GCSE Mathematics Higher

AQA Specification 8300

Part 5

Chapter Nine: Transformations

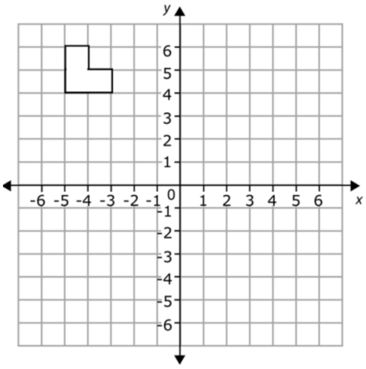
9.1 Transformations

A transformation maps an object to an image. It causes the position and/or size of a shape to change. Except in the case of enlargements, where the size of the shape changes, we say that the original object and the image are **congruent**. This means that they have the same measurements – the easiest way to think of it is that if you were to cut the shapes out and move them they would fit on top of one another perfectly.

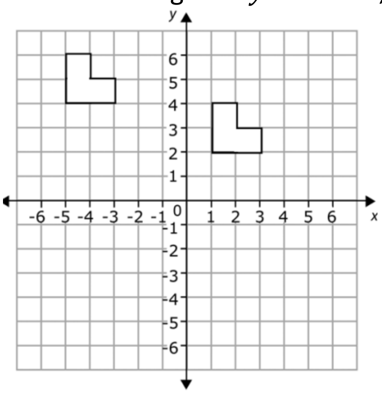
Translations

A translation is described by a vector or by the distance and direction.
All of the points in the object move by the same vector so, when given an object, follow the vector for each vertex and then join the points up to give the image.
A vector is written as $\binom{x}{y}$. The top value tells you how many units the point is moved along the x axis – horizontally – and the bottom value tells you how many units the point is moved along the y axis – vertically.

Example:
Translate this shape by the vector $\binom{6}{-2}$



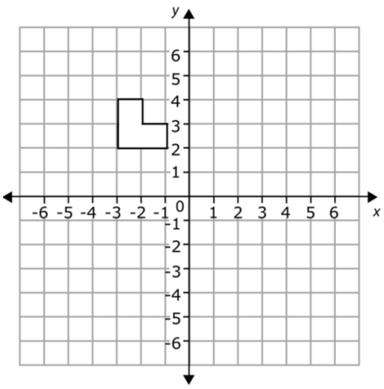
The vector tells us we should move 6 units in the positive x direction, which is to the right, and 2 units in the negative y direction, which is down.



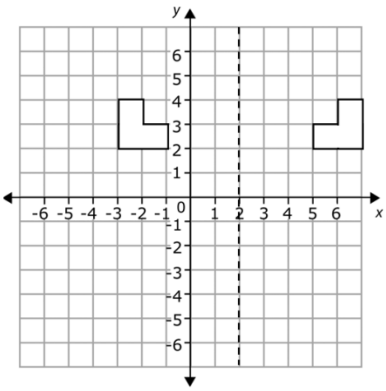
Reflections

A reflection is described by the position of the mirror line. The mirror line is a straight line and is given in the format that you have seen previously. If you are not confident with the equations of straight lines you should revise that section now.

Example:
Reflect this shape in the line $x = 2$



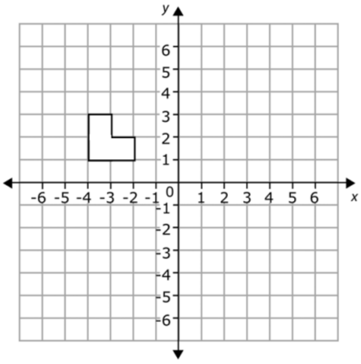
We must first draw the line $x = 2$ and then use this line to reflect the shape. If you find it difficult to visualise you may find it easier to place a mirror along the line but try not to rely on this too much as it is unlikely you'll be allowed a mirror in the exam.
It is a good idea to count the number of squares between the line and the object. Each vertex in the image should be the same number of squares away from the line on the other side.



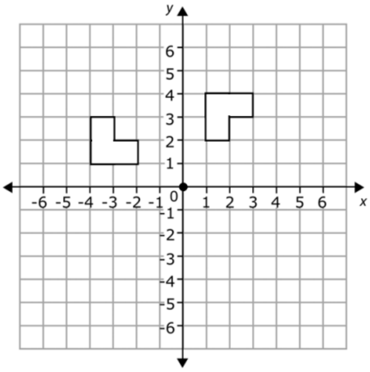
Rotations

Rotation is described using the angle of the rotation, whether it is clockwise or anti-clockwise and the centre of the rotation. The angles of rotation are usually multiples of 90 so that you're working with right angles.
The easiest way to visualise rotations is to use tracing paper. Trace the object and place your pencil, pressing through the tracing paper, onto the centre of rotation. Using your pencil to keep the centre still twist the tracing paper in the required direction. As above try not to rely on this too much because you need to be able to draw rotations in the exam. There is, however, nothing wrong with twisting your exam paper around to help you see what's going on!

Example:
Rotate this shape 90° clockwise about the origin. *Note: the origin is the point (0,0)*



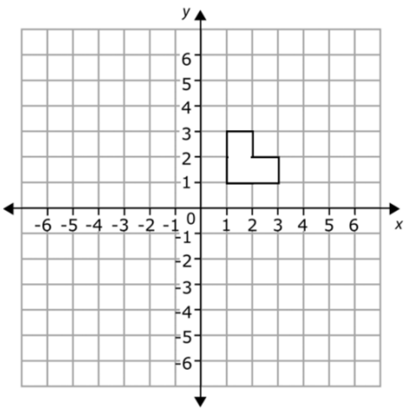
Mark the centre of rotation and twist the shape round one right angle in the clockwise direction



<u>Enlargements</u>

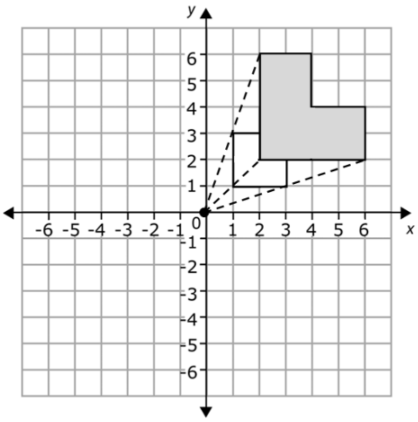
An enlargement is described by the centre of enlargement and the scale factor. If the object gets smaller the scale factor is a fraction.
In an enlargement the object and the image are **similar**. This means that the angles remain the same and the sides all follow the same ratio.

Example:
Enlarge the shape using centre (0, 0) and scale factor 2



Mark the centre of enlargement and draw lines to each of the vertices of the object. Either measure the line from the centre to the vertex or count the number of squares it passes through. Multiply this length by the scale factor and extend the line until it reaches this length. The vertex of the new image should be drawn at the end of the line.

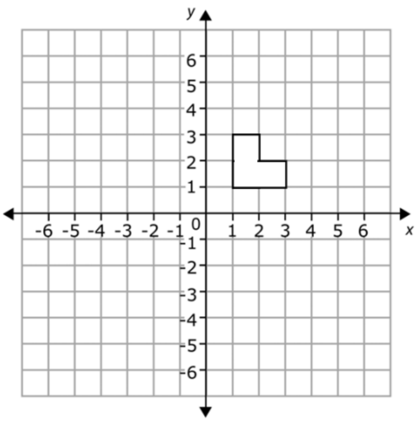
It is usually not necessary to do this for every vertex, particularly with the more complicated shapes. You can do it with the easier ones to reach and work out where the rest of the shape goes from this.



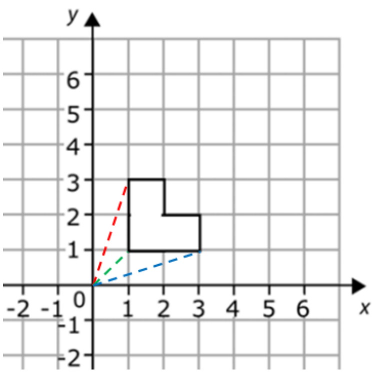
A positive scale factor above 1 makes the shape bigger. A positive scale factor between 0 and 1 makes the shape smaller. A negative scale factor inverts the shape.

When you have a negative scale factor you must deal with it slightly differently. The first thing to do is draw the lines from the centre to the vertices as before. Similarly you determine the length of the lines and multiply this by the scale factor to find the distance the lines should be extended by – at this moment you can ignore the negative sign.
Once this is done you draw the lines in the opposite direction as shown in the following example.

Example:
Enlarge this shape using centre (0,0) and scale factor -2

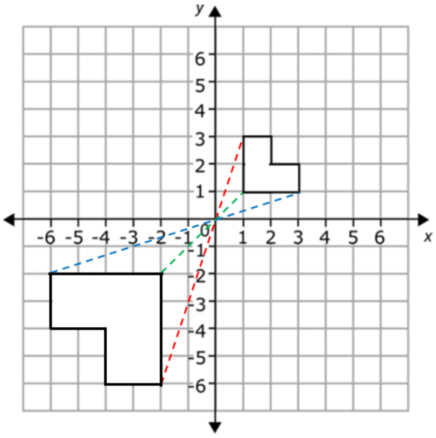


As before mark the centre and draw lines from the centre to the vertices.
Here we have used different coloured lines to illustrate the point.



Now we must work out how long the extensions should be – sometimes this is easier to simply draw the lines rather than carry out any calculations, it's up to you.
Red: 3 vertical, 1 horizontal → 6 vertical, 2 horizontal
Green: 1 vertical, 1 horizontal → 2 vertical, 2 horizontal
Red: 1 vertical, 3 horizontal → 2 vertical, 6 horizontal

These lines should be extended in the opposite direction as shown



Activity 9.1 A

1. On the grid below carry out the following transformations. Label each new shape with the letter of the question.

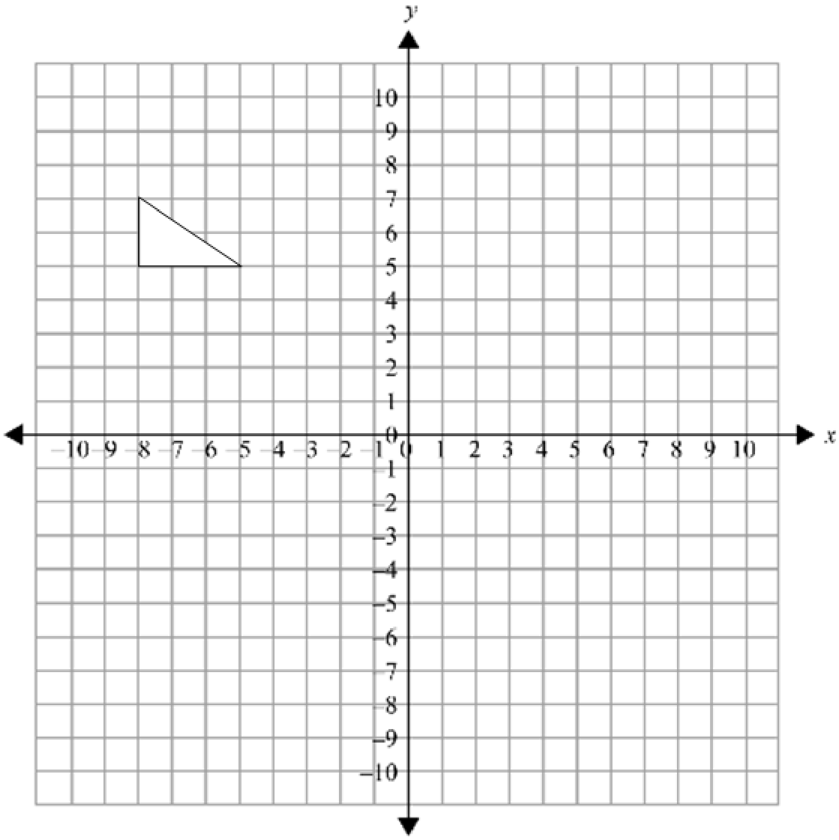
A: Translation of vector $\binom{5}{2}$

B: Reflection in the line $y = 1$

C: Enlargement, centre (9,-10), scale factor 2

D: Rotation about (0,0), 90°, anti clockwise

E: Enlargement centre (-2,7), scale factor -2



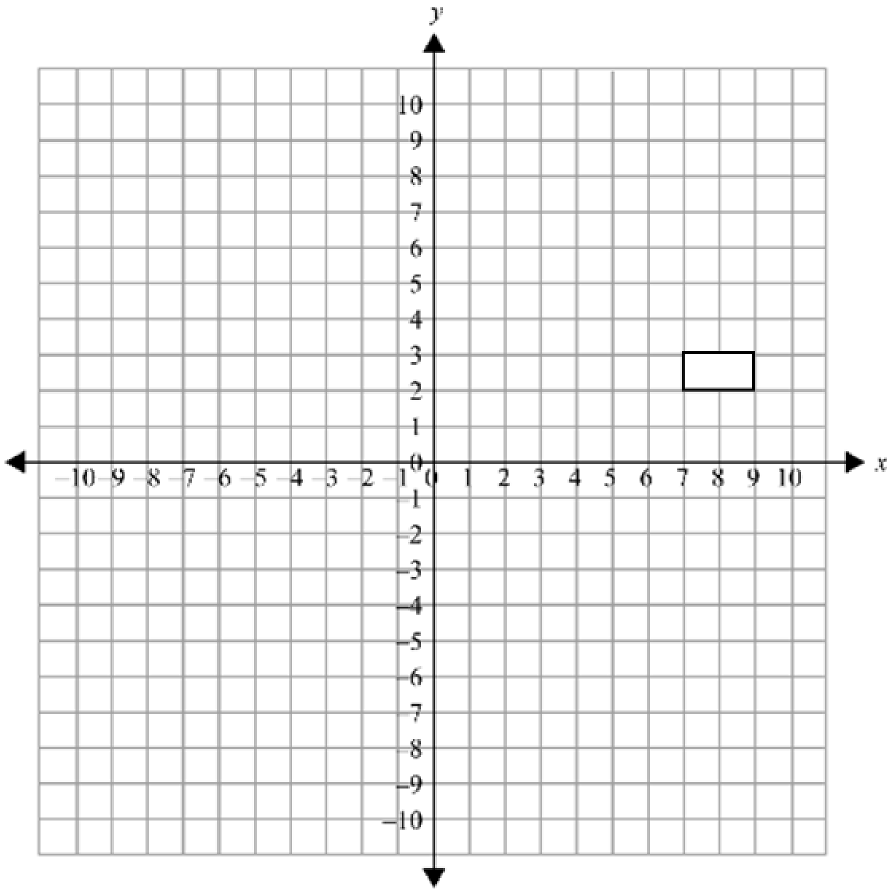
2. On the grid below carry out the following transformations. Label each new shape with the letter of the question.

