## 5 Graph Theory Cheatsheet

Glossary

- \* **Graph** E is an ordered pair  $G = \langle V, E \rangle$ , where  $V = \{v_1, \dots, v_n\}$  is a set of vertices, and  $E = \{e_1, \dots, e_m\}$  is a set of edges.  $\circ$  Given a graph G, the notation V(G) denotes the vertices of G.
  - Given a graph G, the notation F(G) denotes the vertices of G.
    Given a graph G, the notation E(G) denotes the edges of G.
  - In fact,  $V(\cdot)$  and  $E(\cdot)$  functions allow to access "vertices" and "edges" of any object possessing them (e.g., paths).
- \* **Order** of a graph G is the number of vertices in it: |V(G)|.
- \* **Size** of a graph G is the number of edges in it: |E(G)|.
- \* Two graphs are **equal** if their vertex sets and edge sets are equal:  $G_1 = G_2$  iff  $V_1 = V_2$  and  $E_1 = E_2$ .
- \* Two graphs  $G_1 = \langle V_1, E_1 \rangle$  and  $G_2 = \langle V_2, E_2 \rangle$  are called **isomorphic** denoted  $G_1 \simeq G_2$ , if there exists an *edge-preserving* bijection  $f: V_1 \to V_2$ , *i.e.* any two vertices  $u, v \in V_1$  are adjacent in  $G_1$  if and only if f(u) and f(v) are adjacent in  $G_2$ . This means that the graphs are structurally identical *up to vertex renaming*.
- \* Simple **undirected** graphs have  $E \subseteq V^{(2)}$ , *i.e.* each edge  $e_i \in E$  between vertices u and v is denoted by  $\{u, v\} \in V^{(2)}$ . Such *undirected edges* are also called *links* or *lines*.
  - ∘  $A^{(k)} = \{\{x_1, ..., x_k\} \mid x_1 \neq ... \neq x_k \in A\} = \{S \mid S \subseteq A, |S| = k\}$  is the set of *k*-sized subsets of *A*.
- \* Simple **directed** graphs have  $E \subseteq V^2$ , *i.e.* each edge  $e_i \in E$  from vertex u to v is denoted by an ordered pair  $\langle u, v \rangle \in V^2$ . Such *directed edges* are also called *arcs* or *arrows*.

Simple

Null

Multigraph

Pseudograph

- $A^k = A \times \cdots \times A = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in A\}$  is the set of k-tuples (Cartesian k-power of A).
- \* Multi-edges are edges that have the same end nodes.
- \* **Loop**<sup>☑</sup> is an edge that connects a vertex to itself.
- \* Simple graph is a graph without multi-edges and loops.
- \* Multigraph <sup>™</sup> is a graph with multi-edges.
- \* Pseudograph<sup>™</sup> is a multigraph with loops.
- \* Null graph is a "graph" without vertices.
- \* Trivial (singleton) graph is a graph consisting of a single vertex.
- \* Empty (edgeless) graph \(\mathbb{E}\) is a graph without edges.
- \* Complete graph  $K_n$  is a simple graph in which every pair of distinct vertices is connected by an edge.
- \* Weighted graph G = (V, E, w) is a graph in which each edge has an associated numerical value (the weight) represented by the weight function  $w : E \to \text{Num}$ .
- \* **Subgraph** C of a graph  $G = \langle V, E \rangle$  is another graph  $G' = \langle V', E' \rangle$  such that  $V' \subseteq V, E' \subseteq E$ . Designated as  $G' \subseteq G$ .
- \* **Spanning (partial) subgraph** is a subgraph that includes all vertices of a graph.
- \* **Induces subgraph**  $\subseteq$  of a graph  $G = \langle V, E \rangle$  is another graph G' formed from a subset S of the vertices of the graph and all the edges (from the original graph) connecting pairs of vertices in that subset. Formally,  $G' = G[S] = \langle V', E' \rangle$ , where  $S \subseteq V$ ,  $V' = V \cap S$ ,  $E' = \{e \in E \mid \exists v \in S : e \mid v\}$ .
- \* Adjacency is the relation between two vertices connected with an edge.
- \* **Adjacency matrix** is a square matrix  $A_{V\times V}$  of an adjacency relation.
  - ∘ For simple graphs, adjacency matrix is binary, *i.e.*  $A_{ij} \in \{0, 1\}$ .
  - ∘ For directed graphs,  $A_{ij}$  ∈ {0, 1, −1}.
  - ∘ For multigraphs, adjacency matrix contains edge multiplicities, *i.e.*  $A_{ij} \in \mathbb{N}_0$ .
- \* Incidence<sup>™</sup> is a relation between an edge and its endpoints.
- \* **Incidence matrix** is a Boolean matrix  $B_{V \times E}$  of an incidence relation.
- \* **Degree**  $^{\mathbb{Z}}$  deg(v) the number of edges incident to v (loops are counted twice).
  - $\delta(G) = \min_{v} \deg(v)$  is the **minimum degree**.
  - $\Delta(G) = \max_{v \in V} \deg(v)$  is the **maximum degree**.
  - Handshaking Lemma.  $\sum_{v \in V} \deg(v) = 2|E|$ .

