Homework #6 — **Discrete M∀th**
Combinatorics — **Spring 2022**

Do whatever you want, but always explain what you are doing.

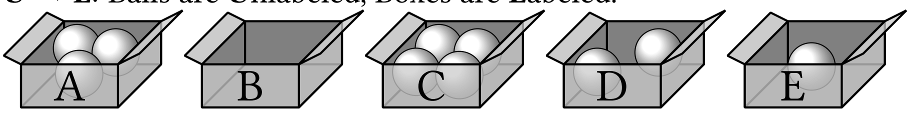
— Konstantin, 2020

1. Find the number of different 5-digit numbers using digits 1–9 under the given constraints. For each case, provide representative examples of (non-)complying numbers (*e.g.*, 12345 and 52814 are suitable for (b), but 44521 and 935 are not) and derive a generic[1] formula. Try to express the formula using standard combinatorial terms, *e.g.*, k-combs C_n^k and k-perms $P(n, k)$.
   (a) Digits *can* be repeated.
   (b) Digits *cannot* be repeated.
   (c) Digits *can* be repeated and must be written in *non-increasing*[2] order.
   (d) Digits *cannot* be repeated and must be written in *strictly increasing* order.
   (e) Digits *can* be repeated, must be written in *non-decreasing* order, and the 4th digit must be 6.

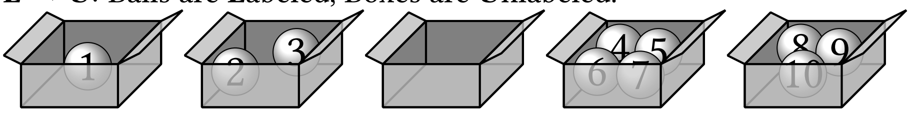
2. One of the classical combinatorial problems is counting the number of arrangements of n balls into k boxes. There are at least 12 variations of this problem: four cases (a–d) with three different constraints (1–3). For each problem (case+constraint), derive the corresponding generic formula. Additionally, pick several (representative) values for n and k and use your derived formulae to find the numbers of arrangements. Visualize several possible arrangements for the chosen n and k.

   Cases with arrangement examples:

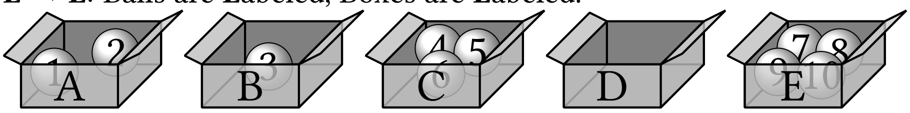
   a. **U** $\longrightarrow$ **L**: Balls are **U**nlabeled, Boxes are **L**abeled.



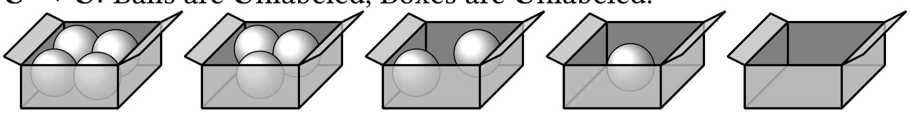
   b. **L** $\longrightarrow$ **U**: Balls are **L**abeled, Boxes are **U**nlabeled.



   c. **L** $\longrightarrow$ **L**: Balls are **L**abeled, Boxes are **L**abeled.



   d. **U** $\longrightarrow$ **U**: Balls are **U**nlabeled, Boxes are **U**nlabeled.



   Constraints:

   1. ≤ 1 ball per box — *injective* mapping.
   2. ≥ 1 ball per box — *surjective* mapping.
   3. Arbitrary number of balls per box.

   Notes:

   * **U**nlabeled means "indistinguishable", and **L**abeled means "distinguishable".
   * Stirling number of the second kind $s_k^{II}(n)$ — number of ways to partition a set of n elements into k non-empty subsets. Use $s_k^{II}(n)$ directly without expanding the closed formula.
   * Partition function $p_k(n)$ — number of ways to partition the integer n into k positive parts, *i.e.* $n = a_1 + \cdots + a_k$, where $a_1 \geq \cdots \geq a_k \geq 1$. Use $p_k(n)$ directly.

[1] Here, "generic formula" means "depending on the input data". In this particular example, $n = 9$ and $k = 5$, but the sought formula must also be valid for all other (adequate) values of n and k.

[2] A sequence (x_n) is said to be *strictly monotonically increasing* if each term is *strictly greater* than the previous one, *i.e.* $x_i < x_{i+1}$. A sequence (x_n) is called *non-increasing* if each term is *less than or equal* to the previous one, *i.e.* $x_i \geq x_{i+1}$.

3. Proof the Generalized Pascal's Formula (for $n \geq 1$ and $k_1, \ldots, k_r \geq 0$ with $k_1 + \cdots + k_r = n$):

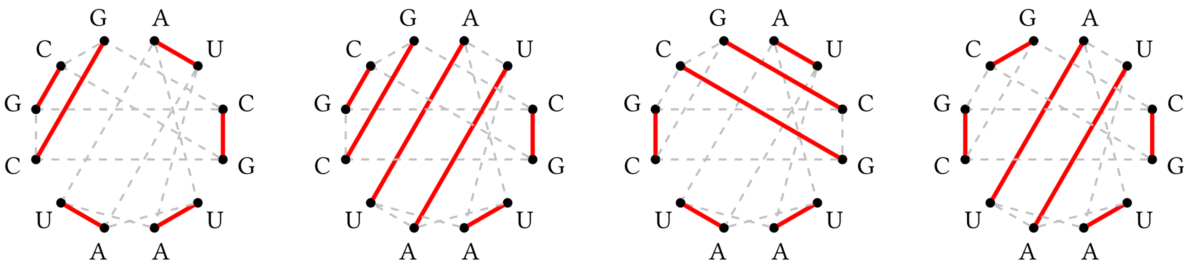
$$\binom{n}{k_1, \ldots, k_r} = \sum_{i=1}^{r} \binom{n-1}{k_1, \ldots, k_i - 1, \ldots, k_r}$$

Count the number of permutations of a multiset $\Sigma^* = \{2 \cdot \triangle, 3 \cdot \square, 1 \cdot$ 🐈$\}$ using GPF.

4. A *non-crossing perfect matching*[3] in a graph is a set of pairwise disjoint edges that cover all vertices and do not intersect with each other. For example, consider a graph on $2n$ vertices numbered from 1 to $2n$ and arranged in a circle. Additionally, assume that edges are straight lines. In this case, edges $\{i, j\}$ and $\{a, b\}$ intersect whenever $i < a < j < b$.

   (a) Count the number of all possible non-crossing perfect matchings in a complete graph K_{2n}.

   (b) Consider a graph on vertices labeled with letters from {`A`, `C`, `G`, `U`}. Each pair of vertices labeled with `A` and `U` is connected with a *basepair edge*. Similarly, `C`–`G` pairs are also connected.
   The picture below illustrates some of possible non-crossing perfect matchings in the graph with 12 vertices `AUCGUAAUCGCG` arranged in a circle. Basepair edges are drawn dashed gray, matching is red.



   Count the number of all possible non-crossing perfect matchings in the graph on 20 vertices arranged in a circle and labeled with `CGUAAUUACGGCAUUAGCAU`.

[3] Credits to Rosalind for this task.