Discrete Mathematics

Combinatorics — Spring 2025

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§1 Combinatorics

Introduction to Combinatorics

Definition 1: Combinatorics is the branch of discrete mathematics that deals with *counting*, *arranging*, and analyzing *discrete structures*.

Three basic problems of Combinatorics:

1. Existence: *Is there at least one arrangement of a particular kind?*
2. Counting: *How many arrangements are there?*
3. Optimization: *Which one is best according to some criteria?*

Discrete structures

- Graphs, sets, multisets, sequences, patterns, coverings, partitions…

Enumeration

- Permutations, combinations, inclusion/exclusion, generating functions, recurrence relations…

Algorithms and optimization

- Sorting, eulerian circuits, hamiltonian cycles, planarity testing, graph coloring, spanning trees, shortest paths, network flows, bipartite matchings, chain partitions…

Discrete Structures

We investigate the *building blocks* of combinatorics:

- Sets and multisets
- Sequences and strings
- Arrangements
- Graphs, networks, trees
- Posets and lattices
- Partitions
- Patterns, coverings, designs, configurations
- Schedules, assignments, distributions

Used in data modeling, logic, cryptography, and the design of data structures.

Enumerative Combinatorics

We learn how to count *without explicit listing*:

- Permutations and combinations
- Inclusion–Exclusion Principle
- Set partitions, integer partitions, Stirling numbers, Catalan numbers
- Recurrence relations
- Generating functions

Used in probability theory, complexity theory, coding theory, computational biology.

Algorithmic and Optimization Methods

Combinatorics powers *algorithm design* and complexity analysis:

- Sorting
- Searching
- Eulerian paths and Hamiltonian cycles
- Planarity, colorings, cliques, coverings
- Spanning trees
- Shortest paths
- Network flows
- Bipartite matchings
- Dilworth's theorem, chain and antichain partitions

Used in logistics, scheduling, routing, and complexity optimization.

§2 Basic Counting Principles

Basic Counting Rules

Product rule: If something can happen in n_1 ways, *and* no matter how the first thing happens, a second thing can happen in n_2 ways, then the two things *together* can happen in $n_1 \cdot n_2$ ways.

Sum rule: If one event can occur in n_1 ways and a second event in n_2 (different) ways, then there are $n_1 + n_2$ ways in which *either* the first event *or* the second event can occur (*but not both*).

Addition Principle

Definition 2: We say a finite set S is *partitioned* into *parts* $S_1, \ldots, S_m$ if the parts are pairwise disjoint and their union is S. In other words, $S_i \cap S_j = \emptyset$ for $i \neq j$ and $S_1 \cup S_2 \cup \ldots \cup S_k = S$. In that case:

$$|S| = |S_1| + |S_2| + \ldots + |S_m|$$

Example: Let S be the set of students attending the combinatorics lecture. It can be partitioned into parts S_1 and S_2 where

$$S_1 = \text{set of students that like easy examples.}$$
$$S_2 = \text{set of students that don't like easy examples.}$$

If $|S_1| = 22$ and $|S_2| = 8$, then we can conclude $|S| = |S_1| + |S_2| = 30$.

Multiplication Principle

Definition 3: If S is a finite set that is the *product* of $S_1, \ldots, S_m$, that is, $S = S_1 \times \ldots \times S_m$, then

$$|S| = |S_1| \times \ldots \times |S_m|$$

Example: TODO: example with car plates

Subtraction Principle

Definition 4: Let S be a subset of a finite set T. We define the *complement* of S as $\overline{S} = T \setminus S$. Then

$$\left|\overline{S}\right| = |T| - |S|$$

Example: If T is the set of students studying at KIT and S the set of students studying neither math nor computer science. If we know $|T| = 23905$ and $|S| = 20178$, then we can compute the number $|S|$ of students studying either math or computer science:

$$|S| = |T| - |S| = 23905 - 20178 = 3727$$

Bijection Principle

Definition 5: If S and T are sets, then

$$|S| = |T| \iff \text{there exists a bijection between } S \text{ and } T$$

Example: Let S be the set of students attending the combinatorics lecture and T the set of homework submissions (unique per student) for the first problem sheet. If the number of students and the number of submissions coincide, then there is a bijection between students and submissions.

