

# Discrete Mathematics

**Combinatorics** — Spring 2025

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# §1 Combinatorics

# Introduction to Combinatorics

**Definition 1:** Combinatorics is the branch of discrete mathematics that deals with *counting*, *arranging*, and analyzing *discrete structures*.

## Three basic problems of Combinatorics:

1. Existence: *Is there at least one arrangement of a particular kind?*
2. Counting: *How many arrangements are there?*
3. Optimization: *Which one is best according to some criteria?*

## Discrete structures

- Graphs, sets, multisets, sequences, patterns, coverings, partitions...

## Enumeration

- Permutations, combinations, inclusion/exclusion, generating functions, recurrence relations...

## Algorithms and optimization

- Sorting, eulerian circuits, hamiltonian cycles, planarity testing, graph coloring, spanning trees, shortest paths, network flows, bipartite matchings, chain partitions...

# Discrete Structures

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We investigate the *building blocks* of combinatorics:

- Sets and multisets
- Sequences and strings
- Arrangements
- Graphs, networks, trees
- Posets and lattices
- Partitions
- Patterns, coverings, designs, configurations
- Schedules, assignments, distributions

*Used in data modeling, logic, cryptography, and the design of data structures.*

# Enumerative Combinatorics

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We learn how to count *without explicit listing*:

- Permutations and combinations
- Inclusion–Exclusion Principle
- Set partitions, integer partitions, Stirling numbers, Catalan numbers
- Recurrence relations
- Generating functions

*Used in probability theory, complexity theory, coding theory, computational biology.*

# Algorithmic and Optimization Methods

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Combinatorics powers *algorithm design* and complexity analysis:

- Sorting
- Searching
- Eulerian paths and Hamiltonian cycles
- Planarity, colorings, cliques, coverings
- Spanning trees
- Shortest paths
- Network flows
- Bipartite matchings
- Dilworth's theorem, chain and antichain partitions

*Used in logistics, scheduling, routing, and complexity optimization.*

## §2 Basic Counting Principles

## Basic Counting Rules

**PRODUCT RULE:** If something can happen in  $n_1$  ways, *and* no matter how the first thing happens, a second thing can happen in  $n_2$  ways, then the two things *together* can happen in  $n_1 \cdot n_2$  ways.

**SUM RULE:** If one event can occur in  $n_1$  ways and a second event in  $n_2$  (different) ways, then there are  $n_1 + n_2$  ways in which *either* the first event *or* the second event can occur (*but not both*).



## Addition Principle

**Definition 2:** We say a finite set  $S$  is *partitioned* into *parts*  $S_1, \dots, S_m$  if the parts are pairwise disjoint and their union is  $S$ . In other words,  $S_i \cap S_j = \emptyset$  for  $i \neq j$  and  $S_1 \cup S_2 \cup \dots \cup S_m = S$ . In that case:

$$|S| = |S_1| + |S_2| + \dots + |S_m|$$

*Example:* Let  $S$  be the set of students attending the combinatorics lecture. It can be partitioned into parts  $S_1$  and  $S_2$  where

$S_1$  = set of students that *like* easy examples.

$S_2$  = set of students that *don't like* easy examples.

If  $|S_1| = 22$  and  $|S_2| = 8$ , then we can conclude  $|S| = |S_1| + |S_2| = 30$ .

## Multiplication Principle

**Definition 3:** If  $S$  is a finite set that is the *product* of  $S_1, \dots, S_m$ , that is,  $S = S_1 \times \dots \times S_m$ , then

$$|S| = |S_1| \times \dots \times |S_m|$$

*Example:* TODO: example with car plates

## Subtraction Principle

**Definition 4:** Let  $S$  be a subset of a finite set  $T$ . We define the *complement* of  $S$  as  $\overline{S} = T \setminus S$ . Then

$$|\overline{S}| = |T| - |S|$$

*Example:* If  $T$  is the set of students studying at KIT and  $S$  the set of students studying neither math nor computer science. If we know  $|T| = 23905$  and  $|S| = 20178$ , then we can compute the number  $|\overline{S}|$  of students studying either math or computer science:

$$|\overline{S}| = |T| - |S| = 23905 - 20178 = 3727$$

## Bijection Principle

**Definition 5:** If  $S$  and  $T$  are sets, then

$$|S| = |T| \iff \text{there exists a bijection between } S \text{ and } T$$

*Example:* Let  $S$  be the set of students attending the combinatorics lecture and  $T$  the set of homework submissions (unique per student) for the first problem sheet. If the number of students and the number of submissions coincide, then there is a bijection between students and submissions.

