# Type-C Stanley Symmetric Functions and Shifted Primed Tableaux

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## Goal

It is known that Stanley symmetric functions are Schur-positive, i.e. the coefficients of the Schur expansion are non-negative. Our goal here is to introduce a crystal structure on the set of unimodal factorizations that is isomorphic to the crystal structure of type-A crystal.
In particular, we use Kraśkiewicz insertion to use the notion of Primed Tableaux and introduce crystal operators on those tableaux instead.

## Background

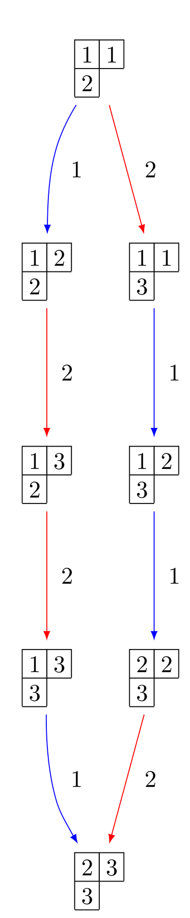
### Stanley Symmetric Functions

- Coxeter group of type $C_n$ is defined to be generated by $\{s_0, s_1, \ldots, s_{n-1}\}$ subject to relations
  - $s_i^2 = 1$ for all $i$,
  - $s_i s_j = s_j s_i$ provided $|i - j| > 1$,
  - $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i > 0$,
  - $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.
- Each Coxeter group element $w = s_{i_1} \ldots s_{i_l}$ is represented by a word $i_1 \ldots i_l$ and many other equivalent words. Among those, the words of shortest length are called *reduced words.*
- A word $i_1 \ldots i_l$ is called unimodal if there exists an index $v$ with $i_1 > \ldots > i_v < \ldots < i_l$. *Unimodal factorization* of a Coxeter group element $w$ is a factorization of its reduced word into unimodal factors. Denote the set of unimodal factorizations of $w$ as $U(w)$.
- For example, given $w = s_2 s_1 s_2 s_0 s_1 s_0$, some of the elements of $U(w)$ are (212)(0)(10), (21)()(201)(0), (1)(2101)(0), ()(12)(01)(01).
- Given unimodal factorization $\mathbf{A}$, define its *weight* $\mathrm{wt}(\mathbf{A})$ to be the vector consisting of the number of elements in each factor, and $\mathrm{nz}(\mathbf{A})$ to be the number of non-empty factors.
- *Type-C Stanley symmetric function* is defined as

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U(w)} 2^{\mathrm{nz}(\mathbf{A})} \mathbf{x}^{\mathrm{wt}(\mathbf{A})}.$$

### Type-A crystals and Schur Functions

An example of a crystal of type $A_2$ is shown to the right.
To define a crystal, one needs to define crystal operators $f_i$, which induce digraph structure, which splits the set into several components.
Each connected component is determined by the highest weight element, and the character of each connected component equal to a symmetric Schur polynomial $s_\lambda(x_1, \ldots, x_{n+1})$ with $\lambda \vdash h$.
In the limit $n \to \infty$ Schur polynomials become Schur functions $s_\lambda(\mathbf{x})$. Schur functions form an orthonormal basis to the vector space of symmetric functions with integer coefficients.



## Unimodal Tableaux and Shifted Primed Tableaux

A *shifted diagram* $\mathcal{S}(\lambda)$ associated to a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_i > \lambda_{i+1}$ is the set of boxes in positions $(i, j)$ satisfying $1 \leqslant i \leqslant \ell$ and $i \leqslant j \leqslant \lambda_i + i - 1$.

### Unimodal Tableaux

- A *unimodal tableau* $\mathbf{P}$ of shape $\lambda$ associated to a Coxeter group element $w$ of type $C_n$ is a filling of a shifted diagram $\mathcal{S}(\lambda)$ with letters from the alphabet $\{0 < 1 < 2 < \cdots < n-1\}$ such that
  - the rows of $\mathbf{P}$, denoted by $P_1, \ldots, P_\ell$, are unimodal words,
  - $P_i$ is the longest unimodal subsequence in a concatenated word $P_{i+1} P_i$,
  - the concatenated word $P_\ell P_{\ell-1} \ldots P_1$ a reduced type-C word that represents $w$.
- For example, unimodal tableau $\begin{array}{|c|c|c|c|c|}\hline 4&3&2&0&1\\\hline \multicolumn{1}{c|}{}&2&1&2\\\cline{2-4}\end{array}$ corresponds to $w = s_2 s_1 s_2 s_4 s_3 s_2 s_0 s_1$.

