Discrete Mathematics

(Not only) Regular Languages — Spring 2025

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§1 Regular Languages

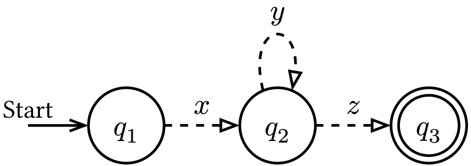
Regular Expressions

Regular languages can be composed from “smaller” regular languages.

- Atomic regular expressions:
  - $\varnothing$, an empty language
  - ε, a singleton language consisting of a single ε word
  - `a`, a singleton language consisting of a single 1-letter word a, for each $a \in \Sigma$
- Compound regular expressions:
  - $R_1 R_2$, the concatenation of R_1 and R_2
  - $R_1 \mid R_2$, the union of R_1 and R_2
  - $R^* = RRR...$, the Kleene star of R
  - (R), just a bracketed expression
  - Operator precedence: `ab*c|d` $\triangleq ((\texttt{a}\ (\texttt{b}^*))\ \texttt{c}) \mid \texttt{d}$

Re-visiting States

- Let D be a DFA with n states.
- Any string w accepted by D that has length at least n must visit some state twice.
- Number of states visited is equal to $|w| + 1$.
- By the pigeonhole principle, some state is "duplicated", i.e. visited more than once.
- The substring of w between those *revisited states* can be removed, duplicated, tripled, etc. without changing the fact that D accepts w.



Informally:

- Let L be a regular language.
- If we have a string $w \in L$ that is "sufficiently long", then we can *split* the string into *three pieces* and *"pump"* the middle.
- We can write $w = xyz$ such that xy^0z, xy^1z, xy^2z, ..., xy^nz, ... are all in L.
  - Notation: y^n means "n copies of y".

Weak Pumping Lemma

Theorem 1 (Weak Pumping Lemma for Regular Languages):

- For any regular language L,
  - There exists a positive natural number n (also called *pumping length*) such that
    - For any $w \in L$ with $|w| \geq n$,
      - There exists strings x, y, z such that
        - For any natural number i,
          - $w = xyz$ (w can be broken into three pieces)
          - $y \neq \varepsilon$ (the middle part is not empty)
          - $xy^i z \in L$ (the middle part can repeated any number of times)

Example: Let $\Sigma = \{0, 1\}$ and $L = \{w \in \Sigma^* \mid w \text{ contains } 00 \text{ as a substring}\}$. Any string of length 3 or greater can be split into three parts, the second of which can be "pumped".

Example: Let $\Sigma = \{0, 1\}$ and $L = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$. The weak pumping lemma still holds for finite languages, because the pumping length n can be longer than the longest word in the language!

Testing Equality

Definition 1: The *equality problem* is, given two strings x and y, to decide whether $x = y$.

Example: Let $\Sigma = \{0, 1, \#\}$. We can *encode* the equality problem as a string of the form $x\#y$.

- "Is *001* equal to *110* ?" would be $001\#110$.
- "Is *11* equal to *11* ?" would be $11\#11$.
- "Is *110* equal to *110* ?" would be $110\#110$.

Let EQUAL $= \{w\#w \mid w \in \{0, 1\}^*\}$.

Question: Is EQUAL a *regular* language?

A typical word in EQUAL looks like this: $001\#001$.

- If the "middle" piece is just a symbol $\#$, then observe that $001\,001 \notin$ EQUAL.
- If the "middle" piece is either completely to the left or completely to the right of $\#$, then observe that any duplication or removal of this piece is not in EQUAL.
- If the "middle" piece includes $\#$ and any symbols from the left/right of it, then, again, observe that any duplication or removal of this piece is not in EQUAL.

Testing Equality [2]

Theorem 2: EQUAL is not a regular language.

Proof: By contradiction. Assume that EQUAL is a regular language.

