



- For each given recurrence relation, find the first five terms, derive the closed-form solution, and check it by substituting it back to the recurrence relation.
  - $a_n = a_{n-1} + n$  with  $a_0 = 2$
  - $a_n = 2a_{n-1} + 2$  with  $a_0 = 1$
  - $a_n = 3a_{n-1} + 2^n$  with  $a_0 = 5$
  - $a_n = 4a_{n-1} + 5a_{n-2}$  with  $a_0 = 1, a_1 = 17$
  - $a_n = 4a_{n-1} - 4a_{n-2}$  with  $a_0 = 3, a_1 = 11$
  - $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$  with  $a_{0,1,2} = 3, 2, 6$
- Solve the following recurrences by applying the [Master theorem](#). For the cases where the Master theorem does not apply, use the [Akra–Bazzi method](#). In cases where neither of these two theorems apply, explain why and solve the recurrence relation by closely examining the recursion tree. Solutions must be in the form  $T(n) \in \Theta(\dots)$ .
  - $T(n) = 2T(n/2) + n$
  - $T(n) = T(3n/4) + T(n/4) + n$
  - $T(n) = 3T(n/2) + n$
  - $T(n) = 2T(n/2) + n/\log n$
  - $T(n) = 6T(n/3) + n^2 \log n$
  - $T(n) = T(3n/4) + n \log n$
  - $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$
  - $T(n) = T(n/2) + T(n/4) + 1$
  - $T(n) = T(n/2) + T(n/3) + T(n/6) + n$
  - $T(n) = 2T(n/3) + 2T(2n/3) + n$
  - $T(n) = \sqrt{2n}T(\sqrt{2n}) + \sqrt{n}$
  - $T(n) = \sqrt{2n}T(\sqrt{2n}) + n$
- Consider a recurrence relation  $a_n = 2a_{n-1} + 2a_{n-2}$  with  $a_0 = a_1 = 2$ . Solve it (i.e. find a closed formula) and show how it can be used to estimate the value of  $\sqrt{3}$  (hint: observe  $\lim_{n \rightarrow \infty} a_n/a_{n-1}$ ). After that, devise an algorithm for constructing a recurrence relation with integer coefficients and initial conditions that can be used to estimate the square root  $\sqrt{k}$  of a given integer  $k$ .
- Find a closed formula for the  $n$ -th term of the sequence with generating function  $\frac{3x}{1-4x} + \frac{1}{1-x}$ .
- Given the generating function  $G(x) = \frac{5x^2+2x+1}{(1-x)^3}$ , decompose it into partial fractions and find the sequence that it represents.
- Pell–Lucas numbers are defined by  $Q_0 = Q_1 = 2$  and  $Q_n = 2Q_{n-1} + Q_{n-2}$  for  $n \geq 2$ . Derive the corresponding generating function and find a closed formula for the  $n$ -th Pell–Lucas number.
- For each given recurrence relation, derive the corresponding generating function and find a closed formula for the  $n$ -th term of the sequence.
  - $a_n = 2a_{n-1} - a_{n-2}$  with  $a_0 = 3, a_1 = 5$
  - $a_n = a_{n-1} + a_{n-2} - a_{n-3}$  with  $a_0 = 1, a_1 = 1, a_2 = 5$
  - $a_n = a_{n-1} + n$  with  $a_0 = 0$
  - $a_n = a_{n-1} + 2a_{n-2} + 2^n$  with  $a_0 = 2, a_1 = 1$
- Find the number of non-negative integer solutions to the Diophantine equation  $3x + 5y = 100$  using generating functions.
- Consider a  $2n$ -digit ticket number to be “lucky” if the sum of its first  $n$  digits is equal to the sum of its last  $n$  digits. Each digit (including the first one!) in a number can take value from 0 to 9. For example, a 6-digit ticket 345 264 is lucky since  $3 + 4 + 5 = 2 + 6 + 4$ .
  - Find the number of lucky 6-digit and 8-digit tickets.
  - Find the generating function for the number of  $2n$ -digit lucky tickets.
  - Find a closed formula for the number of  $2n$ -digit lucky tickets.