1. The graph[1] of Europe $\mathcal{G}^* = \langle V, E\rangle$ is defined as follows: each vertex $v \in V$ is a Europe country[2]; two vertices are adjacent ($\{u, v\} \in E$) if the corresponding countries share a land border. Let $\mathcal{G}$ be the largest connected component of $\mathcal{G}^*$.
   (a) Draw[3] $\mathcal{G}^*$ with the minimum number of intersecting edges[4].
   (b) Find $|V|$, $|E|$, $\delta(\mathcal{G})$, $\Delta(\mathcal{G})$, $\mathrm{rad}(\mathcal{G})$, $\mathrm{diam}(\mathcal{G})$, $\mathrm{girth}(\mathcal{G})$, $\mathrm{center}(\mathcal{G})$, $\varkappa(\mathcal{G})$, $\lambda(\mathcal{G})$.
   (c) Find the minimum vertex coloring $Z : V \longrightarrow \mathbb{N}$ of $\mathcal{G}$.
   (d) Find the minimum edge coloring $X : E \longrightarrow \mathbb{N}$ of $\mathcal{G}$.
   (e) Find the maximum clique $Q \subseteq V$ of $\mathcal{G}$.
   (f) Find the maximum stable set $S \subseteq V$ of $\mathcal{G}$.
   (g) Find the maximum matching $M \subseteq E$ of $\mathcal{G}$.
   (h) Find the minimum vertex cover $R \subseteq V$ of $\mathcal{G}$.
   (i) Find the minimum edge cover $F \subseteq E$ of $\mathcal{G}$.
   (j) Find the shortest closed walk W that visits every vertex of $\mathcal{G}$.
   (k) Find the shortest closed walk U that visits every edge of $\mathcal{G}$.
   (l) Find all biconnected components (blocks) and draw the block-cut tree of $\mathcal{G}^*$.
   (m) Find all 2-edge-connected components of $\mathcal{G}^*$.
   (n) Add the weight function $w : E \longrightarrow \mathbb{R}$ denoting the distance[5] between capitals. Find the minimum (*w.r.t.* the total weight of edges) spanning tree T for the largest connected component of the weighted Europe graph $\mathcal{G}^*_w = (V, E, w)$.
   (o) Find $\mathrm{centroid}(T)$ (*w.r.t.* the edge weight function w).
   (p) Construct the Prüfer code for T.

2. Prove *rigorously* the following theorems:

   Theorem 1 (Triangle Inequality)**.** For any connected graph $G = \langle V, E\rangle$:
   $$\forall x, y, z \in V : \mathrm{dist}(x, y) + \mathrm{dist}(y, z) \geq \mathrm{dist}(x, z)$$

   Theorem 2 (Tree)**.** A connected graph $G = \langle V, E\rangle$ is a tree (*i.e.* acyclic graph) *iff* $|E| = |V| - 1$.

   Theorem 3 (Hall)**.** For any bipartite graph $G = (X, Y, E)$, there exists X-perfect matching (set of disjoint edges covering all vertices in X) *iff* $|N(W)| \geq |W|$ for every $W \subseteq X$.

   Theorem 4 (Whitney)**.** For any graph G: $\varkappa(G) \leq \lambda(G) \leq \delta(G)$.

   Theorem 5 (Chartrand)**.** For a connected graph $G = \langle V, E\rangle$: if $\delta(G) \geq \lfloor |V|/2 \rfloor$, then $\lambda(G) = \delta(G)$.

   Theorem 6 (Menger)**.** For any pair of non-adjacent vertices u and v in an undirected graph, the size of the minimum *vertex cut* is equal to the maximum number of pairwise *internally vertex-disjoint paths* from u to v.

   Theorem 7 (Harary)**.** Every block of a block graph[6] is a clique.

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[1] Hereinafter, "graphs" are "simple, finite, undirected, and unweighted", unless stated otherwise.

[2] Since the absolute geopolitical correctness is not necessary to accomplish this task, you can simply use `https://simple.wikipedia.org/wiki/List_of_European_countries` or any other similar source as a reference.

[3] You may either preserve the original spatial relationships between vertices, or use an automatic layout software. Take a look at Graphviz, Gephi, Cytoscape.

[4] Since $\mathcal{G}^*$ naturally corresponds to a "map", it *has* to be planar. However, due to *some broken countries*, it might be impossible to draw $\mathcal{G}^*$ completely without edge intersections, so you just have to minimize them.

[5] You may choose geodesic or road distance, to your preference. You *have* to use *more-or-less real* data.

[6] A block graph $H = \mathrm{B}(G)$ is an intersection graph of all blocks (biconnected components) of G, *i.e.* each vertex $v \in V(H)$ corresponds to a block of G, and there is an edge $\{v, u\} \in E(H)$ iff "blocks" v and u share a cut vertex.