CDISE
Mobile Robotics Lab
Perception in Robotics Course

L01: the Expectation Operator

Gonzalo Ferrer

1 February 2021

1 Probability review

The random variables x and y are called independent if:

$$p(x, y) = p(x)p(y).$$

Note: In this course $p(x)$ indicates probability density function (PDF). For a review on random variables and PDFs, check out the probability review notes.

Product rule

$$p(x, y) = p(x|y)p(y)$$

Total probability

$$p(x) = \int p(x, y)dy \qquad \text{(marginalization)}$$

$$p(x|z) = \int p(x, y|z)dy$$

Bayes Theorem

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)} = \frac{p(y|x)p(x)}{\int p(x, y)dx}$$

Def: Expectation Operator

$$\mathbb{E}\{x\} = \int_{-\infty}^{+\infty} xp(x)dx = \mu_x$$

2 Properties

$\mathbb{E}\{\cdot\}$ is linear

- $\mathbb{E}\{A\} = A$
- $\mathbb{E}\{Ax\} = A\,\mathbb{E}\{x\}$
- $\mathbb{E}\{A + x\} = A + \mathbb{E}\{x\}$
- $\mathbb{E}\{x + y\} = \mathbb{E}\{x\} + \mathbb{E}\{y\}$:

$$\begin{aligned}
\mathbb{E}\{x + y\} &= \int\int (x + y)p(x, y)dxdy = \\
&= \int\int xp(x, y)dxdy + \int\int yp(x, y)dxdy = \\
&= \int x\left(\int p(x, y)dy\right)dx + \int y\left(\int p(x, y)dx\right)dy = \\
&= \int xp(x)dx + \int yp(y)dy = \mathbb{E}\{x\} + \mathbb{E}\{y\}
\end{aligned}$$

3 Anti-properties

- $\mathbb{E}\{x, y\} \neq \mathbb{E}\{x\}\mathbb{E}\{y\}$ (in general)
- If x, y uncorrelated ($\sigma_{xy} = 0$) $\Rightarrow \mathbb{E}\{x, y\} = \mathbb{E}\{x\}\mathbb{E}\{y\}$
- If x, y independent $\underset{\not\Leftarrow}{\Rightarrow}$ x, y uncorrelated.

4 Expectation of multi-dimensional r.v.

$$\mathbb{E}\left\{\begin{bmatrix} x \\ y \end{bmatrix}\right\} = \begin{bmatrix} \mathbb{E}\{x\} \\ \mathbb{E}\{y\} \end{bmatrix}$$

5 Conditional expectation

$$\mathbb{E}\{x|y\} \triangleq \int_{-\infty}^{+\infty} xp(x|y)dx$$

6 Covariance. Scalar form

Autocovariance or covariance

$$\sigma_{xx}^2 = cov(x, x) = \mathbb{E}\big\{(x - \mathbb{E}\{x\})^2\big\} = \mathbb{E}\{x^2\} - \big(\mathbb{E}\{x\}\big)^2$$

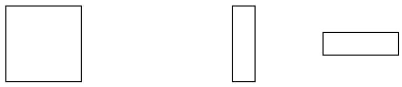
Cross-covariance

$$\sigma_{xy}^2 = cov(x, y) = \mathbb{E}\big\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})\big\}$$

7 Covariance. Vectorial form

Covariance

$$\Sigma_x = cov(x, x) = cov(x) = \mathbb{E}\big\{(x - \mathbb{E}\{x\})(x - \mathbb{E}\{x\})^\top\big\}$$



Cross-covariance

$$\Sigma_{xy} = cov(x, y) = \mathbb{E}\big\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})^\top\big\}$$

Ex.: Expand Σ_{xy}.

$$\begin{aligned}
\Sigma_{xy} &= \mathbb{E}\big\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})^\top\big\} \\
&= \mathbb{E}\big\{xy^\top + x\big(-\mathbb{E}\{y\}\big)^\top - \mathbb{E}\{x\}y^\top + \mathbb{E}\{x\}\mathbb{E}\{y\}^\top\big\} = \\
&= \mathbb{E}\{xy^\top\} - \mathbb{E}\{x\mathbb{E}\{y\}^\top\} - \mathbb{E}\{\mathbb{E}\{x\}y^\top\} + \mathbb{E}\{\mathbb{E}\{x\}\mathbb{E}\{y\}^\top\} = \\
&= \mathbb{E}\{xy^\top\} - \mathbb{E}\{x\}\mathbb{E}\{y\}^\top - \mathbb{E}\{x\}\mathbb{E}\{y^\top\} + \mathbb{E}\{x\}\mathbb{E}\{y^\top\} = \\
&= \mathbb{E}\{xy^\top\} - \mathbb{E}\{x\}\mathbb{E}\{y^\top\} = \Sigma_{xy}
\end{aligned}$$

Col 1: $\Sigma_x = \Sigma_x^\top$ (*is symmetric.* But Σ_{xy} *is* <u>*not*</u> *symmetric*)

$$\Sigma_x = \mathbb{E}\{xx^\top\} = \mathbb{E}\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\right\} = \mathbb{E}\left\{\begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2x_2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3x_3 \end{bmatrix}\right\}$$

Col 2: Σ_x is Positive (Semi)definite or PSD: $v^\top \Sigma_x v \geqslant 0 \ \forall v$

8 Sample mean and sample covariance

$x_i \sim p(x)$ iid

Sample mean

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Sample covariance

$$\overline{\Sigma}_x = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})^\top$$