

## 4 Formal Logic Cheatsheet

### 4.1 Propositional Logic

- \* **Proposition** is a statement which can be either true or false. Truth-bearer
- \* **Alphabet** of propositional logic consists of (1) atomic symbols and (2) operator symbols.
- \* **Atomic formula** (**atom**) is an irreducible formula without logical connectives.
  - Propositional **variables**:  $A, B, C, \dots, Z$ . With indices, if needed:  $A_1, A_2, \dots, Z_1, Z_2, \dots$
  - Logical **constants**:  $\top$  for always true proposition (*tautology*),  $\perp$  for always false proposition (*contradiction*).
- \* **Logical connectives** (**operators**):

Type	Natural meaning	Symbolization
Negation	It is not the case that $\mathcal{P}$ . It is false that $\mathcal{P}$ . It is not true that $\mathcal{P}$ .	$\neg \mathcal{P}$
Conjunction	Both $\mathcal{P}$ and $\mathcal{Q}$ . $\mathcal{P}$ but $\mathcal{Q}$ . $\mathcal{P}$ , although $\mathcal{Q}$ .	$\mathcal{P} \wedge \mathcal{Q}$
Disjunction	Either $\mathcal{P}$ or $\mathcal{Q}$ (or both). $\mathcal{P}$ unless $\mathcal{Q}$ .	$\mathcal{P} \vee \mathcal{Q}$
Exclusive or (Xor)	Either $\mathcal{P}$ or $\mathcal{Q}$ (but not both). $\mathcal{P}$ xor $\mathcal{Q}$ .	$\mathcal{P} \oplus \mathcal{Q}$
Implication (Conditional)	If $\mathcal{P}$ , then $\mathcal{Q}$ . $\mathcal{P}$ only if $\mathcal{Q}$ . $\mathcal{Q}$ if $\mathcal{P}$ .	$\mathcal{P} \rightarrow \mathcal{Q}$
Biconditional	$\mathcal{P}$ , if and only if $\mathcal{Q}$ . $\mathcal{P}$ iff $\mathcal{Q}$ . $\mathcal{P}$ just in case $\mathcal{Q}$ .	$\mathcal{P} \leftrightarrow \mathcal{Q}$

- \* **Sentence** of propositional logic is defined inductively: Well-formed formula (WFF)
  1. Every propositional variable/constant is a sentence.
  2. If  $\mathcal{A}$  is a sentence, then  $\neg \mathcal{A}$  is a sentence.
  3. If  $\mathcal{A}$  and  $\mathcal{B}$  are sentences, then  $(\mathcal{A} \wedge \mathcal{B})$ ,  $(\mathcal{A} \vee \mathcal{B})$ ,  $(\mathcal{A} \rightarrow \mathcal{B})$ ,  $(\mathcal{A} \leftrightarrow \mathcal{B})$  are sentences.
  4. Nothing else is a sentence.

- \* Well-formed formulae grammar: Backus-Naur form (BNF)

$\langle \text{sentence} \rangle ::= \langle \text{constant} \rangle$   
                   |  $\langle \text{variable} \rangle$   
                   |  $\neg \langle \text{sentence} \rangle$   
                   |  $\langle ' \rangle \langle \text{sentence} \rangle \langle \text{binop} \rangle \langle \text{sentence} \rangle \langle ' \rangle$   
 $\langle \text{constant} \rangle ::= \top \mid \perp$   
 $\langle \text{variable} \rangle ::= A \mid \dots \mid Z \mid A_1 \mid \dots \mid Z_n$   
 $\langle \text{binop} \rangle ::= \wedge \mid \vee \mid \oplus \mid \rightarrow \mid \leftarrow \mid \leftrightarrow$

- \* **Literal** is a propositional variable or its negation:  $\mathcal{L}_i = X_i$  (*positive literal*),  $\mathcal{L}_j = \neg X_j$  (*negative literal*).
- \* **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}_{\text{premises}} \quad \therefore \quad \underbrace{C}_{\text{conclusion}}$$

“therefore”

- \* An argument is **valid** if whenever all the premises are true, the conclusion is also true. Validity
- \* An argument is **invalid** if there is a case (*a counterexample*) when all the premises are true, but the conclusion is false.

