

# Discrete Mathematics

**(Not only) Regular Languages** — Spring 2025

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# §1 Regular Languages

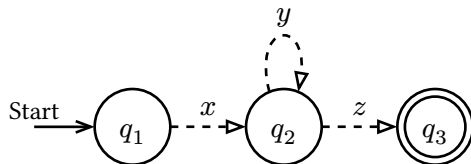
## Regular Expressions

Regular languages can be composed from “smaller” regular languages.

- Atomic regular expressions:
  - $\emptyset$ , an empty language
  - $\varepsilon$ , a singleton language consisting of a single  $\varepsilon$  word
  - $a$ , a singleton language consisting of a single 1-letter word  $a$ , for each  $a \in \Sigma$
- Compound regular expressions:
  - $R_1 R_2$ , the concatenation of  $R_1$  and  $R_2$
  - $R_1 \mid R_2$ , the union of  $R_1$  and  $R_2$
  - $R^* = RRR\dots$ , the Kleene star of  $R$
  - $(R)$ , just a bracketed expression
  - Operator precedence:  $ab^*c \mid d \triangleq ((a (b^*)) c) \mid d$

## Re-visiting States

- Let  $D$  be a DFA with  $n$  states.
- Any string  $w$  accepted by  $D$  that has length at least  $n$  must visit some state twice.
- Number of states visited is equal to  $|w| + 1$ .
- By the pigeonhole principle, some state is “duplicated”, i.e. visited more than once.
- The substring of  $w$  between those *revisited states* can be removed, duplicated, tripled, etc. without changing the fact that  $D$  accepts  $w$ .



Informally:

- Let  $L$  be a regular language.
- If we have a string  $w \in L$  that is “sufficiently long”, then we can *split* the string into *three pieces* and “*pump*” the middle.
- We can write  $w = xyz$  such that  $xy^0z, xy^1z, xy^2z, \dots, xy^nz, \dots$  are all in  $L$ .
  - Notation:  $y^n$  means “ $n$  copies of  $y$ ”.

# Weak Pumping Lemma

**Theorem 1** (Weak Pumping Lemma for Regular Languages):

- For any regular language  $L$ ,
  - ▶ There exists a positive natural number  $n$  (also called *pumping length*) such that
    - For any  $w \in L$  with  $|w| \geq n$ ,
      - There exists strings  $x, y, z$  such that
        - ▶ For any natural number  $i$ ,
          - $w = xyz$  ( $w$  can be broken into three pieces)
          - $y \neq \varepsilon$  (the middle part is not empty)
          - $xy^iz \in L$  (the middle part can repeated any number of times)

*Example:* Let  $\Sigma = \{0, 1\}$  and  $L = \{w \in \Sigma^* \mid w \text{ contains } 00 \text{ as a substring}\}$ . Any string of length 3 or greater can be split into three parts, the second of which can be “pumped”.

*Example:* Let  $\Sigma = \{0, 1\}$  and  $L = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$ . The weak pumping lemma still holds for finite languages, because the pumping length  $n$  can be longer than the longest word in the language!

# Testing Equality

**Definition 1:** The *equality problem* is, given two strings  $x$  and  $y$ , to decide whether  $x = y$ .

*Example:* Let  $\Sigma = \{0, 1, \#\}$ . We can *encode* the equality problem as a string of the form  $x\#y$ .

- “Is *001* equal to *110*?” would be *001#110*.
- “Is *11* equal to *11*?” would be *11#11*.
- “Is *110* equal to *110*?” would be *110#110*.

Let  $\text{EQUAL} = \{w\#w \mid w \in \{0, 1\}^*\}$ .

**Question:** Is EQUAL a *regular* language?

A typical word in EQUAL looks like this: *001#001*.

- If the “middle” piece is just a symbol  $\#$ , then observe that *001001*  $\notin$  EQUAL.
- If the “middle” piece is either completely to the left or completely to the right of  $\#$ , then observe that any duplication or removal of this piece is not in EQUAL.
- If the “middle” piece includes  $\#$  and any symbols from the left/right of it, then, again, observe that any duplication or removal of this piece is not in EQUAL.