- \* A graph is called r-regular if all its vertices have the same degree:  $\forall v \in V : \deg(v) = r$ .
- \* Complement graph G of a graph G is a graph G on the same vertices such that two distinct vertices of G are adjacent iff they are non-adjacent in G.
- \* Intersection graph  $^{\ensuremath{\mathcal{C}}}$  of a family of sets  $F = \{S_i\}$  is a graph  $G = \Omega(F) = \langle V, E \rangle$  such that each vertex  $v_i \in V$  denotes the set  $S_i$ , i.e. V = F, and the two vertices  $v_i$  and  $v_j$  are adjacent whenever the corresponding sets  $S_i$  and  $S_j$  have a non-empty intersection, i.e.  $E = \{\langle v_i, v_j \rangle \mid i \neq j, S_i \cap S_j \neq \emptyset \}$ .

Term

Walk Trail

Path

<sup>1</sup>Can vertices be repeated?

 $^2\mathrm{Can}$  edges be repeated?

V<sup>1</sup> E<sup>2</sup> "Closed" term

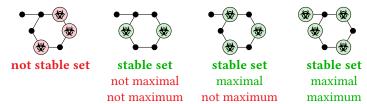
Circuit

Cycle

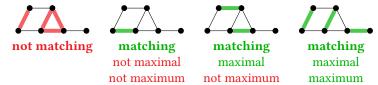
Closed walk

(impossible)

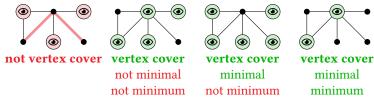
- \* Line graph C of a graph  $G = \langle V, E \rangle$  is another graph  $L(G) = \Omega(E)$  that represents the adjacencies between edges of G. Each vertex of L(G) represents an edge of G, and two vertices of L(G) are adjacent iff the corresponding edges share a common endpoint in G (i.e. edges are "adjacent"/"incident").
- \* **Walk**<sup> $\mathbb{Z}$ </sup> is an alternating sequence of vertices and edges:  $l = v_1 e_1 v_2 \dots e_{n-1} v_n$ .
  - o Trail is a walk with distinct edges.
  - o **Path** is a walk with distinct vertices (and therefore distinct edges).
  - o A walk is **closed** if it starts and ends at the same vertex. Otherwise, it is **open**.
  - o Circuit is a closed trail.
  - Cycle is a closed path.
- \* **Length** of a path (walk, trail)  $l = u \rightsquigarrow v$  is the number of edges in it: |l| = |E(l)|.
- \* **Girth** is the length of the shortest cycle in the graph.
- \* **Distance** dist(u, v) between two vertices is the length of the shortest path  $u \rightsquigarrow v$ .
  - $\varepsilon(v) = \max_{v \in V} \operatorname{dist}(v, u)$  is the **eccentricity** of the vertex v.
  - rad(G) =  $\min_{v \in V} \varepsilon(v)$  is the **radius** of the graph G.
  - diam $(G) = \max_{v \in V} \varepsilon(v)$  is the **diameter** of the graph G.
  - center(G) = { $v \mid \varepsilon(v) = \operatorname{rad}(G)$ } is the **center** of the graph G.
- \* Clique  $Q \subseteq V$  is a set of vertices inducing a complete subgraph.
- \* Stable set  $S \subseteq V$  is a set of independent (pairwise non-adjacent) vertices.



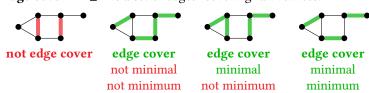
\* **Matching**  $E \subseteq E$  is a set of independent (pairwise non-adjacent) edges.



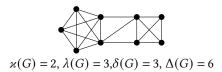
- \* **Perfect matching** is a matching that covers all vertices in the graph.
  - o A perfect matching (if it exists) is always a minimum edge cover (but not vice-versa!).
- \* **Vertex cover**  $^{\bowtie}$   $R \subseteq V$  is a set of vertices "covering" all edges.



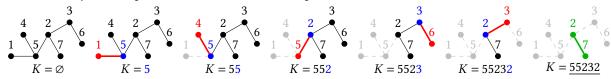
\* Edge cover  $^{\mathbb{Z}}$   $F \subseteq E$  is a set of edges "covering" all vertices.



