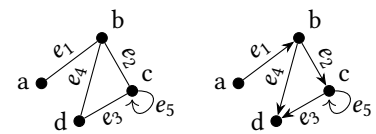
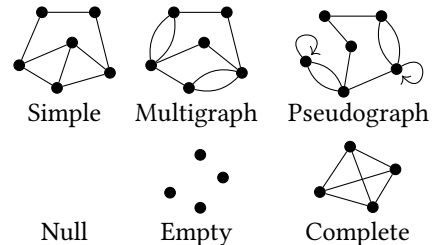


5 Graph Theory Cheatsheet

Glossary

- * **Graph** is an ordered pair $G = \langle V, E \rangle$, where $V = \{v_1, \dots, v_n\}$ is a set of vertices, and $E = \{e_1, \dots, e_m\}$ is a set of edges.
 - Given a graph G , the notation $V(G)$ denotes the vertices of G .
 - Given a graph G , the notation $E(G)$ denotes the edges of G .
 - In fact, $V(\cdot)$ and $E(\cdot)$ functions allow to access “vertices” and “edges” of any object possessing them (e.g., paths).
- * **Order** of a graph G is the number of vertices in it: $|V(G)|$.
- * **Size** of a graph G is the number of edges in it: $|E(G)|$.
- * Two graphs are **equal** if their vertex sets and edge sets are equal: $G_1 = G_2$ iff $V_1 = V_2$ and $E_1 = E_2$.
- * Two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$ are called **isomorphic**, denoted $G_1 \simeq G_2$, if there exists an *edge-preserving* bijection $f: V_1 \rightarrow V_2$, i.e. any two vertices $u, v \in V_1$ are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2 . This means that the graphs are structurally identical *up to vertex renaming*.
- * Simple **undirected** graphs have $E \subseteq V^{(2)}$, i.e. each edge $e_i \in E$ between vertices u and v is denoted by $\{u, v\} \in V^{(2)}$. Such *undirected edges* are also called *links* or *lines*.
 - $A^{(k)} = \{\{x_1, \dots, x_k\} \mid x_1 \neq \dots \neq x_k \in A\} = \{S \mid S \subseteq A, |S| = k\}$ is the set of k -sized subsets of A .
- * Simple **directed** graphs have $E \subseteq V^2$, i.e. each edge $e_i \in E$ from vertex u to v is denoted by an ordered pair $\langle u, v \rangle \in V^2$. Such *directed edges* are also called *arcs* or *arrows*.
 - $A^k = A \times \dots \times A = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in A\}$ is the set of k -tuples (Cartesian k -power of A).
- * **Multi-edges** are edges that have the same end nodes.
- * **Loop** is an edge that connects a vertex to itself.
- * **Simple graph** is a graph without multi-edges and loops.
- * **Multigraph** is a graph with multi-edges.
- * **Pseudograph** is a multigraph with loops.
- * **Null graph** is a “graph” without vertices.
- * **Trivial (singleton) graph** is a graph consisting of a single vertex.
- * **Empty (edgeless) graph** is a graph without edges.
- * **Complete graph** K_n is a simple graph in which every pair of distinct vertices is connected by an edge.
- * **Weighted graph** $G = (V, E, w)$ is a graph in which each edge has an associated numerical value (the *weight*) represented by the **weight function** $w: E \rightarrow \text{Num}$.
- * **Subgraph** of a graph $G = \langle V, E \rangle$ is another graph $G' = \langle V', E' \rangle$ such that $V' \subseteq V, E' \subseteq E$. Designated as $G' \subseteq G$.
- * **Spanning (partial) subgraph** is a subgraph that includes all vertices of a graph.
- * **Induces subgraph** of a graph $G = \langle V, E \rangle$ is another graph G' formed from a subset S of the vertices of the graph and *all* the edges (from the original graph) connecting pairs of vertices in that subset. Formally, $G' = G[S] = \langle V', E' \rangle$, where $S \subseteq V, V' = V \cap S, E' = \{e \in E \mid \exists v \in S: e \text{ I } v\}$.
- * **Adjacency** is the relation between two vertices connected with an edge.
- * **Adjacency matrix** is a square matrix $A_{V \times V}$ of an adjacency relation.
 - For simple graphs, adjacency matrix is binary, i.e. $A_{ij} \in \{0, 1\}$.
 - For directed graphs, $A_{ij} \in \{0, 1, -1\}$.
 - For multigraphs, adjacency matrix contains edge multiplicities, i.e. $A_{ij} \in \mathbb{N}_0$.
- * **Incidence** is a relation between an edge and its endpoints.
- * **Incidence matrix** is a Boolean matrix $B_{V \times E}$ of an incidence relation.
- * **Degree** $\deg(v)$ the number of edges incident to v (loops are counted twice).
 - $\delta(G) = \min_{v \in V} \deg(v)$ is the **minimum degree**.
 - $\Delta(G) = \max_{v \in V} \deg(v)$ is the **maximum degree**.
 - HANDSHAKING LEMMA. $\sum_{v \in V} \deg(v) = 2|E|$.
- * A graph is called **r -regular** if all its vertices have the same degree: $\forall v \in V: \deg(v) = r$.
- * **Complement graph** of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent iff they are non-adjacent in G .
- * **Intersection graph** of a family of sets $F = \{S_i\}$ is a graph $G = \Omega(F) = \langle V, E \rangle$ such that each vertex $v_i \in V$ denotes the set S_i , i.e. $V = F$, and the two vertices v_i and v_j are adjacent whenever the corresponding sets S_i and S_j have a non-empty intersection, i.e. $E = \{\langle v_i, v_j \rangle \mid i \neq j, S_i \cap S_j \neq \emptyset\}$.



