

4 Formal Logic Cheatsheet

4.1 Propositional Logic

- * **Proposition** is a statement which can be either true or false. Truth-bearer
- * **Alphabet** of propositional logic consists of (1) atomic symbols and (2) operator symbols.
- * **Atomic formula** (**atom**) is an irreducible formula without logical connectives.
 - Propositional **variables**: A, B, C, \dots, Z . With indices, if needed: $A_1, A_2, \dots, Z_1, Z_2, \dots$
 - Logical **constants**: \top for always true proposition (*tautology*), \perp for always false proposition (*contradiction*).
- * **Logical connectives** (**operators**):

| Type | Natural meaning | Symbolization |
|------------------------------|---|---|
| Negation | It is not the case that \mathcal{P} . It is false that \mathcal{P} . It is not true that \mathcal{P} . | $\neg \mathcal{P}$ |
| Conjunction | Both \mathcal{P} and \mathcal{Q} . \mathcal{P} but \mathcal{Q} . \mathcal{P} , although \mathcal{Q} . | $\mathcal{P} \wedge \mathcal{Q}$ |
| Disjunction | Either \mathcal{P} or \mathcal{Q} (or both). \mathcal{P} unless \mathcal{Q} . | $\mathcal{P} \vee \mathcal{Q}$ |
| Exclusive or (Xor) | Either \mathcal{P} or \mathcal{Q} (but not both). \mathcal{P} xor \mathcal{Q} . | $\mathcal{P} \oplus \mathcal{Q}$ |
| Implication (Conditional) | If \mathcal{P} , then \mathcal{Q} . \mathcal{P} only if \mathcal{Q} . \mathcal{Q} if \mathcal{P} . | $\mathcal{P} \rightarrow \mathcal{Q}$ |
| Biconditional | \mathcal{P} , if and only if \mathcal{Q} . \mathcal{P} iff \mathcal{Q} . \mathcal{P} just in case \mathcal{Q} . | $\mathcal{P} \leftrightarrow \mathcal{Q}$ |

- * **Sentence** of propositional logic is defined inductively: Well-formed formula (WFF)
 1. Every propositional variable/constant is a sentence.
 2. If \mathcal{A} is a sentence, then $\neg \mathcal{A}$ is a sentence.
 3. If \mathcal{A} and \mathcal{B} are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \rightarrow \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
 4. Nothing else is a sentence.

- * Well-formed formulae grammar: Backus-Naur form (BNF)

$\langle \text{sentence} \rangle ::= \langle \text{constant} \rangle$
 | $\langle \text{variable} \rangle$
 | $\neg \langle \text{sentence} \rangle$
 | $\langle ' \rangle \langle \text{sentence} \rangle \langle \text{binop} \rangle \langle \text{sentence} \rangle \langle ' \rangle$
 $\langle \text{constant} \rangle ::= \top \mid \perp$
 $\langle \text{variable} \rangle ::= A \mid \dots \mid Z \mid A_1 \mid \dots \mid Z_n$
 $\langle \text{binop} \rangle ::= \wedge \mid \vee \mid \oplus \mid \rightarrow \mid \leftarrow \mid \leftrightarrow$

- * **Literal** is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (*positive literal*), $\mathcal{L}_j = \neg X_j$ (*negative literal*).
- * **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}_{\text{premises}} \quad \therefore \quad \underbrace{\mathcal{C}}_{\text{conclusion}}$$

“therefore”

- * An argument is **valid** if whenever all the premises are true, the conclusion is also true. Validity
- * An argument is **invalid** if there is a case (*a counterexample*) when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

- * **Valuation** is any assignment of truth values to propositional variables. Interpretation
- * \mathcal{A} is a **tautology** (valid) iff it is true on *every* valuation. Might be symbolized as “ $\models \mathcal{A}$ ”.
- * \mathcal{A} is a **contradiction** iff it is false on *every* valuation. Might be symbolized as “ $\mathcal{A} \models$ ”.
- * \mathcal{A} is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- * \mathcal{A} is **satisfiable** iff it is true on *some* valuation. Satisfiability
- * \mathcal{A} is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. Falsifiability
- * \mathcal{A} and \mathcal{B} are **equivalent** (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- * $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent (jointly satisfiable)** iff there is *some* valuation which makes them all true. Sentences are **inconsistent (jointly unsatisfiable)** iff there is *no* valuation that makes them all true. Consistency
- * The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false. Semantic entailment
- * If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \therefore C$ is **valid**.