Note: The bijection principle works both for *finite* and *infinite* sets.

Pigeonhole Principle

Definition 6: Let $S_1, \ldots, S_m$ be finite sets that are pairwise disjoint and $|S_1| + |S_2| + \ldots + |S_m| = n$.

$$\exists i \in \{1, \ldots, m\} : |S_i| \geq \left\lfloor \frac{n}{m} \right\rfloor \quad \text{and} \quad \exists j \in \{1, \ldots, m\} : |S_j| \leq \left\lceil \frac{n}{m} \right\rceil$$

Example: Assume there are 5 holes in the wall where pigeons nest. Say there is a set S_i of pigeons nesting in hole i. Assume there are $n = 17$ pigeons in total. Then we know:

- There is some hole with at least $d = 4$ pigeons.
- There is some hole with at most $b = 3$ pigeons.

Double Counting

If we count the same quantity in *two different ways*, then this gives us a (perhaps non-trivial) identity.

Example (Handshaking Lemma): Assume there are n people at a party and everybody will shake hands with everybody else. How many handshakes will occur? We count this number in two ways:

1. Every person shakes $n - 1$ hands and there are n people. However, two people are involved in a handshake so if we just multiply $n \cdot (n - 1)$, then every handshake is counted twice. The total number of handshakes is therefore $\frac{n \cdot (n-1)}{2}$.
2. We number the people from 1 to n. To avoid counting a handshake twice, we count for person i only the handshakes with persons of lower numbers. Then the total number of handshakes is:

$$\sum_{i=1}^{n} (i - 1) = \sum_{i=0}^{\{n-1\}} i = \sum_{i=1}^{n-1} i$$

The identity we obtain is therefore: $\sum_{i=1}^{n-1} i = \frac{n \cdot (n - 1)}{2}$

§3 Arrangements, Permutations, Combinations

Ordered Arrangements

Definition 7: Denote by $[n] = \{1, ..., n\}$ the set of natural numbers from 1 to n.

Hereinafter, let X be a finite set.

Definition 8: An *ordered arrangement* of n elements of X is a *map* $s : [n] \longrightarrow X$.

- Here, $[n]$ is the *domain* of s, and $s(i)$ is the *image* of $i \in [n]$ under s.
- The set $\{x \in X \mid s(i) = x \text{ for some } i \in [n]\}$ is the *range* of s.

Other common names for ordered arrangements are:

- *string* (or *word*), e.g. "Banana"
- *sequence*, e.g. "0815422372"
- *tuple*, e.g. $(3, 5, 2, 5, 8)$

Example:

i	1	2	3	4	5	6	7
$s(i)$	🦀	🥖	🥖	🫙	🥖	🦀	🫙

Permutations

Definition 9: A *permutation* of X is a *bijective* map $\pi : [n] \longrightarrow X$.

Usually, $X = [n]$, and the set of all permutations of $[n]$ is denoted by S_n.

Example:

i	1	2	3	4	5	6	7
$\pi(i)$	2	7	1	3	5	4	6

Definition 10: *k-permutation* of X is an ordered arrangement of k *distinct* elements of X, that is, an *injective* map $\pi : [k] \longrightarrow X$.

The set of all k-permutations of $X = [n]$ is denoted by $P(n, k)$. In particular, $S_n = P(n, n)$.

TODO: circular permutations

Counting Permutations

Theorem 1: For any natural numbers $0 \leq k \leq n$, we have

$$|P(n,k)| = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

Proof: A permutation is an injective map $\pi : [k] \longrightarrow [n]$. We count the number of ways to pick such a map, picking the images one after the other. There are n ways to choose $\pi(1)$. Given a value for $\pi(1)$, there are $(n-1)$ ways to choose $\pi(2)$ (since we may not choose $\pi(1)$ again). Continuing like this, there are $(n-i+1)$ ways to pick $\pi(i)$, and the last value we pick is $\pi(k)$ with $(n-k+1)$ possibilities.