Let n be the pumping length guaranteed by the weak pumping lemma. Let $w = 0^n\#0^n$, which is in EQUAL and $|w| = 2n + 1 \geq n$. By the weak pumping lemma, we can write $w = xyz$ such that $y \neq \varepsilon$ and for any $i \in \mathbb{N}$, $xy^i\#z \in$ EQUAL. Then y cannot contain $\#$, since otherwise if we let $i = 0$, then $xy^0\#z = x\#z$ does not contain $\#$ and would not be in EQUAL. So y is either completely to the left of $\#$ or completely to the right of $\#$.

Let $|y| = k$, so $k > 0$. Since y is completely to the left or right of $\#$, then $y = 0^k$.

Now, we consider two cases:
Case 1: y is to the left of $\#$. Then $xy^2z = 0^{n+k}\#0^n \notin$ EQUAL, contradicting the weak pumping lemma.
Case 2: y is to the right of $\#$. Then $xy^2z = 0^n\#0^{n+k} \notin$ EQUAL, contradicting the weak pumping lemma.

In either case we reach a contradiction, so our assumption was wrong. Thus, EQUAL *is not regular*. □

§2 Non-regular Languages

(Not only) Regular Languages

- The weak pumping lemma describes a property common to *all* regular languages.
- Any language L which does not have this property *cannot be regular*.
- What other languages can we find that are not regular?

Example: Consider the language $L = \{0^n 1^n \mid n \in \mathbb{N}\}$.

- $L = \{\varepsilon, 01, 0011, 000111, 00001111, \ldots\}$
- L is a classic example of a non-regular language.
- **Intuitively:** if you have *only finitely many states* in a DFA, you cannot *"remember"* an arbitrary number of 0s to match *the same* number of 1s.

How would we prove that L is non-regular?

Use the Pumping Lemma to show that L *cannot* be regular.

Pumping Lemma as a Game

The weak pumping lemma can be thought of as a *game* between **you** and an **adversary**.

- **You win** if you can prove that the pumping lemma *fails*.
- **The adversary wins** if the adversary can make a choice for which the pumping lemma *succeeds*.

The game goes as follows:

- The adversary chooses a pumping length n.
- You choose a string w with $|w| \geq n$ and $w \in L$.
- The adversary breaks it into x, y, and z.
- You choose an i such that $xy^iz \notin L$ *(if you can't, you lose!)*.

Pumping Lemma as a Game [2]

$$L = \{0^n 1^n \mid n \in \mathbb{N}\}$$

Adversary	You
Maliciously choose pumping length n	
	Cleverly choose a string $w \in L$, $\lvert w \rvert \geq n$
Maliciously split $w = xyz$, $y \neq \varepsilon$	
	Cleverly choose an i such that $xy^i z \notin L$
Lose	Win

$\{0^n 1^n\}$ is not regular

Formal Proof of Non-regularity

Theorem 3: $L = \{0^n 1^n \mid n \in \mathbb{N}\}$ is not regular.

Proof: By contradiction. Assume that L is regular.

Let n be the pumping length guaranteed by the weak pumping lemma ("there exists n..."). Consider the string $w = 0^n 1^n$. Then $|w| = 2n \geq n$ and $w \in L$, so we can write (split) $w = xyz$ such that $y \neq \varepsilon$ and for any $i \in \mathbb{N}$, we have $xy^i z \in L$.

We consider three cases:
Case 1: y consists solely of 0s. Then $xy^0 z = xz = 0^{n-|y|}1^n$, and since $|y| > 0$, $xz \notin L$.
Case 2: y consists solely of 1s. Then $xy^0 z = xz = 0^n 1^{n-|y|}$, and since $|y| > 0$, $xz \notin L$.
Case 3: y consists of $k > 0$ 0s followed by $m > 0$ 1s. Then $xy^2 z = 0^n 1^m 0^k 1^n$, so $xy^2 z \notin L$.

In all three cases we reach a contradiction, so our assumption was wrong and L is not regular. $\square$

§3 Pumping Lemma

Pumping

Consider the language L over $\Sigma = \{0, 1\}$ of strings $w \in \Sigma^*$ that contain *an equal number* of 0s and 1s.

For example:

- `01` in L
- `11011` not in L
- `110010` in L

Question: Is L a *regular* language?

Let's *use* the weak pumping lemma to show it is by *pumping all the strings* in this language.

