4 Formal Logic Cheatsheet

4.1 Propositional Logic ✓

* **Proposition** is a statement which can be either true or false.

- Truth-bearer
- * Alphabet^[2] of propositional logic consists of (1) atomic symbols and (2) operator symbols.
- * **Atomic formula** (atom) is an irreducible formula without logical connectives.
 - \circ Propositional **variables**: A, B, C, \ldots, Z . With indices, if needed: $A_1, A_2, \ldots, Z_1, Z_2, \ldots$
 - ∘ Logical **constants**: \top for always true proposition (*tautology*), \bot for always false proposition (*contradiction*).
- * Logical connectives (operators):

Type	Natural meaning	Symbolization
[☑] Negation	It is not the case that \mathcal{P} . It is false that \mathcal{P} . It is not true that \mathcal{P} .	$\neg \mathcal{P}$
^E Conjunction	Both \mathcal{P} and \mathcal{Q} . \mathcal{P} but \mathcal{Q} . \mathcal{P} , although \mathcal{Q} .	$\mathscr{P} \wedge Q$
[™] Disjunction	Either \mathcal{P} or Q (or both). \mathcal{P} unless Q .	$\mathscr{P} \lor Q$
Exclusive or (Xor)	Either \mathcal{P} or Q (but not both) \mathcal{P} xor Q .	$\mathcal{P}\oplus Q$
Implication (Conditional)	If \mathcal{P} , then Q . \mathcal{P} only if Q . Q if \mathcal{P} .	$\mathcal{P} \to Q$
[™] Biconditional	\mathcal{P} , if and only if \mathcal{Q} . \mathcal{P} iff \mathcal{Q} . \mathcal{P} just in case \mathcal{Q} .	$\mathcal{P} \leftrightarrow Q$

* **Sentence** of propositional logic is defined inductively:

Well-formed formula (WFF)

- 1. Every propositional variable/constant is a sentence.
- 2. If \mathcal{A} is a sentence, then $\neg \mathcal{A}$ is a sentence.
- 3. If \mathcal{A} and \mathcal{B} are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \to \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
- 4. Nothing else is a sentence.
- * Well-formed formulae grammar:

Backus-Naur form (BNF)

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 \langle sentence \rangle ::= \langle constant \rangle \\ | \langle variable \rangle \\ | \neg \langle sentence \rangle \\ | '(' \langle sentence \rangle \langle binop \rangle \langle sentence \rangle ')' \\ \langle constant \rangle ::= \top | \bot \\ \langle variable \rangle ::= A | ... | Z | A_1 | ... | Z_n \\ \langle binop \rangle ::= \land | \lor | \oplus | \rightarrow | \leftarrow | \leftrightarrow
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- * **Literal**^{\mathcal{L}} is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (positive literal), $\mathcal{L}_j = \neg X_j$ (negative literal).
- * **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}_{\textit{premises}} \quad \vdots \quad \underbrace{C}_{\textit{conclusion}}$$
"therefore"

* An argument is **valid** if whenever all the premises are true, the conclusion is also true.

Validity

* An argument is **invalid** if there is a case (a counterexample) when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

* **Valuation**[™] is any assignment of truth values to propositional variables.

- Interpretation
- * \mathcal{A} is a **tautology** (valid) iff it is true on *every* valuation. Might be symbolized as " $\neq \mathcal{A}$ ".
- * \mathcal{A} is a **contradiction** iff it is false on *every* valuation. Might be symbolized as " $\mathcal{A} \models$ ".
- * \mathcal{A} is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- * \mathcal{A} is **satisfiable** iff it is true on *some* valuation.

Satisfiability

* \mathcal{A} is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation.

Falsifiability

- * \mathcal{A} and \mathcal{B} are **equivalent** (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- * $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. Consistency
- * The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false. Semantic entailment
- * If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \stackrel{.}{.} C$ is **valid**. *Validity check examples*:

			valia	1			valid				invali	d	
A B	$A \longrightarrow B$	\boldsymbol{A}	:.	В	$\neg A$	$A \rightarrow \neg B$	•	$B \longrightarrow A$	$A \rightarrow B$	В	:.	¬ ($B \rightarrow A)$
0 0	1	0		0	1	1 1	✓	1	1	0		0	1
					1			0	1	1	✓	1	0
1 0	0	1	•	0	0	1 1	1	1	0	0		0	1
1 1	1	1	1	1	0	1 0	1	1	1	1	X	0	1