A: Translation of vector $\binom{-7}{0}$

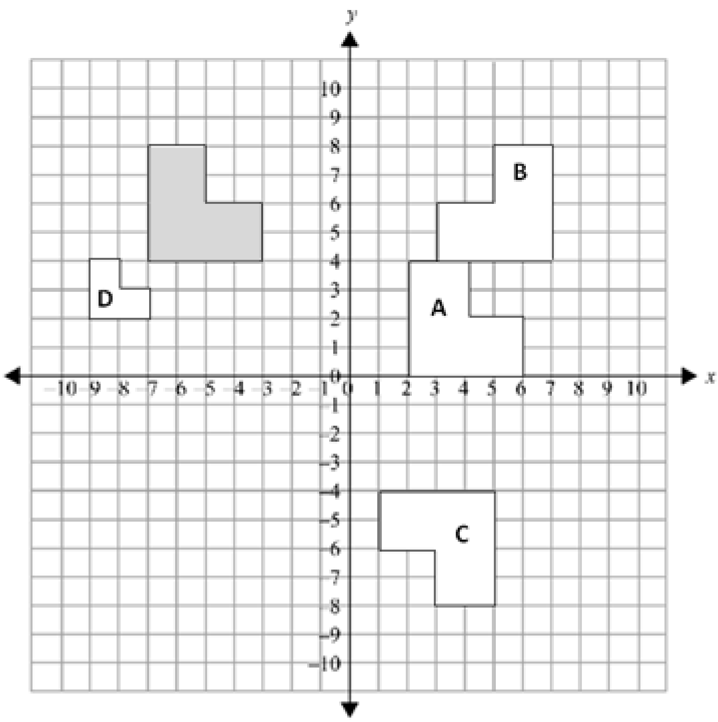
B: Reflection in the line $y = x$

C: Enlargement, centre (8,7), scale factor ½

D: Rotation about (1,2), 180°, clockwise



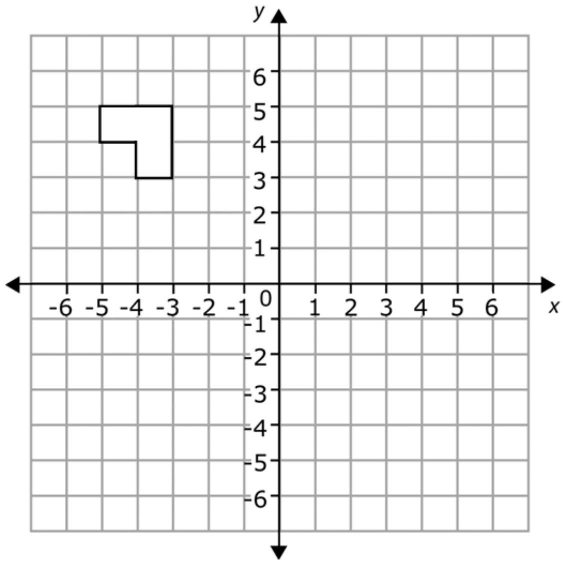
3. Describe the transformation that maps the shaded object onto each of the images



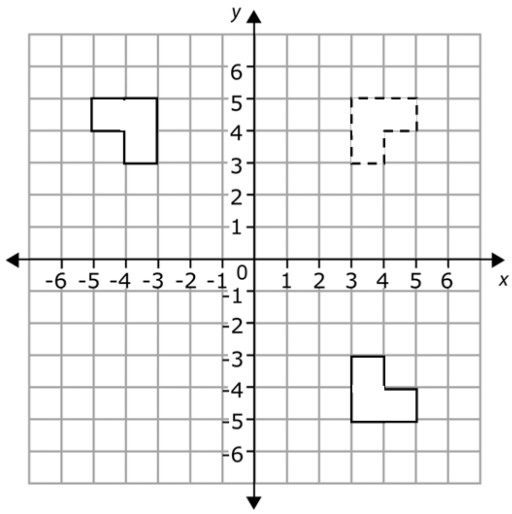
Combinations

Two or more transformations may be combined to map an object onto an image. Sometimes the combination of transformations may be equivalent to a different single transformation.

Example:
Reflect the shape in the y axis then reflect the new shape in the x axis. Describe the single transformation that maps the object onto the image.



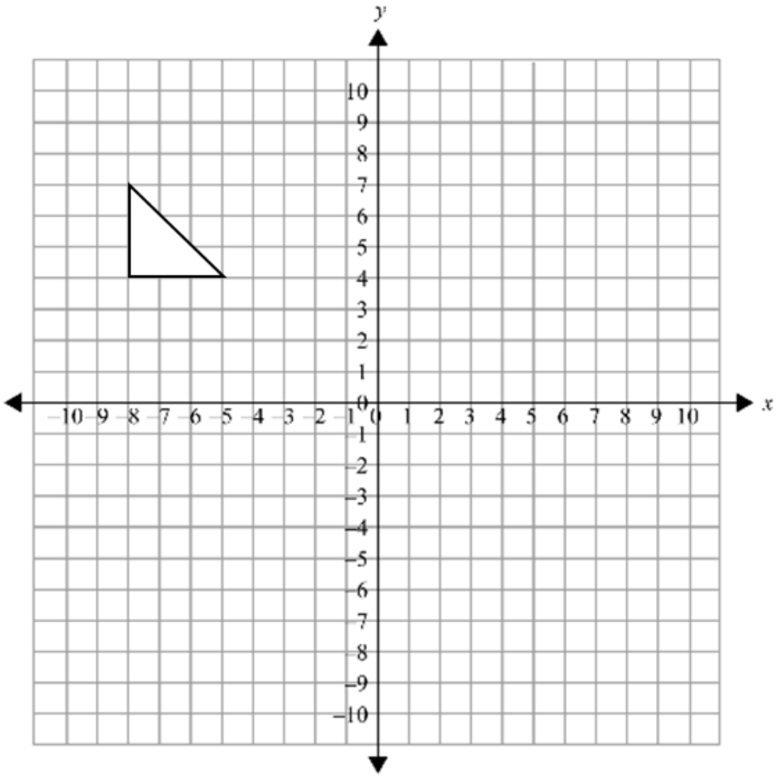
The first reflection is shown with a broken line.



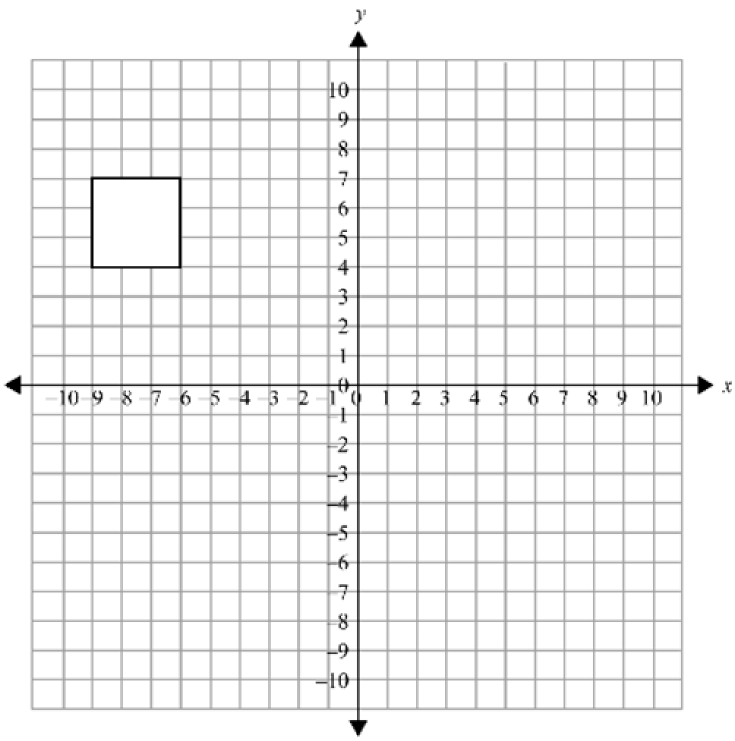
The single transformation is a rotation about the origin of 180° in either direction.

Activity 9.1 B

1. On the grid below rotate the shape 90° clockwise about (-9,3) then translate with vector $\binom{0}{-4}$. Describe the single transformation that maps the object to the image.



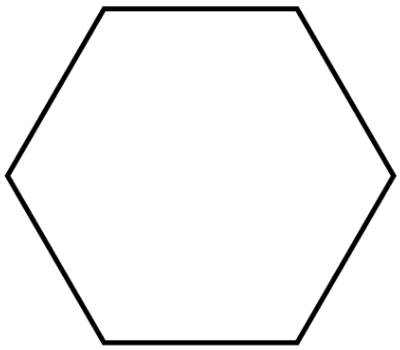
2. On the grid below reflect the shape in the x axis, then rotate anti clockwise 90° about the origin and finally translate in the vector $\binom{-4}{13}$. Describe the single transformation that maps the object to the image.



Rotational Symmetry

The order of rotational symmetry is the number of rotations you can do whereby the shape still looks the same. Given a polygon we generally take the middle to be the centre of rotation.

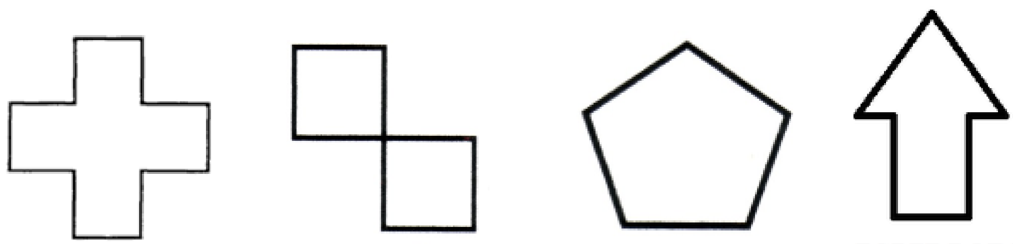
For example, trace this regular hexagon onto a piece of tracing paper and place a pencil in the middle. Rotate the tracing paper round the shape until you have done a full 360°, it is often easier to make a mark at the top of the shape, or draw an upwards facing arrow, so you know when you have turned it all the way around. The traced shape would fit onto its original plan 6 times. So the order of rotational symmetry of the hexagon is 6.



Activity 9.1 C

Find the order of rotational symmetry of each of these shapes

a. b. c. d.



9.2 Vectors

We have just seen how vectors work in terms of translations. You know that they have both size, also called magnitude, and direction. This is in contrast to a scalar which has size but no direction.

Vectors can be represented in a number of ways. You have already seen them written as a column vector in the form $\binom{x}{y}$ and they may also be drawn as an arrow on diagrams. When talking of a vector from A to B we either write $\overrightarrow{AB}$ or **a**. Since it is impractical to write in bold when a letter denotes a vector in handwritten work you should underline it $\underline{a}$.

When adding vectors we add each component and, when multiplying by a scalar, we multiply each component.

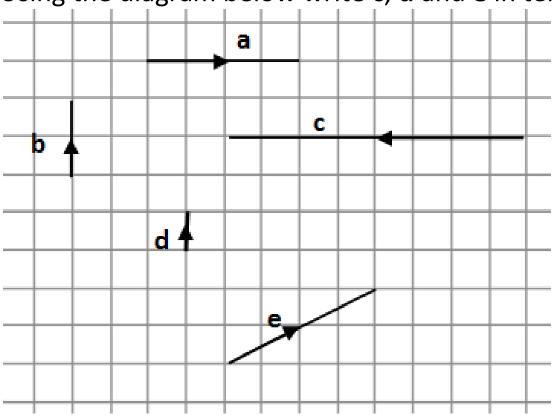
Example:
Given **a** = $\begin{pmatrix}1\\2\end{pmatrix}$ and **b** = $\begin{pmatrix}-2\\3\end{pmatrix}$ find **a** + **b** and 2**a.**

a + **b** = $\begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}1+-2\\2+3\end{pmatrix} = \begin{pmatrix}-1\\5\end{pmatrix}$

2**a** = $2 \times \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}2\times1\\2\times2\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

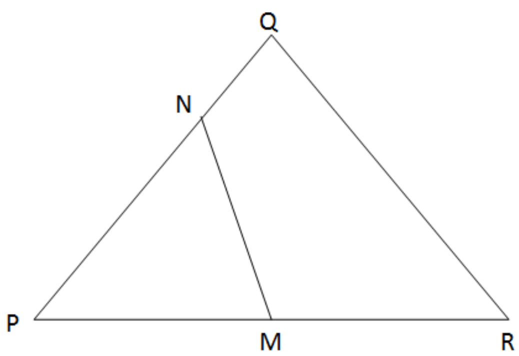
Multiplication gives parallel vectors. Using the last example, 2**a** is parallel to **a** but is twice as long. When finding vectors it is also important to know that **a** and **–a** are the same size but in opposite directions.

Example:
Using the diagram below write **c**, **d** and **e** in terms of **a** and **b**

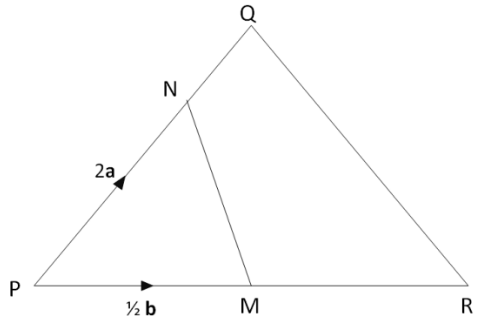


c is twice the length of **a** but in the opposite direction so we have **c** = - 2**a**
d is half the length of **b** and in the same direction so we have **d** = ½**b**
If you were to imagine **e** as part of a triangle and draw in the horizontal and vertical sides you would see that the horizontal side is the same as **a** and the vertical side is the same as **b**, both travelling in the same direction so we have **e** = **a** + **b**.