Proof *(incorrect)*: We are going to show that L satisfies the conditions of the weak pumping lemma. Let $n = 2$. Consider any string $w \in L$ (i.e., w contains the same number of 0s and 1s) with $|w| \geq 2$.

We can split $w = xyz$ such that $x = z = \varepsilon$ and $y = w$, so $y \neq \varepsilon$. Then, for any natural number $i \in \mathbb{N}$, $xy^i z = w^i$, which has the same number of 0s and 1s.

Since L passes the conditions of the weak pumping lemma, L is regular. $\square$

A word of Caution

- The weak and full pumping lemmas describe the *necessary* condition of regular languages.
  - If L is regular, then it passes the conditions of the weak pumping lemma.
- The weak and full pumping lemmas are *not* a *sufficient* condition of regular languages.
  - If L is not regular, then it does *not* pass the conditions of the weak pumping lemma.
- If a language *fails* the pumping lemma, it is *definitely not regular*.
- If a language *passes* the pumping lemma, we *learn nothing* about whether it is regular or not.

The Stronger Pumping Lemma

The language L *can* be proven to be *non-regular* using a *stronger version* of the pumping lemma.

For the intuition behind the "full" pumping lemma, let's revisit our original observation.

- Let D be a DFA with n states.
- Any string w accepted by D of length at least n must visit some state twice *within its first n* symbols.
  - The number of visited states is equal to $n + 1$.
  - By the pigeonhole principle, some state is *duplicated*.
- The substring of w between those *revisited states* can be removed, duplicated, tripled, etc. without changing the fact that D accepts w.

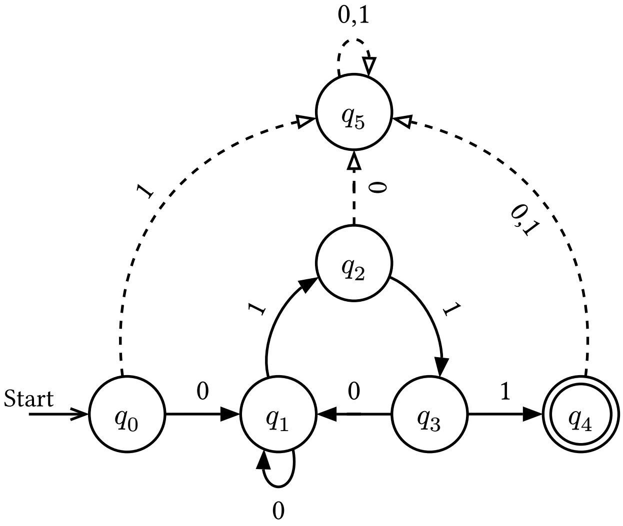
Overall, we can add the following condition to the weak pumping lemma:

$$|xy| \leq n$$

This restriction means that we can limit where the string to pump must be. If we specifically choose the first n characters of the string to pump, we can ensure y (middle part) to have a specific property.

We can then show that y cannot be pumped arbitrarily many times.

The Stronger Pumping Lemma [2]



$$q_0 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{1} q_4$$

Formal Proof of Non-regularity

Theorem 4: $L = \{w \in \{0,1\}^* \mid w \text{ has an equal number of 0s and 1s}\}$ is *not regular*.

Proof: By contradiction. Assume that L is regular.

Let n be the pumping length guaranteed by the weak pumping lemma. Consider the string $w = 0^n 1^n$. Then $|w| = 2n \geq n$ and $w \in L$. Therefore, there exist strings x, y, and z such that $w = xyz$, $|xy| \leq n$, $y \neq \varepsilon$, and for any $i \in \mathbb{N}$, we have $xy^i z \in L$.

Since $|xy| \leq n$, y must consist solely of 0s. But then $xy^2 z = 0^{n+|y|}1^n$, and since $|y| > 0$, $xy^2 z \notin L$.

We have reached a contradiction, so our assumption was wrong and L is not regular. $\square$

Summary of the Pumping Lemma

1. Using the *pigeonhole principle*, we can prove the weak and full *pumping lemma*.
2. These lemmas describe essential properties of the *regular* languages.
3. Any language that *fails* to have these properties *can not be regular*.