqrt{n} \times \sqrt{n} = n$

Because $\sqrt{n} \times \sqrt{n} = \sqrt{n \times n} = \sqrt{n^2} = n$

For example

$\sqrt{3} \times \sqrt{3} = \sqrt{3 \times 3} = \sqrt{9} = 3$

Example:

Rationalise the denominator $\frac{1}{\sqrt{2}}$

To get rid of the $\sqrt{2}$ we must multiply by $\sqrt{2}$. However we can't change the value so we can only multiply by 1.

$\frac{\sqrt{2}}{\sqrt{2}} = 1$

Therefore we do

$\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \;=\; \frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}} \;=\; \frac{\sqrt{2}}{2}$

Activity 3.1A

1. Simplify each of the following as far as possible

a. $\sqrt{80}$	b. $\sqrt{54}$	c. $\sqrt{150}$	d. $\sqrt{192}$	e. $\sqrt{99}$
f. $2\sqrt{50}$	g. $3\sqrt{20}$	h. $2\sqrt{45}$	i. $4\sqrt{125}$	j. $5\sqrt{28}$
k. $\sqrt{20} \times \sqrt{5}$	l. $\sqrt{200} \times \sqrt{2}$	m. $\sqrt{50} \times \sqrt{8}$	n. $\sqrt{162} + \sqrt{18}$	o. $\sqrt{175} - \sqrt{28}$

2. Simplify as far as possible – make sure you rationalise all denominators

a. $\frac{3}{\sqrt{5}}$	b. $\frac{4}{\sqrt{7}}$	c. $\frac{1}{\sqrt{7}}$	d. $\frac{\sqrt{3}}{\sqrt{2}}$	e. $\frac{\sqrt{5}}{\sqrt{3}}$
f. $\frac{3}{2\sqrt{5}}$	g. $\frac{3\sqrt{3}}{\sqrt{2}}$	h. $\frac{\sqrt{3}}{2\sqrt{7}}$	i. $\frac{4\sqrt{3}}{2\sqrt{2}}$	j. $\frac{5\sqrt{5}}{\sqrt{3}}$
k. $\frac{\sqrt{7}\times\sqrt{3}}{\sqrt{5}\times\sqrt{2}}$	l. $\frac{2\sqrt{3}\times\sqrt{2}}{\sqrt{5}}$	m. $\frac{\sqrt{8}+\sqrt{50}}{\sqrt{3}+\sqrt{3}}$		

<u>Indices</u>

An **index** (plural: **indices**) is a power, you met these in section 1.3. As before the index tells you how many times the base is multiplied by itself. In chapter one we mostly dealt with numbers raised to the power of 2 or 3, known as squared and cubed. Here we will deal with exact calculations using numbers and letters raised to higher powers.

If you have numbers or letters raised to a power where the *base* is the same you can multiply and divide them.
When multiplying you add the indices, when dividing you subtract the indices.

Example:
$4^3 \times 4^5 = 4^{3+5} = 4^8$

To help you visualise this, if you write each of calculations out you will see that you end up with 4 multiplied by itself 8 times.
$4^3 \times 4^5 = (4 \times 4 \times 4) \times (4 \times 4 \times 4 \times 4 \times 4) = 4^8$

Example:
$18^5 \times 18^6 = 18^{5+6} = 18^{11}$

Example:
$5^7 \div 5^3 = 5^{7-3} = 5^4$

Example:
$x^{10} \div x^7 = x^{10-7} = x^3$

Example:
$\frac{x^{15}}{x^{12}} = x^{15-12} = x^3$ *Remembering that a fraction means the numerator divided by the denominator.*

If you raising to another power you multiply the indices.

Example:
$(3^2)^4 = 3^{2\times 4} = 3^8$

Example:
$(5^3)^5 = 5^{3\times 5} = 5^8$

Example:
$(y^{10})^2 = y^{10\times 2} = y^{20}$

It is important to remember that these rules only apply when the base is the same.

Any number to the power of one is just the number itself. Any number, except zero, to the power of zero is 1. One to the power of anything is one.
$x^1 = x \; for \; all \; x$
$x^0 = 1 \; for \; all \; x \neq 0$
$1^x = 1 \; for \; all \; x$

Example:
$4^1 = 4$ $\quad$ $5^1 = 5$ $\quad$ $154^1 = 154$ $\quad$ $a^1 = a$ $\quad$ $(9.2)^1 = 9.2$

Example:
$5^0 = 1$ $\quad$ $8^0 = 1$ $\quad$ $456^0 = 1$ $\quad$ $b^0 = 1$ $\quad$ $(23 \times 53 \div 6)^0 = 1$

Example:

$1^4 = 1$ $1^{46} = 1$ $1^{x+y} = 1$

Example:
Find the value of x when $2^{14} = 4^x$

We know that $2^2 = 4$ and that 14 = 2 x 7 so if we rewrite 2^{14} as a power of 4 we have
$2^{14} = 2^{2\times7} = (2^2)^7 = 4^7$
So we have $x = 7$

Activity 3.1 B

1. Calculate the following. Write your answers in index form.
   a. $4^5 \times 4^5$ b. $6^3 \times 6^5$ c. $7^4 \times 7^{10}$ d. $9^3 \times 9^0$ e. $13^6 \times 13^5$

   f. $5^6 \div 5^2$ g. $8^7 \div 8^3$ h. $18^8 \div 18^4$ i. $4^6 \div 4^3$ j. $9^{10} \div 9^2$

2. Calculate the following. Write your answers in index form.
   a. 3×3^4 b. $9^5 \div 9$ c. $6^3 \times 6^2 \times 6$ d. $9^5 \times 9^0 \times 9$

   e. $10 \times 10 \times 10$ f. 3×3^2 g. $8^{11} \times 8$ h. $8 \times 8^7 \times 8 \times 8^3$

3. Evaluate these expressions.
   a. 6^0 b. 890^1 c. 92^1 d. 1728^0 e. 1^{34}
   f. 1^a g. $(182 + 367)^0$ h. $(2783 \times 4^5)^0$

4. Simplify each of these expressions. Write your answers in index form.
   a. $(9^3)^3$ b. $(2^7)^5$ c. $(5^5)^3$ d. $(1^{10})^9$ e. $(3^{19})^0$ f. $(4^8)^3$

5. Calculate the following. Write your answers in index form.
   a.$(4^3 \times 4^6) \div 4^2$ b. $(5^3 \times 5^4) \div (5^2 \times 5^2)$ c. $6^4 \times (6^7 \div 6^5)$

6. Calculate the following. Write your answers in index form.
   a. $\frac{3^4\times3^2}{3^3}$ b. $\frac{8^4\times8^6}{8^2\times8^5}$ c.$\frac{10^6}{10^2\times10^2}$ d.$\frac{6^{10}\div6^2}{6^3\times6^2}$ e. $\frac{9^6}{9^2\times9^3}$

   f.$\left(\frac{6^7}{6^2}\right)^3$ g. $\frac{5^4\times(5^3)^2}{5^2}$ h. $\left(\frac{4^3\times4^5}{4^6\div4^2}\right)^2$ i. $\frac{(7^2)^5}{7^8\div7^3} \times 7$

7. Find the value of the letter in each equation.
   a. $3^4 \times 3^a = 3^7$ b. $3^b = 9$ c. $3^4 \times 9 = 3^c$ d. $5^d = 5^8 \div 25$
   e. $9^6 = 3^e$ f. $4^{10} = 16^f$ g. $(5^g)^5 = 5^{15}$ h. $8^9 \div 8^h = 8^2$

8. If $8^9 = 134217728$, what is $\sqrt[9]{134217728}$?

9. If $4^8 = 65536$, what is $\sqrt[8]{65536}$?

If a base is raised to a negative power it becomes a reciprocal.
$x^{-n} = \frac{1}{x^n}$

For example $x^{-3} = \frac{1}{x^3}$

Example:
Simplify 4^{-2}

$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

If a based is raised to a fractional power it corresponds to a surd
$x^{\frac{1}{n}} = \sqrt[n]{x}$

For example
$x^2 = \sqrt{x}, x^3 = \sqrt[3]{x}$ etc

Example:
Evaluate $8^{\frac{1}{3}}$

$8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
Recall that this is the symbol for the cube root.

If the fraction has a value other than 1 for the numerator either the surd or the value inside the root is raised to the power of the numerator. Either of these options gives the same value.
$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$

Example:
Evaluate $8^{\frac{2}{3}}$

There are two ways to evaluate this
$8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$
Or
$8^{\frac{2}{3}} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4$

Activity 3.1 C

1. Evaluate each of these expressions

a. $100^{\frac{1}{2}}$ b. $27^{\frac{1}{3}}$ c. 3^{-1} d. 5^{-2}

2. Evaluate each of these expressions

a. $27^{\frac{2}{3}}$ b. $9^{\frac{3}{2}}$ c. $125^{\frac{4}{3}}$ d. $4^{-\frac{1}{2}}$

e. $64^{-\frac{1}{3}}$ f. $81^{-0.5}$ g. $169^{-\frac{3}{2}}$ h. $25^{\frac{3}{2}}$

3. Write each of these in index form

a. $\frac{1}{8^2}$ b. $\frac{1}{5}$ c. $\sqrt[4]{9}$ d. $\left(\sqrt[3]{10}\right)^2$ e. $\frac{1}{\sqrt{2}}$ f. $\frac{1}{\sqrt[3]{x}}$

To recap, the rules for using indices are:

$$x^a \times x^b = x^{a+b}$$

$$x^a \div x^b = \frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$x^0 = 1$$

$$x^1 = x$$

$$x^{\frac{1}{a}} = \sqrt[a]{x}$$

$$x^{\frac{a}{b}} = \sqrt[b]{x^a} = \left(\sqrt[b]{x}\right)^a$$

$$x^{-a} = \frac{1}{x^a}$$

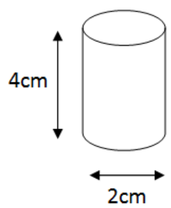
Using Pi (π) in Calculations

With some numbers it is not practical to write them out in full because they are too long or, with some decimals, they don't have an end point. Pi is an example of this because it is irrational – it continues, we believe, forever. At the time of writing pi has been calculated to 12.1 trillion digits so you can see why we would not try to write it out in full!

$$\pi = 3.14159265358979323846264338327950288419716939937510582097494459230 7 \ldots$$

In practice you would round your answers, which we will cover in the next section, so your answers would no longer be exact.
In order to keep your answers exact you would use the symbol π in calculations – this also avoids having to work with horrible decimals!

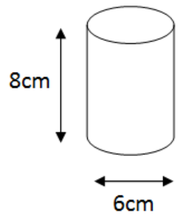
Example:
Find the volume of this cylinder *Note: volume will be covered in more detail in chapter 9*
Volume of a cylinder = $\pi r^2 h$ where *r* is the radius (half of the diameter (width) of the circle) and *h* is the height



Volume = $\pi \times 1^2 \times 4 = 4\pi$

By leaving π in the equation the answer is exact.

