GCSE Mathematics Higher

AQA Specification 8300

Part 3

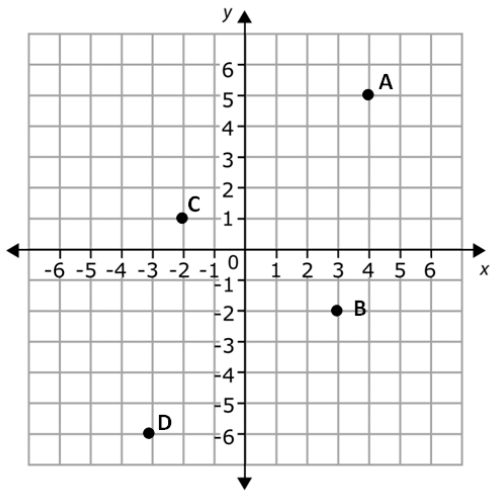
Chapter Five: Graphs

5.1 Straight Line Graphs

Plotting Straight Line Graphs

A coordinate grid uses a set of axes; an $\boldsymbol{x}$ **axis** which is horizontal and an $\boldsymbol{y}$ **axis** that is vertical. These cross at a point called the **origin**. A point on the grid is described in terms of it's x and y coordinates denoted as (x, y).

Example:



The coordinates of the points in the diagram are as follows
A (4 , 5)   B (3 , -2)   C (-2 , 1)   D (-3 , -6)

In order to plot a straight line graph for a given equation first draw an input-output table to work out the coordinates of points that lie on the line, plot these points then join them up with a ruler.

It is advised that you find a minimum of three or four points. That way, if there is an error in calculation it will be highlighted as the points will only form a straight line if the calculations have been done correctly.

Example:
Draw the graph of $y = 3x + 2$

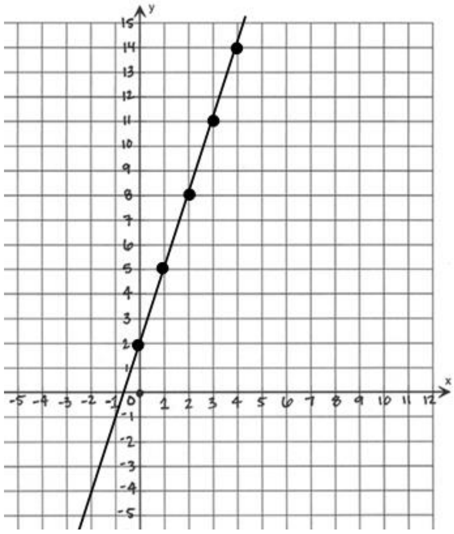
First draw an input-output table as follows. You can choose any values for x.

$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{y}$					

Now substitute each of these values of x into $y = 3x + 2$ to find the values for y
For example $y = 3 \times 0 + 2 = 2$

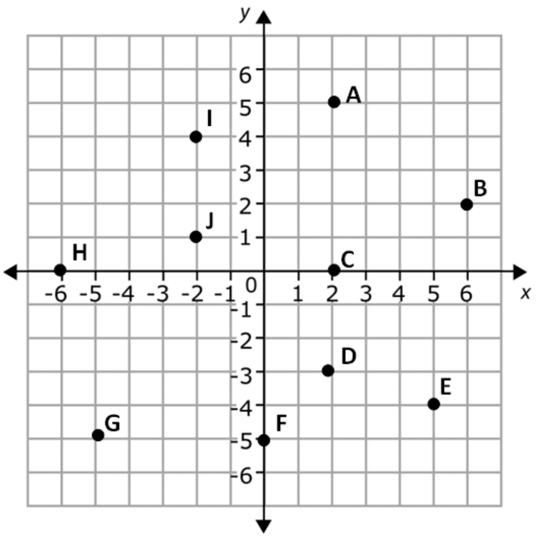
$\boldsymbol{x}$	0	1	2	3	4
$\boldsymbol{y}$	2	5	8	11	14

Next you would plot each of the coordinates, (0,2), (1,5), (2,8)... and join them up. Continue extending the line past the end of the plotted points to give the required graph.



Activity 5.1 A

1. Write down the coordinates of each of the points in the diagram below



2. Draw a set of axes and plot each of the following points

a. (5 , 5)	b. (9 , 3)	c. (5 , -3)	d. (4 , -7)	e. (-4, -3)	f. (-8 , -7)
g. (-2 , 2)	h. (-8 , 5)	i. (9 , -9)	j. (2, 0)	k. (-9 , 0)	l. (0 , 9)

3. Draw each of these graphs

a. $y = x + 2$	b. $y = x - 4$	c. $y = 2x$	d. $y = 2x + 3$	e. $y = 3x - 1$
f. $y = x$	g. $y = 5 - x$	h. $y = 8 - 2x$	i. $y = 2(x + 1)$	j. $y = \frac{1}{2}x$
k. $x = 3$	l. $y = -3$	m. $x = -1$	n. $y = -2x$	o. $y = -3x + 2$

4. By rearranging to make y the subject, or otherwise, draw each of these graphs

a. $2x + y = 4$ b. $x + y = 10$ c. $x - y = 6$ d. $3x - y = 5$ e. $4x + 5y = -20$

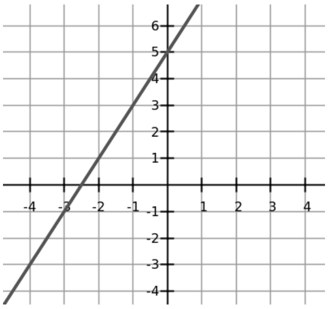
Equation of a Straight Line 1

The equation of a straight line is, generally, given in the form $y = mx + c$.
m gives the **gradient** of the line. The gradient of a line describes how steep it is.
c gives the $\boldsymbol{y}$ **intercept**, this is the point on the y axis that the line passes through.

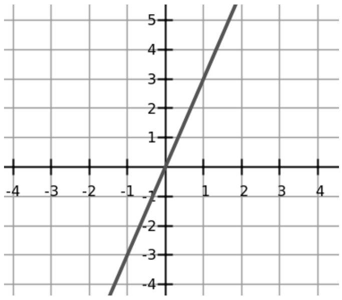
Finding the y intercept:
Simply look at the graph and, by observation, take c to be the point at which the line crosses the y axis.

Example:
Find the y intercept, c, of each of these graphs.

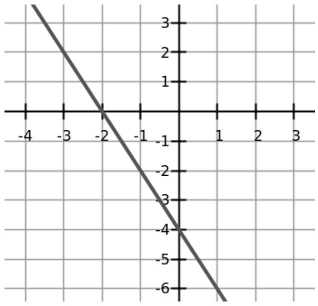
a.



b.



c.



a. $c = 5$

b. $c = 0$

c. $c = -4$

Finding the gradient:
There are two methods for finding the gradient.

Option One: Using the diagram
Place your pencil on any point that the line goes through exactly, draw one space in the positive x direction (to the right) then continue your pencil line up or down until you reach the line again. The number of spaces you have travelled is the gradient – if you went up it is positive, if you went down it is negative.

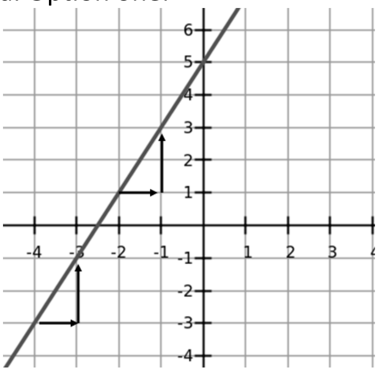
Option Two: Using the equation
Pick any two points on the line, these will be called $(x_{1,}y_1)$ and (x_2, y_2) - this merely denotes that you have two points; the answer will be the same regardless of which points you choose.
Use the values from the points you have chosen in the formula

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Alternatively you can think of this as
(the difference between the y values) ÷ (the difference between the x values)

Example:
Using the same graphs as before, find the gradient, m, of each of the lines.

a. Option one:



The arrows have been drawn in two different places to illustrate that the answer is the same regardless of where you choose to begin.
Our arrows go up two places so the gradient is 2.

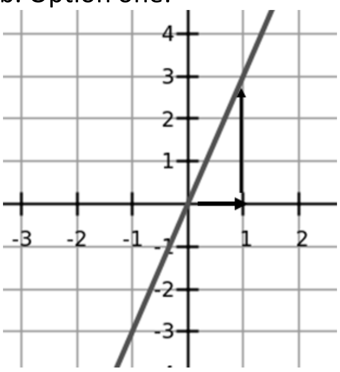
Option two:
Choose 2 points on the line, say (-4, -3) and (-3, -1) using the formula we have

$$\frac{-3 - -1}{-4 - -3} = \frac{-2}{-1} = 2$$

Make sure you remember that the y values are at the top of the fraction

As before we have a gradient of 2. This is true regardless of the points chosen, for example, we could also choose to use (-1, 3) and (0, 5)
The difference between the y values is 2, the difference between the x values is 1 so we have 2÷1=2

b. Option one:

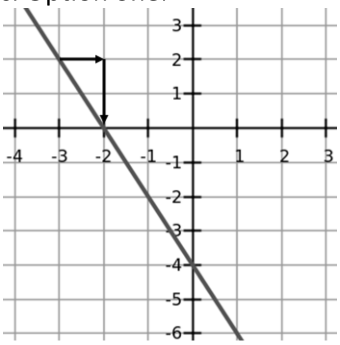


The arrow goes up 3 places so $m = 3$

Option two:
Here we will use the points (0,0) and (1,3)
Any points will work but if it is possible to use positive numbers and zeros it makes the calculations simpler.

$$m = \frac{3 - 0}{1 - 0} = 3$$

c. Option one:



Here the vertical arrow goes down two places so the gradient is -2

Option two:
Here we will choose the points (-3, 2) and (-2, 0)

$$m = \frac{0-2}{-2--3} = \frac{-2}{1} = -2$$

It seems obvious that the simplest option is the first but it is important to know how to use the second as there are occasions where a diagram won't be supplied. Also, as a general rule, the first method is only successful when the gradient is an integer value – the second method will work in every case.

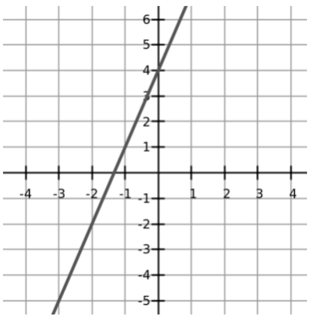
Once the values for m and c - the gradient and the y intercept respectively – have been found it is a simple matter of substituting them into $y = mx + c$ to determine the equation of the line.

Example:
For each of the graphs in the last two examples write down the equation of the line.
a. We found $c = 5$ and $m = 2$ so the equation is $y = 2x + 5$
b. $c = 0$ and $m = 3$ so the equation is $y = 3x$
d. $c = -4$ and $m = -2$ so the equation is $y = -2x - 4$

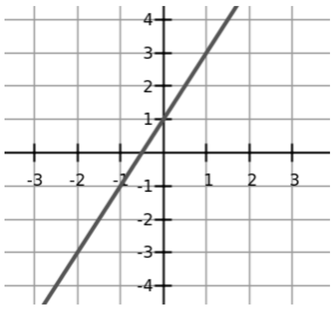
Activity 5.1 B

1.Find the equation of each of these lines in the form $y = mx + c$

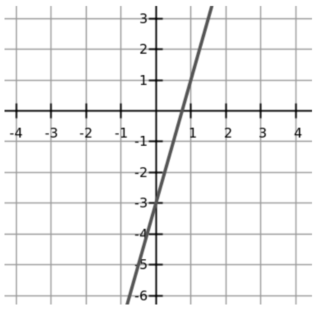
a.



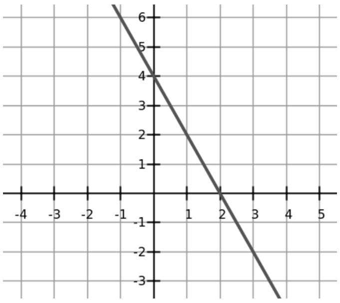
b.



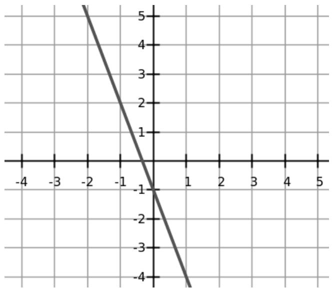
c.



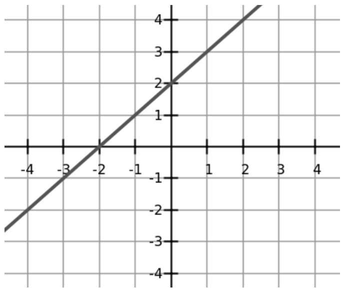
d.



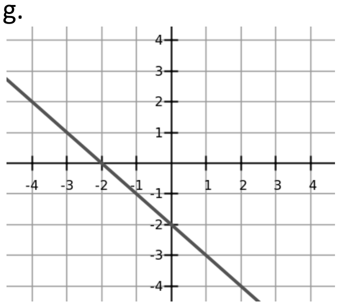
e.



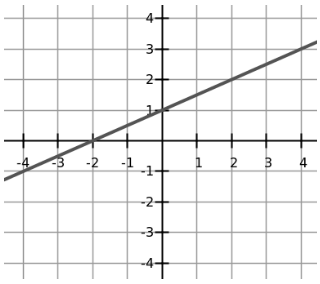
f.



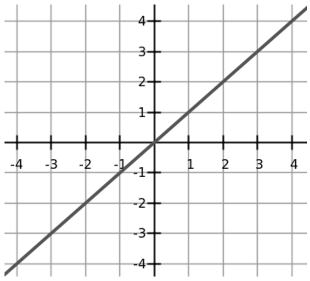
g.



h.



i.



