4 Formal Logic Cheatsheet

4.1 Propositional Logic⬀

* **Proposition**⬀ is a statement which can be either true or false. Truth-bearer
* **Alphabet**⬀ of propositional logic consists of (1) atomic symbols and (2) operator symbols.
* **Atomic formula**⬀ (**atom**) is an irreducible formula without logical connectives.
  * Propositional **variables**: $A, B, C, \ldots, Z$. With indices, if needed: $A_1, A_2, \ldots, Z_1, Z_2, \ldots$.
  * Logical **constants**: $\top$ for always true proposition (*tautology*), $\bot$ for always false proposition (*contradiction*).
* **Logical connectives**⬀ (**operators**):

Type	Natural meaning	Symbolization
⬀Negation	It is not the case that $\mathcal{P}$. It is false that $\mathcal{P}$. It is not true that $\mathcal{P}$.	$\neg\mathcal{P}$
⬀Conjunction	Both $\mathcal{P}$ and $\mathcal{Q}$. $\mathcal{P}$ but $\mathcal{Q}$. $\mathcal{P}$, although $\mathcal{Q}$.	$\mathcal{P} \wedge \mathcal{Q}$
⬀Disjunction	Either $\mathcal{P}$ or $\mathcal{Q}$ (or both). $\mathcal{P}$ unless $\mathcal{Q}$.	$\mathcal{P} \vee \mathcal{Q}$
⬀Exclusive or (Xor)	Either $\mathcal{P}$ or $\mathcal{Q}$ (but not both). $\mathcal{P}$ xor $\mathcal{Q}$.	$\mathcal{P} \oplus \mathcal{Q}$
⬀Implication (Conditional)	If $\mathcal{P}$, then $\mathcal{Q}$. $\mathcal{P}$ only if $\mathcal{Q}$. $\mathcal{Q}$ if $\mathcal{P}$.	$\mathcal{P} \to \mathcal{Q}$
⬀Biconditional	$\mathcal{P}$, if and only if $\mathcal{Q}$. $\mathcal{P}$ iff $\mathcal{Q}$. $\mathcal{P}$ just in case $\mathcal{Q}$.	$\mathcal{P} \leftrightarrow \mathcal{Q}$

* **Sentence** of propositional logic is defined inductively: Well-formed formula (WFF)
  1. Every propositional variable/constant is a sentence.
  2. If $\mathcal{A}$ is a sentence, then $\neg\mathcal{A}$ is a sentence.
  3. If $\mathcal{A}$ and $\mathcal{B}$ are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \to \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
  4. Nothing else is a sentence.
* Well-formed formulae grammar: Backus-Naur form (BNF)

$$
\begin{aligned}
\langle sentence \rangle \; ::= \; & \langle constant \rangle \\
| \; & \langle variable \rangle \\
| \; & \neg \, \langle sentence \rangle \\
| \; & \text{`('} \, \langle sentence \rangle \, \langle binop \rangle \, \langle sentence \rangle \, \text{`)'} \\
\langle constant \rangle ::= \; & \top \mid \bot \\
\langle variable \rangle ::= \; & A \mid \ldots \mid Z \mid A_1 \mid \ldots \mid Z_n \\
\langle binop \rangle \quad ::= \; & \wedge \mid \vee \mid \oplus \mid \to \mid \leftarrow \mid \leftrightarrow
\end{aligned}
$$

* **Literal**⬀ is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (*positive* literal), $\mathcal{L}_j = \neg X_j$ (*negative* literal).
* **Argument**⬀ is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$
\underbrace{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n}_{\textit{premises}} \quad \therefore \quad \underbrace{C}_{\textit{conclusion}}
$$
$$\text{``therefore''}$$

* An argument is **valid** if whenever all the premises are true, the conclusion is also true. Validity
* An argument is **invalid** if there is a case (*a counterexample)* when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

* **Valuation**[🔗] is any assignment of truth values to propositional variables. $\qquad\qquad\qquad$ Interpretation

* $\mathcal{A}$ is a **tautology**[🔗] (**valid**) iff it is true on *every* valuation. Might be symbolized as " $\vDash \mathcal{A}$ ".

* $\mathcal{A}$ is a **contradiction**[🔗] iff it is false on *every* valuation. Might be symbolized as " $\mathcal{A} \vDash$ ".

* $\mathcal{A}$ is a **contingency**[🔗] iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.

