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Crystal Analysis of type C Stanley Symmetric Functions

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Abstract. Combining results of T.K. Lam and J. Stembridge, the type C Stanley symmetric function $F_w^C(\mathbf{x})$, indexed by an element w in the type C Coxeter group, has a nonnegative integer expansion in terms of Schur functions. We provide a crystal theoretic explanation of this fact and give an explicit combinatorial description of the coefficients in the Schur expansion in terms of highest weight crystal elements.

Keywords: Stanley symmetric functions, crystal bases, Kraśkiewicz insertion, mixed Haiman insertion, unimodal tableaux, primed tableaux

1 Introduction

Schubert polynomials of types B and C were independently introduced by Billey and Haiman [1] and Fomin and Kirillov [6]. Stanley symmetric functions [14] are stable limits of Schubert polynomials, designed to study properties of reduced words of Coxeter group elements. In his Ph.D. thesis, T.K. Lam [11] studied properties of Stanley symmetric functions of type B (or similarly C) and D. In particular he showed, using Kraśkiewicz insertion [9, 10], that the type B Stanley symmetric functions have a positive integer expansion in terms of P-Schur functions. On the other hand, Stembridge [15] proved that the P-Schur functions expand positively in terms of Schur functions. Combining these two results, it follows that Stanley symmetric functions of type B (and similarly type C) have a positive integer expansion in terms of Schur functions.

Schur functions $s_\lambda(\mathbf{x})$, indexed by partitions λ, are ubiquitous in combinatorics and representation theory. They are the characters of the symmetric group and can also be interpreted as characters of type A crystals. In [12], this was exploited to provide a combinatorial interpretation in terms of highest weight crystal elements of the coefficients in the Schur expansion of (affine) Stanley symmetric functions in type A. In this paper, we carry out a crystal analysis of the Stanley symmetric functions $F_w^C(\mathbf{x})$ of type C, indexed by a Coxeter group element w. In particular, we use Kraśkiewicz insertion [9, 10] and Haiman's mixed insertion [7] to find a crystal structure on primed tableaux, which in turn implies a crystal structure $\mathcal{B}$ signed unimodal factorizations of w for which $F_w^C(\mathbf{x})$ is a character. Moreover, we present a type A crystal isomorphism $\Phi\colon \mathcal{B} \to \bigoplus_\lambda \mathcal{B}_\lambda^{\oplus g_\lambda}$ for

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some combinatorially defined nonnegative integer coefficients g_λ; here $\mathcal{B}_\lambda$ is the type A highest weight crystal of highest weight λ . This implies the desired decomposition $F_w^C(\mathbf{x}) = \sum_\lambda g_\lambda s_\lambda(\mathbf{x})$.

In Section 2, we review type C Stanley symmetric functions and type A crystals. In Section 3 we describe our crystal isomorphism by combining a slight generalization of the Kraśkiewicz insertion [9, 10] and the mixed insertion first introduced by Haiman [7]. The main result is stated in Theorem 22 and a combinatorial interpretation of the coefficients g_λ in Corollary 25.

2 Background

2.1 Type C Stanley symmetric functions

The *Coxeter group* W_C of type C_n, also known as the hyperoctahedral group or the group of signed permutations, is a finite group generated by $\{s_0, s_1, \ldots, s_{n-1}\}$ subject to the quadratic relations $s_i^2 = 1$ for all $i \in I = \{0, 1, \ldots, n-1\}$, the commutation relations $s_i s_j = s_j s_i$ provided $|i-j| > 1$, and the braid relations $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i > 0$ and $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.

It is often convenient to write down an element of a Coxeter group as a sequence of indices of s_i in the product representation of the element. For example, the element $w = s_2 s_1 s_2 s_1 s_0 s_1 s_0 s_1$ is represented by the word $\mathbf{w} = 2120101$. A word of shortest length is referred to as a *reduced word*. The set of all reduced words of the element w is denoted by $R(w)$.

Example 1. The set of reduced words for $w = s_2 s_1 s_2 s_1 s_0 s_1 s_0 s_1$ is given by

$$R(w) = \{210210, 212010, 121010, 120101, 102101\}.$$

We say that a reduced word $a_1 a_2 \ldots a_\ell$ is *unimodal* if there exists an index v, such that

$$a_1 > a_2 > \cdots > a_v < a_{v+1} < \cdots < a_\ell.$$

Consider a reduced word $\mathbf{a} = a_1 a_2 \ldots a_L$ of a Coxeter group element w. A *unimodal factorization* of $\mathbf{a}$ is a factorization $\mathbf{A} = (a_1 \ldots a_{\ell_1})(a_{\ell_1+1} \ldots a_{\ell_2}) \cdots (a_{\ell_r+1} \ldots a_L)$ such that each factor $(a_{\ell_i+1} \ldots a_{\ell_{i+1}})$ is unimodal. Factors can be empty.

We denote the infinite set of unimodal factorizations of a reduced word $\mathbf{a}$ by $U(\mathbf{a})$. Given a factorization $\mathbf{A} \in U(\mathbf{a})$, define the *weight* of a factorization $\mathrm{wt}(\mathbf{A})$ to be the vector consisting of the number of elements in each factor. Denote by $\mathrm{nz}(\mathbf{A})$ the number of non-empty factors of $\mathbf{A}$.

Example 2. For the factorization $\mathbf{A} = (2102)()(10) \in U(212010)$, we have $\mathrm{wt}(\mathbf{A}) = (4,0,2)$ and $\mathrm{nz}(\mathbf{A}) = 2$.