Adjacency matrix:

$$a \begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad b \begin{bmatrix} a & b & c & d \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Incidence matrix:

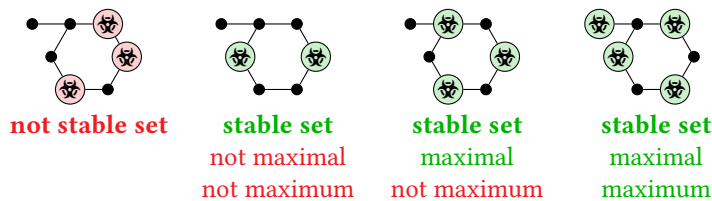
$$a \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad b \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- * **Line graph** $L(G)$ of a graph $G = \langle V, E \rangle$ is another graph $L(G) = \Omega(E)$ that represents the adjacencies between edges of G . Each vertex of $L(G)$ represents an edge of G , and two vertices of $L(G)$ are adjacent iff the corresponding edges share a common endpoint in G (i.e. edges are “adjacent”/“incident”).
- * **Walk** is an alternating sequence of vertices and edges: $l = v_1 e_1 v_2 \dots e_{n-1} v_n$.
 - o **Trail** is a walk with distinct edges.
 - o **Path** is a walk with distinct vertices (and therefore distinct edges).
 - o A walk is **closed** if it starts and ends at the same vertex. Otherwise, it is **open**.
 - o **Circuit** is a closed trail.
 - o **Cycle** is a closed path.
- * **Length** of a path (walk, trail) $l = u \rightsquigarrow v$ is the number of edges in it: $|l| = |E(l)|$.
- * **Girth** is the length of the shortest cycle in the graph.
- * **Distance** $\text{dist}(u, v)$ between two vertices is the length of the shortest path $u \rightsquigarrow v$.
 - o $\varepsilon(v) = \max_{u \in V} \text{dist}(v, u)$ is the **eccentricity** of the vertex v .
 - o $\text{rad}(G) = \min_{v \in V} \varepsilon(v)$ is the **radius** of the graph G .
 - o $\text{diam}(G) = \max_{v \in V} \varepsilon(v)$ is the **diameter** of the graph G .
 - o $\text{center}(G) = \{v \mid \varepsilon(v) = \text{rad}(G)\}$ is the **center** of the graph G .
- * **Clique** $Q \subseteq V$ is a set of vertices inducing a complete subgraph.
- * **Stable set** $S \subseteq V$ is a set of independent (pairwise non-adjacent) vertices.

