

*In der Mathematik ist die Kunst Fragen zu stellen wertvoller als Probleme zu lösen*

— Georg Cantor

1. For each given relation  $R_i \subseteq M_i^2$ , determine whether it is *reflexive*, *irreflexive*, *coreflexive*, *symmetric*, *antisymmetric*, *asymmetric*, *transitive*, *left/right Euclidean*, *connex*. Provide a counterexample for each non-complying property (e.g., “transitivity does not hold for  $x, y, z = (3, 1, 2)$ ”). Organize your answer in a table (e.g., columns—relations, rows—properties).

(a) $M_1 = \mathbb{R}$ $x R_1 y \leftrightarrow  x - y  \leq 1$	(c) $M_3 = \{a, b, c, d\}$ $\ R_3\  = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
(b) $M_2 = \mathcal{P}(\{a, b, c\})$ $R_2 = “\subseteq”$	(d) $M_4 = \{\text{“rock”, “scissors”, “paper”}\}$ $R_4 = \{\langle x, y \rangle \mid x \text{ beats } y\}$

2. Prove (rigorously) or disprove (by providing a counterexample) the following statements about arbitrary homogeneous relations  $R \subseteq M^2$  and  $S \subseteq M^2$ :

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|---|---|
| (a) If $R$ and $S$ are <i>reflexive</i> , then $R \cap S$ is so.  | (d) If $R$ and $S$ are <i>reflexive</i> , then $R \cup S$ is so.  |
| (b) If $R$ and $S$ are <i>symmetric</i> , then $R \cap S$ is so.  | (e) If $R$ and $S$ are <i>symmetric</i> , then $R \cup S$ is so.  |
| (c) If $R$ and $S$ are <i>transitive</i> , then $R \cap S$ is so. | (f) If $R$ and $S$ are <i>transitive</i> , then $R \cup S$ is so. |

3. An equinumerosity relation  $\sim$  over sets is defined as follows:  $A \sim B \leftrightarrow |A| = |B|$ .

- (a) Show that  $\sim$  is an equivalence relation over finite sets.  
 (b) Show that  $\sim$  is an equivalence relation over infinite sets<sup>1</sup>.  
 (c) Find the quotient set of  $\mathcal{P}(\{a, b, c, d\})$  by  $\sim$ .

4. Let  $R_\theta$  be a relation of  $\theta$ -similarity (clearly,  $\theta \in [0; 1] \subseteq \mathbb{R}$ ) of finite non-empty sets defined as follows: a set  $A$  is said to be  $\theta$ -similar to  $B$  iff the Jaccard index  $\text{Jac}(A, B) = \frac{|A \cap B|}{|A \cup B|}$  for these sets is at least  $\theta$ , i.e.  $\langle A, B \rangle \in R_\theta \leftrightarrow \text{Jac}(A, B) \geq \theta$ .

- (a) Determine whether  $\theta$ -similarity is a tolerance relation<sup>2</sup>.  
 (b) Determine whether  $\theta$ -similarity is an equivalence relation.  
 (c) Draw the graph of a relation  $R_\theta \subseteq \{A_i\}^2$ , where  $\theta = 0.25$ ,  $A_1 = \{1, 2, 5, 6\}$ ,  $A_2 = \{2, 3, 4, 5, 7, 9\}$ ,  $A_3 = \{1, 4, 5, 6\}$ ,  $A_4 = \{3, 7, 9\}$ ,  $A_5 = \{1, 5, 6, 8, 9\}$ .

5. Any binary relation  $R \subseteq M^2$  can be represented as a zero-one matrix  $\|R\| = [r_{ij}]$ , where the element  $r_{ij}$  is equal to 1 if  $\langle m_i, m_j \rangle \in R$  and 0 otherwise. Boolean product of two square matrices  $A = [a_{ij}]$  and  $B = [b_{ij}]$  is a matrix  $C = A \odot B = [c_{ij}]$  defined as follows:  $c_{ij} = \bigvee_k (a_{ik} \wedge b_{kj})$ . A composition of relations  $R$  and  $S$  is a relation  $S \circ R$  defined as follows:  $\langle a, b \rangle \in S \circ R \leftrightarrow \exists c : \langle a, c \rangle \in R \wedge \langle c, b \rangle \in S$ . Show that the matrix representation of the composition of relations  $R$  and  $S$  is equal to the Boolean product of the corresponding matrices, i.e.  $\|S \circ R\| = \|R\| \odot \|S\|$ .

6. Find the error in the “proof” of the following “theorem”.

“Theorem”: Let  $R$  be a relation on a set  $A$  that is symmetric and transitive. Then  $R$  is reflexive.

