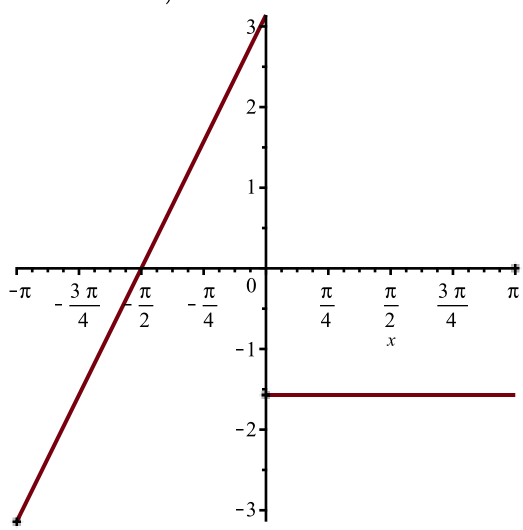
Лабораторная работа №2
Вариант 2
Задание 1:
Для 2π-периодической кусочно-непрерывной функции f(x)
по ее аналитическому определению на главном периоде получите разложение в тригонометрический ряд Фурье. Убедитесь в правильности результата, проведя расчеты в системе Maple.

> $f := x \rightarrow piecewise\left(-\mathrm{Pi} \le x < 0, \mathrm{Pi} + 2 \cdot x, 0 \le x < \mathrm{Pi}, -\frac{\mathrm{Pi}}{2}\right)$

$$f := x \mapsto \begin{cases} \pi + 2\cdot x & -\pi \le x < 0 \\ -\frac{\pi}{2} & 0 \le x < \pi \end{cases} \tag{1}$$

> $plot(f(x), x = -Pi..\mathrm{Pi}, discont = true)$



> $a0 := simplify\left(\frac{1}{\mathrm{Pi}} \cdot int(f(x), x = -\mathrm{Pi}..\mathrm{Pi})\right);$

$$a0 := -\frac{\pi}{2} \tag{2}$$

> $an := simplify\left(\frac{1}{\mathrm{Pi}} \cdot int(f(x) \cdot \cos(n \cdot x), x = -\mathrm{Pi}..\mathrm{Pi})\right)$ assuming $n :: posint;$

$$an := \frac{-2(-1)^n + 2}{\pi n^2} \tag{3}$$

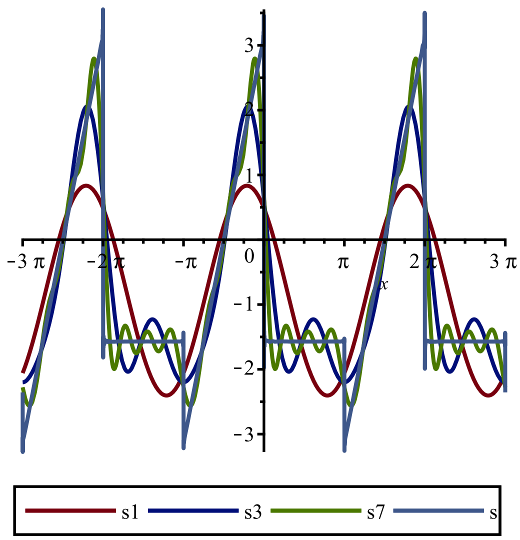
> $bn := simplify\left(\frac{1}{\mathrm{Pi}} \cdot int(f(x) \cdot \sin(n \cdot x), x = -\mathrm{Pi}..\mathrm{Pi})\right)$ assuming $n :: posint;$

$$bn := \frac{-(-1)^n - 3}{2n} \tag{4}$$

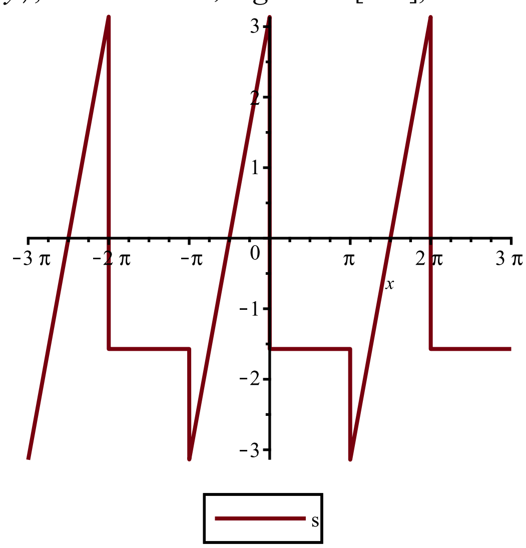
```
> FurieSum :=proc(f, k)
    local a0, an, bn, n;
    a0 := simplify(int(f(x), x = −π..π) / π);
    assume(n::posint);
    an := simplify(int(f(x) * cos(n * x), x = −π..π) / π);
```

```
    bn := simplify(int(f(x) * sin(n * x), x = -π..π) / π);
    return 1/2 * a0 + sum(an * cos(n * x) + bn * sin(n * x), n = 1..k)
end proc:
```

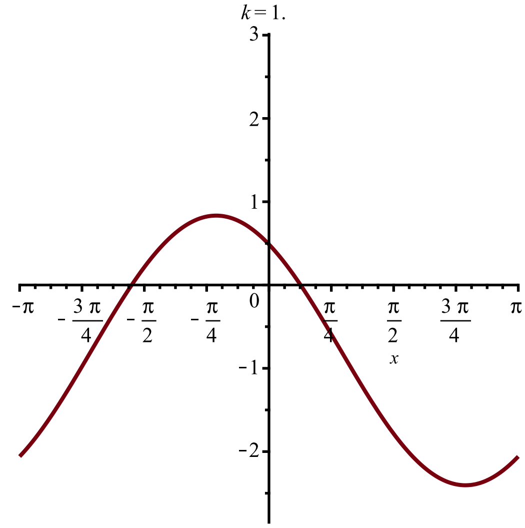
```
> plot([FurieSum(f, 1), FurieSum(f, 3), FurieSum(f, 7), FurieSum(f, 1000)], x =-3π..3 π,
    legend = ["s1", "s3", "s7", "s"], discont = true);
```




```
> plot(FurieSum(f, infinity), x =-3 π..3 π, legend = ["s"], discont = true);
```




```
> plots[animate](plot, [FurieSum(f, k), x =-Pi..Pi], k = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10])
```

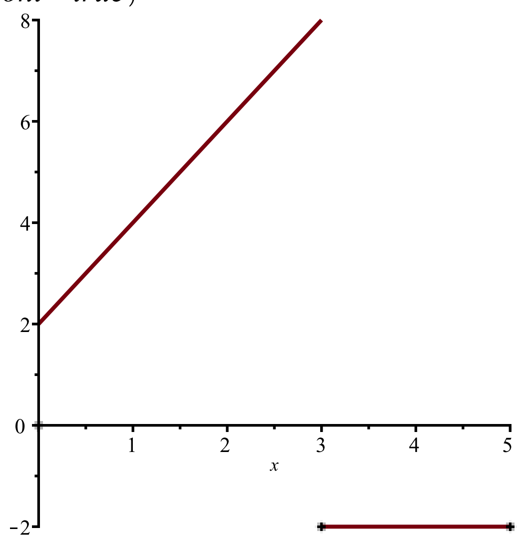


Задание 2:
Разложите в ряд Фурье x2 -периодическую функцию y=f(x), заданную на промежутке (0,x1) формулой y = ax + b, а на [x1,x2] – y = c.