Example:
On the diagram below M is the midpoint of PR. N is $\frac{2}{3}$ of the way from P to Q.
$\overrightarrow{PQ} = 3\mathbf{a}$ $\overrightarrow{PR} = \mathbf{b}$
Find the vector $\overrightarrow{MN}$



Firstly we can see that $\overrightarrow{PM} = \frac{1}{2}\mathbf{b}$ and $\overrightarrow{PN} = 2\mathbf{a}$ (since $\frac{2}{3} \times 3\mathbf{a} = 2\mathbf{a}$)



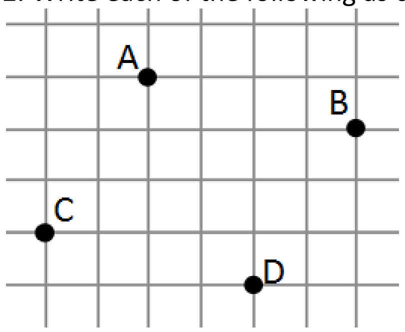
So, from the diagram we can see that, $\overrightarrow{\mathrm{NM}} = \overrightarrow{\mathrm{NP}} + \overrightarrow{\mathrm{PM}} = -2\mathbf{a} + \frac{1}{2}\mathbf{b}$

Activity 9.2

1. Given that **a** = $\binom{3}{5}$, **b** = $\binom{2}{-1}$ and **c** = $\binom{0}{2}$. Find each of the following

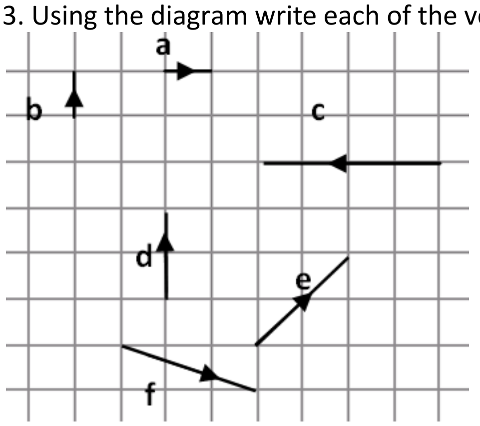
a. **a** + **b** b. **a** + **b** + **c** c. 3**b** d. ½**c** e. 2**a** + **b** f. 5**a** + 2**b**

2. Write each of the following as column vectors



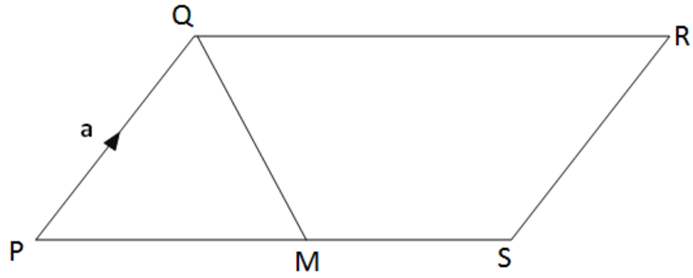
a. $\overrightarrow{AB}$ b. $\overrightarrow{AC}$ c. $\overrightarrow{DB}$ d. $\overrightarrow{BD}$ e. $\overrightarrow{BC}$ f. $\overrightarrow{AD}$

3. Using the diagram write each of the vectors in terms of **a** and **b**.

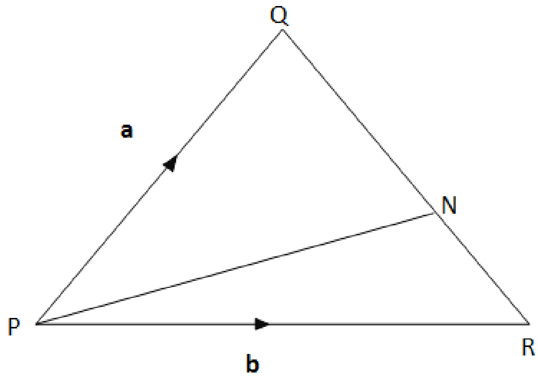


4. The parallelogram PQRS is shown below. M is the midpoint of PS. $\overrightarrow{PS} = 4\mathbf{b}.$

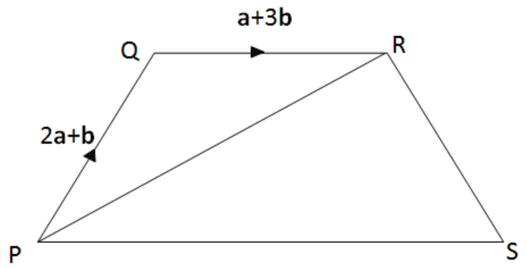
Find a. $\overrightarrow{QR}$ b. $\overrightarrow{RS}$ c. $\overrightarrow{MQ}$



5. PQR is a triangle as shown below. N is the point such that QN:NR = 2:1 *(ratio is covered fully in chapter 11).* Find $\overrightarrow{PN}$



6. Find $\overrightarrow{PR}$



Chapter Ten: Triangles

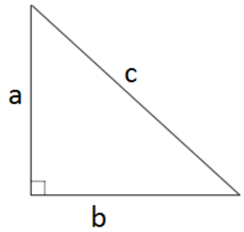
10.1 Pythagoras' Theorem

Pythagoras was an influential Greek mathematician most known for the theorem named after himself. His work was predominantly based around right angled triangles and he pioneered the formula we use to find the value of the sides.

The theorem is quite simply

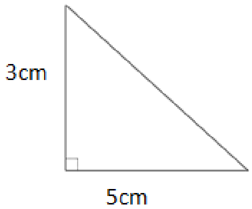
$$a^2 + b^2 = c^2$$

Where a, b and c are the sides of the right angled triangle. It doesn't matter which way round a and b go when you're labelling the triangle but c should always be the **hypotenuse**. The hypotenuse of a right angled triangle is the longest side which is always the one opposite the right angle.

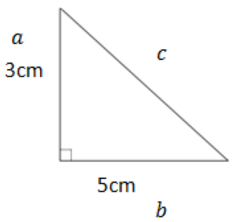


Pythagoras' theorem is used with right angled triangles when you have the length of two sides and you want to find the length of the third.

Example:
Find the length of the unknown side



With any triangle question it is important to add the labels before doing anything else.



Using $a^2 + b^2 = c^2$ we have

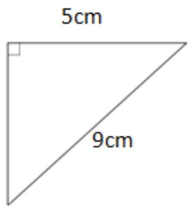
$3^2 + 5^2 = c^2$ *It is okay to represent the unknown as c because the diagram illustrates what this letter represents.*

$9 + 25 = c^2$

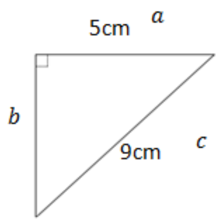
$34 = c^2$

$c = \sqrt{34} = 5.8309518 \ldots. \cong 5.83\text{cm}$

Example:
Find the length of the unknown side



Again the first step is to add the labels to the sides, remembering that *c* must be the hypotenuse and that *a* and *b* can be either of the other two sides.

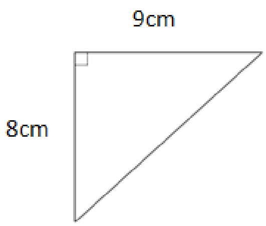


$a^2 + b^2 = c^2$
$5^2 + b^2 = 9^2$
$25 + b^2 = 81$
$b^2 = 81 - 25 = 56$
$b = \sqrt{56} = 7.48331477 \ldots \cong 7.48\text{cm}$

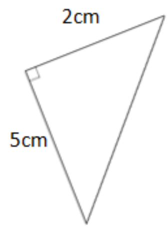
Activity 10.1

1. Find the length of the missing side(s) in each of these triangles.

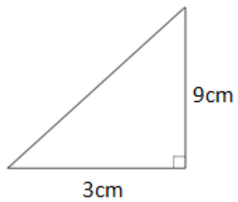
a.



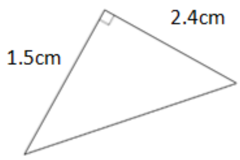
b.



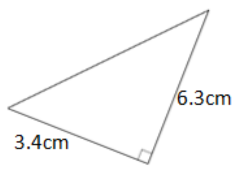
c.



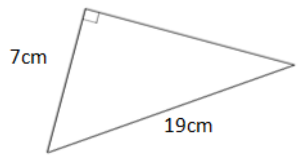
d.



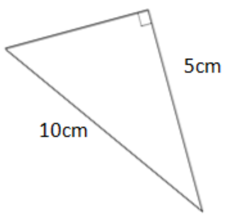
e.



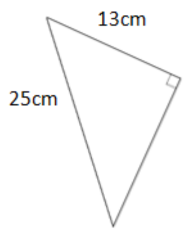
f.



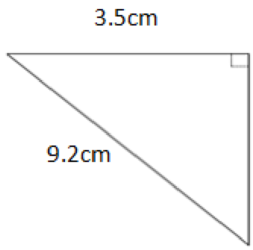
g.



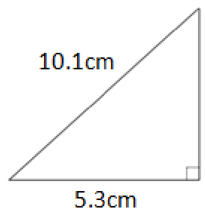
h.



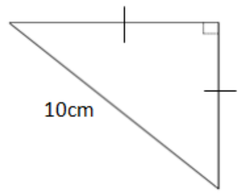
i.



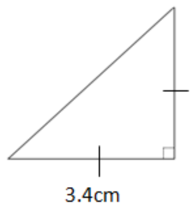
j.



k.



l.



With word questions it is advised that the first thing you do is draw a diagram.

2. A square has sides of length 5cm, find the length of the diagonals.

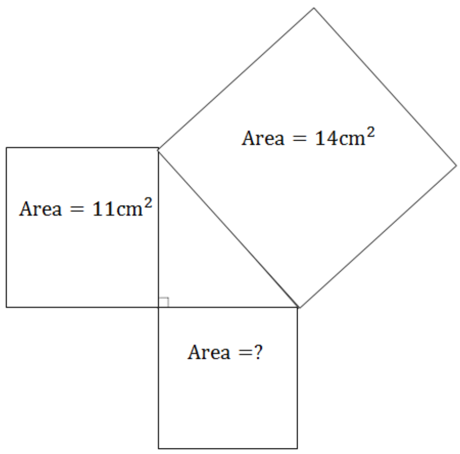
3. A square has sides of length 23cm, find the length of the diagonals.

4. A rectangle has sides of length 4cm and 7cm, find the length of the diagonals.

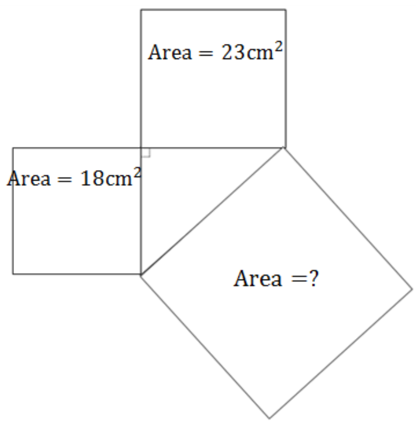
5. A ladder of length 8m is leaning against a wall. The base of the ladder was 3.5m away from the wall. Find how far up the wall the ladder reaches.

6. Find the unknown area

a.



b.



7. A ship travelled 50m North then 40m East. How far is it from its starting point?

8. A circle has a radius of 2.4cm and a chord of 5.6cm. Find the distance from the midpoint of the chord to the centre of the circle.

<u>10.2 Trigonometry</u>

Trigonometry also works with only right angled triangles. It is used when you have two sides and want to find an angle or when you have an angle (other than the right angle) and one side and want to find another side.

There are three trigonometric ratios you need to know sine, cosine and tangent. In calculations these are shortened to sin, cos and tan – you should be able to find these buttons on your calculator.

You should revise the instruction manual or look online for tutorials now if you are not confident with how you should type values into your calculator.

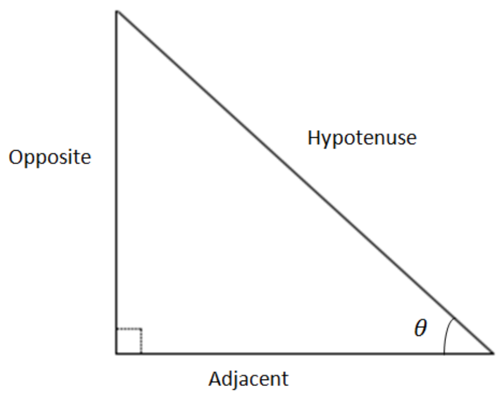
The ratios you need to know are:

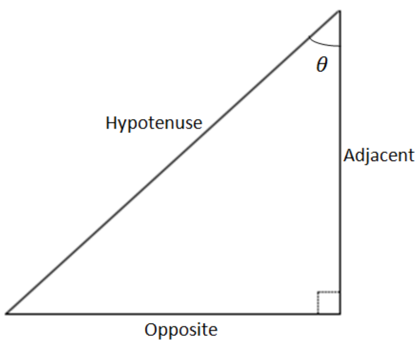
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

These are often shortened to
$S = \frac{O}{H} \quad C = \frac{A}{H} \quad T = \frac{O}{A}$

You will often hear trigonometry referred as sock-a-toe-a because the ratios you need to know spell out SOHCAHTOA. You could also use these letters to form an anagram to help you remember the order they should go in.

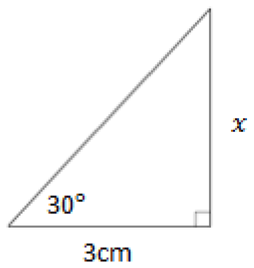
You already know that the hypotenuse is the longest side. The 'opposite' is the side that is opposite the given angle and the adjacent is the remaining side. Some examples are shown below.



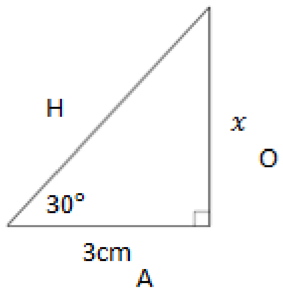


<u>Missing sides</u>

Example:
Find the length of the missing side



As before the first step is to label the triangle. Here we use the letters O, H and A.



The next step is to write out the trigonometric ratios and cross out the values we have – this includes the side we're looking for. The ratio we use is the one that has two parts crossed out.

~~O~~ ~~A~~ ~~O~~

S H C H T ~~A~~

As you can see we have both O and A so we will use tan.

