

In der Mathematik ist die Kunst Fragen zu stellen wertvoller als Probleme zu lösen

— Georg Cantor

1. For each given relation $R_i \subseteq M_i^2$, determine whether it is *reflexive*, *irreflexive*, *coreflexive*, *symmetric*, *antisymmetric*, *asymmetric*, *transitive*, *left/right Euclidean*, *connex*. Provide a counterexample for each non-complying property (e.g., “transitivity does not hold for $x, y, z = (3, 1, 2)$ ”). Organize your answer in a table (e.g., columns—relations, rows—properties).

(a) $M_1 = \mathbb{R}$ $x R_1 y \leftrightarrow x - y \leq 1$	(c) $M_3 = \{a, b, c, d\}$ $\ R_3\ = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
(b) $M_2 = \mathcal{P}(\{a, b, c\})$ $R_2 = “\subseteq”$	(d) $M_4 = \{\text{“rock”, “scissors”, “paper”}\}$ $R_4 = \{\langle x, y \rangle \mid x \text{ beats } y\}$

2. Prove (rigorously) or disprove (by providing a counterexample) the following statements about arbitrary homogeneous relations $R \subseteq M^2$ and $S \subseteq M^2$:

- | | |
|---|---|
| (a) If R and S are <i>reflexive</i> , then $R \cap S$ is so. | (d) If R and S are <i>reflexive</i> , then $R \cup S$ is so. |
| (b) If R and S are <i>symmetric</i> , then $R \cap S$ is so. | (e) If R and S are <i>symmetric</i> , then $R \cup S$ is so. |
| (c) If R and S are <i>transitive</i> , then $R \cap S$ is so. | (f) If R and S are <i>transitive</i> , then $R \cup S$ is so. |

3. An equinumerosity relation \sim over sets is defined as follows: $A \sim B \leftrightarrow |A| = |B|$.

- (a) Show that \sim is an equivalence relation over finite sets.
 (b) Show that \sim is an equivalence relation over infinite sets¹.
 (c) Find the quotient set of $\mathcal{P}(\{a, b, c, d\})$ by \sim .

4. Let R_θ be a relation of θ -similarity (clearly, $\theta \in [0; 1] \subseteq \mathbb{R}$) of finite non-empty sets defined as follows: a set A is said to be θ -similar to B iff the Jaccard index $\text{Jac}(A, B) = \frac{|A \cap B|}{|A \cup B|}$ for these sets is at least θ , i.e. $\langle A, B \rangle \in R_\theta \leftrightarrow \text{Jac}(A, B) \geq \theta$.

- (a) Determine whether θ -similarity is a tolerance relation².
 (b) Determine whether θ -similarity is an equivalence relation.
 (c) Draw the graph of a relation $R_\theta \subseteq \{A_i\}^2$, where $\theta = 0.25$, $A_1 = \{1, 2, 5, 6\}$, $A_2 = \{2, 3, 4, 5, 7, 9\}$, $A_3 = \{1, 4, 5, 6\}$, $A_4 = \{3, 7, 9\}$, $A_5 = \{1, 5, 6, 8, 9\}$.

5. Any binary relation $R \subseteq M^2$ can be represented as a zero-one matrix $\|R\| = [r_{ij}]$, where the element r_{ij} is equal to 1 if $\langle m_i, m_j \rangle \in R$ and 0 otherwise. Boolean product of two square matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is a matrix $C = A \odot B = [c_{ij}]$ defined as follows: $c_{ij} = \bigvee_k (a_{ik} \wedge b_{kj})$. A composition of relations R and S is a relation $S \circ R$ defined as follows: $\langle a, b \rangle \in S \circ R \leftrightarrow \exists c : \langle a, c \rangle \in R \wedge \langle c, b \rangle \in S$. Show that the matrix representation of the composition of relations R and S is equal to the Boolean product of the corresponding matrices, i.e. $\|S \circ R\| = \|R\| \odot \|S\|$.