- \* Cut vertex (articulation point) is a vertex whose removal increases the number of connected components.
- \* **Bridge** is an edge whose removal increases the number of connected components.
- \* **Biconnected graph** is a connected "nonseparable" graph, which means that the removal of any vertex does not make the graph disconnected. Alternatively, this is a graph without *cut vertices*.
- \* **Biconnectivity** can be defined as a relation on edges  $R \subseteq E^2$ :
  - Two edges are called *biconnected* if there exist two *vertex-disjoint* paths between the ends of these edges.
  - o Trivially, this relation is an equivalence relation.
  - Equivalence classes of this relation are called **biconnected components**, also known as **blocks**.
- \* **Edge biconnectivity** can be defined as a relation on vertices  $R \subseteq V^2$ :
  - Two vertices are called *edge-biconnected* if there exist two *edge-disjoint* paths between them.
  - o Trivially, this relation is an equivalence relation.
  - Equivalence classes of this relation are called **edge-biconnected components** (or 2-edge-connected components).
- \* **Vertex connectivity**  $\kappa(G)$  is the minimum number of vertices that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k-vertex-connected.
- - Corollary of Menger's theorem: graph  $G = \langle V, E \rangle$  is k-vertex-connected if, for every pair of vertices  $u, v \in V$ , it is possible to find k vertex-independent (internally vertex-disjoint) paths between u and v.
  - *k*-vertex-connected graphs are also called simply *k*-connected.
  - o 1-connected graphs are called connected, 2-connected are biconnected, 3-connected are triconnected, etc.
  - Note the "exceptions":
    - Singleton graph  $K_1$  has  $\varkappa(K_1) = 0$ , so it is **not** 1-connected, but still considered connected.
    - Graph  $K_2$  has  $\varkappa(K_2)=1$ , so it is **not** 2-connected, but considered biconnected, so it can be a block.
- \* **Edge connectivity**  $^{\mathbb{Z}}$   $\lambda(G)$  is the minimum number of edges that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k-edge-connected.
- \* k-edge-connected graph  $^{\mathbb{Z}}$  is a graph that remains connected after less than k edges are removed, i.e.  $\lambda(G) \geq k$ .
  - Corollary of Menger's theorem: graph  $G = \langle V, E \rangle$  is k-edge-connected if, for every pair of vertices  $u, v \in V$ , it is possible to find k edge-disjoint paths between u and v.
  - o 2-edge-connected are called *edge-biconnected*, 3-edge-connected are *edge-triconnected*, *etc.*
  - Note the "exception":
    - Singleton graph  $K_1$  has  $\lambda(K_1) = 0$ , so it is **not** 2-edge-connected, but considered edge-biconnected, so it can be a 2-edge-connected component.
- \* Whitney's Theorem. For any graph G,  $\varkappa(G) \le \lambda(G) \le \delta(G)$ .



- \* **Tree** is a connected undirected acyclic graph.
- \* Forest is an undirected acyclic graph, *i.e.* a disjoint union of trees.
- \* An **unrooted tree** (**free tree**) is a tree without any designated *root*.
- \* A **rooted tree** is a tree in which one vertex has been designated the *root*.
  - $\circ$  In a rooted tree, the **parent** of a vertex v is the vertex connected to v on the path to the root.
  - A **child** of a vertex v is a vertex of which v is the parent.
  - A **sibling** to a vertex v is any other vertex on the tree which has the same parent as v.
  - A **leaf** is a vertex with no children. Equivalently, **leaf** is a *pendant vertex*, *i.e.* deg(v) = 1.
  - An **internal vertex** is a vertex that is not a leaf.
  - A *k*-ary tree is a rooted tree in which each vertex has at most *k* children. *2-ary trees* are called **binary trees**.
- \* A **labeled tree** is a tree in which each vertex is given a unique *label*, e.g., 1, 2, ..., n.
- \* Cayley's formula. Number of labeled trees on n vertices is  $n^{n-2}$ .
- \* **Prüfer code** is a unique sequence of labels  $\{1, \ldots, n\}$  of length (n-2) associated with the labeled tree on n vertices.
  - **ENCODING** (iterative algorithm for converting tree T labeled with  $\{1, \ldots, n\}$  into a Prüfer sequence K):
    - On each iteration, remove the leaf with *the smallest label*, and extend *K* with *a single neighbour* of this leaf.
    - After (n-2) iterations, the tree will be left with *two adjacent* vertices—there is no need to encode them, because there is only one unique tree on 2 vertices, which requires 0 bits of information to encode.



- **DECODING** (iterative algorithm for converting a Prüfer sequence *K* into a tree *T*):
  - Given a Prüfer code K of length (n-2), construct a set of "leaves"  $W = \{1, \ldots, n\} \setminus K$ .
  - On each iteration:
    - (1) Pop the *first* element of K (denote it as k) and the *minimum* label in W (denote it as w).
    - (2) Connect k and w with an edge  $\langle k, w \rangle$  in the tree T.
    - (3) If  $k \notin K$ , then extend the set of "leaves"  $W := W \cup \{k\}$ .
  - After (n-2) iterations, the sequence K will be empty, and the set W will contain exactly two vertices connect them with an edge.