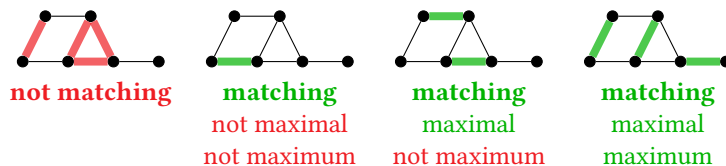
| Term | V ¹ | E ² | “Closed” term |
|-------|----------------|----------------|---------------|
| Walk | + | + | Closed walk |
| Trail | + | - | Circuit |
| Path | - | - | Cycle |
| | - | + | (impossible) |

¹Can vertices be repeated?

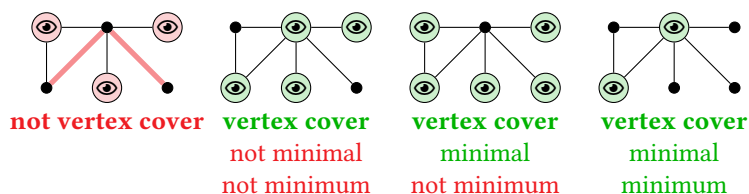
²Can edges be repeated?



- * **Matching** $M \subseteq E$ is a set of independent (pairwise non-adjacent) edges.



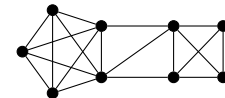
- * **Perfect matching** is a matching that covers all vertices in the graph.
 - o A perfect matching (if it exists) is always a minimum edge cover (*but not vice-versa!*).
- * **Vertex cover** $R \subseteq V$ is a set of vertices “covering” all edges.



- * **Edge cover** $F \subseteq E$ is a set of edges “covering” all vertices.