2. Find the gradient of the line which passes through the points given

a. (2,3) and (4,6) b. (6, -3) and (12, -5) c. (1,2) and (3,4) d. (-3, 5) and (2, 10)
e. (-15, 10) and (5, 20) f. (1, -12) and (-2, -3) g. (1, 12) and (2,2) h. (0,0) and (8, 16)

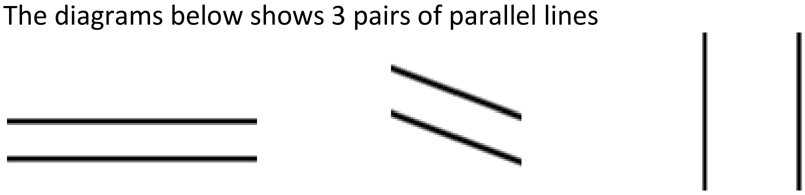
3. Write down the value of the gradient and the y intercept of each of these lines

a. $y = 6x + 2$ b. $y = 2x - 1$ c. $y = 7x + 18$ d. $y = -2x - 5$
e. $y = x + 2$ f. $y = -x - 6$ g. $y = 9 - 2x$ h. $2y = 4x - 10$
i. $x + y = 10$ j. $6x + 3 = y$ k. $8x + 4y = 16$ l. $y = 6x$

Parallel Lines

Parallel lines are the same distance apart everywhere along their length – they have the same gradient.

The diagrams below shows 3 pairs of parallel lines



Because parallel lines have the same gradient any straight lines with the same value for m are parallel, regardless of where they cross the y axis.

Example:
Write down an equation of a line parallel to $y = 8x + 2$

Here we need any equation of the form $y = 8x + c$ where c could be any number. For example $y = 8x - 100$

Example:
Find the equation of the line parallel to $y = 5x - 2$ that has y intercept 10.

$y = 5x + 10$

Activity 5.1 C

1. Write down an equation of a line parallel to $y = 320x - 198$

2. Find the equation of the line parallel to $y = 2x + 10$ that has y intercept 34

3. Find the equation of the line parallel to $y = 8x + 1$ that passes through (0,18)

4. Find the equation of the line parallel to $y = 3x - 23$ that passes through (0,27)

5. Find the equation of the line parallel to $4y = 8x + 12$ that passes through (0,5)

6. Give the equation of the line parallel to $y = 7x + 10$ that has the same y intercept as $y = 2x - 3$

7. Give the equation of the line parallel to $3y = 6x + 12$ that has the same y intercept as $x + y = 3$

8. Line A passes through the point (4,5). Line B also passes through the point (4,5). Are these two lines parallel? How do you know?

Equation of a Straight Line 2

You know how to find the equation of a line from a graph or when you're given the gradient and the y intercept but sometimes you don't have this information. Sometimes you're given two points that the line passes through and other times you're given the gradient and one point that the line passes through.

If you're given two points the first thing you need to do is find the gradient using the same method as before. Once you've done this you can choose one point to work with and discard the other, from then on the two problems are equivalent.

There are two methods for finding the equation of a line with this information. The most common one is to substitute the known values for m, x and y into the equation $y = mx + c$ and work out the value for c.
If you find this difficult you can substitute the values into $y - y_1 = m(x - x_1)$, where x and y remain as unknowns, this can be rearranged into the required form without having to find any more values.
The first is the method you are expected to use although there is nothing wrong with using the second method if you prefer it – the only problem is that it is one more equation to remember!

The following examples will use both techniques.

Example:
Find the equation of the line that has gradient 3 and passes through (2,10)

Option One:
Here we have $m = 3, x = 2$ and $y = 10$
Substituting these into $y = mx + c$ gives
$10 = 3 \times 2 + c$
$10 = 6 + c$
So we have $c = 4$, therefore the equation of the line is $y = 3x + 4$

Option Two:
Here we have $m = 3, x_1 = 2$ and $y_1 = 10$
Substituting these into $y - y_1 = m(x - x_1)$ gives $y - 10 = 3(x - 2)$
Rearranging gives
$y - 10 = 3x - 6$
$y = 3x - 6 + 10$
$y = 3x + 4$

Example:
Find the equation of the line that passes through (2,3) and (4,7)

With either option the first thing you need to do is to find the gradient.
$$m = \frac{change\ in\ y}{change\ in\ x} = \frac{7-3}{4-2} = \frac{4}{2} = 2$$

Option One:
Here we have $m = 2$. We can choose either of the points to give us the x and y values. Here we will use (2,3) simply because the numbers are smaller and easier to work with so we have $x = 2, y = 3$.
Substituting into $y = mx + c$ gives
$3 = 2 \times 2 + c$
$3 = 4 + c$
Therefore $c = -1$ (because, by observation, $3 = 4 - 1$ or, by rearranging, $c = 3 - 4 = -1$) so the equation of the line is $y = 2x - 1$

Option Two:
As before we have $m = 2, x_1 = 2, y_1 = 3$
Substituting these into $y - y_1 = m(x - x_1)$ gives $y - 3 = 2(x - 2)$
Rearranging gives
$y - 3 = 2x - 4$
$y = 2x - 4 + 3$
$y = 2x - 1$

Example:
Find an equation of the line parallel to $y = 5x + 3$ that passes through (3, 15)

Since the line must be parallel to $y = 5x + 3$ the gradients must be the same so $m = 5$

Option One:
Substituting $m = 5, x = 3$ and $y = 15$ into $y = mx + c$ gives

$15 = 5 \times 3 + c$
$15 = 15 + c$
Therefore $c = 0$ so the equation of the line is $y = 5x$

Option Two:
Substituting $m = 5, x_1 = 3$ and $y_1 = 15$ into $y - y_1 = m(x - x_1)$ gives
$y - 15 = 5(x - 3)$
$y - 15 = 5x - 15$
$y = 5x - 15 + 15$
$y = 5x$

Activity 5.1 D

1. Use the information given to find the equation of the line
   a. Gradient is 2, passes through (4, 10)
   b. Gradient is 3, passes through (6,20)
   c. Gradient is 1, passes through (5, 6)
   d. Gradient is 2, passes through (3, 5)
   e. Gradient is -3, passes through (2,7)
   f. Gradient is 5, passes through (-2, -10)

2. Use the information given to find the equation of the line
   a. Parallel to $y = 7x - 19$, passes through (2, 15)
   b. Parallel to $y = 2x + 9$, passes through (3, 8)
   c. Parallel to $y = 8 - 2x$, passes through (5, 5)
   d. Parallel to $y = x$, passes through (6, 4)

3. Find the equation of the line that passes through the points
   a. (1, 2) and (3, 4)
   b. (-1, 5) and (4, -5)
   c. (2, 0) and (4, 1)
   d. (2, 2) and (1, 4)

4. A line passes through (-2, 16) and (2, 8). It also passes through (10, a). Find the value of a.

5. A line passes through the points (8,2), (10,6) and (7, b). Find the value of b.

Perpendicular Bisector

The perpendicular bisector of a line cuts the line through its midpoint and is at right angles to it.

To find the midpoint of a line you need to know the coordinates of the points at either end. You then use the formula
$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

The gradient of the perpendicular bisector is found using the fact that the gradients must multiply together to give -1. That is, the gradient of the perpendicular bisector is the negative reciprocal of the gradient of the straight line. For example, if the gradient of the straight line is 3 the gradient of the perpendicular bisector is $-\frac{1}{3}$

Example:
A line segment passes through (3,4) and (7,10). Find the equation of the perpendicular bisector.

Midpoint: $\left(\frac{3+7}{2}, \frac{4+10}{2}\right) = (5,7)$

Gradient of the line segment: $\frac{y_2-y_1}{x_2-x_1} = \frac{10-4}{7-3} = \frac{6}{4} = \frac{3}{2}$
Gradient of the perpendicular bisector = $-\frac{2}{3}$

Equation of the line is given by
$y - y_1 = m(x - x_1)$
$y - 7 = -\frac{2}{3}(x - 5)$
$y - 7 = -\frac{2}{3}x + \frac{10}{3}$
$y = -\frac{2}{3}x + \frac{10}{3} + 7$

$y = -\frac{2}{3}x + \frac{31}{3}$

Activity 5.1 E

1. Given each pair of coordinates find the midpoint
   a. (3,5) and (6,10) b. (2,7) and (8,4) c. (-2,4) and (0,-5) d. (0,2) and (8,11)

2. Write down the negative reciprocal of each of these
   a. 6 b. 0 c. $\frac{1}{2}$ d. -5 e. -7 f. $-\frac{1}{3}$ g. $-\frac{5}{4}$ h. $\frac{8}{9}$

3. Given each pair of coordinates, find the perpendicular bisector of the line segment
   a. (9,0) and (2,8) b. (2,3) and (0,-1) c. (-9,-2) and (10,2) d. (1,2) and (3,1)

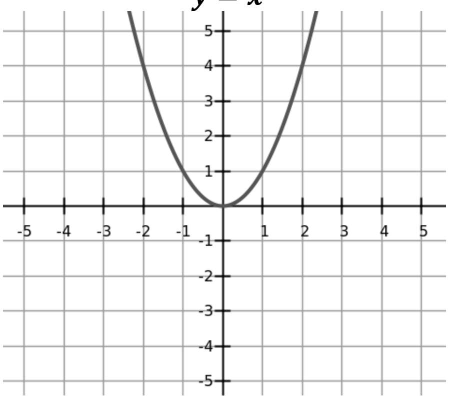
5.2 Quadratic Graphs

Quadratic graphs – those which have an equation where the highest power is 2 – can be thought of as having an n or u shape so they are easy to recognise. A positive x^2 will give you a u shaped graph and a negative $-x^2$ will give an n shaped graph. This is called a **parabola.**

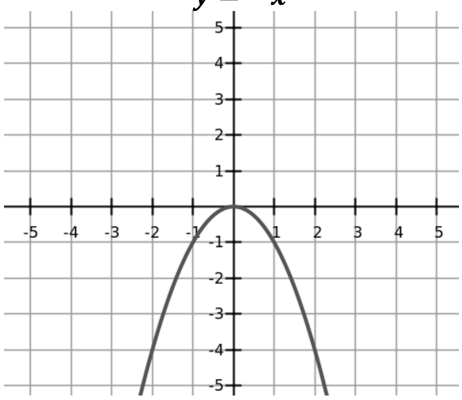
A quadratic function is of the form $y = ax^2 + bx + c$ where a, b and c can take any values, including 0.

You should be able to recognise the following graphs.

$y = x^2$



$y = -x^2$



Plotting Quadratic Graphs

To plot a quadratic graph, and indeed any graph, you use the same technique used to plot straight line graphs. You choose some values for x, set them out in a table to find the values of y and plot the points. In this case, though, you should not use a ruler to join the points instead you should draw a smooth curve.
It is often necessary to plot more points – including negative values - than you would for a straight line graph to get an idea of where the curve goes.

Example:
Plot the graph $y = x^2 - 2x$

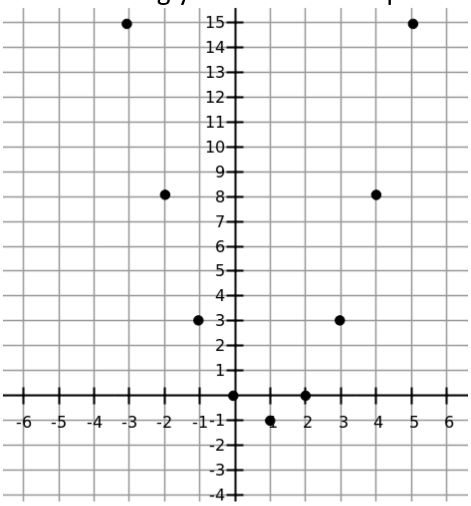
As before the first thing we do is draw the table

x	-4	-3	-2	-1	0	1	2	3	4
y									

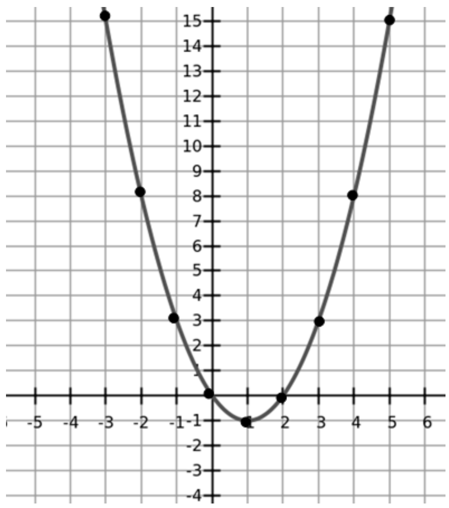
Now we need to find the values for y by substituting in x
For example when $x = -4$, $y = (-4)^2 - (2 \times -4) = 16—8 = 16 + 8 = 24$
Remember that the square of a negative number is positive. When $x = -4, x^2 = -4 \times -4 = 16$

x	-4	-3	-2	-1	0	1	2	3	4	5
y	24	15	8	3	0	-1	0	3	8	15

The next thing you have to do is plot all of these points..



... and join them up in a smooth curve which should continue past your last point – the graph doesn't have an end point.



Properties of Quadratic Graphs

The point at bottom of a u shaped graph, or the top of an n shaped graph, is called the **turning point**.

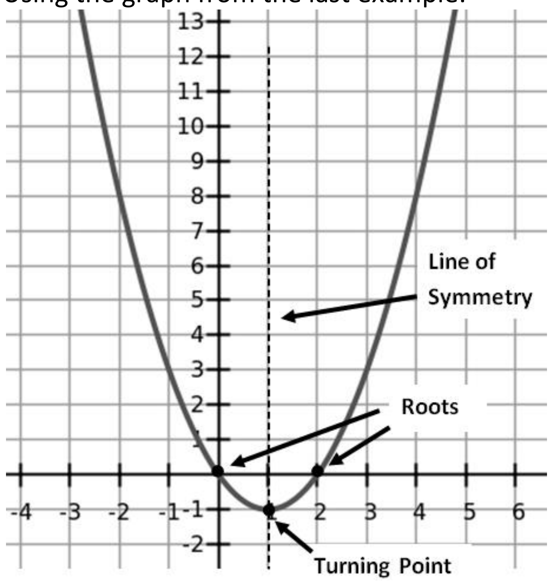
The **roots** of the graph are the $\boldsymbol{x}$**-intercepts**.
The x intercepts of the graph $y = f(x)$ are the solutions to $f(x) = 0$.

$f(x)$ is notation for a function – a function is simply a rule that links one input value with one output value.

For example if you were plotting the graph of $y = 2x^2$ you would simply have $f(x) = 2x^2$
The roots, or the x intercepts, are the solutions to $2x^2 = 0$.

The **line of symmetry** is the vertical line through the turning point.