## Testing Equality [2]

**Theorem 2:** EQUAL is not a regular language.

**Proof:** By contradiction. Assume that EQUAL is a regular language.

Let  $n$  be the pumping length guaranteed by the weak pumping lemma. Let  $w = 0^n \# 0^n$ , which is in EQUAL and  $|w| = 2n + 1 \geq n$ . By the weak pumping lemma, we can write  $w = xyz$  such that  $y \neq \varepsilon$  and for any  $i \in \mathbb{N}$ ,  $xy^i \# z \in \text{EQUAL}$ . Then  $y$  cannot contain  $\#$ , since otherwise if we let  $i = 0$ , then  $xy^0 \# z = x \# z$  does not contain  $\#$  and would not be in EQUAL. So  $y$  is either completely to the left of  $\#$  or completely to the right of  $\#$ .

Let  $|y| = k$ , so  $k > 0$ . Since  $y$  is completely to the left or right of  $\#$ , then  $y = 0^k$ .

Now, we consider two cases:

Case 1:  $y$  is to the left of  $\#$ . Then  $xy^2z = 0^{n+k} \# 0^n \notin \text{EQUAL}$ , contradicting the weak pumping lemma.

Case 2:  $y$  is to the right of  $\#$ . Then  $xy^2z = 0^n \# 0^{n+k} \notin \text{EQUAL}$ , contradicting the weak pumping lemma.

In either case we reach a contradiction, so our assumption was wrong. Thus, EQUAL *is not regular*. □

## §2 Non-regular Languages



## (Not only) Regular Languages

- The weak pumping lemma describes a property common to *all* regular languages.
- Any language  $L$  which does not have this property *cannot be regular*.
- What other languages can we find that are not regular?

*Example:* Consider the language  $L = \{0^n 1^n \mid n \in \mathbb{N}\}$ .

- $L = \{\varepsilon, 01, 0011, 000111, 00001111, \dots\}$
- $L$  is a classic example of a non-regular language.
- **Intuitively:** if you have *only finitely many states* in a DFA, you cannot “*remember*” an arbitrary number of 0s to match *the same* number of 1s.

How would we prove that  $L$  is non-regular?

Use the Pumping Lemma to show that  $L$  *cannot* be regular.

## Pumping Lemma as a Game

The weak pumping lemma can be thought of as a *game* between **you** and an **adversary**.

- **You win** if you can prove that the pumping lemma *fails*.
- **The adversary wins** if the adversary can make a choice for which the pumping lemma *succeeds*.

The game goes as follows:

- **The adversary** chooses a pumping length  $n$ .
- **You** choose a string  $w$  with  $|w| \geq n$  and  $w \in L$ .
- **The adversary** breaks it into  $x$ ,  $y$ , and  $z$ .
- **You** choose an  $i$  such that  $xy^iz \notin L$  (*if you can't, you lose!*).

## Pumping Lemma as a Game [2]

$$L = \{0^n 1^n \mid n \in \mathbb{N}\}$$

**Adversary**

Maliciously choose  
pumping length  $n$

Maliciously split  
 $w = xyz, y \neq \varepsilon$

**Lose**

**You**

Cleverly choose a string  
 $w \in L, |w| \geq n$

Cleverly choose an  $i$   
such that  $xy^i z \notin L$

**Win**

$\{0^n 1^n\}$  is not regular

## Formal Proof of Non-regularity

**Theorem 3:**  $L = \{0^n 1^n \mid n \in \mathbb{N}\}$  is not regular.

**Proof:** By contradiction. Assume that  $L$  is regular.

Let  $n$  be the pumping length guaranteed by the weak pumping lemma (“there exists  $n \dots$ ”). Consider the string  $w = 0^n 1^n$ . Then  $|w| = 2n \geq n$  and  $w \in L$ , so we can write (split)  $w = xyz$  such that  $y \neq \varepsilon$  and for any  $i \in \mathbb{N}$ , we have  $xy^i z \in L$ .