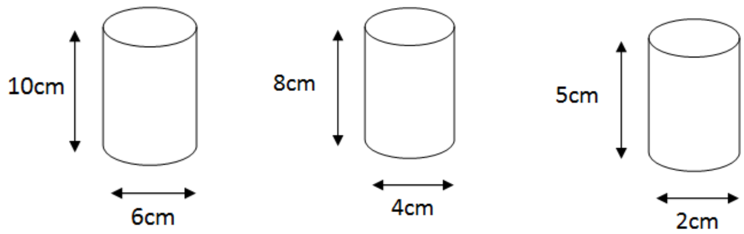
Example:
Find the volume of this cylinder



Volume = $\pi \times 3^2 \times 8 = 72\pi$

Activity 3.1 D

Using the formula $\pi r^2 h$ find the volume of these cylinders. Leave your answers in terms of π.



3.2 Rounding and Estimation

Rounding Integers

You can round a number to the nearest 10, 100, 1000...
To decide whether to round up or down you need to look at the next smaller digit. So if you were rounding to the nearest ten you'd look at the unit, if you were rounding to the nearest hundred you'd look at the ten and so on.

If the next digit is 0, 1, 2, 3 or 4 you round down. If it is 5, 6, 7, 8 or 9 you round up.
So if it is less than 5 you go down, if it is 5 or above you go up.

Example:
Round 48 to the nearest ten

We're rounding to the nearest ten so we look at the next smallest digit which is the unit.
Since the unit digit is 8 we round up. So 48 rounded to the nearest ten is 50.

Example:
Round 721 to the nearest ten

We’re rounding to the nearest ten so we look at the next smallest digit which is the unit.
Since the unit digit is 1 we round down. So 721 rounded to the nearest ten is 720.
The hundreds digit is irrelevant.

Anything that is before the digit you’re rounding to stays the same.

Example:
Round 7817 to the nearest hundred

We’re rounding to the nearest hundred so we look at the next smallest digit which is the tens.
Since the ten digit is 1 we round down. So 7817 rounded to the nearest hundred is 7800.

You can also think of it as 17 is less than 50 so you round down, any number above 50 is rounded up.

Example:
3,593 rounded to the nearest ten is 3,590.
3,593 rounded to the nearest hundred is 3,600.
3,593 rounded to the nearest thousand is 4,000.

Activity 3.2 A

1. Round each of these numbers to the nearest ten.
   a. 27 b. 38 c. 491 d. 29 e. 919 e. 94 f. 375 g. 85 h. 98 1. 802

2. Round each of these numbers to the nearest hundred.
   a. 278 b. 290 c. 510 d. 759 e. 750 f. 1938 g. 3801 h. 2981 i. 280 j. 90

3. Round each of these numbers to the nearest thousand.
   a. 2873 b. 5892 c. 3484 d. 2910 e. 928 f. 1903 g. 3801 h. 2938 i. 8500 j. 6499

<u>Rounding to Decimal Places</u>

The number of decimal places is the number of digits after the decimal point. When rounding to a given number of decimal places you use the same rules as before.
For example if you were rounding to 3 decimal places you’d look to the fourth number after the point to decide whether to round up or down.

Example:
Round 235.2849 to
a. a whole number b. 1 decimal place c. 2 decimal places d. 3 decimal places

a. The number after the decimal point is 2 so we round down: 235
b. The number 2 places after the decimal point is 8 so we round up: 235.3
c. The number 3 places after the decimal point is 4 so we round down: 235.28
d. The number 4 places after the decimal point is 9 so we round up: 235.285

Example:
Round 278.1838 to

a. a. a whole number	b. 1 decimal place	c. 2 decimal places	d. 3 decimal places
a. 278	b. 278.2	c. 278.18	d. 278.184

We shorten "decimal places" to dp. If you round at the end of a question you should make a note of how you rounded it in brackets after your answer.
For example if your answer was 5.38582... and you decided to round to two decimal places s you would write your answer as 5.39 (2dp).

Rounding to Significant Figures

In a number the first digit that isn't zero is called the first **significant figure**. Rounding to significant figures differs from rounding to decimal places because the decimal point is largely irrelevant. The number is rounded from the first non-zero digit so it also takes into account the numbers before the decimal point. The method, however, remains the same.

Example:
Round 0.394 to one significant figure

The number after the first significant figure is 9 so we round up: 0.4

Example:
Round 3720 to one significant figure

The number after the first significant figure is 7 so we round up: 4000

Example:
Round 38.41 to two significant figures

The number after the second significant figure is 4 so we round down: 38

Example:
Round 0.0000283 to one significant figure

The number after the first significant figure is 8 so we round up: 0.00003

Activity 3.2 B

1. Round each of these to the nearest whole number

a. 83.45 b. 382.49 c. 93.92 d. 293.68 e. 294.103

2. Round each of these to i) 1dp ii) 2dp iii)3dp

a. 789.8643	b. 2389.10384	c. 289.29374	d. 0.283475	e. 38.28374
f. 382.38547	g. 0.0583	h. 29.381	i. 39.102784	j. 0.17263

3. Round each of these to i) 1sf ii) 2sf iii) 3sf

a. 283047	b. 93.1093	c. 0.02834	d. 0.0002837	e. 76790
f. 38672	g. 0.0002734	h. 0.017374	i. 2910273	j. 2.000878

4. Round each of these to i)2dp ii)2sf

a. 83.29487	b. 2091.2038	c. 0.023847	d. 29.3018	e. 1928.0283
f. 0.08376	g. 0.0082846	h. 8203.289	i. 900.028	j. 29.03776

All questions regarding money *have* to be rounded to 2dp. Any other rounding would be marked as incorrect. Similarly any questions regarding people, animals, objects etc *have* to be rounded to a whole number. It is not possible to have, for example, 2.5 people!

Estimation

An **approximation** of a number is found by rounding. Approximations are used to estimate an answer. They are useful when an exact answer is not needed or for checking that an answer makes sense.

Unless otherwise stated a useful approximation is usually one significant figure. However the closer your approximation is to the original number the more accurate your answer is so you may choose to use more significant figures when it is appropriate and easy enough to do so.

Example:
Find an approximate answer to 28.3 + 182.92

By rounding each of these to one significant figure the sum becomes 30 + 200 which gives an approximate answer of 230.
By rounding to the nearest ten the sum becomes 30 + 180 = 210.
The exact answer is 211.32 do the second approximation gave a much closer answer.

Example:
Find an approximate answer to 4.23 x 2.92

$4.23 \times 2.92 \cong 4 \times 3 = 12$

≅ means “approximately equal to”. It is sometimes seen with just one line underneath.

Example:
Estimate the answer to $\frac{4.56 \times 5.46}{9.8}$

$$\frac{4.56 \times 5.46}{9.8} \cong \frac{5 \times 5}{10} = \frac{25}{10} = 2.5$$

Example:
Dave hired a car to travel from Southampton to Oxford, a distance of 68.4 miles. He did the return journey the next day. The car cost £49.97 to hire plus 22p per mile. Estimate how much the trip cost him.

The cost of the journey would be found by multiplying the cost per mile by distance travelled and adding this to the initial cost.

Because we are estimating we can round up and say that he travelled 70 miles each way, so the total distance travelled was 140 miles.

The cost per mile is approximately 20p and the initial cost is approximately £50 so we have
(140 x 0.2) + 50 = 78

So the trip cost him approximately £78
Approximations are also used to convert between units as illustrated in the following example.

Example:
Since 1 inch is approximately 2.5cm, estimate how many cm there are in 2 feet. (1 foot = 12 inches)

There are 24 inches in 2 feet so we have 24 x 2.5 = 60
So 2 feet ≅ 60cm

You can also use trial and error to find answers. By estimating initially you narrow down the numbers you need to look at. See the following example.

Example:
Estimate $\sqrt{7}$ to 1dp without using a calculator

You know that $\sqrt{4} = 2$ and $\sqrt{9} = 3$ so $2 < \sqrt{7} < 3$

Because of this we will begin by trying 2.5 which lies in the middle of the interval.
2.5^2 = 6.25

So we need a bigger number. Try 2.6
2.6^2 = 6.76

This is still not large enough so try 2.7
2.7^2= 7.29

We now know the number lies between 2.6 and 2.7. We're only looking for one decimal place but we need to know which of these the correct answer would round to so we still look for two decimal places and then round the answer.

Try 2.65
2.65^2= 7.0225 which rounds, to 1dp, to 7.0

So we know that $2.6 < \sqrt{7} < 2.65$

Therefore we say that $\sqrt{7}$ = 2.6 (1dp)

Activity 3.2 C

1. Estimate answers to each of these calculations
   a. 4.88 + 2.56 b. 4.59 + 7.65 c. 9.23 – 2.28 d. 75.14 – 1.54

2. Estimate answers to each of these calculations
   a. 2.78 x 2.39 b. 3.98 ÷ 2.31 c. 329 x 190 d. 29.302 x 2.14

3. Estimate answers to each of these calculations
   a. $\frac{3.14 \times 4.25}{6.1}$ b. $\frac{29.91 \times 38.2}{3.2 \times 3.8}$ c. $\frac{16.4 \times 0.47862}{0.234 \times 41.698}$ d. $\frac{42.12 \times 1.8^2}{2.046}$

   e. $\frac{4.234 \div 1.892}{9.289 - 7.123}$ f. $\left(9.7^2 + \frac{19.89342}{1.893}\right)$ g. $\sqrt{(8.7 - 4.8)}$ h. $(1.276^2 + 2.98)^2$

4. If John buys a packet of crisps for 47p, a drink for 99p and some fruit for 32p, roughly how much has he spent?

5. Estimate the answer to these square roots to one decimal place
   a. $\sqrt{15}$ b. $\sqrt{10}$ c. $\sqrt{8}$ d. $\sqrt{20}$ e. $\sqrt{90}$

6. Bob says that $\frac{205 \times 19.2}{4.1} \cong 100$. Is he correct? Why?

7. Fred is driving 323 miles. His car does 41 miles for each gallon of petrol. Since 1 gallon is roughly 5 litres estimate how many litres of petrol he will need to complete his journey.

3.3 Using a Calculator

You can use a scientific calculator to work out the exact answers to more complicated calculations. When the calculations involve a fraction you can either use the fraction button and navigate between the top and bottom using the arrow keys or you can divide using brackets as illustrated in the following example.

Example:
Use a calculator to calculate the exact value of $\frac{4.5 \times 5.3}{5 + 0.89}$

You would type (4 . 5 x 5. 3) ÷ (5 + 0 . 8 9)

You calculator will often give your answer as a fraction or as a surd (a number containing the square root symbol.) If this happens, and you want the decimal, press the **S↔D** button.

You will become more familiar with your calculator as the course progresses. If you still feel unsure with how to use it as you're nearing the exam there are a number of helpful guides online specific to your model of calculator.

Activity 3.3 A
For every question write down all of the numbers on your calculator display

1. Use the $\boxed{x^2}$ button to work out the value of
   a. 45^2 b. 31^2 c. 56^2 d. 34^2

2. Use the $\boxed{\sqrt{\ }}$ button to work out the value of
   a. $\sqrt{4225}$ b. $\sqrt{645}$ c. $\sqrt{5041}$ d. $\sqrt{7245}$

Activity 3.3 B

Use your calculator to work out the exact answers to each of the questions in Activity 3.2 C
For every question write down all of the numbers on your calculator display
The approximate answers you found in 3.2 C will give you an indication of whether you have typed the correct sequence into the calculator.