**Note:** The bijection principle works both for *finite* and *infinite* sets.

## Pigeonhole Principle

**Definition 6:** Let  $S_1, \dots, S_m$  be finite sets that are pairwise disjoint and  $|S_1| + |S_2| + \dots + |S_m| = n$ .

$$\exists i \in \{1, \dots, m\} : |S_i| \geq \left\lfloor \frac{n}{m} \right\rfloor \quad \text{and} \quad \exists j \in \{1, \dots, m\} : |S_j| \leq \left\lceil \frac{n}{m} \right\rceil$$

*Example:* Assume there are 5 holes in the wall where pigeons nest. Say there is a set  $S_i$  of pigeons nesting in hole  $i$ . Assume there are  $n = 17$  pigeons in total. Then we know:

- There is some hole with at least  $d = 4$  pigeons.
- There is some hole with at most  $b = 3$  pigeons.

## Double Counting

If we count the same quantity in *two different ways*, then this gives us a (perhaps non-trivial) identity.

*Example (Handshaking Lemma):* Assume there are  $n$  people at a party and everybody will shake hands with everybody else. How many handshakes will occur? We count this number in two ways:

1. Every person shakes  $n - 1$  hands and there are  $n$  people. However, two people are involved in a handshake so if we just multiply  $n \cdot (n - 1)$ , then every handshake is counted twice. The total number of handshakes is therefore  $\frac{n \cdot (n-1)}{2}$ .
2. We number the people from 1 to  $n$ . To avoid counting a handshake twice, we count for person  $i$  only the handshakes with persons of lower numbers. Then the total number of handshakes is:

$$\sum_{i=1}^n (i-1) = \sum_{i=0}^{\{n-1\}} i = \sum_{i=1}^{n-1} i$$

The identity we obtain is therefore:  $\sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$

## §3 Arrangements, Permutations, Combinations

# Ordered Arrangements

**Definition 7:** Denote by  $[n] = \{1, \dots, n\}$  the set of natural numbers from 1 to  $n$ .

Hereinafter, let  $X$  be a finite set.








**Definition 8:** An *ordered arrangement* of  $n$  elements of  $X$  is a *map*  $s : [n] \rightarrow X$ .

- Here,  $[n]$  is the *domain* of  $s$ , and  $s(i)$  is the *image* of  $i \in [n]$  under  $s$ .
- The set  $\{x \in X \mid s(i) = x \text{ for some } i \in [n]\}$  is the *range* of  $s$ .

Other common names for ordered arrangements are:

- *string* (or *word*), e.g. “Banana”
- *sequence*, e.g. “0815422372”
- *tuple*, e.g.  $(3, 5, 2, 5, 8)$

Example:

$i$	1	2	3	4	5	6	7
$s(i)$							



# Permutations

**Definition 9:** A *permutation* of  $X$  is a *bijective* map  $\pi : [n] \rightarrow X$ .

Usually,  $X = [n]$ , and the set of all permutations of  $[n]$  is denoted by  $S_n$ .

Example:

$i$	1	2	3	4	5	6	7
$\pi(i)$	2	7	1	3	5	4	6

**Definition 10:** *k-permutation* of  $X$  is an ordered arrangement of  $k$  *distinct* elements of  $X$ , that is, an *injective* map  $\pi : [k] \rightarrow X$ .

The set of all  $k$ -permutations of  $X = [n]$  is denoted by  $P(n, k)$ . In particular,  $S_n = P(n, n)$ .

TODO: circular permutations

## Counting Permutations

**Theorem 1:** For any natural numbers  $0 \leq k \leq n$ , we have

$$|P(n, k)| = n \cdot (n - 1) \cdot \dots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

**Proof:** A permutation is an injective map  $\pi : [k] \rightarrow [n]$ . We count the number of ways to pick such a map, picking the images one after the other. There are  $n$  ways to choose  $\pi(1)$ . Given a value for  $\pi(1)$ , there are  $(n - 1)$  ways to choose  $\pi(2)$  (since we may not choose  $\pi(1)$  again). Continuing like this, there are  $(n - i + 1)$  ways to pick  $\pi(i)$ , and the last value we pick is  $\pi(k)$  with  $(n - k + 1)$  possibilities.