We consider three cases:

Case 1:  $y$  consists solely of 0s. Then  $xy^0 z = xz = 0^{n-|y|} 1^n$ , and since  $|y| > 0$ ,  $xz \notin L$ .

Case 2:  $y$  consists solely of 1s. Then  $xy^0 z = xz = 0^n 1^{n-|y|}$ , and since  $|y| > 0$ ,  $xz \notin L$ .

Case 3:  $y$  consists of  $k > 0$  0s followed by  $m > 0$  1s. Then  $xy^2 z = 0^n 1^m 0^k 1^n$ , so  $xy^2 z \notin L$ .

In all three cases we reach a contradiction, so our assumption was wrong and  $L$  is not regular. □

## §3 Pumping Lemma

## Pumping

Consider the language  $L$  over  $\Sigma = \{0, 1\}$  of strings  $w \in \Sigma^*$  that contain *an equal number* of 0s and 1s.

For example:

- 01 in  $L$
- 11011 not in  $L$
- 110010 in  $L$

**Question:** Is  $L$  a *regular* language?

Let's *use* the weak pumping lemma to show it is by *pumping all the strings* in this language.

**Proof (incorrect):** We are going to show that  $L$  satisfies the conditions of the weak pumping lemma. Let  $n = 2$ . Consider any string  $w \in L$  (i.e.,  $w$  contains the same number of 0s and 1s) with  $|w| \geq 2$ .

We can split  $w = xyz$  such that  $x = z = \varepsilon$  and  $y = w$ , so  $y \neq \varepsilon$ . Then, for any natural number  $i \in \mathbb{N}$ ,  $xy^iz = w^i$ , which has the same number of 0s and 1s.

Since  $L$  passes the conditions of the weak pumping lemma,  $L$  is regular. □

## A word of Caution

- The weak and full pumping lemmas describe the *necessary* condition of regular languages.
  - If  $L$  is *regular*, then it *passes* the conditions of the pumping lemma.
  - If a language *fails* the pumping lemma, it is *definitely not regular*.
- The weak and full pumping lemmas are *not a sufficient* condition of regular languages.
  - If  $L$  is *not regular*, then it still *may pass* the conditions of the pumping lemma.
  - If a language *passes* the pumping lemma, we *learn nothing* about whether it is regular or not.

## The Stronger Pumping Lemma

The language  $L$  *can* be proven to be *non-regular* using a *stronger version* of the pumping lemma.

For the intuition behind the “full” pumping lemma, let’s revisit our original observation.

- Let  $D$  be a DFA with  $n$  states.
- Any string  $w$  accepted by  $D$  of length at least  $n$  must visit some state twice *within its first  $n$  symbols*.
  - The number of visited states is equal to  $n + 1$ .
  - By the pigeonhole principle, some state is *duplicated*.
- The substring of  $w$  between those *revisited states* can be removed, duplicated, tripled, etc. without changing the fact that  $D$  accepts  $w$ .

Overall, we can add the following condition to the weak pumping lemma:

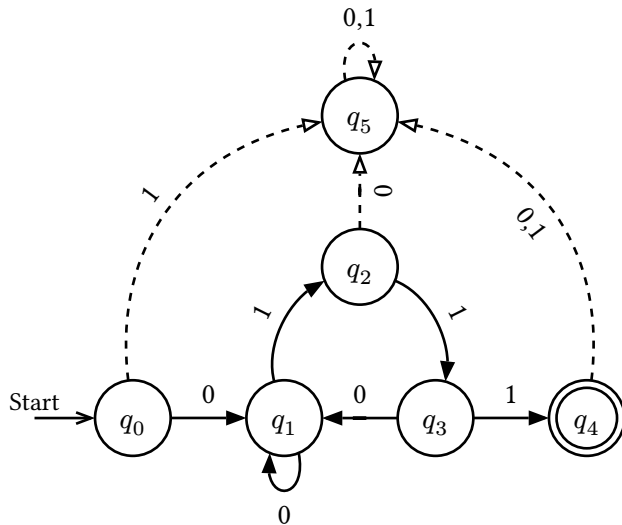
$$|xy| \leq n$$

This restriction means that we can limit where the string to pump must be. If we specifically choose the first  $n$  characters of the string to pump, we can ensure  $y$  (middle part) to have a specific property.

