# Discrete Mathematics

**Combinatorics** — Spring 2025

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# §1 Combinatorics

# Introduction to Combinatorics

**Definition 1**: Combinatorics is the branch of discrete mathematics that deals with *counting*, *arranging*, and analyzing *discrete structures*.

**Three basic problems of Combinatorics:**

1. Existence: *Is there at least one arrangement of a particular kind?*
2. Counting: *How many arrangements are there?*
3. Optimization: *Which one is best according to some criteria?*

### Discrete structures

- Graphs, sets, multisets, sequences, patterns, coverings, partitions…

### Enumeration

- Permutations, combinations, inclusion/exclusion, generating functions, recurrence relations…

### Algorithms and optimization

- Sorting, eulerian circuits, hamiltonian cycles, planarity testing, graph coloring, spanning trees, shortest paths, network flows, bipartite matchings, chain partitions…

# Discrete Structures

We investigate the *building blocks* of combinatorics:

- Sets and multisets
- Sequences and strings
- Arrangements
- Graphs, networks, trees
- Posets and lattices
- Partitions
- Patterns, coverings, designs, configurations
- Schedules, assignments, distributions

*Used in data modeling, logic, cryptography, and the design of data structures.*

# Enumerative Combinatorics

We learn how to count *without explicit listing*:

- Permutations and combinations
- Inclusion–Exclusion Principle
- Set partitions, integer partitions, Stirling numbers, Catalan numbers
- Recurrence relations
- Generating functions

*Used in probability theory, complexity theory, coding theory, computational biology.*

# Algorithmic and Optimization Methods

Combinatorics powers *algorithm design* and complexity analysis:

- Sorting
- Searching
- Eulerian paths and Hamiltonian cycles
- Planarity, colorings, cliques, coverings
- Spanning trees
- Shortest paths
- Network flows
- Bipartite matchings
- Dilworth's theorem, chain and antichain partitions

*Used in logistics, scheduling, routing, and complexity optimization.*

# §2 Basic Counting Principles

# Basic Counting Rules

**Product rule**: If something can happen in $n_1$ ways, *and* no matter how the first thing happens, a second thing can happen in $n_2$ ways, then the two things *together* can happen in $n_1 \cdot n_2$ ways.

**Sum rule**: If one event can occur in $n_1$ ways and a second event in $n_2$ (different) ways, then there are $n_1 + n_2$ ways in which *either* the first event *or* the second event can occur (*but not both*).

# Addition Principle

**Definition 2**: We say a finite set $S$ is *partitioned* into *parts* $S_1, ..., S_k$ if the parts are pairwise disjoint and their union is $S$. In other words, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $S_1 \cup S_2 \cup ... \cup S_k = S$. In that case:

$$|S| = |S_1| + |S_2| + ... + |S_k|$$

*Example*: Let $S$ be the set of students attending the combinatorics lecture. It can be partitioned into parts $S_1$ and $S_2$ where

$$S_1 = \text{set of students that like easy examples.}$$
$$S_2 = \text{set of students that don't like easy examples.}$$

If $|S_1| = 22$ and $|S_2| = 8$, then we can conclude $|S| = |S_1| + |S_2| = 30$.

# Multiplication Principle

**Definition 3**: If $S$ is a finite set that is the *product* of $S_1, ..., S_k$, that is, $S = S_1 \times ... \times S_k$, then

$$|S| = |S_1| \times ... \times |S_k|$$

*Example*: TODO: example with car plates

# Subtraction Principle

**Definition 4**: Let $S$ be a subset of a finite set $T$. We define the *complement* of $S$ as $\overline{S} = T \setminus S$. Then

$$\left|\overline{S}\right| = |T| - |S|$$

*Example*: If $T$ is the set of students studying at KIT and $S$ the set of students studying neither math nor computer science. If we know $|T| = 23905$ and $|S| = 20178$, then we can compute the number $|S|$ of students studying either math or computer science:

$$|S| = |T| - |S| = 23905 - 20178 = 3727$$

# Bijection Principle

**Definition 5**: If $S$ and $T$ are sets, then

$$|S| = |T| \iff \text{there exists a bijection between } S \text{ and } T$$

*Example*: Let $S$ be the set of students attending the combinatorics lecture and $T$ the set of homework submissions (unique per student) for the first problem sheet. If the number of students and the number of submissions coincide, then there is a bijection between students and submissions.

**Note**: The bijection principle works both for *finite* and *infinite* sets.

# Pigeonhole Principle

**Definition 6**: Let $S_1, ..., S_k$ be finite sets that are pairwise disjoint and $|S_1| + |S_2| + ... + |S_k| = n$.

$$\exists i \in \{1, ..., k\} : |S_i| \geq \left\lfloor \frac{n}{k} \right\rfloor \quad \text{and} \quad \exists j \in \{1, ..., k\} : |S_j| \leq \left\lceil \frac{n}{k} \right\rceil$$

*Example*: Assume there are 5 holes in the wall where pigeons nest. Say there is a set $S_i$ of pigeons nesting in hole $i$. Assume there are $n = 17$ pigeons in total. Then we know:

- There is some hole with at least $d = 4$ pigeons.
- There is some hole with at most $b = 3$ pigeons.

# Double Counting

If we count the same quantity in *two different ways*, then this gives us a (perhaps non-trivial) identity.

*Example (Handshaking Lemma)*: Assume there are $n$ people at a party and everybody will shake hands with everybody else. How many handshakes will occur? We count this number in two ways:

1. Every person shakes $n-1$ hands and there are $n$ people. However, two people are involved in a handshake so if we just multiply $n \cdot (n-1)$, then every handshake is counted twice. The total number of handshakes is therefore $\frac{n \cdot (n-1)}{2}$.
2. We number the people from 1 to $n$. To avoid counting a handshake twice, we count for person $i$ only the handshakes with persons of lower numbers. Then the total number of handshakes is:

$$\sum_{i=1}^{n} (i-1) = \sum_{i=0}^{\{n-1\}} i = \sum_{i=1}^{n-1} i$$

The identity we obtain is therefore: $\displaystyle\sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$

# §3 Arrangements, Permutations, Combinations

# Ordered Arrangements

**Definition 7**: Denote by $[n] = \{1, ..., n\}$ the set of natural numbers from 1 to $n$.

Hereinafter, let $X$ be a finite set.

**Definition 8**: An *ordered arrangement* of $n$ elements of $X$ is a *map* $s : [n] \longrightarrow X$.

- Here, $[n]$ is the *domain* of $s$, and $s(i)$ is the *image* of $i \in [n]$ under $s$.
- The set $\{x \in X \mid s(i) = x \text{ for some } i \in [n]\}$ is the *range* of $s$.

Other common names for ordered arrangements are:

- *string* (or *word*), e.g. “Banana”
- *sequence*, e.g. “0815422372”
- *tuple*, e.g. $(3, 5, 2, 5, 8)$

*Example*:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $s(i)$ | 🦀 | 🥖 | 🥖 | 🫙 | 🥖 | 🦀 | 🫙 |

# Permutations

**Definition 9**: A *permutation* of $X$ is a *bijective* map $\pi : [n] \longrightarrow X$.

Usually, $X = [n]$, and the set of all permutations of $[n]$ is denoted by $S_n$.

*Example*:

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\pi(i)$ | 2 | 7 | 1 | 3 | 5 | 4 | 6 |

**Definition 10**: *$k$-permutation* of $X$ is an ordered arrangement of $k$ *distinct* elements of $X$, that is, an *injective* map $\pi : [k] \longrightarrow X$.

The set of all $k$-permutations of $X = [n]$ is denoted by $P(n, k)$. In particular, $S_n = P(n, n)$.

TODO: circular permutations

# Counting Permutations

**Theorem 1**: For any natural numbers $0 \leq k \leq n$, we have

$$|P(n,k)| = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

This formula is also called the *falling factorial* and denoted $n^{\underline{k}}$ or $(n)_k$.

**Proof**: A permutation is an injective map $\pi : [k] \longrightarrow [n]$. We count the number of ways to pick such a map, picking the images one after the other. There are $n$ ways to choose $\pi(1)$. Given a value for $\pi(1)$, there are $(n-1)$ ways to choose $\pi(2)$ (since we may not choose $\pi(1)$ again). Continuing like this, there are $(n-i+1)$ ways to pick $\pi(i)$, and the last value we pick is $\pi(k)$ with $(n-k+1)$ possibilities.

Every $k$-permutation can be constructed like this in *exactly one way*. The total number of $k$-permutations is therefore given as the product:

$$|P(n,k)| = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!} \qquad \square$$

# Counting Circular Permutations

**Theorem 2**: For any natural numbers $0 \leq k \leq n$, we have

$$|P_c(n,k)| = \frac{n!}{k \cdot (n-k)!}$$

**Proof**: We doubly count $P(n,k)$:

1. $|P(n,k)| = \frac{n!}{(n-k)!}$ which we proved before.
2. $|P(n,k)| = |P_c(n,k)| \cdot k$ because every equivalence class in $P_c(n,k)$ contains $k$ permutations from $P(n,k)$ since there are $k$ ways to rotate a $k$-permutation.

From this we get $\dfrac{n!}{(n-k)!} = |P_c(n,k)| \cdot k$, which implies $|P_c(n,k)| = \dfrac{n!}{k \cdot (n-k)!}$. $\square$

# Unordered Arrangements

**Definition 11**: An *unordered arrangement* of $k$ elements of $X$ is a *multiset* $S = \langle X, r \rangle$ of size $k$.

In a multiset, $X$ is the set of *types*, and for each type $x \in X$, $r_x$ is its *repetition number*.