### Primed Tableaux

- A *primed tableau* $\mathbf{T}$ of shape $\lambda$ on $n$ letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet $\{1' < 1 < 2' < 2 < \cdots < m' < m\}$ such that:
  - The entries are weakly increasing along each column and each row of $\mathbf{T}$.
  - Each row contains at most one $i'$ for every $i = 1, \ldots, m$.
  - Each column contains at most one $i$ for every $i = 1, \ldots, m$.
- The weight of a primed tableau, denoted by $\mathrm{wt}(\mathbf{T})$, is the vector with $i$-th coordinate equal to the total number of letters in $\mathbf{T}$ that are either $i$ or $i'$.
- For example, $\begin{array}{|c|c|c|c|c|}\hline 1&1&2'&3'&3\\\hline \multicolumn{1}{c|}{}&2'&2&3'\\\cline{2-4}\end{array}$ is a primed tableau of weight $(2, 3, 3)$.

## Kraśkiewicz insertion

The Kraśkiewicz insertion gives a bijection

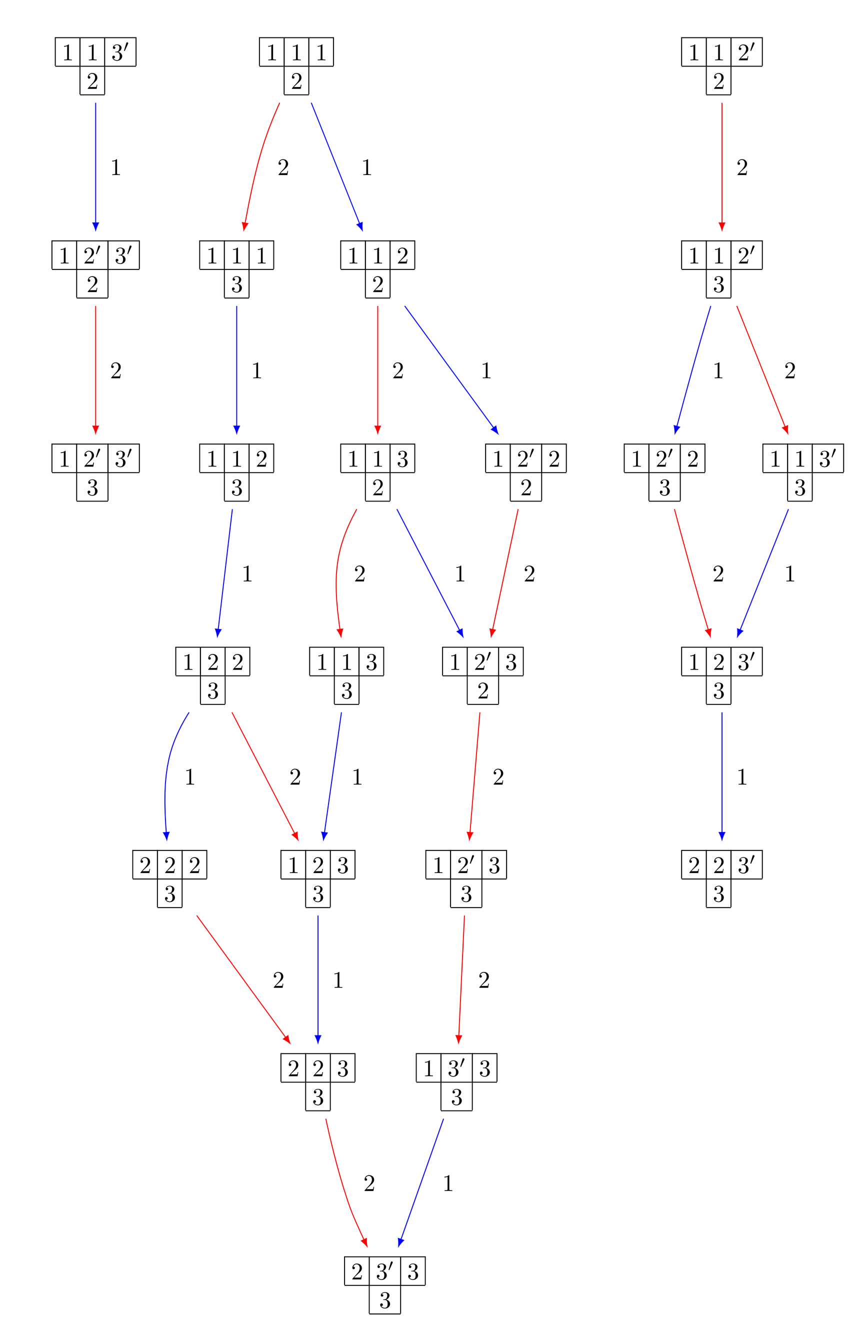
$$\mathrm{KR}\colon\ U^{\pm}(w) \to \bigcup_{\lambda} [\mathcal{UT}_w(\lambda) \times \mathcal{PT}^{\pm}(\lambda)],$$