We can then show that  $y$  cannot be pumped arbitrarily many times.



## The Stronger Pumping Lemma [2]



$q_0 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{0} q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_3 \xrightarrow{1} q_4$

## Formal Proof of Non-regularity

**Theorem 4:**  $L = \{w \in \{0, 1\}^* \mid w \text{ has an equal number of 0s and 1s}\}$  is *not regular*.

**Proof:** By contradiction. Assume that  $L$  is regular.

Let  $n$  be the pumping length guaranteed by the weak pumping lemma. Consider the string  $w = 0^n 1^n$ . Then  $|w| = 2n \geq n$  and  $w \in L$ . Therefore, there exist strings  $x$ ,  $y$ , and  $z$  such that  $w = xyz$ ,  $|xy| \leq n$ ,  $y \neq \varepsilon$ , and for any  $i \in \mathbb{N}$ , we have  $xy^i z \in L$ .

Since  $|xy| \leq n$ ,  $y$  must consist solely of 0s. But then  $xy^2 z = 0^{n+|y|} 1^n$ , and since  $|y| > 0$ ,  $xy^2 z \notin L$ .

We have reached a contradiction, so our assumption was wrong and  $L$  is not regular. □

## Summary of the Pumping Lemma

1. Using the *pigeonhole principle*, we can prove the weak and full *pumping lemma*.
2. These lemmas describe essential properties of the *regular* languages.
3. Any language that *fails* to have these properties *can not be regular*.

## §4 Closure Properties of Regular Languages

## Closure of Regular Languages

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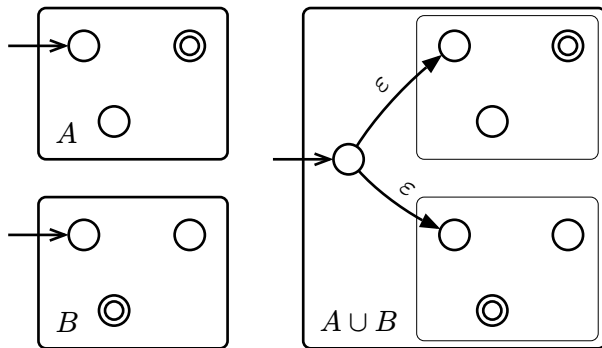
1. The *union* of two regular languages is regular.
2. The *intersection* of two regular languages is regular.
3. The *complement* of a regular language is regular.
4. The *difference* of two regular languages is regular.
5. The *reversal* of a regular language is regular.
6. The *Kleene star* of a regular language is regular.
7. The *concatenation* of regular languages is regular.
8. A *homomorphism* (substitution of strings for symbols) of a regular language is regular.
9. The *inverse homomorphism* of a regular language is regular.

## Closure under Union

**Theorem 5:** If  $L$  and  $M$  are regular languages, then so is their union  $L \cup M$ .

**Proof:** Since  $L$  and  $M$  are regular, they have regular expressions, i.e.  $L = \mathcal{L}(R)$  and  $M = \mathcal{L}(S)$ .

Then  $L \cup M = \mathcal{L}(R + S)$  by the definition of the union (+) operator for regular expressions. □

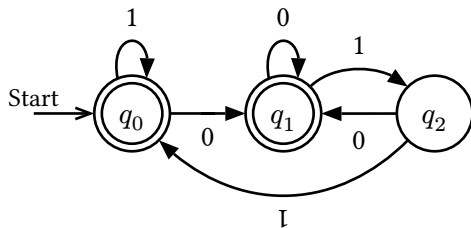
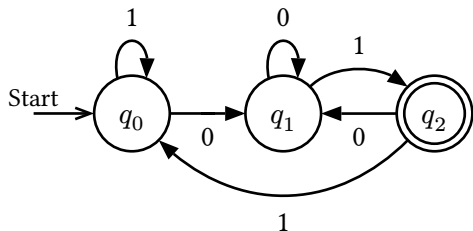


## Closure under Complement

**Theorem 6:** If  $L$  is a regular language over the alphabet  $\Sigma$ , then its complement  $\bar{L} = \Sigma^* - L$  is also a regular language.