Every k-permutation can be constructed like this in *exactly one way*. The total number of k-permutations is therefore given as the product:

$$|P(n,k)| = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

$\square$

Counting Circular Permutations

Theorem 2: For any natural numbers $0 \leq k \leq n$, we have

$$|P_c(n, k)| = \frac{n!}{k! \cdot (n-k)!}$$

Proof: We doubly count $P(n, k)$:

1. $|P(n, k)| = \frac{n!}{(n-k)!}$ which we proved before.
2. $|P(n, k)| = |P_c(n, k)| \cdot k$ because every equivalence class in $P_c(n, k)$ contains k permutations from $P(n, k)$ since there are k ways to rotate a k-permutation.

From this we get $\frac{n!}{(n-k)!} = |P_c(n, k)| \cdot k$ which implies $|P_c(n, k)| = \frac{n!}{k! \cdot (n-k)!}$. $\square$

Unordered Arrangements

Definition 11: An *unordered arrangement* of k elements of X is a *multiset* $S = \langle X, r \rangle$ of size k.

In a multiset, X is the set of *types*, and for each type $x \in X$, r_x is its *repetition number*.

Example: Let $X = \{$👒, 🦭, 🐈, 🪗, 🌵$\}$.

- An unordered arrangement of 7 elements could be $S = \{$👒, 👒, 🦭, 🐈, 🐈, 🐈, 🌵$\}^*$.
- The same multiset could be written as $S = \{2$👒, 1🦭, 3🐈, 0🪗, 1🌵$\}$.

Subsets

The most important special case of unordered arrangements is where all repetitions are 1, i.e., $r_x = 1$ for all $x \in X$. Then S is simply a *subset* of X, denoted $S \subset X$.

Definition 12: A *k-combination* of X is an unordered arrangement of k *distinct* elements of X.

Note: The more standard term is *subset*. The term "combination" is only used to emphasize the selection process.

The set of all k-subsets of X is denoted $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$. If $|X| = n$, then

$$\binom{n}{k} := \left|\binom{X}{k}\right|$$

Example: The set of edges in a simple undirected graph consists of 2-subsets of its vertices: $E \subseteq \binom{V}{2}$.

Counting k-Combinations

Theorem 3: For $0 \leq k \leq n$, we have

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Proof:

$$|P(n,k)| = \frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!$$

$\square$

k-Combinations of a Multiset

Definition 13: Let X be a finite set of types, and let $M = \langle X, r \rangle$ be a finite multiset with repetition numbers $r_1, \ldots, r_{|X|}$. A *k-combination of M* is a multiset $S = \langle X, s \rangle$ with types in X and repetition numbers $s_1, \ldots, s_{|X|}$ such that $s_i \leq r_i$ for all $1 \leq i \leq |X|$, and $\sum_{i=1}^{|X|} s_i = k$.

Example: Consider $M = \{2$🦫, 1🥤, 3🍉, 1💎$\}$.

- $T = \{1$🦫, 2🍉$\}$ is a 3-combination of M.
- $T' = \{3$💎$\}$ is not.

Counting k-combinations of a multiset is not as simple as it might seem…

k-Permutations of a Multiset

Definition 14: Let M be a finite multiset with set of types X. A *k-permutation of M* is an ordered arrangement of k elements of M where different orderings of elements of the same type are *not distinguished*. This is an ordered multiset with types in X and repetition numbers $s_1, ..., s_{|X|}$ such that $s_i \leq r_i$ for all $1 \leq i \leq |X|$, and $\sum_{i=1}^{|X|} s_i = k$.

Note: There might be several elements of the same type compared to a permutation of a set (where each repetition number equals 1).