Every  $k$ -permutation can be constructed like this in *exactly one way*. The total number of  $k$ -permutations is therefore given as the product:

$$|P(n, k)| = n \cdot (n - 1) \cdot \dots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

□

## Counting Circular Permutations

**Theorem 2:** For any natural numbers  $0 \leq k \leq n$ , we have

$$|P_c(n, k)| = \frac{n!}{k! \cdot (n - k)!}$$

**Proof:** We doubly count  $P(n, k)$ :

1.  $|P(n, k)| = \frac{n!}{(n-k)!}$  which we proved before.
2.  $|P(n, k)| = |P_c(n, k)| \cdot k$  because every equivalence class in  $P_c(n, k)$  contains  $k$  permutations from  $P(n, k)$  since there are  $k$  ways to rotate a  $k$ -permutation.

From this we get  $\frac{n!}{(n-k)!} = |P_c(n, k)| \cdot k$  which implies  $|P_c(n, k)| = \frac{n!}{k! \cdot (n-k)!}$ . □

# Unordered Arrangements

**Definition 11:** An *unordered arrangement* of  $k$  elements of  $X$  is a *multiset*  $S = \langle X, r \rangle$  of size  $k$ .

In a multiset,  $X$  is the set of *types*, and for each type  $x \in X$ ,  $r_x$  is its *repetition number*.

*Example:* Let  $X = \{ \text{🍷}, \text{🐼}, \text{🐱}, \text{🎴}, \text{🌵} \}$ .

- An unordered arrangement of 7 elements could be  $S = \{ \text{🍷}, \text{🍷}, \text{🐼}, \text{🐱}, \text{🐱}, \text{🐱}, \text{🌵} \}^*$ .
- The same multiset could be written as  $S = \{ 2 \text{🍷}, 1 \text{🐼}, 3 \text{🐱}, 0 \text{🎴}, 1 \text{🌵} \}$ .

## Subsets

The most important special case of unordered arrangements is where all repetitions are 1, i.e.,  $r_x = 1$  for all  $x \in X$ . Then  $S$  is simply a *subset* of  $X$ , denoted  $S \subset X$ .

**Definition 12:** A *k-combination* of  $X$  is an unordered arrangement of  $k$  *distinct* elements of  $X$ .

**Note:** The more standard term is *subset*. The term “combination” is only used to emphasize the selection process.

The set of all  $k$ -subsets of  $X$  is denoted  $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$ . If  $|X| = n$ , then

$$\binom{n}{k} := \left| \binom{X}{k} \right|$$

*Example:* The set of edges in a simple undirected graph consists of 2-subsets of its vertices:  $E \subseteq \binom{V}{2}$ .

## Counting $k$ -Combinations

**Theorem 3:** For  $0 \leq k \leq n$ , we have

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

**Proof:**

$$|P(n, k)| = \frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!$$

□

## $k$ -Combinations of a Multiset

**Definition 13:** Let  $X$  be a finite set of types, and let  $M = \langle X, r \rangle$  be a finite multiset with repetition numbers  $r_1, \dots, r_{|X|}$ . A  *$k$ -combination of  $M$*  is a multiset  $S = \langle X, s \rangle$  with types in  $X$  and repetition numbers  $s_1, \dots, s_{|X|}$  such that  $s_i \leq r_i$  for all  $1 \leq i \leq |X|$ , and  $\sum_{i=1}^{|X|} s_i = k$ .

*Example:* Consider  $M = \{2 \text{ 🦊}, 1 \text{ 🥤}, 3 \text{ 🍉}, 1 \text{ 💎}\}$ .

- $T = \{1 \text{ 🦊}, 2 \text{ 🍉}\}$  is a 3-combination of  $M$ .
- $T' = \{3 \text{ 💎}\}$  is not.

*Counting  $k$ -combinations of a multiset is not as simple as it might seem...*

## $k$ -Permutations of a Multiset

**Definition 14:** Let  $M$  be a finite multiset with set of types  $X$ . A  $k$ -permutation of  $M$  is an ordered arrangement of  $k$  elements of  $M$  where different orderings of elements of the same type are not distinguished. This is an ordered multiset with types in  $X$  and repetition numbers  $s_1, \dots, s_{|X|}$  such that  $s_i \leq r_i$  for all  $1 \leq i \leq |X|$ , and  $\sum_{i=1}^{|X|} s_i = k$ .