**Proof:** Let  $L = \mathcal{L}(A)$  for some DFA  $A = (Q, \Sigma, \delta, q_0, F)$ . Then  $\bar{L} = \mathcal{L}(B)$ , where  $B$  is the DFA  $(Q, \Sigma, \delta, q_0, Q - F)$ . That is,  $B$  is exactly like  $A$ , but with the accepting states flipped. Then  $w$  is in  $\bar{L}$  if and only if  $\delta(q_0, w)$  is in  $Q - F$ , which occurs if and only if  $w$  is not in  $\mathcal{L}(A)$ .  $\square$

*Example:* The DFA  $A$  presented below on the left accepts only the strings of 0's and 1's that end in 01,  $\mathcal{L}(A) = (0+1)^*01$ . The complement of  $\mathcal{L}(A)$  is therefore all strings of 0's and 1's that *do not* end in 01. Below on the right is the automaton for  $\{0, 1\}^* - \mathcal{L}(A)$ .



## Closure under Intersection

**Theorem 7:** If  $L$  and  $M$  are regular languages, then so is their intersection  $L \cap M$ .

**Proof** (*simple*):  $L \cap M = \overline{\overline{L} \cup \overline{M}}$ . □

**Proof:** We can directly construct a “product” DFA for the intersection of two regular languages.

Let  $L$  and  $M$  be the languages of automata  $A_L = (Q_L, \Sigma, \delta_L, q_L, F_L)$  and  $A_M = (Q_M, \Sigma, \delta_M, q_M, F_M)$ . Note that we assume that the alphabets of both automata are the same (or  $\Sigma$  is their union).

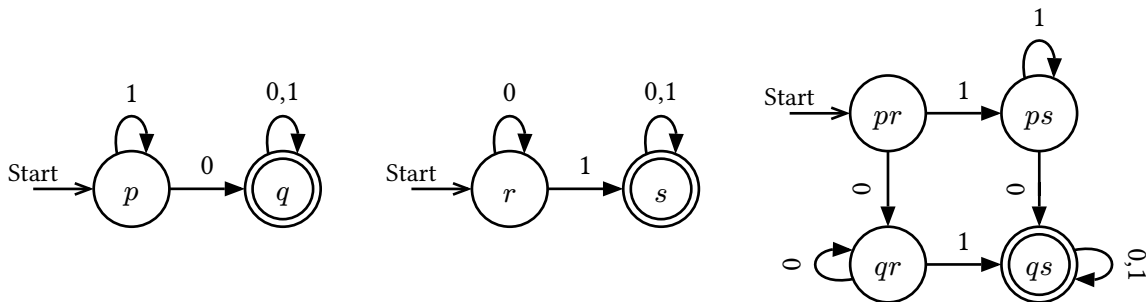
For  $L \cap M$ , we construct the automaton  $A$  that simulates both  $A_L$  and  $A_M$ . The states of  $A$  are the product of the states of  $A_L$  and  $A_M$ . The initial state is  $(q_L, q_M)$ , and the accepting states are  $F_L \times F_M$ . The transitions are defined as  $\delta(\langle p, q \rangle, c) = \langle \delta_L(p, c), \delta_M(q, c) \rangle$ .