Following [1, 6, 11], the *type C Stanley symmetric function* associated to $w \in W_C$ is defined as

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{a} \in R(w)} \sum_{\mathbf{A} \in U(\mathbf{a})} 2^{\mathrm{nz}(\mathbf{A})} \mathbf{x}^{\mathrm{wt}(\mathbf{A})}. \tag{2.1}$$

Here $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ and $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} x_3^{v_3} \cdots$. It is not obvious from the definition why the above functions are symmetric. We refer reader to [2], where this fact follows easily from an alternative definition.

2.2 Type A crystal of words

Crystal bases play an important role in many areas of mathematics. For example, they make it possible to analyze representation theoretic questions using combinatorial tools. Here we only review the crystal of words in type A_n and refer the reader for more background on crystals to [3].

Consider the set of words $\mathcal{B}_n^h$ of length h in the alphabet $\{1, 2, \ldots, n+1\}$. We impose a crystal structure on $\mathcal{B}_n^h$ by defining lowering operators f_i and raising operators e_i for $1 \leq i \leq n$ and a weight function. The weight of $\mathbf{b} \in \mathcal{B}_n^h$ is the tuple $\mathrm{wt}(\mathbf{b}) = (a_1, \ldots, a_{n+1})$, where a_i is the number of letters i in $\mathbf{b}$. The crystal operators f_i and e_i only depend on the letters i and $i+1$ in $\mathbf{b}$. Consider the subword $\mathbf{b}^{\{i,i+1\}}$ of $\mathbf{b}$ consisting only of the letters i and $i+1$. Successively bracket any adjacent pair $i+1$ i and remove it from the word. The resulting word will be of the form $i^a(i+1)^b$ with $a, b \geq 0$. Then f_i changes this subword within $\mathbf{b}$ to $i^{a-1}(i+1)^{b+1}$ if $a > 0$ leaving all other letters unchanged and otherwise annihilates $\mathbf{b}$. The operator e_i changes this subword within $\mathbf{b}$ to $i^{a+1}(i+1)^{b-1}$ if $b > 0$ leaving all other letters unchanged and otherwise annihilates $\mathbf{b}$.

We call an element $\mathbf{b} \in \mathcal{B}_n^h$ *highest weight* if $e_i(\mathbf{b}) = \mathbf{0}$ for all $1 \leq i \leq n$ (meaning that all e_i annihilate $\mathbf{b}$).

Theorem 3. *[8] A word $\mathbf{b} = b_1 \ldots b_h \in \mathcal{B}_n^h$ is highest weight if and only if it is a Yamanouchi word. That is, for any index k with $1 \leq k \leq h$ the weight of a subword $b_k b_{k+1} \ldots b_h$ is a partition.*

Example 4. The word 8 57 44 2346 5 4 333 2222 11111 is highest weight.

Two crystals $\mathcal{B}$ and C are said to be *isomorphic* if there exists a bijective map $\Phi \colon \mathcal{B} \to C$ that preserves the weight function and commutes with the crystal operators e_i and f_i.

Theorem 5. *[8] Each connected component of $\mathcal{B}_n^h$ has a unique highest weight element. Furthermore, if $\mathbf{b}, \mathbf{c} \in \mathcal{B}_n^h$ are highest weight elements such that $\mathrm{wt}(\mathbf{b}) = \mathrm{wt}(\mathbf{c})$, then the connected components generated by $\mathbf{b}$ and $\mathbf{c}$ are isomorphic.*

We denote a connected component with a highest weight element of highest weight λ by $\mathcal{B}_\lambda$. The *character* of the crystal $\mathcal{B}$ is defined to be a polynomial in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_{n+1})$

$$\chi_{\mathcal{B}}(\mathbf{x}) = \sum_{\mathbf{b} \in \mathcal{B}} \mathbf{x}^{\mathrm{wt}(\mathbf{b})}.$$

Theorem 6. *The character of $\mathcal{B}_\lambda$ is equal to the Schur polynomial $s_\lambda(\mathbf{x})$ (or Schur function in the limit $n \to \infty$).*

3 Crystal isomorphism

3.1 Kraśkiewicz insertion

The *Edelman-Greene insertion* [5] is a variant of the RSK insertion from the set of reduced words for an element in the symmetric group S_n to a pair of certain tableaux. Here we describe the special case of inserting k into an increasing word $\mathbf{w} = w_1 \ldots w_\ell$, denoted by $\mathbf{w} \leftsquigarrow k$:

1. If $w_\ell < k$, then $\mathbf{w} \leftsquigarrow k = \mathbf{w}'$, where $\mathbf{w}' = w_1 w_2 \dots w_\ell\, k$.
2. If $w_\ell \geq k$, find the greatest element $w_s \leq k$.
   (a) If $w_s < k$, then $\mathbf{w} \leftsquigarrow k = w_{s+1} \leftsquigarrow \mathbf{w}'$, where $\mathbf{w}' = w_1 \dots w_s\, k\, w_{s+2} \dots w_\ell$ and $w_{s+1} \leftsquigarrow \mathbf{w}'$ is a word $\mathbf{w}'$ together with a "bumped" letter w_{s+1}.
   (b) If $w_s = k$ and $w_{s+1} \neq k+1$ (or w_{s+1} does not exist), return ERROR. Otherwise, if $w_s = k$ and $w_{s+1} = k+1$, then $\mathbf{w} \leftsquigarrow k = k+1 \leftsquigarrow \mathbf{w}$.