Using the graph from the last example:



We can see that the turning point is (1, -1) and the line of symmetry is $x = 1$.

The roots of the graph are (0,0) and (2,0). We therefore say that the roots of $f(x) = 0$ are $x = 0$ and $x = 2$.

You can also find the coordinates of a turning point (maximum/minimum point) by completing the square.
If a function is in the $y = (x + a) + b$ then the turning point is at $(-a, b)$
Notice that the value inside the bracket changes sign whilst the value outside remains the same.

Example
Find the turning point of $y = x^2 - 4x + 5$

If we complete the square we have $y = (x - 2)^2 + 1$ therefore the turning point is at $(2, -1)$

Activity 5.2

1. Plot each of these functions
   a. $y = x^2 + 5$ b. $y = x^2 - 3$ c. $y = 2x^2 + 3x$ d. $y = -2x^2$

2. Find the roots, the turning point and the equation of the line of symmetry for each of the following functions
   a. $y = x^2 - 4x$ b. $y = -x^2 - 2x$ c. $y = 3x^2 - 6x$ d. $y = x^2 + 2x + 1$
   e. $y = x^2 - 1$ f. $y = 2x^2 + 4x + 2$ g. $y = 2x(x - 2)$ h. $y = 4 - x^2$

3. Use a graph to solve each of these equations
   a. $x^2 + 2x - 3 = 0$ b. $x^2 + x - 6 = 0$ c. $2x - x^2 + 8 = 0$ e. $-1 - x^2 = 0$

4. Fred throws a ball in the air, its height is given by $y = 10x - x^2$ where y is the height in metres and x is the time in seconds.
   a. Draw a graph to show the ball's path.
   b. When does the ball hit the ground?
   c. At what time is the ball at its highest point?

5. A stone is dropped from a cliff, its height in metres, h, is given by the equation $h = 16 - 4t^2$ where t is the time in seconds. How long does it take for the stone to hit the ground?

6. A company uses a profit function of $P = (1 - s)(s - 5)$ where P is the profit in £ and s is the selling price, also in £. Sketch the function to find the maximum profit and the prices at which the profit would be zero.

7. A quadratic function has roots $x = 1$ and $x = -3$. It has a y - intercept of 3. Sketch the graph of this function.

5.3 Sketching Functions

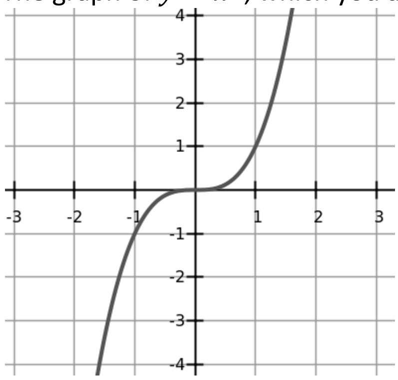
All graphs are sketched using the method learnt previously but you should be able to recognise the graphs of two other types of function.

Cubic Graphs

The highest power in a cubic function is 3.

A cubic function is of the form $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d can be any value, including 0. A function of this sort has an S shape and can have 1, 2 or 3 roots – remembering the roots are the x intercepts.

The graph of $y = x^3$, which you are expected to be able to recognise, is below

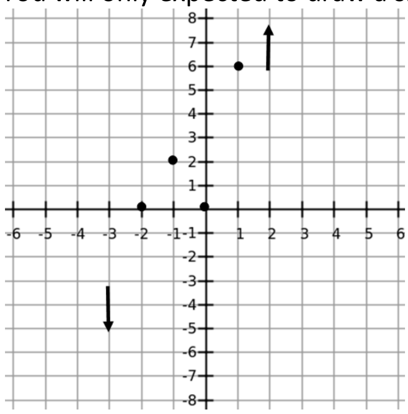


Example:
Use a graph to find the roots of $y = 2x^3 + 4x^2$

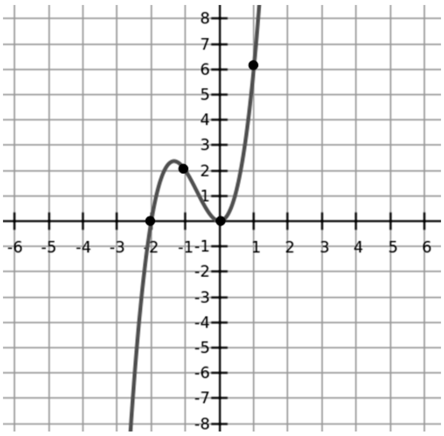
First draw a table to find the values of y at your chosen values of x - make sure you include both positive and negative values.

x	-4	-3	-2	-1	0	1	2	3	4
y	-64	-18	0	2	0	6	32	90	192

As you can see the y value increases and decreases rapidly so, in order to have enough points that are feasible to plot it you can choose to use fractions. However, you know the shape of a cubic graph so it is possible to use just a small number of points in order to draw a sketch.
You will only expected to draw a sketch – it is not possible to draw these graphs accurately by hand.



The lower valued points have been plotted, the arrows have been added to show which way the graph continues – this means you don't have to extend the axes any further.
Joining the points and continuing the curve gives

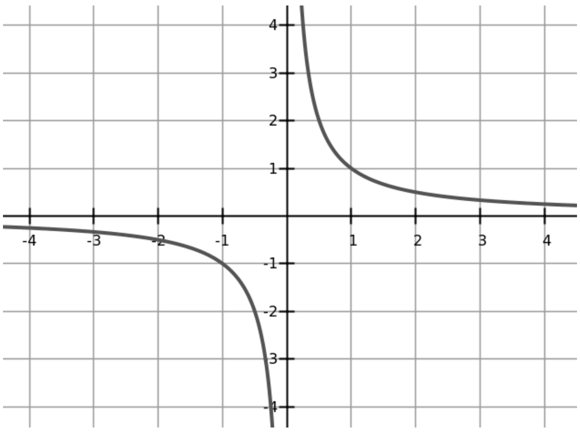


The roots are the x –intercepts so the roots are $x = -2$ and $x = 0$.
Incidentally this could've been read from the table of values where you could see that $y = 0$ at these values of x.

Reciprocal Graphs

A reciprocal function is of the form $f(x) = \frac{c}{x}$ where c is any non zero integer.
The graph of $y = f(x)$ has two separate lines for the positive and the negative values of x. It is undefined at $x = 0$ because it is impossible to divide by zero.

You are expected to be able to recognise the graph of $y = \frac{1}{x}$ which is shown below.



As the lines extend they get closer to the axes but never touch them.

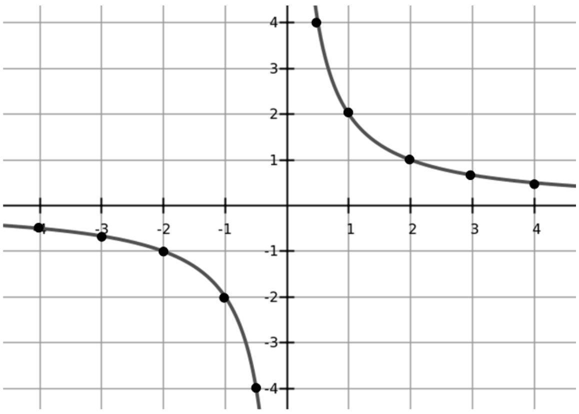
Example:
Plot the graph of $y = \frac{2}{x}$

Here we will use two separate tables, one for the negative values of x and one for the positive, for clarity. You can combine this into one table if you wish.

x	-4	-3	-2	-1	-0.5
y	-0.5	- 2/3	-1	-2	-4

x	0.5	1	2	3	4
y	4	2	1	2/3	0.5

Plotting these points and drawing the curves gives



Activity 5.3 A

1. Sketch each of these functions

a. $y = x^3 + 1$ b. $y = \frac{5}{x}$ c. $y = x^3 - x$ d. $y = -\frac{1}{x}$

e. $y = x^3 - 3x + 1$ f. $y = -x^3$ g. $y = -\frac{2}{x}$ h. $y = x(x-1)(x-2)$

i. $y = \frac{3}{x} - 2$ j. $y = \frac{4}{x-1}$ k. $y = \frac{x}{x+2}$ l. $y = \frac{2}{3x}$

2. A function is written as $f(x) = \frac{1}{x} + 2$. Sketch the graph of $y = f(x)$

3. Use a graph to estimate the solutions of $x^2 + 3x + 2 = \frac{2}{x}$

Exponential Graphs

An exponential function has the form $y = a^x$. The line gets very close to the x axis but never touches it. These are plotted in the same way as other graphs you have seen – by drawing up a table of x and y values.

You need to be familiar with the shape of exponential graphs. Complete question 1 in the activity below in order to see the shape.

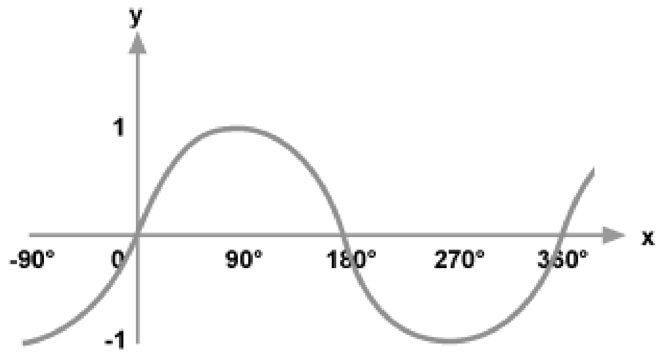
Trigonometric Graphs

Trigonometry will be discussed in chapter 10, here we will just look at the graphs of trigonometric functions.

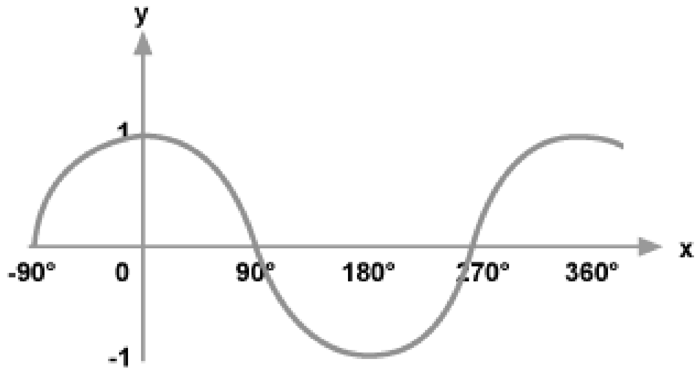
There are three trigonometric functions that you need to be familiar with in this course: sin, cos and tan. The graphs of these functions are **periodic**, this means that they repeat after a fixed interval.

You need to be able to recognise and sketch the graphs of $y = \sin x, y = \cos x$ and $y = \tan x$. They are shown below. Spend some time studying them to become familiar before you move on.

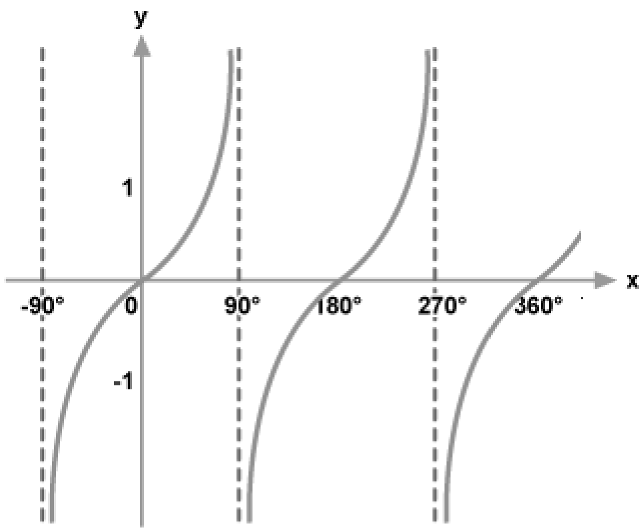
$y = \sin x$



$y = \cos x$



$y = \tan x$



The dotted lines on the tan graph are called **asymptotes**. These are straight lines that the curve continues to approach but never actually touches.

You will find buttons on your calculator saying sin, cos and tan. To find a value of, for example, sin45 you would press the sin button followed by 45.

Activity 5.3 B

1. Draw each of these graphs

a. $y = 2^x$ b. $y = 4^x$ c. $y = 2^{x+1}$ d. $y = \left(\frac{1}{2}\right)^x$

2. Sketch the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ from memory.

3. Use your calculator to draw up a table of values and plot each of these graphs for $0 \leq x \leq 180$
Hint: Use multiples of 45 as values for x

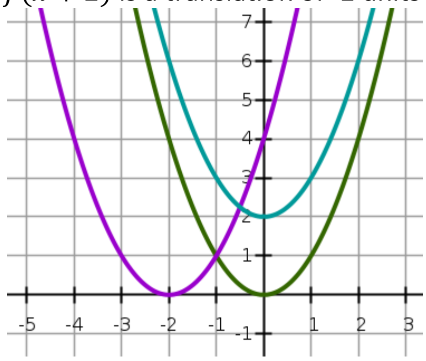
<u>Transformations of Graphs</u>

Once you have drawn a graph you can use transformations in order to draw new ones following the rules below.

$f(x) + a$	A vertical translation of a units
$f(x + a)$	A horizontal translation of – a units
$f(-x)$	A reflection in the y axis

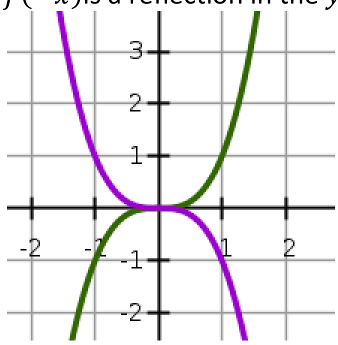
Example:
When $f(x) = x^2$ draw $y = f(x), y = f(x) + 2$ and $y = f(x + 2)$

$y = x^2$ is drawn as usual – shown in green.
$f(x) + 2$ is a translation of 2 units in the vertical direction – shown in blue.
$f(x + 2)$ is a translation of -2 units in the horizontal direction – shown in purple.