For question 6 you need to know that 1 gallon = 4.54609 litres.

3.4 Measures

Standard Units

All measurements should be done in metric units. (The old system in the UK, for example, pounds and ounces, is the imperial system).

Length is a measure of distance. The units used are millimetres (mm), centimetres (cm), metres (m) and kilometres (km). In the UK we still largely use the imperial unit of miles.

The equivalences you need to know are:
10mm = 1cm 100cm = 1m 1000m = 1km

Mass is a measure of the amount of matter in an object, it is usually thought of as an objects weight. The units used are grams (g), kilograms (kg) and tonnes. *Note: this is a metric tonne, it is not the same as an imperial ton which is no longer used.*

The equivalences you need to know are:
1000g = 1 kg 1000kg = 1 tonne

Capacity is a measure of how much fluid a container can hold. The units used are millilitres (ml), centilitres (cl) and litres (l).

The equivalences you need to know are:

10ml = 1cl   100cl = 1l   1000cm^3= 1l

It is also useful to know that 1000ml = 1l

Time is a measure of how long something takes. The units are seconds (s or sec), minutes (m or min) and hours (hr or h). You may also use days, weeks, months or years.

The equivalences you need to know are:

60 seconds = 1 minute   60 minutes = 1 hour

It is assumed that you know 7 days = 1 week, 12 months = 52 weeks = 365 (or 366) days = 1 year

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Activity 3.4 A

1. Decide what unit you would use to measure each of the items below.
    a. The distance from Southampton to London
    b. The height of Big Ben
    c. The mass of a bag of sugar
    d. The thickness of a book
    e. The amount of lemonade in a large bottle
    f. The length of the table
    g. How long it would take to travel to Australia
    h. The weight of a notebook
    i. The amount of water in a glass
    j. The weight of a paperclip
    k. The weight of an elephant

2. Convert each of these measurements to the units shown.

a. 1000ml = ___ l	b. 25kg = ___ g	c. 10kg = ___g	d. 1200g = ___ kg
e. 2.5cm = ___mm	f. 8000m = ___ km	g. 2min = ___seconds	h. 11cm = ___mm
i. 44.36cm = ___mm	j. 22000m = ___km	k. 35.2kg = ___g	l. 83.45kg = ___g
m. 20300g = ___kg	n. 9000ml=___l	o. 2l = ___ ml	p. 23ml = ___cl

3. A bottle holds 250ml of water. How many bottles can be filled if you have 1 litre of water?

4. Which is longer? 500mm or 5cm?

5. How many centimetres are there in 7.5 metres?

6. Write 0.2 hours in minutes.

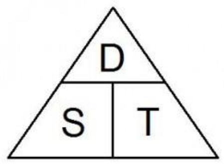
7. Write 180 minutes in hours.

Compound Units

Compound measures describe how one quantity changes in proportion to another.

Speed is measured in metres per second (m/s) or kilometres per hour (km/h). In the UK most cars still use miles per hour (mph).

Speed = Distance ÷ Time
This is written in a **formula triangle** as follows



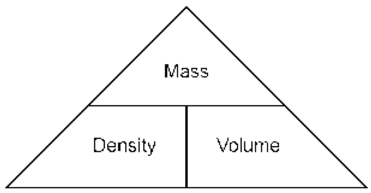
As you might expect S = Speed, D = Distance and T = Time. The equations that can be formed from the triangle are:
$\mathrm{S} = \frac{\mathrm{D}}{\mathrm{T}} \quad \left(\mathrm{Speed} = \frac{\mathrm{Distance}}{\mathrm{Time}}\right)$

$\mathrm{D} = \mathrm{S} \times \mathrm{T} \quad (\mathrm{Distance} = \mathrm{Speed} \times \mathrm{Time})$

$\frac{\mathrm{D}}{\mathrm{S}} = \mathrm{T} \quad \left(\frac{\mathrm{Distance}}{\mathrm{Speed}} = \mathrm{Time}\right)$

Try to see how the equations are formed from the triangle. That way you will only need to remember one triangle rather than three equations.

Density is a measure of the amount of mass per unit of volume, it is measured in grams per centimetre cubed ($\mathrm{g/cm^3}$) or kilograms per metres cubed ($\mathrm{kg/m^3}$). Don't worry too much about the definition, it is the formula triangle and the equations thatt are important.



$$\mathrm{Density} = \frac{\mathrm{Mass}}{\mathrm{Volume}}$$

$$\mathrm{Mass} = \mathrm{Density} \times \mathrm{Volume}$$

$$\frac{\mathrm{Mass}}{\mathrm{Density}} = \mathrm{Volume}$$

When carrying out calculations it is important to make sure that the units match up.

Example:
Calculate the average speed for a journey of 5m in 10 seconds.

The unit we use is metres per second.

$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{5}{10} = 0.5\text{m/s}$

Example:
Calculate the average speed for a journey of 16km that took 30minutes.

Here we use km per hour so we need to convert to hours.
30 minutes = 0.5 hours
$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = \frac{16}{0.5} = 32\text{km/hr}$

(Because $16 \div 0.5 = 16 \div \frac{1}{2} = 16 \times \frac{2}{1} = 32$)

Example:
Calculate the average speed for a journey of 40miles that took 20 minutes.

Here we will use miles per hour. We must multiply 20 minutes by 3 in order to get 1 hour (because 20 x 3 = 60) so we must also multiply 40 by 3 in order to get 120mph.

Or you can use the fact that 20minutes = $\frac{20}{60} = \frac{1}{3}$ hour
$\text{Speed} = \frac{\text{Distance}}{\text{Time}} = 40 \div \frac{1}{3} = 40 \times 3 = 120$

Example:
When the mass is 150kg and the volume is 2m^3, calculate the density.

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}} = \frac{150}{2} = 75\text{kg/m}^3$$

Example:
Calculate the distance travelled when a car travelling at 30km/hr completes a journey in 2 hours.

$\text{Distance} = \text{Speed} \times \text{Time} = 30 \times 2 = 60\text{km}$

Example:
Calculate the time taken to complete a journey of 500m at 5km/hr.

First you need to notice that the units do not match – we have m and km – so we must convert such that they are the same before we begin.

500m = 0.5km

$$\begin{aligned} \text{Time} = \frac{\text{Distance}}{\text{Speed}} &= \frac{0.5}{5} \\ &= 0.1\text{hours} \\ &= 0.1 \times 60 \text{ minutes} \\ &= 6 \text{ minutes} \end{aligned}$$

Activity 3.4 B

1. Calculate the average speed for each of these journeys
   a. 80km in 1 hour
   b. 3km in quarter of an hour
   c. 240 miles in half a day
   d. 20 miles in 8 hours
   e. 23km in 30 minutes

2. Calculate the time taken when 60 miles is travelled at an average speed of 30mph.

3. Find the distance travelled when a journey takes 30minutes with an average speed of 20km/hr

4. Calculate the density of an object with a mass of 250kg and a volume of $5\mathrm{m}^3$

5. Work out the density of an object with a volume of $6\mathrm{cm}^3$ and a mass of 360g

6. Find the mass of an object with a density of $25\mathrm{kg/m}^3$ and a volume of $3\mathrm{m}^3$

<u>3.5 Error Intervals</u>

When measurements are taken they are very rarely exact. The accuracy depends on the precision of the measuring instrument and the skill of the person taking the measurement. It also largely depends on whether the answer has been rounded.

We dealt with rounding earlier in the chapter; 3.141562 is far more accurate than 3.14 yet we always round answers to an appropriate degree of accuracy. This is determined by the question. For example a measurement of 5.4cm has an *implied accuracy* of 1 decimal place whereas a measurement of 6.39cm has an implied accuracy of 2 decimal places. You should give your answers to the same number of decimals places that have been used in the question.

Quantities are, therefore, assumed to be measured within an error interval such that

$$\text{measurement} - \frac{1}{2}\text{error interval} \leq \text{true value} < \text{measurement} + \frac{1}{2}\text{error interval}$$

Example:
The length of an object is given as 78.3cm, find the limits of accuracy of the length.

Here the implied accuracy is 1 decimal place, that is, the length is correct to the nearest 0.1cm.

$\frac{1}{2}\text{error interval} = \frac{1}{2} \times 0.1 = 0.05$

78.3 – 0.05 = 78.25

78.3 + 0.05 = 78.35

So the limits are 78.25 and 78.35. That is $78.25 \leq \text{length} < 78.35$

Example:
The weight of an object is given as 94.23kg, find the error interval of the weight.

Here the implied accuracy is 2 decimal places, that is, the weight is correct to the nearest 0.01kg.

$\frac{1}{2}\text{error interval} = \frac{1}{2} \times 0.01 = 0.005$

94.23 – 0.005 = 94.225

94.23 + 0.005 = 94.235

So the limits are 94.225 and 94.235. That is $94.225 \leq \text{weight} < 94.235$

When combining two different units that have been rounded, use the following table

	Maximum	Minimum
$X + Y$	$X_{max} + Y_{max}$	$X_{min} + Y_{min}$
$X - Y$	$X_{max} - Y_{min}$	$X_{min} - Y_{max}$
$X \times Y$	$X_{max} \times Y_{max}$	$X_{min} \times Y_{min}$
$X \div Y$	$X_{max} \div Y_{min}$	$X_{min} \div Y_{max}$

Example:
A bird travels 4.2m in 3.1 seconds. Both of these measurements have been rounded to 1dp. Find the upper and lower bound for the speed of the bird.

Speed = Distance ÷ Time

4.15≤distance<4.25
3.05≤time<3.15

Therefore the upper bound for the speed is $D_{max} \div T_{min} = 4.25 \div 3.05 = 1.39344 \ldots \text{m/s}$

The lower bound is $D_{min} \div T_{max} = 4.15 \div 3.15 = 1.31746 \ldots \text{m/s}$

Activity 3.5

1. Find the error intervals for each of these measurements
   a. 9.3m/s b. 6.3s c. 16.0km d. 0.867kg e. 7.896km
   f. 749.085m g. 86.9cm h. 4.78km i. 0.005kg j.7.432m

2. An object travels at 4.5m/s for 8.8 seconds. Both of these values have been rounded to 1dp, find the upper and lower bound for the distance travelled.

3. Given that X and Y have been rounded to 2dp with X = 9.21 and Y = 1.23 find the upper and lower bounds for each of these calculations
   a. X + Y b. X – Y c. X x Y d. X ÷ Y

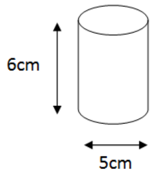
ASSIGNMENT THREE

Answers to these questions are not provided. You should send your work to your tutor for marking. No calculators are allowed unless specified – show all of your working.

