

# Discrete Mathematics

**Network Flows** — Spring 2025

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# §1 Network Flows

## Motivation

TODO: a picture with a graph and a question about the flow in it

# Flow Network

**Definition 1:** A *flow network* is a directed graph  $G = \langle V, E \rangle$  with:

- a *source*  $s \in V$ , a vertex without incoming edges,
- a *sink*  $t \in V$ , a vertex without outgoing edges,
- a *capacity* function  $c : E \rightarrow \mathbb{R}_+$  that assigns a non-negative capacity to each edge  $e \in E$ .

The flow network is denoted as  $N = \langle V, E, s, t, c \rangle$ .

**Note:** We require that  $E$  never contains both edges  $(u, v)$  and  $(v, u)$  for any  $u, v \in V$ .

**Note:** If  $(u, v) \notin E$ , then  $c(u, v) = 0$ .

**Note:** The graph is connected, i.e., every node has at least one incident edge.

## Flow Network Example

*Example:* Very meaningful example of a flow network with annotated capacities:



# Flow

**Definition 2:** Given a flow network  $N$ , a *flow* is a function  $f : E \rightarrow \mathbb{R}_+$  that satisfies the following *feasibility* conditions:

1. *Capacity constraint*:  $0 \leq f(e) \leq c(e)$  for each edge  $e \in E$ .
2. *Flow conservation (balance constraint)*: for each node  $v \in V$ , except for  $s$  and  $t$ ,

$$\underbrace{\sum_{e \in \text{in}(v)} f(e)}_{\text{flow into } v} = \underbrace{\sum_{e \in \text{out}(v)} f(e)}_{\text{flow out of } v}$$

**Note:** If  $(u, v) \notin E$ , then  $f(u, v) = 0$ .

## Flow Value

**Definition 3:** The *value*  $|f|$  of a flow  $f$  is the total amount of flow that leaves the source  $s$ :

$$|f| = \sum_{e \in \text{out}(s)} f(e) - \underbrace{\sum_{e \in \text{in}(s)} f(e)}_{\text{commonly 0}}$$

**Note:**  $f^{\text{in}}(v) := \sum_{e \in \text{in}(v)} f(e)$

**Note:**  $f^{\text{out}}(v) := \sum_{e \in \text{out}(v)} f(e)$

**Definition 4** (Maximum Flow Problem): Given a flow network  $N$ , the *maximum flow problem* is to find a flow  $f$  that maximizes the value  $|f|$ .

# Max Flow Example

*Example:* Yet another meaningful example.





## Flow Conservation

**Theorem 1:** For any feasible flow  $f$ , the net flow out of  $s$  is equal to the net flow into  $t$ :

$$|f| = \sum_{e \in \text{out}(s)} f(e) = \sum_{e \in \text{in}(t)} f(e)$$

**Proof:** This follows directly from the flow conservation condition.

$$\begin{aligned} |f| &= \sum_{e \in \text{out}(s)} f(e) = \\ &= \sum_{e \in \text{out}(s)} f(e) - \sum_{v \in V \setminus \{s, t\}} \left[ \sum_{e \in \text{in}(v)} f(e) - \sum_{e \in \text{out}(v)} f(e) \right] = \\ &= \sum_{e \in \text{in}(t)} f(e) \end{aligned}$$

□

## Residual Capacity

**Definition 5:** The *skew-symmetry* convention defines the flow in the opposite direction of an edge  $e = (u, v)$  as  $f(v, u) = -f(u, v)$ .

**Definition 6:** Given a flow  $f$  in a flow network  $N$ , the *residual capacity*  $c_f$  of an edge  $e$  is the amount of flow that can be sent through the edge in addition to the flow already in it:

$$c_f(e) := c(e) - f(e)$$

## Residual Network

**Definition 7:** The *residual network*  $N_f$  for a flow  $f$  is a flow network with the same vertices as  $N$ , constructed as follows:

- *Forward edges:* For each edge  $e = (u, v)$  of  $N$ , if  $f(e) < c(e)$ , add an edge  $e' = (u, v)$  to  $N_f$  with capacity  $c(e) - f(e)$ .
- *Backward edges:* For each edge  $e = (u, v)$  in  $N$ , if  $f(e) > 0$ , add a reversed edge  $e' = (v, u)$  to  $N_f$  with capacity  $f(e)$ .

In other words, a residual network is a directed graph with *all* edges with *positive* residual capacity.

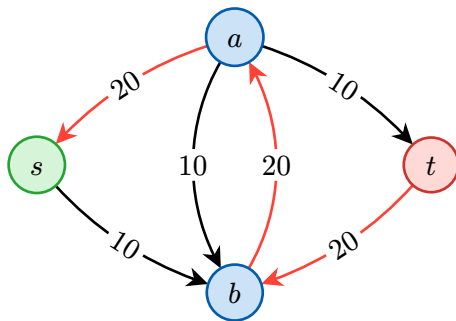
## Residual Network Example

- *Remaining capacity*: If  $f(e) < c(e)$ , add edge  $e$  to  $N_f$  with capacity  $c(e) - f(e)$ .
- *Can erase up to  $f(e)$  capacity*: If  $f(u, v) > 0$ , add reversed edge  $(v, u)$  to  $N_f$  with capacity  $f(e)$ .

Network  $N$  with flow  $f$



Residual Network  $N_f$



## Augmenting Paths

**Definition 8:** An *augmenting path* in the residual network  $N_f$  is an  $s$ - $t$  path (a path from  $s$  to  $t$ ) such that all edges in the path have positive capacity. The *bottleneck* of an augmenting path is the minimum capacity of the edges in the path.