Example:
When $f(x) = x^3$ draw $y = f(x), y = f(-x)$

Again, $y = x^3$ is drawn as usual – shown in green.
$f(-x)$is a reflection in the y axis – shown in purple



Activity 5.3 C

1. Given $f(x) = x^2$, draw, on the same axes
   a. $y = f(x)$ b. $y = f(x+3)$ c. $y = f(x) + 5$ d. $y = f(x-2)$

2. Given $f(x) = 2^x$ draw, on the same axes
   a. $y = f(x)$ b. $y = f(x-1)$ c. $y = f(x) + 3$ d. $y = f(-x)$

5.4 Real Life Graphs

Many real life situations can be modelled using line graphs.

Distance – Time Graphs

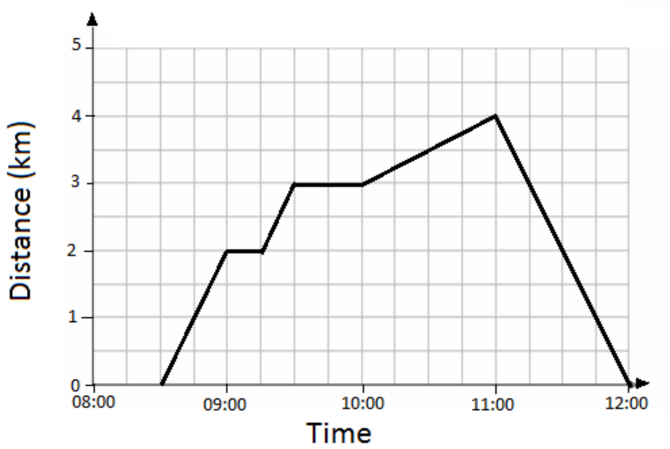
A distance-time graph is used to show information about a journey. They always take the same format – time is displayed along the x axis and distance is displayed along the y axis.

The gradient of a distance-time graph shows you the speed travelled.
In chapter 3 you learnt that $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$, you also know that $\text{Gradient} = \frac{\text{Change in y}}{\text{Change in x}}$
Since distance is on the y axis and time is on the x axis these two equations become the same.

You also need to be able to calculate the average speed. This is done by using the equation

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

Example:
Kathryn went horse riding with Rusty, the distance-time graph shows information about their journey.



a. What time did they leave the yard? *08:30*
b. What time did they arrive back at the yard? *12:00*
c. What happened between 09:00 and 09:15? *They were not travelling*

d. How far did they travel in the first 30 minutes? *2km*
e. How far did they travel in total? *8km – they travelled 4km in one direction then 4km back*
f. Find their speed between 08:30 and 09:00. $\frac{distance}{time} = \frac{2}{0.5} = 2 \div \frac{1}{2} = 4km/h$
g. Find their speed between 11:00 and 12:00. $\frac{distance}{time} = \frac{4}{1} = 4km/h$
h. What was their total journey time? *3 ½ hours*
i. At what time were they 3.5km away from the start? *10:30 (read across from 3.5km to the line, then down to the axis)*
j. Find their average speed. $\frac{Total\ distance}{Total\ time} = \frac{8}{3.5} = 8 \div 3\frac{1}{2} = 8 \div \frac{7}{2} = 8 \times \frac{2}{7} = \frac{16}{7} km/h$

Information from distance-time graphs can also be shown on speed-time graphs. The time remains on the x axis and the speed is shown on the y axis.

Example:
Draw a speed-time graph for Kathryn and Rusty's journey.

First find the speed at each of the time intervals in the example above using the formula
$speed = \frac{distance}{time}$

08:30 – 09:00 $4km/h$

09:00 – 09:15 $0km/h$

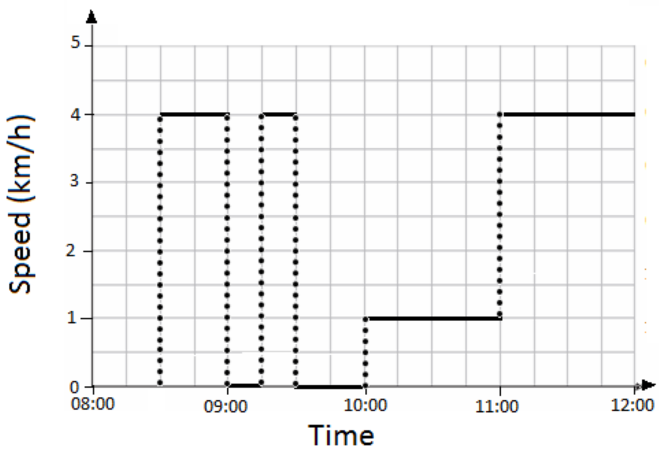
09:15 – 09:30 $\frac{1}{0.25} = 1 \div \frac{1}{4} = 4km/h$

09:30 – 10:00 $0km/h$

10:00 – 11:00 $\frac{1}{1} = 1km/h$

11:00 – 12:00 $4km/h$

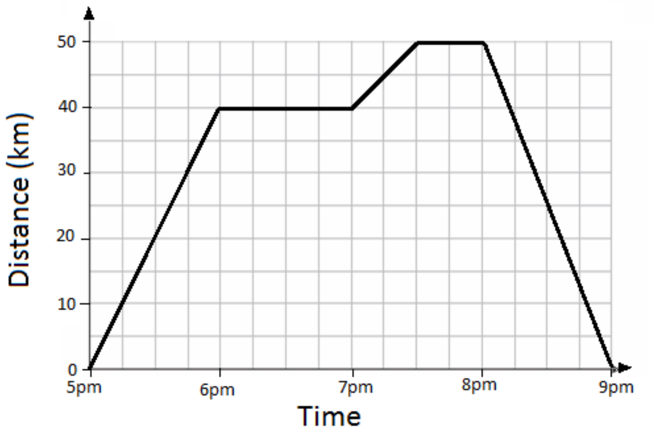
This information is shown on a speed-time graph as follows



In reality the speed would not jump around like this, there would be a smooth curve as they accelerated or decelerated.

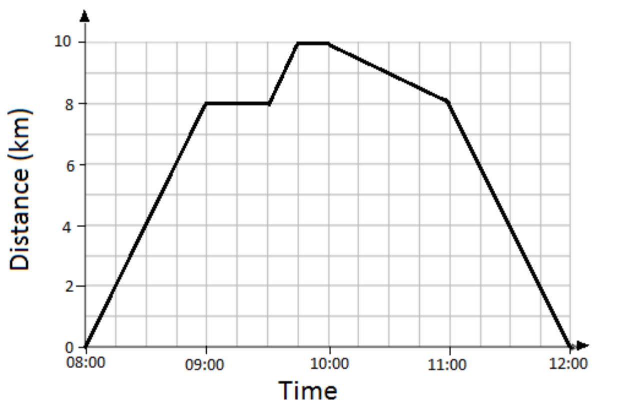
Activity 5.4 A

1. Thomas is travelling in his car, the information from his journey is shown in the distance-time graph below



a. How far does he travel between 5pm and 6pm?
b. How long does it take to travel 50km from the start?
c. What speed is he travelling on his return journey?
d. What speed is he travelling between 6pm and 7pm?
e. What is the total distance travelled?
f. How long did the journey take?
g. Use your answers to part e and f to find the average speed for the journey.

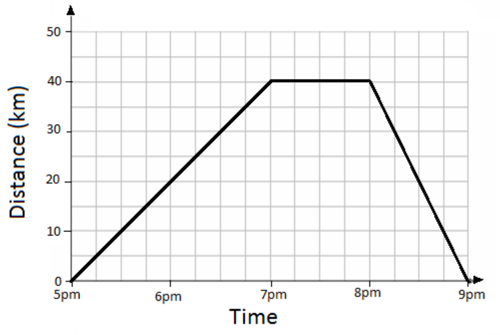
2. Russell goes for a bike ride, information about his journey is shown in the distance-time graph below



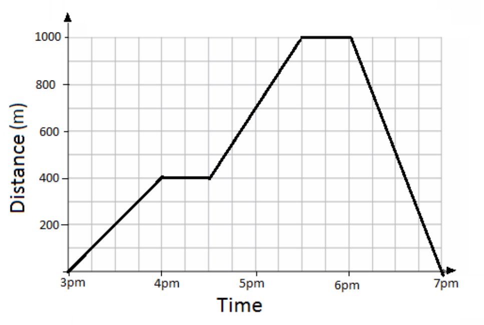
a. How far does he travel between 09:30 and 09:45?
b. How long does it take to travel 8km from the start?
c. What speed is he travelling between 10:00 and 11:00?
d. What speed is he travelling between 11:00 and 12:00?
e. What is the total distance travelled?
f. How long did the journey take?
g. Use your answers to part e and f to find the average speed for the journey.

3. For each of the following distance-time graphs
i. Work out the average speed for the journey
ii. Draw a speed-time graph

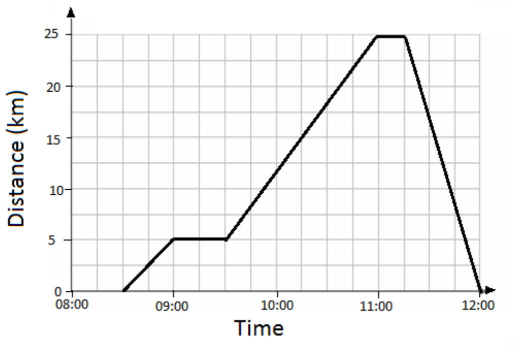
a.



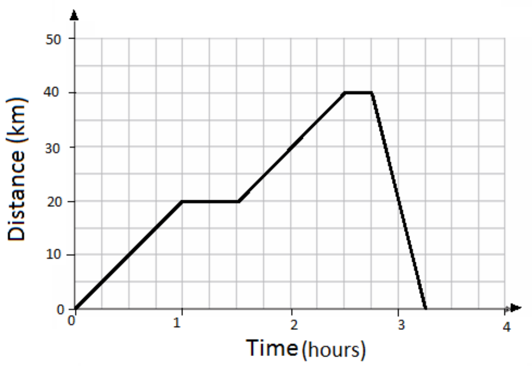
b.



c.



d.



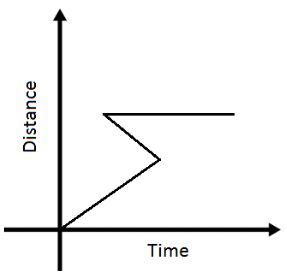
4. Jo flies an aeroplane in a straight line for 3 hours at a speed of 200km/h. Draw a distance time graph to show this journey.

5. Daisy walks 10km in two hours, she has a half hour break before walking home at a constant speed of 5km/h. Draw a distance time graph to show her journey.

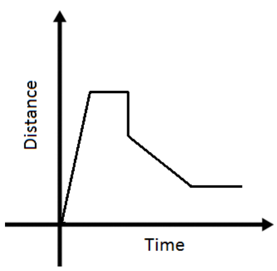
6. Graham leaves his house at 9am, he travels 100m in 15minutes before realising he's forgotten his phone. It takes him 10 minutes to rush back home. After spending 5 minutes at home looking for it he leaves again and travels at a speed of 0.5m/s for 10 minutes until he reaches his destination. He remains there for half an hour before travelling home. He arrives home at half past 10. Draw a distance-time graph for this journey.

7. Explain why each of these distance-time graphs are impossible.

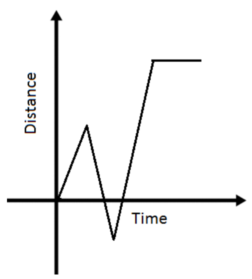
a.



b.



c.



Straight Line Graphs – Real Life Problems

Straight line graphs can be used to solve real life problems. In a given problem identify the variables and write down an equation that connects them. You can then use this equation to draw the graph and use this to answer the question.

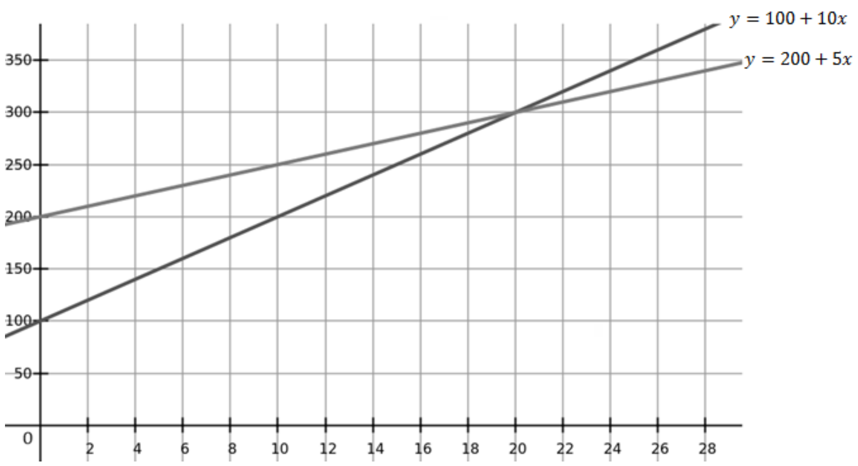
Example:
You are given two options for your weekly pay.
A £100 plus £10 per sale B £200 plus £5 per sale

Which option would you choose if:
a. You make 15 sales?
b. You want to earn £350?
c. Is there a number of sales where it makes no difference which option you choose? If yes, what is it?

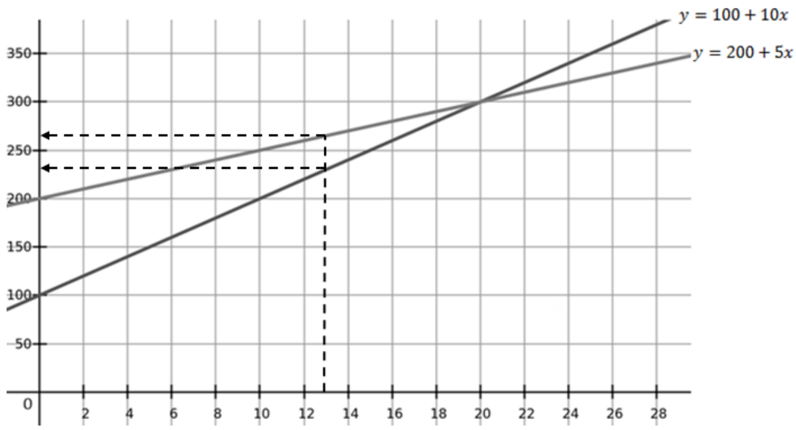
First choose and label your variables
Let x= the number of sales and y= the total amount earned.
Now we need to set up the equations so we have
A: $y = 100 + 10x$
B: $y = 200 + 5x$

a. Since the question doesn't specify that you have to use a graph this can be solved without one if you prefer, simply substitute in $x = 15$ and see which equation gives you the highest value of y.