1. Evaluate $(x^5)^3 \times x^{-3} \times x^{-2}$ Write your answer as a power of x (3)

2. Find the value of x when $4^2 \times 4^x \times 4^5 = 4^{11} \div 4^2$ (2)

3. Use the formula $\pi r^2 h$ to find the volume of this cylinder. Leave your answer in terms of pi. (1)



4. Here are some numbers:
$\pi = 3.141592653 \ldots$ $e = 2.718281828 \ldots$
Suzie noticed that $\pi \cong 3.14$ (2dp) and $\pi \cong 3.14$ (3sf), she concludes that rounding to 2 decimals places and 3 significant figures is the same thing. Is she correct? Explain how you know. (1)

5. Calculate the distance travelled if the average speed is 30mph the journey takes 2½ hours. (2)

6. An object has a density of 2kg/m^3, the volume is 6.2m^3
Assuming that density was rounded to the nearest integer and volume was rounded to 1dp, find the upper and lower bound of the mass. *You may use a calculator* (3)

7. A bag of sugar contained 90g of sugar. 35g was removed. Assuming both measurements are given to the nearest gram, what is the largest and smallest amount that could be left in the bag? (3)

8. A car travels 315 miles in 7 hours 30 minutes using 47 litres of petrol. What was the average speed? (2)

9. A calculator displays 84.2356. If the unit is £, what would your answer be? (1)

10. The number of tea bags in a box is 240 to the nearest 4. Give the error interval. (2)

11. A plane travels for 7hours at 350mph. How far has it flown? (2)

12. Simplify each of these as far as possible, make sure denominators are rationalised

a. $4\sqrt{5} + \sqrt{200} + 2\sqrt{20}$ b. $\frac{5}{\sqrt{8}}$ c. $\sqrt{147} + \sqrt{12}$ (5)

Total 27 marks

End of Section Revision Quiz

These questions are based upon everything you have covered in the number section of the course. It is not compulsory to complete this quiz but it is recommended that you use it in order to make sure you have a firm understanding of all of the topics covered.
Answers are provided to these questions – if you're finding a topic particularly difficult you should go back to that section in your notes and go over it to make sure you understand it fully.
The majority of these questions are exam style questions so you should familiarise yourself with them now. Do not use a calculator unless specifically told to.

1. How many centimetres are there in 4.6 metres?

2. Circle the fraction that is not equivalent to $\frac{5}{8}$

$\frac{25}{40}$ $\frac{15}{24}$ $\frac{30}{48}$ $\frac{10}{14}$ $\frac{20}{32}$

3. Rationalise and simplify $\frac{2}{3\sqrt{5}}$

4. Here are some numbers. Write the numbers in pairs so that the sum of each pair is the same: 7.5 12.4 9.7 12.6 17.5 15.3

5. Write 160g as a fraction of 2kg

6. Write $\frac{2}{x^2}$ in index form

7. Which of these numbers has exactly three factors? 7 9 10 17 20

8. Which of these numbers is 8 less than – 1.3? - 6.7 - 9.3 - 7.3 9.3 6.7 -7

9. Lucy says that 5 is odd and 2 is even so when you add a multiple of 5 to a multiple of 2 you always get an odd number. Is she correct? Write a calculation to support your answer.

10. Elliot earns £8.20 an hour. He works 23 hours a week for 45 weeks of the year. If he earns over £10,000 a year he has to pay tax – work out if he needs to pay tax or not.

11. Miss Smith works 4 days a week. If she drives to work she drives a distance of 10.3 miles a day in a car that uses 1 gallon of petrol for every 41.2 miles. Petrol costs £1.19 a litre – there are 4.5 litres in a gallon. If she catches the bus she can buy a weekly ticket for £5.25. Is it cheaper for her to drive or catch the bus?

12. Work out the calculation. Write your answer in its simplest form. $2\frac{3}{4} \times 1\frac{3}{8}$

13. 150 men and 70 women were asked if they drove to work. $\frac{1}{4}$ of the people asked said yes. $\frac{1}{5}$ of the men said yes. How many women said yes?

14. x^2 is always greater than x. True or false? Why?

15. Work out the square root of 100 million

16. Round 8188.08689 to the nearest hundred

17. Simplify as far as possible. All denominators should be rationalised.
$\frac{5\sqrt{2}}{\sqrt{32}} \times \frac{10}{2\sqrt{8}} + \frac{1}{\sqrt{2}}$

18. Given that X and Y have been rounded to 1dp with X = 4.2 and Y = 6.7 find the upper and lower bounds for each of these calculations

a. X + Y b. X – Y c. X x Y d. X ÷ Y

19. Write down all the factors of 42

20. Simplify $\frac{\sqrt{18} \times \sqrt{12}}{\sqrt{3}}$

21. Write down the value of the 4 in the answer to 546 x 10^3

22. Use each of the numbers 2, 4 and 6 only once to write down a calculation that equals 3

23. Work out $\frac{7 + 8 \times 2 - 1}{3 \times (2+1)}$

24. Work out $\frac{(-3) \times 7 + 4}{-2 \times -1}$

25. Write, in its simplest form, the fraction and percentage that is equivalent to 0.45

26. Write, in its simplest form, the fraction and decimal that is equivalent to 32%

27. Work out three fifths of forty five

28. Work out $\frac{3}{4} \div \frac{5}{6} + \frac{6}{7}$

29. Here are some test results. Which subject had the better mark?
English: 12 out of 20 Maths: $\frac{23}{50}$ Science 122 out of 200

30. Find the HCF and LCM of 110 and 132

31. Estimate the value of $20.43^2 - \frac{\sqrt[3]{1007}}{4.823}$ then use your calculator to find the exact answer

32. Given that $\sqrt{2} \cong 1.414$ and $\sqrt{5} \cong 2.236$ estimate $\sqrt{2} + 2\sqrt{5}$ to 3sf.

33. Write $0.4\dot{8}$ as a fraction in its lowest terms.

34. Put these in order from smallest to largest: 25% $\frac{1}{3}$ 0.28 $\frac{3}{5}$ 0.526 47%

35. Use a calculator to work out $\frac{3.92 \times 4.58}{4.869 + 0.15}$

a. Round your answer to 2 dp
b. Round your answer to 2 sf

36. Simplify $\frac{x^2 \times (x^3)^4}{x}$

Section Two: Algebra

Chapter Four: Notation, Expressions and Formulae

4.1 Notation and Expressions

Throughout this section you will come across words in bold – these are all words that you need to know the meaning of so pay extra attention to these.

Terminology

In maths we often come across unknown values known as **variables** these are represented by letters in what is known as algebra.

An **expression** is a collection of mathematical symbols, numbers and/or letters without an equal sign. This is in contrast to an **equation** – which we will deal with in more detail in chapter 6 – which does contain an equal sign.
For example $2x + 7$ is an expression whereas $2x + 7 = 10$ is an equation.

The **terms** in an expression or equation are the groups of symbols/numbers/letters that are separated by the +, - or = signs.
In the example above the terms are $2x$, 7 and 10.

The **coefficient** in a term is the number before the variable.
For example, the coefficient of $2x$ is 2. The coefficient of $10xy$ is 10.

Notation

It's important to remember that, in algebra, multiplication signs aren't written. If two values are to be multiplied together they are simply written next to one another. *Note: this is not true if you are using two numbers for example 4 x 5 would simplified to 20 before proceeding.*
It is common practice to write the letters in alphabetical order with the number at the beginning although it makes no difference to the calculation that is carried out.

The rules of notation are much better understood when seen so take a look through the following examples which illustrate the rules you need to know.

Example:
Multiplication
$a \times b = ab$
$a \times c \times b = abc$
$4 \times x \times y \times z = 4xyz$
$y \times 5 \times x \times z = 5xyz$

Repeated Addition
$m + m + m = 3m$
$a + a + a + a + a = 5a$
$a + a + a + b + b = 3a + 2b$

Powers/Indices
$x \times x = x^2$
$y \times y \times y \times y = y^4$
$aaa = a^3$

$a \times a \times b \times b = a^2b^2$
$a \times b \times b \times b \times b = ab^4$
$x \times y \times y \times x \times x = x^3y^2$

It is important to remember the difference between addition and multiplication
$x + x + x = 3x$
$x \times x \times x = x^3$

In algebra we do not use decimals - although you may convert to them when you're doing your working out if you prefer calculating with them. All coefficients should be written as fractions. Fractions are also used in place of a division sign.

Example:
$a \div b = \frac{a}{b}$

$x \div y = \frac{x}{y}$

$n \div 2 = \frac{n}{2} \; or \; \frac{1}{2}n$

$x \div 5 = \frac{x}{5} \; or \; \frac{1}{5}x$

$3 \div n = \frac{3}{n}$

$10 \div x = \frac{10}{x}$

Example:
$3 \div 4n = \frac{3}{4n}$

$3n \div 4 = \frac{3n}{4} = \frac{3}{4}n$

Algebraic logic is frequently used to solve word problems so converting from words to expressions is an important skill.

Example:
$n \text{ add } 4 = n + 4$

$n \text{ multiplied by } 8 = 8n$

$x \text{ divided by } 10 = \frac{x}{10}$

m multiplied by itself $= m^2$

x multiplied by 3 add $2 = 3x + 2$

Activity 4.1 A

1. Write each of these expressions correctly in algebraic notation

a. $a \times b \times c \times d \times e$ b. $b \div a$ c. $d \times f \times e \times 4$ d. $a \times c \times 4b$

e. $h + h + h + h$ f. $5x \times yz$ g. $x \times x \times y$ h. $8 \div v$

2. Write algebraic expressions for each of these descriptions

a. Half of x b. 8 plus y c. s times 7 d. 6 multiplied by w
e. Double t f. y divided by 2 g. Three quarters of x h. k multiplied by itself
i. 6 multiplied by r add 7 j. e divided by 6 subtract 8

3. Bob has x sweets. Dave has twice as many sweets as Bob. Fred has 5 less sweets than Dave. Write an expression for the number of sweets each person has.

4. Tea costs 40p, Coffee costs 50p, Cake costs 70p and Fruit costs 20p.
Write an expression for the cost of:

a. x teas and y coffees
b. a cakes, b coffees and c teas
c. x pieces of fruit and y cakes

Simplifying Expressions – Collecting Like Terms

Expressions, and indeed equations, are simplified by collecting like terms. Like terms are terms whereby the letter or symbol are the same.

Example:
Simplify each of the following expressions
a. $2a + 5a$ b. $8\pi + 5\pi + \pi$ c. $6y - 4y$ d. $10a + 5a - 3a$

a. $7a$ b. 14π c. $2y$ d. $12a$

The sign stays with the term it is in front of. For example $x - y = -y + x$

Example:
Simplify $3a + 4b - 2a + 5b$

Watch out for minus signs!

Collect *a* and *b* terms separately:
$3a - 2a = a$ *When the coefficient is 1 is it omitted* $1a = a$
$4b + 5b = 9b$

So we have $a + 9b$

Example:
Simplify $6x + 4y + 2x - x - 5y$

$6x + 2x - x = 7x$
$4y - 5y = -y$

The simplified expression is $7x - y$

Terms can only be collected together if they are alike. For example $2a + 4b - 5$ can't be simplified because there are no like terms

Example:
Simplify $3a + 4b - 5ab + 2a + 4b$

$3a + 2a = 5a$
$4b + 4b = 8b$

Note that *ab* is completely separate to *a* and *b*

We have $5a + 8b - 5ab$

As above where a, b and ab are separate terms are only alike if they have the same base and power i.e. x^2 and x^3 are not alike because the powers are different, similarly x^2 and y^2 are note like terms because the bases are different.

