

3 Boolean Algebra Cheatsheet

3.1 Definitions

- * **Boolean function** is a function of the form $f: \mathbb{B}^n \rightarrow \mathbb{B}$, where $n \geq 0$ is the *arity* of the function and $\mathbb{B} = \{0, 1\} = \{\perp, \top\} = \{F, T\}$ is a Boolean domain.
- * There are multiple ways to represent a Boolean function (all examples represent the same function):
 1. Truth table, e.g., $f = (1010)$, where LSB corresponds to $\mathbb{1}$, MSB to $\mathbb{0}$. Least/Most Significant Bit
 2. Analytically (as a sentence of propositional logic), e.g., $f(A, B) = \neg B$. Propositional logic
 3. Sum of minterms, e.g., $f = \sum m(0, 2) = m_0 + m_2$. Minterms
 4. Product of maxterms, e.g., $f = \prod M(1, 3) = M_1 \cdot M_3$. Maxterms
 5. Boolean function number, e.g., $f_{10}^{(2)}$ is the 10-th 2-ary function.
 Note that Wolfram's "Boolean operator number" is a slightly different term, which uses the reversed truth table.
 10-th Boolean function $f_{10}^{(2)}$ with the truth table (1010) can be obtained via the query "5th Boolean function of 2 variables" (note: not 10th!) in WolframAlpha, since $\text{rev}(1010_2) = 0101_2 = 5_{10}$.

3.2 Normal Forms

- * **Disjunctive forms:**
 - **Cube** is a conjunction of literals: $\mathcal{T} = \bigwedge_i \mathcal{L}_i$.
 - Formula is in **disjunctive normal form (DNF)** if it is a disjunction of terms: $\text{DNF} = \bigvee_i \mathcal{T}_i$.
 - **Minterm** is conjunction of literals, where *each* variable appears *once*, e.g., $m_6 = (A \wedge B \wedge \neg C)$.
 - Formula is in **canonical DNF (CDNF)** if it is a disjunction of minterms: $\text{CDNF} = \bigvee_i m_i$.
- * **Conjunctive forms:**
 - **Clause** is a disjunction of literals: $C = \bigvee_i \mathcal{L}_i$
 - Formula is in **conjunctive normal form (CNF)** if it is a conjunction of clauses: $\text{CNF} = \bigwedge_i C_i$.
 - **Maxterm** is disjunction of literals, where *each* variable appears *once*, e.g., $M_6 = (\neg A \vee \neg B \vee C)$.
 - Formula is in **canonical CNF (CCNF)** if it is a conjunction of maxterms: $\text{CCNF} = \bigwedge_i M_i$.
- * Some other normal forms:
 - Formula is in **negation normal form (NNF)** if the negation operator (\neg) is only applied to variables and the only other allowed Boolean operators are conjunction (\wedge) and disjunction (\vee).
 - Formula f is in **Blake canonical form (BCF)** if it is a disjunction of *all* the *prime implicants* of f .
 - Formula is in **prenex normal form (PNF)** if it consists of *prefix*—quantifiers and bound variables, and *matrix*—quantifier-free part.
 - Formula is in **Skolem normal form (SNF)** if it is in prenex normal form with only universal first-order quantifiers.
 - **Zhegalkin polynomial** is a formula in the following form (**algebraic normal form (ANF)**):
 - $f(X_1, \dots, X_n) = a_0 \oplus \bigoplus_{\substack{1 \leq i_1 \leq \dots \leq i_k \leq n \\ 1 \leq k \leq n}} (a_{i_1, \dots, i_k} \wedge X_{i_1} \wedge \dots \wedge X_{i_k})$, where $a_0, a_{i_1, \dots, i_k} \in \mathbb{B}$
 - $f(x_1, \dots, x_n) = a_0 \oplus (a_1 x_1 \oplus \dots \oplus a_n x_n) \oplus (a_{1,2} x_1 x_2 \oplus \dots \oplus a_{n-1,n} x_{n-1} x_n) \oplus \dots \oplus a_{1, \dots, n} x_1 \dots x_n$

3.3 Conversion to CNF/DNF

In order to convert *arbitrary* (i.e. any) Boolean formula to *equivalent* CNF/DNF:

1. Eliminate equivalences, implications and other "non-standard" operations (i.e. rewrite using only $\{\wedge, \vee, \neg\}$):

$$\mathcal{A} \leftrightarrow \mathcal{B} \rightsquigarrow (\mathcal{A} \rightarrow \mathcal{B}) \wedge (\mathcal{B} \rightarrow \mathcal{A})$$

$$\mathcal{A} \rightarrow \mathcal{B} \rightsquigarrow \neg \mathcal{A} \vee \mathcal{B}$$
2. Push negation downwards:

$$\neg(\mathcal{A} \vee \mathcal{B}) \rightsquigarrow \neg \mathcal{A} \wedge \neg \mathcal{B}$$

$$\neg(\mathcal{A} \wedge \mathcal{B}) \rightsquigarrow \neg \mathcal{A} \vee \neg \mathcal{B}$$
3. Eliminate double negation:

$$\neg \neg \mathcal{A} \rightsquigarrow \mathcal{A}$$

Note that after the recursive application of 1–3 the formula is in NNF.
4. Push disjunction (for CNF) / conjunction (for DNF) downward:

$$(\mathcal{A} \wedge \mathcal{B}) \vee \mathcal{C} \rightsquigarrow_{\text{CNF}} (\mathcal{A} \vee \mathcal{C}) \wedge (\mathcal{B} \vee \mathcal{C})$$

$$(\mathcal{A} \vee \mathcal{B}) \wedge \mathcal{C} \rightsquigarrow_{\text{DNF}} (\mathcal{A} \wedge \mathcal{C}) \vee (\mathcal{B} \wedge \mathcal{C})$$
5. Eliminate \top and \perp :

$$\mathcal{A} \wedge \top \rightsquigarrow \mathcal{A} \qquad \mathcal{A} \wedge \perp \rightsquigarrow \perp$$

$$\mathcal{A} \vee \top \rightsquigarrow \top \qquad \mathcal{A} \vee \perp \rightsquigarrow \mathcal{A}$$

$$\neg \top \rightsquigarrow \perp \qquad \neg \perp \rightsquigarrow \top$$

3.4 Functional Completeness

- * A set S is called **closed** under some operation “ \bullet ” if the result of the operation applied to any elements in the set is also contained in this set, i.e. $\forall x, y \in S : (x \bullet y) \in S$. Closed set
- * The **closure** S^* of a set S is the minimal *closed* superset of S . Closure
- * A set of Boolean functions F is called **functionally complete** if it can be used to express all possible Boolean functions. Formally, $F^* = \mathbb{F}$, where F^* is a *functional closure* of F , and $\mathbb{F} = \bigcup_{n \in \mathbb{N}} \{f : \mathbb{B}^n \rightarrow \mathbb{B}\}$.

Post’s Functional Completeness Theorem. A set of Boolean functions F is functionally complete iff it contains:

- at least one function that does *not* preserve zero, i.e. $\exists f \in F : f \notin T_0$, and
 - at least one function that does *not* preserve one, i.e. $\exists f \in F : f \notin T_1$, and
 - at least one function that is *not* self-dual, i.e. $\exists f \in F : f \notin S$, and
 - at least one function that is *not* monotonic, i.e. $\exists f \in F : f \notin M$, and
 - at least one function that is *not* linear function, i.e. $\exists f \in F : f \notin L$.
- * A function f is **zero-preserving** iff it is False on the zero-valuation ($\mathbf{0} = (0, 0, \dots, 0)$):
 $f \in T_0 \leftrightarrow f(\mathbf{0}) = 0$
 - * A function f is **one-preserving** iff it is True on the one-valuation ($\mathbf{1} = (1, 1, \dots, 1)$):
 $f \in T_1 \leftrightarrow f(\mathbf{1}) = 1$
 - * A function f is **self-dual** iff it is dual to itself:
 $f \in S \leftrightarrow \forall x_1, \dots, x_n \in \mathbb{B} : f(x_1, \dots, x_n) = \bar{f}(\bar{x}_1, \dots, \bar{x}_n)$.
 - * A function f is **monotonic** iff for every increasing valuations, the function does not decrease:
 $f \in M \leftrightarrow \forall a, b \in \mathbb{B}^n : a \leq b \rightarrow f(a) \leq f(b)$.
 Comparison of valuations $a, b \in \mathbb{B}^n$ is defined as follows:

$$a \leq b \leftrightarrow \bigwedge_{1 \leq i \leq n} (a_i \leq b_i)$$
 - * A function f is **linear** iff its Zhegalkin polynomial is linear (i.e. has a degree at most 1):
 $f \in L \leftrightarrow \deg f_{\oplus} \leq 1$