To see why  $\mathcal{L}(A) = \mathcal{L}(A_L) \cap \mathcal{L}(A_M)$ , first observe that  $\hat{\delta}(\langle q_L, q_M \rangle, w) = \langle \hat{\delta}_L(q_L, w), \hat{\delta}_M(q_M, w) \rangle$ . But  $A$  accepts  $w$  if and only if  $\hat{\delta}(q_0, w)$  is in  $F_L \times F_M$ , which occurs if and only if  $\hat{\delta}_L(q_L, w)$  is in  $F_L$  and  $\hat{\delta}_M(q_M, w)$  is in  $F_M$ . Or rather,  $A$  accepts  $w$  if and only if both  $A_L$  and  $A_M$  accept  $w$ . Thus,  $A$  accepts the intersection of  $L$  and  $M$ . □



## Closure under Intersection [2]

*Example:* The first automaton on the left accepts all strings that *have a 0*. The second automaton in the middle accepts all strings that *have a 1*. On the right, we show the *product* of these two automata. Its states are labelled by the pairs of states of the two automata. It is easy to see that this automaton accepts the *intersection* of the two languages: all strings that *have both a 0 and a 1*.



## Closure under Difference

**Theorem 8:** If  $L$  and  $M$  are regular languages, then so is their difference  $L - M$ .

**Proof:** Observe that  $L - M = L \cap \overline{M}$ . By previous theorems,  $\overline{M}$  is regular, and  $L \cap \overline{M}$  is also regular.  $\square$

## Closure under Reversal

**Definition 2:** The *reversal* of a string  $w = a_1 a_2 \dots a_n$  is the string  $w^R = a_n a_{n-1} \dots a_1$ .

*Example:*  $0010^R = 0100$  and  $\varepsilon^R = \varepsilon$ .

**Definition 3:** The *reversal* of a language  $L$  is the language  $L^R = \{w^R \mid w \in L\}$ .

*Example:* Let  $L = \{001, 10, 111\}$ , then  $L^R = \{001^R, 10^R, 111^R\} = \{100, 01, 111\}$ .

**Theorem 9:** If  $L$  is a regular language, then so its reversal  $L^R$ .

**Proof:** Assume  $L$  is defined by regular expression  $E$ . The proof is a structural induction on the size of  $E$ . We show that there is another regular expression  $E^R$  such that  $\mathcal{L}(E^R) = (\mathcal{L}(E))^R$ , that is, the language of  $E^R$  is the reversal of the language of  $E$ .

*Basis:* If  $E$  is  $\varepsilon$ ,  $\emptyset$ , or  $a$  for some symbol  $a$ , then  $E^R$  is the same as  $E$ .

## Closure under Reversal [2]

*Induction:* There are three cases, depending on the form of  $E$ .

1.  $E = E_1 + E_2$ . Then  $E^R = E_1^R + E_2^R$ .

The justification is that the reversal of the union of two languages is obtained by computing the reversals of the two languages and taking the union of those languages.

2.  $E = E_1 E_2$ . Then  $E^R = E_2^R E_1^R$ . Note that we reverse the order of the two languages, as well as reversing the languages themselves. For example, if  $\mathcal{L}(E_1) = \{0, 1111\}$  and  $\mathcal{L}(E_2) = \{00, 10\}$ , then  $\mathcal{L}(E_1 E_2) = \{0100, 0110, 11100, 11110\}$ . The reversal of the latter language is

$$\{0010, 0110, 00111, 01111\}$$

If we concatenate the reversals of  $\mathcal{L}(E_2)$  and  $\mathcal{L}(E_1)$ , we get

$$\{00, 01\}\{10, 111\} = \{0010, 00111, 0110, 01111\}$$

which is the same language as  $(\mathcal{L}(E_1 E_2))^R$ . In general, if a word  $w$  in  $\mathcal{L}(E)$  is the concatenation of  $w_1$  from  $\mathcal{L}(E_1)$  and  $w_2$  from  $\mathcal{L}(E_2)$ , then  $w^R = w_2^R w_1^R$ .

## Closure under Reversal [3]

3.  $E = E_1^*$ . Then  $E^R = (E_1^R)^*$ .