Example:
Simplify a. $x^2 + 3x^2$ b. $x^3 + x^2 + 2x^3 - x^2$
a. $4x^2$ b. $x^3 + 2x^3 = 3x^3$ $x^2 - x^2 = 0$ So it simplifies to $3x^3$

Example:
Simplify $4x^2 + y^2 - 2x^2 + x + 3y^2 + x^3$

$4x^2 - 2x^2 = 2x^2$
$y^2 + 3y^2 = 4y^2$
x and x^3 have no like terms

So we have $x^3 + 2x^2 + x + 4y^2$

Division is always written as a fraction and simplified as normal.

Example:
Simplify $2x \div 10$

$2x \div 10 = \frac{2x}{10} = \frac{x}{5}$

Example:

Simplify $\frac{3x^3 \times 4y^2}{2xy}$

Treat the numbers and letters separately. The rules of index notation you met in chapter three apply.

$$\frac{3x^3 \times 4y^2}{2xy} = \frac{12x^3y^2}{2xy} = 6x^2y$$

Because: $\frac{12}{2} = 6$, $\frac{x^3}{x} = x^3 \div x = x^{3-1} = x^2$ and $\frac{y^2}{y} = y^3 \div y = y^{2-1} = y$

Activity 4.1 B

1. Simplify each of the following expressions
   a. $4x + 4x + 3x$ b. $a + 4a + 2a$ c. $2x + 3y + x - y$ d. $a + b + a + b$
   e. $7x + 8y + 10xy - 3x$ f. $4 + 3a + 2 + 4b$ g. $3xy - 4x - 2xy + 9x$
   h. $b + 3a + b$ i. $7a - 6ab + b + 3a$ j. $f + f + g + 5 + g$

2. Daisy says $3a + 4b + 2a + 3b = 12ab$ is she correct?

3. Simplify each of the following expressions
   a. $x^2 + x^2 + 2x^2$ b. $x + x^2 + 3x - x^2$ c. $a + a^2 + 3a + 5a^2$
   d. $x^2 + y^2 + x + 2y + 4x^2 - 2y^2 + 4x - 3$

4. Abbie says $x + x + x = x^3$. Amy says $x + x + x = 3x$. Which one of them is correct?

5. Simplify each of the following expressions
   a. $a \times a \times a$ b. $x \times y \times 3$ c. $x \times x \times x \times y$ d. $3 \times a \times 2 \times b$
   e. $\frac{4x^2y}{2xy}$ f. $\frac{10x^6y^6z^2}{2x^2y^4z}$ g. $10x \div 5x$ h. $4x \div 10xy$

6. Simplify each of these algebraic fractions. All usual fraction rules apply.
   a. $\frac{x}{4} + \frac{x}{4}$ b. $\frac{2y}{3} + \frac{4y}{3}$ c. $\frac{x}{2} + \frac{x}{4}$ d. $\frac{2}{x} + \frac{3}{x}$ e. $\frac{7}{y} - \frac{2}{y}$
   f. $\frac{3}{2x} + \frac{1}{x}$ g. $\frac{1}{x} + \frac{1}{x^2}$ h. $\frac{a}{y} + \frac{b}{y^2}$ i. $\frac{3}{a} + \frac{2}{b}$ j. $\frac{a}{x} - \frac{b}{y}$
   k. $\frac{2}{x} + \frac{3}{y^2}$ l. $\frac{5}{x} \times \frac{2}{y}$ m. $\frac{1}{x} \div \frac{x}{3}$ n. $\frac{x}{y} \div \frac{y}{x}$ o. $\frac{2}{a} \times \frac{1}{2} \times \frac{3}{y}$

4.2 Expanding and Factorising 1

Expressions that contain a bracket always have the coefficient written in front. For example you would write $3(x+2)$ not $(x+2)3$.
In this section we will deal with single brackets only, double brackets will be covered in section 4.5 of this chapter.

Expanding Single Brackets

An expression of this form can be expanded (or "multiplied out") by multiplying everything outside the bracket by everything inside.

Example:
Expand $6(x+3)$

Multiply what's outside the bracket by everything inside the bracket

$$6(x+3)$$

So we do $6 \times x = 6x$ and $6 \times 3 = 18$
Therefore $6(x+3) = 6x + 18$

Example:
Expand each of the following expressions
a. $5(x-8)$ b. $x(x+2)$ c. $7x(x-1)$ d. $a(a+b)$ e. $2(x+y+1)$

a. $5 \times x - 5 \times 8 = 5x - 40$
b. $x \times x + x \times 2 = x^2 + 2x$
c. $7x \times x - 7x \times 1 = 7x^2 - 7x$
d. $a \times a + a \times b = a^2 + ab$
e. $2 \times x + 2 \times y + 2 \times 1 = 2x + 2y + 2$

Only the terms that are "connected" to the bracket are multiplied by the contents as illustrated in the following two examples.

Example:
Expand and simplify $5 - 4(x+3)$

Be careful of the minus sign!
$4(x+3) = 4x + 12$
So we have
$5 - (4x + 12)$ *You're subtracting the whole bracket, not just the first term. It is not* $5 - 4x + 12$
$= 5 - 4x - 12$
$= -4x - 7$

Or you can think of this as having a -4 outside the bracket so
$-4(x+3) = -4x - 12$
So we have $5 - 4x - 12 = -4x - 7$ as before

Example:
Expand and simplify $5(x+1) - 8(x-2)$

$5x + 5 - 8x + 16$ *Notice it is +16 because -8 x -2 = +16*
$= 3x + 21$

Factorising Single Brackets

Factorising an expression means putting it into brackets, this is done by taking out the highest common factor. This can be a number, a letter or a combination of both.
Essentially, expanding is the opposite of factorising.

Example:
Factorise $3x + 9$

Here we can see than both terms are divisible by 3 so we divide each term by 3
$3x \div 3 = x$
$9 \div 3 = 3$

These answers remain in the bracket whilst the 3 goes on the outside so,
$3x + 9 = 3(x + 3)$

If you're unsure about your answer, try expanding it again to see if you get the expression you started with.

Example:
Factorise $x^2 + 4x$

The highest common factor in this expression is x
$x^2 \div x = x$
$4x \div x = 4$

So we have $x^2 + 4x = x(x + 4)$

Example:
Factorise $8x^2y^3 - 16xy^2$

To find the highest common factor deal with each part separately
Numbers: The HCF of 8 and 16 is 8
x: The HCF of x^2 and x is x
y: The HCF of y^3 and y^2 is y^2 (the lower of the two powers)

So the highest common factor is $8xy^2$
$8x^2y^3 \div 8xy^2 = xy$
Again, you would do each part separately:

$8 \div 8 = 1$
$x^2 \div x = x$
$y^3 \div y^2 = y$

$16xy^2 \div 8xy^2 = 2$
Because $16 \div 8 = 2$, $x \div x = 1$ and $y^2 \div y^2 = 1$

So we have $8x^2y^3 - 16xy^2 = 8xy^2(xy - 2)$

Example:
Factorise each of these expressions
a. $7x^2 + 21x$ b. $2x + 2$ c. $4a^4b - 2ab$ d. $3x^2y + 6xy - 9x^2$

a. $7x(x + 3)$ b. $2(x + 1)$ c. $2ab(2a^3 - 1)$ d. $3x(xy + 2y - 3x)$

Activity 4.2

1. Expand each of the following expressions

a. $5(x + 2)$ b. $10(x + 3)$ c. $2(3x - 1)$ d. $3(6x - 2)$ e. $x(x + 2)$

f. $x(x + 5)$ g. $x(x + y)$ h. $2x(x - 4)$ i. $ab(a - b)$ j. $3x(2x - y)$

k. $-2(x + 4)$ l. $-3(x - 3)$ m. $-x(x + 2)$ n. $-2x(2x - 3)$ o. $x^2(x + 3)$

2. Expand and simplify each of these expressions

a. $3(x + 3) + 2(x + 1)$ b. $2(2x - 1) + 8(x + 3)$ c. $5(3x + 4) - 9(x + 1)$

d. $7(x + 5) - 6(x - 6)$ e. $4x(x - 2) - 2x(x - 1)$ f. $a(a + b) - b(a - b)$

g. $x^2(x + 1) - x(x^2 + 1)$ h. $x(y - 2) + x^2(x + y^2)$ i. $a^2b(a + 2) + ab(a + b^2)$

3. Factorise each of these expressions

a. $5p + 10$ b. $6x - 3$ c. $2x + 2$ d. $10x - 100$ e. $88x + 22$ f. $6x - 4$

g. $5x + 2xy$ h. $3x^2 - 2x$ i. $x^2 + x$ j. $6x^2 - 4x$ k. $abc - 3a$ l. $3x^3 - x^2$

m. $2xy + 4x^2$ n. $5ab^2c^3 - 10a^5b^4c$ o. $x^2y - xy^2$ p. $mickey + minnie$

4. Three friends are doing their maths homework and are trying to factorise $16x^2 + 4x$. Amelia says it is $2x(8x^2 + 2)$, Yasmine says $4x(4x + 1)$ and Callie says $2(x^2 + 2x)$. Who is correct?

5. Simplify each of these algebraic fractions. Multiply out all brackets.

a. $\frac{3}{x} + \frac{5}{x+3}$ b. $\frac{1}{y} - \frac{x}{x+1}$ c. $\frac{x+2}{x+1} \times \frac{3}{x} \times \frac{1}{y}$ d. $\frac{3}{x+4} + \frac{4}{xyz}$ e. $\frac{y+2}{x+2} - \frac{x}{y}$

4.3 Substitution

Formulae

A **formula** is an equation involving one or more variable. They usually, although not always, have a real life application. Formulae are often used to find the value of one variable when given the values of one or more others.

When you are given values to substitute into a formula write the formula out again replacing the variables you know the value of with their numerical equivalent before you set about finding the value of the remaining one.

Example:
The cost of printing, C, is calculated by the formula $C = 25 + 5n$ where n is the number of leaflets. Find the cost of producing 100 leaflets.

$C = 25 + 5 \times n$ First write the formula out
$C = 25 + 5 \times 100$ then write it again with the known values substituted in
$C = 25 + 500$
$C = 525$

Example:
Using the formula $N = 2x + y^2$ find the value of N when $x = 3$ and $y = -2$

$N = 2 \times x + y^2$
$N = 2 \times 3 + (-2)^2$
$N = 6 + 4$
$N = 10$

Example:
Using the formula $p = qr$ find the value of r when $p = 10$ and $q = 5$

$p = q \times r$
$10 = 5 \times r$
$\therefore r = 2$

Note: capital and lower case letters represent different values. For example it can not be assumed that A and a are the same value.