*Example*: Let $X = \{$👒, 🦭, 🐈, 🪗, 🌵$\}$.

- An unordered arrangement of 7 elements could be $S = \{$👒, 👒, 🦭, 🐈, 🐈, 🐈, 🌵$\}^*$.
- The same multiset could be written as $S = \{2$👒, 1🦭, 3🐈, 0🪗, 1🌵$\}$.

# Subsets

The most important special case of unordered arrangements is where all repetitions are 1, that is, $r_x = 1$ for all $x \in X$. Then $S$ is simply a *subset* of $X$, denoted $S \subseteq X$.

**Definition 12**: A *k-combination* of $X$ is an unordered arrangement of $k$ *distinct* elements of $X$.

**Note**: The more standard term is *subset*. The term "combination" is only used to emphasize the selection process.

The set of all $k$-subsets of $X$ is denoted $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$. If $|X| = n$, then

$$\binom{n}{k} := \left|\binom{X}{k}\right|$$

*Example*: The set of edges in a simple undirected graph consists of 2-subsets of its vertices: $E \subseteq \binom{V}{2}$.

# Counting $k$-Combinations

**Theorem 3**: For $0 \leq k \leq n$, we have

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

**Proof**: $|P(n,k)| = \frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!$ $\square$

# §4 Multisets

# Multiset

**Definition 13**: A *multiset* is a modification of the concept of a set that allows for *repetitions* of its elements. Formally, it is denoted as a pair $M = \langle X, r \rangle$, where $X$ is the *groundset* (the set of *types*) and $r : X \longrightarrow \mathbb{N}_0$ is the *multiplicity function*.

*Example*: When the multiset is defined by enumeration, it is advisable to use the notation with the star:

$$M = \{a, b, a, a, b\}^* = \{3 \cdot a, 2 \cdot b\} \quad X = \{a, b\} \quad r_a = 3, r_b = 2$$

*Example*: Prime factorization of a natural number $n$ is a multiset, e.g. $120 = 2^3 \cdot 3^1 \cdot 5^1$.

# $k$-Combinations of a Multiset

**Definition 14**: Let $X$ be a finite set of types, and let $M = \langle X, r \rangle$ be a finite multiset with repetition numbers $r_1, ..., r_{|X|}$. A *k-combination of M* is a multiset $S = \langle X, s \rangle$ with types in $X$ and repetition numbers $s_1, ..., s_{|X|}$ such that $s_i \leq r_i$ for all $1 \leq i \leq |X|$, and $\sum_{i=1}^{|X|} s_i = k$.

*Example*: Consider $M =$ {2🦫, 1🥤, 3🍉, 1💎}.

- $T =$ {1🦫, 2🍉} is a 3-combination of $M$.
- $T' =$ {3💎} is not.

*Counting k-combinations of a multiset is not as simple as it might seem…*

# $k$-Permutations of a Multiset

**Definition 15**: Let $M$ be a finite multiset with set of types $X$. A *$k$-permutation of $M$* is an ordered arrangement of $k$ elements of $M$ where different orderings of elements of the same type are *not distinguished*. This is an ordered multiset with types in $X$ and repetition numbers $s_1, \ldots, s_{|X|}$ such that $s_i \leq r_i$ for all $1 \leq i \leq |X|$, and $\sum_{i=1}^{|X|} s_i = k$.

**Note**: There might be several elements of the same type compared to a permutation of a set (where each repetition number equals 1).

*Example*: Let $M = \{2$🦫, $1$🥤, $3$🍉, $1$💎$\}$, then $T = ($💎, 🍉, 🍉, 🦫$)$ is a 4-permutation of multiset $M$.

# Binomial Theorem

**Theorem 4**: The expansion of any non-negative integer power $n$ of the binomial $(x + y)$ is a sum

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k}$$

where each $\binom{n}{k}$ is a positive integer known as a *binomial coefficient*, defined as

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot 2 \cdot 1}$$

# Multinomial Theorem

**Theorem 5**: The generalization of the binomial theorem:

$$(x_1 + \ldots + x_r)^n = \sum_{\substack{0 \leq k_1, \ldots, k_r \leq n \\ k_1 + \ldots + k_r = n}}^{n} \binom{n}{k_1, \ldots, k_r} \cdot x_1^{k_1} \cdot \ldots \cdot x_r^{k_r}$$

*Multinomial coefficients* are defined as

$$\binom{n}{k_1, \ldots, k_r} = \frac{n!}{k_1! \cdot \ldots \cdot k_r!}$$

**Note**: Binomial coefficients are special cases of multinomial coefficients ($r = 2$):

$$\binom{n}{k} = \binom{n}{k_1, k_2} = \binom{n}{k, n-k} = \frac{n!}{k! \cdot (n-k)!}$$

**Proof**: TODO $\square$

# Permutations of a Multiset

**Theorem 6**: Let $S$ be a finite multiset with $k$ different types and repetition numbers $r_1, \ldots, r_k$. Let the size of $S$ be $n = r_1 + \ldots + r_k$. Then the number of $n$-permutations of $S$ equals

$$\binom{n}{r_1, \ldots, r_k}$$

**Proof**: In an $n$-permutation there are $n$ positions that need to be assigned a type.

First, choose the $r_1$ positions for the first type, there are $\binom{n}{r_1}$ ways to do so. Then, assign $r_2$ positions for the second type, out of the $(n - r_1)$ positions that are still available, there are $\binom{n-r_1}{r_2}$ ways to do so. Continue for all $k$ types. The total number of choices will be:

$$\binom{n}{r_1} \cdot \binom{n - r_1}{r_2} \cdot \ldots \cdot \binom{n - r_1 - r_2 - \ldots - r_{k-1}}{r_k} = \binom{n}{r_1, \ldots, r_k}$$

$\square$

## $k$-Combinations of an *Infinite* Multiset

*Example*: Suppose you have a *sufficiently large* amount of each type of fruit (🍌 , 🍎 , 🍐 ) in the supermarket, and you want to buy *two* fruits. How many choices do you have?

There are exactly *six* combinations: {🍌 , 🍌 }, {🍌 , 🍎 }, {🍌 , 🍐 }, {🍎 , 🍎 }, {🍎 , 🍐 }, {🍐 , 🍐 }.

Note that your selection is *not ordered*, so {🍐 , 🍎 } and {🍎 , 🍐 } are considered the *same* choice.

# $k$-Combinations of an *Infinite* Multiset [2]

**Theorem 7**: Let $k, s \in \mathbb{N}$ and let $S$ be a multiset with $s$ types and large repetition numbers (each $r_1, ..., r_s$ is *at least $k$*), then the number of $k$-combinations of $S$ equals

$$\binom{k+s-1}{k} = \binom{k+s-1}{s-1}$$

**Proof**: Let $S = \langle X, r_\infty \rangle = \{\infty$🍌, $\infty$🍎, $\infty$🍐$\}$ with $r_x = \infty$ and $|X| = s = 3$.

- Let $k = 5$ (as an example). Consider a 5-combination of $S$: {🍌, 🍎, 🍌, 🍐, 🍐}.
- Reorder and group: {🍌 🍌 | 🍎 | 🍐 🍐}.
- Convert to *dots* and *bars*: $\bullet \bullet | \bullet | \bullet \bullet$
- Represent as a 2-type multiset: $M = \{\, k \cdot \bullet, (s-1) \cdot | \,\}$
- Observe: each *permutation* of $k$ dots and $(s-1)$ bars corresponds *uniquely* to a *$k$-combination* of $S$.
- Permute the 2-type multiset: $\binom{k+s-1}{k, s-1}$ ways, by Theorem 5.

This method is also known as *Stars and Bars*. □

# §5 Compositions

# Weak Compositions

**Definition 16**: A *weak composition* of a non-negative integer $k \geq 0$ into $s$ parts is a *solution* to the equation $b_1 + \ldots + b_s = k$, where each $b_i \geq 0$.

*Example*: Let $k = 5$, $s = 3$. Possible non-negative integer solutions for $b_1 + b_2 + b_3 = 5$ are:

- $(b_1, b_2, b_3) = (1, 1, 3)$
- $(b_1, b_2, b_3) = (1, 3, 1)$
- $(b_1, b_2, b_3) = (2, 0, 3)$
- $(b_1, b_2, b_3) = (0, 5, 0)$
- … (total 21 solutions)

**Note**: If $M$ is a multiset over groundset $\{1, \ldots, s\}$ with all multiplicities infinite ($r_i = \infty$), then for $k \geq 0$, the number of sub-multisets of $M$ of size $k$ is exactly the number of weak compositions of $k$ into $s$ parts.

# Counting Weak Compositions

**Theorem 8**: There are $\binom{k+s-1}{k,s-1}$ *weak compositions* of $k > 0$ into $s$ parts.

**Proof**: Observe that $k = \overbrace{\underbrace{1+1}_{b_1} + \underbrace{...}_{b_i} + \underbrace{1+1}_{b_s}}^{k \text{ ones}}$.

Use the *stars-and-bars* method to count the number of $s$ groups composed of $k$ "ones". □

*Example*: Let $k = 3$. There are $\binom{3+3-1}{3,3-1} = \binom{5}{3} = \binom{5}{2} = 10$ ways to decompose $k = 3$ into $s = 3$ parts:

$$\begin{aligned} k = 3 = & \\ & = 0+1+2 = 0+2+1 \\ & = 1+0+2 = 1+2+0 = 1+1+1 \\ & = 2+0+1 = 2+1+0 \\ & = 3+0+0 = 0+3+0 = 0+0+3 \end{aligned}$$

# Compositions

**Definition 17**: A *composition* of a positive integer $k \geq 1$ into $s$ *positive* parts is a *solution* to the equation $b_1 + \ldots + b_s = k$, where each $b_i > 0$.

