4 Formal Logic Cheatsheet

4.1 Propositional Logic[⬀]

* **Proposition**[⬀] is a statement which can be either true or false. Truth-bearer
* **Alphabet**[⬀] of propositional logic consists of (1) atomic symbols and (2) operator symbols.
* **Atomic formula**[⬀] (**atom**) is an irreducible formula without logical connectives.
  ○ Propositional **variables**: $A, B, C, \ldots, Z$. With indices, if needed: $A_1, A_2, \ldots, Z_1, Z_2, \ldots$.
  ○ Logical **constants**: $\top$ for always true proposition (*tautology*), $\bot$ for always false proposition (*contradiction*).
* **Logical connectives**[⬀] (**operators**):

Type	Natural meaning	Symbolization
[⬀]Negation	It is not the case that $\mathcal{P}$. It is false that $\mathcal{P}$. It is not true that $\mathcal{P}$.	$\neg\mathcal{P}$
[⬀]Conjunction	Both $\mathcal{P}$ and $\mathcal{Q}$. $\mathcal{P}$ but $\mathcal{Q}$. $\mathcal{P}$, although $\mathcal{Q}$.	$\mathcal{P} \wedge \mathcal{Q}$
[⬀]Disjunction	Either $\mathcal{P}$ or $\mathcal{Q}$ (or both). $\mathcal{P}$ unless $\mathcal{Q}$.	$\mathcal{P} \vee \mathcal{Q}$
[⬀]Exclusive or (Xor)	Either $\mathcal{P}$ or $\mathcal{Q}$ (but not both). $\mathcal{P}$ xor $\mathcal{Q}$.	$\mathcal{P} \oplus \mathcal{Q}$
[⬀]Implication (Conditional)	If $\mathcal{P}$, then $\mathcal{Q}$. $\mathcal{P}$ only if $\mathcal{Q}$. $\mathcal{Q}$ if $\mathcal{P}$.	$\mathcal{P} \rightarrow \mathcal{Q}$
[⬀]Biconditional	$\mathcal{P}$, if and only if $\mathcal{Q}$. $\mathcal{P}$ iff $\mathcal{Q}$. $\mathcal{P}$ just in case $\mathcal{Q}$.	$\mathcal{P} \leftrightarrow \mathcal{Q}$

* **Sentence** of propositional logic is defined inductively: Well-formed formula (WFF)
  1. Every propositional variable/constant is a sentence.
  2. If $\mathcal{A}$ is a sentence, then $\neg\mathcal{A}$ is a sentence.
  3. If $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \rightarrow \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
  4. Nothing else is a sentence.
* Well-formed formulae grammar: Backus-Naur form (BNF)

$$\langle sentence \rangle ::= \langle constant \rangle$$
$$| \quad \langle variable \rangle$$
$$| \quad \neg \langle sentence \rangle$$
$$| \quad \text{`('} \langle sentence \rangle \langle binop \rangle \langle sentence \rangle \text{`)'}$$
$$\langle constant \rangle ::= \top \mid \bot$$
$$\langle variable \rangle ::= A \mid \ldots \mid Z \mid A_1 \mid \ldots \mid Z_n$$
$$\langle binop \rangle ::= \wedge \mid \vee \mid \oplus \mid \rightarrow \mid \leftarrow \mid \leftrightarrow$$

* **Literal**[⬀] is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (*positive* literal), $\mathcal{L}_j = \neg X_j$ (*negative* literal).
* **Argument**[⬀] is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n}_{premises} \quad \therefore \quad \underbrace{C}_{conclusion}$$
"therefore"

* An argument is **valid** if whenever all the premises are true, the conclusion is also true. Validity
* An argument is **invalid** if there is a case (*a counterexample)* when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

∗ **Valuation**⬈ is any assignment of truth values to propositional variables. Interpretation

∗ $\mathcal{A}$ is a **tautology**⬈ (**valid**) iff it is true on *every* valuation. Might be symbolized as "$\vDash \mathcal{A}$".

∗ $\mathcal{A}$ is a **contradiction**⬈ iff it is false on *every* valuation. Might be symbolized as "$\mathcal{A} \vDash$ ".

∗ $\mathcal{A}$ is a **contingency**⬈ iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.