> $f := x \rightarrow piecewise(0 < x < 3,\ 2 \cdot x + 2,\ 3 \le x \le 5, -2)$;

$$f := x \mapsto \begin{cases} 2\cdot x + 2 & 0 < x < 3 \\ -2 & 3 \le x \le 5 \end{cases} \quad (5)$$

> $plot(f(x), x = 0..5, discont = true)$



> $l := \frac{5}{2}$:

> $a0 := simplify\left(\frac{1}{l} \cdot int(f(x), x = 0..2\cdot l)\right)$;

$$a0 := \frac{22}{5} \quad (6)$$

> $an := simplify\left(\frac{1}{l} \cdot int\left(f(x) \cdot \cos\left(\frac{\pi \cdot n \cdot x}{l}\right), x = 0..2\cdot l\right)\right)$ assuming $n :: posint$

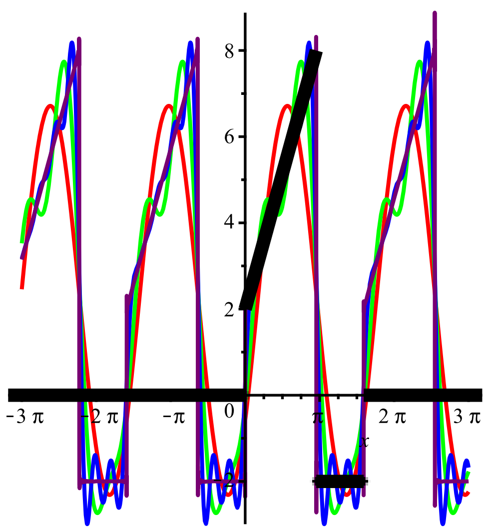
(7)

$$an := \frac{10\,\pi\,n\,\sin\left(\frac{6\,\pi\,n}{5}\right) + 5\cos\left(\frac{6\,\pi\,n}{5}\right) - 5}{\pi^2\,n^2} \tag{7}$$

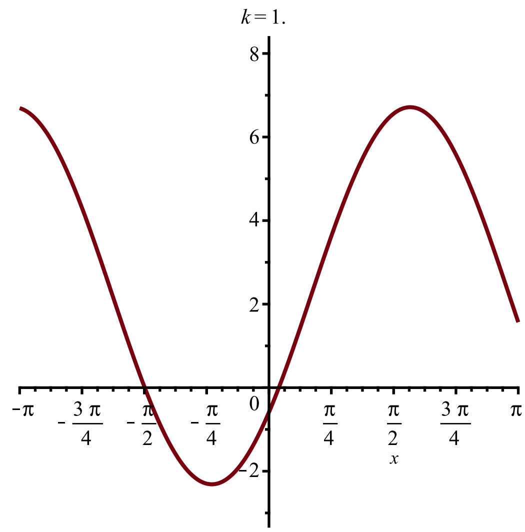
> $bn := simplify\left(\frac{1}{l}\cdot int\left(f(x)\cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right), x=0..2\cdot l\right)\right)$ assuming $n :: posint$

$$bn := \frac{-10\,\pi\,n\cos\left(\frac{6\,\pi\,n}{5}\right) + 4\,\pi\,n + 5\sin\left(\frac{6\,\pi\,n}{5}\right)}{\pi^2\,n^2} \tag{8}$$

```
> FurieSumModify :=proc(f, k, x1, x2)
    local a0, an, bn, n, l;
    l := 1/2*x2 − 1/2*x1;
    a0 := simplify(int(f(x), x=0..2*l)/l);
    assume(n::posint);
    an := simplify(int(f(x)*cos(π*n*x/l), x=0..2*l)/l);
    bn := simplify(int(f(x)*sin(π*n*x/l), x=0..2*l)/l);
    return 1/2*a0 + sum(an*cos(π*n*x/l) + bn*sin(π*n*x/l), n=1..k)
  end proc:
> fur := plot([FurieSumModify(f, 1, 0, 5), FurieSumModify(f, 3, 0, 5), FurieSumModify(f, 7,
    0, 5), FurieSumModify(f, 1000, 0, 5)], x = -3·Pi..3·Pi, discont = true, color = [red, green,
    blue, purple]) :
> func := plot(f(x), x = -10..10, discont = true, color = black, thickness = 5) :
> plots[display](fur, func)
```




```
> plots[animate](plot, [FurieSumModify(f, k, 0, 5), x = -Pi..Pi], k = [1, 2, 3, 4, 5, 6, 7, 8, 9, 10])
```



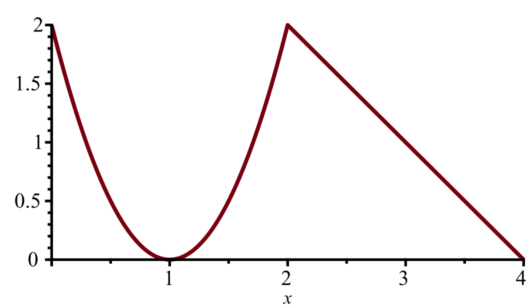
Задание 3:
Для графически заданной на промежутке функции как комбинации квадратичной и линейной постройте три разложения в тригонометрический ряд Фурье, считая, что функция определена:
- на полном периоде;
- на полупериоде (является четной);
- на полупериоде (является нечетной).

> $restart;$

> $f := x \rightarrow piecewise\left(0 \le x \le 2,\ 2 \cdot (x-1)^2,\ 2 < x < 4,\ 4 - x\right);$

$$f := x \mapsto \begin{cases} 2\cdot(x-1)^2 & 0 \le x \le 2 \\ 4 - x & 2 < x < 4 \end{cases} \tag{9}$$

> $plot(f(x), x = 0..4, scaling = constrained)$



> $l := 2$

$$l := 2 \tag{10}$$

> $a0 := simplify\left(\frac{1}{l} \cdot int(f(x), x = 0..2\cdot l)\right);$

$$a0 := \frac{5}{3} \tag{11}$$

> $an := simplify\left(\frac{1}{l}\cdot int\left(f(x)\cdot\cos\left(\frac{\pi\cdot n\cdot x}{l}\right), x=0..2\cdot l\right)\right)$ assuming $n :: posint$

$$an := \frac{6+10\,(-1)^n}{\pi^2\,n^2} \qquad \textbf{(12)}$$

> $bn := simplify\left(\frac{1}{l}\cdot int\left(f(x)\cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right), x=0..2\cdot l\right)\right)$ assuming $n :: posint$

$$bn := \frac{2\,\pi^2\,n^2+16\,(-1)^n-16}{\pi^3\,n^3} \qquad \textbf{(13)}$$

> $S := k\rightarrow \frac{a0}{2} + sum\left(an\cdot\cos\left(\frac{\mathrm{Pi}\cdot n\cdot x}{l}\right)+bn\cdot\sin\left(\frac{\mathrm{Pi}\cdot n\cdot x}{l}\right), n=1..k\right)$

$$S := k \mapsto \frac{a0}{2} + \sum_{n=1}^{k}\left(an\cdot\cos\left(\frac{\pi\cdot n\cdot x}{l}\right)+bn\cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right)\right) \qquad \textbf{(14)}$$

> $plot(S(10000), x=-12..12, y=0..2.5, discont=true)$

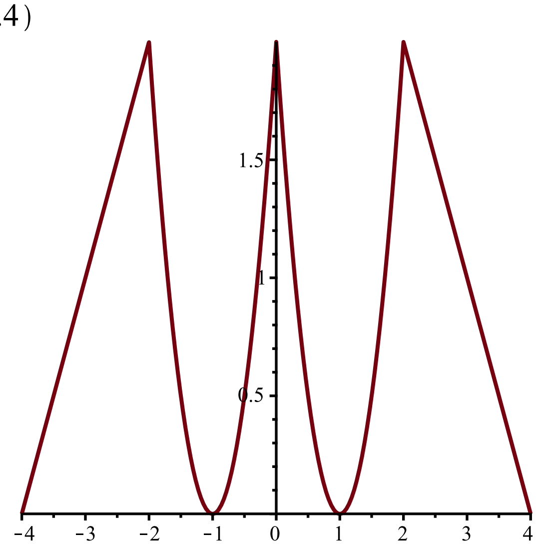
> $f(x)$

$$\begin{cases} 2\,(-1+x)^2 & 0\le x\le 2\\ 4-x & 2<x<4\end{cases} \qquad \textbf{(15)}$$