Note that the ERROR is returned when the concatenation of $\mathbf{w}$ and k is not actually a reduced word.

Next we consider a reduced unimodal word $\mathbf{a} = a_1 a_2 \dots a_\ell$ with $a_1 > a_2 > \cdots > a_v < a_{v+1} < \cdots < a_\ell$. The *Kraśkiewicz row insertion* [9, 10] of k into $\mathbf{a}$ (denoted similarly as $\mathbf{a} \leftsquigarrow k$), is performed as follows:

1. If $k = 0$ and there is a subword 101 in $\mathbf{a}$, then $\mathbf{a} \leftsquigarrow 0 = 0 \leftsquigarrow \mathbf{a}$.
2. If $k \neq 0$ or there is no subword 101 in $\mathbf{a}$, denote the decreasing part $a_1 \dots a_v$ as $\mathbf{d}$ and the increasing part $a_{v+1} \dots a_\ell$ as $\mathbf{g}$. Perform the Edelman-Greene insertion of k into $\mathbf{g}$.
   (a) If there is no bumped letter (that is, if $a_\ell < k$) and $\mathbf{g} \leftsquigarrow k = \mathbf{g}'$, then $\mathbf{a} \leftsquigarrow k = \mathbf{dg} \leftsquigarrow k = \mathbf{d}\,\mathbf{g}' =: \mathbf{a}'$.
   (b) If there is a bumped letter and $\mathbf{g} \leftsquigarrow k = k' \leftsquigarrow \mathbf{g}'$, negate all the letters in $\mathbf{d}$ (call the resulting word $-\mathbf{d}$) and perform the Edelman-Greene insertion $-\mathbf{d} \leftsquigarrow -k'$. Note that there will always be a bumped letter, and so $-\mathbf{d} \leftsquigarrow -k' = -k'' \leftsquigarrow -\mathbf{d}'$ for some decreasing word $\mathbf{d}'$. The result of Kraśkiewicz insertion is: $\mathbf{a} \leftsquigarrow k = \mathbf{d}[\mathbf{g} \leftsquigarrow k] = \mathbf{d}[k' \leftsquigarrow \mathbf{g}'] = -[-\mathbf{d} \leftsquigarrow -k']\,\mathbf{g}' = [k'' \leftsquigarrow \mathbf{d}']\mathbf{g}' = k'' \leftsquigarrow \mathbf{a}'$, where $\mathbf{a}' = \mathbf{d}'\mathbf{g}'$.
3. If one of the Edelman-Greene insertions above returns ERROR, the Kraśkiewicz insertion also returns ERROR.

Example 7. $31012 \leftsquigarrow 0 = 0 \leftsquigarrow 31012, \quad 31012 \leftsquigarrow 1 = 3 \leftsquigarrow 21012, \quad 31012 \leftsquigarrow 3 = 310123.$

Note that the Kraśkiewicz insertion returns an ERROR if and only if $a_1 a_2 \dots a_\ell\, k$ is not a reduced word. The insertion is constructed to "commute" a unimodal word with a letter: If $\mathbf{a} \leftsquigarrow k = k' \leftsquigarrow \mathbf{a}'$, the two elements of the B type Coxeter group corresponding to words $\mathbf{a}\,k$ and $k'\mathbf{a}'$ are the same.

The type C Stanley symmetric functions (**??**) are defined in terms of unimodal factorizations. To put the formula on a completely combinatorial footing, we need to treat the powers of 2 by introducing signed unimodal factorizations. A *signed unimodal factorization* of a reduced word $\mathbf{a}$ is a unimodal factorization $\mathbf{A}$ of $\mathbf{a}$, in which every non-empty factor is assigned either a $+$ or $-$ sign. For a fixed Coxeter group element w, consider all reduced words $R(w)$, and denote the set of all signed unimodal factorizations for reduced words in $R(w)$ as $U^{\pm}(w)$.

For a signed unimodal factorization $\mathbf{A} \in U^{\pm}(w)$, define $\mathrm{wt}(\mathbf{A})$ to be the vector with i-th coordinate equal to the number of letters in the i-th factor of $\mathbf{A}$. Notice from (**??**) that

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U^{\pm}(w)} \mathbf{x}^{\mathrm{wt}(\mathbf{A})}. \tag{3.1}$$

We will use the Kraśkiewicz insertion to construct a map between signed unimodal factorizations of a Coxeter group element w and pairs of certain types of tableaux $(\mathbf{P}, \mathbf{T})$. We define these types of tableaux next.

A *shifted diagram* $\mathcal{S}(\lambda)$ associated to a strictly decreasing partition λ is the set of boxes in positions $\{(i, j) \mid 1 \le i \le \ell(\lambda),\ i \le j \le \lambda_i + i - 1\}$. Here, we use English notation, where the box $(1, 1)$ is always top-left.

A *unimodal tableau* $\mathbf{P}$ of shape λ on n letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet $\{1 < 2 < \cdots < n\}$ such that the word P_i obtained by reading the ith row from the top of $\mathbf{P}$ from left to right, is a unimodal word, and P_i is the longest unimodal subword in the concatenated word $(P_{i+1}P_i)$ [2] (cf. also with decomposition tableaux [13, 4]). The *reading word* of a unimodal tableau $\mathbf{P}$ is given by $\pi_{\mathbf{P}} = (P_\ell P_{\ell-1} \dots P_1)$. A unimodal tableau is called *reduced* if $\pi_{\mathbf{P}}$ is a type C reduced word corresponding to the Coxeter group element $w_{\mathbf{P}}$. Given a fixed Coxeter group element w, denote the set of reduced unimodal tableaux $\mathbf{P}$ of shape λ with $w_{\mathbf{P}} = w$ as $\mathcal{UT}_w(\lambda)$.