Validity check examples:

| A | B | $A \rightarrow B$ | A | \therefore | B | $\neg A \rightarrow \neg B$ | \therefore | $B \rightarrow A$ | $A \rightarrow B$ | B | \therefore | $\neg (B \rightarrow A)$ | | | |
|-----|-----|-------------------|-----|--------------|-----|-----------------------------|--------------|-------------------|-------------------|-----|--------------|--------------------------|---|---|---|
| 0 | 0 | 1 | 0 | · | 0 | 1 | 1 | ✓ | 1 | 1 | 0 | · | 0 | 1 | |
| 0 | 1 | 1 | 0 | · | 1 | 1 | 0 | 0 | 1 | 1 | ✓ | 1 | 1 | 0 | |
| 1 | 0 | 0 | 1 | · | 0 | 0 | 1 | 1 | ✓ | 1 | 0 | 0 | · | 0 | 1 |
| 1 | 1 | 1 | 1 | ✓ | 1 | 0 | 1 | 0 | ✓ | 1 | 1 | 1 | ✗ | 0 | 1 |

| R | S | T | $R \vee S$ | $S \vee T$ | $\neg R$ | \therefore | $S \wedge T$ | $(R \wedge S) \rightarrow T$ | \therefore | $R \rightarrow (S \rightarrow T)$ |
|-----|-----|-----|------------|------------|----------|--------------|--------------|------------------------------|--------------|-----------------------------------|
| 0 | 0 | 0 | 0 | 0 | 1 | . | 0 | 0 | 1 | 0 |
| 0 | 0 | 1 | 0 | 1 | 1 | . | 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | ✗ | 0 | 0 | 1 | 0 |
| 0 | 1 | 1 | 1 | 1 | 1 | ✓ | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 | 0 | . | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | . | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 1 | 1 | 0 | . | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 0 | . | 1 | 1 | 1 | 1 |

- * **Soundness**: $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$ “Every provable statement is in fact true”
- * **Completeness**: $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$ “Every true statement has a proof”

4.3 Natural Deduction Rules

Reiteration

| | | |
|--------------|---------------|-------|
| m | \mathcal{A} | |
| \therefore | \mathcal{A} | R m |

Explosion

| | | |
|--------------|---------------|-------|
| m | \perp | |
| \therefore | \mathcal{A} | X m |

Conditional

| | | |
|--------------|---------------------------------------|-----------------------|
| i | \mathcal{A} | |
| j | \mathcal{B} | |
| \therefore | $\mathcal{A} \rightarrow \mathcal{B}$ | \rightarrow I $i-j$ |

Modus ponens

| | | |
|--------------|---------------------------------------|-----------|
| i | $\mathcal{A} \rightarrow \mathcal{B}$ | |
| j | \mathcal{A} | |
| \therefore | \mathcal{B} | MP i, j |

Conjunction

| | | |
|--------------|----------------------------------|-------------------|
| i | \mathcal{A} | |
| j | \mathcal{B} | |
| \therefore | $\mathcal{A} \wedge \mathcal{B}$ | \wedge I i, j |

| | | |
|--------------|----------------------------------|----------------|
| m | $\mathcal{A} \wedge \mathcal{B}$ | |
| \therefore | \mathcal{A} | \wedge E m |
| \therefore | \mathcal{B} | \wedge E m |

Contraposition

| | | |
|--------------|---|------------|
| m | $\mathcal{A} \rightarrow \mathcal{B}$ | |
| \therefore | $\neg \mathcal{B} \rightarrow \neg \mathcal{A}$ | Contra m |

Modus tollens

| | | |
|--------------|---------------------------------------|-----------|
| i | $\mathcal{A} \rightarrow \mathcal{B}$ | |
| j | $\neg \mathcal{B}$ | |
| \therefore | $\neg \mathcal{A}$ | MT i, j |

Biconditional

| | | |
|--------------|---|--------------------------------|
| i | \mathcal{A} | |
| j | \mathcal{B} | |
| k | \mathcal{B} | |
| l | \mathcal{A} | |
| \therefore | $\mathcal{A} \leftrightarrow \mathcal{B}$ | \leftrightarrow I $i-j, k-l$ |

