

4 Formal Logic Cheatsheet

4.1 Propositional Logic

* **Proposition** is a statement which can be either true or false.

Truth-bearer

* **Alphabet** of propositional logic consists of (1) atomic symbols and (2) operator symbols.

* **Atomic formula** (**atom**) is an irreducible formula without logical connectives.

◦ Propositional **variables**: A, B, C, \dots, Z . With indices, if needed: $A_1, A_2, \dots, Z_1, Z_2, \dots$

◦ Logical **constants**: \top for always true proposition (*tautology*), \perp for always false proposition (*contradiction*).

* **Logical connectives** (**operators**):

Type	Natural meaning	Symbolization
Negation	It is not the case that \mathcal{P} . It is false that \mathcal{P} . It is not true that \mathcal{P} .	$\neg \mathcal{P}$
Conjunction	Both \mathcal{P} and \mathcal{Q} . \mathcal{P} but \mathcal{Q} . \mathcal{P} , although \mathcal{Q} .	$\mathcal{P} \wedge \mathcal{Q}$
Disjunction	Either \mathcal{P} or \mathcal{Q} (or both). \mathcal{P} unless \mathcal{Q} .	$\mathcal{P} \vee \mathcal{Q}$
Exclusive or (Xor)	Either \mathcal{P} or \mathcal{Q} (but not both). \mathcal{P} xor \mathcal{Q} .	$\mathcal{P} \oplus \mathcal{Q}$
Implication (Conditional)	If \mathcal{P} , then \mathcal{Q} . \mathcal{P} only if \mathcal{Q} . \mathcal{Q} if \mathcal{P} .	$\mathcal{P} \rightarrow \mathcal{Q}$
Biconditional	\mathcal{P} , if and only if \mathcal{Q} . \mathcal{P} iff \mathcal{Q} . \mathcal{P} just in case \mathcal{Q} .	$\mathcal{P} \leftrightarrow \mathcal{Q}$

* **Sentence** of propositional logic is defined inductively:

Well-formed formula (WFF)

1. Every propositional variable/constant is a sentence.
2. If \mathcal{A} is a sentence, then $\neg \mathcal{A}$ is a sentence.
3. If \mathcal{A} and \mathcal{B} are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \rightarrow \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
4. Nothing else is a sentence.

* Well-formed formulae grammar:

Backus-Naur form (BNF)

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<sentence> ::= <constant>
              | <variable>
              |  $\neg$  <sentence>
              | '(' <sentence> <binop> <sentence> ')'
<constant> ::=  $\top$  |  $\perp$ 
<variable> ::=  $A$  | ... |  $Z$  |  $A_1$  | ... |  $Z_n$ 
<binop>      ::=  $\wedge$  |  $\vee$  |  $\oplus$  |  $\rightarrow$  |  $\leftarrow$  |  $\leftrightarrow$ 

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* **Literal** is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (*positive literal*), $\mathcal{L}_j = \neg X_j$ (*negative literal*).

* **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}_{\text{premises}} \therefore \underbrace{C}_{\text{conclusion}}$$

“therefore”

* An argument is **valid** if whenever all the premises are true, the conclusion is also true.

Validity

* An argument is **invalid** if there is a case (*a counterexample*) when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

- * **Valuation** is any assignment of truth values to propositional variables. Interpretation
- * \mathcal{A} is a **tautology** (valid) iff it is true on *every* valuation. Might be symbolized as “ $\models \mathcal{A}$ ”.
- * \mathcal{A} is a **contradiction** iff it is false on *every* valuation. Might be symbolized as “ $\mathcal{A} \models$ ”.
- * \mathcal{A} is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- * \mathcal{A} is **satisfiable** iff it is true on *some* valuation. Satisfiability
- * \mathcal{A} is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. Falsifiability
- * \mathcal{A} and \mathcal{B} are **equivalent** (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- * $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent (jointly satisfiable)** iff there is *some* valuation which makes them all true. Sentences are **inconsistent (jointly unsatisfiable)** iff there is *no* valuation that makes them all true. Consistency
- * The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false. Semantic entailment
- * If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \therefore C$ is **valid**.

Validity check examples:

A	B	$A \rightarrow B$	A	\therefore	B	$\neg A \rightarrow \neg B$	\therefore	$B \rightarrow A$	$A \rightarrow B$	B	\therefore	$\neg(B \rightarrow A)$
0	0	1	0	.	0	1	1	✓	1	0	.	0
0	1	1	0	.	1	1	0	.	0	1	✓	1
1	0	0	1	.	0	0	1	✓	1	0	.	0
1	1	1	1	✓	1	0	1	✓	1	1	✗	0