A *signed primed tableau* $\mathbf{T}$ of shape λ on n letters (cf. semistandard Q-tableau [11]) is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet $\{1' < 1 < 2' < 2 < \cdots < n' < n\}$ such that:

1. The entries are weakly increasing along each column and each row of $\mathbf{T}$.
2. Each row contains at most one i' for every $i = 1, \dots, n$.
3. Each column contains at most one i for every $i = 1, \dots, n$.

Denote the set $\mathcal{PT}^{\pm}(\lambda)$. Given an element $\mathbf{T} \in \mathcal{PT}^{\pm}(\lambda)$, define the weight of the tableau $wt(\mathbf{T})$ as a vector with i-th coordinate equal to the number of letters i' and i in $\mathbf{T}$.

Example 8. $\left(\begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 2 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 1 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array}\ ,\ \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2' & 2 & 3' & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-3} \end{array} \right)$ is a pair consisting of unimodal and signed primed tableau of shifted shape $\mathcal{S}(5, 3, 1)$.

For a reduced unimodal tableau $\mathbf{P}$ with rows $P_\ell, P_{\ell-1}, \dots, P_1$, the Kraśkiewicz insertion of a letter k into tableau $\mathbf{P}$ (denoted again by $\mathbf{P} \leftsquigarrow k$) is performed as follows:

1. Perform Kraśkiewicz insertion of a letter k into the unimodal word P_1. If there is no bumped letter and $P_1 \leftsquigarrow k = P_1'$, the algorithm terminates and the new tableau $\mathbf{P}'$ consists of rows $P_\ell, P_{\ell-1}, \dots, P_2, P_1'$. If there is a bumped letter and $P_1 \leftsquigarrow k = k' \leftsquigarrow P_1'$, continue the algorithm by inserting k' into the unimodal word P_2.

2. Repeat the previous step for the rows of $\mathbf{P}$ until either the algorithm terminates, and then the new tableau $\mathbf{P}'$ consists of rows $P_\ell, \ldots, P_{s+1}, P'_s, \ldots, P'_1$, or we manage to get to row P_ℓ. In this case the insertion bumps a letter k_e from P'_ℓ and we put (k_e) on a new row of the shifted shape of $\mathbf{P}'$, so that the resulting tableau $\mathbf{P}'$ consists of rows $(k_e), P'_\ell, \ldots, P'_1$.

Example 9. $\begin{array}{ccccc} 4 & 3 & 2 & 0 & 1 \\ & 2 & 1 & 2 & \\ & & 0 & & \end{array} \leftsquigarrow 0 = \begin{array}{ccccc} 4 & 3 & 2 & 1 & 0 \\ & 2 & 1 & 0 & \\ & & 0 & 1 & \end{array}$.

Lemma 1. *[9] Let $\mathbf{P}$ be a reduced unimodal tableau with a reading word $\mathbf{w}$ and let k be a letter such that $\mathbf{w}k$ is a reduced word. Then the tableau $\mathbf{P}' = \mathbf{P} \leftsquigarrow k$ is a reduced unimodal tableau with reading word $\mathbf{w}k$.*

Let $\mathbf{A} \in U^{\pm}(w)$ be a signed unimodal factorization with unimodal factors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$. We recursively construct a sequence $(\emptyset, \emptyset) = (\mathbf{P}_0, \mathbf{T}_0)$, $(\mathbf{P}_1, \mathbf{T}_1), \ldots, (\mathbf{P}_n, \mathbf{T}_n) = (\mathbf{P}, \mathbf{T})$ of tableaux, where $\mathbf{P}_s \in \mathcal{UT}_{(\mathbf{a}_1\mathbf{a}_2\ldots\mathbf{a}_s)}(\lambda_s)$ and $\mathbf{T}_s \in \mathcal{PT}^{\pm}(\lambda_s)$ are tableaux of the same shape λ_s.

To obtain *insertion tableau* $\mathbf{P}_s$, insert the letters of $\mathbf{a}_s$ one by one from left to right, into $\mathbf{P}_{s-1}$. Denote the shifted shape of $\mathbf{P}_s$ by λ_s. Now, enumerate the boxes in the skew shape λ_s/λ_{s-1} in the order they appear in $\mathbf{P}_s$. Let these boxes be $(x_1, y_1), \ldots, (x_{\ell(s)}, y_{\ell(s)})$.

Lemma 2. *[11] There exists an index v, such that $y_1 < \cdots < y_v \geq \cdots \geq y_{\ell(s)}$.*

The *recording tableau* $\mathbf{T}_s$ is a primed tableau obtained by adding boxes $(x_1, y_1), \ldots, (x_{v-1}, y_{v-1})$ with letters s', and boxes $(x_{v+1}, y_{v+1}), \ldots, (x_{\ell(s)}, y_{\ell(s)})$ with letters s, to a signed primed tableau $\mathbf{T}_{s-1}$. The special case is a box (x_v, y_v), which could contain either s' or s. The letter is determined by the sign of the factor $\mathbf{a}_s$: If the sign is $-$, fill the box with letter s', and if the sign is $+$, fill the box with a letter s.