> $fchetn := x\rightarrow piecewise(-4<x<-2, 4+x, -2\le x\le 0, 2\cdot(-x-1)^2, 0\le x\le 2, 2\cdot(x-1)^2, 2<x<4, 4-x);$

$$fchetn := x \mapsto \begin{cases} 4+x & -4<x<-2\\ 2\cdot(-x-1)^2 & -2\le x\le 0\\ 2\cdot(x-1)^2 & 0\le x\le 2\\ 4-x & 2<x<4\end{cases} \qquad \textbf{(16)}$$

> $plot(fchetn(x), x=-4..4)$



> $l := 4$

$$l := 4 \qquad \textbf{(17)}$$

> $a0 := simplify\left(\frac{1}{l}\cdot int(fchetn(x), x=-l..l)\right);$

$$a0 := \frac{5}{3} \tag{18}$$

> $an := simplify\left(\frac{1}{l}\cdot int\left(fchetn(x)\ \cdot\cos\left(\frac{\pi\cdot n\cdot x}{l}\right), x=-l..l\right)\right)$ assuming $n :: posint$

$$an := \frac{-8\,\pi\,(-1)^n\,n + 40\,\pi\,n\cos\left(\frac{\pi\,n}{2}\right) + 32\,\pi\,n - 128\sin\left(\frac{\pi\,n}{2}\right)}{\pi^3\,n^3} \tag{19}$$

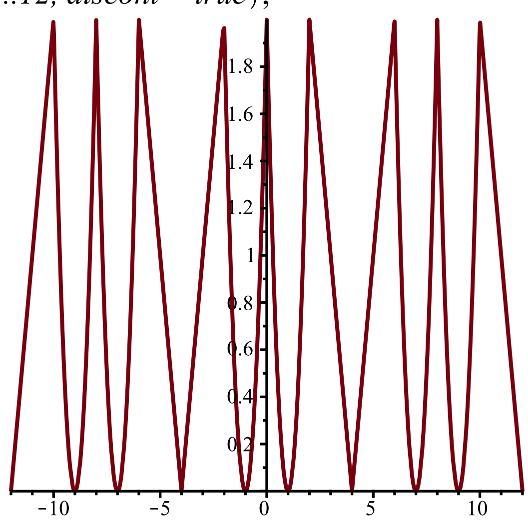
> $bn := simplify\left(\frac{1}{l}\cdot int\left(fchetn(x)\ \cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right), x=-l..l\right)\right)$ assuming $n :: posint$

$$bn := 0 \tag{20}$$

> $S := k\rightarrow\frac{a0}{2} + sum\left(an\cdot\cos\left(\frac{\mathrm{Pi}\cdot n\cdot x}{l}\right) + bn\cdot\sin\left(\frac{\mathrm{Pi}\cdot n\cdot x}{l}\right), n=1..k\right)$

$$S := k \mapsto \frac{a0}{2} + \sum_{n=1}^{k}\left(an\cdot\cos\left(\frac{\pi\cdot n\cdot x}{l}\right) + bn\cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right)\right) \tag{21}$$

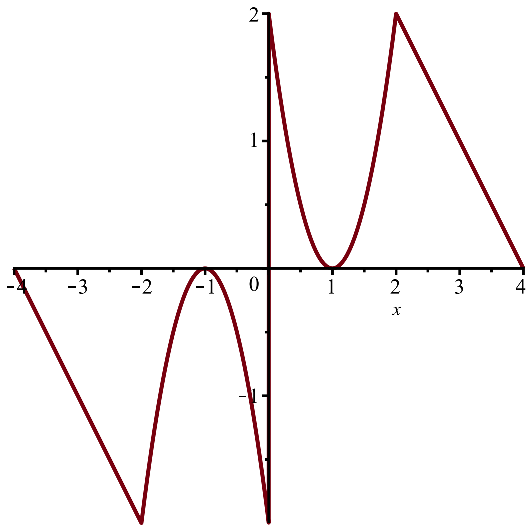
> $plot(S(10000), x=-12..12, discont=true);$



> $fnech := x\rightarrow piecewise(-4 < x < -2, -(4 + x), -2 \le x \le 0, -2\cdot(-x-1)^2, 0 \le x \le 2, 2\cdot(x-1)^2, 2 < x < 4, (4-x));$

$$fnech := x \mapsto \begin{cases} -4 - x & -4 < x < -2 \\ -2\cdot(-x-1)^2 & -2 \le x \le 0 \\ 2\cdot(x-1)^2 & 0 \le x \le 2 \\ 4 - x & 2 < x < 4 \end{cases} \tag{22}$$

> $plot(fnech(x), x=-4..4)$



> $a0 := simplify\left(\frac{1}{l}\cdot int(\mathit{fnech}(x), x=-l..l)\right);$

$$a0 := 0 \tag{23}$$

> $an := simplify\left(\frac{1}{l}\cdot int\left(\mathit{fnech}(x)\cdot\cos\left(\frac{\pi\cdot n\cdot x}{l}\right), x=-l..l\right)\right)$ assuming $n :: posint$

$$an := 0 \tag{24}$$

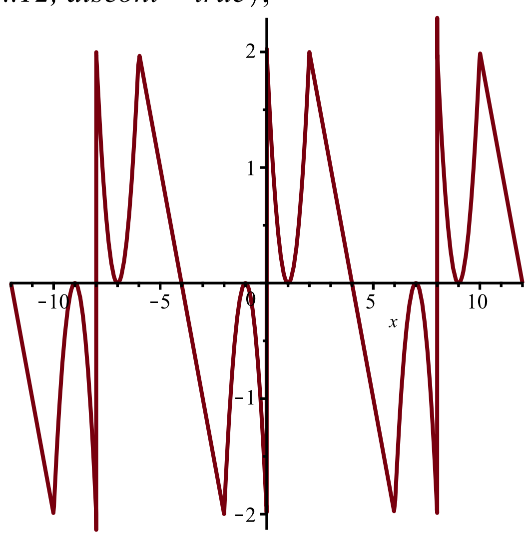
> $bn := simplify\left(\frac{1}{l}\cdot int\left(\mathit{fnech}(x)\cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right), x=-l..l\right)\right)$ assuming $n :: posint$

$$bn := \frac{4\,\pi^2\,n^2 + 40\,\pi\,n\,\sin\left(\frac{\pi\,n}{2}\right) + 128\cos\left(\frac{\pi\,n}{2}\right) - 128}{\pi^3\,n^3} \tag{25}$$

> $S := k \rightarrow \frac{a0}{2} + sum\left(an\cdot\cos\left(\frac{\mathrm{Pi}\cdot n\cdot x}{l}\right) + bn\cdot\sin\left(\frac{\mathrm{Pi}\cdot n\cdot x}{l}\right), n=1..k\right)$

$$S := k \mapsto \frac{a0}{2} + \sum_{n=1}^{k}\left(an\cdot\cos\left(\frac{\pi\cdot n\cdot x}{l}\right) + bn\cdot\sin\left(\frac{\pi\cdot n\cdot x}{l}\right)\right) \tag{26}$$

> $plot(S(10000), x=-12..12, discont=true);$

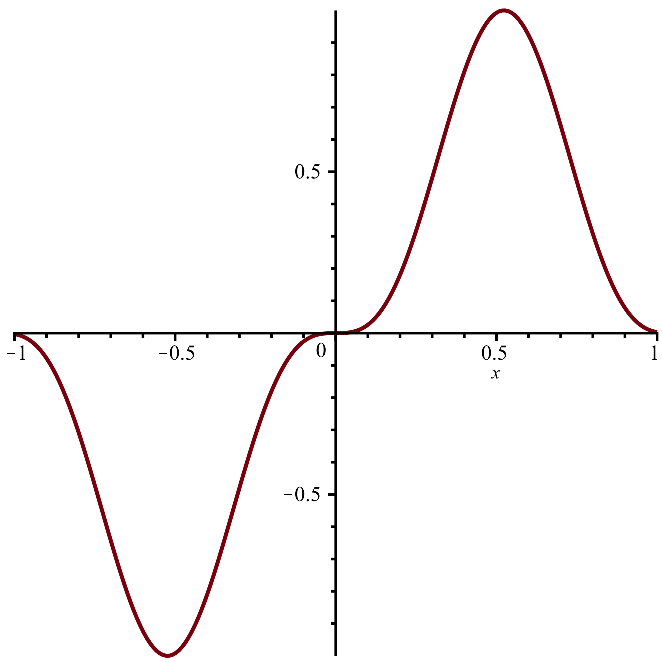


Задание 4:
Разложите функцию в ряд Фурье по многочленам Лежандра и Чебышёва на промежутке [-1, 1]. Создайте пользовательские процедуры, осуществляющие построение частичной суммы ряда для абсолютно интегрируемой функции по этим ортогональным полиномам.

