

Type-C Stanley Symmetric Functions and Shifted Primed Tableaux

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Goal

It is known that Stanley symmetric functions are Schur-positive, i.e. the coefficients of the Schur expansion are non-negative. Our goal here is to introduce a crystal structure on the set of unimodal factorizations that is isomorphic to the crystal structure of type-A crystal. In particular, we use Kraśkiewicz insertion to use the notion of Primed Tableaux and introduce crystal operators on those tableaux instead.

Background

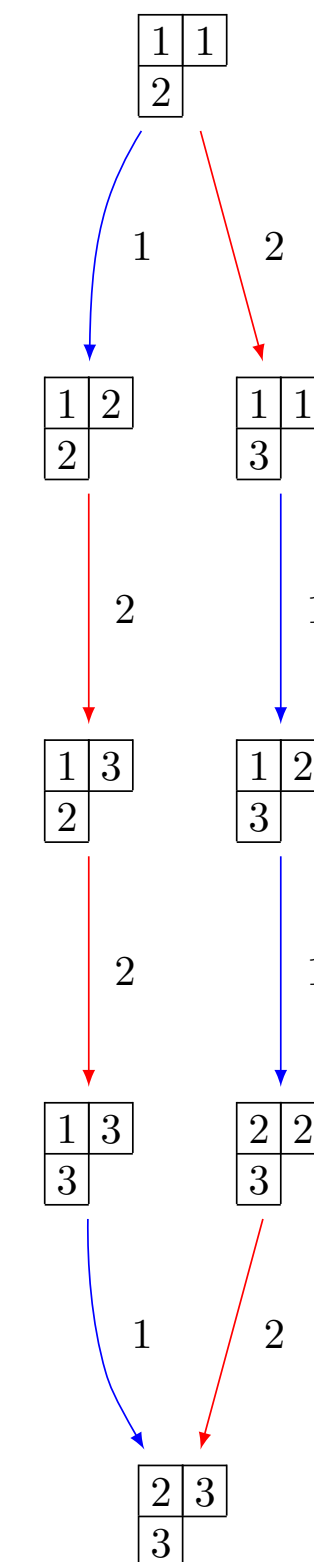
Stanley Symmetric Functions

- Coxeter group of type C_n is defined to be generated by $\{s_0, s_1, \dots, s_{n-1}\}$ subject to relations
 - $s_i^2 = 1$ for all i ,
 - $s_i s_j = s_j s_i$ provided $|i - j| > 1$,
 - $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all $i > 0$,
 - $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$.
- Each Coxeter group element $w = s_{i_1} \dots s_{i_l}$ is represented by a word $i_1 \dots i_l$ and many other equivalent words. Among those, the words of shortest length are called *reduced words*.
- A word $i_1 \dots i_l$ is called unimodal if there exists an index ν with $i_1 > \dots > i_\nu < \dots < i_l$. *Unimodal factorization* of a Coxeter group element w is a factorization of its reduced word into unimodal factors. Denote the set of unimodal factorizations of w as $U(w)$.
- For example, given $w = s_2 s_1 s_2 s_0 s_1 s_0$, some of the elements of $U(w)$ are $(212)(0)(10)$, $(21)()(201)(0)$, $(1)(2101)(0)$, $()(12)(01)(01)$.
- Given unimodal factorization \mathbf{A} , define its *weight* $\text{wt}(\mathbf{A})$ to be the vector consisting of the number of elements in each factor, and $\text{nz}(\mathbf{A})$ to be the number of non-empty factors.
- *Type-C Stanley symmetric function* is defined as

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U(w)} 2^{\text{nz}(\mathbf{A})} \mathbf{x}^{\text{wt}(\mathbf{A})}.$$

Type-A crystals and Schur Functions

An example of a crystal of type A_2 is shown to the right. To define a crystal, one needs to define crystal operators f_i , which induce digraph structure, which splits the set into several components. Each connected component is determined by the highest weight element, and the character of each connected component equal to a symmetric Schur polynomial $s_\lambda(x_1, \dots, x_{n+1})$ with $\lambda \vdash h$. In the limit $n \rightarrow \infty$ Schur polynomials become Schur functions $s_\lambda(\mathbf{x})$. Schur functions form an orthonormal basis to the vector space of symmetric functions with integer coefficients.



Unimodal Tableaux and Shifted Primed Tableaux

A *shifted diagram* $\mathcal{S}(\lambda)$ associated to a partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ with $\lambda_i > \lambda_{i+1}$ is the set of boxes in positions (i, j) satisfying $1 \leq i \leq \ell$ and $i \leq j \leq \lambda_i + i - 1$.

Unimodal Tableaux

- A *unimodal tableau* \mathbf{P} of shape λ associated to a Coxeter group element w of type C_n is a filling of a shifted diagram $\mathcal{S}(\lambda)$ with letters from the alphabet $\{0 < 1 < 2 < \dots < n - 1\}$ such that
 - the rows of \mathbf{P} , denoted by P_1, \dots, P_ℓ , are unimodal words,
 - P_i is the longest unimodal subsequence in a concatenated word $P_{i+1} P_i$,
 - the concatenated word $P_\ell P_{\ell-1} \dots P_1$ a reduced type-C word that represents w .
- For example, unimodal tableau $\begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 2 & 0 & 1 \\ \hline & 2 & 1 & 2 & \\ \hline \end{array}$ corresponds to $w = s_2 s_1 s_2 s_4 s_3 s_2 s_0 s_1$.