The above algorithm gives the *Kraśkiewicz map*

$$KR\colon U^{\pm}(w) \to \bigcup_{\lambda} \left[\mathcal{UT}_w(\lambda) \times \mathcal{PT}^{\pm}(\lambda)\right]$$

from the signed unimodal factorizations of w to pairs of reduced unimodal and signed primed tableaux of the same shape.

Example 10. Given a signed unimodal factorization $\mathbf{A} = (-0)(+212)(-43201)$, the sequence of tableaux is

$$(\emptyset, \emptyset), \quad \left(\,\boxed{0}\,, \boxed{1'}\,\right), \quad \left(\begin{array}{ccc} 2 & 1 & 2 \\ & 0 & \end{array}, \begin{array}{ccc} 1' & 2' & 2 \\ & 2 & \end{array}\right), \quad \left(\begin{array}{ccccc} 4 & 3 & 2 & 0 & 1 \\ & 2 & 1 & 2 & \\ & & 0 & & \end{array}, \begin{array}{ccccc} 1' & 2' & 2 & 3' & 3 \\ & 2 & 3' & 3 & \\ & & 3' & & \end{array}\right).$$

If $KR(\mathbf{A}) = (\mathbf{P}, \mathbf{T})$, denote $P_{KR}(\mathbf{A}) = \mathbf{P}$ and $Q_{KR}(\mathbf{A}) = \mathbf{T}$. Slightly modifying a theorem of Kraśkiewicz [9, 10] to the semistandard case yields the following result.

Theorem 11. *[9, 10] The map KR is a bijection.*

The consequence of Theorem 11 and Equation (3.1) is the following relation:

$$F_w^C(\mathbf{x}) = \sum_{\lambda} \left|\mathcal{UT}_w(\lambda)\right| \sum_{\mathbf{T}\in\mathcal{PT}^{\pm}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}. \tag{3.2}$$

Remark 12. The sum $\sum_{\mathbf{T}\in\mathcal{PT}^{\pm}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}$ is also known as the Q-Schur function. The expansion (3.2), with a slightly different interpretation of Q-Schur function, was shown in [1].

At this point, we are halfway there to expand $F_w^C(\mathbf{x})$ in terms of Schur functions. In the next section we introduce a crystal structure on the set $\mathcal{PT}(\lambda)$ of unsigned primed tableaux.

3.2 Mixed insertion

Set $\mathcal{B}^h = \mathcal{B}^h_\infty$. Similar to the well-known RSK-algorithm, mixed insertion [7] gives a bijection between $\mathcal{B}^h$ and the set of pairs of tableaux $(\mathbf{T}, \mathbf{Q})$, but in this case $\mathbf{T}$ is an (unsigned) primed tableau of shifted shape λ and $\mathbf{Q}$ is a standard shifted tableau of the same shape.

An *(unsigned) primed tableau* of shape λ (cf. semistandard P-tableau [11] or semistandard marked shifted tableau [4]) is a signed primed tableau $\mathbf{T}$ of shape λ with only unprimed elements on the main diagonal. Denote the set as $\mathcal{PT}(\lambda)$. Given $\mathbf{T} \in \mathcal{PT}(\lambda)$, the weight $\mathrm{wt}(\mathbf{T})$ is defined to be the vector with i-th coordinate equal to the number of letters i' and i in $\mathbf{T}$. We can simplify (3.2) as

$$F_w^C(\mathbf{x}) = \sum_{\lambda} 2^{\ell(\lambda)} \left|\mathcal{UT}_w(\lambda)\right| \sum_{\mathbf{T}\in\mathcal{PT}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}. \tag{3.3}$$

Remark 13. The sum $\sum_{\mathbf{T}\in\mathcal{PT}(\lambda)} \mathbf{x}^{\mathrm{wt}(\mathbf{T})}$ is also known as a P-Schur function.

A *standard shifted tableau* of shape λ is a filling of $\mathcal{S}(\lambda)$ with letters $1, 2, \ldots, |\lambda|$ such that each letter appears exactly once, each row filling is increasing and each column filling is increasing. Denote the set as $\mathcal{ST}(\lambda)$.

Given a word $b_1 b_2 \ldots b_h$, we recursively construct a sequence of tableaux $(\emptyset, \emptyset) = (\mathbf{T}_0, \mathbf{Q}_0)$, $(\mathbf{T}_1, \mathbf{Q}_1), \ldots, (\mathbf{T}_h, \mathbf{Q}_h) = (\mathbf{T}, \mathbf{Q})$. To get the tableau $\mathbf{T}_s$, insert the letter b_s into $\mathbf{T}_{s-1}$ as follows. First, insert b_s into the first row of $\mathbf{T}_{s-1}$, bumping out the smallest element y that is strictly greater than b_i in the alphabet $\{1' < 1 < 2' < 2 < \cdots\}$.

1. If y is not on the main diagonal and y is not primed, then insert it in the next row, bumping out the smallest element that is strictly greater than y from that row.

2. If y is not on the main diagonal and y is primed, then insert it to the next column to the right, bumping out the smallest element that is strictly greater than y from that column.

3. If y is on the main diagonal, then it must be unprimed. Prime it and insert it to the column on the right.

The insertion process terminates either by placing a letter at the end of a row, bumping no new element, or forming a new row with the last bumped element. The shapes of $\mathbf{T}_{s-1}$ and $\mathbf{T}_s$ differ by one box. Add that box to $\mathbf{Q}_{s-1}$ with a letter s in it, to obtain the standard tableau $\mathbf{Q}_s$.

