4 Formal Logic Cheatsheet

4.1 Propositional Logic[™]

* **Proposition** [™] is a statement which can be either true or false.

Truth-bearer

- * Alphabet^[2] of propositional logic consists of (1) atomic symbols and (2) operator symbols.
- * **Atomic formula** (atom) is an irreducible formula without logical connectives.
 - \circ Propositional **variables**: A, B, C, \ldots, Z . With indices, if needed: $A_1, A_2, \ldots, Z_1, Z_2, \ldots$
 - ∘ Logical **constants**: ⊤ for always true proposition (*tautology*), ⊥ for always false proposition (*contradiction*).
- * Logical connectives (operators):

Type	Natural meaning	Symbolization
[⊄] Negation	It is not the case that \mathcal{P} . It is false that \mathcal{P} . It is not true that \mathcal{P} .	$\neg \mathcal{P}$
^E Conjunction	Both \mathcal{P} and Q . \mathcal{P} but Q . \mathcal{P} , although Q .	$\mathscr{P}\wedge Q$
[™] Disjunction	Either \mathcal{P} or Q (or both). \mathcal{P} unless Q .	$\mathcal{P} \lor Q$
Exclusive or (Xor)	Either \mathcal{P} or Q (but not both) \mathcal{P} xor Q .	$\mathcal{P}\oplus Q$
Implication (Conditional)	If \mathcal{P} , then Q . \mathcal{P} only if Q . Q if \mathcal{P} .	$\mathcal{P} \to Q$
[☑] Biconditional	\mathcal{P} , if and only if Q . \mathcal{P} iff Q . \mathcal{P} just in case Q .	$\mathcal{P} \leftrightarrow Q$

* **Sentence** of propositional logic is defined inductively:

Well-formed formula (WFF)

- 1. Every propositional variable/constant is a sentence.
- 2. If \mathcal{A} is a sentence, then $\neg \mathcal{A}$ is a sentence.
- 3. If \mathcal{A} and \mathcal{B} are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \to \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
- 4. Nothing else is a sentence.
- * Well-formed formulae grammar:

Backus-Naur form (BNF)