$\tan\theta = \frac{O}{A}$

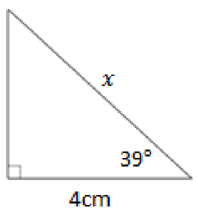
$\tan 30 = \frac{x}{3}$

$x = 3 \times \tan 30 = 1.7320508 \ldots \cong 1.73\text{cm}$

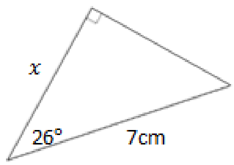
Activity 10.2 A

1. Find the missing side

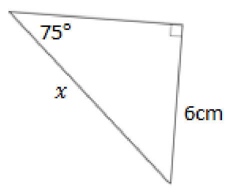
a.



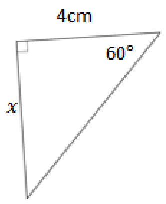
b.



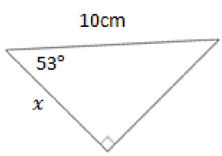
c.



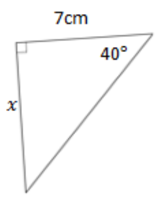
d.



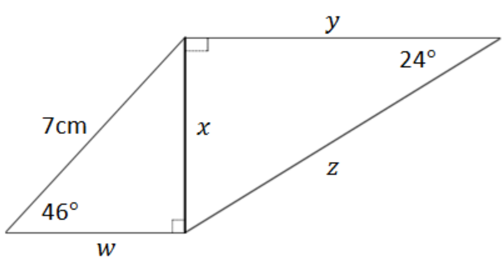
e.



f.



2. Find the lengths of the missing sides



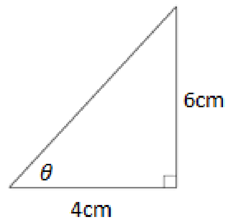
3. A ladder is placed against a wall at an angle of 35°. The base is 75cm away from the wall. How high up the wall does the ladder reach?

4. A straight road of 40km reaches the top of a hill which makes an angle of 34° with the horizontal. Assuming the road starts at the bottom, how high is the hill?

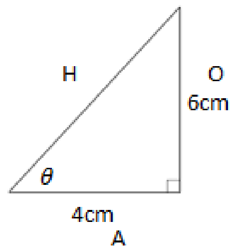
<u>Missing Angles</u>

The same ratios can also be used to find missing angles as long as the triangle contains a right angle and you have the lengths of two sides. In order to do this we need to use the inverse trigonometric functions. You will sometimes see these written as arcsin, arccos and arctan but for the GCSE course the notation used is $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$. In the case of trigonometric functions the power is merely notation, it is not the same as putting a value to the power of negative one.
These inverse functions can be found on your calculator. They are usually above the standard sin, cos and tan buttons and are accessed by pressing the shift button first but this may vary depending on the model of your calculator.

Example:
Find the value of the missing angle



As before we label the sides and cross out the values we have from the SOHCAHTOA ratios.



~~O~~ ~~A~~ ~~O~~
S H C H T ~~A~~

So we have

$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$

$\tan\theta = \frac{6}{4}$

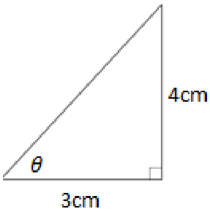
Now we must use the inverse of tan to find the value of θ. The easiest way to think of it is that the θ and the value of the right hand side swap places when tan is replaced by $\tan^{-1}$.

$\theta = \tan^{-1}\left(\frac{6}{4}\right) = 56.30993247 \ldots \cong 56.31°$

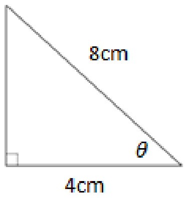
Activity 10.2 B

1. Find the value of the missing angle

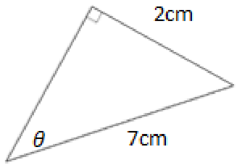
a.



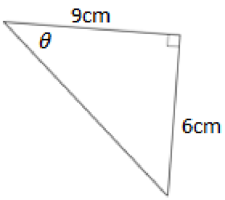
b.



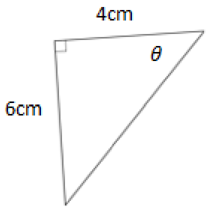
c.



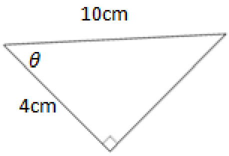
d.



e.



f.



2. A ladder of 5m is leaning against a vertical wall. The bottom of the ladder is 2m away from the wall. It is known that the ladder is safe to use when the angle it makes with the horizontal is more than 70°. Is the ladder safe to use?

3. An old building has steps outside which need to be replaced with a ramp. The owners decide to place the ramp over the top of the steps. Each step is 50cm wide and 20cm high. Find the angle of the ramp.

It is important to know which rule to use in different situations. Below are some mixed questions to help you gain an understanding of this.

4. For each of the scenarios below write down whether you would use trigonometry or Pythagoras. Assume that in each case you have a right angled triangle.

   a. You know the two shorter sides and you're looking for the hypotenuse

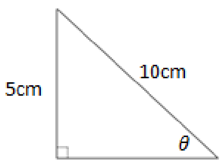
   b. You know an angle and the hypotenuse and you're looking for one of the shorter sides

   c. You know the two shorter sides and you're looking for an angle

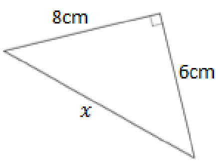
   d. You know one shorter side and the hypotenuse, you're looking for the remaining side

5. In each of these triangles find the value of the missing side or angle.

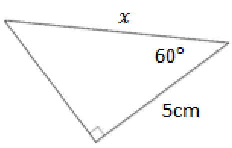
a.



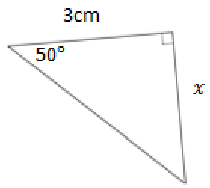
b.



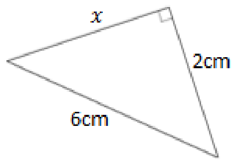
c.



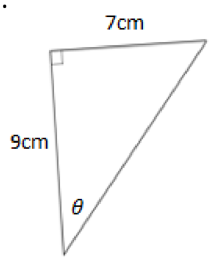
d.



e.

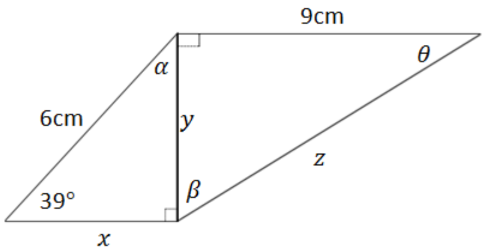


f.



6. A person out for a walk ends up 43km away from where they started. They initially travelled 25km North and then turned and went West. How far did they travel West?

7. Find all of the missing sides and angles in this diagram



<u>Ratios</u>

There are a number of specific values for which you need to know the value of the ratios. As a general rule trigonometry questions will only come up on a calculator paper but, when they're using these values, there is a chance that they will be used on a non-calculator paper so you need to learn these.

You should know the value of sin, cos and tan at 0°, 30°, 45° and 60°. You should also know the value of sin and cos at 90°.
$\tan 90 = \infty$ so this value cannot be used and, as a result, it is not so important that you remember it.

$$\sin 0 = 0 \qquad \cos 0 = 1 \qquad \tan 0 = 0$$

$$\sin 30 = \frac{1}{2} \qquad \cos 30 = \frac{\sqrt{3}}{2} \qquad \tan 30 = \frac{1}{\sqrt{3}}$$

$$\sin 45 = \frac{1}{\sqrt{2}} \qquad \cos 45 = \frac{1}{\sqrt{2}} \qquad \tan 45 = 1$$

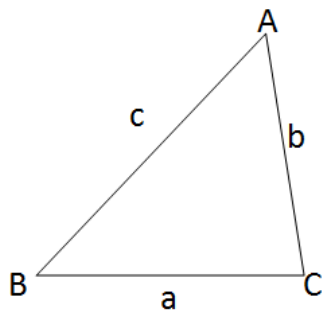
$$\sin 60 = \frac{\sqrt{3}}{2} \qquad \cos 60 = \frac{1}{2} \qquad \tan 60 = \sqrt{3}$$

$$\sin 90 = 1 \qquad \cos 90 = 0 \qquad \tan 90 = \infty$$

10.3 Further Trigonometry

As well as the standard ratios you need to use there are also two rules you should be familiar with. These rules enable you to find missing sides and angles of triangles without a right angle.

The standard notation for labelling triangles is to use capital letters for the angles and lower case letters for the sides. The corresponding letters should be opposite each other, ie side a is opposite angle A.



Sine Rule

The sine rule can be written one of two ways

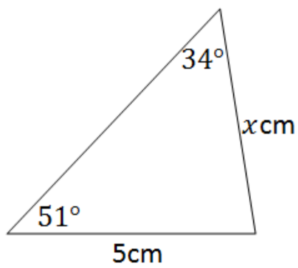
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

or

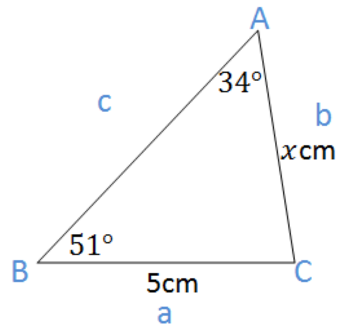
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

To use the sine rule the first step is to substitute in any values for a,b,c, A, B or C that are present. You only need to keep two of the fractions so cross out the one that has no values substituted in. Provided that you have one fraction containing all numbers and one with only one unknown you can simply rearrange and solve.

Example:
Find the value of x



The first step with triangle questions is always to add labels.



Now we substitute the values we have into

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This gives

$$\frac{5}{\sin 34} = \frac{x}{\sin 51} = \frac{c}{\sin C}$$

Therefore the equation we're working with is

$$\frac{5}{\sin 34} = \frac{x}{\sin 51}$$

This gives us

$$x = \frac{5}{\sin 34} \times \sin 51 = 6.9488 \ldots$$

Cosine Rule

The cosine rule for finding a side is given by

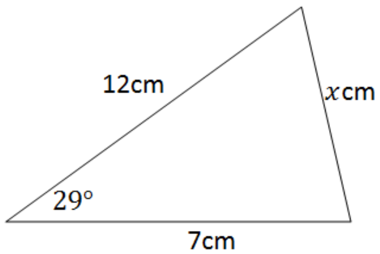
$$a^2 = b^2 + c^2 - 2bc\cos A$$

This can be rearranged to give the following formula for finding an angle

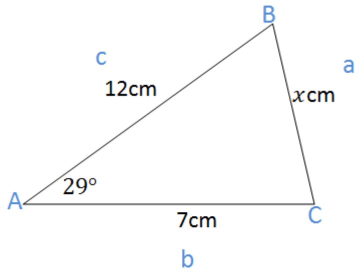
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

The examples and activities in this course will only use the first formula and will rearrange it where necessary. You will need to be able to find both sides and angles using the cosine rule however, since you need to remember these formulae, if you are comfortable with rearranging the first one you do not need to remember both.

Example:
Find the value of x

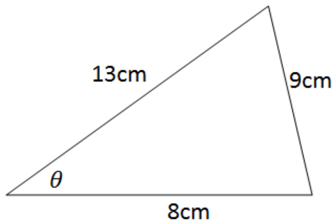


Again the first thing we need to do is add labels. When using the cosine rule you should label it such that the unknown becomes a or A as appropriate.

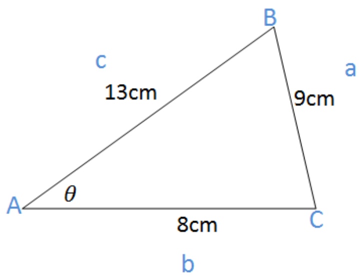


Substituting into $a^2 = b^2 + c^2 - 2bc\cos A$
Gives $x^2 = 7^2 + 12^2 - 2 \times 7 \times 12 \times \cos 29$
$x^2 = 193 - 168\cos 29$
$x^2 = 46.063889\ldots$
$x = \sqrt{46.063889\ldots} = 6.787038\ldots$

Example:
Find the value of θ



Adding labels gives



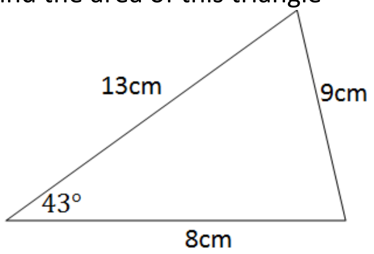
Substituting into $a^2 = b^2 + c^2 - 2bc\cos A$
Gives $9^2 = 8^2 + 13^2 - 2 \times 8 \times 13 \times \cos\theta$
$81 = 233 - 208\cos\theta$
$-152 = -208\cos\theta$
$\cos\theta = \frac{152}{208}$
$\theta = \cos^{-1}\left(\frac{152}{208}\right) = 43.0490\ldots$

<u>Area of a Triangle</u>

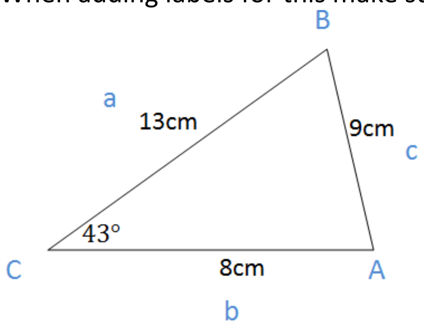
You have already seen how to find the area of a right angled triangle, the formula for finding the area when the triangle doesn't have a right angle is given by

$$\text{Area} = \frac{1}{2}ab\sin C$$

Example:
Find the area of this triangle



When adding labels for this make sure the angle you have is labelled as C



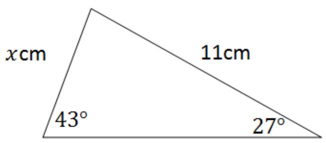
So now we have
$\text{Area} = \frac{1}{2} \times 13 \times 8 \times \sin 43 = 35.4639 \ldots \text{cm}^2$

Activity 10.3 A

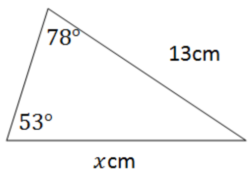
1. Prove that $a^2 = b^2 + c^2 - 2bc\cos A$ can be rearranged to give $\cos A = \frac{b^2+c^2-a^2}{2bc}$

2. Use the sine rule to find the missing side/angle

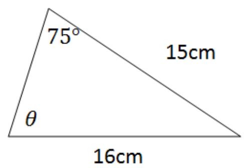
a.



b.