6. Find the error in the “proof” of the following “theorem”.

“Theorem”: Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.

“Proof”: Let $a \in A$. Take an element $b \in A$ such that $\langle a, b \rangle \in R$. Because R is symmetric, we also have $\langle b, a \rangle \in R$. Now using the transitive property, we can conclude that $\langle a, a \rangle \in R$ because $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$.

7. Give an example of a relation R on the set $\{a, b, c\}$ such that the symmetric closure of the reflexive closure of the transitive closure of R is not transitive.

¹ For infinite sets, $|A| = |B|$ means there is a bijection between A and B .

² A tolerance relation is a *reflexive* and *symmetric* relation.

8. Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either *red* or *blue* so that $\langle C_1, C_2 \rangle$, where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.
 - (a) Show that R is an equivalence relation.
 - (b) What are the equivalence classes of R ?
9. Consider two relations $R \subseteq A \times B$ and $S \subseteq B \times C$. Prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.
10. Prove or disprove the following statements about the functions f and g :
 - (a) If f and g are injections, then $g \circ f$ is also an injection.
 - (b) If f and g are surjections, then $g \circ f$ is also a surjection.
 - (c) If f and $f \circ g$ are injections, then g is also an injection.
 - (d) If f and $f \circ g$ are surjections, then g is also a surjection.
11. Let $H = \{1, 2, 4, 5, 10, 12, 20\}$. Consider a divisibility relation $R \subseteq H^2$ defined as follows: $x R y \leftrightarrow y : x$.
 - (a) Sort R (as a set of pairs) lexicographically³.
 - (b) Show that R is a partial order.
 - (c) Determine whether R is a linear (total) order.
 - (d) Draw the Hasse diagram for a graded poset $\langle H, R, \rho \rangle$, where $\rho: H \rightarrow \mathbb{N}_0$ is a grading function which maps a number $n \in H$ to the sum of all exponents appearing in its prime factorization, e.g., $\rho(20) = \rho(2^2 \cdot 5^1) = 2 + 1 = 3$, so the number 20 would have the 3rd rank (bottom-up).
 - (e) Find the minimal, minimum (least), maximal and maximum (greatest) elements in the poset $\langle H, R \rangle$. If there are multiple or none, explain why.
 - (f) Perform a topological sort⁴ of the poset $\langle H, R \rangle$.
12. Prove that the transitive closure R^+ is in fact transitive.

Definition. $R^+ = \bigcup_{n \in \mathbb{N}^+} R^n$ is a transitive closure of $R \subseteq M^2$, where

 - * $R^{k+1} = R^k \circ R$ is a compositional (functional) power⁴,
 - * $R^1 = R$,
 - * $S \circ R = \{\langle x, y \rangle \mid \exists z : (x R z) \wedge (z S y)\}$ is a composition (relative product) of relations R and S .
13. Given a set S and two partitions P_1 and P_2 of S , we define the relation $P_1 \preceq P_2$ as follows: partition P_1 is considered a *refinement* of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2 . Show that the set of all partitions of a set S with the refinement relation \preceq is a lattice.
14. A poset $\langle R, \preceq \rangle$ is *well-founded* if there is no infinite decreasing sequence of elements in the poset, that is, elements x_1, x_2, \dots, x_n such that $\dots \prec x_n \prec \dots \prec x_2 \prec x_1$. Determine whether the set of strings of lowercase English letters with lexicographic order is well-founded.

³ Lexicographic order for pairs: $\langle a, b \rangle \preceq \langle a', b' \rangle \leftrightarrow (a < a') \vee ((a = a') \wedge (b \leq b'))$. For example, $\langle 1, 2 \rangle \preceq \langle 1, 3 \rangle \preceq \langle 2, 1 \rangle$.

⁴ Note: this is *not* a Cartesian power, despite of the same notation R^n . Another possible notation for compositional power is $R^{\circ n}$, but it is too wild to use it here.