Example: Let $M = \{2$🦫, 1🥤, 3🍉, 1💎$\}$, then $T = ($💎, 🍉, 🍉, 🦫$)$ is a 4-permutation of multiset M.

Binomial Theorem

Theorem 4: The expansion of any non-negative integer power n of the binomial $(x + y)$ is a sum

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \cdot y^{n-k}$$

where each $\binom{n}{k}$ is a positive integer known as a *binomial coefficient*, defined as

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n(n-1)(n-2)...(n-k+1)}{k(k-1)(k-2)... \cdot 2 \cdot 1}$$

Multinomial Theorem

Theorem 5: The generalization of the binomial theorem:

$$(x_1 + \dots + x_r)^n = \sum_{\substack{0 \le k_1, \dots, k_r \le n \\ k_1 + \dots + k_r = n}}^{n} \binom{n}{k_1, \dots, k_r} \cdot x_1^{k_1} \cdot \dots \cdot x_r^{k_r}$$

Multinomial coefficients are defined as

$$\binom{n}{k_1, \dots, k_r} = \frac{n!}{k_{1!} \cdot \dots \cdot k_{r!}}$$

Note: Binomial coefficients are special cases of multinomial coefficients ($r = 2$):

$$\binom{n}{k} = \binom{n}{k_1, k_2} = \binom{n}{k, n-k} = \frac{n!}{k! \cdot (n-k)!}$$

Proof: TODO □

Permutations of a Multiset

Theorem 6: Let S be a finite multiset with k different types and repetition numbers $r_1, \dots, r_k$. Let the size of S be $n = r_1 + \dots + r_k$. Then the number of n-permutations of S equals

$$\binom{n}{r_1, \dots, r_k}$$

Proof: In an n-permutation there are n positions that need to be assigned a type.

First, choose the r_1 positions for the first type, there are $\binom{n}{r_1}$ ways to do so. Then, assign r_2 positions for the second type, out of the $(n - r_1)$ positions that are still available, there are $\binom{n-r_1}{r_2}$ ways to do so. Continue for all k types. The total number of choices will be:

$$\binom{n}{r_1} \cdot \binom{n - r_1}{r_2} \cdot \dots \cdot \binom{n - r_1 - r_2 - \dots - r_{k-1}}{r_k} = \binom{n}{r_1, \dots, r_k}$$

$\square$

k-Combinations of an *Infinite* Multiset

Example: Suppose you have a *sufficiently large* amount of each type of fruit (🍌, 🍎, 🍐) in the supermarket, and you want to buy *two* fruits. How many choices do you have?

There are exactly *six* combinations: {🍌, 🍌}, {🍌, 🍎}, {🍌, 🍐}, {🍎, 🍎}, {🍎, 🍐}, {🍐, 🍐}.

Note that your selection is *not ordered*, so {🍐, 🍎} and {🍎, 🍐} are considered the *same* choice.

k-Combinations of an *Infinite* Multiset [2]

Theorem 7: Let $k, s \in \mathbb{N}$ and let S be a multiset with s types and large repetition numbers (each $r_1, \ldots, r_s$ is *at least k*), then the number of k-combinations of S equals

$$\binom{k+s-1}{k} = \binom{k+s-1}{s-1}$$

Proof: Let $S = \{\infty$🍌, ∞🍎, ∞🍐$\}$, so $s = 3$.

- Suppose $k = 5$.
- Consider a 5-combination of S: {🍌, 🍎, 🍌, 🍐, 🍐}.
- Reorder and group: {🍌 🍌 | 🍎 | 🍐 🍐}.
- Convert to *dots* and *bars*: $\bullet\bullet \mid \bullet \mid \bullet\bullet$
- Represent as a multiset: $M = \{k \cdot \bullet, (s-1) \cdot \mid\ \}$
- Permute the 2-type multiset: $\binom{k+s-1}{k,s-1}$ ways, by Theorem 5.

□