### **Boolean Algebra Cheatsheet** 3

## 3.1 Definitions

- \* Boolean function is a function of the form  $f: \mathbb{B}^n \to \mathbb{B}$ , where  $n \ge 0$  is the arity of the function and  $\mathbb{B} = \{0, 1\} = \{\bot, \top\} = \{F, T\}$  is a Boolean domain.
- \* There are multiple ways to represent a Boolean function (all examples represent the same function):
  - 1. Truth table, *e.g.*, f = (1010), where LSB corresponds to 1, MSB to 0.

Least/Most Significant Bit

2. Analytically (as a sentence of propositional logic), e.g.,  $f(A, B) = \neg B$ .

Propositional logic

3. Sum of minterms, e.g.,  $f = \sum m(0, 2) = m_0 + m_2$ .

**Minterms** 

**Maxterms** 

4. Product of maxterms, e.g.,  $f = \prod M(1,3) = M_1 \cdot M_3$ .

5. Boolean function number  $^{e}$ , e.g.,  $f_{10}^{(2)}$  is the 10-th 2-ary function. Note that Wolfram's "Boolean operator number" is a slightly different term, which uses the reversed truth table. 10-th Boolean function  $f_{10}^{(2)}$  with the truth table (1010) can be obtained via the query "5th Boolean function of 2 variables" (note: not 10th!) in WolframAlpha<sup>[2]</sup>, since  $rev(1010_2) = 0101_2 = 5_{10}$ .

## 3.2 Normal Forms

- \* Disjunctive forms:
  - **Cube** is a conjunction of literals:  $\mathcal{T} = \bigwedge_i \mathcal{L}_i$ .
  - Formula is in **disjunctive normal form (DNF)** if it is a disjunction of terms: DNF =  $\bigvee_i \mathcal{T}_{i}$ .
  - **Minterm** is conjunction of literals, where *each* variable appears *once*, *e.g.*,  $m_6 = (A \land B \land \neg C)$ .
  - Formula is in **canonical DNF (CDNF)** if it is a disjunction of minterms: CDNF =  $\bigvee_i m_i$ .
- \* Conjunctive forms:
  - Clause  $^{\mathbb{Z}}$  is a disjunction of literals:  $C = \bigvee_i \mathcal{L}_i$
  - Formula is in **conjunctive normal form** (CNF)  $^{\text{LC}}$  if it is a conjunction of clauses: CNF =  $\bigwedge_i C_i$ .
  - **Maxterm** is disjunction of literals, where *each* variable appears *once*, *e.g.*,  $M_6 = (\neg A \lor \neg B \lor C)$ .
  - Formula is in **canonical CNF** (CCNF) if it is a conjunction of maxterms: CCNF =  $\bigwedge_i M_i$ .
- \* Some other normal forms:
  - Formula is in **negation normal form (NNF)** if the negation operator  $(\neg)$  is only applied to variables and the only other allowed Boolean operators are conjunction ( $\wedge$ ) and disjunction ( $\vee$ ).
  - Formula f is in **Blake canonical form (BCF)** if it is a disjunction of all the prime implicants of f.
  - Formula is in **prenex normal form (PNF)** if it consists of *prefix* quantifiers and bound variables, and *matrix* quatifier-free part.
  - ∘ Formula is in **Skolem normal form (SNF)** if it is in prenex normal form with only universal first-order quantifiers.
  - **Zhegalkin polynomial** is a formula in the following form (**algebraic normal form (ANF)**):
    - $f(X_1, \ldots, X_n) = a_0 \bigoplus_{\substack{1 \le i_1 \le \cdots \le i_k \le n \\ 1 \le k \le n}} (a_{i_1, \ldots, i_k} \wedge X_{i_1} \wedge \cdots \wedge X_{i_k})$ , where  $a_0, a_{i_1, \ldots, i_k} \in \mathbb{B}$
    - $f(x_1,\ldots,x_n)=a_0\oplus (\overline{a_1x_1}\oplus\cdots\oplus a_nx_n)\oplus (a_{1,2}x_1x_2\oplus\cdots\oplus a_{n-1,n}x_{n-1}x_n)\oplus\cdots\oplus a_{1,\ldots,n}x_1\ldots x_n$

## 3.3 Conversion to CNF/DNF

In order to convert arbitrary (i.e. any) Boolean formula to equivalent CNF/DNF:

1. Eliminate equivalences, implications and other "non-standard" operations (i.e. rewrite using only  $\{\land,\lor,\neg\}$ ):

$$\begin{array}{ccc} \mathcal{A} \leftrightarrow \mathcal{B} & \leadsto & (\mathcal{A} \rightarrow \mathcal{B}) \land (\mathcal{B} \rightarrow \mathcal{A}) \\ \mathcal{A} \rightarrow \mathcal{B} & \leadsto & \neg \mathcal{A} \lor \mathcal{B} \\ \end{array}$$

2. Push negation downwards:

$$\neg (\mathcal{A} \lor \mathcal{B}) \rightsquigarrow \neg \mathcal{A} \land \neg \mathcal{B}$$
$$\neg (\mathcal{A} \land \mathcal{B}) \rightsquigarrow \neg \mathcal{A} \lor \neg \mathcal{B}$$

3. Eliminate double negation:

$$\neg\neg\mathcal{A} \rightsquigarrow \mathcal{A}$$

Note that after the recursive application of 1–3 the formula is in NNF.

4. Push disjunction (for CNF) / conjunction (for DNF) downward:

$$(\mathcal{A} \wedge \mathcal{B}) \vee C \rightsquigarrow_{\mathsf{CNF}} (\mathcal{A} \vee C) \wedge (\mathcal{B} \vee C)$$

 $(\mathcal{A} \vee \mathcal{B}) \wedge C \rightsquigarrow_{\mathsf{DNF}} (\mathcal{A} \wedge C) \vee (\mathcal{B} \wedge C)$ 

5. Eliminate  $\top$  and  $\bot$ :

# 3.4 Functional Completeness

- \* A set *S* is called **closed** under some operation "•" if the result of the operation applied to any elements in the set is also contained in this set, *i.e.*  $\forall x, y \in S : (x \cdot y) \in S$ .
- \* The **closure**  $S^*$  of a set S is the minimal *closed* superset of S.

Closure

\* A set of Boolean functions F is called **functionally complete** if it can be used to express all possible Boolean functions. Formally,  $F^* = \mathbb{F}$ , where  $F^*$  is a *functional closure* of F, and  $\mathbb{F} = \bigcup_{n \in \mathbb{N}} \{ f : \mathbb{B}^n \to \mathbb{B} \}$ .

**Post's Functional Completeness Theorem**  $^{\mbox{\sc e}}$ . A set of Boolean functions F is functionally complete iff it contains:

- at least one function that does *not* preserve zero, *i.e.*  $\exists f \in F : f \notin T_0$ , and
- at least one function that does *not* preserve one, i.e.  $\exists f \in F : f \notin T_1$ , and
- at least one function that is *not* self-dual, *i.e.*  $\exists f \in F : f \notin S$ , and
- at least one function that is *not* monotonic, *i.e.*  $\exists f \in F : f \notin M$ , and
- at least one function that is *not* linear function, *i.e.*  $\exists f \in F : f \notin L$ .
- \* A function f is  $\mathbf{zero\text{-}preserving}^{\mathbf{L}}$  iff it is False on the zero-valuation  $(\mathbb{0}=(0,0,\ldots,0))$ :

$$f \in T_0 \leftrightarrow f(\mathbb{O}) = 0$$

\* A function f is **one-preserving** iff it is True on the one-valuation ( $\mathbb{1} = (1, 1, ..., 1)$ ):  $f \in T_1 \leftrightarrow f(\mathbb{1}) = 1$ 

\* A function f is **self-dual**  $^{\mbox{\'e}}$  iff it is dual to itself:

$$f \in S \leftrightarrow \forall x_1, \ldots, x_n \in \mathbb{B} : f(x_1, \ldots, x_n) = \overline{f}(\overline{x}_1, \ldots, \overline{x}_n).$$

\* A function f is **monotonic** iff for every increasing valuations, the function does not decrease:

$$f \in M \leftrightarrow \forall a, b \in \mathbb{B}^n : a \leq b \rightarrow f(a) \leq f(b).$$

Comparison of valuations  $a, b \in \mathbb{B}^n$  is defined as follows:

$$a \preceq b \leftrightarrow \bigwedge_{1 \leq i \leq n} (a_i \leq b_i)$$

\* A function *f* is **linear** iff its Zhegalkin polynomial is linear (*i.e.* has a degree at most 1):

$$f \in L \leftrightarrow \deg f_{\oplus} \le 1$$