R S T	$R \vee S$	$S \vee T$	$\neg R$	nvali	${}^{\mathbf{d}}S \wedge T$	$(R \land S) \to T \stackrel{valid}{::} R \to (S \to T)$
0 0 0	0	0	1		0	0 10 1 0 1
0 0 1	0	1	1		0	0 11 1 01 1
0 1 0	1	1	1	X	0	0 10 / 01 0
0 1 1	1	1	1	✓	1	0 11 / 01 1
1 0 0	1	0	0		0	0 10 11 1
1 0 1	1	1	0		0	0 11 1 1
1 1 0	1	1	0		0	1 00 · 10 0
1 1 1	1	1	0		1	1 11 🗸 11 1

- * **Soundness** $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$ "Every provable statement is in fact true"
- * **Completeness:** $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$ "Every true statement has a proof"

4.3 Natural Deduction Rules

Reiteration

 \mathcal{A} m

 \mathcal{A} Rm

Modus ponens

 $\mathcal{A} \to \mathcal{B}$ \mathcal{A}

j

 ${\mathcal B}$ MP i, j

Modus tollens

 $\mathcal{A} \to \mathcal{B}$

 $\neg \mathcal{B}$ j

 $\neg \mathcal{A}$

MT i, j

Negation

 $\neg \mathcal{A}$

j A

1 $\neg E i, j$

i

 \perp

 $\neg I i - j$

Indirect proof

j

 ${\mathcal A}$

IP i-j

Double negation

 $\neg\neg\mathcal{A}$

A

 $\neg \neg E m$

Excluded middle

 ${\mathcal A}$ \mathcal{B} j

k $\neg \mathcal{A}$

l \mathcal{B}

LEM i-j, k-l

Explosion

 \perp

 \mathcal{A} X m

Conjunction

 \mathcal{A}

 \mathcal{B}

 $\mathcal{A} \wedge \mathcal{B}$ \wedge I i, j

 $\mathcal{A} \wedge \mathcal{B}$

 \mathcal{A}

 $\wedge E m$

 $\wedge E m$

 \mathcal{B}

Disjunction

 \mathcal{A}

 $\mathcal{A}\vee\mathcal{B}$ \vee I m

 $\mathcal{B} \vee \mathcal{A}$ $\forall I m$

 $\mathcal{A} \vee \mathcal{B}$

i $\mathcal A$

 \mathcal{C}

 \mathcal{B} k l

CC

 \forall E m, i-j, k-l

Disjunctive syllogism

 $\mathcal{A}\vee\mathcal{B}$

 $\neg \mathcal{A}$

DS i, j

 $\mathcal{A} \vee \mathcal{B}$ i

 $\neg \mathcal{B}$ j

A

DS i, j

Hypothetical syllogism

 $\mathcal{A} \to \mathcal{B}$

 $\mathcal{B} \to \mathcal{C}$

 $\mathcal{A} \to \mathcal{C}$

HS i, j

Conditional

 ${\mathcal A}$

j

 ${\mathcal B}$ $\mathcal{A} \to \mathcal{B}$

 \rightarrow I i-j

Contraposition

 $\mathcal{A} \to \mathcal{B}$

 $\neg \mathcal{B} \to \neg \mathcal{A}$

Contra m

Biconditional

A \mathcal{B} j

k ${\mathcal B}$

l $\mathcal A$

> $\mathcal{A} \leftrightarrow \mathcal{B}$ \leftrightarrow I i-j, k-l

i $\mathcal{A} \leftrightarrow \mathcal{B}$

 \mathcal{A} j

 \leftrightarrow E i, j

i $\mathcal{A} \leftrightarrow \mathcal{B}$

 \mathcal{B} j

 ${\mathcal A}$

 \leftrightarrow E i, j

De Morgan Rules

 $\neg (\mathcal{A} \lor \mathcal{B})$

 $\neg \mathcal{A} \wedge \neg \mathcal{B}$

 $\neg \mathcal{A} \wedge \neg \mathcal{B}$ m

 $\neg (\mathcal{A} \lor \mathcal{B})$

 $\mathrm{DeM}\;m$

DeM m

 $\neg (\mathcal{A} \land \mathcal{B})$

 $\neg \mathcal{A} \vee \neg \mathcal{B}$

 ${
m DeM}\ m$

 $\neg \mathcal{A} \lor \neg \mathcal{B}$

 $\neg (\mathcal{A} \land \mathcal{B})$

DeM m

Green: basic rules.

Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).