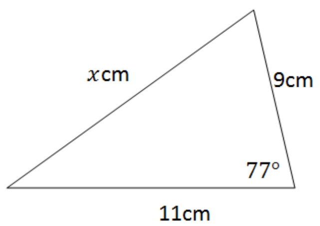


c.

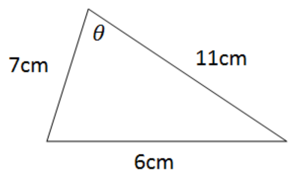


3. Use the cosine rule to find the missing side/angle

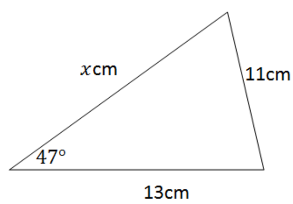
a.



b.

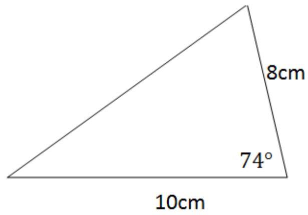


c.

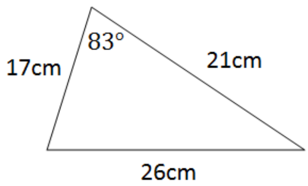


4. Find the area of each of these triangles

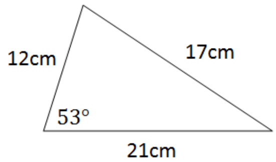
a.



b.

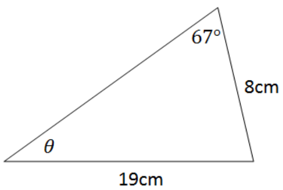


c.



5. Use an appropriate rule to find the missing side/angle

a.



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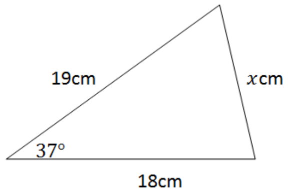
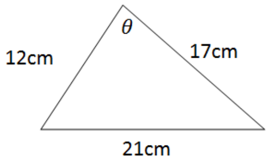
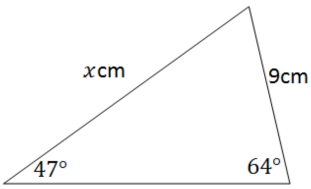


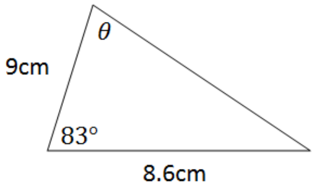
c.



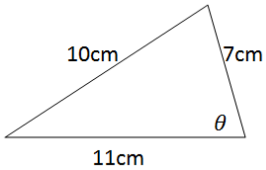
d.



e.



f.

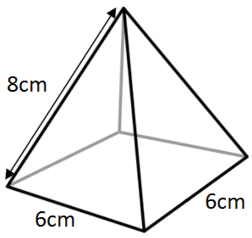


Problems in 3D

We have seen that there are rules for finding missing sides and angles in both right angled and non-right angled triangles when working in 2D. These rules can also enable you to solve problems in 3D.

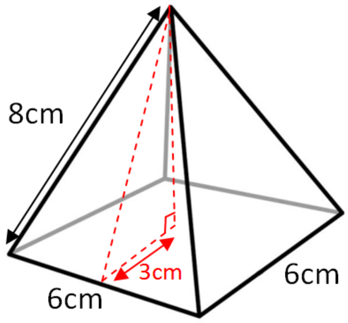
The rules are applied in the same way but visualising the 3D shapes can take some getting used to!

Example:
A square based pyramid is shown below. The length of each slanted edge is 8cm, the length of each side of the base is 6cm. Find the height of the pyramid and the angle between the base and the slanted face.

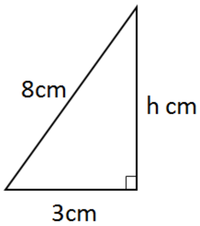


The height of the pyramid is measured from the tip to the middle of the base.
We have a right angled triangle made up of one slanted side, the height and the joining line from the outside of the base to the centre.

This is shown on the diagram below.



So we have the following triangle



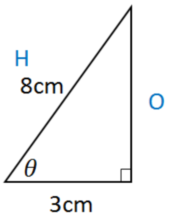
Using Pythagoras, $a^2 + b^2 = c^2$ we have

$h^2 + 3^2 = 8^2$

$h^2 = 55$

$h = 7.416198487 \ldots \cong 7.42$ cm

To find the angle between the slanted face and the base we can use the same triangle.



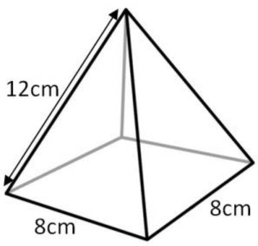
$\cos\theta = \frac{3}{8}$

$\theta = \cos^{-1}\left(\frac{3}{8}\right) = 67.975687 \ldots \cong 67.98°$

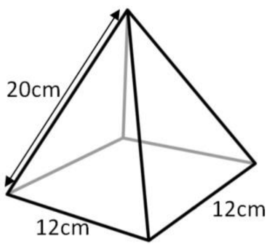
Activity 10.3 B

1. Given the square based pyramids below, find the height and the angle between the slanted face and the base

a.

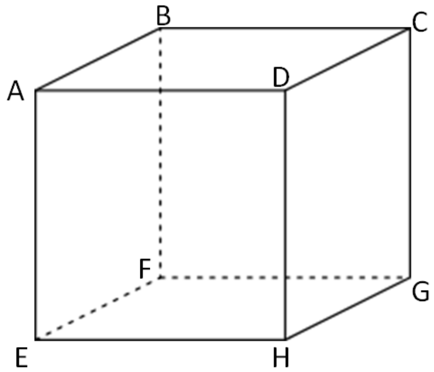


b.



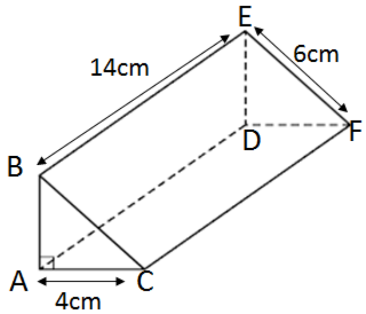
2. Below is a regular cube where each edge is 12cm. Find each of the following lengths.

a. AC b. EG c. EC



3. Below is a triangular prism. Find the following

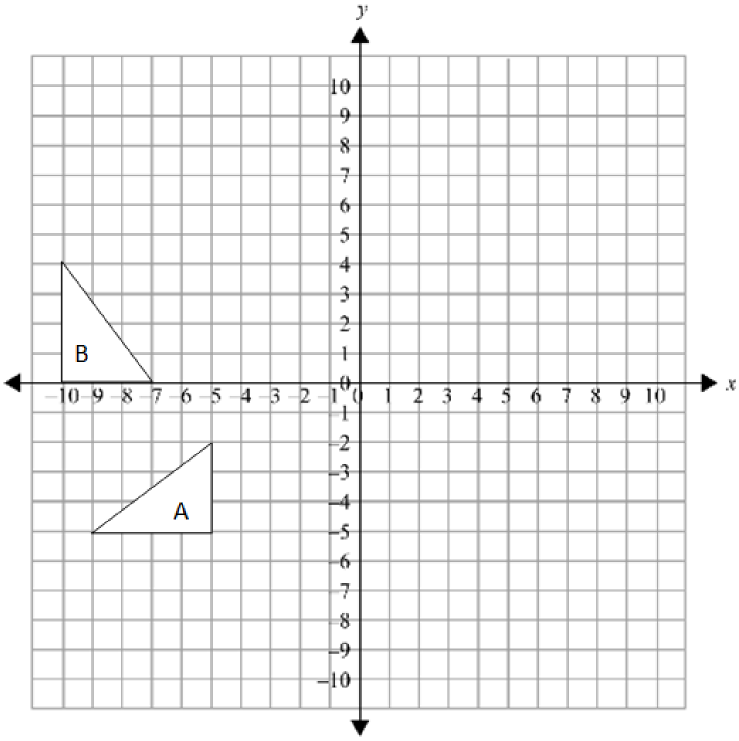
a. the length AB b. the length BF c. angle ACB d. angle BFA



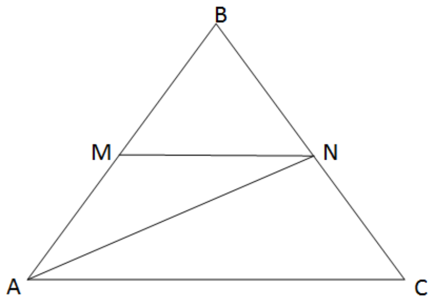
ASSIGNMENT EIGHT

Answers to these questions are not provided. You should send your work to your tutor for marking.

1. Describe the transformation that maps A onto B. On the same graph reflect A in the y axis, label this shape C, and translate B in the vector $\binom{10}{-2}$, label this shape D. (5)

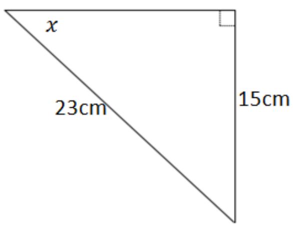


2. Given the diagram below $\overrightarrow{AC} = 3\mathbf{a}$ and $\overrightarrow{AB} = 5\mathbf{b}$. M is the midpoint of AB, N is the midpoint of BC. MN is $\frac{2}{3}$ of the length of AC. Find the vector $\overrightarrow{NA}$. (3)

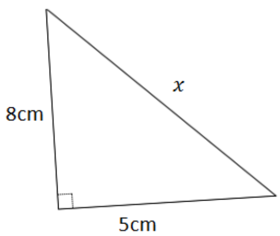


3. Find the value of x in each of these triangles (12)

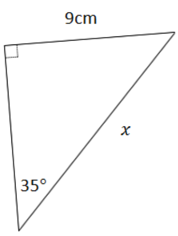
a.



b.



c.



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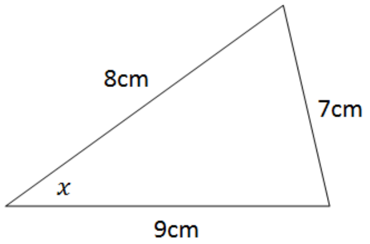
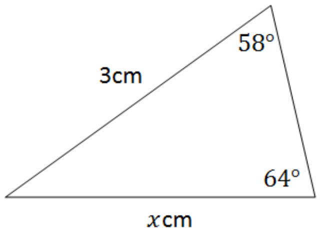
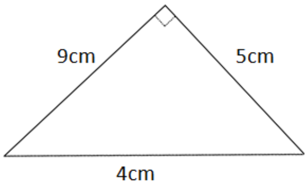


e.



4. Explain what is wrong with this triangle (1)



5. A ladder of 7m makes an angle of 20° with the wall. The base of the ladder lies on a horizontal floor. What is the distance between the base of the ladder and the wall? (2)

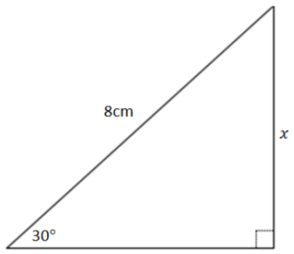
Total marks 23

<u>End of Section Revision Quiz</u>

These questions are based upon everything you have covered in the geometry section of the course. It is not compulsory to complete this quiz but it is recommended that you use it in order to make sure you have a firm understanding of all of the topics covered.
Answers are provided to these questions – if you're finding a question or group of questions particularly difficult you should go back to that section in your notes and go over it to make sure you understand it fully.
The majority of these questions are exam style questions so you should familiarise yourself with them now.

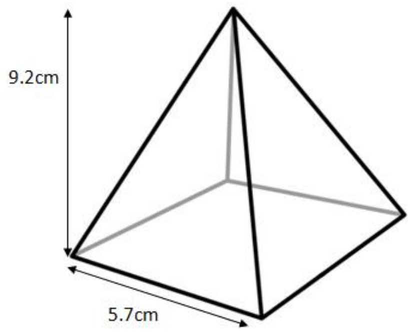
1. A semi circle has a radius of 4cm. Find the area of the semi circle in terms of pi.

2. Without using a calculator find the value of x

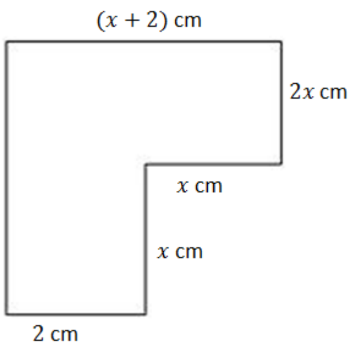


3. A map has a scale of 1cm:3.5km. On this map the distance between two towns is 3cm. Find the actual distance.

4. Find the volume and the angle between the slanted face and the base of this square based pyramid

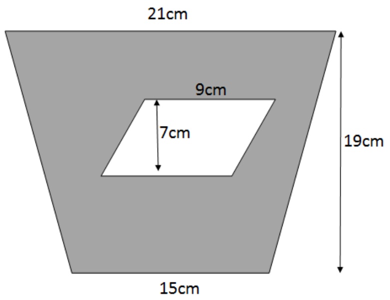


5. The perimeter of this shape is 28cm. Find the value of x

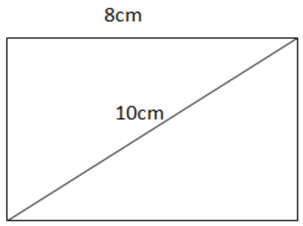


6. The points (1,4) and (3,2) are diagonally opposite corners of a square. Find the coordinates of the other two corners.

7. Find the area of the shaded region



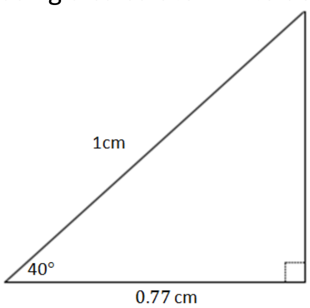
8. A rectangle of width 8cm is shown below. The diagonal is 10cm. Find the area of the rectangle.

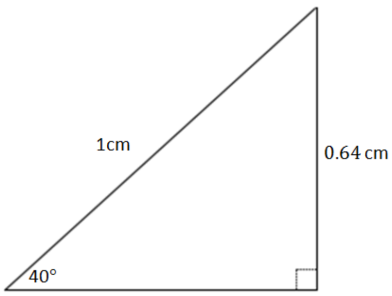


9. Is it possible to have a regular polygon with the given angle as an exterior angle?
If yes write down the number of sides the polygon would have.

a. 8° b. 10° c. 9° d. 5° e. 11°

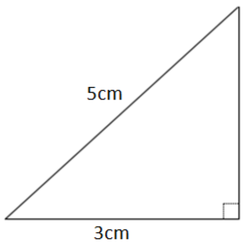
10. Two triangles are shown below. The values have been rounded to 2 decimal places. Without using a calculator write down the value of sin40 to two decimal places.





