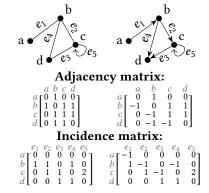
5 Graph Theory Cheatsheet

Glossary

- * **Graph** E is an ordered pair $G = \langle V, E \rangle$, where $V = \{v_1, \dots, v_n\}$ is a set of vertices, and $E = \{e_1, \dots, e_m\}$ is a set of edges. \circ Given a graph G, the notation V(G) denotes the vertices of G.
 - Given a graph G, the notation V(G) denotes the vertices of G.
 Given a graph G, the notation E(G) denotes the edges of G.
 - \circ In fact, $V(\cdot)$ and $E(\cdot)$ functions allow to access "vertices" and "edges" of any object possessing them (e.g., paths).
- * **Order** of a graph G is the number of vertices in it: |V(G)|.
- * **Size** of a graph G is the number of edges in it: |E(G)|.
- * Simple **undirected** graphs have $E \subseteq V^{(2)}$, *i.e.* each edge $e_i \in E$ between vertices u and v is denoted by $\{u, v\} \in V^{(2)}$. Such *undirected edges* are also called *links* or *lines*.
 - $\circ A^{(k)} = \{\{x_1, \dots, x_k\} \mid x_1 \neq \dots \neq x_k \in A\} = \{S \mid S \subseteq A, |S| = k\} \text{ is the set of } k\text{-sized subsets of } A.$
- * Simple **directed** graphs have $E \subseteq V^2$, *i.e.* each edge $e_i \in E$ from vertex u to v is denoted by an ordered pair $\langle u, v \rangle \in V^2$. Such *directed edges* are also called *arcs* or *arrows*.
 - ∘ $A^k = A \times \cdots \times A = \{(x_1, \dots, x_k) \mid x_1, \dots, x_k \in A\}$ is the set of *k*-tuples (Cartesian *k*-power of *A*).
- * Multi-edges are edges that have the same end nodes.
- * **Loop** [™] is an edge that connects a vertex to itself.
- * **Simple graph** is a graph without multi-edges and loops.
- * **Multigraph**[™] is a graph with multi-edges.
- * **Pseudograph** [™] is a multigraph with loops.
- * Null graph is a "graph" without vertices.
- * Trivial (singleton) graph is a graph consisting of a single vertex.
- * Complete graph K_n is a simple graph in which every pair of distinct vertices is connected by an edge.

Null

- * **Subgraph** of a graph $G = \langle V, E \rangle$ is another graph $G' = \langle V', E' \rangle$ such that $V' \subseteq V, E' \subseteq E$. Designated as $G' \subseteq G$.
- * **Spanning (partial) subgraph** is a subgraph that includes all vertices of a graph.
- * **Induces subgraph** \subseteq of a graph $G = \langle V, E \rangle$ is another graph G' formed from a subset S of the vertices of the graph and *all* the edges (from the original graph) connecting pairs of vertices in that subset. Formally, $G' = G[S] = \langle V', E' \rangle$, where $S \subseteq V$, $V' = V \cap S$, $E' = \{e \in E \mid \exists v \in S : e \mid E \mid E \mid v \in S : e \mid E$
- * **Adjacency** is the relation between two vertices connected with an edge.
- * **Adjacency matrix** is a square matrix $A_{V\times V}$ of an adjacency relation.
 - For simple graphs, adjacency matrix is binary, *i.e.* $A_{ij} \in \{0, 1\}$.
 - ∘ For directed graphs, A_{ij} ∈ {0, 1, −1}.
 - ∘ For multigraphs, adjacency matrix contains edge multiplicities, *i.e.* A_{ij} ∈ \mathbb{N}_0 .
- * **Incidence** is a relation between an edge and its endpoints.
- * **Incidence matrix** $^{\mathbb{Z}}$ is a Boolean matrix $B_{V \times E}$ of an incidence relation.
- * **Degree** $^{\mathbb{Z}}$ deg(v) the number of edges incident to v (loops are counted twice).
 - $\circ \ \delta(G) = \min_{v \in V} \deg(v) \text{ is the } \mathbf{minimum } \mathbf{degree}.$
 - $\Delta(G) = \max_{v \in V} \deg(v)$ is the **maximum degree**.
 - Handshaking Lemma. $\sum_{v \in V} \deg(v) = 2|E|$.