**Theorem 9**: There are $\binom{k-1}{s-1}$ *compositions* of $k > 0$ into $s$ positive parts.

**Theorem 10**: The total number of compositions of $k > 0$ into *some* number of positive parts is

$$\sum_{s=1}^{k} \binom{k-1}{s-1} = 2^{k-1}$$

# Parallel Summation Identity

**Q**: How many integer solutions are there to the *inequality* $b_1 + \ldots + b_s \leq k$, where each $b_i \geq 0$?

**Theorem 11**: $\sum_{m=0}^{k} \binom{m+s-1}{m} = \binom{k+s}{k}$

**Proof** *(hint)*: Introduce a "dummy" variable $b_{s+1}$ to take up the *slack* between $b_1 + \ldots + b_s$ and $k$. Construct a bijection with the solutions to $b_1 + \ldots + b_s + b_{s+1} = k$, where $b_i \geq 0$. □

# §6 Set Partitions

# Set Partitions

**Definition 18**: A *partition* of a set $X$ is a set of non-empty subsets of $X$ such that every element of $X$ belongs to exactly one of these subsets.

Equivalently, a family of sets $P$ is a partition of $X$ iff:

1. The family $P$ does not contain the empty set: $\varnothing \notin P$.
2. The union of $P$ is $X$, that is, $\bigcup_{A \in P} A = X$. The sets in $P$ are said to *cover* $X$.
3. The intersection of any two distinct sets in $P$ is empty: $\forall A, B \in P.\, (A \neq B) \to (A \cap B = \varnothing)$.
   The sets in $P$ are said to be *pairwise disjoint* or *mutually exclusive*.

The sets in $P$ are called *blocks*, *parts*, or *cells*, of the partition.

The block in $P$ containing an element $x \in X$ is denoted by $[x]$.

# Examples of Set Partitions

*Example*: The empty set $X = \emptyset$ has exactly one partition, $P = \emptyset$.

*Example*: Any singleton set $X = \{x\}$ has exactly one partition, $P = \{\{x\}\}$.

*Example*: For any non-empty proper subset $A \subset U$, the set $A$ and its complement form a partition of $U$, namely $P = \{A, U - A\}$.

*Example*: The set $X = \{1, 2, 3\}$ has five partitions:

1. $\{\{1\}, \{2\}, \{3\}\}$ or $1 \mid 2 \mid 3$
2. $\{\{1\}, \{2, 3\}\}$ or $1 \mid 2\ 3$
3. $\{\{1, 2\}, \{3\}\}$ or $1\ 2 \mid 3$
4. $\{\{1, 3\}, \{2\}\}$ or $1\ 3 \mid 2$
5. $\{\{1, 2, 3\}\}$ or $1\ 2\ 3$

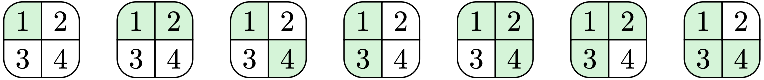
*Example*: The following are *not* partitions of $\{1, 2, 3\}$:

- $\{\{\}, \{1, 3\}, \{2\}\}$, because it contains the empty set.
- $\{\{1, 2\}, \{2, 3\}\}$, because the element 2 is contained in more than one block.
- $\{\{1\}, \{3\}\}$, because no block contains the element 3.

# Counting Set Partitions

**Definition 19**: The number of partitions of a set $X$ (of size $n = |X|$) into $k$ non-empty blocks (“unlabeled subsets”) is called a *Stirling number of the second kind* and denoted $S(n, k)$ or $\left\{ n \atop k \right\}$.

*Example*: Let $X = \{1, 2, 3, 4\}$, $k = 2$. There are 7 possible partitions:



**Theorem 12**: Let $\left\{ n \atop 0 \right\} = 0$ for $n \geq 1$, $\left\{ 0 \atop k \right\} = 0$ for $k \geq 1$, and $\left\{ 0 \atop 0 \right\} = 1$. For $n, k \geq 1$, we have:

$$\left\{ n \atop k \right\} = \left\{ n - 1 \atop k - 1 \right\} + k \cdot \left\{ n - 1 \atop k \right\}$$

**Proof** *(informal)*: TODO □

# Bell Numbers

**Definition 20**: The total number of partitions of a set $X$ of size $n = |X|$ (into an arbitrary number of non-empty blocks) is called a *Bell number* and denoted $B_n$.

$$B_n = \sum_{k=0}^{n} \left\{ n \atop k \right\}$$

**Note**: Consider the special case of $n = 0$. There is exactly *one* partition of $\varnothing$ into non-empty parts: $\varnothing = \bigcup_{A \in \varnothing} A \in \varnothing$. Every $A \in \varnothing$ is non-empty, since no such $A$ exists. Thus, we have $B_0 = S(0, 0) = 1$.

# Bell Numbers [2]

**Theorem 13**: For $n \geq 1$, we have the recursive identity for Bell numbers:

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

**Proof**: Every partition of $[n]$ has one part that contains the number $n$. In addition to $n$, this part also contains $k$ other numbers (for some $0 \leq k \leq n-1$). The remaining $n-1-k$ elements are partitioned arbitrarily. From this correspondence, we obtain the desired identity:

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{n-1-k} B_{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

$\square$

# §7 Integer Partitions

# Integer Partitions

**Definition 21**: An *integer partition* of a positive integer $n \geq 1$ into $k$ *positive* parts is a *solution* to the equation $n = a_1 + \ldots + a_k$, where $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$.

- The number of integer partitions of $n$ into $k$ positive non-decreasing parts is denoted $p_k(n)$ and defined recursively:

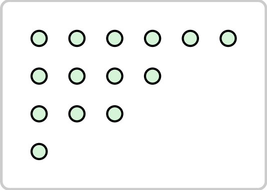
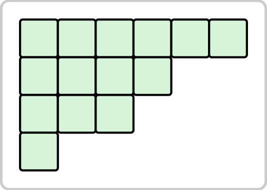
$$p_k(n) = \begin{cases} 0 & \text{if } k > n \\ 0 & \text{if } n \geq 1 \text{ and } k = 0 \\ 1 & \text{if } n = k = 0 \\ p_k(n-k) + p_{k-1}(n-1) & \text{if } 1 \leq k \leq n \end{cases}$$

- The number of partitions of $n$ (into an arbitrary number of parts) is the *partition function* $p(n)$:

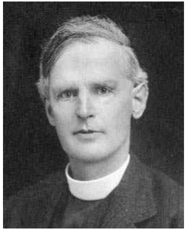
$$p(n) = \sum_{k=0}^{n} p_k(n)$$

# Ferrer Diagrams and Yound Tableaux

*Example*: Consider an integer partition: $14 = 6 + 4 + 3 + 1$.

| **Ferrer Diagram** | **Young Tableaux** |
|---|---|
|  |  |

Norman Ferrer



Alfred Young

# §8 Inclusion–Exclusion

# The Inclusion–Exclusion Principle

TODO: small example of PIE with 2 or 3 sets

# Principle of Inclusion–Exclusion (PIE)

**Theorem 14**: Let $X$ be a finite set and $P_1, \ldots, P_m$ *properties*.

- Define $X_i = \{x \in X \mid x \text{ has } P_i\}$, i.e. the set of all elements from $X$ having a property $P_i$.
- Define for $S \subseteq [m]$ the set $N(S) = \{x \in X \mid \forall i \in S : x \text{ has } P_i\}$. Observe: $N(S) = \bigcap_{i \in S} X_i$.

The number of elements of $X$ that satisfy *none* of the properties $P_1, \ldots, P_m$ is given by

$$|X \setminus (X_1 \cup \ldots \cup X_m)| = \sum_{S \subseteq [m]} (-1)^{|S|} |N(S)| \tag{1}$$

**Proof**: Consider any $x \in X$. If $x \in X$ has none of the properties, then $x \in N(\emptyset)$ and $x \notin N(S)$ for any other $S \neq \emptyset$. Hence *$x$ contributes 1* to the sum (1).

If $x \in X$ has exactly $k \geq 1$ of the properties, call this set $T \in \binom{[m]}{k}$. Then $x \in N(S)$ iff $S \subseteq T$.
The *contribution of $x$* to the sum (1) is $\sum_{S \subseteq T} (-1)^{|S|} = \sum_{i=0}^{k} \binom{k}{i} (-1)^i = 0$, i.e. *zero*. $\square$

**Note**: In the last step, we used that for any $y \in \mathbb{R}$ we have $(1-y)^k = \sum_{i=0}^{k} \binom{k}{i} (-y)^i$ which implies (for $y = 1$) that $0 = \sum_{i=0}^{k} \binom{k}{i} (-1)^i$.

# Very Useful Corollary of PIE

**Corollary 14.1:** 🙀

$$\left|\bigcup_{i\in[m]} X_i\right| = |X| - \sum_{S\subseteq[m]} (-1)^{|S|}|N(S)| = \sum_{\emptyset \neq S\subseteq[m]} (-1)^{|S|-1}|N(S)|$$

# Applications of PIE

Let's state the principle of inclusion-exclusion using a rigid pattern:

1. *Define "bad" properties.*

   Identify the things to count as the elements of some universe $X$ except for the whose having *at least one* of the "bad" properties $P_1, \ldots, P_m$. In other words, we want to count $X \setminus (X_1 \cup \ldots \cup X_m)$.

2. *Count $N(S)$.*

   For each $S \subseteq [m]$, determine $N(S)$, the number of elements of $X$ having *all* bad properties $P_i$ for $i \in S$.

3. *Apply PIE*.

