

1. The graph<sup>1</sup> of Europe  $\mathcal{G}^* = \langle V, E \rangle$  is defined as follows: each vertex  $v \in V$  is a Europe country<sup>2</sup>; two vertices are adjacent ( $\{u, v\} \in E$ ) if the corresponding countries share a land border. Let  $\mathcal{G}$  be the largest connected component of  $\mathcal{G}^*$ .
  - (a) Draw<sup>3</sup>  $\mathcal{G}^*$  with the minimum number of intersecting edges<sup>4</sup>.
  - (b) Find  $|V|$ ,  $|E|$ ,  $\delta(\mathcal{G})$ ,  $\Delta(\mathcal{G})$ ,  $\text{rad}(\mathcal{G})$ ,  $\text{diam}(\mathcal{G})$ ,  $\text{girth}(\mathcal{G})$ ,  $\text{center}(\mathcal{G})$ ,  $\kappa(\mathcal{G})$ ,  $\lambda(\mathcal{G})$ .
  - (c) Find the minimum vertex coloring  $Z : V \rightarrow \mathbb{N}$  of  $\mathcal{G}$ .
  - (d) Find the minimum edge coloring  $X : E \rightarrow \mathbb{N}$  of  $\mathcal{G}$ .
  - (e) Find the maximum clique  $Q \subseteq V$  of  $\mathcal{G}$ .
  - (f) Find the maximum stable set  $S \subseteq V$  of  $\mathcal{G}$ .
  - (g) Find the maximum matching  $M \subseteq E$  of  $\mathcal{G}$ .
  - (h) Find the minimum vertex cover  $R \subseteq V$  of  $\mathcal{G}$ .
  - (i) Find the minimum edge cover  $F \subseteq E$  of  $\mathcal{G}$ .
  - (j) Find the shortest closed walk  $W$  that visits every vertex of  $\mathcal{G}$ .
  - (k) Find the shortest closed walk  $U$  that visits every edge of  $\mathcal{G}$ .
  - (l) Find all biconnected components (blocks) and draw the block-cut tree of  $\mathcal{G}^*$ .
  - (m) Find all 2-edge-connected components of  $\mathcal{G}^*$ .
  - (n) Add the weight function  $w : E \rightarrow \mathbb{R}$  denoting the distance<sup>5</sup> between capitals. Find the minimum (w.r.t. the total weight of edges) spanning tree  $T$  for the largest connected component of the weighted Europe graph  $\mathcal{G}_w^* = (V, E, w)$ .
  - (o) Find  $\text{centroid}(T)$  (w.r.t. the edge weight function  $w$ ).
  - (p) Construct the Prüfer code for  $T$ .
2. Prove *rigorously* the following theorems:

**Theorem 1** (TRIANGLE INEQUALITY). For any connected graph  $G = \langle V, E \rangle$ :

$$\forall x, y, z \in V : \text{dist}(x, y) + \text{dist}(y, z) \geq \text{dist}(x, z)$$

**Theorem 2** (TREE). A connected graph  $G = \langle V, E \rangle$  is a tree (i.e. acyclic graph) iff  $|E| = |V| - 1$ .

**Theorem 3** (HALL). For any bipartite graph  $G = (X, Y, E)$ , there exists  $X$ -perfect matching (set of disjoint edges covering all vertices in  $X$ ) iff  $|N(W)| \geq |W|$  for every  $W \subseteq X$ .

**Theorem 4** (WHITNEY). For any graph  $G$ :  $\kappa(G) \leq \lambda(G) \leq \delta(G)$ .

**Theorem 5** (CHARTRAND). For a connected graph  $G = \langle V, E \rangle$ : if  $\delta(G) \geq \lfloor |V|/2 \rfloor$ , then  $\lambda(G) = \delta(G)$ .

**Theorem 6** (Menger). For any pair of non-adjacent vertices  $u$  and  $v$  in an undirected graph, the size of the minimum *vertex cut* is equal to the maximum number of pairwise *internally vertex-disjoint paths* from  $u$  to  $v$ .

**Theorem 7** (HARARY). Every block of a block graph<sup>6</sup> is a clique.

<sup>1</sup> Hereinafter, “graphs” are “simple, finite, undirected, and unweighted”, unless stated otherwise.

<sup>2</sup> Since the absolute geopolitical correctness is not necessary to accomplish this task, you can simply use [https://simple.wikipedia.org/wiki/List\\_of\\_European\\_countries](https://simple.wikipedia.org/wiki/List_of_European_countries) or any other similar source as a reference.

<sup>3</sup> You may either preserve the original spatial relationships between vertices, or use an automatic layout software. Take a look at Graphviz, Gephi, Cytoscape.

<sup>4</sup> Since  $\mathcal{G}^*$  naturally corresponds to a “map”, it *has* to be planar. However, due to *some broken countries*, it might be impossible to draw  $\mathcal{G}^*$  completely without edge intersections, so you just have to minimize them.

<sup>5</sup> You may choose geodesic<sup>6</sup> or road distance, to your preference. You *have* to use *more-or-less real data*.

<sup>6</sup> A block graph  $H = B(G)$  is an intersection graph of all blocks (biconnected components) of  $G$ , i.e. each vertex  $v \in V(H)$  corresponds to a block of  $G$ , and there is an edge  $\{v, u\} \in E(H)$  iff “blocks”  $v$  and  $u$  share a cut vertex.