Example 14. For a word 332332123, some of the tableaux in the sequence $(\mathbf{T}_i, \mathbf{Q}_i)$ are

$$\left(\begin{ytableau} 2 & 3' \\ \none & 3 \end{ytableau}, \begin{ytableau} 1 & 2 \\ \none & 3 \end{ytableau}\right), \quad \left(\begin{ytableau} 2 & 2 & 3' & 3 \\ \none & 3 & 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 4 & 5 \\ \none & 3 & 6 \end{ytableau}\right), \quad \left(\begin{ytableau} 1 & 2' & 2 & 3' & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}, \begin{ytableau} 1 & 2 & 4 & 5 & 9 \\ \none & 3 & 6 & 8 \\ \none & \none & 7 \end{ytableau}\right).$$

Theorem 15. *[7] The construction above gives a bijection*

$$HM \colon \mathcal{B}^h \to \bigcup_{\lambda} \left[\mathcal{PT}(\lambda) \times \mathcal{ST}(\lambda)\right].$$

That bijection is called a *mixed insertion*. If $HM(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$, denote $P_{HM}(\mathbf{b}) = \mathbf{T}$ and $Q_{HM}(\mathbf{b}) = \mathbf{Q}$. In the next section, we will provide the following result:

Theorem 16. *There exist operators f_i, e_i on the set $\mathcal{PT}(\lambda)$, such that for any $\mathbf{b} \in \mathcal{B}^h$ with $HM(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$,*

$$HM(e_i(\mathbf{b})) = (e_i(\mathbf{T}), \mathbf{Q}) \quad \textit{and} \quad HM(f_i(\mathbf{b})) = (f_i(\mathbf{T}), \mathbf{Q}),$$

given the left-hand side is well-defined.

Before we describe the crystal operators e_i and f_i explicitly on $\mathcal{PT}(\lambda)$, let us look at the consequences of Theorem 16.

Corollary 17. *The recording tableau $Q_{HM}(\mathbf{b})$ is constant on each connected component of the crystal $\mathcal{B}^h$.*

Let us fix a recording tableau $\mathbf{Q}_\lambda \in \mathcal{ST}(\lambda)$. Define a map $\Psi_\lambda \colon \mathcal{PT}(\lambda) \to \mathcal{B}^h$ as $\Psi_\lambda(\mathbf{T}) = HM^{-1}(\mathbf{T}, \mathbf{Q}_\lambda)$. By Corollary 17, the set $\mathrm{Im}(\Psi_\lambda)$ consists of several connected components of $\mathcal{B}^h$. The map Ψ_λ is a crystal isomorphism, since by Theorem 16

$$e_i(\mathbf{T}) = (\Psi_\lambda^{-1} \circ e_i \circ \Psi_\lambda)(\mathbf{T}), \quad f_i(\mathbf{T}) = (\Psi_\lambda^{-1} \circ f_i \circ \Psi_\lambda)(\mathbf{T}), \quad \mathrm{wt}(\mathbf{T}) = (\mathrm{wt} \circ \Psi_\lambda)(\mathbf{T}).$$

Notice that, given Corollary 17, the above relations could serve as a definition of crystal maps on $\mathcal{PT}(\lambda)$.

Example 18.

$$e_1\left(\begin{ytableau} 1 & 2' & 2 & 3' & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}\right) = \begin{ytableau} 1 & 1 & 2 & 3' & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau} \quad \text{and} \quad e_2\left(\begin{ytableau} 1 & 2' & 2 & 3' & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}\right) = \begin{ytableau} 1 & 2' & 2 & 2 & 3 \\ \none & 2 & 3' & 3 \\ \none & \none & 3 \end{ytableau}.$$

To summarize, we obtain a crystal isomorphism between the crystal $(\mathcal{PT}(\lambda), e_i, f_i, \mathrm{wt})$, denoted again as $\mathcal{PT}(\lambda)$, and a direct sum $\bigoplus_\mu \mathcal{B}_\mu^{\oplus h_{\lambda\mu}}$. We will provide a combinatorial description of coefficients $h_{\lambda\mu}$ in the next section. This implies the relation on characters of the corresponding crystals $\chi_{\mathcal{PT}(\lambda)} = \sum_\mu h_{\lambda\mu} s_\mu$. Thus we can rewrite (3.3) one last time

$$F_w^C(\mathbf{x}) = \sum_\lambda 2^{\ell(\lambda)} \left|\mathcal{UT}_w(\lambda)\right| \sum_\mu h_{\lambda\mu} s_\mu = \sum_\mu \Big(\sum_\lambda 2^{\ell(\lambda)} \left|\mathcal{UT}_w(\lambda)\right| h_{\lambda\mu} \Big) s_\mu.$$

3.3 Explicit crystal operators on shifted primed tableaux

We consider the alphabet $\{1 < 2' < 2 < 3' < \cdots\}$ of primed and unprimed letters. It is useful to think about the letter $(i+1)'$ as a number $i+0.5$. Thus, we say that letters i and $(i+1)'$ differ by half a unit and letters i and $(i+1)$ differ by a whole unit.

How come $1'$ is not in the list here? — Anne

Given an (unsigned) primed tableau $\mathbf{T}$, we construct the *reading word* $\mathrm{rw}(\mathbf{T})$ as follows:

1. List all primed letters in the tableau, column by column, in decreasing order within each column, moving from the rightmost column to the left, and with all the primes removed (i.e. all letters are increased by half a unit). (Call that part of the word a *prime reading word*.)

2. Then list all unprimed elements, row by row, in increasing order within each row, moving from the bottommost row to the top. (Call that part of the word an *unprimed reading word*.)