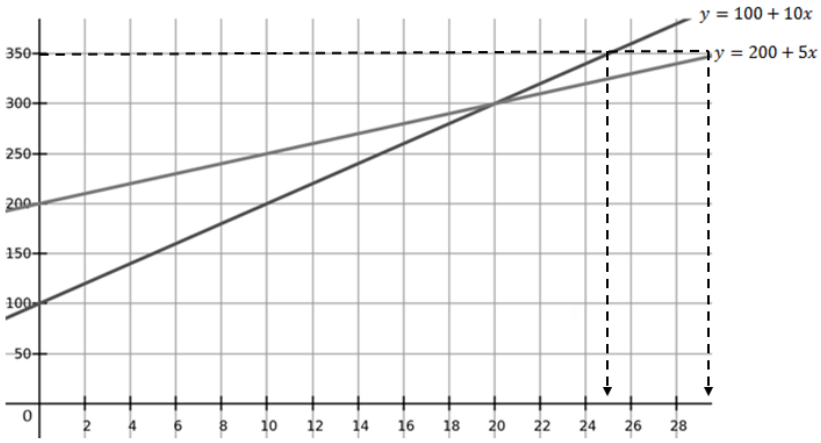
Plotting the graphs gives



Reading up from $x = 15$ and across shows that the equation $y = 200 + 5x$ gives the higher value for y so the best option in this case is B.



b. If you want to earn £350 read across from $y = 350$ and down.



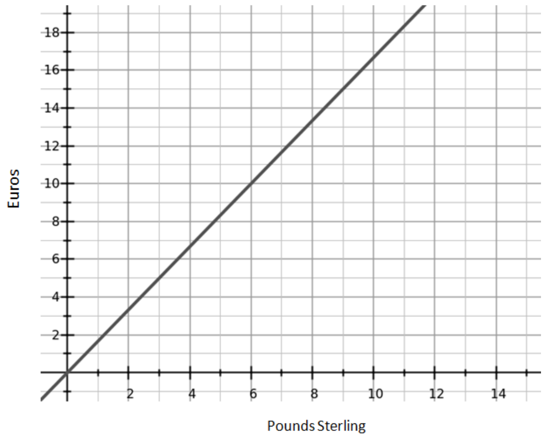
You can see that the option with the lowest value of x, that is, the one that earns £350 with the least amount of sales is option A.

c. The number of sales where it makes no difference which option you choose is the point at which the lines cross. Since the lines cross at (20,300) it is at 20 sales that you would earn the same amount regardless of which option you used.

Real Life Graphs

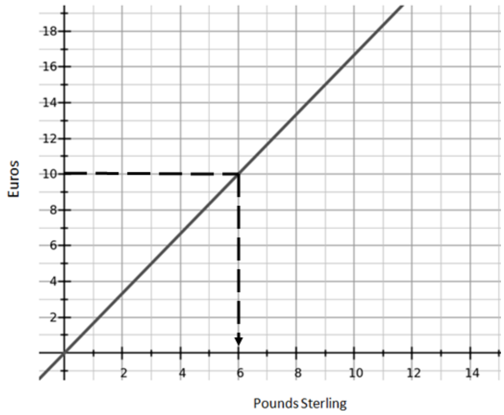
Line graphs – different to straight line graphs – are used to represent a variety of real life situations. They are frequently used to convert from one quantity to another.

Example:
The graph below is a conversion graph between pounds sterling and Euros.



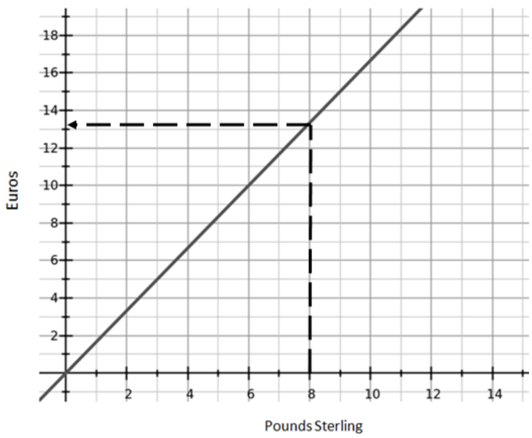
a. Convert 10 Euros to Pounds Sterling.
b. Convert £8 to Euros

a. We draw a line across from 10 Euros until we reach the line and then draw down to the axis



So we can see that 10 Euros is £6

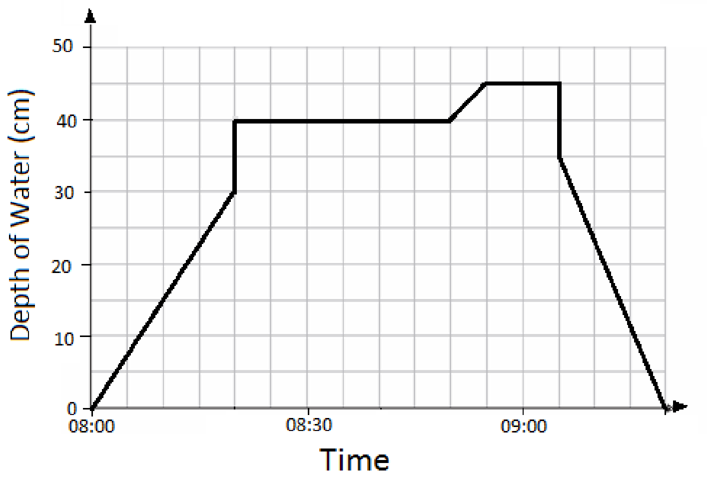
b. We draw a line up from £8 until we reach the line and then draw across to the axis



So we can see that £8 is just above 13 Euros, in this case we will say it is 13 Euros rounded to the nearest Euro.

Line graphs can also be used to model situations in a similar way to distance-time graphs.

Example:
The graph below shows the depth of water in Zara's bath



a. How long did it take to fill the bath?
b. What happened at 08:20? Why do you think this is?
c. What happened between 08:50 and 08:55? Why do you think this is?
d. How long was Zara in the bath?
e. What happened at five past nine? Why?
f. What rate was the water emptied from the bath? Give your answer in cm/min.

a. The bath was filling between 08:00 and 08:20 so it took 20 minutes.
b. The depth of the water suddenly increased by 10cm – Zara got in the bath.
c. The depth of the water increased by 5cm – more hot water could have been added.
d. She was in the bath from 08:20 until 09:05 so 45 minutes.
e. The depth of the water suddenly decreased by 10cm – she got out of the bath.
f. It took 15 minutes for 35cm to empty so the rate is $\frac{35cm}{15min} = \frac{7}{3} cm/min$

Activity 5.4 B

1. The cost of water is calculated using the formula
Total Cost = Fixed Charge + Cost per Unit
Customers can choose two different options:
A: Fixed Charge = £10, Cost per Unit = £2
B: Fixed Charge = £35, Cost per Unit = £1

a. Draw a graph illustrating each of the options.
b. Using the graph to find the total cost of each option when 20 units are used.
c. If you used 30 units, which option should you choose?
d. Abbie used option A, Amy used option B. They both paid the same amount. How many units did they use?

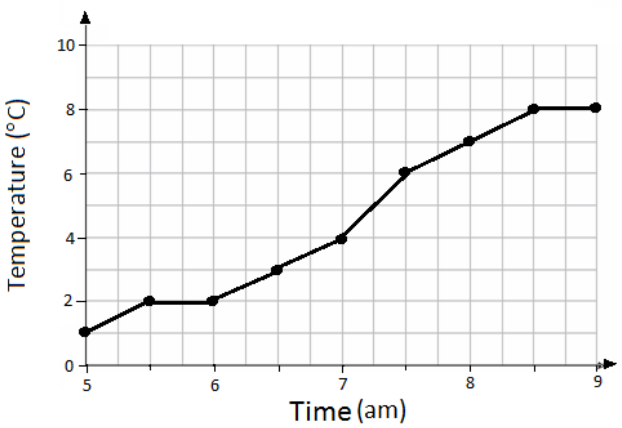
2. Two companies have the following charges for organising a party.

	Fixed Charge	Price per Guest
Let's Party	£80	£5
AB Events	£40	£6

Both companies use the same formula to calculate a total cost. We can assume that both companies are of an equal quality, the difference between them is based entirely on their prices.

a. If there are 20 guests then AB Events charges £40 + £6 x 20 = £160. How much would Let's Party charge for this number of guests?
b. Draw a graph to show the costs of each company for up to 60 guests.
c. Use the graph to find the cost of a party with 15 guests with each of the two companies.
d. If you had £200 to spend which company would you use? Why?
e. If you wanted a party with 40 guests, which company would you use?

3. The line graph below shows the hourly temperatures on one day.

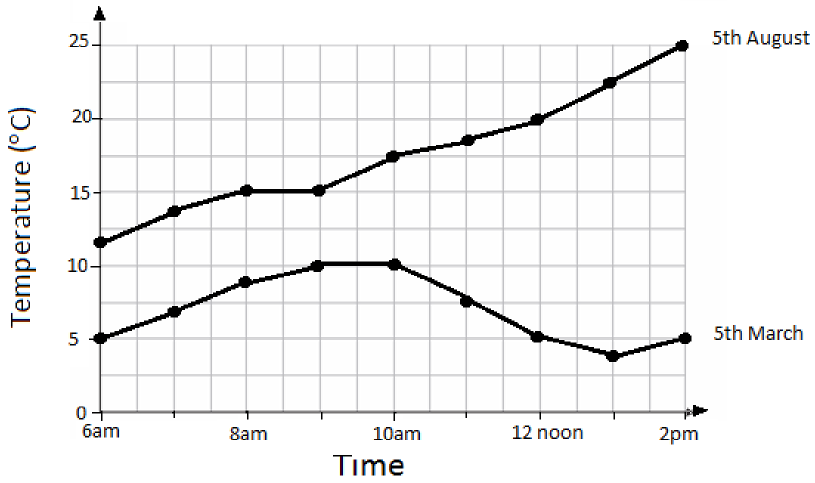


a. What was the temperature at 8am?
b. At what time was the lowest temperature recorded?
c. What was the difference in temperature between 9am and 6am?
d. What was the temperature at 5.15am?
e. What can you say about the temperature over the time span shown on the graph?
f. What can you say about the temperature after 9am?

4. This table shows the number of times Marc uses his van each day of the week, draw a line graph to represent this data.

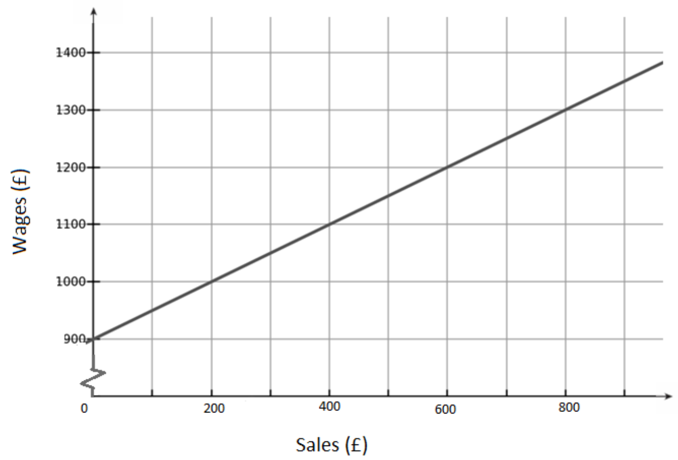
Day of the Week	1	2	3	4	5	6	7
Frequency of Use	4	6	2	5	8	3	1

5. The graph below shows the temperatures recorded on two different days.



a. What was the difference in temperature at 12noon on each day?
b. What was the temperature at 11am on 5th March?
c. On 5th March how much did the temperature drop between 9am and 12noon?

6. Theresa's wages depend on how many sales she makes. The graph below shows the relationship between her wages and the amount she's made in sales.



The zigzag line on the y axis is just to illustrate the point that a section of the scale, in this case from 0-800, has been missed out to condense the graph.

a. How much will she earn if she makes £400 worth of sales?
b. How much will she earn if she makes £700 worth of sales?
c. How much did she make in sales if her wages totalled £1300?
d. Theresa wants to earn at least £1200, what it the minimum amount she can make in sales?

7. Keith is testing the engine in his new motor home. The table shows the relationship between the torque and the amount of power generated.

Torque (N/m)	1	2	3	4	5	6	7
Power (W)	28	50	63	70	67	58	40

a. Plot the data on a graph
b. What type of function could be used to model the data?
c. Estimate the power when a torque of 1.5N/m is applied.

8. The relationship between the temperature and number of volts is shown in the table below

Temperature °C	-30	-25	-15	-5	10	25
Voltage	35	25	15	10	5	5

a. Plot the data on a graph
b. What type of function best describes the data?

As you have seen in the last few questions, real life graphs aren't always straight lines. Regardless of this, the gradient is still used to show the rate of change.

In order to find the gradient of a curve between two points a **chord** should be drawn. This is a straight line that connects the two points, the gradient of this line gives an estimate for the gradient of the curve between the points.

To find the gradient of a curve at a single point a **tangent** should be drawn. This is a line that touches the curve.

Activity 5.4 C

1. In the scenario of question 7 in activity 5.4 find an estimate for the rate of change as the torque changed from 2 to 4.

2. In the scenario of question 8 in activity 5.4 find an estimate for the rate of change as the torque changed from -5 to 10.

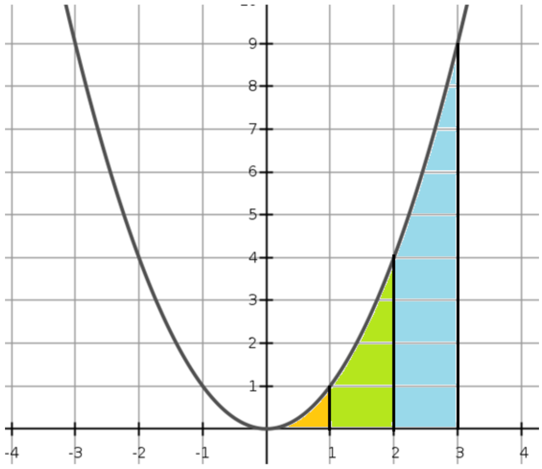
5.5 Areas Under Graphs

In some situations it is useful to be able to find the area under a graph – this usually refers to the area under the line of the graph and above the x axis. For example, in a speed-time graph the area under the line gives the distance travelled.