The justification is that any string  $w$  in  $\mathcal{L}(E)$  can be written as  $w_1 w_2 \dots w_n$ , where each  $w_i$  is in  $\mathcal{L}(E_1)$ . Then  $w^R = w_n^R w_{n-1}^R \dots w_1^R$ . Each  $w_i^R$  is in  $\mathcal{L}(E_1^R)$ , so  $w^R$  is in  $\mathcal{L}((E_1^R)^*)$ .

Conversely, any string in  $\mathcal{L}((E_1^R)^*)$  is of the form  $w_1 w_2 \dots w_n$ , where each  $w_i$  is the reversal of a string in  $\mathcal{L}(E_1)$ . The reversal of this string,  $w_n^R w_{n-1}^R \dots w_1^R$ , is therefore a string in  $\mathcal{L}(E_1^*)$ , which is  $\mathcal{L}(E)$ .

We have thus shown that a string is in  $\mathcal{L}(E)$  if and only if its reversal is in  $\mathcal{L}((E_1^R)^*)$ .

□

*Example:* Let  $L$  be defined by the regular expression  $(0+1)0^*$ . Then  $L^R$  is the language of  $(0^*)^R(0+1)^R$ .

If we apply the rules for Kleene star and union to the two parts, and then apply the basis rule that says the reversals of 0 and 1 are unchanged, we find that  $L^R$  has regular expression  $0^*(0+1)$ .

## §5 Decision Properties of Regular Languages

## Fundamental Questions about Languages

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1. Is the language *empty*?
2. Is the language *finite*?
3. Is the particular string  $w$  *in* the language?
4. Is the language a *subset* of another language?
5. Are the two languages *equivalent*?

# Decision Procedures

## Converting among representations

- $\varepsilon$ -closure:  $O(n^3)$
- $\varepsilon$ -NFA to DFA:  $n^3 2^n$
- DFA to  $\varepsilon$ -NFA:  $O(n)$
- $\varepsilon$ -NFA to RegEx:  $O(n^3 4^n)$
- RegEx to  $\varepsilon$ -NFA:  $O(n)$

## Testing *emptiness* of a regular language

- Given an automaton, we can determine whether the accepting states are reachable, in  $O(n^2)$  time.
- Given a regular expression, we can construct an  $\varepsilon$ -NFA and then determine the reachability of the accepting states, in  $O(n)$  time. Alternatively, we can inspect the regex directly.

## Testing *membership* in a regular language

- Given an automaton with  $s$  states and a string  $w$  of size  $n$ , we can simulate the automaton for  $w$  to determine whether it accepts  $w$ .
  - For DFA, this can be done in  $O(n)$  time.
  - For NFA or  $\varepsilon$ -NFA, in  $O(ns^2)$ .



## Emptiness, Finiteness, Infiniteness

**Theorem 10:** The language  $L$  accepted by a finite automaton with  $n$  states is *non-empty* iff the finite automaton accepts a word of length less than  $n$ .

**Theorem 11:** The language  $L$  accepted by a finite automaton  $M$  with  $n$  states is *infinite* iff the automaton accepts some word of length  $l$ , where  $n \leq l < 2n$ .

**Proof:** If  $w$  is in  $\mathcal{L}(M)$  and  $n \leq |w| < 2n$ , then by the Pumping lemma,  $\mathcal{L}(M)$  is infinite. That is,  $w = xyz$ , and for all  $i$ ,  $xy^iz$  is in  $L$ . Conversely, if  $\mathcal{L}(M)$  is infinite, then there exists  $w$  in  $\mathcal{L}(M)$ , where  $|w| \geq n$ . If  $|w| < 2n$ , we are done. If no word is of length between  $n$  and  $2n - 1$ , let  $w$  be of length at least  $2n$ , but as short as any word in  $\mathcal{L}(M)$  whose length is greater than or equal to  $2n$ . Again by the Pumping lemma, we can write  $w = xyz$  with  $1 \leq |y| \leq n$  and  $xz \in \mathcal{L}(M)$ . Either  $w$  was not the shortest word of length  $2n$  or more, or  $|xz|$  is between  $n$  and  $2n - 1$ , a contradiction in either case.  $\square$