To find the letter on which the crystal operator f_i acts, apply the bracketing rule for letters i and $i+1$ within the reading word $\mathrm{rw}(\mathbf{T})$. If all letters i are bracketed in $\mathrm{rw}(\mathbf{T})$, then $f_i(\mathbf{T}) = \mathbf{0}$. Otherwise, let the rightmost unbracketed letter i correspond to the box x in $\mathbf{T}$. Denote the content (or entry) of a box x as $c(x)$ and the box to the right of x as x_E.

Crystal operators on primed tableau:
We have two cases depending on the content of x, namely either $c(x) = i$ or i'. First assume $c(x) = i$. Consider the content $c(x_E)$ of the box to its right and notice that $c(x_E) \geq (i+1)'$.

1. If $c(x_E) = (i+1)'$, the box x must lie outside of the main diagonal and the box right below x_E cannot have content equal to $(i+1)'$. Then the operator f_i acts on $\mathbf{T}$ by changing $c(x)$ to $(i+1)'$ and changing $c(x_E)$ to $(i+1)$ (i.e. increase the content of x and x_E by half a unit).

2. If $c(x_E) \neq (i+1)'$ or there are no boxes to the right of x (i.e. x_E does not exist), then there is a maximal connected ribbon (expanding in South and West directions) with the following properties:

   (a) The North-Eastern most box of the ribbon (the tail of the ribbon) is x.

   (b) The contents of all boxes within a ribbon besides the tail are either $(i+1)'$ or $(i+1)$.

   Denote the South-Western most box of the ribbon (the head) as x_H.

   (a) If $x_H = x$, the operator f_i acts on $\mathbf{T}$ by changing $c(x)$ to $(i+1)$ (i.e. increase the content of x by a whole unit).

   (b) If $x_H \neq x$ and x_H is on the main diagonal, then x cannot be on the main diagonal and the operator f_i acts on $\mathbf{T}$ by changing $c(x)$ to $(i+1)'$ (i.e. increase the content of x by half a unit).

   (c) Otherwise, the content $c(x_H)$ must be $(i+1)'$ due to the bracketing rule and the operator f_i acts on $\mathbf{T}$ by changing $c(x)$ to $(i+1)'$ and changing $c(x_H)$ to $(i+1)$ (i.e. increase the content of x and x_H by half a unit).

Example 19.

$$f_2\left(\begin{ytableau} 1 & 2' & 2 & 3' \\ \none & 2 & 3' & 3 \end{ytableau}\right) = \mathbf{0}, \quad f_2\left(\begin{ytableau} 1 & 2' & 2 & 3' \\ \none & 2 & 3' & 4 \end{ytableau}\right) = \begin{ytableau} 1 & 2' & 3' & 3 \\ \none & 2 & 3' & 4 \end{ytableau}, \quad f_2\left(\begin{ytableau} 1 & 1 & 2 & 2 \\ \none & 3 & 4' & 4 \end{ytableau}\right) = \begin{ytableau} 1 & 1 & 2 & 3 \\ \none & 3 & 4' & 4 \end{ytableau},$$

$$f_2\left(\begin{ytableau} 1 & 1 & 2' & 2 & 3 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 & 3 \end{ytableau}\right) = \begin{ytableau} 1 & 1 & 2' & 3' & 3 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 & 3 \end{ytableau}, \quad f_2\left(\begin{ytableau} 1 & 1 & 1 & 2 & 3 \\ \none & 2 & 2 & 3' \\ \none & \none & 3 & 4' \end{ytableau}\right) = \begin{ytableau} 1 & 1 & 1 & 3' & 3 \\ \none & 2 & 2 & 3 \\ \none & \none & 3 & 4' \end{ytableau}.$$

Now, if $c(x) = i'$, we can apply the rules above for a conjugate of $\mathbf{T}$ instead.

The *conjugate* of a primed tableau $\mathbf{T}$ is obtained by reflecting the tableau over the main diagonal, changing all primed entries i' to i and changing all unprimed elements i to $(i+1)'$ (i.e. increase the content of all boxes by half a unit). The main diagonal is now the North-East boundary of the tableau. Denote the resulting tableau as $\mathbf{T}^*$.

To apply f_i on $\mathbf{T}$ with $c(x) = i'$, conjugate $\mathbf{T}$, apply the operator rules on the primed tableau $\mathbf{T}^*$ with a cell x^* of content i, and then reverse the conjugation. Note that for the operator rules on the conjugate tableau $\mathbf{T}^*$, in case 1 (i.e. when $c(x_E^*) = (i+1)'$) the box x_E^* cannot be on the main diagonal, and in case 2 the ribbon's head x_H^* is not on the main diagonal, so case 2b can never happen.

Example 20.

$$\text{Let } \mathbf{T} = \begin{ytableau} 1 & 2' & 2 & 3 \\ \none & 3 & 4' \\ \none & \none & 4 \end{ytableau}, \quad \text{then } \mathbf{T}^* = \begin{ytableau} 2' \\ 2 & 4' \\ 3' & 4 & 5' \\ 4' \end{ytableau} \quad \text{and } f_2(\mathbf{T}) = \begin{ytableau} 1 & 2 & 3' & 3 \\ \none & 3 & 4' \\ \none & \none & 4 \end{ytableau}.$$

We define $e_i(\mathbf{T})$ similarly. Apply the bracketing rule for letters i and $(i+1)$ within the reading word of $\mathbf{T}$. If there are no unbracketed $(i+1)$ left, we set $e_i(\mathbf{T}) = \mathbf{0}$. Otherwise, let the leftmost unbracketed letter $(i+1)$ correspond to the box x in $\mathbf{T}$.