> $f := \sin^3(3 \cdot x)$

$$f := \sin(3\,x)^3 \tag{27}$$

> $funcplot := plot(f, x = -1 \,..1)$



> $with(orthopoly)$

$$[G, H, L, P, T, U] \tag{28}$$

> **for** n **from** 0 **to** 9 **do** $c[n] := \dfrac{\int_{-1}^{1} f \cdot P(n, x)\, \mathrm{d}x}{\int_{-1}^{1} P(n, x)^2 \mathrm{d}x}$; **end do**

$$c_0 := 0$$

$$c_1 := -\frac{\sin(3)^2 \cos(3)}{3} - \frac{2\cos(3)}{3} + \frac{\sin(3)^3}{27} + \frac{2\sin(3)}{9}$$

$$c_2 := 0$$

$$c_3 := -\frac{154 \sin(3)^2 \cos(3)}{243} + \frac{322 \cos(3)}{243} + \frac{1568 \sin(3)}{729} + \frac{1099 \sin(3)^3}{2187}$$

$$c_4 := 0$$

$$c_5 := \frac{407 \sin(3)^2 \cos(3)}{2187} - \frac{126940 \sin(3)}{6561} - \frac{6116 \cos(3)}{2187} + \frac{26620 \sin(3)^3}{19683}$$

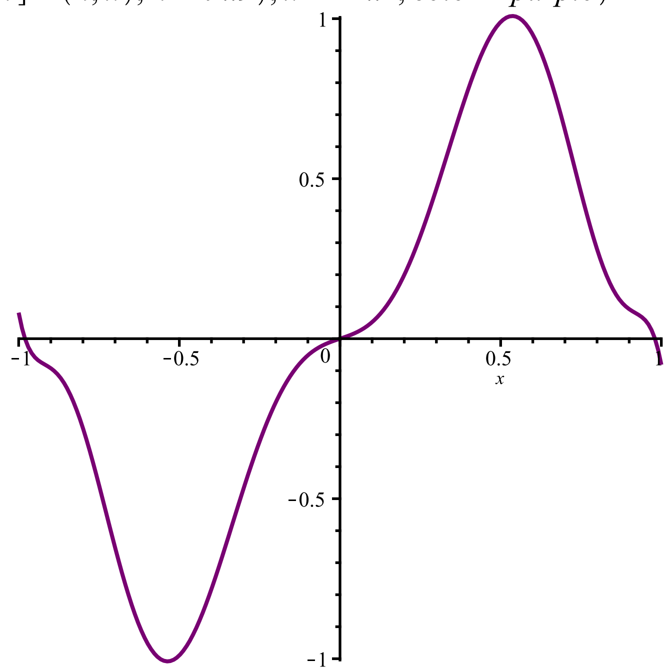
$$c_6 := 0$$

$$c_7 := \frac{126005 \sin(3)^2 \cos(3)}{59049} - \frac{161105 \sin(3)^3}{531441} + \frac{58195340 \sin(3)}{177147} + \frac{2732320 \cos(3)}{59049}$$

$$c_8 := 0$$

$$c_9 := -\frac{12876034 \sin(3)^2 \cos(3)}{4782969} - \frac{117607055 \sin(3)^3}{43046721} - \frac{154663846930 \sin(3)}{14348907} - \frac{7347111938 \cos(3)}{4782969} \qquad \textbf{(29)}$$

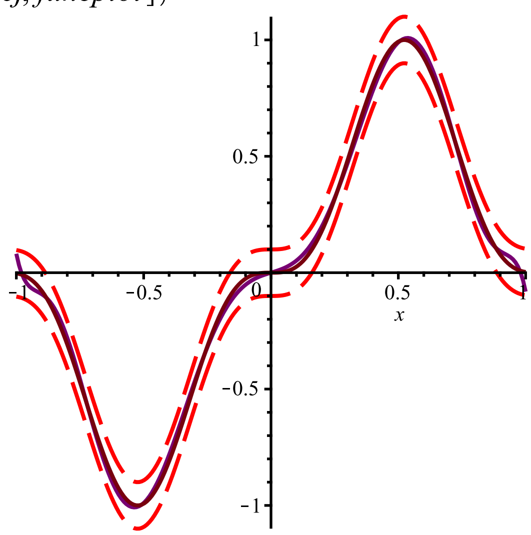
> $lej := plot(add(c[n] \cdot P(n, x), n = 0 .. 9), x = -1 .. 1, color = purple)$



> $f1 := plot(f + 0.1, x = -1 .. 1, linestyle = dash, color = red)$:

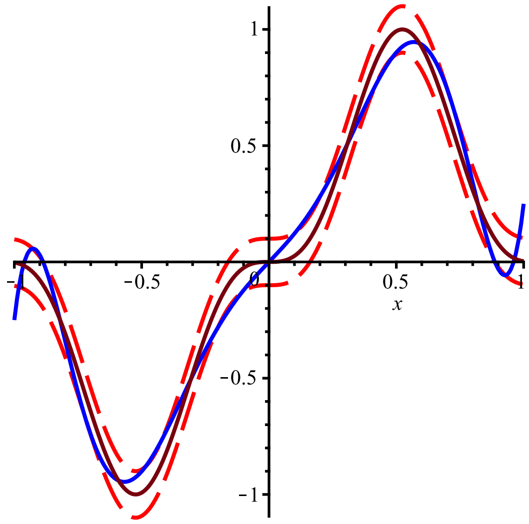
> $f2 := plot(f - 0.1, x = -1 .. 1, linestyle = dash, color = red)$:

> $plots[display]([f1, f2, lej, funcplot])$



> $nmin := plot(add(c[n] \cdot P(n, x), n = 0 .. 8), x = -1 .. 1, color = blue)$:

> $plots[display](f1, f2, nmin, funcplot)$



> **for** n **from** 0 **to** 9 **do** $c[n] := \dfrac{2}{\pi}\displaystyle\int_{-1}^{1} \dfrac{f \cdot T(n,x)}{\sqrt{1-x^2}}\, \mathrm{d}x$; **end do**

$$c_0 := 0$$

$$c_1 := \frac{2\left(\displaystyle\int_{-1}^{1} \frac{\sin(3\,x)^3\,x}{\sqrt{-x^2+1}}\,\mathrm{d}x\right)}{\pi}$$

$$c_2 := 0$$

$$c_3 := \frac{2\left(\displaystyle\int_{-1}^{1} \frac{\sin(3\,x)^3\,\left(4\,x^3-3\,x\right)}{\sqrt{-x^2+1}}\,\mathrm{d}x\right)}{\pi}$$

$$c_4 := 0$$

$$c_5 := \frac{2\left(\displaystyle\int_{-1}^{1} \frac{\sin(3\,x)^3\,\left(16\,x^5-20\,x^3+5\,x\right)}{\sqrt{-x^2+1}}\,\mathrm{d}x\right)}{\pi}$$

