

1. The graph¹ of Europe $\mathcal{G}^* = \langle V, E \rangle$ is defined as follows: each vertex $v \in V$ is a Europe country²; two vertices are adjacent ($\{u, v\} \in E$) if the corresponding countries share a land border. Let \mathcal{G} be the largest connected component of \mathcal{G}^* .
 - (a) Draw³ \mathcal{G}^* with the minimum number of intersecting edges⁴.
 - (b) Find $|V|$, $|E|$, $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$, $\text{rad}(\mathcal{G})$, $\text{diam}(\mathcal{G})$, $\text{girth}(\mathcal{G})$, $\text{center}(\mathcal{G})$, $\kappa(\mathcal{G})$, $\lambda(\mathcal{G})$.
 - (c) Find the minimum vertex coloring $Z : V \rightarrow \mathbb{N}$ of \mathcal{G} .
 - (d) Find the minimum edge coloring $X : E \rightarrow \mathbb{N}$ of \mathcal{G} .
 - (e) Find the maximum clique $Q \subseteq V$ of \mathcal{G} .
 - (f) Find the maximum stable set $S \subseteq V$ of \mathcal{G} .
 - (g) Find the maximum matching $M \subseteq E$ of \mathcal{G} .
 - (h) Find the minimum vertex cover $R \subseteq V$ of \mathcal{G} .
 - (i) Find the minimum edge cover $F \subseteq E$ of \mathcal{G} .
 - (j) Find the shortest closed walk W that visits every vertex of \mathcal{G} .
 - (k) Find the shortest closed walk U that visits every edge of \mathcal{G} .
 - (l) Find all biconnected components (blocks) and draw the block-cut tree of \mathcal{G}^* .
 - (m) Find all 2-edge-connected components of \mathcal{G}^* .
 - (n) Add the weight function $w : E \rightarrow \mathbb{R}$ denoting the distance⁵ between capitals. Find the minimum (w.r.t. the total weight of edges) spanning tree T for the largest connected component of the weighted Europe graph $\mathcal{G}_w^* = (V, E, w)$.
 - (o) Find $\text{centroid}(T)$ (w.r.t. the edge weight function w).
 - (p) Construct the Prüfer code for T .

2. Prove *rigorously* the following theorems:

Theorem 1 (TRIANGLE INEQUALITY). For any connected graph $G = \langle V, E \rangle$:

$$\forall x, y, z \in V : \text{dist}(x, y) + \text{dist}(y, z) \geq \text{dist}(x, z)$$

Theorem 2 (TREE). A connected graph $G = \langle V, E \rangle$ is a tree (i.e. acyclic graph) iff $|E| = |V| - 1$.

Theorem 3 (HALL). For any bipartite graph $G = (X, Y, E)$, there exists X -perfect matching (set of disjoint edges covering all vertices in X) iff $|N(W)| \geq |W|$ for every $W \subseteq X$.

Theorem 4 (WHITNEY). For any graph G : $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Theorem 5 (CHARTRAND). For a connected graph $G = \langle V, E \rangle$: if $\delta(G) \geq \lfloor |V|/2 \rfloor$, then $\lambda(G) = \delta(G)$.

Theorem 6 (Menger). For any pair of non-adjacent vertices u and v in an undirected graph, the size of the minimum *vertex cut* is equal to the maximum number of pairwise *internally vertex-disjoint paths* from u to v .

Theorem 7 (HARARY). Every block of a block graph⁶ is a clique.

¹ Hereinafter, “graphs” are “simple, finite, undirected, and unweighted”, unless stated otherwise.

² Since the absolute geopolitical correctness is not necessary to accomplish this task, you can simply use https://simple.wikipedia.org/wiki/List_of_European_countries or any other similar source as a reference.

³ You may either preserve the original spatial relationships between vertices, or use an automatic layout software. Take a look at Graphviz, Gephi, Cytoscape.

⁴ Since \mathcal{G}^* naturally corresponds to a “map”, it *has* to be planar. However, due to *some broken countries*, it might be impossible to draw \mathcal{G}^* completely without edge intersections, so you just have to minimize them.

⁵ You may choose geodesic⁶ or road distance, to your preference. You *have* to use *more-or-less real data*.

⁶ A block graph $H = B(G)$ is an intersection graph of all blocks (biconnected components) of G , i.e. each vertex $v \in V(H)$ corresponds to a block of G , and there is an edge $\{v, u\} \in E(H)$ iff “blocks” v and u share a cut vertex.