   Use Theorem 14 to obtain a closed formula for $|X \setminus (X_1 \cup \ldots \cup X_m)|$.

# Counting Surjections via PIE

**Theorem 15**: The number of surjections from $[k]$ to $[n]$ is given by

$$\left|\left\{f : [k] \xrightarrow[\text{surj.}]{} [n]\right\}\right| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$

**Proof**: Let $X$ be the set of all maps from $[k]$ to $[n]$.

1. *Define bad properties:* Define the "bad" property $P_i$ for $i \in [n]$ as "$i$ is not in the image of $f$", i.e.

   $$f : [k] \longrightarrow [n] \text{ has property } P_i \quad \leftrightarrow \quad \forall j \in [k] : f(j) \neq i$$

   The *surjective* functions are exactly those functions that *do not* have any of the "bad" properties.

2. *Count $N(S)$:* We claim $N(S) = (n - |S|)^k$ for any $S \subseteq [n]$. To see this, observe that $f$ has all properties with indices from $S$ if and only if $f(i) \notin S$ for all $i \in [k]$. In other words, $f$ must be a function from $[k]$ to $[n] \setminus S$, and there are $(n - |S|)^k$ of those.

# Counting Surjections via PIE [2]

3. *Apply PIE:* Using Theorem 14, the number of surjections from $[k]$ to $[n]$ is

$$\begin{aligned}|X \setminus (X_1 \cup \ldots \cup X_n)| &\overset{\text{PIE}}{=} \sum_{S \subseteq [n]} (-1)^{|S|} |N(S)| \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)^k \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)^k\end{aligned}$$

In the last step, we used that $(-1)^{|S|}(n - |S|)^k$ only depends on the size of $S$, and there are $\binom{n}{i}$ sets $S \subseteq [n]$ of size $i$.

$\square$

# More Useful Corollaries

**Corollary 15.1**: Consider the case $n = k$. A function from $[n]$ to $[n]$ is a *surjection* iff it is a *bijection*. Since there are $n!$ bijections on $[n]$ (namely, all permutations), we have the following identity:

$$n! = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^n$$

**Corollary 15.2**: A surjection from $[k]$ to $[n]$ can be seen as a partition of $[k]$ into $n$ non-empty distinguishable (labeled) parts (the map assigns a part to each $i \in [k]$).

Since the partition of $[k]$ into $n$ non-empty indistinguishable parts is denoted $s_n^{\mathrm{II}}(k)$, and there are $n!$ ways to assign labels to $n$ parts, we obtain that the number of surjections is equal to $n! s_n^{\mathrm{II}}(k)$:

$$n! s_n^{\mathrm{II}}(k) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$$

# Derangements

**Theorem 16**: The *derangements* $D_n$ on $n$ elements are permutations of $[n]$ without fixed points.

The number of derangements is given by

$$|D_n| = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)!$$

**Proof**: Let $X$ be the set of all permutations of $[n]$.

1. Define the "bad" property $P_i$ to mean "$\pi$ has a fixpoint $i$" ($i \in [n]$):

$$\pi \in X \text{ has property } P_i \quad \leftrightarrow \quad \pi(i) = i$$

2. We claim $N(S) = (n - |S|)!$ for any $S \subseteq [n]$.

   Indeed, $\pi \in X$ has all properties with indices from $S$ if and only if all $i \in S$ are fixed points of $\pi$. On the other elements, i.e. on $[n] \setminus S$, $\pi$ may be an arbitrary bijection, so there are $(n - |S|)!$ choices for $\pi$.

# Derangements [2]

3. Using Theorem 14, the number of derangements is given by

$$\begin{aligned} |X \setminus (X_1 \cup \ldots \cup X_n)| &\overset{\text{PIE}}{=} \sum_{S \subseteq [n]} (-1)^{|S|} |N(S)| \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! \end{aligned}$$

In the last step, we used that $(-1)^{|S|}(n - |S|)!$ only depends on the size of $S$, and there are $\binom{n}{i}$ sets $S \subseteq [n]$ of size $i$.

$\square$

# §9 Generating Functions

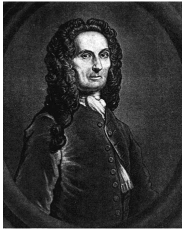
# Generating Functions

*A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.*

— George Pólya, Mathematics and Plausible Reasoning [1]

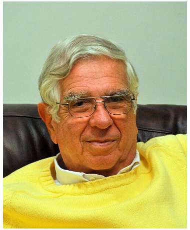
*A generating function is a clothesline on which we hang up a sequence of numbers for display.*

— Herbert Wilf, generatingfunctionology [2]



Abraham de Moivre

George Pólya



Herbert Wilf

# Ordinary Generating Functions

**Definition 22**: An *ordinary generating function* (OGF) of a sequence $a_n$ is a *power series*

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n$$

*Example*: The *sequence* $a_n = (a_0, a_1, a_2, ...)$ is *generated* by the OGF $G(x) = a_0 + a_1 x + a_2 x^2 + ...$

*Example*: $G(x) = 3 + 8x^2 + x^3 + \frac{1}{7}x^5 + 100x^6 + ...$ *generates* the sequence $\left(3, 0, 8, 1, 0, \frac{1}{7}, 100, 0, ...\right)$

*Example*: Consider a long division of 1 by $(1 - x)$, the result is an infinite power series

$$\frac{1}{1 - x} = 1 + x^1 + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$$

Note that all coefficients are 1. Thus, the generating function of $(1, 1, 1, ...)$ is $G(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

# The Core Generating Function

Another proof that $(1, 1, 1, \ldots)$ is generated by $G(x) = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = S$:

$$\begin{array}{rcl} S & = & 1 + x + x^2 + x^3 + \ldots \\ x \cdot S & = & \phantom{1 +{}} x + x^2 + x^3 + \ldots \\ \hline S - x \cdot S & = & 1 \end{array}$$

Thus, $S = \dfrac{1}{1 - x}$.

The generating function $G(x) = 1 + x + x^2 + \ldots$ is also known as the *Maclaurin series* of $\dfrac{1}{1 - x}$.

# More Examples of Generating Functions

| Formula | Power series | Sequence | Description |
|---|---|---|---|
| $\frac{1}{1-x}$ | $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$ | $(1, 1, 1, \ldots)$ | constant 1 |
| $\frac{2}{1-x}$ | $\sum_{n=0}^{\infty} 2x^n = 2 + 2x + 2x^2 + 2x^3 + \ldots$ | $(2, 2, 2, \ldots)$ | constant 2 |
| $\frac{x}{1-x}$ | $\sum_{n=1}^{\infty} x^n = 0 + x + x^2 + x^3 + \ldots$ | $(0, 1, 1, 1, \ldots)$ | right shift |
| $\frac{1}{1+x}$ | $\sum_{n=0}^{\infty} (-1)^n x^n = 0 + 1 - x + x^2 - x^3 + \ldots$ | $(1, -1, 1, \ldots)$ | sign-alternating 1′s |
| $\frac{1}{1-3x}$ | $\sum_{n=0}^{\infty} 3^n x^n = 1 + 3x + 9x^2 + 27x^3 + \ldots$ | $(1, 3, 9, \ldots)$ | powers of 3 |

# More Examples of Generating Functions [2]

| Formula | Power series | Sequence | Description |
|---|---|---|---|
| $\frac{1}{1-x^2}$ | $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots$ | $(1, 0, 1, 0, \ldots)$ | regular gaps |
| $\frac{1}{(1-x)^2}$ | $\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \ldots$ | $(1, 2, 3, 4, \ldots)$ | natural numbers |

$$\begin{aligned}
\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} &= \\
&\triangleq (1, 1, 1, \ldots) - (\underbrace{0, 0, \ldots, 0}_{n+1 \text{ zeros}}, 1, 1, \ldots) = \\
&= (\underbrace{1, 1, \ldots, 1}_{n+1 \text{ ones}}, 0, 0, \ldots) = \\
&\triangleq 1 + x + x^2 + \ldots + x^n
\end{aligned}$$

# Exercises

*Example*: Find GF for odd numbers: $(1, 3, 5, ...)$.

*Example*: Find GF for $(1, 3, 7, 15, 31, 63)$, which satisfies $a_n = 3a_{n-1} - 2a_{n-2}$ with $a_0 = 1$, $a_1 = 3$.

# Solving Combinatorial Problems via Generating Functions

*Example*: Find the number of integer solutions to $y_1 + y_2 + y_3 = 12$ with $0 \leq x_i \leq 6$.

- Possible values for $y_1$ are $0 \leq y_1 \leq 6$.
  - There is a *single* way to select $y_1 = 0$. The same for other values among $1, ..., 6$.
  - There are *no* ways to select any value of $y_1$ higher than 6.
  - The *number of ways to select $y_1$ to be equal to $n$* forms a sequence $(1, 1, 1, 1, 1, 1, 1, 0, ...)$.
  - Write this sequence as a polynomial $x^0 + x^1 + ... + x^6$.
  - Do the same for $y_2$ and $y_3$ (*in isolation*!).
- Since all combinations of $y_1$, $y_2$ and $y_3$ are valid non-conflicting solutions, we can multiply those polynomials and obtain the *generating function* $G(x) = \left(1 + x + x^2 + ... + x^6\right)^3$.
  - For each $n$, the coefficient of $x^n$ in $G(x)$ is the number of integer solutions to $x_1 + x_2 + x_3 = n$.
  - In particular, we are interested in the coefficient of $x^{12}$ in $G(x)$, denoted $\left[x^{12}\right]G(x)$.
  - Use ~~pen and paper~~ Wolfram Alpha to expand $G(x)$:

$$G(x) = x^{18} + 3x^{17} + 6x^{16} + ... + \underbrace{28x^{12}} + ... + 6x^2 + 3x + 1$$

- The *answer* is $\left[x^{12}\right]G(x) = 28$ solutions.