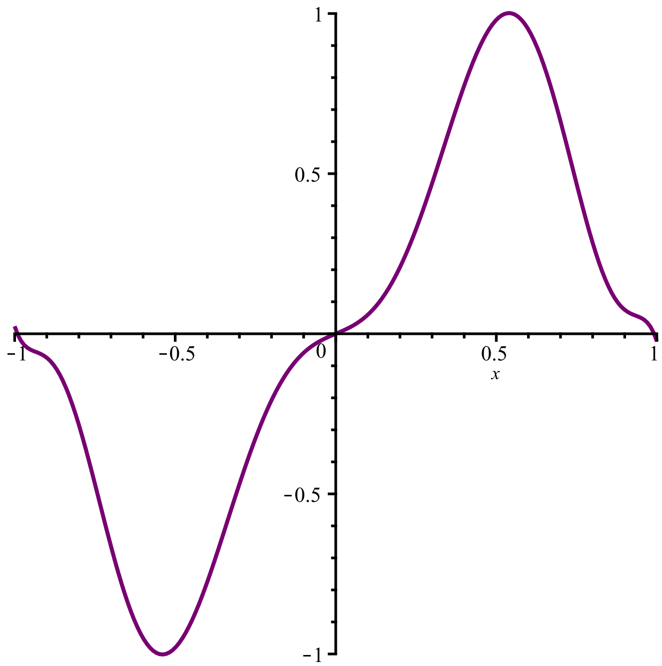
$$c_6 := 0$$

$$c_7 := \frac{2\left(\displaystyle\int_{-1}^{1} \frac{\sin(3\,x)^3\,\left(64\,x^7-112\,x^5+56\,x^3-7\,x\right)}{\sqrt{-x^2+1}}\,\mathrm{d}x\right)}{\pi}$$

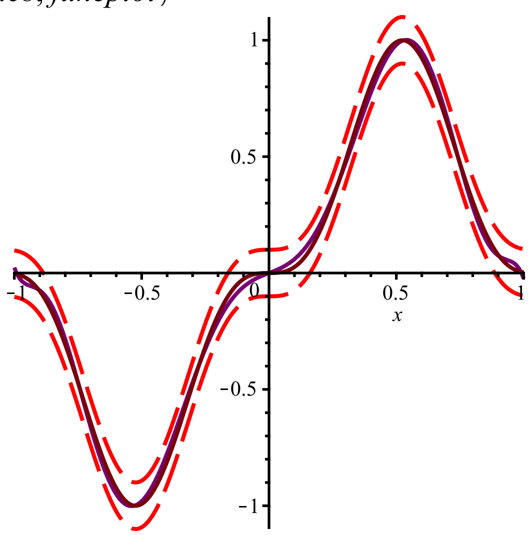
$$c_8 := 0$$

$$c_9 := \frac{2\left(\int_{-1}^{1} \frac{\sin(3\,x)^3\,\left(256\,x^9 - 576\,x^7 + 432\,x^5 - 120\,x^3 + 9\,x\right)}{\sqrt{-x^2+1}}\,dx\right)}{\pi} \tag{30}$$

> $\text{cheb} := plot\left(\frac{c[0]}{2} + add(c[n]\cdot T(n,x), n=1..9), x=-1..1, color=purple\right)$

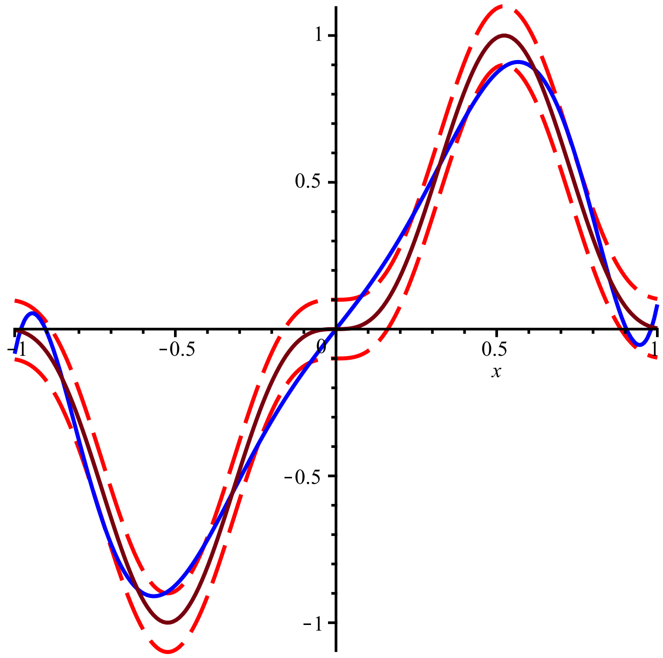


> $plots[display](f1, f2, cheb, funcplot)$



> $nmin := plot\left(\frac{c[0]}{2} + add(c[n]\cdot T(n,x), n=1..8), x=-1..1, color=blue\right):$

> $plots[display](f1, f2, nmin, funcplot)$



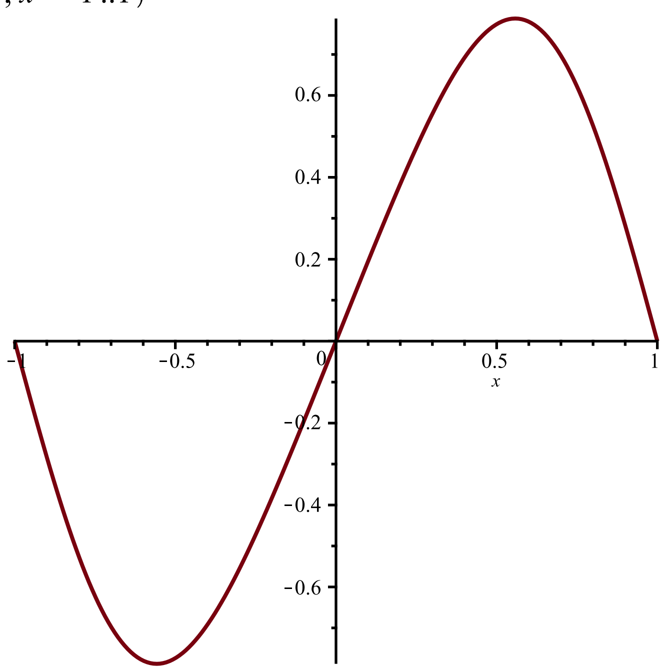
> $bn := simplify(int(\mathrm{f}\cdot\sin(\mathrm{Pi}\cdot nn\cdot x), x=-1..1))$ assuming $nn :: posint$

$$bn := \frac{\pi\,(-1)^{nn}\,nn\,\left(\sin(9)\,\pi^2\,nn^2 - 3\,\sin(3)\,\pi^2\,nn^2 - 9\,\sin(9) + 243\,\sin(3)\right)}{2\,\pi^4\,nn^4 - 180\,\pi^2\,nn^2 + 1458} \tag{31}$$