- * A graph is called r-regular if all its vertices have the same degree: $\forall v \in V : \deg(v) = r$.
- * Complement graph \mathcal{C} of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent iff they are non-adjacent in G.
- * **Intersection graph** $^{\bowtie}$ of a family of sets $F = \{S_i\}$ is a graph $G = \Omega(F) = \langle V, E \rangle$ such that each vertex $v_i \in V$ denotes the set S_i , *i.e.* V = F, and the two vertices v_i and v_j are adjacent whenever the corresponding sets S_i and S_j have a non-empty intersection, *i.e.* $E = \{\langle v_i, v_j \rangle \mid i \neq j, S_i \cap S_j \neq \emptyset \}$.
- * Line graph $^{\ensuremath{\mbox{\it E}}}$ of a graph $G = \langle V, E \rangle$ is another graph $L(G) = \Omega(E)$ that represents the adjacencies between edges of G. Each vertex of L(G) represents an edge of G, and two vertices of L(G) are adjacent iff the corresponding edges share a common endpoint in G (*i.e.* edges are "adjacent"/"incident").

Term

Walk

Trail

Path

+

¹Can vertices be repeated?

²Can edges be repeated?

Center

•

Centroid

Center

V¹ E² "Closed" term

Circuit

Cycle

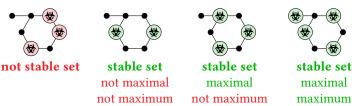
Closed walk

(impossible)

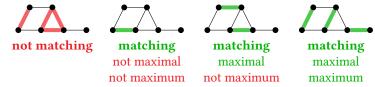
Centroid

Center

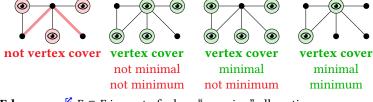
- * **Walk**^l is an alternating sequence of vertices and edges: $l = v_1 e_1 v_2 \dots e_{n-1} v_n$.
 - o **Trail** is a walk with distinct edges.
 - o Path is a walk with distinct vertices (and therefore distinct edges).
 - A walk is **closed** if it starts and ends at the same vertex. Otherwise, it is **open**.
 - Circuit is a closed trail.
 - Cycle is a closed path.
- * **Length** of a path (walk, trail) $l = u \rightsquigarrow v$ is the number of edges in it: |l| = |E(l)|.
- * **Girth**^{L'} is the length of the shortest cycle in the graph.
- * **Distance** dist(u, v) between two vertices is the length of the shortest path $u \rightsquigarrow v$.
 - $\varepsilon(v) = \max_{v} \operatorname{dist}(v, u)$ is the **eccentricity** of the vertex v.
 - $rad(G) = \min_{v \in V} \varepsilon(v)$ is the **radius** of the graph G.
 - o diam(G) = $\max_{v \in V} \varepsilon(v)$ is the **diameter** of the graph G. o center(G) = $\{v \mid \varepsilon(v) = \operatorname{rad}(G)\}$ is the **center** of the graph G.
- * Clique $Q \subseteq V$ is a set of vertices inducing a complete subgraph.
- * Stable set $S \subseteq V$ is a set of independent (pairwise non-adjacent) vertices.



* **Matching** $E \subseteq E$ is a set of independent (pairwise non-adjacent) edges.



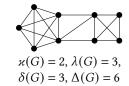
- * **Perfect matching** is a matching that covers all vertices in the graph.
 - o A perfect matching (if it exists) is always a minimum edge cover (but not vice-versa!).
- * **Vertex cover** $\mathbb{Z}^{\mathbb{Z}}$ $R \subseteq V$ is a set of vertices "covering" all edges.



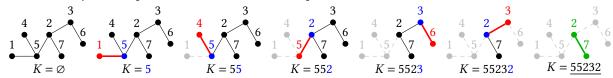
* Edge cover $F \subseteq E$ is a set of edges "covering" all vertices.