“Proof”: Let  $a \in A$ . Take an element  $b \in A$  such that  $\langle a, b \rangle \in R$ . Because  $R$  is symmetric, we also have  $\langle b, a \rangle \in R$ . Now using the transitive property, we can conclude that  $\langle a, a \rangle \in R$  because  $\langle a, b \rangle \in R$  and  $\langle b, a \rangle \in R$ .

7. Give an example of a relation  $R$  on the set  $\{a, b, c\}$  such that the symmetric closure of the reflexive closure of the transitive closure of  $R$  is not transitive.

<sup>1</sup> For infinite sets,  $|A| = |B|$  means there is a bijection between  $A$  and  $B$ .

<sup>2</sup> A tolerance relation is a *reflexive* and *symmetric* relation.

8. Let  $R$  be the relation on the set of all colorings of the  $2 \times 2$  checkerboard where each of the four squares is colored either *red* or *blue* so that  $\langle C_1, C_2 \rangle$ , where  $C_1$  and  $C_2$  are  $2 \times 2$  checkerboards with each of their four squares colored blue or red, belongs to  $R$  if and only if  $C_2$  can be obtained from  $C_1$  either by rotating the checkerboard or by rotating it and then reflecting it.
  - (a) Show that  $R$  is an equivalence relation.
  - (b) What are the equivalence classes of  $R$ ?
9. Consider two relations  $R \subseteq A \times B$  and  $S \subseteq B \times C$ . Prove that  $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ .
10. Prove or disprove the following statements about the functions  $f$  and  $g$ :
  - (a) If  $f$  and  $g$  are injections, then  $g \circ f$  is also an injection.
  - (b) If  $f$  and  $g$  are surjections, then  $g \circ f$  is also a surjection.
  - (c) If  $f$  and  $f \circ g$  are injections, then  $g$  is also an injection.
  - (d) If  $f$  and  $f \circ g$  are surjections, then  $g$  is also a surjection.
11. Let  $H = \{1, 2, 4, 5, 10, 12, 20\}$ . Consider a divisibility relation  $R \subseteq H^2$  defined as follows:  $x R y \leftrightarrow y : x$ .
  - (a) Sort  $R$  (as a set of pairs) lexicographically<sup>3</sup>.
  - (b) Show that  $R$  is a partial order.
  - (c) Determine whether  $R$  is a linear (total) order.
  - (d) Draw the Hasse diagram for a graded poset  $\langle H, R, \rho \rangle$ , where  $\rho: H \rightarrow \mathbb{N}_0$  is a grading function which maps a number  $n \in H$  to the sum of all exponents appearing in its prime factorization, e.g.,  $\rho(20) = \rho(2^2 \cdot 5^1) = 2 + 1 = 3$ , so the number 20 would have the 3rd rank (bottom-up).
  - (e) Find the minimal, minimum (least), maximal and maximum (greatest) elements in the poset  $\langle H, R \rangle$ . If there are multiple or none, explain why.
  - (f) Perform a topological sort<sup>4</sup> of the poset  $\langle H, R \rangle$ .
12. Prove that the transitive closure  $R^+$  is in fact transitive.
 

**Definition.**  $R^+ = \bigcup_{n \in \mathbb{N}^+} R^n$  is a transitive closure of  $R \subseteq M^2$ , where

  - \*  $R^{k+1} = R^k \circ R$  is a compositional (functional) power<sup>4</sup>,
  - \*  $R^1 = R$ ,
  - \*  $S \circ R = \{\langle x, y \rangle \mid \exists z : (x R z) \wedge (z S y)\}$  is a composition (relative product) of relations  $R$  and  $S$ .
13. Given a set  $S$  and two partitions  $P_1$  and  $P_2$  of  $S$ , we define the relation  $P_1 \preceq P_2$  as follows: partition  $P_1$  is considered a *refinement* of the partition  $P_2$  if every set in  $P_1$  is a subset of one of the sets in  $P_2$ . Show that the set of all partitions of a set  $S$  with the refinement relation  $\preceq$  is a lattice.
14. A poset  $\langle R, \preceq \rangle$  is *well-founded* if there is no infinite decreasing sequence of elements in the poset, that is, elements  $x_1, x_2, \dots, x_n$  such that  $\dots \prec x_n \prec \dots \prec x_2 \prec x_1$ . Determine whether the set of strings of lowercase English letters with lexicographic order is well-founded.

<sup>3</sup> Lexicographic order for pairs:  $\langle a, b \rangle \preceq \langle a', b' \rangle \leftrightarrow (a < a') \vee ((a = a') \wedge (b \leq b'))$ . For example,  $\langle 1, 2 \rangle \preceq \langle 1, 3 \rangle \preceq \langle 2, 1 \rangle$ .

<sup>4</sup> Note: this is *not* a Cartesian power, despite of the same notation  $R^n$ . Another possible notation for compositional power is  $R^{\circ n}$ , but it is too wild to use it here.