# Slightly More Complex Combinatorial Problem

*Example*: Suppose we have marbles of three different colors (🔴 , 🟢 , 🔵 ), and we want to *count* the number of ways to select 20 marbles, such that:

- There are an even number of 🔴 : $1 + x^2 + x^4 + \ldots + x^{20}$.
- There are at least 12 🟢 : $x^{12} + x^{13} + \ldots + x^{20}$.
- There are at most 5 🔵 : $1 + x + x^2 + x^3 + x^4 + x^5$.

Multiply polynomials and find $[x^{20}]G(x)$:

$$
\begin{aligned}
&[x^{20}](1 + x^2 + x^4 + \ldots + x^{20})(x^{12} + x^{13} + \ldots + x^{20})(1 + x + x^2 + x^3 + x^4 + x^5) = \\
&= [x^{20}]\left(x^{45} + 2x^{44} + \ldots + \underbrace{21x^{20}} + \ldots + 2x^{13} + x^{12}\right) \\
&= 21
\end{aligned}
$$

# Using Power Series in Combinatorial Problems

*Example*: Find the number of integer solutions to $a_1 + a_2 + a_3 = 12$ with $a_1 \geq 2$, $3 \leq a_2 \leq 6$, $0 \leq a_3 \leq 9$.

- Compose the generating function:

$$G(x) = (x^2 + x^3 + \ldots) \cdot (x^3 + x^4 + x^5 + x^6) \cdot (1 + x + x^2 + \ldots + x^9)$$

- Substitute the power series with the corresponding simple forms:

$$G(x) = \left(x^2 \cdot \frac{1}{1-x}\right) \cdot \left(x^3 \cdot \frac{1-x^4}{1-x}\right) \cdot \left(\frac{1-x^{10}}{1-x}\right)$$

- Expand the series:

$$G(x) = x^5 + 3x^6 + 6x^7 + 10x^8 + 14x^9 + 18x^{10} + 22x^{11} + \underbrace{26x^{12}} + 30x^{13} + 34x^{14} + 37x^{15} + 39x^{16} + 40x^{17} + \ldots + 40x^n + \ldots$$

- Sequence: $(g_n) = (0, 0, 0, 0, 0, 1, 3, 6, 10, 14, 18, 22, 26, 30, 34, 37, 39, \overline{40}, \ldots)$
- Answer for $n = 12$ is $[x^{12}]G(x) = 26$.

# Operations on Generating Functions

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be ordinaty generating functions.

| **Operation** | **Result** |
|---|---|
| Differentiate $F(x)$ term-wise | $F'(x) = \sum\limits_{n=0}^{\infty} (n+1) a_{n+1} x^n$ |
| Multiply $F(x)$ by a scalar $\lambda \in \mathbb{R}$ term-wise | $\lambda F(x) = \sum\limits_{n=0}^{\infty} \lambda a_n x^n$ |
| Add $F(x)$ and $G(x)$ term-wise | $F(x) + G(x) = \sum\limits_{n=0}^{\infty} (a_n + b_n) x^n$ |
| Multiply $F(x)$ and $G(x)$ term-wise (*Cauchy product*, or *convolution*) | $F(x) \cdot G(x) = \sum\limits_{n=0}^{\infty} \left( \sum\limits_{k=0}^{n} a_k b_{n-k} \right) x^n$ |

# Well-Formed Paranthesis Expressions

*Example*: Find the number of *well-formed parenthesis expressions* with $n$ pairs of parenthesis.

For example, “(()())()” is a well-formed parenthesis expression with 4 pairs of parenthesis.

Formally, a permutation of the multiset $\{n \cdot "(", n \cdot ")"\}$ is *well-formed* if reading it from left to right and counting “+1” for every opening parenthesis “(“ and “−1” for every closing parenthesis “)” never yields a negative number at any time.

Every well-formed expression with $n \geq 1$ pairs of paranthesis starts with “(“ and there is a unique matching “)” such that the sequence in between and the sequence after are well-formed. For example:

$$\Big(()\Big)()() \quad \Big(()(())\Big) \quad \Big(\Big)((()(())))$$

In other words, a well-formed expression with $n$ pairs of parenthesis is obtained by putting a well-formed expression with $k$ pairs in between “(“ and “)” and then appending a well-formed expression with $n-k-1$ pairs of parenthesis. This gives the equation:

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1}$$

# Well-Formed Paranthesis Expressions [2]

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1}$$

Let $F(x)$ be a generating function for $a_n$, then we know:

$$F(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} a_k a_{n-k-1} \right) x^n = 1 + \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^{n+1}$$

$$= 1 + x \cdot \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k a_{n-k} \right) x^n = 1 + x \cdot F(x)^2$$

# Newton's Binomial Theorem

Let's revisit the binomial theorem:

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k \quad \forall n \in \mathbb{N}$$

where $\binom{n}{k} = 0$ for $k > n$.

**Note**: This shows that $(1+x)^n$ is the generating function for the series $(a_k)_{k\in\mathbb{N}}$ with $a_k = \binom{n}{k}$.

We can extend this result from natural numbers $n \in \mathbb{N}$ to any *real* number $n \in \mathbb{R}$.

# Binomial Coefficients for Real Numbers

**Definition 23**: Let $p(n, k) = n \cdot (n-1) \cdot \ldots \cdot (n-k+1)$, also called the *falling factorial* $n^{\underline{k}}$.

Extend the definition of binomial coefficients for real numbers $n, k \in \mathbb{R}$:

$$\binom{n}{k} = \frac{p(n, k)}{k!}$$

**Note**: This definition aligns with the definition of binomial coefficients for natural numbers:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!}$$

*Example*: Consider the number "$-7/2$ choose 5":

$$\binom{-7/2}{5} = \frac{-\frac{7}{2} \cdot -\frac{9}{2} \cdot -\frac{11}{2} \cdot -\frac{13}{2} \cdot -\frac{15}{2}}{5!} = -\frac{9009}{256}$$

**Note**: $p(n, 0) = 1$ and for $k \geq 1$, we have $p(n, k) = (n-k+1) \cdot p(n, k-1) = n \cdot p(n-1, k-1)$. $\quad (\star)$

# Extended Newton's Binomial Theorem

**Theorem 17**: For all non-zero $n \in \mathbb{R}$, we have:

$$(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

*Example*: Let $n = 1/2$, then we have an identity for $\sqrt{1 + x}$:

$$\sqrt{1 + x} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k$$

To actually *use* this fact, we need some *lemma*...

**Lemma 18**: For any integer $n \geq 1$, we have:

$$\binom{1/2}{n} = (-1)^{n+1} \cdot \binom{2n - 2}{n - 1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{1}{n}$$

# Extended Newton's Binomial Theorem [2]

**Proof**: By induction on $n$.

**Base**: $n = 1$.

$$\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2} = 1 \cdot 1 \cdot \frac{1}{2} \cdot 1 = \underbrace{(-1)^2}_{1} \cdot \underbrace{\binom{2-2}{1-1}}_{1} \cdot \underbrace{\frac{1}{2^{2-1}}}_{\frac{1}{2}} \cdot \underbrace{\frac{1}{1}}_{1}$$

**Induction step**: $n$ to $n+1$ for $n > 1$. We use the recusion $(\star)$ $p(n,k) = n \cdot p(n-1, k-1)$:

$$\begin{aligned}
\binom{1/2}{n+1} &= \frac{p(1/2, n+1)}{(n+1)!} = \frac{(1/2 - (n+1) + 1) \cdot p(1/2, n)}{(n+1) \cdot n!} = -\frac{n - 1/2}{n+1}\binom{1/2}{n} \\
&\overset{\text{IH}}{=} -\frac{n-1/2}{n+1}(-1)^{n+1} \cdot \binom{2n-2}{n-1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{1}{n} \\
&= \frac{2n}{2n} \cdot \frac{2n-1}{2n} \cdot (-1)^{n+2} \cdot \binom{2n-2}{n-1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{1}{n+1} \\
&= (-1)^{n+2} \cdot \underbrace{\frac{(2n-2)! \cdot (2n-1) \cdot (2n)}{(n-1)! \cdot (n-1) \cdot n \cdot n}}_{\binom{2n}{n}} \cdot \frac{1}{2^{2n+1}} \cdot \frac{1}{n+1}
\end{aligned}$$

$\square$

# Catalan Numbers

**Proposition 19**: Now we can expand $\sqrt{1+n}$ into the following series:

$$\sqrt{1+n} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \sum_{n=1}^{\infty} -2 \cdot \binom{2n-2}{n-1} \cdot (-1)^n \cdot \frac{1}{2^{2n}} \cdot \frac{1}{n} \cdot x^n$$

*Example*: Going back to the example with the number of well-formed paranthesis expressions, we get:

$$F(x) = \frac{1-\sqrt{1-4x}}{2x} = \frac{1}{2x} \sum_{n=1}^{\infty} 2 \cdot \binom{2n-2}{n-1} \cdot (-1)^n \cdot \frac{1}{2^{2n}} \cdot \frac{1}{n} \cdot (-4x)^n$$

$$= \frac{1}{x} \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{1}{n} x^n = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^n$$

The numbers $C_n := \binom{2n}{n} \frac{1}{n+1}$ are called *Catalan numbers*.