R	S	T	$R \vee S$	$S \vee T$	$\neg R$	\therefore	$S \wedge T$	$(R \wedge S) \rightarrow T$	\therefore	$R \rightarrow (S \rightarrow T)$
0	0	0	0	0	1	.	0	0	1	0
0	0	1	0	1	1	.	0	0	1	1
0	1	0	1	1	1	✗	0	0	1	0
0	1	1	1	1	1	✓	1	0	1	1
1	0	0	1	0	0	.	0	0	1	0
1	0	1	1	1	0	.	0	0	1	1
1	1	0	1	1	0	.	0	1	0	0
1	1	1	1	1	0	.	1	1	1	1

- * **Soundness**: $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$ “Every provable statement is in fact true”
- * **Completeness**: $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$ “Every true statement has a proof”

4.3 Natural Deduction Rules

Reiteration

m	\mathcal{A}	
\therefore	\mathcal{A}	R m

Explosion

m	\perp	
\therefore	\mathcal{A}	X m

Conditional

i	\mathcal{A}	
j	\mathcal{B}	
\therefore	$\mathcal{A} \rightarrow \mathcal{B}$	\rightarrow I $i-j$

Modus ponens

i	$\mathcal{A} \rightarrow \mathcal{B}$	
j	\mathcal{A}	
\therefore	\mathcal{B}	MP i, j

Conjunction

i	\mathcal{A}	
j	\mathcal{B}	
\therefore	$\mathcal{A} \wedge \mathcal{B}$	\wedge I i, j

m	$\mathcal{A} \wedge \mathcal{B}$	
\therefore	\mathcal{A}	\wedge E m
\therefore	\mathcal{B}	\wedge E m

Contraposition

m	$\mathcal{A} \rightarrow \mathcal{B}$	
\therefore	$\neg \mathcal{B} \rightarrow \neg \mathcal{A}$	Contra m

Modus tollens

i	$\mathcal{A} \rightarrow \mathcal{B}$	
j	$\neg \mathcal{B}$	
\therefore	$\neg \mathcal{A}$	MT i, j

Biconditional

i	\mathcal{A}	
j	\mathcal{B}	
k	\mathcal{B}	
l	\mathcal{A}	
\therefore	$\mathcal{A} \leftrightarrow \mathcal{B}$	\leftrightarrow I $i-j, k-l$

Negation

i	$\neg \mathcal{A}$	
j	\mathcal{A}	
\therefore	\perp	\neg E i, j

i	\mathcal{A}	
j	\perp	
\therefore	$\neg \mathcal{A}$	\neg I $i-j$

Disjunction

m	\mathcal{A}	
\therefore	$\mathcal{A} \vee \mathcal{B}$	\vee I m

m	\mathcal{A}	
\therefore	$\mathcal{B} \vee \mathcal{A}$	\vee I m

m	$\mathcal{A} \vee \mathcal{B}$	
i	\mathcal{A}	
j	\mathcal{C}	
k	\mathcal{B}	
l	\mathcal{C}	
\therefore	\mathcal{C}	\vee E $m, i-j, k-l$

i	$\mathcal{A} \leftrightarrow \mathcal{B}$	
j	\mathcal{A}	
\therefore	\mathcal{B}	\leftrightarrow E i, j

i	$\mathcal{A} \leftrightarrow \mathcal{B}$	
j	\mathcal{B}	
\therefore	\mathcal{A}	\leftrightarrow E i, j

Indirect proof

i	$\neg \mathcal{A}$	
j	\perp	
\therefore	\mathcal{A}	IP $i-j$

Disjunctive syllogism

i	$\mathcal{A} \vee \mathcal{B}$	
j	$\neg \mathcal{A}$	
\therefore	\mathcal{B}	DS i, j

i	$\mathcal{A} \vee \mathcal{B}$	
j	$\neg \mathcal{B}$	
\therefore	\mathcal{A}	DS i, j

De Morgan Rules

m	$\neg(\mathcal{A} \vee \mathcal{B})$	
\therefore	$\neg \mathcal{A} \wedge \neg \mathcal{B}$	DeM m

m	$\neg \mathcal{A} \wedge \neg \mathcal{B}$	
\therefore	$\neg(\mathcal{A} \vee \mathcal{B})$	DeM m

m	$\neg(\mathcal{A} \wedge \mathcal{B})$	
\therefore	$\neg \mathcal{A} \vee \neg \mathcal{B}$	DeM m

m	$\neg \mathcal{A} \vee \neg \mathcal{B}$	
\therefore	$\neg(\mathcal{A} \wedge \mathcal{B})$	DeM m

Double negation

m	$\neg \neg \mathcal{A}$	
\therefore	\mathcal{A}	$\neg \neg$ E m

Law of excluded middle

i	\mathcal{A}	
j	\mathcal{B}	
k	$\neg \mathcal{A}$	
l	\mathcal{B}	
\therefore	\mathcal{B}	LEM $i-j, k-l$

Hypothetical syllogism

i	$\mathcal{A} \rightarrow \mathcal{B}$	
j	$\mathcal{B} \rightarrow \mathcal{C}$	
\therefore	$\mathcal{A} \rightarrow \mathcal{C}$	HS i, j

Green: basic rules.

Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).