

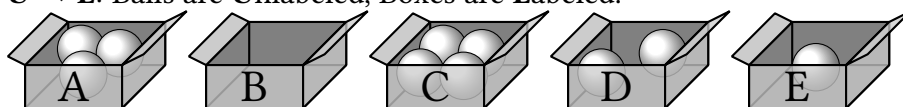
Do whatever you want, but always explain what you are doing.

— KONSTANTIN, 2020

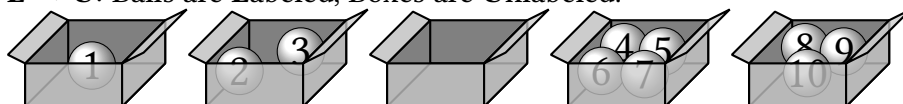
- Find the number of different 5-digit numbers using digits 1–9 under the given constraints. For each case, provide representative examples of (non-)complying numbers (e.g., 12345 and 52814 are suitable for (b), but 44521 and 935 are not) and derive a generic¹ formula. Try to express the formula using standard combinatorial terms, e.g., k -combs C_n^k and k -perms $P(n, k)$.
 - Digits *can* be repeated.
 - Digits *cannot* be repeated.
 - Digits *can* be repeated and must be written in *non-increasing*² order.
 - Digits *cannot* be repeated and must be written in *strictly increasing* order.
 - Digits *can* be repeated, must be written in *non-decreasing* order, and the 4th digit must be 6.
- One of the classical combinatorial problems is counting the number of arrangements of n balls into k boxes. There are at least 12 variations of this problem: four cases (a–d) with three different constraints (1–3). For each problem (case+constraint), derive the corresponding generic formula. Additionally, pick several (representative) values for n and k and use your derived formulae to find the numbers of arrangements. Visualize several possible arrangements for the chosen n and k .

Cases with arrangement examples:

- a. $U \rightarrow L$: Balls are Unlabeled, Boxes are Labeled.



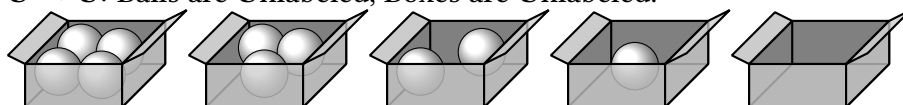
- b. $L \rightarrow U$: Balls are Labeled, Boxes are Unlabeled.



- c. $L \rightarrow L$: Balls are Labeled, Boxes are Labeled.



- d. $U \rightarrow U$: Balls are Unlabeled, Boxes are Unlabeled.



Constraints:

- ≤ 1 ball per box — *injective* mapping.
- ≥ 1 ball per box — *surjective* mapping.
- Arbitrary number of balls per box.

Notes:

- * Unlabeled means “indistinguishable”, and Labeled means “distinguishable”.
- * **Stirling number of the second kind** $s_k^{II}(n)$ — number of ways to partition a set of n elements into k non-empty subsets. Use $s_k^{II}(n)$ directly without expanding the closed formula.
- * **Partition function** $p_k(n)$ — number of ways to partition the integer n into k positive parts, i.e. $n = a_1 + \dots + a_k$, where $a_1 \geq \dots \geq a_k \geq 1$. Use $p_k(n)$ directly.

¹ Here, “generic formula” means “depending on the input data”. In this particular example, $n = 9$ and $k = 5$, but the sought formula must also be valid for all other (adequate) values of n and k .

² A sequence (x_n) is said to be *strictly monotonically increasing* if each term is *strictly greater* than the previous one, i.e. $x_i < x_{i+1}$. A sequence (x_n) is called *non-increasing* if each term is *less than or equal* to the previous one, i.e. $x_i \geq x_{i+1}$.

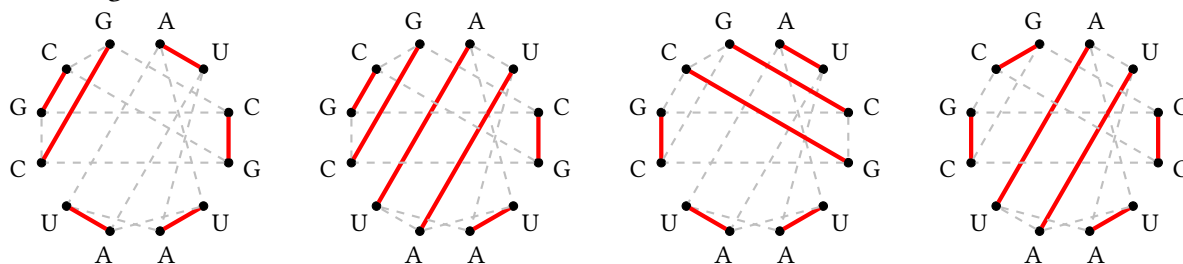
3. Prove the Generalized Pascal's Formula (for $n \geq 1$ and $k_1, \dots, k_r \geq 0$ with $k_1 + \dots + k_r = n$):

$$\binom{n}{k_1, \dots, k_r} = \sum_{i=1}^r \binom{n-1}{k_1, \dots, k_i-1, \dots, k_r}$$

Count the number of permutations of a multiset $\Sigma^* = \{2 \cdot \triangle, 3 \cdot \square, 1 \cdot \blacklozenge\}$ using GPF.

4. A *non-crossing perfect matching*³ in a graph is a set of pairwise disjoint edges that cover all vertices and do not intersect with each other. For example, consider a graph on $2n$ vertices numbered from 1 to $2n$ and arranged in a circle. Additionally, assume that edges are straight lines. In this case, edges $\{i, j\}$ and $\{a, b\}$ intersect whenever $i < a < j < b$.

- (a) Count the number of all possible non-crossing perfect matchings in a complete graph K_{2n} .
- (b) Consider a graph on vertices labeled with letters from $\{A, C, G, U\}$. Each pair of vertices labeled with A and U is connected with a *basepair edge*. Similarly, C-G pairs are also connected. The picture below illustrates some of possible non-crossing perfect matchings in the graph with 12 vertices AUCGUAUUCGCG arranged in a circle. Basepair edges are drawn dashed gray, matching is red.



Count the number of all possible non-crossing perfect matchings in the graph on 20 vertices arranged in a circle and labeled with CGUAAUACGGCAUUAAGCAU.

³ Credits to Rosalind for this task.