## 4.2 Semantics of Propositional Logic

- \* **Valuation** is any assignment of truth values to propositional variables. Interpretation
- \*  $\mathcal{A}$  is a **tautology** (valid) iff it is true on *every* valuation. Might be symbolized as “ $\models \mathcal{A}$ ”.
- \*  $\mathcal{A}$  is a **contradiction** iff it is false on *every* valuation. Might be symbolized as “ $\mathcal{A} \models$ ”.
- \*  $\mathcal{A}$  is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- \*  $\mathcal{A}$  is **satisfiable** iff it is true on *some* valuation. Satisfiability
- \*  $\mathcal{A}$  is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. Falsifiability
- \*  $\mathcal{A}$  and  $\mathcal{B}$  are **equivalent** (symbolized as  $\mathcal{A} \equiv \mathcal{B}$ ) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- \*  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  are **consistent (jointly satisfiable)** iff there is *some* valuation which makes them all true. Sentences are **inconsistent (jointly unsatisfiable)** iff there is *no* valuation that makes them all true. Consistency
- \* The sentences  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  **entail** the sentence  $C$  (symbolized as  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$ ) if there is no valuation which makes all of  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$  true and  $C$  false. Semantic entailment
- \* If  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$ , then the argument  $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \therefore C$  is **valid**.

Validity check examples:

$A$	$B$	$A \rightarrow B$	$A$	$\therefore$	$B$	$\neg A \rightarrow \neg B$	$\therefore$	$B \rightarrow A$	$A \rightarrow B$	$B$	$\therefore$	$\neg (B \rightarrow A)$			
0	0	1	0	·	0	1	1	✓	1	1	0	·	0	1	
0	1	1	0	·	1	1	0	0	1	1	✓	1	1	0	
1	0	0	1	·	0	0	1	1	✓	1	0	0	·	0	1
1	1	1	1	✓	1	0	1	0	✓	1	1	1	✗	0	1

$R$	$S$	$T$	$R \vee S$	$S \vee T$	$\neg R$	$\therefore$	$S \wedge T$	$(R \wedge S) \rightarrow T$	$\therefore$	$R \rightarrow (S \rightarrow T)$
0	0	0	0	0	1	.	0	0	1	0
0	0	1	0	1	1	.	0	0	1	1
0	1	0	1	1	1	✗	0	0	1	0
0	1	1	1	1	1	✓	1	0	1	1
1	0	0	1	0	0	.	0	0	1	1
1	0	1	1	1	0	.	0	0	1	1
1	1	0	1	1	0	.	0	1	0	0
1	1	1	1	1	0	.	1	1	1	1

- \* **Soundness**:  $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$  “Every provable statement is in fact true”
- \* **Completeness**:  $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$  “Every true statement has a proof”

## 4.3 Natural Deduction Rules

## Reiteration

$m$	$\mathcal{A}$	
$\therefore$	$\mathcal{A}$	R $m$

## Explosion

$m$	$\perp$	
$\therefore$	$\mathcal{A}$	X $m$

## Conditional

$i$	$\mathcal{A}$	
$j$	$\mathcal{B}$	
$\therefore$	$\mathcal{A} \rightarrow \mathcal{B}$	$\rightarrow$ I $i-j$

## Modus ponens

$i$	$\mathcal{A} \rightarrow \mathcal{B}$	
$j$	$\mathcal{A}$	
$\therefore$	$\mathcal{B}$	MP $i, j$

## Conjunction

$i$	$\mathcal{A}$	
$j$	$\mathcal{B}$	
$\therefore$	$\mathcal{A} \wedge \mathcal{B}$	$\wedge$ I $i, j$

$m$	$\mathcal{A} \wedge \mathcal{B}$	
$\therefore$	$\mathcal{A}$	$\wedge$ E $m$
$\therefore$	$\mathcal{B}$	$\wedge$ E $m$