where $U^{\pm}(w)$ is the set of all unimodal factorizations of $w$ with a sign assigned to each non-zero factor, $\mathcal{UT}_w(\lambda)$ is the set of all unimodal tableaux of shape $\lambda$ associated with $w$, and $\mathcal{PT}^{\pm}(\lambda)$ is the set of all primed tableaux of shape $\lambda$.
Moreover, the weight of a unimodal factorization is equal to the weight of a primed tableau in the image.

## Lowering operator $f_i$ on Primed Tableau

We can introduce a crystal structure on the set $\mathcal{PT}(\lambda)$ instead and induce the structure on $U^{\pm}(w)$ via the Kraśkiewicz insertion. In particular, we obtain the Schur expansion of the characteristic function of $\mathcal{PT}^{\pm}(\lambda)$, also known as $Q$-Schur function, and get the expansion of type-C Stanley symmetric function by summing over all elements of $\mathcal{UT}_w(\lambda)$.
It is convenient to define the crystal operators on the subset of $\mathcal{PT}^{\pm}(\lambda)$ with no primed elements on the diagonal (denoted by $\mathcal{PT}(\lambda)$), and extend the action on the whole set afterwards. Consider a primed tableau $\mathbf{T}$.

- Construct the *reading word* $\mathrm{rw}(\mathbf{T})$ as follows:
  1. List all primed letters in the tableau, column by column, in decreasing order within each column, moving from the rightmost column to the left, and with all the primes removed.
  2. Then list all unprimed elements, row by row, in increasing order within each row, moving from the bottommost row to the top.
- Apply a bracketing rule for letters $i$ and $i + 1$ in $\mathrm{rw}(\mathbf{T})$ to find an $i$ that operator $f_i$ would act on. If no such $i$ exist, the operator $f_i$ is not defined for $\mathbf{T}$.
- As an example, crystal structure for $\mathcal{PT}((3, 1))$ is shown below.



Schur expansion of the characteristic function of $\mathcal{PT}((3, 1))$, also known as $P$-Schur function, is

$$P_{(3,1)} = s_{(2,1,1)} + s_{(3,1)} + s_{(2,2)}.$$

Characteristic function of $\mathcal{PT}^{\pm}((3, 1))$ is

$$Q_{(3,1)} = 4 P_{(3,1)}.$$

And, finally, taking long Coxeter group element

$$w = s_0 s_1 s_0 s_1,$$

characteristic function of $U_w^{\pm}$ is

$$F_w^C = 4 s_{(2,1,1)} + 4 s_{(3,1)} + 4 s_{(2,2)},$$

since there is only one unimodal tableau corresponding to $w$, namely

$$\begin{array}{|c|c|c|}\hline 1&0&1\\\hline \multicolumn{1}{c|}{}&0\\\cline{2-2}\end{array}.$$

In general, the highest weight elements of $\mathcal{PT}(\lambda)$ are exactly the ones with Yamanouchi reading words, and Schur expansion of $F_w^C$ can be found similarly.