Let $c(x) = (i+1)$. We can construct the tableau $e_i(\mathbf{T})$ using the operator rules again on a negation tableau $-\mathbf{T}$.

The *negation* of a primed tableau $\mathbf{T}$ is obtained by rotating the tableau around the origin box by 180 degrees, changing all unprimed entries i to $(-i)$ and changing all primed entries $(i+1)'$ to $(-i)'$. Denote the resulting tableau as $-\mathbf{T}$.

To apply the operator e_i on the primed tableau $\mathbf{T}$ with $c(x) = (i+1)$, negate the tableau, apply the operator rules on tableau $-\mathbf{T}$ with a cell $-x$ with content $-(i+1)$, and then reverse the negation.

Similarly, to apply the operator e_i on $\mathbf{T}$ with $c(x) = (i+1)'$, first conjugate $\mathbf{T}$, then negate $\mathbf{T}^*$, apply the operator rules on tableau $-\mathbf{T}^*$ with a cell $-x^*$ with content $-(i+1)$, and reverse the negation and conjugation. Note that the case 2b is possible now.

Example 21.

Something still needs to be explained here. I assume you are saying to apply the algorithm in "Crystal operators on primed tableau". The cases are, however, not written with negative numbers, so some guess work is necessary. Cases 1 and 2 are now $c(x_E) =$

Let $\mathbf{T} = \begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 \\ \cline{2-3}\end{array}$, then $-(\mathbf{T}^*) = \begin{array}{|c|c|}\hline -2' & -2 \\ \hline -2' & -1' \\ \hline \multicolumn{1}{c|}{} & -1' \\ \cline{2-2}\end{array}$, case 2b applies and $e_1(\mathbf{T}) = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 \\ \cline{2-3}\end{array}$.

Now, we are ready to state the main theorem.

Theorem 22. *For any $\mathbf{b} \in \mathcal{B}^h$ with $HM(\mathbf{b}) = (\mathbf{T}, \mathbf{Q})$, the operators f_i and e_i defined above satisfy the following relations:*

$$HM(e_i(\mathbf{b})) = (e_i(\mathbf{T}), \mathbf{Q}) \quad \textit{and} \quad HM(f_i(\mathbf{b})) = (f_i(\mathbf{T}), \mathbf{Q}),$$

given the left-hand side is well-defined.

The consequence of Theorem 22, as discussed in the Section 3.2, is a crystal isomorphism $\Psi_\lambda \colon \mathcal{PT}(\lambda) \to \bigoplus \mathcal{B}_\mu^{\oplus h_{\lambda\mu}}$. Now, to determine the nonnegative integer coefficients $h_{\lambda\mu}$, it is enough to count the highest weight elements in $\mathcal{PT}(\lambda)$ with the given weight μ.

Proposition 23. *A primed tableau $\mathbf{T} \in \mathcal{PT}(\lambda)$ is a highest weight element if and only if its reading word $\mathrm{rw}(\mathbf{T})$ is a Yamanouchi word. That is, for any suffix of $\mathrm{rw}(\mathbf{T})$, its weight is a partition.*

Thus we define $h_{\lambda\mu}$ to be the number of primed tableaux $\mathbf{T}$ of shifted shape $\mathcal{S}(\lambda)$ and weight μ such that $\mathrm{rw}(\mathbf{T})$ is Yamanouchi.

Example 24. Let $\lambda = (5,3,2)$ and $\mu = (4,3,2,1)$. There are three primed tableaux of shifted shape $\mathcal{S}((5,3,2))$ and weight $(4,3,2,1)$ with a Yamanouchi reading word, namely

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 3' & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 4' & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array}\,, \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 3' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 4' & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array} \qquad \text{and} \qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 1 & 4' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array}\,.$$

Therefore $h_{(5,3,2)(4,3,2,1)} = 3$.

We summarize our results for the type C Stanley symmetric functions as follows.

Corollary 25. *The expansion of $F_w^C(\mathbf{x})$ in terms of Schur symmetric functions is*

$$F_w^C(\mathbf{x}) = \sum_\lambda g_\lambda s_\lambda(\mathbf{x}), \quad \textit{where} \quad g_\lambda = \sum_\mu 2^{\ell(\mu)} \left|\mathcal{UT}_w(\mu)\right| h_{\mu\lambda}\,.$$

Example 26. Consider the word $w = 0101 = 1010$. There is only one unimodal tableau corresponding to w, namely $\mathbf{P} = \begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline \multicolumn{1}{c|}{} & 0 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$, which belongs to $\mathcal{UT}_{0101}(3,1)$. Thus, $g_\lambda = 4h_{(3,1)\lambda}$. There are only three possible highest weight primed tableaux of shape $(3,1)$, namely $\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$, $\begin{array}{|c|c|c|}\hline 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$ and $\begin{array}{|c|c|c|}\hline 1 & 1 & 3' \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{1}{c}{} \\ \cline{2-2}\end{array}$, which implies that $h_{(3,1)(3,1)} = h_{(3,1)(2,2)} = h_{(3,1)(2,1,1)} = 1$ and $h_{(3,1)\lambda} = 0$ for other weights λ. The expansion of $F_{0101}^C(\mathbf{x})$ is thus

$$F_{0101}^C = 4s_{(3,1)} + 4s_{(2,2)} + 4s_{(2,1,1)}.$$

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