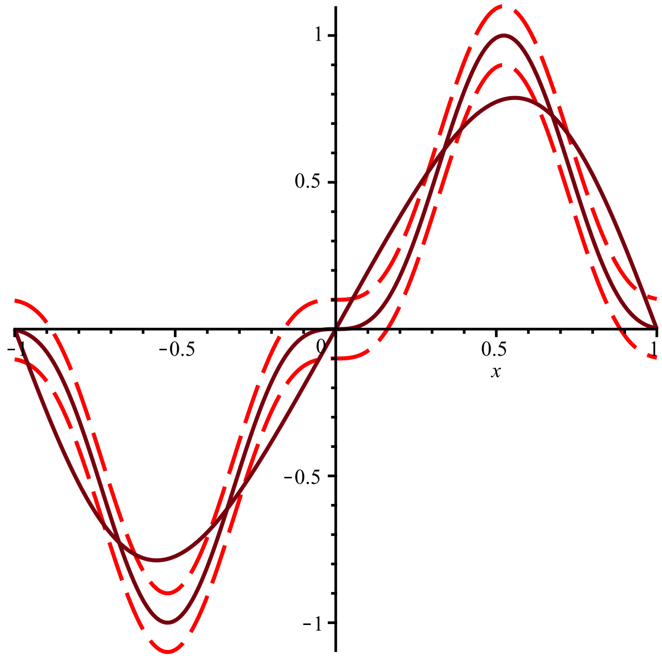
> $Sm := \mathrm{k} \rightarrow sum(bn\cdot\sin(\pi\cdot\mathrm{nn}\cdot\mathrm{x}), nn = 1..k)$

$$Sm := k \mapsto \sum_{nn=1}^{k} bn\cdot\sin(\pi\cdot nn\cdot x) \tag{32}$$

> $fur := plot(Sm(2), x=-1..1)$



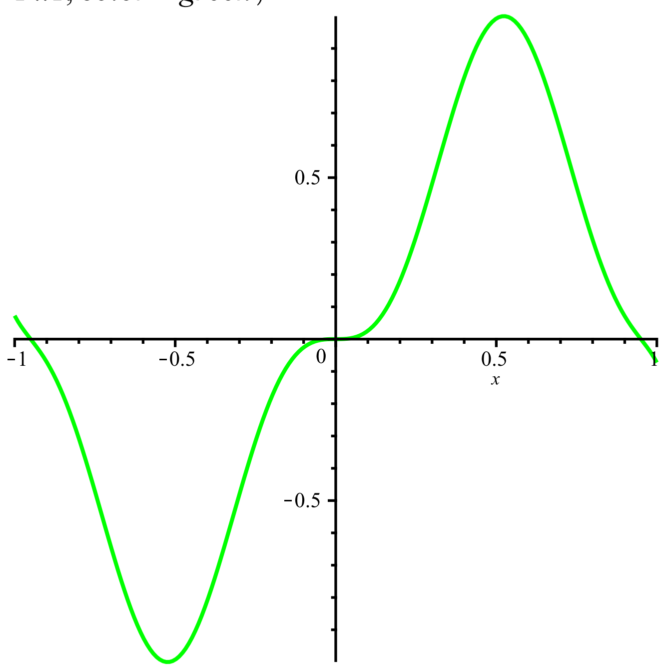
> $plots[display](f1, f2, fur, funcplot)$



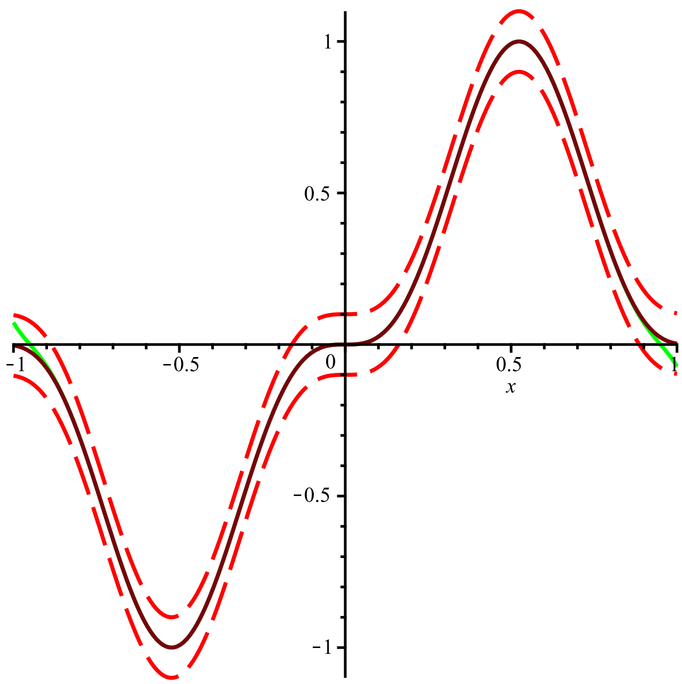
> $St := convert(taylor(f, x = 0, 22), polynom)$

$$St := 27x^3 - \frac{243}{2}x^5 + \frac{9477}{40}x^7 - \frac{29889}{112}x^9 + \frac{4402431}{22400}x^{11} - \frac{1436859}{14080}x^{13} + \frac{35303684679}{896896000}x^{15} - \frac{4205292633}{358758400}x^{17} + \frac{14885130969}{5361664000}x^{19} - \frac{1263422855673}{2359717068800}x^{21} \quad (33)$$

> $StF := plot(St, x = -1..1, color = green)$



> $plots[display](f1, f2, StF, funcplot)$

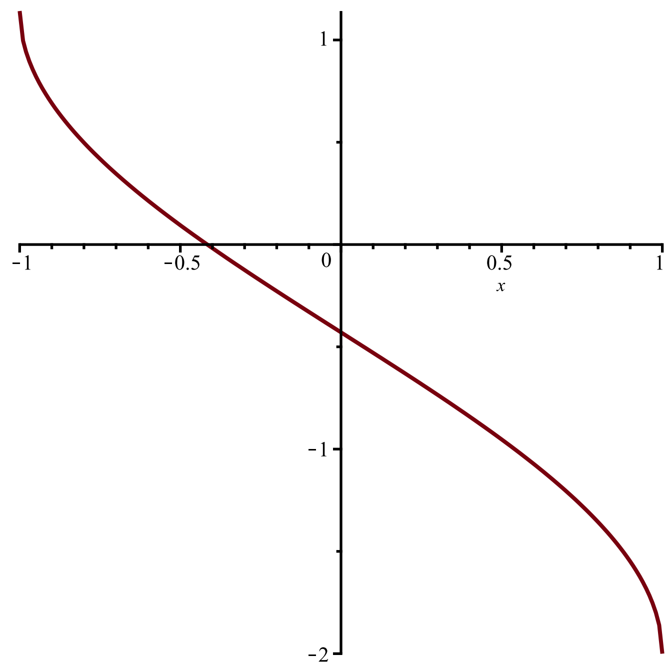


> *restart*

> $f := \arccos(x) - 2$

$$f := \arccos(x) - 2 \quad \textbf{(34)}$$

> $funcplot := plot(f, x = -1..1)$



> $with(orthopoly)$

$$[G, H, L, P, T, U] \quad \textbf{(35)}$$

> **for** n **from** 0 **to** 7 **do** $c[n] := \dfrac{\displaystyle\int_{-1}^{1} f \cdot P(n, x)\, \mathrm{d}x}{\displaystyle\int_{-1}^{1} P(n, x)^2 \mathrm{d}x}$; **end do**