Activity 4.3 A

1. Using the formula $v = u + at$, find the value of v, when:
a. $u = 2, a = 3, t = 1$ b. $u = 5, a = 4, t = 2$ c. $u = 10, a = -2, t = 2$

2. Find the value of A, using the formula, $A = -a + b$ when:
a. $a = 2, b = 4$ b. $a = 5, b = -3$ c. $a = -5, b = 3$ d. $a = -1, b = -4$

3. Use the following formulae to find the value of y when $x = 4$

a. $y = 3 + 2x$ b. $y = x^2$ c. $y = 2x^2$ d. $y = \frac{5x}{10}$ e. $y = \sqrt{x}$

f. $y = \frac{x+2}{x-1}$ g. $y = \frac{\sqrt{x}}{x}$ h. $2y = x + 4$ i. $y = \frac{2x-3}{x^2}$ j. $y = x^2 + 3x - 1$

4. The formula $F = \frac{9}{5}C + 32$ is used to convert temperatures from °F to °C. Convert 20°C to °F.

5. Use the following formulae to find the value of x when $y = 10$

a. $y = x + 8$ b. $y = 2x$ c. $y = \sqrt{x}$ d. $y = \frac{x}{4}$ e. $y = \frac{60}{x}$

Identities

An **identity** is an equation that is true for every value of the unknowns. For example, $2x = y$ is not an identity because it is true for some values of x and y but not for all of them. On the other hand, $4x + 2x + y = 6x + y$ is an identity because it is true for all values of x and y.
In these cases the = sign is replaced by ≡ which means "identical to".

Questions involving identities usually involve proving that they are true or false. In order to prove that an identity is false it is sufficient to provide just one example of a case in which it does not hold but algebraic logic, rather than examples, must be used to prove that an identity is true.

Example:
Prove that $5x + 2x \equiv 7x$ is true

$$5x + 2x \equiv (x + x + x + x + x) + (x + x)$$
$$\equiv 7x$$

Or
$$5x + 2x \equiv (5 + 2)x$$
$$\equiv 7x$$

Example:
Prove that $2x + 4 \equiv 4x + 2$ is false

When $x = 2$ the left hand side gives $2 \times 2 + 4 = 4 + 4 = 8$,
the right hand side gives $4 \times 2 + 2 = 8 + 2 = 10$

$8 \neq 10 \therefore 2x + 4 \not\equiv 4x + 2$ $\not\equiv$ means "is not identical to"

Activity 4.3 B

1. Decide whether each of these identities are true or false and prove it.

a. $6x + 2y \equiv 8xy$ b. $2x + 2x \equiv 4x^2$ c. $4a + 2b - 2a \equiv 2(a + b)$

d. $3(x + 1) \equiv 3x + 3$ e. $15x - 10 \equiv 5x$ f. $x(x + 3) \equiv x^2 + 3$

2. Decide whether each of these statements are true or false and prove it.

a. The sum of two prime numbers is always odd

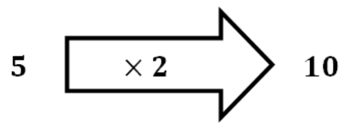
b. The product of two consecutive numbers is always even.
c. Any even number squared gives a multiple of 4.
d. Adding any integer to 9 gives an odd number
e. The sum of two odd numbers is even

4.4 Rearranging

Function Machines

A function can be thought of as a machine that converts an input value into an output value.

Example:

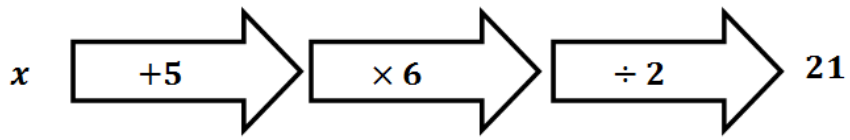


This represents the sum 5 x 2 = 10

Function machines are useful when trying to "work backwards" to find the original number or when trying to rearrange an equation to change the **subject**. The subject of an equation is the variable that is alone on one side, for example, in $y = mx + c$ the subject is y.

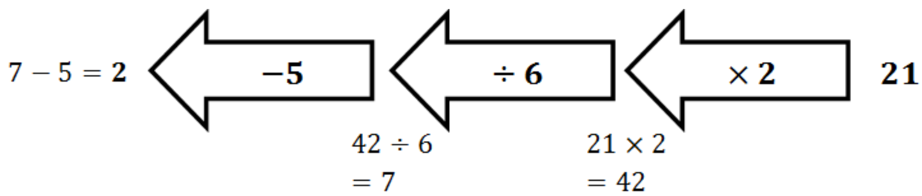
Example:
I'm thinking of a number, I add 5, multiply by 6 then divide by 2. The answer is 21. What number did I start with?

First draw a function machine, the unknown number can be represented by an empty box or any letter that you choose. Here we will use x.



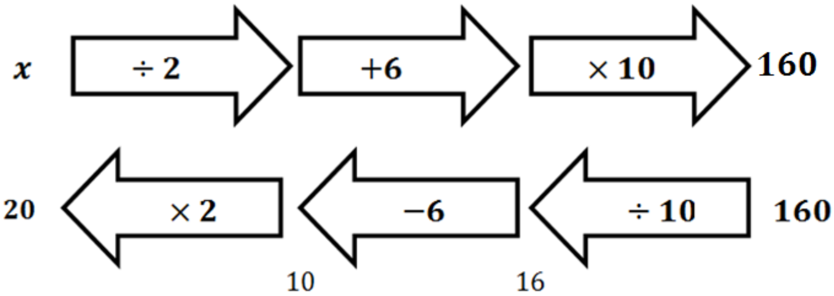
The next step is to repeat the function machine backwards where each step is the opposite of the original.

You need to know that add and subtract are inverses of each other, as are multiply and divide.



So the original number was 2.

Example:
I'm thinking of a number. I half it, add 6 then multiply by 10. My answer is 160, what number did I start with?



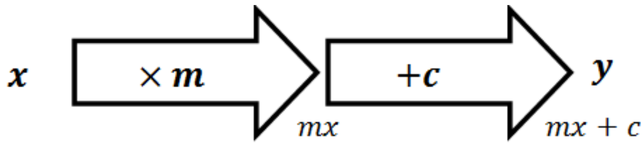
So the original number was 20.

Rearranging Equations

Function machines can also be used to rearrange equations – this means changing the equation to make the desired variable the subject.

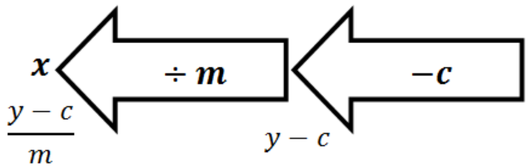
Example:
Rearrange $y = mx + c$ to make x the subject.

Before you start you have to do is work out what order the operations are carried out using BIDMAS. In this case the first thing you do to x is multiply it by m to give mx, the next thing is adding c which gives $mx + c$ which is an expression equal to y



The steps are written underneath so you can see the construction of the equation, it is not necessary to write these on the first function machine.

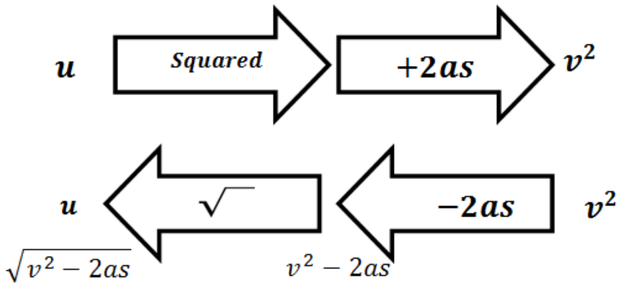
We now go the opposite direction using the inverses as before.



So the rearranged equation is $x = \frac{y-c}{m}$

Example:
Make u the subject of $v^2 = u^2 + 2as$

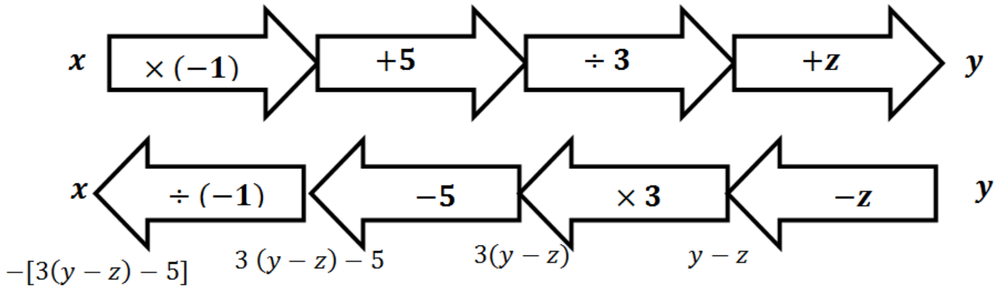
The first thing you do to u is square it to get u^2, the second thing is to add $2as$ to give $u^2 + 2as$. Drawing the function machine in both directions gives:



So the rearranged equation is $u = \sqrt{v^2 - 2as}$

Example:
Make x the subject of the equation $y = \frac{5-x}{3} + z$

Here you have to be careful to notice that the x is negative. In order to achieve $5 - x$ you are actually multiplying x by -1 and then adding 5; $5 - x$ is the same as $-x + 5$, writing expressions in a different order has no effect on their value as long as the sign remains the same.



Square brackets are used when there are already regular brackets in the expression.
$x = -[3(y - z) - 5]$
$x = -3(y - z) + 5 \ \ or \ \ 5 - 3(y - z)$

As you have seen function machines can be used to rearrange equations, provided you can determine the order of operations. Another method is algebraic rearranging whereby you “get rid of” the terms you don’t need leaving the new subject alone on one side of the equation.
Either method will work for all equations, it is up to you to choose which one you prefer. See the examples below which rearrange the same equations as above using a different method.

Example:
Rearrange $y = mx + c$ to make x the subject.

The x is on the right hand side so, for now, we shall leave it there.
The first thing you need to do is remove any terms without an x, in this case that is the c.
On the RHS it is $+c$ so we’ll do the opposite $(-c)$ when we move it to the LHS.

$y - c = mx$

Now the only thing left that we don't want is the m. This is connected to x by multiplication so we will do the opposite $(\div m)$ on the LHS.

$$\frac{y-c}{m} = x$$

So now we have a rearranged equation, $x = \frac{y-c}{m}$ as before.

Example:
Make u the subject of $v^2 = u^2 + 2as$

As above first we remove the terms without u in, in these case this is $+2as$ so we will subtract this from the LHS
$v^2 - 2as = u^2$

Now all that remains is to remove the power, since it is squared we will square root the LHS. (Note: dividing by u will leave just a single u on the RHS as required but there will also be one on the LHS so it is not the subject of the equation.)

$$\sqrt{v^2 - 2as} = u$$

So we have $u = \sqrt{v^2 - 2as}$ as before

Example:
Make x the subject of the equation $y = \frac{5-x}{3} + z$

There are two ways to deal with this equation, both will be illustrated here.
Option 1:
As before remove all terms without an x in first, in this case the $+z$

$$y - z = \frac{5-x}{3}$$

Now we need to get rid of the fraction before we can get to the x, since everything on the RHS is divided by 3 we multiply everything on the LHS by 3 to give

$3(y-z) = 5 - x$ or $3y - 3z = 5 - x$

We now have another term on the LHS that doesn't contain an x so the $+5$ has to be moved giving

$3(y-z) - 5 = -x$ or $3y - 3z - 5 = -x$

Remember the sign stays with the term but we wanted just x not $-x$ so the negative sign must be removed. Technically this is done by dividing by -1 but the easiest way to think of it is to simply change the sign of every term in the equation, remembering that $3(y-z)$ is one term so the signs inside the bracket remain the same.

