4 Formal Logic Cheatsheet

4.1 Propositional Logic ✓

* **Proposition** is a statement which can be either true or false.

- Truth-bearer
- * Alphabet^[2] of propositional logic consists of (1) atomic symbols and (2) operator symbols.
- * **Atomic formula** (atom) is an irreducible formula without logical connectives.
 - \circ Propositional **variables**: A, B, C, \ldots, Z . With indices, if needed: $A_1, A_2, \ldots, Z_1, Z_2, \ldots$
 - ∘ Logical **constants**: ⊤ for always true proposition (*tautology*), ⊥ for always false proposition (*contradiction*).
- * Logical connectives (operators):

Type	Natural meaning	Symbolization
[☑] Negation	It is not the case that \mathcal{P} . It is false that \mathcal{P} . It is not true that \mathcal{P} .	$\neg \mathcal{P}$
^E Conjunction	Both \mathcal{P} and \mathcal{Q} . \mathcal{P} but \mathcal{Q} . \mathcal{P} , although \mathcal{Q} .	$\mathscr{P}\wedge Q$
[™] Disjunction	Either \mathcal{P} or Q (or both). \mathcal{P} unless Q .	$\mathcal{P} \lor Q$
Exclusive or (Xor)	Either \mathcal{P} or Q (but not both) \mathcal{P} xor Q .	$\mathcal{P}\oplus Q$
Implication (Conditional)	If \mathcal{P} , then Q . \mathcal{P} only if Q . Q if \mathcal{P} .	$\mathcal{P} \to Q$
[☑] Biconditional	\mathcal{P} , if and only if Q . \mathcal{P} iff Q . \mathcal{P} just in case Q .	$\mathcal{P} \leftrightarrow Q$

* **Sentence** of propositional logic is defined inductively:

Well-formed formula (WFF)

- 1. Every propositional variable/constant is a sentence.
- 2. If \mathcal{A} is a sentence, then $\neg \mathcal{A}$ is a sentence.
- 3. If \mathcal{A} and \mathcal{B} are sentences, then $(\mathcal{A} \wedge \mathcal{B})$, $(\mathcal{A} \vee \mathcal{B})$, $(\mathcal{A} \to \mathcal{B})$, $(\mathcal{A} \leftrightarrow \mathcal{B})$ are sentences.
- 4. Nothing else is a sentence.
- * Well-formed formulae grammar:

Backus-Naur form (BNF)

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 \langle sentence \rangle ::= \langle constant \rangle \\ | \langle variable \rangle \\ | \neg \langle sentence \rangle \\ | '(' \langle sentence \rangle \langle binop \rangle \langle sentence \rangle ')' \\ \langle constant \rangle ::= \top | \bot \\ \langle variable \rangle ::= A | ... | Z | A_1 | ... | Z_n \\ \langle binop \rangle ::= \land | \lor | \oplus | \rightarrow | \leftarrow | \leftrightarrow
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- * **Literal**^{\mathcal{L}} is a propositional variable or its negation: $\mathcal{L}_i = X_i$ (positive literal), $\mathcal{L}_j = \neg X_j$ (negative literal).
- * **Argument** is a set of logical statements, called *premises*, intended to support or infer a claim (*conclusion*):

$$\underbrace{\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n}_{\textit{premises}} \quad \vdots \quad \underbrace{C}_{\textit{conclusion}}$$
 "therefore"

* An argument is **valid** if whenever all the premises are true, the conclusion is also true.

Validity

* An argument is **invalid** if there is a case (a counterexample) when all the premises are true, but the conclusion is false.

4.2 Semantics of Propositional Logic

* **Valuation**[™] is any assignment of truth values to propositional variables.

- Interpretation
- * \mathcal{A} is a **tautology** (valid) iff it is true on *every* valuation. Might be symbolized as " $\neq \mathcal{A}$ ".
- * \mathcal{A} is a **contradiction** iff it is false on *every* valuation. Might be symbolized as " $\mathcal{A} \models$ ".
- * \mathcal{A} is a **contingency** iff it is true on some valuation and false on another. In other words, a **contingent** proposition is neither a tautology nor a contradiction.
- * \mathcal{A} is **satisfiable** iff it is true on *some* valuation.

Satisfiability

* \mathcal{A} is **falsifiable** iff it is not valid, *i.e.* it is false on *some* valuation.

Falsifiability

- * \mathcal{A} and \mathcal{B} are **equivalent** (symbolized as $\mathcal{A} \equiv \mathcal{B}$) iff, for every valuation, their truth values agree, *i.e.* there is no valuation in which they have opposite truth values. Equivalence check
- * $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ are **consistent** (**jointly satisfiable**) iff there is *some* valuation which makes them all true. Sentences are **inconsistent** (**jointly unsatisfiable**) iff there is *no* valuation that makes them all true. Consistency
- * The sentences $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ **entail** the sentence C (symbolized as $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$) if there is no valuation which makes all of $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ true and C false.

