4 Formal Logic Cheatsheet

4.1 Propositional Logic

* **Proposition** is a statement which can be either true or false. Truth-bearer
* **Alphabet** of propositional logic consists of (1) atomic symbols and (2) operator symbols.
* **Atomic formula** (**atom**) is an irreducible formula without logical connectives.
  * Propositional **variables**: $A, B, C, \dots, Z$. With indices, if needed: $A_1, A_2, \dots, Z_1, Z_2, \dots$.
  * Logical **constants**: $\top$ for always true proposition (*tautology*), $\bot$ for always false proposition (*contradiction*).
* **Logical connectives** (**operators**):

Type	Natural meaning	Symbolization
Negation	It is not the case that $\mathcal{P}$. It is false that $\mathcal{P}$. It is not true that $\mathcal{P}$.	$\neg\mathcal{P}$
Conjunction	Both $\mathcal{P}$ and $\mathcal{Q}$. $\mathcal{P}$ but $\mathcal{Q}$. $\mathcal{P}$, although $\mathcal{Q}$.	$\mathcal{P} \land \mathcal{Q}$
Disjunction	Either $\mathcal{P}$ or $\mathcal{Q}$ (or both). $\mathcal{P}$ unless $\mathcal{Q}$.	$\mathcal{P} \lor \mathcal{Q}$
Exclusive or (Xor)	Either $\mathcal{P}$ or $\mathcal{Q}$ (but not both). $\mathcal{P}$ xor $\mathcal{Q}$.	$\mathcal{P} \oplus \mathcal{Q}$
Implication (Conditional)	If $\mathcal{P}$, then $\mathcal{Q}$. $\mathcal{P}$ only if $\mathcal{Q}$. $\mathcal{Q}$ if $\mathcal{P}$.	$\mathcal{P} \to \mathcal{Q}$
Biconditional	$\mathcal{P}$, if and only if $\mathcal{Q}$. $\mathcal{P}$ iff $\mathcal{Q}$. $\mathcal{P}$ just in case $\mathcal{Q}$.	$\mathcal{P} \leftrightarrow \mathcal{Q}$

* **Sentence** of propositional logic is defined inductively: Well-formed formula (WFF)
  1. Every propositional variable/constant is a sentence.
  2. If $\mathcal{A}$ is a sentence, then $\neg\mathcal{A}$ is a sentence.
  3. If $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $(\mathcal{A} \land \mathcal{B})$, $(\mathcal{A} \lor \mathcal{B})$, $(\mathcal{A} \to \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
  4. Nothing else is a sentence.
* Well-formed formulae grammar: Backus-Naur form (BNF)

$$
\begin{array}{lcl}
\langle \textit{sentence} \rangle & ::= & \langle \textit{constant} \rangle \\
 & \mid & \langle \textit{variable} \rangle \\
 & \mid & \neg \, \langle \textit{sentence} \rangle \\
 & \mid & \text{`('} \, \langle \textit{sentence} \rangle \, \langle \textit{binop} \rangle \, \langle \textit{sentence} \rangle \, \text{`)'} \\
\langle \textit{constant} \rangle & ::= & \top \mid \bot \\
\langle \textit{variable} \rangle & ::= & A \mid \dots \mid Z \mid A_1 \mid \dots \mid Z_n \\
\langle \textit{binop} \rangle & ::= & \land \mid \lor \mid \oplus \mid \to \mid \leftarrow \mid \leftrightarrow
\end{array}
$$

* **Literal** is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (*positive* literal), $\mathcal{L}_j = \neg X_j$ (*negative* literal).
* **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n}_{\textit{premises}} \;\therefore\; \underbrace{C}_{\textit{conclusion}}$$

"therefore"

* An argument is **valid** if whenever all the premises are true, the conclusion is also true. Validity
* An argument is **invalid** if there is a case (*a counterexample)* when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

- **Valuation** is any assignment of truth values to propositional variables. Interpretation
- $\mathcal{A}$ is a **tautology** (**valid**) iff it is true on *every* valuation. Might be symbolized as "$\vDash \mathcal{A}$".
- $\mathcal{A}$ is a **contradiction** iff it is false on *every* valuation. Might be symbolized as "$\mathcal{A} \vDash$ ".
- $\mathcal{A}$ is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- $\mathcal{A}$ is **satisfiable** iff it is true on *some* valuation. Satisfiability
- $\mathcal{A}$ is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. Falsifiability
- $\mathcal{A}$ and $\mathcal{B}$ are **equivalent** (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. Consistency
- The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vDash C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false. Semantic entailment
- If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \vDash C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \therefore C$ is **valid**.

  Validity check examples:

A	B	$A \to B$	A	$\overset{valid}{\therefore}$	B	$\neg A$	$\to$	$\neg B$	$\overset{valid}{\therefore}$	$B \to A$	$A \to B$	B	$\overset{invalid}{\therefore}$	$\neg$	$(B \to A)$
0	0	1	0	·	0	1	1	1	✓	1	1	0	·	0	1
0	1	1	0	·	1	1	0	0	·	0	1	1	✓	1	0
1	0	0	1	·	0	0	1	1	✓	1	0	0	·	0	1
1	1	1	1	✓	1	0	1	0	✓	1	1	1	✗	0	1