$$c_0 := -2 + \frac{\pi}{2}$$

$$c_1 := -\frac{3\pi}{8}$$

$$c_2 := 0$$

$$c_3 := -\frac{7\pi}{128}$$

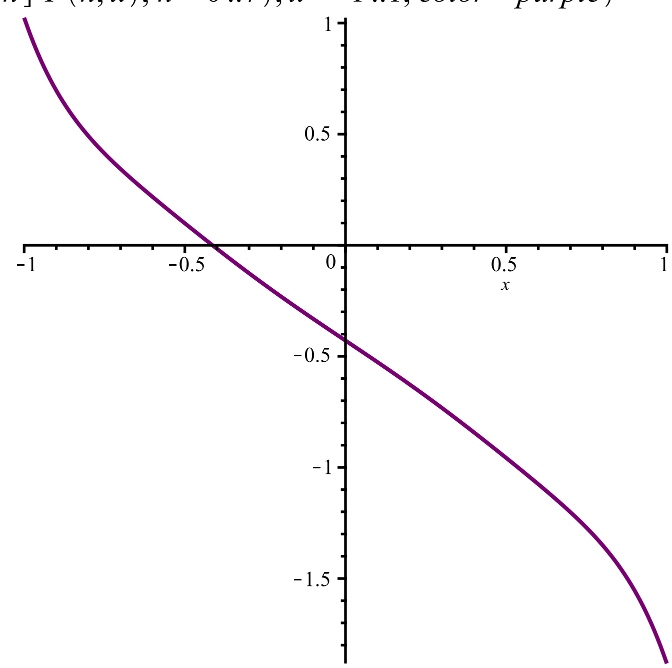
$$c_4 := 0$$

$$c_5 := -\frac{11\pi}{512}$$

$$c_6 := 0$$

$$c_7 := -\frac{375\pi}{32768} \tag{36}$$

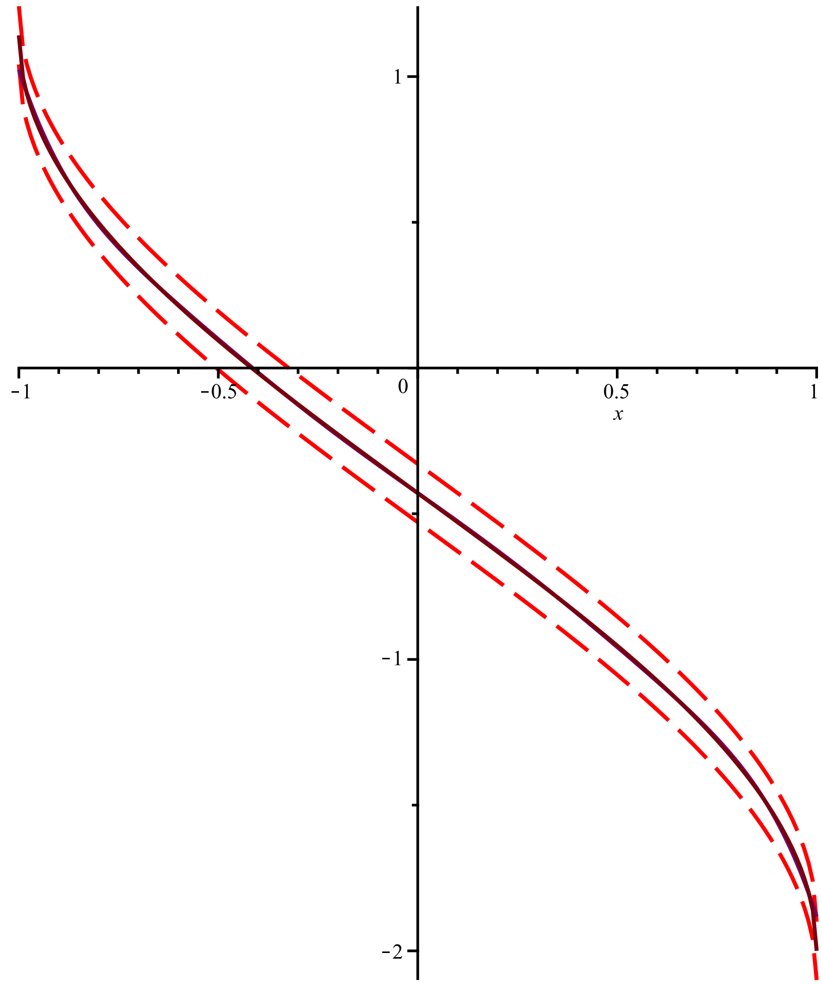
> $lej := plot(add(c[n] \cdot P(n, x), n = 0..7), x = -1..1, color = purple)$



> $f1 := plot(f + 0.1, x = -1..1, linestyle = dash, color = red)$:

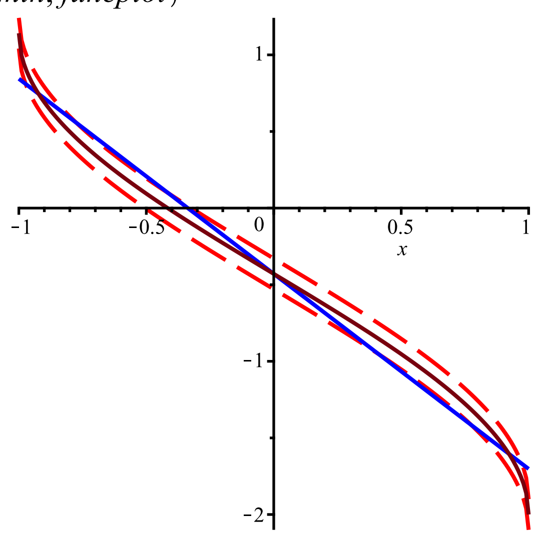
> $f2 := plot(f - 0.1, x = -1..1, linestyle = dash, color = red)$:

> $plots[display]([f1, f2, lej, funcplot])$



> $nmin := plot(add(c[n] \cdot P(n, x), n = 0..6), x = -1..1, color = blue)$:

> $plots[display](f1, f2, nmin, funcplot)$



> **for** n **from** 0 **to** 3 **do** $c[n] := \dfrac{2}{\pi}\displaystyle\int_{-1}^{1} \frac{f \cdot T(n, x)}{\sqrt{1 - x^2}}\, \mathrm{d}x$; **end do**

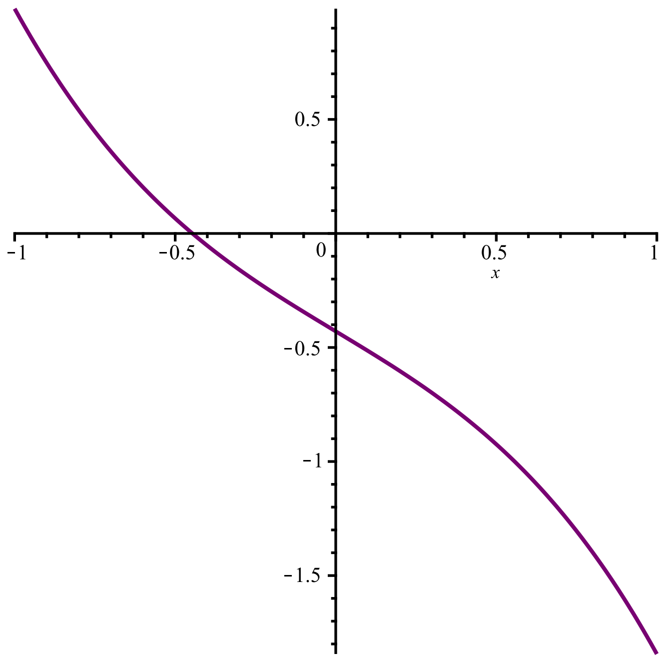
$$c_0 := \frac{2\left(\frac{1}{2}\pi^2 - 2\pi\right)}{\pi}$$

$$c_1 := -\frac{4}{\pi}$$

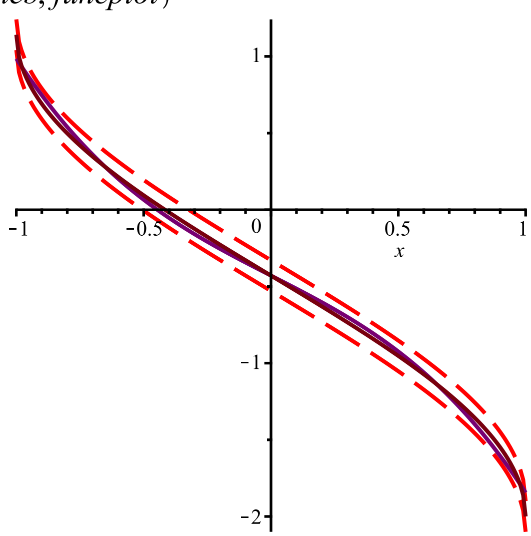
$$c_2 := 0$$

$$c_3 := -\frac{4}{9\pi} \tag{37}$$

> cheb $:= plot\left(\frac{c[0]}{2} + add(c[n]\cdot T(n, x), n = 1..3), x = -1..1, color = purple\right)$