**Note:** There might be several elements of the same type compared to a permutation of a set (where each repetition number equals 1).

*Example:* Let  $M = \{2 \text{ 🦊}, 1 \text{ 🥤}, 3 \text{ 🍉}, 1 \text{ 💎}\}$ , then  $T = (\text{💎}, \text{🍉}, \text{🍉}, \text{🦊})$  is a 4-permutation of multiset  $M$ .



## Binomial Theorem

**Theorem 4:** The expansion of any non-negative integer power  $n$  of the binomial  $(x + y)$  is a sum

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} \cdot x^k \cdot y^{n-k}$$

where each  $\binom{n}{k}$  is a positive integer known as a *binomial coefficient*, defined as

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots \cdot 2 \cdot 1}$$

# Multinomial Theorem

**Theorem 5:** The generalization of the binomial theorem:

$$(x_1 + \dots + x_r)^n = \sum_{\substack{0 \leq k_1, \dots, k_r \leq n \\ k_1 + \dots + k_r = n}}^n \binom{n}{k_1, \dots, k_r} \cdot x_1^{k_1} \cdot \dots \cdot x_r^{k_r}$$

*Multinomial coefficients* are defined as

$$\binom{n}{k_1, \dots, k_r} = \frac{n!}{k_1! \cdot \dots \cdot k_r!}$$

**Note:** Binomial coefficients are special cases of multinomial coefficients ( $r = 2$ ):

$$\binom{n}{k} = \binom{n}{k_1, k_2} = \binom{n}{k, n-k} = \frac{n!}{k! \cdot (n-k)!}$$

**Proof:** TODO



## Permutations of a Multiset

**Theorem 6:** Let  $S$  be a finite multiset with  $k$  different types and repetition numbers  $r_1, \dots, r_k$ . Let the size of  $S$  be  $n = r_1 + \dots + r_k$ . Then the number of  $n$ -permutations of  $S$  equals

$$\binom{n}{r_1, \dots, r_k}$$

**Proof:** In an  $n$ -permutation there are  $n$  positions that need to be assigned a type.

First, choose the  $r_1$  positions for the first type, there are  $\binom{n}{r_1}$  ways to do so. Then, assign  $r_2$  positions for the second type, out of the  $(n - r_1)$  positions that are still available, there are  $\binom{n-r_1}{r_2}$  ways to do so. Continue for all  $k$  types. The total number of choices will be:

$$\binom{n}{r_1} \cdot \binom{n-r_1}{r_2} \cdot \dots \cdot \binom{n-r_1-r_2-\dots-r_{k-1}}{r_k} = \binom{n}{r_1, \dots, r_k}$$

□

## *k*-Combinations of an *Infinite Multiset*

*Example:* Suppose you have a *sufficiently large* amount of each type of fruit (🍌, 🍎, 🍐) in the supermarket, and you want to buy *two* fruits. How many choices do you have?

There are exactly *six* combinations: {🍌, 🍌}, {🍌, 🍎}, {🍌, 🍐}, {🍎, 🍎}, {🍎, 🍐}, {🍐, 🍐}.

Note that your selection is *not ordered*, so {🍐, 🍎} and {🍎, 🍐} are considered the *same* choice.

## $k$ -Combinations of an *Infinite Multiset* [2]

**Theorem 7:** Let  $k, s \in \mathbb{N}$  and let  $S$  be a multiset with  $s$  types and large repetition numbers (each  $r_1, \dots, r_s$  is *at least*  $k$ ), then the number of  $k$ -combinations of  $S$  equals

$$\binom{k+s-1}{k} = \binom{k+s-1}{s-1}$$

**Proof:** Let  $S = \{\infty \text{ 🍌}, \infty \text{ 🍎}, \infty \text{ 🍏}\}$ , so  $s = 3$ .

- Suppose  $k = 5$ .
- Consider a 5-combination of  $S$ : { 🍌, 🍎, 🍌, 🍏, 🍏 }.
- Reorder and group: { 🍌 🍌 | 🍎 | 🍏 🍏 }.
- Convert to *dots* and *bars*:  $\bullet\bullet \mid \bullet \mid \bullet\bullet$
- Represent as a multiset:  $M = \{k \cdot \bullet, (s-1) \cdot \mid\}$
- Permute the 2-type multiset:  $\binom{k+s-1}{k, s-1}$  ways, by Theorem 5.

□