# §10 Recurrence Relations

# Recurrence Relations

*Example*:

- *Recurrent relation* defining a sequence $(a_n)$:

$$a_n = \begin{cases} a_0 = \text{const} & \text{if } n = 0 \\ a_{n-1} + d & \text{if } n > 0 \end{cases}$$

- *Solving* it results in a non-recursive *closed* formula:

$$a_n = a_0 + n \cdot d$$

- *Checking* it confirms that the formula is correct:

$$a_n = \underbrace{a_{n-1}} + d = \underbrace{a_0 + (n-1)d}_{a_{n-1}} + d = a_0 + n \cdot d \quad \blacksquare$$

# Linear Homogeneous Recurrence Relations

**Definition 24**: A *linear homogeneous* recurrence relation *of degree $k$* with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where $c_1, c_2, \dots, c_k$ are constants (real or complex numbers), and $c_k \neq 0$.

*Examples*:

- $b_n = 2.71 b_{n-1}$ is a linear homogeneous recurrence relation of degree 1.
- $F_n = F_{n-1} + F_{n-2}$ is a linear homogeneous recurrence relation of degree 2.
- $g_n = 2 g_{n-5}$ is a linear homogeneous recurrence relation of degree 5.
- The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is *not linear*.
- The recurrence relation $H_n = 2H_{n-1} + 1$ is *not homogeneous*.
- The recurrence relation $B_n = nB_{n-1}$ does *not* have *constant* coefficients.

# Characteristic Equations

Hereinafter, $(*)$ denotes a linear homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.

**Theorem 20**: $a_n = r^n$ is a solution to $(*)$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}$.

**Definition 25**: A *characteristic equation* for $(*)$ is the algebraic equation in $r$ defined as:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$$

The sequence $(a_n)$ with $a_n = r^n$ (with $r_n \neq 0$) is a solution if and only if $r$ is a solution of the characteristic equation. Such solutions are called *characteristic roots* of $(*)$.

# Distinct Roots Case

**Theorem 21**: Let $c_1$ and $c_2$ be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two *distinct* roots $r_1$ and $r_2$. Then the sequence $(a_n)$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$, where $\alpha_1$ and $\alpha_2$ are constants.

**Proof** *(sketch)*: Since $r_1$ and $r_2$ are roots, then $r_1^2 = c_1 r_1 + c_2$ and $r_2^2 = c_1 r_2 + c_2$. Next, we can see:

$$\begin{aligned}
c_1 a_{n-1} + c_2 a_{n-2} &= c_1\left(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}\right) + c_2\left(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2}\right) \\
&= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2) \\
&= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2 \\
&= \alpha_1 r_1^n + \alpha_2 r_2^n \\
&= a_n
\end{aligned}$$

To show that every solution $(a_n)$ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for some constants $\alpha_1$ and $\alpha_2$, suppose that the initial condition are $a_0 = C_0$ and $a_1 = C_1$, and show that there exist constants $\alpha_1$ and $\alpha_2$ such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies the same initial conditions. $\square$

# Solving Recurrence Relations using Characteristic Equations

*Example*: Solve $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

- The characteristic equation is $r^2 - r - 2 = 0$.
- It has two distinct roots $r_1 = 2$ and $r_2 = -1$.
- The sequence $(a_n)$ is a solution iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$ and some constants $\alpha_1$ and $\alpha_2$.

$$\begin{cases} a_0 = 2 = \alpha_1 + \alpha_2 \\ a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1) \end{cases}$$

- Solving these two equations gives $\alpha_1 = 3$ and $\alpha_2 = -1$.
- Hence, the *solution* to the recurrence equation with given initial conditions is the sequence $(a_n)$ with

$$a_n = 3 \cdot 2^n - (-1)^n$$

# Fibonacci Numbers

*Example*: Find the closed formula for Fibonacci numbers.

- The recurrence relation is $F_n = F_{n-1} + F_{n-2}$.
- The characteristic equation is $r^2 - r - 1 = 0$.
- The roots are $r_1 = (1 + \sqrt{5})/2$ and $r_2 = (1 - \sqrt{5})/2$.
- Therefore, the solution is $F_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$ for some constants $\alpha_1$ and $\alpha_2$.
- Using the initial conditions $F_0 = 0$ and $F_1 = 1$, we get

$$\begin{cases} F_0 = \alpha_1 + \alpha_2 = 0 \\ F_1 = \alpha_1 \cdot (\frac{1+\sqrt{5}}{2}) + \alpha_2 \cdot (\frac{1-\sqrt{5}}{2}) = 1 \end{cases}$$

- Solving these two equations gives $\alpha_1 = 1/\sqrt{5}$ and $\alpha_2 = -1/\sqrt{5}$.
- Hence, the *closed formula* (also known as Binet's formula) for Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\varphi}^n - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\psi}^n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

# Single Root Case

**Theorem 22**: Let $c_1$ and $c_2$ be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has a *single* root $r_0$. A sequence $(a_n)$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \ldots$, where $\alpha_1$ and $\alpha_2$ are constants.

*Example*: Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$.

The characteristic equation is $r^2 - 6r + 9 = 0$ with a single (repeated) root $r_0 = 3$. Hence, the solutions is of the form $a_n = \alpha_1 3^n + \alpha_2 n 3^n$.

$$\begin{cases} a_0 = 1 = \alpha_1 \\ a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = 1 \\ \alpha_2 = 1 \end{cases}$$

Thus, the *solution* is $a_n = 3^n + n 3^n$.

# Generic Case

TODO

# Linear Non-Homogeneous Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$$

*Example*: $a_n = 3a_{n-1} + 2n$ is non-homogeneous.

**Definition 26**: An *associated homogeneous recurrence relation* is the relation without the term $F(n)$.

# Solving Non-Homogeneous Recurrence Relations

**Theorem 23**: If $(a_n^{(p)})$ is a *particular* solution of the non-homogeneous linear recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + F(n)$, then *every solution* is of the form $(a_n^{(p)} + a_n^{(h)})$, where $(a_n^{(h)})$ is a solution of the associated homogeneous recurrence relation.

*Example*: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

- First, solve the associated homogeneous recurrence relation $a_n = 3a_{n-1}$.
- It has a general solution $a_n^{(h)} = \alpha 3^n$, where $\alpha$ is a constant.
- To find a particular solution, observe that $F(n) = 2n$ is a polynomial in $n$ of degree 1, so a reasonable trial solution is a linear function in $n$, for example, $p_n = cn + d$, where $c$ and $d$ are constants.
- Thus, the equation $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.
- Simplify and reorder: $(2 + 2c)n + (2d - 3c) = 0$.

$$\begin{cases} 2 + 2c = 0 \\ 2d - 3c = 0 \end{cases} \implies \begin{cases} c = -1 \\ d = -3/2 \end{cases}$$

- Thus, $a_n^{(p)} = -n - 3/2$ is a *particular* solution.

# Solving Non-Homogeneous Recurrence Relations [2]

- By Theorem 23, all solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n,$$

  where $\alpha$ is a constant.
- To find the solution with $a_1 = 3$, let $n = 1$ in the formula: $3 = -1 - 3/2 + 3\alpha$, thus $\alpha = 11/6$.
- The *solution* is $a_n = -n - 3/2 + (11/6)3^n$.

# §11 Annihilators

# Operators

**Definition 27**: *Operators* are higher-order functions that transform functions into other functions.

For example, differential and integral operators $\frac{d}{dx}$ and $\int dx$ are core operators in calculus.

In combinatorics, we are interested in the following three operators:

- *Sum*: $(f + g)(n) := f(n) + g(n)$
- *Scale*: $(\alpha \cdot f)(n) := \alpha \cdot f(n)$
- *Shift*: $(\mathbf{E}\, f)(n) := f(n + 1)$

*Examples*:

- Scale and Shift operators are *linear*: $\mathbf{E}(f - 3(g - h)) = \mathbf{E}\, f + (-3)\, \mathbf{E}\, g + 3\, \mathbf{E}\, h$
- Operators are *composable*: $(\mathbf{E} - 2) f := \mathbf{E}\, f + (-2) f$
- $\mathbf{E}^2\, f = \mathbf{E}(\mathbf{E}\, f)$
- $\mathbf{E}^k\, f(n) = f(n + k)$
- $(\mathbf{E} - 2)^2 = (\mathbf{E} - 2)(\mathbf{E} - 2)$
- $(\mathbf{E} - 1)(\mathbf{E} - 2) = \mathbf{E}^2 - 3\, \mathbf{E} + 2$

# Applying Operators

*Examples*: Below are the results of applying different operators to $f(n) = 2^n$:

$$
\begin{aligned}
2f(n) &= 2 \cdot 2^n = 2^{n+1} \\
3f(n) &= 3 \cdot 2^n \\
\mathbf{E}\, f(n) &= 2^{n+1} \\
\mathbf{E}^2\, f(n) &= 2^{n+2} \\
(\mathbf{E} - 2) f(n) &= \mathbf{E}\, f(n) - 2f(n) = 2^{n+1} - 2^{n+1} = 0 \\
(\mathbf{E}^2 - 1) f(n) &= \mathbf{E}^2\, f(n) - f(n) = 2^{n+2} - 2^n = 3 \cdot 2^n
\end{aligned}
$$

# Compound Operators

The compound operators can be seen as polynomials in “variable” $\mathbf{E}$.