Finding the exact area under a curve is not covered in the GCSE course, if you go on to study at A Level it will be covered then. The area under a curve can be estimated by splitting the shape into triangles and trapeziums.

Example:
Estimate the area under the graph of $y = x^2$ between $x = 0$ and $x = 3$

The first thing you should do is draw the graph so that you're able to visualise what you're working with. You should also draw in the lines of the shapes you're using to find the area.



Orange: Area of a triangle = ½ x 1 x 1 = ½

Green: Area of trapezium $= \frac{1+4}{2} \times 1 = \frac{5}{2}$

Blue: Area of trapezium $= \frac{4+9}{2} \times 1 = \frac{13}{2}$

Total area = $\frac{1}{2} + \frac{5}{2} + \frac{13}{2} = \frac{19}{2}$

Activity 5.5

1. Write down an estimate for the area under $y = x^2$ between $x = -3$ and $x = 0$

2. Find an estimate for the area under $y = x^3$ between $x = 0$ and $x = 2$

3. Find an estimate for the area under $y = x^2 + 1$ between $x = -2$ and $x = 1$

4. Find an estimate for the area between the line $y = 2 - x^2$ and the x axis

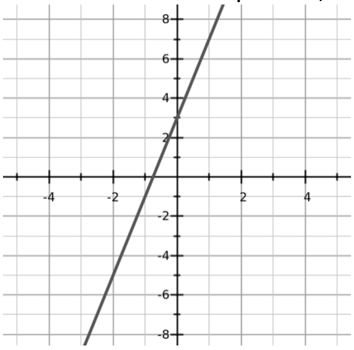
---

ASSIGNMENT FIVE

Answers to these questions are not provided. You should send your work to your tutor for marking.
No calculators are allowed - show all of your working.
When asked to complete a graph or table, as in questions 11, 12 and 13, you may submit a copy of this question sheet or draw it out for yourself. When asked to use your graph to find values you may draw on your original diagram, there is no need to redraw it for each part of the question.

1. Plot the graphs of $2x - y = 6$ and $y = 4$. Write down the coordinates of the point where the lines intersect. (5)

2. Write down an equation, in the form $y = mx + c$, for the line shown in the diagram



(2)

3. In the form $y = mx + c$ give the equation of the line that passes through (6, -3) and (12, -5) (2)

4. Find an equation of the line parallel to $y = 3x + 4$ that passes through (2,3) (2)

5. A=(5,6), B=(4,2). Find the perpendicular bisector of AB. (5)

6. Does the point (4,5) lie on the line $2x - y = 10$? Show/Explain your reasoning. (1)

7. Find the roots of the function $y = x^2 + 3x - 4$ (4)

8. The height above the ground of a football is modelled by the function $h = 3 + 2t - t^2$ where h is the height in metres and t is the time in seconds. Find the maximum height of the ball. (4)

9. A function is written as $f(x) = (x + 1)(x + 2)(x - 3)$

a. Sketch the graph of $y = f(x)$ (3)

b. What type of function is this? (1)

c. Sketch the graph of $y = f(x + 3)$ on the same axis (1)

10. The cost of hiring a car is a fixed price of £20 plus 50p for every kilometre travelled.

a. Complete the table of values

Kilometres	0	10	20	30	40	50	60
Total Cost (£)							

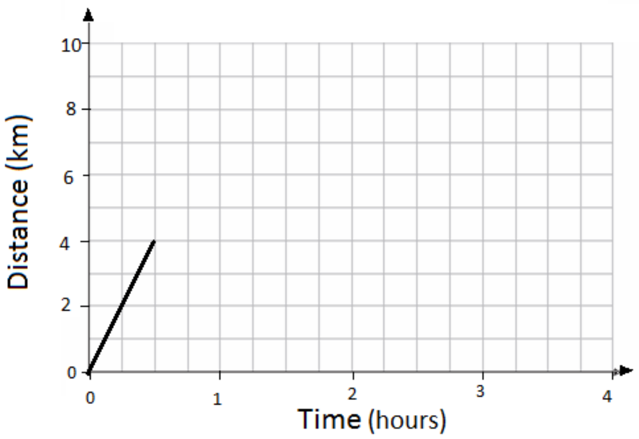
(2)

b. Plot this data on a graph (1)

c. What type of function best describes the data? (1)

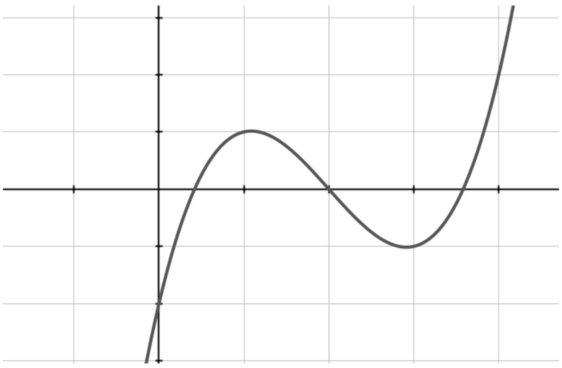
d. Use your graph to find the total cost when the distance travelled is 32km (1)

11. Here is part of a distance-time graph showing Russell's journey from his house and back.



a. Work out the speed Russell travelled for the first part of his journey. (2)
b. Russell stopped for half an hour before travelling another 2km at a speed of 4km/h he then stayed at the cafe for an hour before taking 90 minutes to travel home. Draw this journey on the graph. (3)

12. On the diagram label the turning point(s) and the root(s).
You may draw a copy yourself; the function need not be identical as long as the key features are the same.



(2)

Total marks 42

Chapter Six: Solving Equations and Inequalities

6.1 Solving Linear Equations

A **linear** equation is one in which the highest power is 1. An equation is solved by finding the value of the unknown(s). Rarely this can be done by observation, for example, when $x + 7 = 10$ it is easy to see that $x = 3$ but it is usually done with algebraic manipulation or by plotting graphs.

Solving equations is essentially the same as rearranging to change the subject. The methods you have learnt to rearrange equations – such as function machines or removing the unwanted terms – can be used here. We will focus predominantly on the second of these methods but if you find it easier to draw a function machine then there is no problem with that.

Linear Equations with Unknowns on One Side

Example:
Solve $2x - 5 = 15$

We want to find a value for x whereby this equation is true, in order to do that we need to rearrange it so that the x is on its own on one side of the equals sign.

The first thing we must do is remove the -5 by adding 5.
$2x = 15 + 5$
$2x = 20$

Now we must remove the 2 by dividing by 2
$x = 20 \div 2$
$x = 10$

So the solution to the equation is $x = 10$

Example:
Solve $3(a + 2) = 18$

Expand the bracket: $3a + 6 = 18$
Subtract 6: $3a = 12$
Divide by 3: $a = 4$

Example:
Solve $\frac{x}{2} + 5 = 10$

Subtract 5: $\frac{x}{2} = 5$
Multiply by 2: $x = 10$

As you will have noticed in these examples it is important to always do the same thing to both sides of the equation.

Activity 6.1 A

1. Solve each of these equations

a. $3 + x = 5$ b. $x - 3 = 1$ c. $x + 3 = 12$ d. $5x = 30$ e. $\frac{x}{5} = 5$

f. $2x = 8$ g. $\frac{12}{x} = 4$ h. $8 - x = 2$ i. $x + 10 = 20$ j. $\frac{20}{x} = 10$

k. $3y + 2 = 17$ l. $5n - 3 = 7$ m. $7n + 3 = 24$ n. $-8 + 3x = 25$

o. $\frac{x}{8} - 3 = 3$ p. $5 = \frac{x}{8} + 1$ q. $2 + \frac{x}{17} = 5$ r. $3x + 22 = 76$

2. Solve each of these equations

a. $4(x + 2) = 44$ b. $2(2 - x) = 8$ c. $-12 = 4(x - 7)$ d. $10 = 3(x + 4)$

3. Solve each of these equations

a. $\frac{3x+4}{4} = 3$ b. $5 = \frac{5x-4}{6}$ c. $\frac{2x-1}{-2} = 3$ d. $\frac{2(x-1)}{3} = 6$

4. By plotting the graph of $y = 4x - 3$ solve each of the following equations

a. $4x - 3 = 5$ b. $1 = 4x - 3$ c. $4x - 3 = -3$

5. David has 4 boxes of crayons, each box contains x crayons.

a. Write an expression for the number of crayons David has.

b. David has a total of 24 crayons, write an equation for the number of crayons he has.

c. Use your equation to work out the number of crayons in each box.

6. Solve each of these equations

a. $3x + 3 - 2x = 4$ b. $5x - 3 - 2x = 6$ c. $\frac{10x}{5x} = 6$ d. $4x - 2x + 3x - 2 = 8$

7. Yasmine, Amelia and Elliot bring cakes to a party. There are a total of 11 cakes. Yasmine brings y cakes, Elliot brings twice as many as Yasmine and Amelia brings 4 less than Elliot. Work out how many each of them bring.

8. Four people go out for a meal, all of their meals cost the same amount. The total bill comes to £60, this included £12 worth of drinks. How much did one meal cost?

Linear Equations with Unknowns on Both Sides

If an equation has unknowns on both sides of the equals sign you can either use algebra to find an exact solution or a graph to find an approximate one. Unless you are specifically told to use graphs you should use algebra to solve equations of this form.

The first step is to gather all of the unknowns on the same side. Once you have done this you can continue to solve the equation in the same way as before.

Example:
Solve $10x + 2 = 7x + 14$

Subtract $7x$: $10x + 2 - 7x = 14$

$3x + 2 = 14$

Subtract 2: $3x = 14 - 2$

$3x = 12$

Divide by 3: $x = 12 \div 3$

$x = 4$

Note, you could move the $10x$ but this would result in a negative number so it's easier to move the $7x$

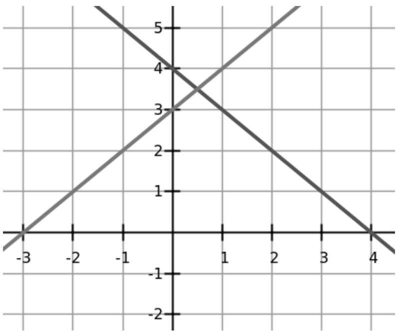
Example:
Solve $6 - 2x = 8 - x$

Add $2x$: $6 = 8 + x$

Subtract 8: $x = -2$

Example:
Use a graph to find an approximate solution of $4 - x = 3 + x$

Plot the graphs of $y = 4 - x$ and $y = 3 + x$, the solution is the x value at the point at which the lines intersect.



The point of intersection is (0.5, 3.5) so the solution is $x = 0.5$

Activity 6.1 B

1. Solve each of these equations

a. $7x + 3 = 3x + 27$	b. $5n + 4 = 2n + 22$	c. $11x - 9 = 5x + 27$
d. $3x + 3 = 2x + 8$	e. $10y + 17 = 3y + 52$	f.$4x - 91 = 130 - 9x$
g. $-31 - 5x = 17 - x$	h. $-69 - 11x = -4x + 99$	i. $p - 2 = 3p - 3$
j. $3y = y + 6$	k. $\frac{x}{2} + 4 = x - 6$	l. $\frac{1}{2}x + 7 = 10 - \frac{1}{2}x$

2. Explain why $3 - 7x = 5 - 7x$ doesn't have a solution.

3. Use a graph to find an approximate solution for $6 - x = 5x - 2$

4. Use a graph to find an approximate solution for $2x + 3 = 4 - x$

5. Solve each of these equations

a. $2(x - 1) = 4 - 3(x + 2)$ b. $-13 + 5x = 5 - 3(2x - 5)$

6. Katie has n sweets, Fay has two more sweets than Katie. If Fay multiplies the number of sweets she has by 3 and Katie multiplies the number of sweets she has by 4 their answers would be the same. Find the value of n.

7. Solve each of these equations

a. $\frac{x+2}{3} = \frac{x-1}{2}$ b. $\frac{2x}{5} = x + 2$ c. $\frac{5x-2}{4} = \frac{x}{2}$ d. $\frac{2x+8}{3} = 3x$ e. $\frac{3}{x} = \frac{4}{x+1}$ f. $\frac{3}{x+3} = 2$

g. $\frac{4}{x+1} = \frac{5}{2x+4}$ h. $\frac{2}{3x-4} = \frac{5}{x-3}$ i. $3x + 4 = \frac{x}{3}$ j. $\frac{2}{x-4} - \frac{5}{2x+1} = 0$

6.2 Solving Quadratic Equations

Factorising

In chapter 5 we dealt with how to solve quadratic equations using a graph – the roots, found from reading the x intercepts, are the solutions $f(x) = 0$ for some quadratic function $f(x)$. However, as with linear equations, using a graph rarely gives exact solutions. Here we will deal with how to solve quadratic equations algebraically.

When presented with a quadratic equation you should rearrange it into the form $ax^2 + bx + c = 0$ before factorising the left hand side. Once you have done this it is easy to read the solutions from each of your brackets. It's important to remember that, in the majority of cases, quadratic equations have two solutions.

If you are unsure about factorising it is recommended that you revise sections 4.2 and 4.5.