11. A circle has a radius of 4.5cm. Find the area and the circumference.

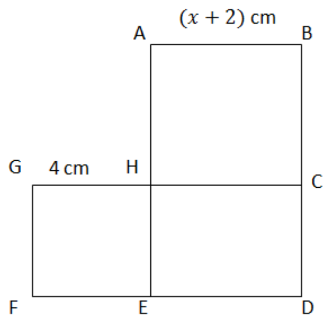
12. Find the area of this triangle



13. Find **a** – **b** when **a** = $\binom{3}{-5}$ and **b** = $\binom{5}{1}$

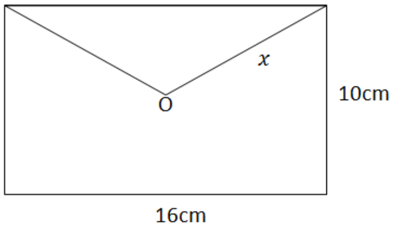
14. A square has sides of length 9cm. Find the length of the diagonal.

15. ABCH and EFGH are both squares. Show that the area of the whole shape is $x^2 + 8x + 28$

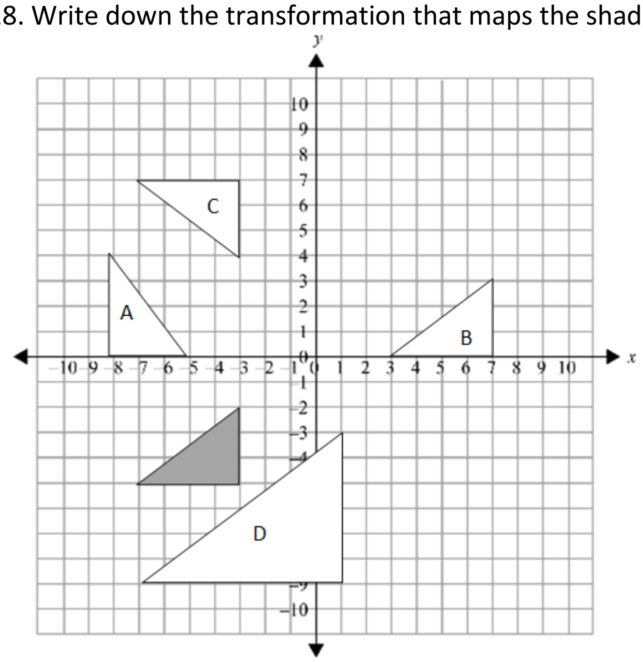


16. A cylinder has a height of 11cm and radius of 2.5cm. Find the volume.

17. O is the centre of the rectangle. Find the value of x.



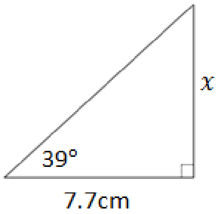
18. Write down the transformation that maps the shaded object onto each of the images.



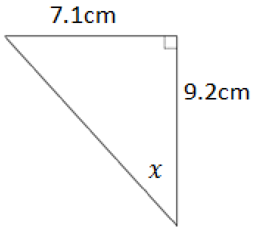
19. A circle with centre (0,0) passes through the point (0,5). Write down the equation of the circle.

20. Find the value of x in each of these triangles

a.



b.



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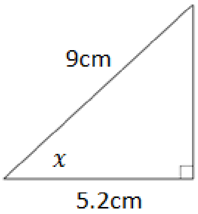

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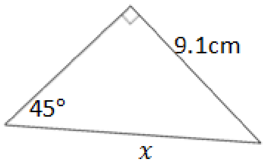


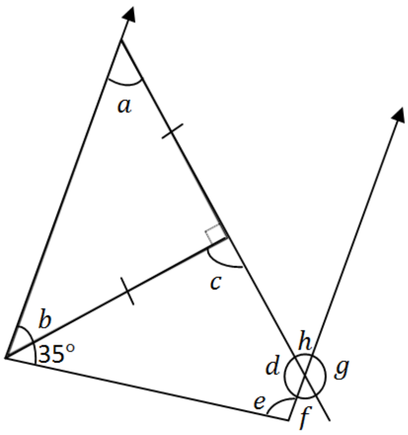
21. A circle has an area of approximately 153.94. Find the circumference.

22. Write down the value of x in each of these cases. Do not use a calculator.

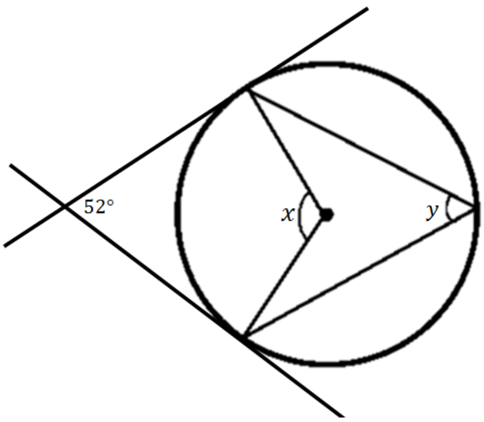
$\sin 30 = x \qquad \cos 45 = x \qquad \sin 60 = x \qquad \cos x = \frac{1}{2} \qquad \tan 60 = x \qquad \sin x = 1 \qquad \tan 0 = x$

23. Find the size of an interior angle in a regular 53 sided polygon.

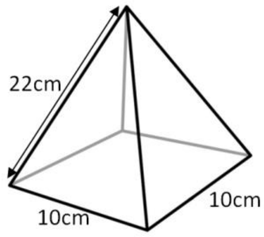
24. Find the missing angles in the diagram below



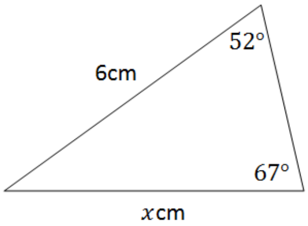
25. Find the value of x and y



26. Find the height of this pyramid



27. Find the value of x



Section Four: Proportion and Statistics

Chapter Eleven: Ratio, Proportion and Change

11.1 Ratio

A ratio allows us to compare the size of two quantities. For example the ratio 2:5 means 2 parts of one quantity and 5 parts of another.
You have already come across one type of ratio earlier in the course. A ratio in the form 1:n is called a scale.

A ratio is in its simplest form when the numbers have no common divisors. They are simplified in the same way as fractions – by dividing each number in the ratio by their common factors.

Example:
Write 10:20 in its simplest form.

Clearly both of these numbers are in the ten times table so dividing by 10 gives 1:2, this can not be simplified any further.

Example:
Write 30:90:60 in its simplest form

Again each of these numbers are in the then times table so dividing by 10 gives 3:9:6, this is not in its simplest form yet as we can also divide by 3 to give 1:3:2

There are three types of ratio question. One type is where you are given the total and asked to divide it into a ratio, for example, divide £50 into the ratio 2:3. The second type is where you are given one side of the ratio and asked to find the total or the other side, for example, red and white paint is in the ratio 3:4, you have 30 litres of red paint, how much white paint do you have?
The final type of question is where you're given a ratio and told how much more one person gets than the other, for example, Sophie and Paul share money in the ratio 2:5, Paul gets £90 more than Sophie, how much do each of them get?

We will deal with each of these question types in turn. It is important that the first thing you do when confronted with a ratio problem is identify which type you have as they are each dealt with differently.

Example:
John and Ian share £50 in the ratio 2:3. Find out how much each person gets.

The first step is to add the numbers in the ratio together: 2 + 3 = 5
Now divide the total by this value: 50 ÷ 5 = 10
This gives you "one part", the ratio tells you that John gets 2 parts and Ian gets 3 parts so you need to multiply either side of the ratio by this value.
John: 2 x 10 = 20
Ian: 3 x 10 = 30
So John gets £20 and Ian gets £30.

Example:
Sam is 12 years old, Abbie is 5 years old and TJ is 8 years old. They have a bag of 50 sweets that they are sharing between them in the ratio of their ages. Find out how many sweets each of them get.

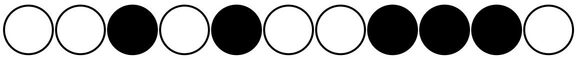
Here the ratio is 12:5:8, it cannot be simplified as there is no number that all of them can be divided by.
Add the numbers in the ratio together: 12 + 5 + 8 = 25
Divide the total by this value: 50 ÷ 25 = 2
Multiply this new value by each side of the ratio:
Sam: 12 x 2 = 24
Abbie: 5 x 2 = 10
TJ: 8 x 2 = 16
So Sam gets 24 sweets, Abbie gets 10 and TJ gets 16.

Activity 11.1 A

1. Write each of these ratios in their simplest form

a. 8:4	b. 5:15	c. 9:18	d. 36:72	e. 100:500
f. 6:3:12	g. 90:50:25	h. 8:16:26	i. 9:36:90	j. 80:72:88

2. What is the ratio of black circles to white circles?



3. Divide £90 into the ratio 3:7

4. Divide £500 into the ratio 2:5:3

5. On the plan of a house a door measures 3cm by 8cm. The scale is 1cm represents 25cm. Calculate the dimensions of the real door.

6. Emma and Kate share £40 in the ratio 3:5 How much do Emma & Kate get each?

7. £7,000 is shared between three schools in the ratio of 2:3:5 How much will each school get?

8. Sophie, Lisa and Charlotte share £250 in the ratio 2:5:3. How much do each of them get?

9. A pot of paint is made up of blue and white paint in the ratio 3:5. How much blue paint is in a pot of 4 litres of paint?

We will now look at how to deal with the second type of ratio question you will encounter, this is when you are given just one side and asked to find the other or the total. Each of these scenarios is dealt with in the same way. The first step is to find the other side(s), then if you are asked to find the total you simply add all of the parts of the ratio together at the end.

Example:
Red and white paint is in the ratio 3:4, you have 30 litres of red paint, how much white paint do you have?

With questions of this sort you write the original ratio out, usually with initials above it to remind you which side is which. You then fill in the side you know about underneath, putting a question mark in any gaps

R : W
3 : 4
30 : ?

Now we have to work out what we need to multiply by to get from the first ratio to the second.

R : W
30 ÷ 3 = 10 3 : 4
x 10
30 : ?

You can see that we are multiplying by 10 to get from one ratio to the other therefore we need to multiply by 10 on the other side to find the value of the question mark.
4 x 10 = 40 so we have 40 litres of white paint.

Example:
Amy, Bethany and Claire share some money in the ratio 4:5:2. Amy gets £2. How much do Bethany and Claire get?

Again we write the ratio out and the values we have, using question marks in place of the numbers we don't know.

A : B : C
4 : 5 : 2
2 : ? : ?

The next step is to see what we are multiplying by

A : B : C
2 ÷ 4 = ½ 4 : 5 : 2
x ½
2 : ? : ?

Note that in this case you may also notice that it is dividing by 2 rather than using multiplying by ½ , either option will work.
Bethany: 5 x ½ = £2.50
Claire: 2 x ½ = £1

Activity 11.1 B

1. The ratio of the length of a car to the length of a van is 2:3. The car has a length of 360cm, what is the length of the van?

2. The ratio of rabbits to foxes is 200:3. If there are 1000 rabbits how many foxes are there?

3. In a pet shop there are dogs and cats in the ratio 3:4.
    a. How many cats are there if there are 12 dogs?
    b. How many dogs are there if there are 8 cats?

4. Fred and John both run trampoline clubs, in a competition the ratio of medals won by each club is 7:3. If John's club won 6 medals how many did Fred's club win?

5. Mary, Wendy and Natalie share the cost of a holiday in the ratio 5:6:3. Mary pays £112.

a. How much does Natalie pay?
b. How much does Wendy pay?
c. How much does the holiday cost altogether?

Finally we will deal with the third type of ratio question, this is where you're given the difference between the amounts. As with the last type you may be asked to find either side or the total, as before these are dealt with in a similar way.

Example:
Sophie and Paul share money in the ratio 2:5, Paul gets £90 more than Sophie, how much do each of them get?

Paul gets 5 – 2 = 3 parts more than Sophie, we also know that he gets £90 more than Sophie so it follows that 3 parts = £90
By dividing by 3 we can see that 1 part = £30 so we can now work out how much each of them get
Sophie: 2 x £30 = £60
Paul: 5 x £30 = £150

Activity 11.1 C

1. Daisy and Zara share sweets in the ratio 3:8. Zara gets 15 more sweets than Zara. How many sweets did they have to begin with?

2. Dave and Dan share money in the ratio 4:9. Dave gets £35 less than Dan. How much do each of them get?

3. A bag contains blue and red balls in the ration 5:6. There are 7 more red balls than blue balls. How many balls are in the bag?

The following questions are mixed so it is up to you to determine which technique to use

4. Simon and Alec share money in the ratio 5:3. Simon receives £16,034, how much does Alec get?

5. There are 40 balls in a bag. These are coloured red, green and blue in the ratio 3:1:4. Find the number of each colour.

6. A bag contains £1 and £2 coins in the ratio 2:5. There are 6 more £2 coins than £1 coins. How many coins are in the bag?

7. Abbie and Amy share £600 in the ratio 3:7. Abbie gives one quarter of her share to Elliot, Amy gives one half of her share to Elliot. What fraction of the total does Elliot receive?

8. A regular polygon has internal and external angles in the ratio 1:2. How many sides does this polygon have?

9. The ratio of pink flowers to purple flowers is 12:24. If there are 3 pink flowers how many purple flowers are there?

10. Oliver and Charlie share £10 in the ratio 9:27, how much do each of them get?

11.2 Percentages

In chapter 2 we saw that percentage means parts per one hundred, it is equivalent to a fraction with a denominator of 100.

Finding a percentage of a number

There are many different ways to calculate percentages. You may find 1% or other easy to find percentages, multiply by the fraction or multiply by the decimal equivalent.

For some percentages there are quick ways of finding them:
10% - divide by 10
1% - divide by 100
50% - divide by 2
25% - divide by 4

Each of the techniques are illustrated for the following examples.