*Example*: The compound operators $\mathbf{E}^2 - 3\,\mathbf{E} + 2$ and $(\mathbf{E} - 1)(\mathbf{E} - 2)$ are equivalent:

$$\begin{aligned}
\text{Let } g(n) &:= (\mathbf{E} - 2) f(n) = f(n+1) - 2f(n) \\
\text{Then } (\mathbf{E} - 1)(\mathbf{E} - 2) f(n) &= (\mathbf{E} - 1) g(n) \\
&= g(n+1) - g(n) \\
&= [f(n+2) - 2f(n-1)] - [f(n+1) - 2f(n)] \\
&= f(n+2) - 3f(n+1) + 2f(n) \\
&= (\mathbf{E}^2 - 3\,\mathbf{E} + 2) f(n) \quad \checkmark
\end{aligned}$$

# Operators Summary

| Operator | Definition |
|---|---|
| addition | $(f+g)(n) := f(n) + g(n)$ |
| subtraction | $(f-g)(n) := f(n) - g(n)$ |
| multiplication | $(\alpha \cdot f)(n) := \alpha \cdot f(n)$ |
| shift | $\mathbf{E}\, f(n) := f(n+1)$ |
| k-fold shift | $\mathbf{E}^k\, f(n) := f(n+k)$ |
| composition | $(X+Y)f := X\, f + Y\, f$ |
| | $(X-Y)f := X\, f - Y\, f$ |
| | $X\, Y\, f := X(Y\, f) = Y(X\, f)$ |
| distribution | $X(f+g) = X\, f + X\, g$ |

# Annihilators

**Definition 28**: An *annihilator* of a function $f$ is any non-trivial operator that transforms $f$ into zero.

TODO: examples!

# Annihilators Summary

| Operator | Functions annihilated |
|---|---|
| $\mathbf{E}-1$ | $\alpha$ |
| $\mathbf{E}-a$ | $\alpha a^n$ |
| $(\mathbf{E}-a)(\mathbf{E}-b)$ | $\alpha a^n + \beta b^n$ [if $a \neq b$] |
| $(\mathbf{E}-a_0)(\mathbf{E}-a_1)\ldots(\mathbf{E}-a_k)$ | $\sum_{i=0}^{k} \alpha_i a_i^n$ [if $a_i$ are distinct] |
| $(\mathbf{E}-1)^2$ | $\alpha n + \beta$ |
| $(\mathbf{E}-a)^2$ | $(\alpha n + \beta)a^n$ |
| $(\mathbf{E}-a)^2(\mathbf{E}-b)$ | $(\alpha n + \beta)a^n + \gamma b^n$ [if $a \neq b$] |
| $(\mathbf{E}-a)^d$ | $\left(\sum_{i=0}^{d-1} \alpha_i n^i\right)a^n$ |

# Properties of Annihilators

**Theorem 24**: If $X$ annihilates $f$, then $X$ also annihilates $\alpha f$ for any constant $\alpha$.

**Theorem 25**: If $X$ annihilates both $f$ and $g$, then $X$ also annihilates $f \pm g$.

**Theorem 26**: If $X$ annihilates $f$, then $X$ also annihilates $\mathbf{E} f$.

**Theorem 27**: If $X$ annihilates $f$ and $Y$ annihilates $g$, then $X Y$ annihilates $f \pm g$.

# Annihilating Recurrences

1. Write the recurrence in the *operator form*.
2. Find the *annihilator* for the recurrence.
3. *Factor* the annihilator, if necessary.
4. Find the *generic solution* from the annihilator.
5. Solve for coefficients using the *initial conditions*.

*Example*: $r(n) = 5r(n-1)$ with $r(0) = 3$.

1. $r(n+1) - 5r(n) = 0$
   $(\mathbf{E} - 5)r(n) = 0$
2. $(\mathbf{E} - 5)$ annihilates $r(n)$.
3. $(\mathbf{E} - 5)$ is already factored.
4. $r(n) = \alpha 5^n$ is a generic solution.
5. $r(0) = \alpha = 3 \quad \Longrightarrow \quad \alpha = 3$

Thus, $r(n) = 3 \cdot 5^n$.

# Annihilating Recurrences [2]

*Example*: $T(n) = 2T(n-1) + 1$ with $T(0) = 0$

1. $T(n+1) - 2T(n) = 1$
   $(\mathbf{E} - 2)T(n) = 1$
2. $(\mathbf{E} - 2)$ does *not* annihilate $T(n)$: the residue is 1.
   $(\mathbf{E} - 1)$ annihilates the residue 1.
   Thus, $(\mathbf{E} - 1)(\mathbf{E} - 2)$ annihilates $T(n)$.
3. $(\mathbf{E} - 1)(\mathbf{E} - 2)$ is already factored.
4. $T(n) = \alpha 2^n + \beta$ is a generic solution.
5. Find the coefficients $\alpha, \beta$ using $T(0) = 0$ and $T(1) = 2T(0) + 1 = 1$:

$$\left.\begin{array}{l} T(0) = 0 = \alpha \cdot 2^0 + \beta \\ T(1) = 1 = \alpha \cdot 2^1 + \beta \end{array}\right\} \implies \left\{\begin{array}{l} \alpha = 1 \\ \beta = -1 \end{array}\right.$$

Thus, $T(n) = 2^n - 1$.

# Annihilating Recurrences [3]

*Example*: $T(n) = T(n-1) + 2T(n-2) + 2^n - n^2$

1. Operator form:
   $(\mathbf{E}^2 - \mathbf{E} - 2)T(n) = \mathbf{E}^2(2^n - n^2)$
2. Annihilator:
   $(\mathbf{E}^2 - \mathbf{E} - 2)(\mathbf{E} - 2)(\mathbf{E} - 1)^3$
3. Factorization:
   $(\mathbf{E} + 1)(\mathbf{E} - 2)^2(\mathbf{E} - 1)^3$
4. Generic solution:
   $T(n) = \alpha(-1)^n + (\beta n + \gamma)2^n + \delta n^2 + \varepsilon n + n$
5. There are no initial conditions. We can only provide an asymptotic bound.

Thus, $T(n) \in \Theta(n2^n)$

# §12 Asymptotic Analysis

# Asymptotics 101

**Definition 29** (*Big-O notation*): The notation $f \in O(g)$ means that the function $f(n)$ is *asymptotically bounded from above* by the function $g(n)$, up to a constant factor.

$$f(n) \in O(g(n)) \quad \leftrightarrow \quad \exists c > 0.\, \exists n_0.\, \forall n > n_0 : |f(n)| \leq c \cdot g(n)$$

**Definition 30** (*Small-o notation*): The notation $f \in o(g)$ means that the function $f(n)$ is *asympotically dominated* by $g(n)$, up to a constant factor.

$$f(n) \in o(g(n)) \quad \leftrightarrow \quad \forall c > 0.\, \exists n_0.\, \forall n > n_0 : |f(n)| \leq c \cdot g(n)$$

**Note**: The difference is only in the $\exists c$ and $\forall c$ quantifier.

**Note**: Flip $\leq$ to $\geq$ in the above definitions to obtain the dual notations: $f \in \Omega(g)$ and $f \in \omega(g)$.

**Definition 31** (*Theta notation*): $f \in \Theta(g)$ iff $f \in O(g)$ and $g \in O(f)$.

# Limits

| Notation | Name | Description | Limit definition |
|---|---|---|---|
| $f \in o(g)$ | Small Oh | $f$ is dominated by $g$ | $\lim\limits_{n \longrightarrow \infty} \dfrac{f(n)}{g(n)} = 0$ |
| $f \in O(g)$ | Big Oh | $f$ is bounded above by $g$ | $\limsup\limits_{n \longrightarrow \infty} \dfrac{\lvert f(n) \rvert}{g(n)} < \infty$ |
| $f \sim g$ | Equivalence | $f$ is asympotically equal to $g$ | $\lim\limits_{n \longrightarrow \infty} \dfrac{f(n)}{g(n)} = 1$ |
| $f \in \Omega(g)$ | Big Omega | $f$ is bounded below by $g$ | $\liminf\limits_{n \longrightarrow \infty} \dfrac{f(n)}{g(n)} > 0$ |
| $f \in \omega(g)$ | Small Omega | $f$ dominates $g$ | $\lim\limits_{n \longrightarrow \infty} \dfrac{f(n)}{g(n)} = \infty$ |

# Asymptotic Equivalence

**Definition 32**: The notation $f \sim g$ means that functions $f(n)$ and $g(n)$ are *asymptotically equivalent*.

$$f \sim g \quad \leftrightarrow \quad \forall \varepsilon > 0.\, \exists n_0.\, \forall n > n_0 : \left| \frac{f(n)}{g(n)} - 1 \right| \leq \varepsilon \quad \leftrightarrow \quad \lim_{n \longrightarrow \infty} \frac{f(n)}{g(n)} = 1$$

**Note**: $f \sim g$ and $g \sim f$ are equivalent, since $\sim$ is an equivalence relation.

**Note**: $f \sim g$ and $f \in \Theta(g)$ are *different* notions!

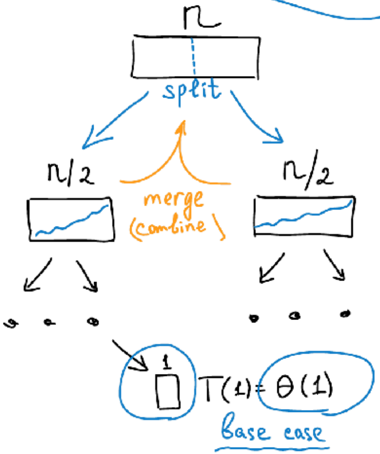
# Some Properties of Asymptotics

$$
\begin{aligned}
f \in O(g) \text{ and } f \in \Omega(g) &\quad\leftrightarrow\quad f \in \Theta(g) \\
f \in O(g) &\quad\leftrightarrow\quad g \in \Omega(f) \\
f \in o(g) &\quad\leftrightarrow\quad g \in \omega(f) \\
f \in o(g) &\quad\rightarrow\quad f \in O(g) \\
f \in \omega(g) &\quad\rightarrow\quad f \in \Omega(g) \\
f \sim g &\quad\rightarrow\quad f \in \Theta(g)
\end{aligned}
$$

# Divide-and-Conquer Algorithms Analysis

Divide-and-Conquer

$$T(n) = \underbrace{2\,T(n/2)}_{\text{recursive work}} + \underbrace{\Theta(n)}_{\text{split / merge work}}$$




$$\Sigma = 2 \cdot 2 \cdot \underbrace{T(n/4)}_{\text{merge}} = n$$

$$\Sigma \approx 2^{i+1}\, T(n/i) = n$$

$$\Sigma = n \cdot \Theta(1) = n$$

$$\underbrace{\Theta(n)}_{\text{work on each level of recursion tree}} \cdot \underbrace{\Theta(\log n)}_{\text{height of recursion tree}} = \Theta(n \log n)$$

# Divide-and-Conquer Recurrence

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- $T(n)$ is the *cost* of the recursive algorithm
- $a$ is the number of *parts* (*sub-problems*)
- $n/b$ is the *size* of each part
- $T\left(\frac{n}{b}\right)$ is the cost of each *sub-problem*
- $f(n)$ is the cost of *splitting* and *merging* the solutions of the subproblems

Hereinafter, $c_{\text{crit}} = \log_b a$ is a *critical constant*.