- * Cut vertex (articulation point) is a vertex whose removal increases the number of connected components.
- * **Bridge** is an edge whose removal increases the number of connected components.
- * **Biconnected graph** is a connected "nonseparable" graph, which means that the removal of any vertex does not make the graph disconnected. Alternatively, this is a graph without *cut vertices*.
- * **Biconnectivity** can be defined as a relation on edges $R \subseteq E^2$:
 - Two edges are called *biconnected* if there exist two *vertex-disjoint* paths between the ends of these edges.
 - o Trivially, this relation is an equivalence relation.
 - Equivalence classes of this relation are called **biconnected components**, also known as **blocks**.
- * **Edge biconnectivity** can be defined as a relation on vertices $R \subseteq V^2$:
 - Two vertices are called *edge-biconnected* if there exist two *edge-disjoint* paths between them.
 - o Trivially, this relation is an equivalence relation.
 - Equivalence classes of this relation are called **edge-biconnected components** (or 2-edge-connected components).
- * **Vertex connectivity** $\kappa(G)$ is the minimum number of vertices that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k-vertex-connected.
- * k-vertex-connected graph $^{\mathbb{Z}}$ is a graph that remains connected after less than k vertices are removed, i.e. $\varkappa(G) \ge k$.
 - Corollary of Menger's theorem: graph $G = \langle V, E \rangle$ is k-vertex-connected if, for every pair of vertices $u, v \in V$, it is possible to find k vertex-independent (internally vertex-disjoint) paths between u and v.
 - *k*-vertex-connected graphs are also called simply *k*-connected.
 - o 1-connected graphs are called connected, 2-connected are biconnected, 3-connected are triconnected, etc.
 - Note the "exceptions":
 - Singleton graph K_1 has $\varkappa(K_1) = 0$, so it is **not** 1-connected, but still considered connected.
 - Graph K_2 has $\varkappa(K_2) = 1$, so it is **not** 2-connected, but considered biconnected, so it can be a block.
- * **Edge connectivity** $\lambda(G)$ is the minimum number of edges that has to be removed in order to make the graph disconnected or trivial (singleton). Equivalently, it is the largest k for which the graph G is k-edge-connected.
- * k-edge-connected graph $^{\mathbb{Z}}$ is a graph that remains connected after less than k edges are removed, i.e. $\lambda(G) \geq k$.
 - Corollary of Menger's theorem: graph $G = \langle V, E \rangle$ is k-edge-connected if, for every pair of vertices $u, v \in V$, it is possible to find k edge-disjoint paths between u and v.
 - o 2-edge-connected are called *edge-biconnected*, 3-edge-connected are *edge-triconnected*, *etc.*
 - Note the "exception":
 - Singleton graph K_1 has $\lambda(K_1) = 0$, so it is **not** 2-edge-connected, but considered edge-biconnected, so it can be a 2-edge-connected component.
- * Whitney's Theorem. For any graph G, $\varkappa(G) \leq \lambda(G) \leq \delta(G)$.



- * **Tree** [☑] is a connected undirected acyclic graph.
- * Forest is an undirected acyclic graph, *i.e.* a disjoint union of trees.
- * An **unrooted tree** (**free tree**) is a tree without any designated *root*.
- * A **rooted tree** is a tree in which one vertex has been designated the *root*.
 - \circ In a rooted tree, the **parent** of a vertex v is the vertex connected to v on the path to the root.
 - A **child** of a vertex v is a vertex of which v is the parent.
 - A **sibling** to a vertex v is any other vertex on the tree which has the same parent as v.
 - A **leaf** is a vertex with no children. Equivalently, **leaf** is a pendant vertex, i.e. deg(v) = 1.
 - An **internal vertex** is a vertex that is not a leaf.
 - A *k*-ary tree is a rooted tree in which each vertex has at most *k* children. *2-ary trees* are called **binary trees**.
- * A **labeled tree** is a tree in which each vertex is given a unique *label*, e.g., 1, 2, ..., n.
- * Cayley's formula. Number of labeled trees on n vertices is n^{n-2} .
- * **Prüfer code** is a unique sequence of labels $\{1, \ldots, n\}$ of length (n-2) associated with the labeled tree on n vertices.
 - **ENCODING** (iterative algorithm for converting tree T labeled with $\{1, \ldots, n\}$ into a Prüfer sequence K):
 - On each iteration, remove the leaf with *the smallest label*, and extend *K* with *a single neighbour* of this leaf.
 - After (n-2) iterations, the tree will be left with *two adjacent* vertices—there is no need to encode them, because there is only one unique tree on 2 vertices, which requires 0 bits of information to encode.



- **DECODING** (iterative algorithm for converting a Prüfer sequence K into a tree T):
 - Given a Prüfer code K of length (n-2), construct a set of "leaves" $W = \{1, \ldots, n\} \setminus K$.
 - On each iteration:
 - (1) Pop the *first* element of K (denote it as k) and the *minimum* label in W (denote it as w).
 - (2) Connect k and w with an edge $\langle k, w \rangle$ in the tree T.
 - (3) If $k \notin K$, then extend the set of "leaves" $W := W \cup \{k\}$.
 - After (n-2) iterations, the sequence K will be empty, and the set W will contain exactly two vertices connect them with an edge.