Example:
Solve $(x + 2)(x - 3) = 0$

Here the equation is already factorised. An equation of this form means the first bracket multiplied by the second:
$(x + 2) \times (x - 3) = 0$
Since anything multiplied by zero equals zero we need to find numbers such that one of the brackets equals 0. That is
$x + 2 = 0$
Or
$x - 3 = 0$

So we have $x = -2 \; or \; x = 3$

Example:
Solve $x^2 + 9x = 0$

Factorising gives $x(x + 9) = 0$
So we need either $x = 0$ or $x + 9 = 0$

The solutions to this equation are therefore $x = 0 \; or \; x = -9$

Example:
Solve $x^2 - 3x = 10$

First we need to rearrange this to give $x^2 - 3x - 10 = 0$
Factorising then gives $(x-5)(x+2) = 0$
So we need $x - 5 = 0 \; or \; x + 2 = 0$

The solutions are therefore $x = 5 \; or \; x = -2$

Example:
Solve $2x^2 - 3x - 2 = 0$

Factorising gives: $(2x+1)(x-2) = 0$

So either $2x + 1 = 0$ which gives $x = -\frac{1}{2}$
Or $x - 2 = 0$ which gives $x = 2$

Activity 6.2 A

1. Solve each of these equations
   a. $x^2 = 16$ b. $x^2 = 100$ c. $n^2 = 25$ d. $y^2 = 49$

2. Solve each of these equations
   a. $(x+3)(x-6) = 0$ b. $(y-5)(y+8) = 0$ c. $(n+16)(n-367) = 0$
   d. $(x+4)^2 = 0$ e. $x(x-3) = 0$ f. $(x-36)^2 = 0$
   g. $(2x-1)(3x+4) = 0$ h. $(x+3)(4x-5) = 0$ i. $(7x-3)(4x+1) = 0$

3. Solve each of these equations
   a. $x^2 + 4x = 0$ b. $x^2 + 5x = 0$ c. $x^2 + 13x = 0$ d. $2x^2 + 8x = 0$
   e. $7x^2 + 14x = 0$ f. $2x^2 + 5x = 0$ g. $x^2 = 3x$ h. $x^2 = -2x$

4. Solve each of these equations
   a. $x^2 - 3x + 2 = 0$ b. $x^2 + 8x + 12 = 0$ c. $x^2 - 9x - 10 = 0$
   d. $x^2 + 14x + 40 = 0$ e. $x^2 - 5x + 6 = 0$ f. $x^2 - 10x + 24 = 0$

   g. $2x^2 + 6x + 4 = 0$ h. $2x^2 + 2x - 24 = 0$ i. $2x^2 - 14x + 24 = 0$
   j. $4x^2 + 5x + 1 = 0$ k. $5x^2 - 13x - 6 = 0$ l. $6x^2 + x - 1 = 0$

   m. $x^2 - 1 = 0$ n. $x^2 - 49 = 0$ o. $x^2 - 25 = 0$

5. Solve each of these equations
   a. $x^2 + 8x = -15$ b. $x^2 - 8x = -12$ c. $x^2 + 3x = 10$
   d. $x^2 - 40 = -3x$ e. $x^2 = x + 2$ f. $x^2 + x = 2$

   g. $6x^2 + 7x = 3$ h. $-x - 10 = -2x^2$ i. $6x^2 = -7x + 20$

6. When x is added to x^2 the total is 12. Find the value of x.

7. When y is subtracted from y^2 the answer is 12. Find the value of y.

8. Explain why $x^2 + 16 = 0$ does not have a solution.

9. Rearrange and solve $x + 21 = \frac{9(x-3)}{x}$

10. Solve $6x(x + 2) = 24 - 6x$

11. Solve $3x^3 + 10x^2 + 8x = 0$

Quadratic Formula

Sometimes it is not possible to factorise quadratic equations however it is still possible to solve them. When equations can't be factorised we use the quadratic formula.
If $ax^2 + bx + c = 0$ then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You will have to be confident using your calculator in order to complete questions on this topic.

In the formula you will see the sign $\pm$. This is read "plus or minus".
This means that there are two solutions to the equation, as in the last section, and they are found by calculating $x = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$
The $\pm$ sign is used to save writing it out in full twice!

Example:
Solve the equation $x^2 + x - 4 = 0$

This cannot be factorised as there is not a pair of numbers that multiply to make -4 and sum to make 1, therefore we will use the formula.
$a = 1, b = 1, c = -4$

$$\begin{aligned} x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{-1\pm\sqrt{1^2-4\times1\times(-4)}}{2\times1} \\ &= \frac{-1\pm\sqrt{1+16}}{2} \end{aligned}$$

So $x = \frac{-1+\sqrt{17}}{2} \cong 1.56$ or $x = \frac{-1-\sqrt{17}}{2} \cong -2.56$

You will notice that the first term of the numerator is $-b$. You have to be particularly careful of this, especially when b is already negative. For example if $b = -1$ then $-b = 1$.

Example:
Use the quadratic formula to solve $3x^2 - 2x - 2 = 0$

$a = 3, b = -2, c = -2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{2 \pm \sqrt{(-2)^2 - 4 \times 3 \times (-2)}}{2 \times 3}$$
$$= \frac{2 \pm \sqrt{4 + 24}}{6}$$

So $x = \frac{2+\sqrt{28}}{6} \cong 1.22$ or $x = \frac{2-\sqrt{28}}{6} \cong -0.54$

Activity 6.2 B

1. Use the formula to solve each of these equations

a. $x^2 + 2x + 1 = 0$	b. $2x^2 - 5x - 6 = 0$	c. $9x^2 + x - 1 = 0$
d. $3x^2 + 4x - 3 = 0$	e. $x^2 - x - 1 = 0$	f. $x^2 + x - 1 = 0$
g. $10x^2 - 6x - 23 = 0$	h. $5x^2 + 6x - 7 = 0$	i. $3x^2 + 4x - 9 = 0$

2. Explain why $3x^2 - 2x + 2 = 0$ does not have a solution

Completing the Square

Sometimes, when an equation cannot be solved by factorising, you can use the method of completing the square rather than using the formula.
In order to do this the first thing you need to do is divide throughout by a such that there is no coefficient of x^2
(Recall that a quadratic equation is in the form $ax^2 + bx + c = 0$)

To complete the square you can either remember a formula or a technique.

The formula is
$\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c$

Example:
Complete the square for $x^2 + 4x - 3$

Using the formula gives
$(x + 2)^2 - \frac{16}{4} - 3 = (x + 2)^2 - 7$

The technique for completing the square is:
1. Halve the value of b then put this into the brackets $(x + \_\_\_)^2$
2. Multiply out the brackets
3. Compare this expression with the required one and add/subtract a value as necessary to make them the same

Example:

Complete the square for $x^2 + 4x - 3$

1. Halving b gives 2 so we have $(x + 2)^2$
2. Multiplying this out gives $x^2 + 4x + 4$
3. Compare with the original, the x^2 and the $4x$ are the same so we can ignore them. The original expression has -3 and the multiplied out one has $+4$.
To get from $+4$ to -3 you have to subtract 7 so we have
$(x + 2)^2 - 7$

Completing the square enables you to solve equations as shown in the following example

Example:
Solve $x^2 + 4x - 3 = 0$ by completing the square.

We already know that $x^2 + 4x - 3 = (x + 2)^2 - 7$
So we have $(x + 2)^2 - 7 = 0$
This gives
$(x + 2)^2 = 7$
$x + 2 = \sqrt{7}$
$x = \sqrt{7} - 2$
$\therefore x = 0.645..or - 4.645$ … *(Remembering to use the both $\pm\sqrt{7}$)*

Activity 6.2 C

1. Complete the square for each of these expressions

a. $x^2 + 2x + 5$ b. $x^2 - 4x + 9$ c. $x^2 - 6x + 1$ d. $x^2 - 3x - 4$

2. Solve each of these equations by completing the square

a. $x^2 - 6x + 6 = 0$ b. $x^2 - 7x + 3 = 0$ c. $2x^2 + 4x - 7 = 0$
d. $x^2 + 7x - 10 = 0$ e. $3x^2 - 6x + 8 = 0$ f. $2x^2 + 3x - 4 = 0$

3. Use whichever method you prefer to solve each of these equations

a. $x^2 + 5x - 4 = 0$ b. $6x^2 + 6x - 7 = 0$ c. $7x^2 + 8x - 10 = 0$

d. $x^2 - 2x = 9$ e. $5x^2 + x - 8 = 2x + 2$ f. $x(x + 3) = 2x$

g. $2x = \frac{x+1}{x-3}$ h. $\frac{3x}{2} = x^2 + 1$ i. $\frac{x+3}{2x-4} = \frac{2x+1}{x}$

6.3 Solving Simultaneous Equations

Simultaneous equations are a set of equations that have the same solutions. At GCSE level you will be expected to solve a pair of equations with two unknowns.

As with other equations we have seen simultaneous equations may be solved using algebra or graphs.

Solving Simultaneous Equations Graphically

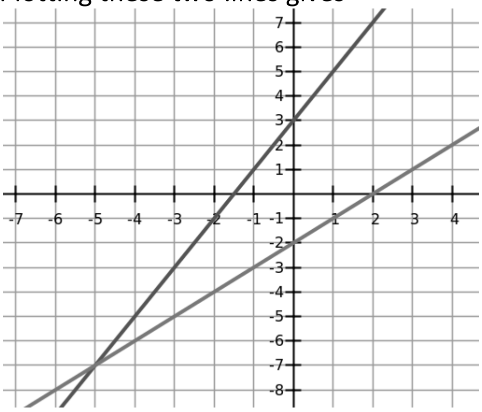
We will look at how to solve simultaneous equations using a graph first as it is a method you will find familiar. However, as with other equations, unless you are expressly told to use a graph it is advised that you use an algebraic method as you are not guaranteed to find an exact solution using a graph.

In order to solve simultaneous equations using a graph you should rearrange each of them into the form $y = mx + c$ and plot the graph as you have done before. The solutions are the values of x and y at the point where the graphs intersect.

Example:
Use a graph to solve these simultaneous equations
$y = 2x + 3$
$2y = 2x - 4$

The first equation is already in the correct form, the second can be rearranged, by dividing by 2, to give $y = x - 2$

Plotting these two lines gives



The lines intersect at (-5, -7) so the solution is $x = -5, y = -7$

This can be checked by substituting these values back into the original equations
$y = 2x + 3 \Rightarrow$ RHS: -7 LHS: $2 \times (-5) + 3 = -10 + 3 = -7$
$2y = 2x - 4 \Rightarrow$ RHS: $2 \times (-7) = -14$ LHS: $2 \times (-5) - 4 = -10 - 4 = -14$

Solving Simultaneous Equations Algebraically - Elimination

There are two methods for solving simultaneous equations using algebra: elimination and substitution. It is up to you to decide which method you prefer but it is helpful to be relatively comfortable with both techniques.

The first method introduced here is elimination. All of the terms in one or both of the equations should be multiplied by some number such that the two equations have a term that is the same. Once you have done this you add or subtract the equations to eliminate one of the variables. The resulting equation can be solved as before and this solution is then substituted back in to one of the equations to find the value of the other variable.
The following three examples illustrate this technique, they each present slightly different situations so work through all of them.

Example:
Solve this pair of simultaneous equations
$6x + y = 25$
$2x + y = 13$

In this pair one of the terms is already the same so no multiplication is necessary, we want to eliminate the y term so we will subtract the equations.

$$\begin{array}{rl} & 6x + y = 25 \\ - & \underline{2x + y = 13} \\ & 4x \qquad = 12 \end{array}$$

Solving $4x = 12$ gives us $x = 3$
We need to substitute this back into one of the original equations to find the value of y
$6x + y = 25$
$6 \times 3 + y = 25$
$18 + y = 25$
$y = 25 - 18 = 7$

You can check your answer by substituting these values into the second equation
LHS: $2x + y = 2 \times 3 + 7 = 6 + 7 = 13$
RHS: 13
So we can assume our answer is correct and that the solution is $x = 3, y = 7$

Example:
Solve this pair of simultaneous equations
$x + 3y = 10$
$2x - 3y = 2$

As before we have a term that is the same in each equation so there is no multiplication needed. In this case we need to add the equations, if we subtract we will end up with the sum $3y - -3y = 6y$ so the term has not been eliminated. If we add instead we get $3y + -3y = 0$ as required. Alternatively you could multiply all of the terms in one of the equations by -1 so that the signs become the same, you could then subtract the two equations as in the last example.

$$
\begin{array}{rl}
& x + 3y = 10 \\
+ & \underline{2x - 3y = 2} \\
& 3x \quad\quad = 12
\end{array}
$$

Solving $3x = 12$ gives $x = 4$, substituting this into the first equation gives
$x + 3y = 10$
$4 + 3y = 10$
$3y = 6$
$y = 2$

Checking the answer with the second equation gives
LHS: $2x - 3y = 2 \times 4 - 3 \times 2 = 8 - 6 = 2$
RHS: 2
So the solution is $x = 4, y = 2$

Example:
Solve this pair of simultaneous equations
$2x + 7y = 16$
$4x + 3y = 10$

In this pair none of the terms are the same. It is up to you what you multiply the equations by and which terms you choose to make equal. Here we will multiply the first equation by 2 so that the x terms become the same.

Multiplying the first equation by 2 gives $4x + 14y = 32$
Make sure you remember to multiply all the terms in the equation.

Now we can subtract the equations to eliminate the x terms

$$
\begin{array}{rl}
& 4x + 14y = 32 \\
- & \underline{4x + 3y \;\; = 10} \\
& \quad\quad 11y = 22
\end{array}
$$

Solving $11y = 22$ gives $y = 2$
Substituting this into the original first equation gives
$2x + 7y = 16$
$2x + 7 \times 2 = 16$
$2x + 14 = 16$
$2x = 2 \qquad (-14)$
$x = 1 \qquad (\div 2)$

Substituting these values into the second equation to check the answer gives
LHS: $4 \times 1 + 3 \times 2 = 4 + 6 = 10$
RHS: 10
So the solution is $x = 1, y = 2$

Solving Simultaneous Equations Algebraically – Substitution

The second algebraic method for solving simultaneous equations involves rearranging one of the equations such that it is equal to one of the variables. In the majority of cases you need one side of

the equation to be the variable on its own with no coefficient but, if the same term is present in the second equation, there is no need to remove the coefficient.
Once you have rearranged one of the equations you substitute it into the second one and rearrange to find the value of one variable. The value of the second variable is found in the same way as before.

The following examples illustrate this technique using the same equations as before.