**Theorem 2:** If *bottleneck* is positive, then the flow can be increased by that amount along the path.

## Ford-Fulkerson Algorithm

**INPUT:** A flow network  $N$  with source  $s$  and sink  $t$ .

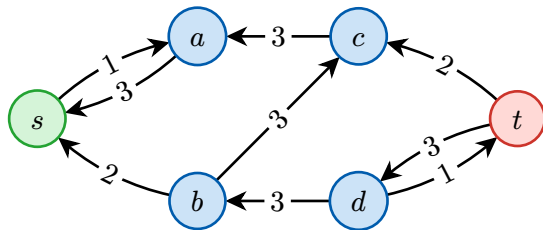
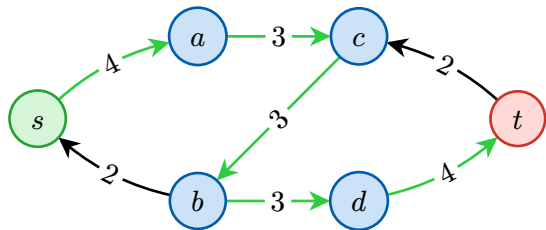
**OUTPUT:** Maximum flow  $f$  from  $s$  to  $t$ .

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1 Initialize  $f(e) = 0$  for all  $e \in E$ 
2 while there is an augmenting path  $P$  in the residual network  $N_f$  do
3   | Let  $b = \min_{e \in P} c'(e)$  in  $N_f$  along  $P$ 
4   | for each edge  $e \in P$  do
5   |   | Update flow:  $f(e) := f(e) + b$ 
6   |   | Rebuild the residual network  $N_f$ 
7 return  $f$ 
```

## Example



Networks (above) and residual networks (below) after pushing the flow with  $|f| = 2$  along the path  $s - b - c - t$  (left), and then after pushing the flow with  $|f| = 3$  along the path  $s - a - c - b - d - t$  (right).



# Cuts

**Definition 9:** An *s-t cut* is a set of edges whose removal disconnects  $t$  from  $s$ .

Formally, a *cut* is a partition of the vertices  $V = A \cup B$  such that  $s \in A$  and  $t \in B$ . The edges of the cut are the edges that go from  $A$  to  $B$ .

**Definition 10:** The *capacity* of a cut  $(A, B)$  is the sum of the capacities of the edges leaving  $A$ .

$$c(A, B) = \sum_{a \in A, b \in B} c(a, b)$$

**Definition 11:** Given a flow  $f$  in  $N$ , the *net flow* across a cut  $(A, B)$  is defined as

$$f(A, B) = \sum_{e \in \text{out}(A)} f(e) - \sum_{e \in \text{in}(A)} f(e)$$



## Cut Theorem 1

**Theorem 3:** Let  $f$  be a flow and  $(A, B)$  be an  $s$ - $t$  cut. Then:

$$f(A, B) = |f|$$

## Cut Theorem 2

**Theorem 4:** Let  $f$  be a flow and  $(A, B)$  be an  $s$ - $t$  cut. Then:

$$f(A, B) \leq c(A, B)$$

**Proof:**

$$\begin{aligned} f(A, B) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &\leq f^{\text{out}}(A) \\ &= \sum_{e \in \text{out}(A)} f(e) \\ &\leq \sum_{e \in \text{out}(A)} c(e) \\ &= c(A, B) \end{aligned}$$

□

# Max-Flow Min-Cut Theorem

**Theorem 5:** Given a flow network  $N$  and a flow  $f$ , the following are equivalent:

1.  $f$  is a *maximum flow* in  $N$ .
2. There is *no augmenting path* in the residual network  $N_f$ .
3.  $|f| = c(A, B)$  for some  $s$ - $t$  cut  $(A, B)$  in  $N$ .

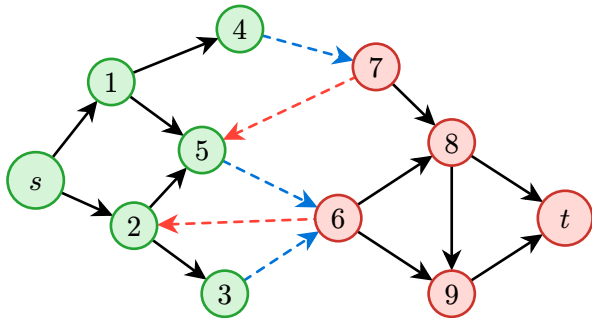
If one (and hence all) of these conditions hold, then  $(A, B)$  is a *minimum cut*.

**Proof** ( $1 \rightarrow 2$ ): An augmenting path in the residual network  $N_f$  would allow us to increase the flow  $f$ .  $\square$

**Proof** ( $3 \rightarrow 1$ ): No flow can exceed the capacity of a cut (by Theorem 4)  $\square$

**Proof** ( $2 \rightarrow 3$ ): Let  $S$  be the set of vertices reachable from  $s$  in the residual network  $N_f$ . Since there is no augmenting path in  $N_f$ ,  $S$  does not contain  $t$ . Then  $(S, T)$  is a cut of  $N$ , where  $T = V \setminus S$ . Moreover, for any  $u \in S$  and  $v \in T$ , the residual capacity  $c_f(e)$  must be zero (otherwise, the path  $s \rightsquigarrow u$  in  $N_f$  could be extended to a path  $s \rightsquigarrow u \rightarrow v$  in  $N_f$ ). Thus,  $f(S, T) = f^{\text{out}}(S) - f^{\text{in}}(S) = f^{\text{out}}(S) - 0 = c(S, T)$ .  $\square$

## Max-Flow Min-Cut Theorem [2]



- Cut  $(S, T)$  with  $s \in S, t \in T$ .
- Blue edges must be saturated.
- Red edges must be empty (zero).