 Semantic entailment
- * If $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \models C$, then the argument $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n \stackrel{.}{.} C$ is **valid**. *Validity check examples*:

A B	$A \rightarrow B$	A	valid	В	$\neg A$	$\rightarrow \neg B$	valid	$B \rightarrow A$	$A \rightarrow B$	В	invali	^d ¬ ($B \rightarrow A)$
0 0	1	0		0	1	1 1	✓	1	1	0		0	1
0 1	1	0		1	1	0 0	•	0	1	1	✓	1	0
1 0	0	1		0	0	1 1	1	1	0	0		0	1
1 1	1	1	1	1	0	1 0	1	1	1	1	X	0	1

R S T	$R \vee S$	$S \vee T$	$\neg R$	invali	${}^{d}S \wedge T$	$ (R \land S) \rightarrow T \stackrel{valid}{::} R \rightarrow (S \rightarrow T) $
0 0 0	0	0	1		0	0 10 🗸 01 1
0 0 1	0	1	1		0	0 11 1 1
0 1 0	1	1	1	X	0	0 10 1 0 1 0
0 1 1	1	1	1	✓	1	0 11 1 01 1
1 0 0	1	0	0	•	0	0 10 11 1
1 0 1	1	1	0	•	0	0 11 1 1
1 1 0	1	1	0	•	0	1 00 · 10 0
1 1 1	1	1	0	•	1	1 11 11 1

- * **Soundness** $\Gamma \vdash \mathcal{A} \rightarrow \Gamma \models \mathcal{A}$ "Every provable statement is in fact true"
- * **Completeness:** $\Gamma \models \mathcal{A} \rightarrow \Gamma \vdash \mathcal{A}$ "Every true statement has a proof"

4.3 Natural Deduction Rules

Reiteration

m	A
	l

$$\therefore$$
 \mathcal{A} R m

Modus ponens

$$\begin{array}{c|c} i & \mathcal{A} \to \mathcal{B} \\ j & \mathcal{A} \end{array}$$

$$\therefore \mid \mathcal{B}$$

Modus tollens

$$i \mid \mathcal{A} \to \mathcal{B}$$

$$j \mid \neg \mathcal{B}$$

MT
$$i, j$$

Negation

$$j \mid \mathcal{A}$$

$$\therefore \mid \bot \qquad \neg E i, j$$

$$i$$
 \mathcal{A}

$$j$$
 \perp

Indirect proof

$$i$$
 j
 $\neg \mathcal{A}$

Double negation

$$m \mid \neg \neg \mathcal{A}$$

$$\neg \neg E m$$

Law of excluded middle

$$i$$
 j
 \mathcal{A}
 \mathcal{B}

$$k \mid | \neg \mathcal{A}|$$

$$l \mid \mathcal{B}$$

LEM
$$i-j$$
, $k-l$

Explosion

$$m \mid \bot$$

Conjunction

$$j \mid \mathcal{B}$$

$$\therefore \mid \mathcal{A} \wedge \mathcal{B} \mid$$

$$\wedge$$
I i, j

 $\wedge E m$

$$m \mid \mathcal{A} \wedge \mathcal{B}$$

$$\mathcal{B}$$
 $\wedge E m$

Disjunction

$$m \mid \mathcal{A}$$

$$\therefore \quad | \mathcal{A} \vee \mathcal{B} \qquad \forall I m$$

$$\therefore \quad \mathcal{B} \vee \mathcal{A} \qquad \forall \mathbf{I} \ m$$

$$m \mid \mathcal{A} \vee \mathcal{B}$$

$$i$$
 j
 \mathcal{A}
 C

$$k \mid \mathcal{B}$$

$$\vee$$
E $m, i-j, k-l$

Disjunctive syllogism

$$i \mid \mathcal{A} \vee \mathcal{B}$$

$$j \mid \neg \mathcal{A}$$

DS
$$i, j$$

$$i \mid \mathcal{A} \vee \mathcal{B}$$

$$j \mid \neg \mathcal{B}$$

DS
$$i, j$$

Hypothetical syllogism

$$i \mid \mathcal{A} \to \mathcal{B}$$

$$j \mid \mathcal{B} \to C$$

$$\therefore \mid \mathcal{A} \to C$$

Conditional

$$i \mid \mathcal{A}$$
 $j \mid \mathcal{B}$

$$\therefore \mid \mathcal{A} \to \mathcal{B}$$

$$\rightarrow$$
I $i-j$

Contraposition

$$m \mid \mathcal{A} \to \mathcal{B}$$

$$\therefore \quad | \neg \mathcal{B} \to \neg \mathcal{A}$$

Biconditional

$$i$$
 j
 \mathcal{A}
 \mathcal{B}

$$k \mid \mathcal{B}$$

$$\therefore \quad | \mathcal{A} \leftrightarrow \mathcal{B}$$

$$\leftrightarrow$$
I $i-j$, $k-l$

$$i \mid \mathcal{A} \leftrightarrow \mathcal{B}$$

$$j \mid \mathcal{A}$$

$$\therefore \mid \mathcal{B}$$

$$\leftrightarrow$$
E i, j

$$i \mid \mathcal{A} \leftrightarrow \mathcal{B}$$

$$j \mid \mathcal{B}$$

$$\leftrightarrow$$
E i, j

DeM m

 $\mathrm{DeM}\;m$

 ${
m DeM}\ m$

DeM m

De Morgan Rules

$$m \mid \neg(\mathcal{A} \vee \mathcal{B})$$

$$\neg \mathcal{A} \wedge \neg \mathcal{B}$$

$$m \mid \neg \mathcal{A} \wedge \neg \mathcal{B}$$

$$. \qquad \neg (\mathcal{A} \vee \mathcal{B})$$

$$n \mid \neg(\mathcal{A} \land \mathcal{B})$$

$$\therefore \quad | \neg \mathcal{A} \lor \neg \mathcal{B} |$$

$$i \mid \neg \mathcal{A} \lor \neg \mathcal{B}$$

$$\neg (\mathcal{A} \land \mathcal{B})$$

Green: basic rules.

Orange: derived rules.

More rules can be found in the "forall x: Calgary" book (p. 406).