∗ $\mathcal{A}$ is **satisfiable** iff it is true on *some* valuation. Satisfiability

∗ $\mathcal{A}$ is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. Falsifiability

∗ $\mathcal{A}$ and $\mathcal{B}$ are **equivalent**⬈ (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check

∗ $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. Consistency

∗ The sentences $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ true and C false. Semantic entailment

∗ If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \therefore C$ is **valid**.

Validity check examples:

A	B	$A\to B$	A	$\therefore$ (valid)	B	$\neg A \to \neg B$	$\therefore$ (valid)	$B\to A$	$A\to B$	B	$\therefore$ (invalid)	$\neg(B\to A)$
0	0	1	0	·	0	1 1 1	✓	1	1	0	·	0 1
0	1	1	0	·	1	1 0 0	·	0	1	1	✓	1 0
1	0	0	1	·	0	0 1 1	✓	1	0	0	·	0 1
1	1	1	1	✓	1	0 1 0	✓	1	1	1	✗	0 1

R	S	T	$R\vee S$	$S\vee T$	$\neg R$	$\therefore$ (invalid)	$S\wedge T$	$(R\wedge S)\to T$	$\therefore$ (valid)	$R\to(S\to T)$
0	0	0	0	0	1	·	0	0 1 0	✓	0 1 1
0	0	1	0	1	1	·	0	0 1 1	✓	0 1 1
0	1	0	1	1	1	✗	0	0 1 0	✓	0 1 0
0	1	1	1	1	1	✓	1	0 1 1	✓	0 1 1
1	0	0	1	0	0	·	0	0 1 0	✓	1 1 1
1	0	1	1	1	0	·	0	0 1 1	✓	1 1 1
1	1	0	1	1	0	·	0	1 0 0	·	1 0 0
1	1	1	1	1	0	·	1	1 1 1	✓	1 1 1

∗ **Soundness**⬈: $\Gamma \vdash \mathcal{A} \to \Gamma \vDash \mathcal{A}$ "Every provable statement is in fact true"

∗ **Completeness:**⬈ $\Gamma \vDash \mathcal{A} \to \Gamma \vdash \mathcal{A}$ "Every true statement has a proof"

4.3 Natural Deduction Rules

Reiteration

$$
\begin{array}{l|l}
m & \mathcal{A} \\
\therefore & \mathcal{A} \qquad \text{R } m
\end{array}
$$

Modus ponens

$$
\begin{array}{l|l}
i & \mathcal{A} \to \mathcal{B} \\
j & \mathcal{A} \\
\therefore & \mathcal{B} \qquad \text{MP } i, j
\end{array}
$$

Modus tollens

$$
\begin{array}{l|l}
i & \mathcal{A} \to \mathcal{B} \\
j & \neg \mathcal{B} \\
\therefore & \neg \mathcal{A} \qquad \text{MT } i, j
\end{array}
$$

Negation

$$
\begin{array}{l|l}
i & \neg \mathcal{A} \\
j & \mathcal{A} \\
\therefore & \bot \qquad \neg\text{E } i, j
\end{array}
$$

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$$
\begin{array}{l|l}
i & \quad \mathcal{A} \\
j & \quad \bot \\
\therefore & \neg \mathcal{A} \qquad \neg\text{I } i\text{--}j
\end{array}
$$

Indirect proof

$$
\begin{array}{l|l}
i & \quad \neg \mathcal{A} \\
j & \quad \bot \\
\therefore & \mathcal{A} \qquad \text{IP } i\text{--}j
\end{array}
$$

Double negation

$$
\begin{array}{l|l}
m & \neg\neg \mathcal{A} \\
\therefore & \mathcal{A} \qquad \neg\neg\text{E } m
\end{array}
$$

Law of excluded middle

$$
\begin{array}{l|l}
i & \quad \mathcal{A} \\
j & \quad \mathcal{B} \\
k & \quad \neg \mathcal{A} \\
l & \quad \mathcal{B} \\
\therefore & \mathcal{B} \qquad \text{LEM } i\text{--}j, k\text{--}l
\end{array}
$$

Explosion

$$
\begin{array}{l|l}
m & \bot \\
\therefore & \mathcal{A} \qquad \text{X } m
\end{array}
$$

Conjunction

$$
\begin{array}{l|l}
i & \mathcal{A} \\
j & \mathcal{B} \\
\therefore & \mathcal{A} \wedge \mathcal{B} \qquad \wedge\text{I } i, j
\end{array}
$$