* $\mathcal{A}$ is **satisfiable** iff it is true on *some* valuation. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ Satisfiability

* $\mathcal{A}$ is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation. $\qquad\qquad\qquad\qquad$ Falsifiability

* $\mathcal{A}$ and $\mathcal{B}$ are **equivalent**[🔗] (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. $\qquad\qquad\qquad\qquad\qquad$ Equivalence check

* $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. $\qquad\qquad$ Consistency

* The sentences $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ true and C false. $\qquad\qquad$ Semantic entailment

* If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \vDash C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n \therefore C$ is **valid**.

  Validity check examples:

A B	$A \to B$	A	$\overset{valid}{\therefore}$	B	$\neg A \to \neg B$	A	$\overset{valid}{\therefore}$	$B \to A$	$A \to B$	B	$\overset{invalid}{\therefore}$	$\neg (B \to A)$
0 0	**1**	**0**	·	**0**	1	**1** 1	✓	**1**	**1**	**0**	·	**0** 1
0 1	**1**	**0**	·	**1**	1	**0** 0	·	**0**	**1**	**1**	✓	**1** 0
1 0	**0**	**1**	·	**0**	0	**1** 1	✓	**1**	**0**	**0**	·	**0** 1
1 1	**1**	**1**	✓	**1**	0	**1** 0	✓	**1**	**1**	**1**	✗	**0** 1

R S T	$R \vee S$	$S \vee T$	$\neg R$	$\overset{invalid}{\therefore}$	$S \wedge T$	$(R \wedge S) \to T$	$\overset{valid}{\therefore}$	$R \to (S \to T)$
0 0 0	**0**	**0**	1	·	**0**	0	**1** 0 ✓	0 1	1
0 0 1	**0**	**1**	1	·	**0**	0	**1** 1 ✓	0 1	1
0 1 0	**1**	**1**	1	✗	**0**	0	**1** 0 ✓	0 1	0
0 1 1	**1**	**1**	1	✓	**1**	0	**1** 1 ✓	0 1	1
1 0 0	**1**	**0**	0	·	**0**	0	**1** 0 ✓	1 1	1
1 0 1	**1**	**1**	0	·	**0**	0	**1** 1 ✓	1 1	1
1 1 0	**1**	**1**	0	·	**0**	1	**0** 0 ·	1 **0**	0
1 1 1	**1**	**1**	0	·	**1**	1	**1** 1 ✓	1 1	1

* **Soundness**[🔗]: $\Gamma \vdash \mathcal{A} \to \Gamma \vDash \mathcal{A}$ $\quad$ "Every provable statement is in fact true"

* **Completeness:**[🔗] $\Gamma \vDash \mathcal{A} \to \Gamma \vdash \mathcal{A}$ $\quad$ "Every true statement has a proof"

4.3 Natural Deduction Rules

Reiteration

$$
\begin{array}{l|l}
m & \mathcal{A} \\
\therefore & \mathcal{A} \qquad \text{R } m
\end{array}
$$

Modus ponens

$$
\begin{array}{l|l}
i & \mathcal{A} \to \mathcal{B} \\
j & \mathcal{A} \\
\therefore & \mathcal{B} \qquad \text{MP } i, j
\end{array}
$$

Modus tollens

$$
\begin{array}{l|l}
i & \mathcal{A} \to \mathcal{B} \\
j & \neg \mathcal{B} \\
\therefore & \neg \mathcal{A} \qquad \text{MT } i, j
\end{array}
$$

Negation

$$
\begin{array}{l|l}
i & \neg \mathcal{A} \\
j & \mathcal{A} \\
\therefore & \bot \qquad \neg \text{E } i, j
\end{array}
$$

$$
\begin{array}{l|l}
i & \mathcal{A} \\
j & \bot \\
\therefore & \neg \mathcal{A} \qquad \neg \text{I } i-j
\end{array}
$$

Indirect proof

$$
\begin{array}{l|l}
i & \neg \mathcal{A} \\
j & \bot \\
\therefore & \mathcal{A} \qquad \text{IP } i-j
\end{array}
$$

Double negation

$$
\begin{array}{l|l}
m & \neg\neg \mathcal{A} \\
\therefore & \mathcal{A} \qquad \neg\neg \text{E } m
\end{array}
$$

Law of excluded middle

$$
\begin{array}{l|l}
i & \mathcal{A} \\
j & \mathcal{B} \\
k & \neg \mathcal{A} \\
l & \mathcal{B} \\
\therefore & \mathcal{B} \qquad \text{LEM } i-j, k-l
\end{array}
$$