Example:
Find 15% of 40

Option One
Find 10% by dividing by 10
10% of 40 = 40 ÷ 10 = 4
Find 5% by halving 10%
5% of 40 = 4 ÷ 2 = 2
Find 15% by adding these together
15% of 40 = 10% + 5% = 4 + 2 = 6

Option Two
Convert the percentage to a fraction and multiply by this
$15\% = \frac{15}{100}$
$\frac{15}{100} \times 40 = \frac{600}{100} = 6$

Option Three
Convert the percentage to a decimal and multiply by this
15% = 0.15
0.15 x 40 = 6

Example:
Find 34% of 85

Option One
Find 1% by dividing by 100
1% of 85 = 85 ÷ 100 = 0.85
Find 34% by multiplying the value for 1% by 34
34% of 85 = 34 x 0.85 = 28.9

Option Two
Converting to a fraction gives 34% = $\frac{34}{100}$
$\frac{34}{100} \times 85 = \frac{2890}{100} = 28.9$

Option Three
Converting to a decimal gives 34% = 0.34
0.34 x 85 = 28.9

It is entirely up to you which technique you use. There are many different ways of finding percentages so try them all and stick with the one you feel most comfortable with.

<u>Simple Interest</u>

Banks and savings account often pay interest on money saved, this is usually given as a percentage. To calculate simple interest you find the interest for one year then multiply this by the number of years.

Example:
£3851 is placed into a bank with a simple interest rate of 3% per year. Calculate how much money there is after 5 years.

Find 3% of 3851 using the preferred method:
1% of 3851 = 3851 ÷ 100 = 38.51 3% of 3851 = 38.51 x 3 =115.53
or
$\frac{3}{100}$ x 3851 = $\frac{11553}{100}$ = 115.53
or
0.03 x 3851 = 115.53

So the interest for one year is £115.53 therefore the interest over 5 years is £115.53 x 5 = £577.65
The total money in the account after 5 years is £3851 + £577.65 = £4428.65

Activity 11.2 A

1. Calculate each of these percentages
   a. 50% of 73 b. 25% of 92 c. 10% of 73 d. 1% of 83 e. 50% of 88
   f. 25% of 82 g. 10% of 743 h. 1% of 719 i. 75% of 829 g. 20% of 829

2. Calculate each of these percentages
   a. 3% of 382 b. 27% of 829 c. 36% of 9000 d. 24% of 810 e. 24% of 289

f. 89% of 89 g. 99% of 800 h. 28% of 801 i.83% of 938 f. 28% of 183

3. £4702 is placed into a bank with a simple interest rate of 5% per year. Calculate how much money there is after 4 years.

4. Calculate the amount of interest gained over 10 years when £4780 is placed into a bank with a simple interest rate of 2%

<u>Percentage Change</u>

Percentages are used in real life to show how much an amount has increased or decreased. It's not uncommon, for example, to see signs declaring "30% off" outside a shop.

To calculate a percentage increase you work out the percentage and then add it on to the original value. To calculate a percentage decrease you work out the percentage and then subtract it from the original value.

Example:
Increase 50 by 20%

First find the percentage
20% of 50 = 10
Then add it on
50 + 10 = 60

Example:
Decrease 67 by 45%

45% of 67 = 30.15
67 – 30.15 = 36.85

You can also calculate percentage increases and decreases by multiplying by decimals.
To increase by a x% you multiply by the decimal equivalent of $(100+x)$%

Example:
Increase 29 by 37%

We need to multiply by the decimal equivalent of 137%, this is 1.37
29 x 1.37 = 39.73

Example:
Increase 24 by 2%

Here we would multiply by 1.02 to give 24 x 1.02 = 24.48

To calculate a percentage decrease of x% using this method you multiply by the decimal equivalent of $(100-x)$%.

Example:
Decrease 23 by 23%

We need to multiply by the decimal equivalent of 100 – 23 = 77% which is 0.77
23 x 0.77 = 17.71

Example:
Decrease 28 by 95%

100 – 95 = 5
28 x 0.05 = 1.4

As before it is entirely up to you whether you decide to find the percentage and then add/subtract as appropriate or you use decimals. When choosing your technique with percentages, though, it is a good idea to become relatively comfortable with more than one and make sure you don't rely too heavily on the calculator to work out your answers.

With regards to percentage change you will sometimes have to find the percentage that a value has increased or decreased by. To find the percentage change you find the difference, put this as a fraction of the original and then convert to a percentage by multiplying by 100%. (This is essentially the same thing as multiplying by 100 then writing % after the answer!).

Example:
A value is increased from 20 to 80, find the percentage change.

First we need to find the difference: 80 – 20 = 60
Now we put this as a fraction of the original and convert to a percentage – you may do this by finding an equivalent fraction with a denominator of 100, as was seen in chapter 2, or you can use the technique described above which is illustrated below
$\frac{60}{20} \times 100\%$ = 300% increase

Don't be put off by the fact that this is over 100%, we will illustrate increasing 20 by 300% below to check the answer is correct.
100% of 20 = 20
300% of 20 = 20 x 3 = 60
Increasing by 300% gives 20 + 60 = 80

Example:
A value is decreased from 85 to 70, find the percentage change.

The difference is 85 – 70 = 15
Putting this as a fraction of the original and then converting to a percentage gives
$\frac{15}{85}$ x 100% ≅ 17.65% decrease

Activity 11.2 B

1. Calculate these percentage changes
    a. Increase 36 by 40%   b. Increase 26 by 75%   c. Decrease 24 by 22%

d. Decrease 34 by 10% e. Increase 28 by 38% f. Decrease 294 by 23%

2. A pair of jeans costing £50 are reduced by 20% in a sale, what price are they now?

3. A TV costs £300 plus 20% VAT. What is the price when VAT is added on?

4. If I multiply by 1.56 am I increasing or decreasing? By what percentage?

5. If I multiply by 0.73 am I increasing or decreasing? By what percentage?

6. If I multiply by 1.02 am I increasing or decreasing? By what percentage?

7. Lily got 34 out of 40 on a test, what percentage did she get?

8. In a class of 40 there are 23 girls, what percentage are boys?

9. A t-shirt was reduced from £5 to £4, what percentage was it reduced by?

10. A sofa was increased from £400 to £450, what percentage was it increased by?

11. A value was increased from 80 to 100, find the percentage change.

12. The cost of an item changed from £50 to £44.99, find the percentage change.

13. A TV original cost £500, it was reduced by 30% then by a further 20%. Find the new price and the percentage change between the original price and the final sale price.

14. In a sale an item was reduced by 8.5%. It was originally £20, find the new price.

15. If I multiply by 1.765 am I increasing or decreasing? By what percentage?

<u>Original Value</u>

Original value problems are sometimes called reverse percentage problems. In these situations you are given an amount and told what the percentage change was, you have to find the original amount.

As is the norm with percentages there are a number of different ways of solving problems of this sort, two of which will be explained here. You can either work out what percentage of the original amount you have and use this to find 1% which then enables you to find 100%, which would be the original amount, or you can work out what you would have multiplied by to bring about the percentage change and divide by this.

Example:
A pair of trousers are in a 20% off sale. They now cost £65, how much did they cost originally?

Option One
The value has 20% off so £65 is 80% of the original price, that is
80% of original price = £65

Dividing by 80 we are able to find 1% (note, you could also work from 10% instead of 1%)
1% of original price = £65 ÷ 80 = £0.8125
We can now move from this to 100% by multiplying by 100
100% of original price = £0.8125 x 100 = £81.25

Option Two
The trousers have been reduced by 20%, in order to do this you would have multiplied by 0.8 therefore the original value is found by "undoing" this and dividing by 0.8
£65 ÷ 0.8 = £81.25

Example:
A price of a TV is increased by 5% to £126, what was the original price.

Option One
The price has increased so we have
105% of original price = £126
1% of original price = £126 ÷ 105 = £1.20
100% of original price = £1.20 x 100 = £120

Option Two
To increase by 5% you would have multiplied by 1.05 so we have
£126 ÷ 1.05 = £120

Activity 11.2 C

1. I buy a jumper for £40 in a sale where it was 20% off. What was the original price?

2. The price of a TV, including VAT of 20%, is £144. What is the price without VAT?

3. A price increased by 25% to £1.50. What was the original price?

4. The price of an item increased by 15% to £172.50. What was the original price?

5. In a class there are 9 people absent. This is 20% of the class. How many are there in the class when no-one is off ill?

6. A book was reduced by 12% to £8.79, what did it cost originally?

11.3 Proportion

Direct Proportion

Two variables are in direct proportion with one another if they change at the same rate. The graph illustrating their change will be a straight line that passes through the origin with the equation $y = kx$ for some integer *k* that is called the rate of change.

$y \propto x$ is used to denote "y is proportional to x"

In order to calculate one value from the other you need to know the value of the rate of change. They can also be solved as ratio problems in the way that you learnt previously.

If you are told that $y \propto x$ you rewrite this as $y = kx$ and solve for *k* in the way illustrated below.

Example:
A 500ml bottle of lemonade costs £2.50. Assuming the amount of lemonade is directly proportional to the cost find the price of a 300ml bottle.

Using the equation $y = kx$ we have 500 = *k* x 2.5 which we have to solve for *k*

k = 500 ÷ 2.5 = 200
So our equation becomes $y = kx$ where y is the amount of lemonade and x is the cost so, with a 300ml bottle, we have
$300 = 200x$
Solving for x gives
$x =$ 300 ÷ 200 = 1.5
So a 300ml bottle costs £1.50

Activity 11.3 A

In each of the questions below assume the variables given are in direct proportion with one another

1. Given that 400g costs £2 find
   a. The cost of 100g
   b. The cost of 300g
   c. How much you could get for £3.50

2. Amy buys 35g of chalk for £1.40. Abbie buys 20g of chalk, how much does it cost her?

3. Given that 1kg = 2.2lbs find each of these unknowns
   a. 5kg = ___lbs b. 10kg = ___lbs c. 3kg = ___lbs
   d. 8lbs = ___kg e. 10lbs = ___kg f. 0lbs = ___kg
   g. Use some of the values you have found to draw a conversion graph.

4. A pack of 24 crayons costs £2.70. A second pack of 30 crayons costs £3.10. Which one is the better value?

5. Given that £1 = $1.30 draw a conversion graph and use this to find
a. The value of £3 in $
b. The value of $6 in £

6. In store A a packet of tissues costs 50p for 100. Store B sells the same brand of tissues in packs of 150 for 70p. Which is the better value?

Inverse Proportion

Two variables are in inverse proportion with one another if one increases at the same rate as the other decreases. The graph illustrating their change will be a reciprocal curve with the equation $y = \frac{k}{x}$ for some integer *k* that is the constant of proportionality.
As a general rule we only require the positive curve.

You met reciprocal graphs in chapter 5. If you are not confident with how to plot them you should revise the appropriate section now.

Inverse proportion problems are dealt with in a similar way to direct proportion ones. You should use values you have been given to find the value of *k*, this can then be used in the equation $y = \frac{k}{x}$ to find other values and plot the graph if needed.

$y \propto \frac{1}{x}$ is used to denote "y is proportional to $\frac{1}{x}$" or "y is inversely proportional to x"

Example:
The number of builders is inversely proportional to the amount of time a job takes.
If 10 builders take 5 days to complete a job, how long would the same job take 20 builders?

If we substitute the values we have into the equation $y = \frac{k}{x}$ we have
$10 = \frac{k}{5}$
Solving for *k* gives *k* = 50 therefore the equation becomes $y = \frac{50}{x}$

To find out how many days it would take 20 builders we have $20 = \frac{50}{x}$
Solving for x gives us $x = \frac{5}{2} = 2.5$
Therefore the job would take 20 builders 2 and a half days.

Activity 11.3 B

1. Given that $y \propto \frac{1}{x}$ and that $y = 5$ when $x = 6$ find
   a. The value of x when $y = 2$
   b. The value of y when $x = 3$

2. If y is inversely proportional to the square of x and $y = 4$ when $x = 3$ find
   a. The value of x when $y = 3$
   b. The value of y when $x = 6$

3. A journey takes 4 hours if it is travelled at a constant speed of 20km/h, how long will the same journey take if it is travelled at a constant speed of 30km/h.

4. A hotel has enough food to feed 400 guests from 3 weeks. How long would the same amount of food last 500 guests?

5. Given that $y \propto \frac{1}{x}$ and that $y = 1$ when $x = 2$. Sketch a graph to illustrate the rate of change.

Growth and Decay

You have already seen how to increase or decrease amounts by percentages as well as how to calculate simple interest. You should revise section 11.2 now if you are not confident with percentage change.

As we saw before simple interest is an amount that is calculated from the initial amount and then added on with each subsequent year. **Compound interest**, on the other hand, is where the interest is calculated each year. These means we calculate the percentage of the new number – which is the original amount plus any interest already earned.

To calculate compound interest you can simply work it out for each year as shown in the following example.

Example:
£5000 is placed into a bank with a compound interest rate of 2%. Find the amount in the bank after 3 years.

Initial amount: £5000
After 1 year: £5000 x 1.02 = £5100
After 2 years: £5100 x 1.02 = £5202
After 3 years: £5202 x 1.02 = £5306.04

As you can see if the problem referred to a large number of years this method would be time consuming and impractical. In order to overcome this problem we raise the multiplier to the power n where n is the length of time.
If a value continues to increase by the same percentage it is called exponential growth.

Example: (Same question as above)
£5000 is placed into a bank with a compound interest rate of 2%. Find the amount in the bank after 3 years.

The multiplier is 1.02 (multiplier refers to the number you need to multiply by in order to calculate the percentage change). The power is the number of years which is 3. So we have
£5000 x 1.02^3 = £5306.04

The same method can be used for decay, this refers to situations where the amount decreases over time. If a value continues to decrease by the same percentage it is called exponential decay.

Example:
A car costs £10,000 new. It decreases in value by 1% each month. Find the value of the car after a year.

Here the multiplier is 0.99 and the power is 12.

£10,000 x $0.99^{12} = 8863.848717 \ldots \cong £8863.85$

Activity 11.3 C

1. £3000 is placed into a bank with a compound interest rate of 3%. Find the amount in the bank after 10 years.

2. £9000 is placed into a bank with a compound interest rate of 2%. Find the amount in the bank after 13 years.

3. £11,000 is placed into a bank with a compound interest rate of 1.5%. Find the amount in the bank after 9 years.

4. The number of rabbits in a given area reduces by 4% each year. If there are 3,000 initially how many are there after 7 years?