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 \langle sentence \rangle ::= \langle constant \rangle \\ | \langle variable \rangle \\ | \neg \langle sentence \rangle \\ | '(' \langle sentence \rangle \langle binop \rangle \langle sentence \rangle ')' \\ \langle constant \rangle ::= \top | \bot \\ \langle variable \rangle ::= A | ... | Z | A_1 | ... | Z_n \\ \langle binop \rangle ::= \land | \lor | \oplus | \rightarrow | \leftarrow | \leftrightarrow
```

- * **Literal**^{\mathcal{L}} is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (positive literal), $\mathcal{L}_j = \neg X_j$ (negative literal).
- * **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}_{\textit{premises}} \quad \vdots \quad \underbrace{C}_{\textit{conclusion}}$$
"therefore"

* An argument is **valid** if whenever all the premises are true, the conclusion is also true.

Validity

* An argument is **invalid** if there is a case (a counterexample) when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

* **Valuation**[™] is any assignment of truth values to propositional variables.

- Interpretation
- * \mathcal{A} is a **tautology** (valid) iff it is true on *every* valuation. Might be symbolized as " $\models \mathcal{A}$ ".
- * \mathcal{A} is a **contradiction** iff it is false on *every* valuation. Might be symbolized as " $\mathcal{A} \models$ ".
- * \mathcal{A} is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- * \mathcal{A} is **satisfiable** iff it is true on *some* valuation.

Satisfiability

* \mathcal{A} is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation.

Falsifiability

- * \mathcal{A} and \mathcal{B} are **equivalent** (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- * $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. Consistency
- * The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false. Semantic entailment
- * If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \stackrel{.}{.} C$ is **valid**. *Validity check examples*:

			valia	1			valid				invali	d	
A B	$A \longrightarrow B$	\boldsymbol{A}	:.	В	$\neg A$	$A \rightarrow \neg B$	•	$B \longrightarrow A$	$A \rightarrow B$	В	:.	¬ ($B \rightarrow A)$
0 0	1	0		0	1	1 1	✓	1	1	0		0	1
					1			0	1	1	✓	1	0
1 0	0	1	•	0	0	1 1	1	1	0	0		0	1
1 1	1	1	1	1	0	1 0	1	1	1	1	X	0	1

R S T	$R \vee S$	$S \vee T$	$\neg R$	invali	$\frac{d}{S} \wedge T$	$ (R \land S) \rightarrow T \stackrel{valid}{::} R \rightarrow (S \rightarrow T) $
0 0 0	0	0	1		0	0 10 1 0 1
0 0 1	0	1	1		0	0 11 1 01 1
0 1 0	1	1	1	X	0	0 10 10 0
0 1 1	1	1	1	1	1	0 11 1 01 1
1 0 0	1	0	0		0	0 10 11 1
1 0 1	1	1	0		0	0 11 1 1
1 1 0	1	1	0		0	1 00 · 10 0
1 1 1	1	1	0		1	1 11 1 1

- * **Soundness** $^{\mbox{\sc E}}$: $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$ "Every provable statement is in fact true"
- * **Completeness:** $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$ "Every true statement has a proof"

4.3 Natural Deduction Rules

Reiteration

m	A	
<i>:</i> .	Я	R <i>m</i>

Modus ponens

i	$\mathcal{A} \to \mathcal{B}$	
<i>j</i> ∴	Я	
<i>:</i> .	\mathcal{B}	MP i, j

Modus tollens

i	$\mathcal{A} \to \mathcal{B}$	
j	$\neg \mathcal{B}$	
\therefore	$\neg \mathcal{A}$	MT i, j

Negation

i	$\neg \mathcal{A}$	
j	A ⊥	
<i>i j</i> ∴	1	$\neg E i, j$
i	A	
j	<i>A</i> ⊥	
<i>:</i> .	$\neg \mathcal{A}$	¬I <i>i−j</i>

Indirect proof

$$\begin{array}{c|ccc}
i & \neg \mathcal{A} \\
j & \bot \\
\vdots & \mathcal{A} & \text{IP } i-j
\end{array}$$

Double negation

$$\begin{array}{c|cc}
m & \neg \neg \mathcal{A} \\
\therefore & \mathcal{A} & \neg \neg \mathbf{E} m
\end{array}$$

Law of excluded middle

$$\begin{array}{c|c}
i & \mathcal{A} \\
j & \mathcal{B} \\
k & \neg \mathcal{A} \\
l & \mathcal{B} \\
\therefore & \mathcal{B} & \text{LEM } i-j, k-l
\end{array}$$

Explosion

$$\begin{array}{c|c}
m & \bot \\
\therefore & \mathcal{A} & X m
\end{array}$$

Conjunction

i	$\mathcal A$	
j	${\cal B}$	
<i>i j</i> ∴	\mathcal{B} $\mathcal{A} \wedge \mathcal{B}$	\wedge I i, j
	· 	
m	$\mathcal{A} \wedge \mathcal{B}$	
<i>m</i> ∴	$egin{array}{c} \mathcal{A} \wedge \mathcal{B} \ \mathcal{A} \ \mathcal{B} \end{array}$	\wedge E m

Disjunction

m ∴	\mathcal{A} $\mathcal{A} \vee \mathcal{B}$	∨I m
m ∴	$ig _{\mathcal{B}ee\mathcal{A}}$	∨I m
m i j k l ∴	$ \begin{vmatrix} \mathcal{A} \lor \mathcal{B} \\ \mathcal{A} \\ C \\ \mathcal{B} \\ C \\ C \end{vmatrix} $	\vee E $m, i-j, k-l$

Disjunctive syllogism

$$i \mid \mathcal{A} \vee \mathcal{B}$$

$$j \mid \neg \mathcal{A}$$

$$\therefore \mid \mathcal{B} \qquad \text{DS } i, j$$

$$i \mid \mathcal{A} \vee \mathcal{B}$$

$$j \mid \neg \mathcal{B}$$

$$\therefore \mid \mathcal{A} \qquad \text{DS } i, j$$

Hypothetical syllogism

$$\begin{array}{c|c} i & \mathcal{A} \to \mathcal{B} \\ j & \mathcal{B} \to C \\ \therefore & \mathcal{A} \to C & \text{HS } i, j \end{array}$$

Conditional

$$\begin{array}{c|cc}
i & & \mathcal{A} \\
j & & \mathcal{B} \\
\therefore & \mathcal{A} \to \mathcal{B} & \to I i - j
\end{array}$$

Contraposition

m	$\mathcal{A} \to \mathcal{B}$	
<i>:</i> .	$\neg \mathcal{B} \to \neg \mathcal{A}$	Contra m

Biconditional

i j k	$egin{array}{ c c c c c c c c c c c c c c c c c c c$	
<i>l</i> ∴.	$\begin{array}{ c c } \hline \mathcal{A} \\ \mathcal{A} \leftrightarrow \mathcal{B} \\ \hline \end{array}$	↔I i−j, k−l
i	$\mathcal{A} \leftrightarrow \mathcal{B}$	
j	$ \begin{array}{c c} \mathcal{A} \leftrightarrow \mathcal{B} \\ \mathcal{A} \\ \mathcal{B} \end{array} $	
∴.	\mathcal{B}	\leftrightarrow E i, j
i	$\mathcal{A} \leftrightarrow \mathcal{B}$	
i j ∴.	$ \begin{array}{c} \mathcal{A} \leftrightarrow \mathcal{B} \\ \mathcal{B} \\ \mathcal{A} \end{array} $	
∴	\mathcal{A}	\leftrightarrow E i, j

De Morgan Rules $m \mid \neg(\mathcal{A} \vee \mathcal{B})$

	$\mid \neg \mathcal{A} \wedge \neg \mathcal{B} \mid$	DeM m
m ∴		DeM m
m ∴		DeM m
m ∴		DeM m

Green: basic rules. **Orange:** derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).