# Master Theorem

The master theorem [3] applies to divide-and-conquer recurrences of the form

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

| **Case** | **Description** | **Condition** | **Bound** |
|---|---|---|---|
| Case I | "merge" $\ll$ "recursion" | $f(n) \in O(n^c)$ where $c < c_{\text{crit}}$ | $T(n) \in \Theta(n^{c_{\text{crit}}})$ |
| Case II | "merge" $\approx$ "recursion" | $f(n) \in \Theta\left(n^{c_{\text{crit}}} \log^k n\right)$ where $k \geq 0$ | $T(n) \in \Theta\left(n^{c_{\text{crit}}} \log^{k+1} n\right)$ |
| Case III | "merge" $\gg$ "recursion" | $f(n) \in \Omega(n^c)$ where $c > c_{\text{crit}}$ | $T(n) \in \Theta(f(n))$ |

**Note**: Case III also requires the *regularity condition* to hold: $af(n/b) \leq kf(n)$ for some constant $k < 1$ and all sufficiently large $n$.

**Note**: There is an *extended* Case II, with three sub-cases (IIa, IIb, IIc) for other values of $k$. See wiki.

# Examples of Master Theorem Application

*Examples*: Determine the case of Master Theorem and the bound of $T(n)$ for the following recurrences.

1. $T(n) = 3T(n/9) + \sqrt{n}$
2. $T(n) = 2T(n/4) + n^{0.51}$
3. $T(n) = 5T(n/25) + n^{0.49}$
4. $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil)$
5. $T(n) = 3T(n/9) + \frac{\sqrt{n}}{\log n}$
6. $T(n) = 6T(n/36) + \frac{\sqrt{n}}{\log^2 n}$
7. $T(n) = 4T(n/16) + \sqrt{\frac{n}{\log n}}$

# Akra–Bazzi Method

The Akra–Bazzi method [4] is a *generalization* of the master theorem to recurrences of the form

$$T(n) = f(n) + \sum_{i=1}^{k} a_i T\left(b_i n + \underbrace{h_i(n)}_{*}\right)$$

- $k$ is a constant
- $a_i > 0$
- $0 < b_i < 1$
- $h_i(n) \in O\left(\frac{n}{\log^2 n}\right)$ is a *small perturbation*

Bound of $T(n)$ by Akra–Bazzi method:

$$T(n) \in \Theta\left(n^p \cdot \left(1 + \int_1^n \frac{f(x)}{x^{p+1}} dx\right)\right)$$

where $p$ is the solution for the equation $\sum_{i=1}^{k} a_i b_i^p = 1$

# Example of Akra–Bazzi Method Application

*Example*: Suppose the runtime of an algorithm is expressed by the following recurrence relation:

$$T(n) = \begin{cases} 1 \text{ for } 0 \leq n \leq 3 \\ n^2 + \frac{7}{4} T\left(\left\lfloor \frac{1}{2} n \right\rfloor\right) + T\left(\left\lceil \frac{3}{4} n \right\rceil\right) \text{ for } n > 3 \end{cases}$$

- Note that the Master Theorem *is not* applicable here, since there are *two* different recursive terms.
- Let's apply the Akra–Bazzi method. First, solve the equation $\frac{7}{4}\left(\frac{1}{2}\right)^p + \left(\frac{3}{4}\right)^p = 1$. This gives us $p = 2$.
- Next, use the formula from AB-method to obtain the bound:

$$\begin{aligned} T(x) \in \Theta\left(x^p \left(1 + \int_1^x \frac{f(u)}{u^{p+1}} du\right)\right) &= \\ = \Theta\left(x^2 \left(1 + \int_1^x \frac{u^2}{u^3} du\right)\right) &= \\ = \Theta(x^2 (1 + \ln x)) &= \\ = \Theta(x^2 \log x) \end{aligned}$$

# §13 Advanced Topics

# Gamma Function

**Definition 33**: *Gamma function* $\Gamma(x)$ is the most common *extension* of the factorial function to real and complex numbers. It is defined for all complex numbers $z \in \mathbb{C}$ (except non-positive integers) as

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

For positive integers $z = n$, it is defined as

$$\Gamma(n) = (n-1)!$$

**Motivation**: The factorial is defined for positive integers as $n! = 1 \cdot 2 \cdot \ldots \cdot n = (n-1)! \cdot n$.
We want to *extend* this definition to *all real numbers* and capture its *recursive* nature.
Overall, we are looking for a *smooth* function $\Gamma(x)$ such that:

- $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$, matching the factorial.
- $\Gamma(x+1) = x \cdot \Gamma(x)$, satisfying a *recursive* property.
- $\Gamma(x)$ is defined for all *real* numbers $x > 0$.

# Definition of a Gamma Function

The main definition of a gamma function is known as *Euler integral of the second kind*:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Gauss proposed a function $\Gamma(x)$ defined by the *limit*

$$\Gamma(x) := \lim_{n \longrightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \ldots \cdot (x+n)} = \lim_{n \longrightarrow \infty} \frac{n! \cdot n^x}{\prod_{k=0}^{n} (x+k)} \quad \text{for } x > 0$$

# Integral Definition

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Let's check that the integral definition is indeed a suitable definition of a gamma function.

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty t^z e^{-t} dt \\ &= \left[-t^z e^{-t}\right]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} dt \\ &= \lim_{t \longrightarrow \infty} \left(-t^z e^{-t}\right) - \left(-0^z e^{-0}\right) + z \int_0^\infty t^{z-1} e^{-t} dt\end{aligned}$$

Note that $-t^z e^{-t} \longrightarrow 0$ as $t \longrightarrow \infty$, so:

$$\Gamma(z+1) = z \int_0^\infty t^{z-1} e^{-t} dt = z \cdot \Gamma(z)$$

# Limit Definition

$$\Gamma(x) = \lim_{n \longrightarrow \infty} \frac{n! \cdot n^x}{\prod_{k=0}^{n} (x+k)}$$

Let's check that the limit definition is indeed a suitable definition of a gamma function.

**Step 1**. Write $\Gamma(x+1)$.

$$\Gamma(x+1) = \lim_{n \longrightarrow \infty} \frac{n! \cdot n^{x+1}}{\prod_{k=0}^{n} (x+1+k)} = \lim_{n \longrightarrow \infty} \frac{n! \cdot n^{x+1}}{\prod_{k=1}^{n+1} (x+k)}$$

**Step 2**. Multiply both numerator and denominator by $x$ and rearrange:

$$= \lim_{n \longrightarrow \infty} \frac{n! \cdot n^{x+1}}{(x+1) \cdot \ldots \cdot (x+n+1)} = \lim_{n \longrightarrow \infty} \frac{n! \cdot n^{x}}{x \cdot (x+1) \cdot \ldots \cdot (x+n)} \cdot \frac{n}{x+n+1} \cdot x$$

**Step 3**. Take the limit. As $n \longrightarrow \infty$, the ratio $\frac{n}{x+n+1}$ approaches 1.

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

# Equivalence of Definitions

Let's prove the equivalence of two definitions: integral and limit.

We claim:

$$\int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt \stackrel{?}{=} \lim_{n\longrightarrow\infty} \frac{n!\cdot n^x}{x\cdot(x+1)\cdot\ldots\cdot(x+n)}$$

Note that as $n \longrightarrow \infty$, the integrand $\left(1-\frac{t}{n}\right)^n$ approaches $e^{-t}$, so this integral approximates $\Gamma(x)$.

Substitute $u = \frac{t}{n}$:

$$\int_0^n t^{x-1}\left(1-\frac{t}{n}\right)^n dt = n^x \int_0^1 u^{x-1}(1-u)^n du = n^x \cdot \mathrm{B}(x, n+1)$$

where $\mathrm{B}(x, n+1)$ is the Beta function:

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$$

# Equivalence of Definitions [2]

Then:

$$I_n = n^x \cdot \mathrm{B}(x, n+1) = n^x \cdot \frac{\Gamma(x) \cdot \Gamma(n+1)}{\Gamma(x+n+1)} = \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \ldots \cdot (x+n)}$$

Take the limit on both sides. Since $\lim\limits_{n \longrightarrow \infty} I_n = \Gamma(x)$, we have:

$$\Gamma(x) = \lim_{n \longrightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \ldots \cdot (x+n)}$$

# Using the Gamma Function

$$n! = \Gamma(n+1)$$

$$\binom{r}{k} = \frac{\Gamma(r+1)}{\Gamma(k+1) \cdot \Gamma(r-k+1)}$$

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

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