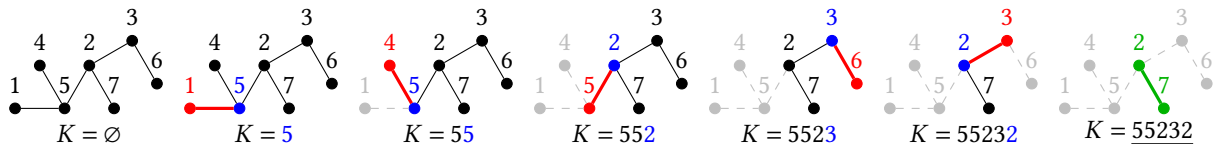


- * **Cut vertex (articulation point)** is a vertex whose removal increases the number of connected components.
- * **Bridge** is an edge whose removal increases the number of connected components.
- * **Biconnected graph** is a connected “nonseparable” graph, which means that the removal of any vertex does not make the graph disconnected. Alternatively, this is a graph without *cut vertices*.
- * **Biconnectivity** can be defined as a relation on edges $R \subseteq E^2$:
 - Two edges are called *biconnected* if there exist two *vertex-disjoint* paths between the ends of these edges.
 - Trivially, this relation is an equivalence relation.
 - Equivalence classes of this relation are called **biconnected components**, also known as **blocks**.
- * **Edge biconnectivity** can be defined as a relation on vertices $R \subseteq V^2$:
 - Two vertices are called *edge-biconnected* if there exist two *edge-disjoint* paths between them.
 - Trivially, this relation is an equivalence relation.
 - Equivalence classes of this relation are called **edge-biconnected components** (or *2-edge-connected components*).
- * **Vertex connectivity** $\kappa(G)$ is the minimum number of vertices that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k -vertex-connected.
- * **k -vertex-connected graph** is a graph that remains connected after less than k vertices are removed, i.e. $\kappa(G) \geq k$.
 - Corollary of Menger’s theorem: graph $G = \langle V, E \rangle$ is k -vertex-connected if, for every pair of vertices $u, v \in V$, it is possible to find k *vertex-independent* (*internally vertex-disjoint*) paths between u and v .
 - k -vertex-connected graphs are also called simply *k -connected*.
 - 1-connected graphs are called *connected*, 2-connected are *biconnected*, 3-connected are *triconnected*, etc.
 - Note the “exceptions”:
 - Singleton graph K_1 has $\kappa(K_1) = 0$, so it is **not** *1-connected*, but still considered *connected*.
 - Graph K_2 has $\kappa(K_2) = 1$, so it is **not** *2-connected*, but considered *biconnected*, so it can be a block.
- * **Edge connectivity** $\lambda(G)$ is the minimum number of edges that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k -edge-connected.
- * **k -edge-connected graph** is a graph that remains connected after less than k edges are removed, i.e. $\lambda(G) \geq k$.
 - Corollary of Menger’s theorem: graph $G = \langle V, E \rangle$ is k -edge-connected if, for every pair of vertices $u, v \in V$, it is possible to find k *edge-disjoint* paths between u and v .
 - 2-edge-connected are called *edge-biconnected*, 3-edge-connected are *edge-triconnected*, etc.
 - Note the “exception”:
 - Singleton graph K_1 has $\lambda(K_1) = 0$, so it is **not** *2-edge-connected*, but considered *edge-biconnected*, so it can be a *2-edge-connected component*.
- * WHITNEY’S THEOREM. For any graph G , $\kappa(G) \leq \lambda(G) \leq \delta(G)$.



$$\kappa(G) = 2, \lambda(G) = 3, \delta(G) = 3, \Delta(G) = 6$$

- * **Tree** is a connected undirected acyclic graph.
- * **Forest** is an undirected acyclic graph, *i.e.* a disjoint union of trees.
- * An **unrooted tree (free tree)** is a tree without any designated *root*.
- * A **rooted tree** is a tree in which one vertex has been designated the *root*.
 - In a rooted tree, the **parent** of a vertex v is the vertex connected to v on the path to the root.
 - A **child** of a vertex v is a vertex of which v is the parent.
 - A **sibling** to a vertex v is any other vertex on the tree which has the same parent as v .
 - A **leaf** is a vertex with no children. Equivalently, **leaf** is a *pendant vertex*, *i.e.* $\deg(v) = 1$.
 - An **internal vertex** is a vertex that is not a leaf.
 - A **k -ary tree** is a rooted tree in which each vertex has at most k children. *2-ary trees* are called **binary trees**.
- * A **labeled tree** is a tree in which each vertex is given a unique *label*, *e.g.*, $1, 2, \dots, n$.
- * **CAYLEY'S FORMULA**. Number of labeled trees on n vertices is n^{n-2} .
- * **Prüfer code** is a unique sequence of labels $\{1, \dots, n\}$ of length $(n-2)$ associated with the labeled tree on n vertices.
 - **ENCODING** (iterative algorithm for converting tree T labeled with $\{1, \dots, n\}$ into a Prüfer sequence K):
 - On each iteration, remove the leaf with *the smallest label*, and extend K with *a single neighbour* of this leaf.
 - After $(n-2)$ iterations, the tree will be left with *two adjacent* vertices—there is no need to encode them, because there is only one unique tree on 2 vertices, which requires 0 bits of information to encode.



- **DECODING** (iterative algorithm for converting a Prüfer sequence K into a tree T):
 - Given a Prüfer code K of length $(n-2)$, construct a set of “leaves” $W = \{1, \dots, n\} \setminus K$.
 - On each iteration:
 - (1) Pop the *first* element of K (denote it as k) and the *minimum* label in W (denote it as w).
 - (2) Connect k and w with an edge $\langle k, w \rangle$ in the tree T .
 - (3) If $k \notin K$, then extend the set of “leaves” $W := W \cup \{k\}$.
 - After $(n-2)$ iterations, the sequence K will be empty, and the set W will contain exactly two vertices—connect them with an edge.