## Contraposition

$m$	$\mathcal{A} \rightarrow \mathcal{B}$	
$\therefore$	$\neg \mathcal{B} \rightarrow \neg \mathcal{A}$	Contra $m$

## Modus tollens

$i$	$\mathcal{A} \rightarrow \mathcal{B}$	
$j$	$\neg \mathcal{B}$	
$\therefore$	$\neg \mathcal{A}$	MT $i, j$

## Biconditional

$i$	$\mathcal{A}$	
$j$	$\mathcal{B}$	
$k$	$\mathcal{B}$	
$l$	$\mathcal{A}$	
$\therefore$	$\mathcal{A} \leftrightarrow \mathcal{B}$	$\leftrightarrow$ I $i-j, k-l$

## Negation

$i$	$\neg \mathcal{A}$	
$j$	$\mathcal{A}$	
$\therefore$	$\perp$	$\neg$ E $i, j$

$i$	$\mathcal{A}$	
$j$	$\perp$	
$\therefore$	$\neg \mathcal{A}$	$\neg$ I $i-j$

## Disjunction

$m$	$\mathcal{A}$	
$\therefore$	$\mathcal{A} \vee \mathcal{B}$	$\vee$ I $m$

$m$	$\mathcal{A}$	
$\therefore$	$\mathcal{B} \vee \mathcal{A}$	$\vee$ I $m$

$m$	$\mathcal{A} \vee \mathcal{B}$	
$i$	$\mathcal{A}$	
$j$	$\mathcal{C}$	
$k$	$\mathcal{B}$	
$l$	$\mathcal{C}$	
$\therefore$	$\mathcal{C}$	$\vee$ E $m, i-j, k-l$

$i$	$\mathcal{A} \leftrightarrow \mathcal{B}$	
$j$	$\mathcal{A}$	
$\therefore$	$\mathcal{B}$	$\leftrightarrow$ E $i, j$

$i$	$\mathcal{A} \leftrightarrow \mathcal{B}$	
$j$	$\mathcal{B}$	
$\therefore$	$\mathcal{A}$	$\leftrightarrow$ E $i, j$

## Indirect proof

$i$	$\neg \mathcal{A}$	
$j$	$\perp$	
$\therefore$	$\mathcal{A}$	IP $i-j$

## Disjunctive syllogism

$i$	$\mathcal{A} \vee \mathcal{B}$	
$j$	$\neg \mathcal{A}$	
$\therefore$	$\mathcal{B}$	DS $i, j$

$i$	$\mathcal{A} \vee \mathcal{B}$	
$j$	$\neg \mathcal{B}$	
$\therefore$	$\mathcal{A}$	DS $i, j$

## De Morgan Rules

$m$	$\neg(\mathcal{A} \vee \mathcal{B})$	
$\therefore$	$\neg \mathcal{A} \wedge \neg \mathcal{B}$	DeM $m$

$m$	$\neg \mathcal{A} \wedge \neg \mathcal{B}$	
$\therefore$	$\neg(\mathcal{A} \vee \mathcal{B})$	DeM $m$

$m$	$\neg(\mathcal{A} \wedge \mathcal{B})$	
$\therefore$	$\neg \mathcal{A} \vee \neg \mathcal{B}$	DeM $m$

$m$	$\neg \mathcal{A} \vee \neg \mathcal{B}$	
$\therefore$	$\neg(\mathcal{A} \wedge \mathcal{B})$	DeM $m$

## Double negation

$m$	$\neg \neg \mathcal{A}$	
$\therefore$	$\mathcal{A}$	$\neg \neg$ E $m$

## Law of excluded middle

$i$	$\mathcal{A}$	
$j$	$\mathcal{B}$	
$k$	$\neg \mathcal{A}$	
$l$	$\mathcal{B}$	
$\therefore$	$\mathcal{B}$	LEM $i-j, k-l$

## Hypothetical syllogism

$i$	$\mathcal{A} \rightarrow \mathcal{B}$	
$j$	$\mathcal{B} \rightarrow \mathcal{C}$	
$\therefore$	$\mathcal{A} \rightarrow \mathcal{C}$	HS $i, j$

Green: basic rules.

Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).