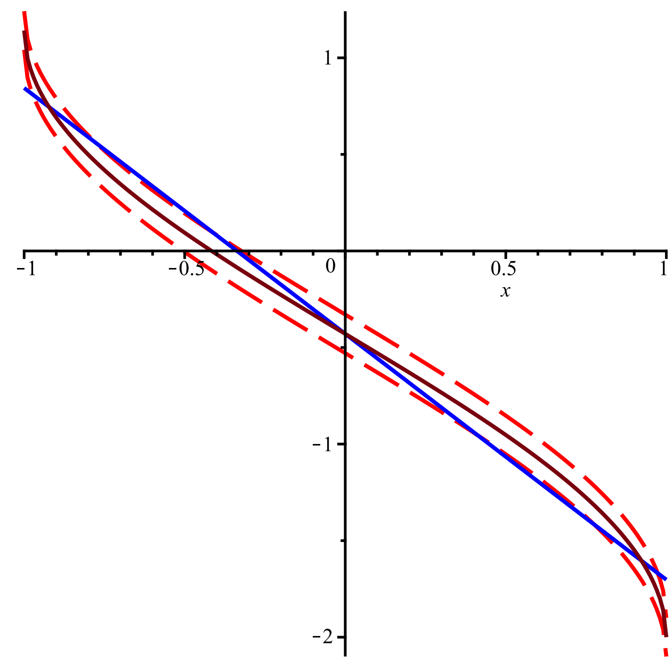


> $plots[display](f1, f2, cheb, funcplot)$



> $nmin := plot\left(\frac{c[0]}{2} + add(c[n]\cdot T(n, x), n = 1..2), x = -1..1, color = blue\right):$

> $plots[display](f1, f2, nmin, funcplot)$



> $a0 := simplify(int(f, x=-1..1))$

$$a0 := -4 + \pi \tag{38}$$

> $an := simplify(int(\mathrm{f}\cdot\cos(\mathrm{Pi}\cdot nn\cdot x), x=-1..1))$ assuming $nn :: posint$

$$an := 0 \tag{39}$$

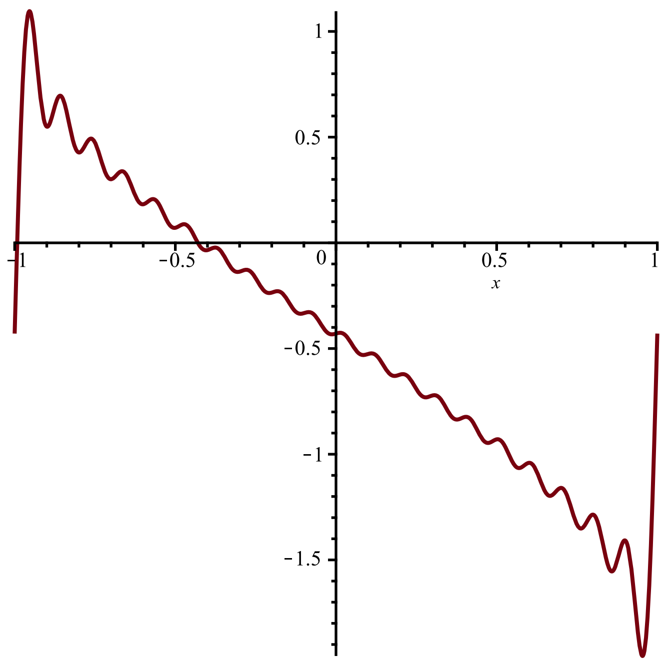
> $bn := simplify(int(\mathrm{f}\cdot\sin(\mathrm{Pi}\cdot nn\cdot x), x=-1..1))$ assuming $nn :: posint$

$$bn := \int_{-1}^{1} (\arccos(x) - 2)\sin(\pi\, nn\, x)\, \mathrm{d}x \tag{40}$$

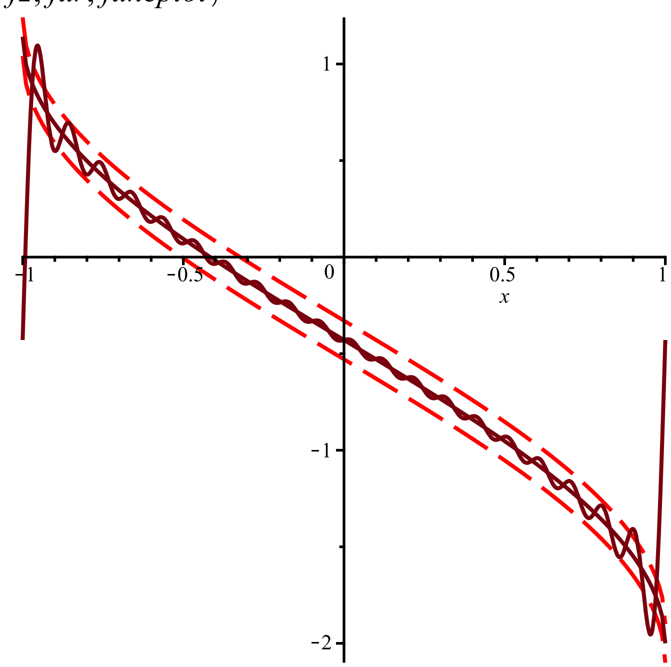
> $Sm := \mathrm{k} \rightarrow \frac{a0}{2} + sum(bn\cdot\sin(\pi\cdot\mathrm{nn}\cdot\mathrm{x}), nn=1..k)$

$$Sm := k \mapsto \frac{a0}{2} + \left(\sum_{nn=1}^{k} bn\cdot\sin(\pi\cdot nn\cdot x)\right) \tag{41}$$

> $fur := plot(\mathrm{Sm}(20), x=-1..1, discont=true)$



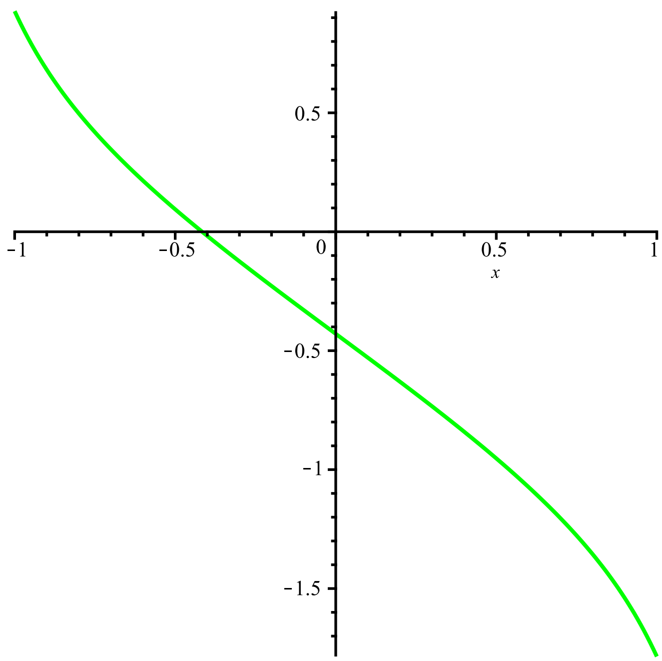
> $plots[display](f1, f2, fur, funcplot)$



> $St := convert(taylor(f, x=0, 14), polynom)$

$$St := -2 + \frac{1}{2}\pi - x - \frac{1}{6}x^3 - \frac{3}{40}x^5 - \frac{5}{112}x^7 - \frac{35}{1152}x^9 - \frac{63}{2816}x^{11} - \frac{231}{13312}x^{13} \qquad \textbf{(42)}$$

> $StF := plot(St, x=-1..1, color=green)$



> $plots[display](f1, f2, StF, funcplot)$