R	S	T	$R \lor S$	$S \lor T$	$\neg R$	$\overset{invalid}{\therefore}$	$S \land T$	$(R \land S)$	$\to$	T	$\overset{valid}{\therefore}$	R	$\to$	$(S \to T)$
0	0	0	0	0	1	·	0	0	1	0	✓	0	1	1
0	0	1	0	1	1	·	0	0	1	1	✓	0	1	1
0	1	0	1	1	1	✗	0	0	1	0	✓	0	1	0
0	1	1	1	1	1	✓	1	0	1	1	✓	0	1	1
1	0	0	1	0	0	·	0	0	1	0	✓	1	1	1
1	0	1	1	1	0	·	0	0	1	1	✓	1	1	1
1	1	0	1	1	0	·	0	1	0	0	·	1	0	0
1	1	1	1	1	0	·	1	1	1	1	✓	1	1	1

- **Soundness**: $\Gamma \vdash \mathcal{A} \to \Gamma \vDash \mathcal{A}$ "Every provable statement is in fact true"
- **Completeness:** $\Gamma \vDash \mathcal{A} \to \Gamma \vdash \mathcal{A}$ "Every true statement has a proof"

4.3 Natural Deduction Rules

Reiteration

m	$\mathcal{A}$	
$\therefore$	$\mathcal{A}$	R m

Modus ponens

i	$\mathcal{A} \to \mathcal{B}$	
j	$\mathcal{A}$	
$\therefore$	$\mathcal{B}$	MP i, j

Modus tollens

i	$\mathcal{A} \to \mathcal{B}$	
j	$\neg\mathcal{B}$	
$\therefore$	$\neg\mathcal{A}$	MT i, j

Negation

i	$\neg\mathcal{A}$	
j	$\mathcal{A}$	
$\therefore$	$\bot$	$\neg$E i, j

i	$\quad \mathcal{A}$	
j	$\quad \bot$	
$\therefore$	$\neg\mathcal{A}$	$\neg$I i–j

Indirect proof

i	$\quad \neg\mathcal{A}$	
j	$\quad \bot$	
$\therefore$	$\mathcal{A}$	IP i–j

Double negation

m	$\neg\neg\mathcal{A}$	
$\therefore$	$\mathcal{A}$	$\neg\neg$E m

Law of excluded middle

i	$\quad \mathcal{A}$	
j	$\quad \mathcal{B}$	
k	$\quad \neg\mathcal{A}$	
l	$\quad \mathcal{B}$	
$\therefore$	$\mathcal{B}$	LEM i–j, k–l

Explosion

m	$\bot$	
$\therefore$	$\mathcal{A}$	X m

Conjunction

i	$\mathcal{A}$	
j	$\mathcal{B}$	
$\therefore$	$\mathcal{A} \land \mathcal{B}$	$\land$I i, j

m	$\mathcal{A} \land \mathcal{B}$	
$\therefore$	$\mathcal{A}$	$\land$E m
$\therefore$	$\mathcal{B}$	$\land$E m

Disjunction

m	$\mathcal{A}$	
$\therefore$	$\mathcal{A} \lor \mathcal{B}$	$\lor$I m

m	$\mathcal{A}$	
$\therefore$	$\mathcal{B} \lor \mathcal{A}$	$\lor$I m

m	$\mathcal{A} \lor \mathcal{B}$	
i	$\quad \mathcal{A}$	
j	$\quad \mathcal{C}$	
k	$\quad \mathcal{B}$	
l	$\quad \mathcal{C}$	
$\therefore$	$\mathcal{C}$	$\lor$E m, i–j, k–l

Disjunctive syllogism

i	$\mathcal{A} \lor \mathcal{B}$	
j	$\neg\mathcal{A}$	
$\therefore$	$\mathcal{B}$	DS i, j

i	$\mathcal{A} \lor \mathcal{B}$	
j	$\neg\mathcal{B}$	
$\therefore$	$\mathcal{A}$	DS i, j

Hypothetical syllogism

i	$\mathcal{A} \to \mathcal{B}$	
j	$\mathcal{B} \to \mathcal{C}$	
$\therefore$	$\mathcal{A} \to \mathcal{C}$	HS i, j

Conditional

i	$\quad \mathcal{A}$	
j	$\quad \mathcal{B}$	
$\therefore$	$\mathcal{A} \to \mathcal{B}$	$\to$I i–j

Contraposition

m	$\mathcal{A} \to \mathcal{B}$	
$\therefore$	$\neg\mathcal{B} \to \neg\mathcal{A}$	Contra m

Biconditional

i	$\quad \mathcal{A}$	
j	$\quad \mathcal{B}$	
k	$\quad \mathcal{B}$	
l	$\quad \mathcal{A}$	
$\therefore$	$\mathcal{A} \leftrightarrow \mathcal{B}$	$\leftrightarrow$I i–j, k–l

i	$\mathcal{A} \leftrightarrow \mathcal{B}$	
j	$\mathcal{A}$	
$\therefore$	$\mathcal{B}$	$\leftrightarrow$E i, j

i	$\mathcal{A} \leftrightarrow \mathcal{B}$	
j	$\mathcal{B}$	
$\therefore$	$\mathcal{A}$	$\leftrightarrow$E i, j

De Morgan Rules

m	$\neg(\mathcal{A} \lor \mathcal{B})$	
$\therefore$	$\neg\mathcal{A} \land \neg\mathcal{B}$	DeM m

m	$\neg\mathcal{A} \land \neg\mathcal{B}$	
$\therefore$	$\neg(\mathcal{A} \lor \mathcal{B})$	DeM m

m	$\neg(\mathcal{A} \land \mathcal{B})$	
$\therefore$	$\neg\mathcal{A} \lor \neg\mathcal{B}$	DeM m

m	$\neg\mathcal{A} \lor \neg\mathcal{B}$	
$\therefore$	$\neg(\mathcal{A} \land \mathcal{B})$	DeM m

Green: basic rules.
Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).