Example:
Solve this pair of simultaneous equations
$6x + y = 25$
$2x + y = 13$

As y doesn't have a coefficient it is simplest to make this the subject – either of the equations can be rearranged, here we will rearrange the first so it becomes
$y = 25 - 6x$

Now that we have an equation for y we can substitute this into the second equation.
$2x + y = 13$
$2x + (25 - 6x) = 13$
$2x + 25 - 6x = 13$
$-4x + 25 = 13$ $\quad (Collect\ like\ terms)$
$-4x = -12$ $\quad (-25)$
$4x = 12$ $\quad (Change\ signs)$
$x = 3$ $\quad (\div 4)$

Next we need to substitute this back into one of the original equations to find the value of y as before
$6x + y = 25$
$6 \times 3 + y = 25$
$18 + y = 25$
$y = 25 - 18 = 7$

Checking the answer by substituting these values into the second equation gives
LHS: $2x + y = 2 \times 3 + 7 = 6 + 7 = 13$
RHS: 13
So we can assume our answer is correct and that the solution is $x = 3, y = 7$

Example:
Solve this pair of simultaneous equations
$x + 3y = 10$
$2x - 3y = 2$

Here we could rearrange the first equation to equal $3y$ or x, we will choose to rearrange to equal $3y$ as this term is present in the second equation.
$3y = 10 - x$

Substituting this into the second equation gives
$2x - 3y = 2$
$2x - (10 - x) = 2$
$2x - 10 + x = 2$ *Be careful of the minus signs*

$3x - 10 = 2$ $\qquad$ $(Collect\ like\ terms)$
$3x = 12$ $\qquad$ $(+10)$
$x = 4$ $\qquad$ $(\div 3)$

Substituting this into the first equation gives
$x + 3y = 10$
$4 + 3y = 10$
$3y = 6$
$y = 2$

Checking the answer with the second equation gives
LHS: $2 \times 4 - 3 \times 2 = 8 - 6 = 2$
RHS: 2
So the solution is $x = 4, y = 2$

Example:
Solve this pair of simultaneous equations
$2x + 7y = 16$
$4x + 3y = 10$

Here you could rearrange the first equation to equal $2x$, remembering that $4x = 2 \times 2x$. However, since questions sometimes don't have terms that match up so nicely we will rearrange to equal x so you're familiar with how to use this method in all situations.

The first equation becomes
$2x = 16 - 7y$
$x = 8 - \frac{7}{2}y$ $\qquad$ *Remember that we leave coefficients as fractions but, during your working, you may use decimals if you feel more comfortable.*

Substituting this into the second equation gives
$4x + 3y = 10$
$4\left(8 - \frac{7}{2}y\right) + 3y = 10$

$32 - 14y + 3y = 10$ $\qquad$ $(Expand\ brackets)$
$32 - 11y = 10$ $\qquad$ $(Collect\ like\ terms)$
$-11y = -22$ $\qquad$ (-32)
$11y = 22$ $\qquad$ $(Change\ signs)$
$y = 2$ $\qquad$ $(\div 11)$

Substituting this into the original first equation gives
$2x + 7y = 16$
$2x + 7 \times 2 = 16$
$2x + 14 = 16$
$2x = 2$
$x = 1$

Substituting these values into the second equation to check the answer gives
LHS: $4 \times 1 + 3 \times 2 = 4 + 6 = 10$
RHS: 10
So the solution is $x = 1, y = 2$

When you have a quadratic equation you need to use substitution in order to solve them so make sure you are comfortable with using this method.
You will find equations of this type in question 9 below.

Activity 6.3

1. Use a graph to solve each pair of simultaneous equations. Some answers will have to be approximations.

a. $y + 2x = 5$ $y = x - 4$	b. $3x + 2y = 12$ $2x + y = 7$	c. $y = 2x - 3$ $x + y = 4$	d. $2x + 4y = 8$ $3y = 4 + x$

2. Solve each pair of simultaneous equations algebraically. It is ultimately up to you whether you use substitution or elimination – you may find you prefer one method for some questions and the other method for other questions. Make sure you try both techniques.

a. $4x + y = 43$ $3x + y = 33$	b. $x + 6y = 60$ $x + 4y = 40$	c. $3x + y = 43$ $x + y = 19$	d. $x + 8y = 155$ $x + 5y = 110$
e. $2x + 5y = 29$ $x + 2y = 14$	f. $x + 3y = 16$ $3x + 2y = 14$	g. $x + 4y = 13$ $3x + 5y = 18$	h. $4x - 4y = 20$ $x - 4y = 2$
i. $4x + 2y = 26$ $x - 2y = 4$	j. $8x + 3y = 8$ $5x - 3y = 5$	k. $x - 12y = 16$ $5x + 12y = 8$	l. $x + 6y = 5$ $3(x + 2y) = 3$
m. $5x - y = 17$ $2x + 3y = 0$	n. $x + 3y = 11$ $2x + 5y = 19$	o. $5x + 3y = 24$ $x + 5y = -4$	p. $x + 14y = -2$ $2x + 3y = 21$

3. Solve each pair of simultaneous equations algebraically. When using elimination you will have to multiply both equations by some number (it can be a different number for each equation) in order to make terms equal.

a. $7y - 3x = 2$ $5y - 2x = 2$	b. $4x - 2y = -6$ $5x + 3y = 20$	c. $10x + 4y = 2$ $8x + 3y = 1$	d. $5x - 8y = 12$ $4x - 7y = 9$

4. Two numbers have a sum of 15 and a difference of 3. Find the value of these two numbers.

5. The sum of the ages of Elaine and Paul is 135 years. The difference between their ages is 11 years. Elaine is older than Paul, find out how old each of them are.

6. Six bananas and four pears cost £1.90. Eight bananas and two pears cost £1.80. Find the cost of one banana and the cost of one pear.

7. Two people walk into a shop. Gary buys four pints and milk and three boxes of cereal, he pays with a £10 note and gets £6.03 in change. Miranda buys five pints of milk and two boxes of cereal, it costs her £3.44. Find the cost of one pint of milk and the cost of one box of cereal.

8. Find the value of x and y

$$\frac{3(x-y)}{5} = x - 3y = x - 6$$

9. Find the value of x and y

a. $x + y = 2$
$y = x^2$

b. $x + y = 6$
$y = 2x^2$

c. $x - 2y = 10$
$x^2 + y^2 = 60$

d. $x^2 + 2xy = 200$
$x = 2y$

10. By writing the right hand side as a product of its prime factors solve this pair of simultaneous equations
$xy^3 = 24$ $\quad$ $xy^5 = 96$

6.4 Solving Inequalities

An inequality is a statement in which two expressions are not equal. For example
$3x > 3\ for\ all\ x > 1$ means that $3x$ is bigger than 3 any time that x is bigger than 1.

You met the greater than and less than signs in chapter one.

Linear inequalities are solved in a similar way to linear equations the only difference is that when a negative number or sign moves from one side to the other you must turn the sign around.
If you multiply or divide an inequality by a negative number the sign needs to be reversed.

You should also remember that if you decide to write the inequality the other way around the sign should also be changed. For example, when working with equations, you may choose to write $8 = x$ as $x = 8$. If you have the inequality $8 > x$ you could also write $x < 8$, notice that the smaller end of the sign stays by the x and the wider end stays by the 8. Similarly $9 < x$ can also be written as $x > 9$.

Example:
Solve $3x < 12$

In the same way as you would solve $3x = 12$ you simply divide by 3 to get $x < 4$

Example:
Solve $3x + 2 \geq 26$

Subtract 2: $\quad 3x \geq 24$
Divide by 3: $\quad x \geq 8$

Example:
Solve $5 < 2x - 3 < 9$

Add 3: $\quad 8 < 2x < 12$
Divide by 2: $\quad 4 < x < 6$

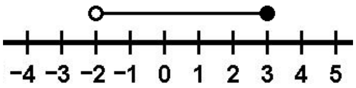
Example:
Solve $-2x < 10$

Divide by -2 (don't forget to reverse the sign): $\quad x > -5$

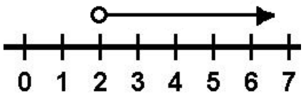
Inequalities can be represented on a number line. If the sign is greater than or less then place an empty circle above the number, if the sign includes an equals to then place a coloured circle above the number. You should then join the circles with a line if you have two numbers or draw an arrow in the appropriate direction if you just have one.

Example:

$-2 < n \leq 3$



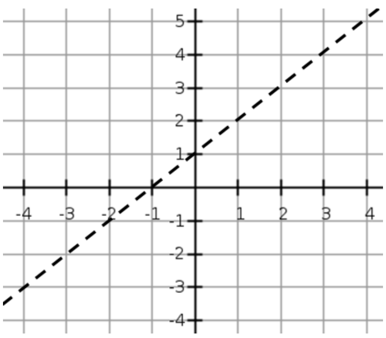
$n > 2$



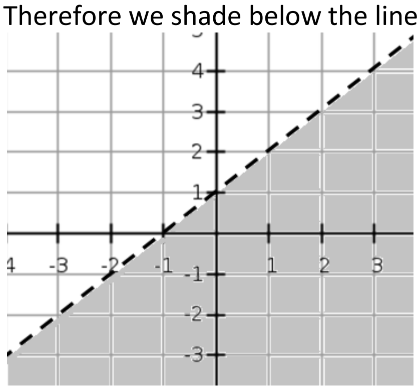
Inequalities can also be represented on graphs. The line is plotted as usual, using a dashed line if the sign is greater than or less than or a solid line if the sign includes an equals to. Once the line is drawn the appropriate side of it is shaded.
To decide which side to shade it is usually easier to pick a point on either side and see if the values satisfy the inequality.

Example:
Show the region defined by $y < x + 1$

Here we have to plot the line $y = x + 1$ using a dashed line.



Now we must decide which side should be shaded.
Picking any point above the line, say (0,2). This gives $y = 2$ and $x + 1 = 1$ so $y > x + 1$
Picking a point below the line: (0,0). This gives $y = 0$ and $x + 1 = 1$ so $y < x + 1$
Therefore we shade below the line to give



Activity 6.4

1. List all of the integer values that satisfy each of these inequalities

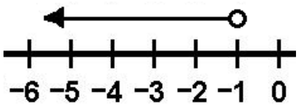
a. $-2 < n \leq 0$ b. $-8 \leq n \leq 0$ c. $3 < n < 10$ d. $2 < n < 4$

e. $-1 \leq n < 3$ f. $-12 < 3n \leq 15$ g. $-4 < 2n < 10$ h. $-10 < 5n \leq 15$

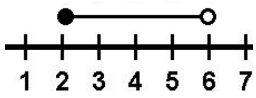
2. Draw each of the inequalities from question 1 on number lines.

3. Write down an inequality to represent the solution shown on the number lines

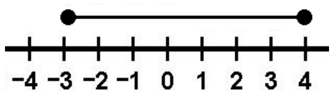
a.



b.



c.



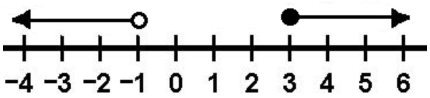
4. Solve each of these inequalities

a. $2x > 16$ b. $x + 3 < 9$ c. $2 \leq x + 2 \leq 8$ d. $12 < 2n \leq 18$

e. $2x + 1 \leq 15$ f. $20 > x + 3$ g. $-3 \leq 2x - 1 < 15$ h. $2(x - 3) > 18$

i. $-7x < 49$ j. $27 - 6x > 39$ k. $110 < 30 - 2x$ l. $-5 \leq 10 + 5x$

5. Write down the inequality represented on this number line



6. Solve each of these inequalities

a. $x^2 > 9$ b. $x^2 > 49$ c. $x^2 - 5 > 20$

d. $x^2 - x - 2 \leq 0$ e. $x^2 + 4x + 3 > 0$ f. $x^2 + x - 6 < 0$

7. Find the range of values that satisfy both $2x + 4 < 12$ and $-3x < 3$

8. The height of a table is less than 100cm. The height of the table is known to be $x - 2$. Find the possible values of x, write you answer as an inequality.

9. Show the region defined by each of these inequalities

a. $x \leq 2$ b. $y \geq 1$ c. $y \leq x$ d. $-2 < x \leq 3$ e. $y > x - 2$

10. Show the region defined by each of these pairs of inequalities.

a. $y \leq x, x \geq 2$ b. $y \leq 3, x > 1$ c. $y \geq x + 3, x \leq 2$

<u>6.5 Iteration</u>

So far we have dealt with equations that are relatively easy to solve this, however, is not always the case. Approximate solutions can be found by using iteration. This is a process of finding a sequence of approximate solutions whereby one is used to find the next until the sequence appears to be converging.

We begin with a value denoted by x_1 and then find $f(x_1)$. Once this is done we label our solution to $f(x_1)$ as x_2 and repeat the process.
As a general rule we can say $x_{n+1} = f(x_n)$

Example:
Use the iterative formula $x_{n+1} = \sqrt{2x_n + 1}$ to find a solution to 2dp. Start with $x_1 = 2$.

$x_1 = 2$
$x_2 = \sqrt{2 \times 2 + 1} = \sqrt{5} = 2.23606797 \ldots$ *(Don't round your answer until the end!)*
$x_3 = \sqrt{2 \times 2.2360679 \ldots + 1} = 2.339259702 \ldots$
$x_4 = \sqrt{2 \times 2.339259702 \ldots + 1} = 2.382964415 \ldots$
$x_5 = \sqrt{2 \times 2.382.96441 \ldots + 1} = 2.401234855 \ldots$
$x_6 = \sqrt{2 \times 2.401234855.. + 1} = 2.408831607 \ldots$
$x_7 = \sqrt{2 \times 2.408831607 \ldots + 1} = 2.411983253 \ldots$
$x_8 = \sqrt{2 \times 2.411983253 \ldots + 1} = 2.413289561 \ldots$

So, to 2dp, the series has converged to $x \cong 2.41$
Note: we couldn't stop at x_6 as, even though we had 2.40... twice the second rounded to 2.41

Activity 6.5

1. Use the iterative formula $x_{n+1} = \sqrt{3x_n + 5}$ to find a solution to $x^2 - 3x - 5 = 0$ to 2dp. Start with $x_1 = 4$

2. Iterate $x_{n+1} = \sqrt[4]{3x - 1}$ up to x_6, begin with $x_1 = 1$

3. Use the iterative formula $x_{n+1} = \sqrt[3]{2x_n + 5}$ to find a solution to $x^3 - 2x - 5 = 0$ to 2dp. Start with $x_1 = 2$

4. a. Rearrange $x^2 - 5x - 2 = 0$ to give an iterative formula
b. Use your formula to find a solution to 2dp, start with $x_1 = 5$

5. a. Rearrange $x^3 - 2x + 3 = 0$ to give an iterative formula
b. Use your formula to find a solution to 3dp, start with $x_1 = -2$
