

- For each given relation $R_i \subseteq M_i^2$, determine whether it is *reflexive*, *irreflexive*, *coreflexive*, *symmetric*, *antisymmetric*, *asymmetric*, *transitive*, *antitransitive*, *semiconnex*, *connex*, *left/right Euclidean*, *dense*. Provide a counterexample for each non-complying property (e.g., “transitivity does not hold for $x, y, z = (3, 1, 2)$ ”). Organize your answer in a table (e.g., columns – relations, rows – properties).
 - $M_1 = \mathbb{R}$
 $x R_1 y \leftrightarrow |x - y| \leq 1$
 - $M_2 = \mathcal{P}(\{a, b, c\})$
 $R_2 = “\subseteq”$
 - $M_3 = \{a, b, c, d\}$ $\|R_3\| = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
 - $M_4 = \{\text{“rock”}, \text{“scissors”}, \text{“paper”}\}$
 $R_4 = \{\langle x, y \rangle \mid x \text{ beats } y\}$
- Prove (rigorously) or disprove (by providing a counterexample) the following statements about arbitrary homogeneous relations $R \subseteq M^2$ and $S \subseteq M^2$:
 - If R and S are *reflexive*, then $R \cap S$ is so.
 - If R and S are *symmetric*, then $R \cap S$ is so.
 - If R and S are *transitive*, then $R \cap S$ is so.
 - If R and S are *reflexive*, then $R \cup S$ is so.
 - If R and S are *symmetric*, then $R \cup S$ is so.
 - If R and S are *transitive*, then $R \cup S$ is so.
- An equinumerosity relation R over sets is defined as follows: $A R B \leftrightarrow |A| = |B|$.
 - Show that R is an equivalence relation.
 - Find the quotient set of $\mathcal{P}(\{a, b, c, d\})$ by R .
- Let R_θ be a relation of θ -similarity of finite non-empty sets defined as follows: a set A is said to be θ -similar to B iff the Jaccard index² $\text{Jac}(A, B)$ for these sets is at least θ , i.e. $A R_\theta B \leftrightarrow \text{Jac}(A, B) \geq \theta$. Obviously, $\theta \in [0; 1] \subseteq \mathbb{R}$.
 - Draw the graph of a relation $R_\theta \subseteq \{A_i\}^2$, where $\theta = 0.25$, $A_1 = \{1, 2, 5, 6\}$, $A_2 = \{2, 3, 4, 5, 7, 9\}$, $A_3 = \{1, 4, 5, 6\}$, $A_4 = \{3, 7, 9\}$, $A_5 = \{1, 5, 6, 8, 9\}$.
 - Determine whether θ -similarity is a tolerance relation.
 - Determine whether θ -similarity is an equivalence relation.
- Let $H = \{1, 2, 4, 5, 10, 12, 20\}$. Consider a divisibility relation $R \subseteq H^2$ defined as follows: $x R y \leftrightarrow y : x$.
 - Sort R (as a set of pairs) lexicographically¹.
 - Show that R is a partial order.
 - Determine whether R is a linear (total) order.
 - Draw the Hasse diagram for a *graded poset* $\langle H, R, \rho \rangle$, where $\rho : H \rightarrow \mathbb{N}_0$ is a grading function which maps a number $n \in H$ to the sum of all exponents appearing in its prime factorization, e.g., $\rho(20) = \rho(2^2 \cdot 5^1) = 2 + 1 = 3$, so the number 20 would have the 3rd rank (bottom-up).
 - Find the minimal, minimum (least), maximal and maximum (greatest) elements in the poset $\langle H, R \rangle$. If there are multiple or none, explain why.
 - Perform a topological sort² of the poset $\langle H, R \rangle$.
- Prove that the transitive closure R^+ is in fact transitive.

Definition. $R^+ = \bigcup_{n \in \mathbb{N}^+} R^n$ is a transitive closure of $R \subseteq M^2$, where

* $R^{k+1} = R^k \circ R$ is a compositional (functional) power²,

* $R^1 = R$,

* $S \circ R = \{\langle x, y \rangle \mid \exists z : (x R z) \wedge (z S y)\}$ is a composition (relative product) of relations R and S .

¹ Lexicographic order for pairs: $\langle a, b \rangle \leq \langle a', b' \rangle \leftrightarrow (a < a') \vee ((a = a') \wedge (b \leq b'))$

² Note: this is not a Cartesian power, despite of the same notation R^n . Another possible notation for compositional power is $R^{\circ n}$, but it is too wild to use it here.