𝓗omework #2 — **𝓓iscrete 𝓜∀th**
Binary Relations — **Fall 2022**

1. For each given relation $R_i \subseteq {M_i}^2$, determine whether it is *reflexive*, *irreflexive*, *coreflexive*, *symmetric*, *antisymmetric*, *asymmetric*, *transitive*, *antitransitive*, *semiconnex*, *connex*, *left/right Euclidean*, *dense*. Provide a counterexample for each non-complying property (*e.g.*, "transitivity does not hold for $x, y, z = (3, 1, 2)$"). Organize your answer in a table (*e.g.*, columns — relations, rows — properties).
   (a) $M_1 = \mathbb{R}$
   $x \; R_1 \; y \leftrightarrow |x - y| \leq 1$
   (b) $M_2 = \mathcal{P}(\{a, b, c\})$
   $R_2 = \text{“}\subseteq\text{”}$
   (c) $M_3 = \{a, b, c, d\} \quad \|R_3\| = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
   (d) $M_4 = \{$“rock”, “scissors”, “paper”$\}$
   $R_4 = \{\langle x, y\rangle \mid x \text{ beats } y\}$

2. Prove (rigorously) or disprove (by providing a counterexample) the following statements about arbitrary homogeneous relations $R \subseteq M^2$ and $S \subseteq M^2$:
   (a) If R and S are *reflexive*, then $R \cap S$ is so.
   (b) If R and S are *symmetric*, then $R \cap S$ is so.
   (c) If R and S are *transitive*, then $R \cap S$ is so.
   (d) If R and S are *reflexive*, then $R \cup S$ is so.
   (e) If R and S are *symmetric*, then $R \cup S$ is so.
   (f) If R and S are *transitive*, then $R \cup S$ is so.

3. An equinumerosity relation R over sets is defined as follows: $A \; R \; B \leftrightarrow |A| = |B|$.
   (a) Show that R is an equivalence relation.
   (b) Find the quotient set of $\mathcal{P}(\{a, b, c, d\})$ by R.

4. Let R_θ be a relation of θ-similarity of finite non-empty sets defined as follows: a set A is said to be θ-similar to B *iff* the Jaccard index $\text{Jac}(A, B)$ for these sets is at least θ, *i.e.* $A \; R_\theta \; B \leftrightarrow \text{Jac}(A, B) \geq \theta$. Obviously, $\theta \in [0; 1] \subseteq \mathbb{R}$.
   (a) Draw the graph of a relation $R_\theta \subseteq \{A_i\}^2$, where $\theta = 0.25$, $A_1 = \{1, 2, 5, 6\}$, $A_2 = \{2, 3, 4, 5, 7, 9\}$, $A_3 = \{1, 4, 5, 6\}$, $A_4 = \{3, 7, 9\}$, $A_5 = \{1, 5, 6, 8, 9\}$.
   (b) Determine whether θ-similarity is a tolerance relation.
   (c) Determine whether θ-similarity is an equivalence relation.

5. Let $H = \{1, 2, 4, 5, 10, 12, 20\}$. Consider a divisibility relation $R \subseteq H^2$ defined as follows: $xRy \leftrightarrow y \mathrel{\vdots} x$.
   (a) Sort R (as a set of pairs) lexicographically[1].
   (b) Show that R is a partial order.
   (c) Determine whether R is a linear (total) order.
   (d) Draw the Hasse diagram for a *graded poset* $\langle H, R, \rho\rangle$, where $\rho : H \longrightarrow \mathbb{N}_0$ is a grading function which maps a number $n \in H$ to the sum of all exponents appearing in its prime factorization, *e.g.*, $\rho(20) = \rho(2^2 \cdot 5^1) = 2 + 1 = 3$, so the number 20 would have the 3rd rank (bottom-up).
   (e) Find the minimal, minimum (least), maximal and maximum (greatest) elements in the poset $\langle H, R\rangle$. If there are multiple or none, explain why.
   (f) Perform a topological sort of the poset $\langle H, R\rangle$.

6. Prove that the transitive closure R^+ is in fact transitive.

   Definition. *$R^+ = \bigcup\limits_{n \in \mathbb{N}^+} R^n$ is a transitive closure of $R \subseteq M^2$, where*
   * *$R^{k+1} = R^k \circ R$ is a compositional (functional) power[2],*
   * *$R^1 = R$,*
   * *$S \circ R = \{\langle x, y\rangle \mid \exists z : (x \; R \; z) \land (z \; S \; y)\}$ is a composition (relative product) of relations R and S.*

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[1] Lexicographic order for pairs: $\langle a, b\rangle \leq \langle a', b'\rangle \leftrightarrow (a < a') \lor ((a = a') \land (b \leq b'))$

[2] Note: this *is not a Cartesian power*, despite of the same notation R^n. Another possible notation for compositional power is $R^{\circ n}$, but it is too wild to use it here.