$-3(y-z) + 5 = x$ or $-3y + 3z + 5 = x$

Option 2:
Another way to deal with this equation is to get rid of the fraction first – many people prefer this method because it gets dealing with the fractions over and done with at the beginning!

The denominator is 3 so multiplying by 3 will get rid of it but you need to make sure you remember to multiply every term by 3.

$3y = (5 - x) + 3z$ Notice that the numerator remains the same: $\frac{5-x}{3} \times 3 = 5 - x$

Now we proceed as before, $3y = 5 - x + 3z$ has two terms without an x, the $+3z$ and the $+5$ so these are subtracted from the other side.

$3y - 3z - 5 = -x$

As before we change the sign of the x by changing the sign of every term to give
$-3y + 3z + 5 = x$

Pay extra attention to questions 5 and 6 in the following exercise as they illustrate what you should do if the desired subject appears more than once.

Activity 4.4

1. Write down the inverse operation to each of these

a. $+5$ b. -10 c. $\times 18$ d. $\div 29$ e. $\times -5$ f. $\sqrt{\ }$

2. I'm thinking of a number, I double it, add 50 then subtract 12. My answer was 58. What was my number?

3. I think of a number, I halve it, add 97 then double it. My answer is 200. What was my number?

4. Rearrange each of these equations to make x the subject. You can use whichever method you prefer but it is recommended that you try using both until you decide which works best for you.

a. $y = 3x + 2$ b. $y = \frac{2x}{z}$ c. $y = x + z^2$ d. $y = \frac{x}{2} + 3$

e. $y = z - x$ f. $3y = 8x + 1$ g. $y = \frac{x}{4} + \frac{1}{2}$ h. $10y = x + 4$

i. $y = \frac{x+3}{5}$ j. $5x = y + 2$ k. $y = \frac{2x-5}{8}$ l. $y = \frac{4-x}{9}$

m. $y = x^2 + 8$ n. $y^2 = x^2 - 7$ o. $y = \frac{x^2}{8}$ p. $y = \frac{x^2-9}{10}$

q. $y = \frac{z-x^2}{9}$ r. $y = \sqrt{x}$ s. $y = \sqrt{x+2}$ t. $y = \sqrt{2x-3}$

*5. Make x the subject of these formulae

a. $xy - wx = yz$ b. $5x + xy = 7$ c. $4xy + 5x - xz = y$

*6. Make x the subject of these formulae

a. $xy + z = 2x$ b. $x + 2 = 4x - 4$ c. $xy + 4 = xz - 1$ d. $xy - z = t - px$

e. $y = \frac{x+1}{x-1}$ f. $y(x - z) = z(x + y)$ g. $\frac{x+y}{x+1} = z$ h. $\sqrt{\frac{x}{y}} = z$

i. $y = 2z^2 + 2z\sqrt{x^2 + z^2}$

<u>4.5 Expanding and Factorising 2</u>

We dealt with expanding and factorising a single bracket in section 4.2, here we deal with double brackets.

<u>Expanding Double Brackets</u>

There are several ways of expanding double brackets, the grid method is arguably the easiest. The examples below illustrate the two main technique work through each of them so you can decide for yourself which you prefer.

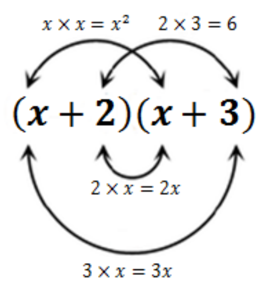
Example:
Expand and simplify $(x + 2)(x + 3)$

Option 1 – <u>Grid Method</u>

$\times$	x	$+2$
x	x^2	$+2x$
$+3$	$+3x$	$+6$

$$(x + 2)(x + 3) = x^2 + 2x + 3x + 6$$
$$= x^2 + 5x + 6$$

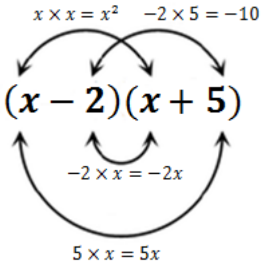
Option 2 – <u>Smiley Face</u>



As before $(x + 2)(x + 3) = x^2 + 5x + 6$

Example:
Expand and simplify $(x-2)(x+5)$
Using either method we have

$\times$	x	-2
x	x^2	$-2x$
$+5$	$+5x$	-10

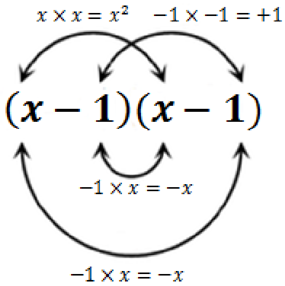


$$\begin{aligned}(x-2)(x+5) &= x^2 - 2x + 5x - 10\\ &= x^2 + 3x - 10\end{aligned}$$

Example:
Expand and simplify $(x-1)^2$

$(x-1)^2 = (x-1)(x-1)$ so continuing as before, using either method, we have

$\times$	x	-1
x	x^2	$-x$
-1	$-x$	$+1$



$$\begin{aligned}(x-1)^2 &= x^2 - x - x + 1\\ &= x^2 - 2x + 1\end{aligned}$$

<u>Factorising Double Brackets</u>

Quadratic expressions (those where the highest power is squared) are factorised into two brackets by finding a pair of numbers that multiply to make the result of the coefficient of x^2 multiplied by the number with no variable and which sum to make the coefficient of the x term. (*Note: x could be any letter, the rules remain the same.)*

Quadratic expressions should be rearranged so that they're of the form $ax^2 + bx + c$ where a, b and c can be any numbers.
The pair of numbers you're looking for multiply to make ac and sum to make b.

When $a = 1$ (that is, when there is no coefficient of x^2) $ac = c$ (since $1 \times c = c$) so you're looking for a pair of numbers that multiply to make c and sum to make b.

The pair of numbers are placed into the brackets:

$$(x + ?)(x + ?)$$

It is worth remembering that these could be positive or negative and that it makes no difference which way round you write the brackets.

Example:
Factorise $x^2 + 3x + 2$

We're looking for numbers that multiply to make 2 and sum to make 3.

There is only one pair of numbers that multiply to make 2:
1 x 2 = 2
These also sum to make 3:
1 + 2 = 3

So we place the numbers 1 and 2 into the brackets to get
$x^2 + 3x + 2 = (x + 1)(x + 2)$

Example:
Factorise $x^2 - 11x + 10$

Here we need a pair of numbers that multiply to make 10 and sum to make -11

The numbers that multiply to make 10 are 1x10 or 2x5
However neither 1 + 10 nor 2 + 5 equal -11

The pair that looks most likely to be helpful is 1 and 10 because we have 1 + 10 = 11 so we'll turn our attention to this pair.
In order for them to equal -11 we would need them to both be negative $(-1) + (-10) = -11$
Since $(-1) \times (-10) = 10$ as required this is the pair of numbers we need

Therefore we have $x^2 - 11x + 10 = (x - 1)(x - 10)$

You can check your answers by expanding the brackets, if your expansion is the expression you started with then you can assume your answer is correct.

Example:
Factorise $y^2 + 2y - 3$

Here we need a pair of numbers that multiply to make -3 and sum to make 2.

There is only one pair of numbers that multiply to make 3:
1 x 3 = 3
However there are two pairs that multiply to make -3:
$-1 \times 3 = -3$ or $1 \times -3 = -3$

Turning our attention to the sum of the numbers we have:
$-1 + 3 = 2$ and $1 + -3 = -2$

So the pair we need are -1 and 3 which gives

$y^2 + 2y - 3 = (y - 1)(y + 3)$

When there is a coefficient of x^2 you're looking for a pair of numbers that multiply to make ac and add to make b. Unfortunately in these cases it isn't as easy as simple as placing the numbers inside the brackets. You should replace bx with the values you have found, insert brackets and then factorise to see what the contents of the two brackets should be.
The method is illustrated in the examples below.

Example:
Factorise $2x^2 + 5x + 3$

Here we're looking for a pair of numbers that add to make 5 and multiply to make 6
(because ac = 2 x 3 = 6)

Therefore we will use 2 and 3
These have to replace the value in the middle, it doesn't matter which way round you put them in.
$2x^2 + 2x + 3x + 3$
Now we insert brackets around the first two terms and the second two terms
$(2x^2 + 2x) + (3x + 3)$
The next step is to factorise each of these brackets.
$2x(x + 1) + 3(x + 1)$
You know you're on the right lines if each of your brackets are the same, if they're different go back and figure out why!
Now we compose our pair of brackets. One will be the brackets from the expression, the other will be the terms on the outside
$(2x + 3)(x + 1)$

Example:
Factorise $2x^2 + 9x + 10$

We need a pair of numbers that add to make 9 and multiply to make 20.
4+5=9 4x5=20
Putting these in instead of the middle term gives
$2x^2 + 5x + 4x + 10$
Insert brackets
$(2x^2 + 5x) + (4x + 10)$
Factorise
$x(2x + 5) + 2(2x + 5)$
Compose the double brackets
$(x + 2)(2x + 5)$

When there is a negative sign included you have to be careful when it comes to adding the brackets in. If the sign in front of the second bracket is negative you need to change the sign inside the bracket. The easiest way to think of this is you need to change the sign in the second bracket to make sure it matches the sign in the first bracket.

Example:
Factorise $3x^2 - 10x + 8$

We need a pair of numbers that add to make – 10 and multiply to make 24
This is – 6 and – 4.

Putting these values in gives
$3x^2 - 6x - 4x + 8$
When adding the brackets be careful to make sure the signs are correct – the second bracket should have the same sign as the first bracket.
$(3x^2 - 6x) - (4x - 8)$
Factorise
$3x(x - 2) - 4(x - 2)$
$(3x - 4)(x - 2)$

Difference of Two Squares

Finding the difference of two squares also involves factorising into two brackets but it relies upon you being familiar enough with all of the square numbers so that you can recognise the expression.

The difference of two squares refers to an expression with two squared terms separated by a subtraction sign. For example $x^2 - 16$ is the difference of two squares because both x^2 and 16 are square numbers.

These are factorised by finding the square root of each of the two terms and putting them into brackets of the form
$(\boldsymbol{x} + ?)(\boldsymbol{x} - ?)$
Notice the different sign in each bracket.