Explosion

$$
\begin{array}{l|l}
m & \bot \\
\therefore & \mathcal{A} \qquad \text{X } m
\end{array}
$$

Conjunction

$$
\begin{array}{l|l}
i & \mathcal{A} \\
j & \mathcal{B} \\
\therefore & \mathcal{A} \wedge \mathcal{B} \qquad \wedge \text{I } i, j
\end{array}
$$

$$
\begin{array}{l|l}
m & \mathcal{A} \wedge \mathcal{B} \\
\therefore & \mathcal{A} \qquad \wedge \text{E } m \\
\therefore & \mathcal{B} \qquad \wedge \text{E } m
\end{array}
$$

Disjunction

$$
\begin{array}{l|l}
m & \mathcal{A} \\
\therefore & \mathcal{A} \vee \mathcal{B} \qquad \vee \text{I } m
\end{array}
$$

$$
\begin{array}{l|l}
m & \mathcal{A} \\
\therefore & \mathcal{B} \vee \mathcal{A} \qquad \vee \text{I } m
\end{array}
$$

$$
\begin{array}{l|l}
m & \mathcal{A} \vee \mathcal{B} \\
i & \mathcal{A} \\
j & C \\
k & \mathcal{B} \\
l & C \\
\therefore & C \qquad \vee \text{E } m, i-j, k-l
\end{array}
$$

Disjunctive syllogism

$$
\begin{array}{l|l}
i & \mathcal{A} \vee \mathcal{B} \\
j & \neg \mathcal{A} \\
\therefore & \mathcal{B} \qquad \text{DS } i, j
\end{array}
$$

$$
\begin{array}{l|l}
i & \mathcal{A} \vee \mathcal{B} \\
j & \neg \mathcal{B} \\
\therefore & \mathcal{A} \qquad \text{DS } i, j
\end{array}
$$

Hypothetical syllogism

$$
\begin{array}{l|l}
i & \mathcal{A} \to \mathcal{B} \\
j & \mathcal{B} \to C \\
\therefore & \mathcal{A} \to C \qquad \text{HS } i, j
\end{array}
$$

Conditional

$$
\begin{array}{l|l}
i & \mathcal{A} \\
j & \mathcal{B} \\
\therefore & \mathcal{A} \to \mathcal{B} \qquad \to \text{I } i-j
\end{array}
$$

Contraposition

$$
\begin{array}{l|l}
m & \mathcal{A} \to \mathcal{B} \\
\therefore & \neg \mathcal{B} \to \neg \mathcal{A} \qquad \text{Contra } m
\end{array}
$$

Biconditional

$$
\begin{array}{l|l}
i & \mathcal{A} \\
j & \mathcal{B} \\
k & \mathcal{B} \\
l & \mathcal{A} \\
\therefore & \mathcal{A} \leftrightarrow \mathcal{B} \qquad \leftrightarrow \text{I } i-j, k-l
\end{array}
$$

$$
\begin{array}{l|l}
i & \mathcal{A} \leftrightarrow \mathcal{B} \\
j & \mathcal{A} \\
\therefore & \mathcal{B} \qquad \leftrightarrow \text{E } i, j
\end{array}
$$

$$
\begin{array}{l|l}
i & \mathcal{A} \leftrightarrow \mathcal{B} \\
j & \mathcal{B} \\
\therefore & \mathcal{A} \qquad \leftrightarrow \text{E } i, j
\end{array}
$$

De Morgan Rules

$$
\begin{array}{l|l}
m & \neg(\mathcal{A} \vee \mathcal{B}) \\
\therefore & \neg \mathcal{A} \wedge \neg \mathcal{B} \qquad \text{DeM } m
\end{array}
$$

$$
\begin{array}{l|l}
m & \neg \mathcal{A} \wedge \neg \mathcal{B} \\
\therefore & \neg(\mathcal{A} \vee \mathcal{B}) \qquad \text{DeM } m
\end{array}
$$

$$
\begin{array}{l|l}
m & \neg(\mathcal{A} \wedge \mathcal{B}) \\
\therefore & \neg \mathcal{A} \vee \neg \mathcal{B} \qquad \text{DeM } m
\end{array}
$$

$$
\begin{array}{l|l}
m & \neg \mathcal{A} \vee \neg \mathcal{B} \\
\therefore & \neg(\mathcal{A} \wedge \mathcal{B}) \qquad \text{DeM } m
\end{array}
$$

Green: basic rules.
Orange: derived rules.

More rules can be found in the
"forall x: Calgary" book (p. 406).