Negation

| | | |
|--------------|--------------------|-----------------|
| i | $\neg \mathcal{A}$ | |
| j | \mathcal{A} | |
| \therefore | \perp | \neg E i, j |

| | | |
|--------------|--------------------|----------------|
| i | \mathcal{A} | |
| j | \perp | |
| \therefore | $\neg \mathcal{A}$ | \neg I $i-j$ |

Disjunction

| | | |
|--------------|--------------------------------|--------------|
| m | \mathcal{A} | |
| \therefore | $\mathcal{A} \vee \mathcal{B}$ | \vee I m |

| | | |
|--------------|--------------------------------|--------------|
| m | \mathcal{A} | |
| \therefore | $\mathcal{B} \vee \mathcal{A}$ | \vee I m |

| | | |
|--------------|--------------------------------|------------------------|
| m | $\mathcal{A} \vee \mathcal{B}$ | |
| i | \mathcal{A} | |
| j | \mathcal{C} | |
| k | \mathcal{B} | |
| l | \mathcal{C} | |
| \therefore | \mathcal{C} | \vee E $m, i-j, k-l$ |

| | | |
|--------------|---|----------------------------|
| i | $\mathcal{A} \leftrightarrow \mathcal{B}$ | |
| j | \mathcal{A} | |
| \therefore | \mathcal{B} | \leftrightarrow E i, j |

| | | |
|--------------|---|----------------------------|
| i | $\mathcal{A} \leftrightarrow \mathcal{B}$ | |
| j | \mathcal{B} | |
| \therefore | \mathcal{A} | \leftrightarrow E i, j |

Indirect proof

| | | |
|--------------|--------------------|----------|
| i | $\neg \mathcal{A}$ | |
| j | \perp | |
| \therefore | \mathcal{A} | IP $i-j$ |

Disjunctive syllogism

| | | |
|--------------|--------------------------------|-----------|
| i | $\mathcal{A} \vee \mathcal{B}$ | |
| j | $\neg \mathcal{A}$ | |
| \therefore | \mathcal{B} | DS i, j |

| | | |
|--------------|--------------------------------|-----------|
| i | $\mathcal{A} \vee \mathcal{B}$ | |
| j | $\neg \mathcal{B}$ | |
| \therefore | \mathcal{A} | DS i, j |

De Morgan Rules

| | | |
|--------------|--|---------|
| m | $\neg(\mathcal{A} \vee \mathcal{B})$ | |
| \therefore | $\neg \mathcal{A} \wedge \neg \mathcal{B}$ | DeM m |

| | | |
|--------------|--|---------|
| m | $\neg \mathcal{A} \wedge \neg \mathcal{B}$ | |
| \therefore | $\neg(\mathcal{A} \vee \mathcal{B})$ | DeM m |

| | | |
|--------------|--|---------|
| m | $\neg(\mathcal{A} \wedge \mathcal{B})$ | |
| \therefore | $\neg \mathcal{A} \vee \neg \mathcal{B}$ | DeM m |

| | | |
|--------------|--|---------|
| m | $\neg \mathcal{A} \vee \neg \mathcal{B}$ | |
| \therefore | $\neg(\mathcal{A} \wedge \mathcal{B})$ | DeM m |

Double negation

| | | |
|--------------|-------------------------|-------------------|
| m | $\neg \neg \mathcal{A}$ | |
| \therefore | \mathcal{A} | $\neg \neg$ E m |

Law of excluded middle

| | | |
|--------------|--------------------|----------------|
| i | \mathcal{A} | |
| j | \mathcal{B} | |
| k | $\neg \mathcal{A}$ | |
| l | \mathcal{B} | |
| \therefore | \mathcal{B} | LEM $i-j, k-l$ |

Hypothetical syllogism

| | | |
|--------------|---------------------------------------|-----------|
| i | $\mathcal{A} \rightarrow \mathcal{B}$ | |
| j | $\mathcal{B} \rightarrow \mathcal{C}$ | |
| \therefore | $\mathcal{A} \rightarrow \mathcal{C}$ | HS i, j |

Green: basic rules.

Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).