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$$
\begin{array}{l|l}
m & \mathcal{A} \wedge \mathcal{B} \\
\therefore & \mathcal{A} \qquad \wedge\text{E } m \\
\therefore & \mathcal{B} \qquad \wedge\text{E } m
\end{array}
$$

Disjunction

$$
\begin{array}{l|l}
m & \mathcal{A} \\
\therefore & \mathcal{A} \vee \mathcal{B} \qquad \vee\text{I } m
\end{array}
$$

- - - - -

$$
\begin{array}{l|l}
m & \mathcal{A} \\
\therefore & \mathcal{B} \vee \mathcal{A} \qquad \vee\text{I } m
\end{array}
$$

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$$
\begin{array}{l|l}
m & \mathcal{A} \vee \mathcal{B} \\
i & \quad \mathcal{A} \\
j & \quad C \\
k & \quad \mathcal{B} \\
l & \quad C \\
\therefore & C \qquad \vee\text{E } m, i\text{--}j, k\text{--}l
\end{array}
$$

Disjunctive syllogism

$$
\begin{array}{l|l}
i & \mathcal{A} \vee \mathcal{B} \\
j & \neg \mathcal{A} \\
\therefore & \mathcal{B} \qquad \text{DS } i, j
\end{array}
$$

- - - - -

$$
\begin{array}{l|l}
i & \mathcal{A} \vee \mathcal{B} \\
j & \neg \mathcal{B} \\
\therefore & \mathcal{A} \qquad \text{DS } i, j
\end{array}
$$

Hypothetical syllogism

$$
\begin{array}{l|l}
i & \mathcal{A} \to \mathcal{B} \\
j & \mathcal{B} \to C \\
\therefore & \mathcal{A} \to C \qquad \text{HS } i, j
\end{array}
$$

Conditional

$$
\begin{array}{l|l}
i & \quad \mathcal{A} \\
j & \quad \mathcal{B} \\
\therefore & \mathcal{A} \to \mathcal{B} \qquad \to\text{I } i\text{--}j
\end{array}
$$

Contraposition

$$
\begin{array}{l|l}
m & \mathcal{A} \to \mathcal{B} \\
\therefore & \neg \mathcal{B} \to \neg \mathcal{A} \qquad \text{Contra } m
\end{array}
$$

Biconditional

$$
\begin{array}{l|l}
i & \quad \mathcal{A} \\
j & \quad \mathcal{B} \\
k & \quad \mathcal{B} \\
l & \quad \mathcal{A} \\
\therefore & \mathcal{A} \leftrightarrow \mathcal{B} \qquad \leftrightarrow\text{I } i\text{--}j, k\text{--}l
\end{array}
$$

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$$
\begin{array}{l|l}
i & \mathcal{A} \leftrightarrow \mathcal{B} \\
j & \mathcal{A} \\
\therefore & \mathcal{B} \qquad \leftrightarrow\text{E } i, j
\end{array}
$$

- - - - -

$$
\begin{array}{l|l}
i & \mathcal{A} \leftrightarrow \mathcal{B} \\
j & \mathcal{B} \\
\therefore & \mathcal{A} \qquad \leftrightarrow\text{E } i, j
\end{array}
$$

De Morgan Rules

$$
\begin{array}{l|l}
m & \neg(\mathcal{A} \vee \mathcal{B}) \\
\therefore & \neg \mathcal{A} \wedge \neg \mathcal{B} \qquad \text{DeM } m
\end{array}
$$

- - - - -

$$
\begin{array}{l|l}
m & \neg \mathcal{A} \wedge \neg \mathcal{B} \\
\therefore & \neg(\mathcal{A} \vee \mathcal{B}) \qquad \text{DeM } m
\end{array}
$$

- - - - -

$$
\begin{array}{l|l}
m & \neg(\mathcal{A} \wedge \mathcal{B}) \\
\therefore & \neg \mathcal{A} \vee \neg \mathcal{B} \qquad \text{DeM } m
\end{array}
$$

- - - - -

$$
\begin{array}{l|l}
m & \neg \mathcal{A} \vee \neg \mathcal{B} \\
\therefore & \neg(\mathcal{A} \wedge \mathcal{B}) \qquad \text{DeM } m
\end{array}
$$

Green: basic rules.
Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).