

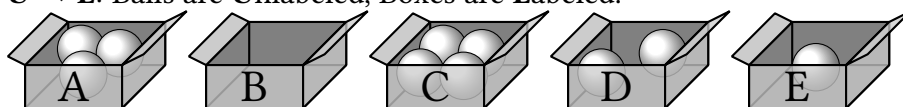
*Do whatever you want, but always explain what you are doing.*

— KONSTANTIN, 2020

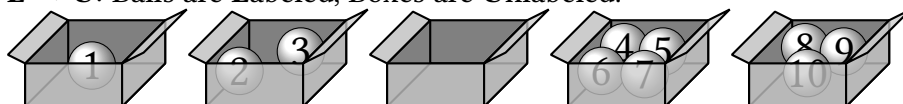
- Find the number of different 5-digit numbers using digits 1–9 under the given constraints. For each case, provide representative examples of (non-)complying numbers (e.g., 12345 and 52814 are suitable for (b), but 44521 and 935 are not) and derive a generic<sup>1</sup> formula. Try to express the formula using standard combinatorial terms, e.g.,  $k$ -combs  $C_n^k$  and  $k$ -perms  $P(n, k)$ .
  - Digits *can* be repeated.
  - Digits *cannot* be repeated.
  - Digits *can* be repeated and must be written in *non-increasing*<sup>2</sup> order.
  - Digits *cannot* be repeated and must be written in *strictly increasing* order.
  - Digits *can* be repeated, must be written in *non-decreasing* order, and the 4th digit must be 6.
- One of the classical combinatorial problems is counting the number of arrangements of  $n$  balls into  $k$  boxes. There are at least 12 variations of this problem: four cases (a–d) with three different constraints (1–3). For each problem (case+constraint), derive the corresponding generic formula. Additionally, pick several (representative) values for  $n$  and  $k$  and use your derived formulae to find the numbers of arrangements. Visualize several possible arrangements for the chosen  $n$  and  $k$ .

Cases with arrangement examples:

- a.  $U \rightarrow L$ : Balls are Unlabeled, Boxes are Labeled.



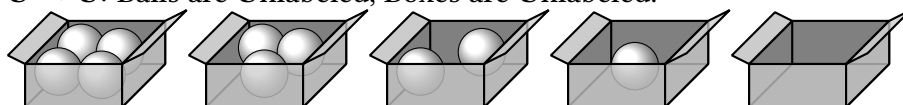
- b.  $L \rightarrow U$ : Balls are Labeled, Boxes are Unlabeled.



- c.  $L \rightarrow L$ : Balls are Labeled, Boxes are Labeled.



- d.  $U \rightarrow U$ : Balls are Unlabeled, Boxes are Unlabeled.



Constraints:

- $\leq 1$  ball per box — *injective* mapping.
- $\geq 1$  ball per box — *surjective* mapping.
- Arbitrary number of balls per box.

Notes:

- \* Unlabeled means “indistinguishable”, and Labeled means “distinguishable”.
- \* **Stirling number of the second kind**  $s_k^{II}(n)$  — number of ways to partition a set of  $n$  elements into  $k$  non-empty subsets. Use  $s_k^{II}(n)$  directly without expanding the closed formula.
- \* **Partition function**  $p_k(n)$  — number of ways to partition the integer  $n$  into  $k$  positive parts, i.e.  $n = a_1 + \dots + a_k$ , where  $a_1 \geq \dots \geq a_k \geq 1$ . Use  $p_k(n)$  directly.

<sup>1</sup> Here, “generic formula” means “depending on the input data”. In this particular example,  $n = 9$  and  $k = 5$ , but the sought formula must also be valid for all other (adequate) values of  $n$  and  $k$ .

<sup>2</sup> A sequence  $(x_n)$  is said to be *strictly monotonically increasing* if each term is *strictly greater* than the previous one, i.e.  $x_i < x_{i+1}$ . A sequence  $(x_n)$  is called *non-increasing* if each term is *less than or equal* to the previous one, i.e.  $x_i \geq x_{i+1}$ .

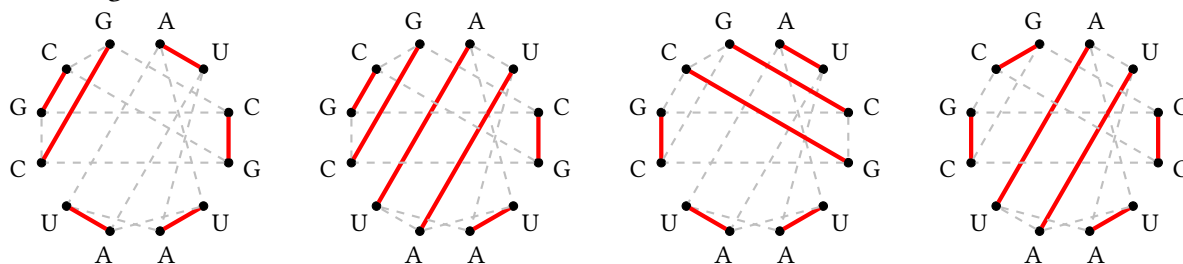
3. Prove the Generalized Pascal's Formula (for  $n \geq 1$  and  $k_1, \dots, k_r \geq 0$  with  $k_1 + \dots + k_r = n$ ):

$$\binom{n}{k_1, \dots, k_r} = \sum_{i=1}^r \binom{n-1}{k_1, \dots, k_i-1, \dots, k_r}$$

Count the number of permutations of a multiset  $\Sigma^* = \{2 \cdot \triangle, 3 \cdot \square, 1 \cdot \clubsuit\}$  using GPF.

4. A *non-crossing perfect matching*<sup>3</sup> in a graph is a set of pairwise disjoint edges that cover all vertices and do not intersect with each other. For example, consider a graph on  $2n$  vertices numbered from 1 to  $2n$  and arranged in a circle. Additionally, assume that edges are straight lines. In this case, edges  $\{i, j\}$  and  $\{a, b\}$  intersect whenever  $i < a < j < b$ .

- (a) Count the number of all possible non-crossing perfect matchings in a complete graph  $K_{2n}$ .
- (b) Consider a graph on vertices labeled with letters from  $\{A, C, G, U\}$ . Each pair of vertices labeled with A and U is connected with a *basepair edge*. Similarly, C-G pairs are also connected. The picture below illustrates some of possible non-crossing perfect matchings in the graph with 12 vertices AUCGUAUUCGCG arranged in a circle. Basepair edges are drawn dashed gray, matching is red.



Count the number of all possible non-crossing perfect matchings in the graph on 20 vertices arranged in a circle and labeled with CGUAAUACGGCAUUAAGCAU.

<sup>3</sup> Credits to Rosalind for this task.