Example:
Factorise $x^2 - 16$

$\sqrt{x^2} = x$
$\sqrt{16} = 4$

So we have
$x^2 - 16 = (x + 4)(x - 4)$

Example:
Factorise $4x^2 - 49$

$\sqrt{4x^2} = 2x \ \ because\ \sqrt{4} = 2\ and\ \sqrt{x^2}$
$\sqrt{49} = 7$

So $4x^2 - 49 = (2x - 7)(2x + 7)$

Activity 4.5

1. Expand and simplify
   a. $(x + 2)(x + 6)$ b. $(x + 5)(x + 3)$ c. $(y + 4)(y + 10)$ d. $(x - 1)(x + 3)$

e. $(x+3)(x-4)$ f. $(y-3)(y+1)$ g. $(y-2)(y+4)$ h. $(x-2)(x-10)$

i. $(x-3)(x-4)$ j. $(x-5)(x-4)$ k. $(c-2)(c-4)$ l. $(x-10)(x-5)$

m. $(x+2)^2$ n. $(x+4)^2$ o. $(x-2)^2$ p. $(x-10)^2$

2. Expand and simplify

a. $(2x+1)(x+3)$ b. $(3x+2)(x+5)$ c. $(x-2)(4x+1)$ d. $(x+2)(2x-1)$

e. $(2x+1)(3x+3)$ f. $(3x+4)(10x+2)$ g. $(3y-2)(2y-2)$ h. $(2a+1)(a-1)$

3. Expand and simplify

a. $(x+y)(x+2)$ b. $(2x+3)(x-y)$ c. $(x+y)(x+z)$ d. $(y-x)(x-y)$

4. Expand and simplify. *Hint: deal with one pair first then think about the third bracket*

a. $(x+2)(x+1)(x+3)$ b. $(x-2)(x+3)(x+4)$ c. $(x-2)^3$

d. $(2x+1)(x+2)(x+3)$ e. $(x+y)(x-y)(x+1)$ f. $(x+1)^2(x+2)$

5. Expand and simplify

a. $(2-\sqrt{2})(4+\sqrt{2})$ b. $(3+\sqrt{5})(1-\sqrt{5})$ c. $(2+\sqrt{7})(1-2\sqrt{7})$ d. $(3-2\sqrt{3})^2$

6. Factorise

a. $x^2+8x+15$ b. $x^2+9x+14$ c. x^2+5x+4 d. $y^2+10y+25$

e. x^2+4x-5 f. x^2+x-12 g. y^2-5y+6 h. $x^2-7x+12$

i. $x^2-3x-40$ j. $x^2-4x-12$ k. $a^2-9a-10$ l. $x^2+4x-21$

m. b^2-b-2 n. x^2+9+6x o. $8x+x^2+16$ p. $y^2-24y+23$

q. x^2-9 r. n^2-25 s. y^2-81 t. $36-x^2$ u. a^2-b^2 v. x^2-1

7. Factorise

a. $2x^2+5x+2$ b. $2x^2+7x+3$ c. $2x^2+13x+20$ d. $5x^2+13x+6$

e. $7x^2+15x+2$ f. $3x^2+14x+15$ g. $4x^2-20x+25$ h. $6x^2-13x+6$

i. $7x^2-5x-150$ j. $3-5x-2x^2$ k. $2x^2-11x+5$ l. $3x^2-10x+8$

m. x^3-25x n. x^4-7x^2+12 o. $10xy+20x^2y^2$ p. $x^2-\frac{1}{4}$

8. Rearrange to make x the subject

a. $\frac{2}{x+2}=\frac{y}{x-1}$ b. $\frac{2y-1}{4}=\frac{x}{x+2}$ c. $\frac{3}{x^2-y}=\frac{2}{y^2-2x^2}$

9. Simplify as far as possible

a. $\frac{5}{2x^2+5x-3}+\frac{3}{x^2+3x+2}$ b. $\frac{2x^2-17x+21}{4x^2-9}$ c. $\frac{5x^2+17x-12}{5x^2+7x-6}$

4.6 Functions

Notation and Terminology

A function is a rule that connects an input to an output. The **domain** of a function is the set of values that are input into the function; the **range** of a function are the outputs.

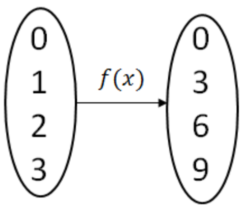
The notation for functions was introduced by one of the most prolific ever mathematicians, Leonard Euler. We use $f(x)$ to denote a function. Different letters are used to denote different functions, a second one would usually be denoted by $g(x)$

Example:
Given that $f(x) = 3x + 2$ find the value of $f(2)$

In $f(2)$ we have $x = 2$ so we simply use the method of substitution that we saw earlier.
$f(2) = 3 \times 2 + 2 = 6 + 2 = 8$

Functions are often shown in **mapping diagrams**

Example:
Draw the mapping diagram for the function $f(x) = 3x$ and domain $0 \leq x \leq 3$



Activity 4.6 A

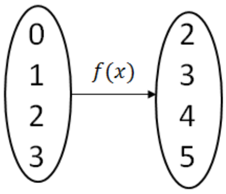
1. Given $f(x) = 4x - 1$, find
   a. $f(1)$ b. $f(4)$ c. $f(0)$ d. $f(-1)$

2. Given $f(x) = \frac{1}{x+3}$, find
   a. $f(3)$ b. $f(1)$ c. $f(0)$ d. $f(-2)$

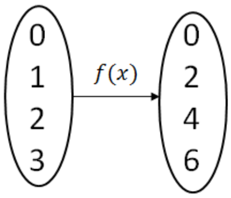
3. Draw the mapping diagram for each of these functions
   a. $f(x) = 5x$, domain $0 \leq x \leq 3$
   b. $g(x) = 3x + 1$, domain $1 \leq x \leq 4$
   c. $h(x) = x^2$, domain $-1 \leq x \leq 2$

4. Find the function for each of these mapping diagrams

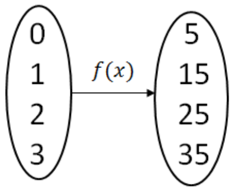
a.



b.



c.



Inverse Functions

The inverse of the function $f(x)$ is written as $f^{-1}(x)$. By "inverse" we generally mean "opposite". You saw this concept earlier when we looked at function machines; the inverse is the operation that is done when working backwards.

Example:
Given that $f(x) = 2x + 1$ find $f^{-1}(x)$.

In $2x + 1$ we multiply by 2 then add 1. Doing the opposite and working backwards we would subtract 1 then divide by 2.
So $f^{-1}(x) = \frac{x-1}{2}$

Composite Functions

Two or more functions can be combined to form composite functions. When this happens you have to be very careful to complete them in the right order – you work from the inside out.
For example $fg(x) = f(g(x))$
So, in the case of $fg(x)$ you would perform the function of g first then f.

Example:
Given $f(x) = x + 3$ and $g(x) = x^2$ find $fg(2)$ and $gf(2)$

With $fg(x)$ we do g first, then f.
$fg(2) = f(4)$ *Because* $g(2) = 2^2 = 4$
$= 7$

With $gf(x)$, f is done first
$gf(2) = g(5) = 25$

It is also possible to write composite functions in terms of x, this is often easier to do when a function machine is drawn.

Example:
Given $f(x) = x + 3$ and $g(x) = x^2$ find $fg(x)$ and $gf(x)$

$fg(x): x \rightarrow x^2 \rightarrow x^2 + 3$

$gf(x): x \rightarrow x + 3 \rightarrow (x + 3)^2$

Example:
Given that $f(x) = 2x - 3$, $fg(x) = 10x + 2$ and $g(x) = mx + c$ find the values m and c

$fg(x)$ is constructed by performing g then f.
$g(x) = mx + c$
$fg(x) = f(mx + c) = 2(mx + c) - 3 = 2mx + 2c - 3$
$= 10x + 2$
Therefore $2mx = 10x$ and $2c - 3 = 2$
So
$2m = 10 \rightarrow m = 5$
$2c - 3 = 2 \rightarrow 2c = 5 \rightarrow c = \frac{5}{2}$
If you struggle with this leave question 4 in the activity below for now and return to it after studying chapter 6.

Activity 4.6 B

1. Find the inverse of each of these functions

a. $f(x) = x + 5$ b. $g(x) = x - 1$ c. $f(x) = \frac{x}{5}$ d. $g(x) = 5x - 3$
e. $h(x) = \frac{x}{2} - 1$ f. $f(x) = 4x$ g. $h(x) = \frac{1}{x}$ h. $g(x) = 3 - x$

2. Given $f(x) = 2x - 1$ and $g(x) = x + 3$, find

a. $ff(x)$ b. $gg(x)$ c. $f^{-1}(x)$ d. $g^{-1}(x)$ e. $fg(x)$ f. $gf(x)$
g. $fg^{-1}(x)$ h. $gf^{-1}(x)$ i. $g^{-1}f^{-1}(x)$ j. $f^{-1}g^{-1}(x)$ k. $(gf)^{-1}(x)$ l. $ggf(x)$

3. Given $f(x) = \frac{x}{2}$ and $g(x) = x^2$, find

a. $ff(x)$ b. $gg(x)$ c. $f^{-1}(x)$ d. $g^{-1}(x)$ e. $fg(x)$ f. $gf(x)$
g. $fg^{-1}(x)$ h. $gf^{-1}(x)$ i. $g^{-1}f^{-1}(x)$ j. $f^{-1}g^{-1}(x)$ k. $(gf)^{-1}(x)$ l. $ggf(x)$

4. Given that $g(x) = mx + c$, find the values of m and c

a. $f(x) = 2x + 1$ $fg(x) = 7x + 2$
b. $f(x) = 2 - 3x$ $fg(x) = 2x - 1$
c. $f(x) = \frac{x}{2}$ $fg(x) = x + 3$

ASSIGNMENT FOUR

Answers to these questions are not provided. You should send your work to your tutor for marking. No calculators are allowed – show all of your working.

1. In a month Fay sends x text messages. Callie sends four times as many as Fay. Amy sends 5 more than Callie. Write an expression for the number of text messages sent by Amy. (1)

2. Simplify $\frac{3x}{4} + \frac{y}{2} - \frac{x}{4}$ (1)

3. Simplify $\frac{4ab + 6ab}{5a} - \frac{2(a+3)}{b}$ (2)

4. Fill in the missing value so the equation is correct $2(x+1) + \square(3x-1) = 20x - 4$ (3)

5. Factorise $2x^2y^3 - 4xy^4$ (2)

6. Expand and simplify $3x^2(x-4) - 2x(2-x^2) + xy(x+2)$ (3)

7. Susan wants to use a taxi to travel 6 miles. Where C is the cost in £ and n is the number of miles Fred's Taxi's charges $C = 5 + \frac{3}{2}n$ and Paula's Taxi's charges $C = 2 + 3n$. Which taxi company is the cheapest? (3)

8. A catering company charges $C = na + b$ where C is the total cost, n is the number of guests, a is the cost per guest and b is the baseline charge.
When there are 50 guests at £3 each find the baseline charge when the total cost is £160. (2)

9. Decide whether this identity is true or false and prove it: $x(x - 1 + y) \equiv x^2 + xy$ (2)

10. Make L the subject of the formula $T = 2\pi\sqrt{\frac{L}{g}}$ (3)

11. Expand and simplify $(4x+1)(2x-10)$ (2)

12. Factorise
a. $25x^2 - y^2$ b. $x^2 + 28x - 60$ c. $2x^2 + x - 6$ (6)

13. Simplify as far as possible (6)

$$\frac{x^2 - 4x - 12}{x^3 - 36x} \div \frac{x^2 - 2x - 8}{2x + 12}$$

14. Given that $f(x) = \frac{2x+1}{2}$ and $g(x) = 3x + 4$ find $gf^{-1}(2)$ (2)

Total 38 marks