5. The value of a car falls by 13% each year. It cost £43,000 new, how much is it worth after 5 years?

6. The estimated population of a city is given by 450×0.95^n where n is the number of years from now.
   a. What is the population now?
   b. What is the estimated population after 10 years?
   c. What does this formula tell you about the assumptions made?

7. It is estimated that the population of a city increases by 2.5% each year. The population is initially 21,352
   a. What is the population of the city after 15 years?
   b. Why might your answer to part a be wrong?

<u>Converting Units</u>

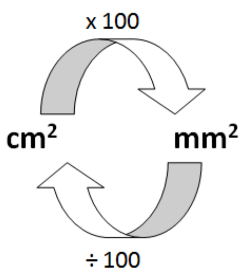
You have already seen how the standard metric units relate to one another. If you are unsure about this take a moment to revise section 3.4.

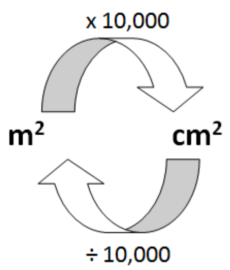
As well as converting between units for length you also have to be able to convert between units used for area and volume.

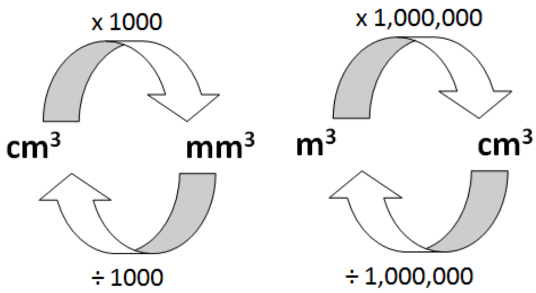
This is done as follows
$1cm^2 = 100mm^2$ (since $1cm^2 = 10 \times 10\ mm^2$)
$1m^2 = 10{,}000cm^2$ (since $1m^2 = 100 \times 100cm^2$)

$1cm^3 = 1000mm^3$ (since $1cm^3 = 10 \times 10 \times 10mm^3$)
$1m^3 = 1{,}000{,}000cm^3$ (since $1m^3 = 100 \times 100 \times 100cm^3$)

So we have







Example:
Convert from $3m^2$ to cm^2

$3m^2 = 3 \times 10{,}000\ cm^2 = 30{,}000cm^2$

Activity 11.3 D

1. Convert each of these to mm
   a. 3cm b. 8cm c. 1.1cm d. 7.2cm e. 8m

2. Convert each of these to cm
   a. 90m b. 85mm c. 7m d. 10mm e. 100m

3. Convert each of these to m
   a. 1700cm b.725cm c.9km d.8000mm e.19cm

4. Convert these measurements to cm^2
   a. $28mm^2$ b. $90m^2$ c. $100mm^2$ d. $5m^2$ e. $10m^2$

5. Convert these measurements to m^2
   a. $80cm^2$ b.$67cm^2$ c.$800mm^2$ d. $10cm^2$ e.$80mm^2$

6. Convert these measurements to cm^3
   a. $190mm^3$ b.$80m^3$ c.$5m^3$ d.$100mm^3$ e.$3.2m^3$

7. Convert these measurements to m^3
   a.$90cm^3$ b.$2cm^3$ c.$5.6cm^3$ d. $50mm^3$ e.$1000mm^3$

8. A rectangle has a height of 5cm and a width of 6cm. Find the area in mm^2

9. A square has sides of 200mm. Find the area in m^2

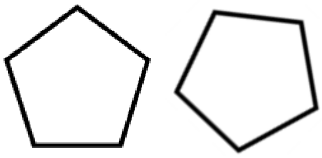
11.4 Congruence and Similarity

In section 9.1 we briefly encountered the concept of congruent and similar shapes.

Congruence

Congruent shapes have the same size angles and sides. If they were to be cut out they'd fit on top of each other perfectly.

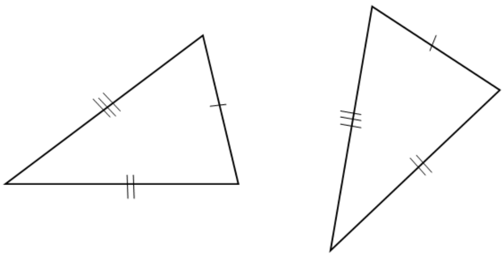
For example, these two shapes are congruent despite the fact that they are not "the same way up"



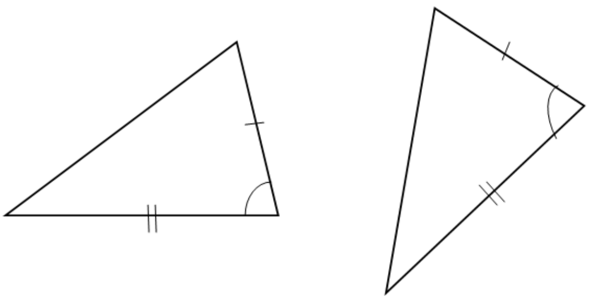
There is a shorthand often used to denote different pairs of triangles. Each of these can be used to show that triangles are congruent.
It is advised that you learn what each of these stand for.

SSS	Three equal sides
SAS	Two sides and the angle between them is equal
ASA	Two angles and a corresponding side are equal
RHS	A right angle, the hypotenuse and another side equal

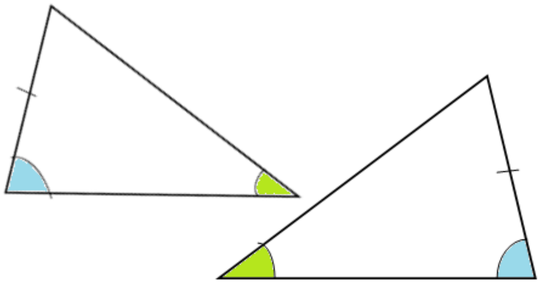
SSS



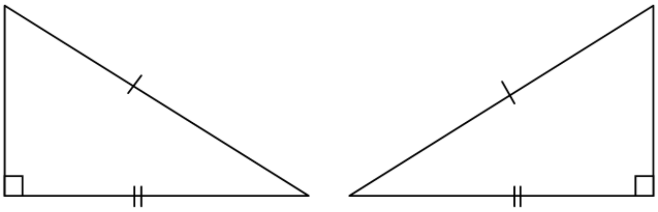
SAS



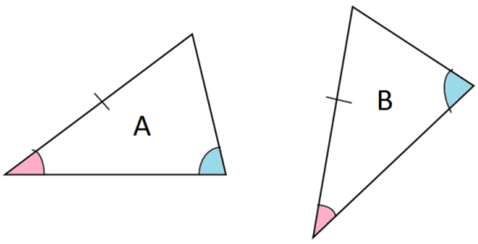
ASA



RHS

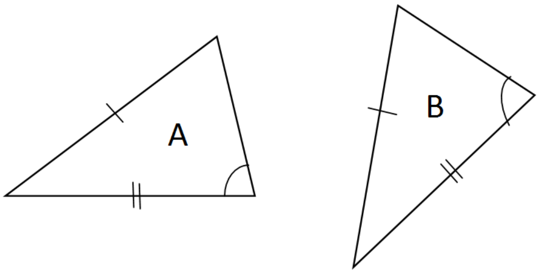


Example:
Is A congruent to B?



Yes. They are ASA triangles. They have a pair of equal angles and the equal sides are corresponding because they are both opposite the blue angle.

Example:
Are A and B congruent?

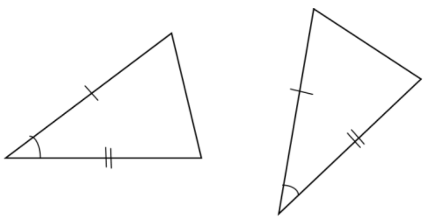


No. At first glance they look like SAS triangles but the angle is not between the equal sides.

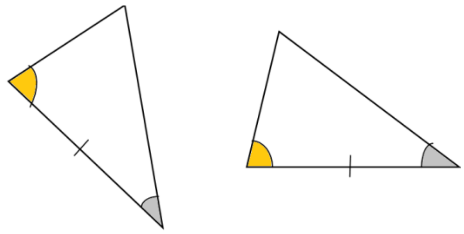
Activity 11.4 A

1. Write down whether each pair of triangles are congruent. If they are state which type they are.

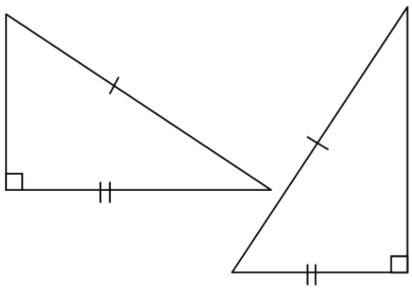
a.



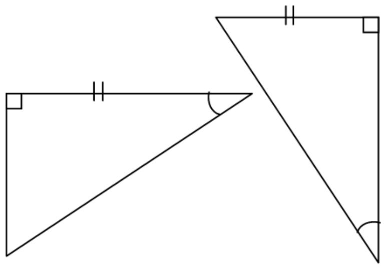
b.



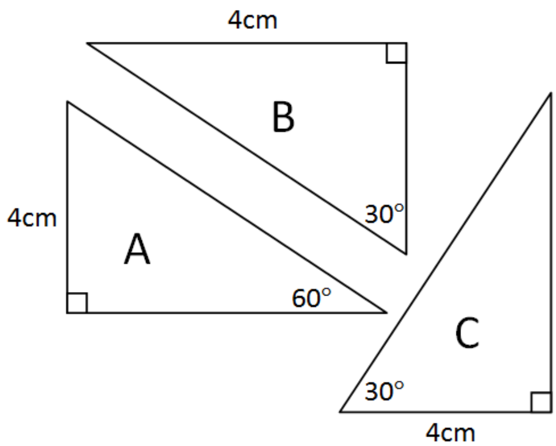
c.



d.



2. Which two triangles are congruent. Give your reasons.

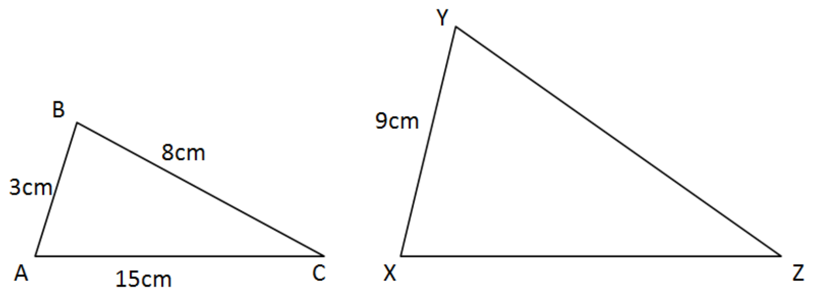


<u>Similarity</u>

Similar shapes are different in size but are the same shape and have the same angles. Their sides have been multiplied or divided by the same scale factor.

Example:

These two triangles are similar. Find the length of XZ and YZ.



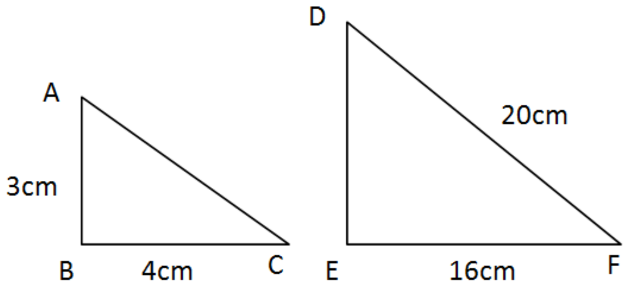
The scale factor is found by dividing the length of a side in one triangle by the length of the corresponding side in the other triangle.

In this example the scale factor is $\frac{XY}{AB} = \frac{9}{3} = 3$
This means that the lengths of the sides in the larger triangle are found by multiplying the sides in the smaller triangle by 3.

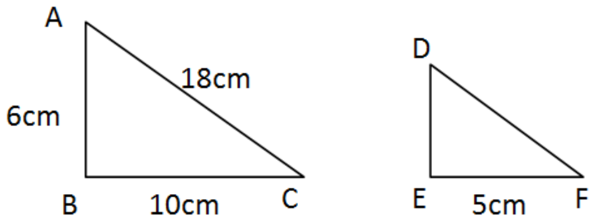
Therefore XZ = 15 x 3 = 45cm
YZ = 8 x 3 = 24cm

Activity 11.4 B

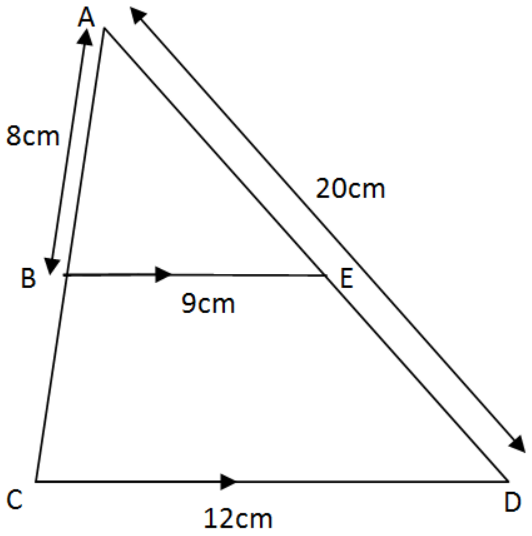
1. These triangles are similar. Find the length of DE and AC



2. These triangles are similar. Find the length of DE and DF

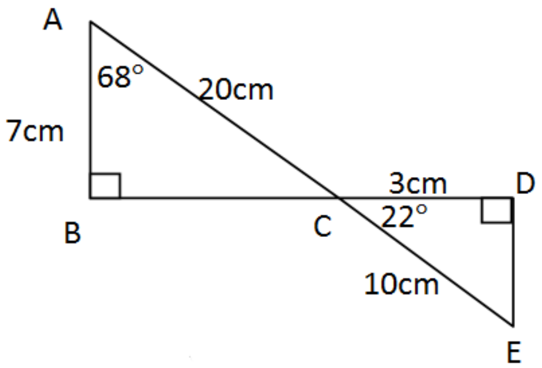


3.



a. Show that triangle ABC is similar to triangle ADE
b. Find the length of AE, ED and BC

4.



a. Show that triangle ABC is similar to triangle CDE
b. Find the length of BC and DE

5. A box of 1kg of pasta has a height of 20cm, a width of 18cm and a depth of 12cm. A similar box holds 250g. Find the dimensions of this smaller box.