Primed Tableaux

- A *primed tableau* \mathbf{T} of shape λ on n letters is a filling of $\mathcal{S}(\lambda)$ with letters from the alphabet $\{1' < 1 < 2' < 2 < \dots < m' < m\}$ such that:
 - The entries are weakly increasing along each column and each row of \mathbf{T} .
 - Each row contains at most one i' for every $i = 1, \dots, m$.
 - Each column contains at most one i for every $i = 1, \dots, m$.
- The weight of a primed tableau, denoted by $\text{wt}(\mathbf{T})$, is the vector with i -th coordinate equal to the total number of letters in \mathbf{T} that are either i or i' .
- For example, $\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2' & 3' & 3 \\ \hline & 2' & 2 & 3' & \\ \hline \end{array}$ is a primed tableau of weight $(2, 3, 3)$.

Kraśkiewicz insertion

The Kraśkiewicz insertion gives a bijection

$$\text{KR}: U^\pm(w) \rightarrow \bigcup_{\lambda} [\mathcal{UT}_w(\lambda) \times \mathcal{PT}^\pm(\lambda)],$$

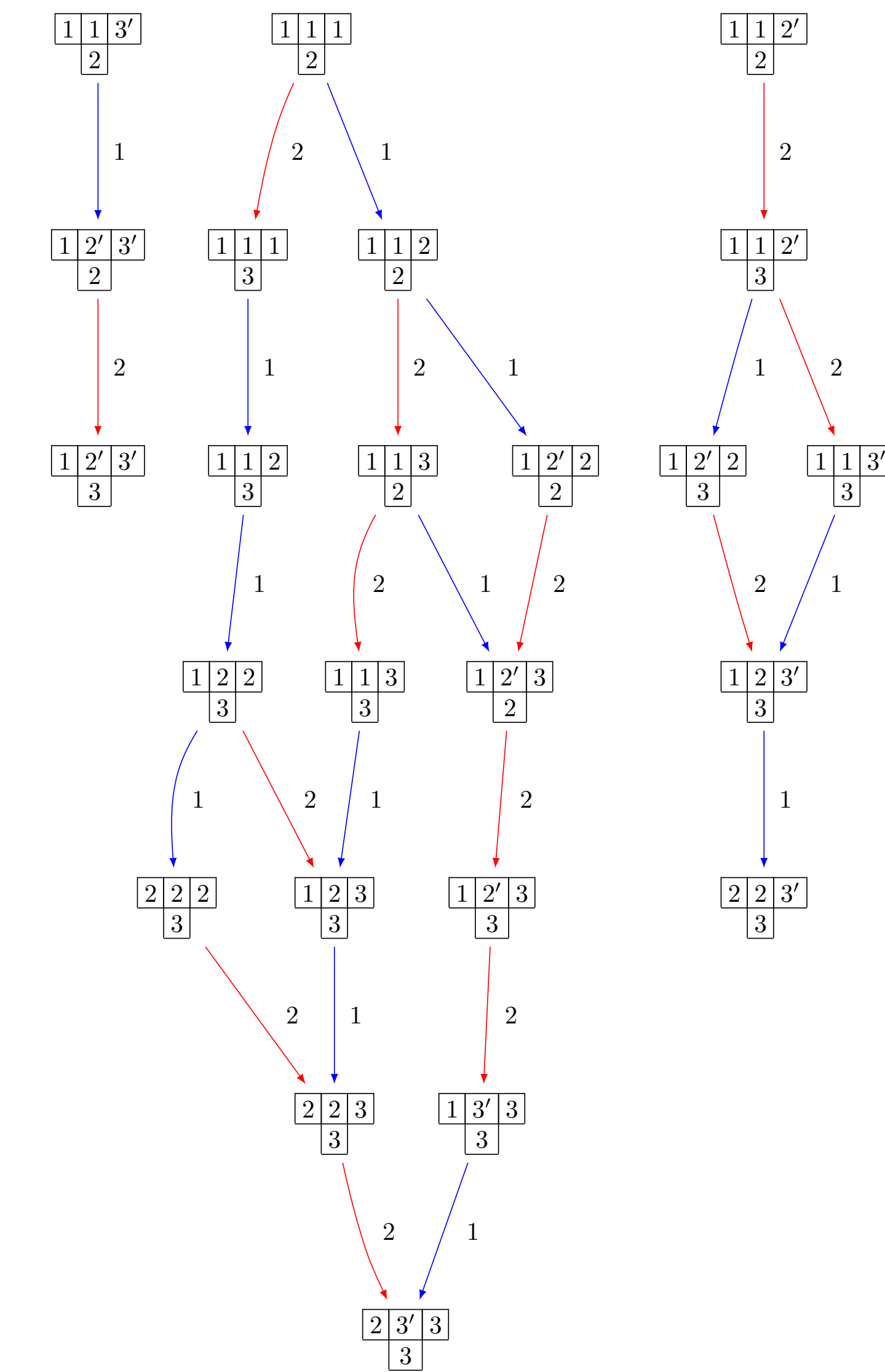
where $U^\pm(w)$ is the set of all unimodal factorizations of w with a sign assigned to each non-zero factor, $\mathcal{UT}_w(\lambda)$ is the set of all unimodal tableaux of shape λ associated with w , and $\mathcal{PT}^\pm(\lambda)$ is the set of all primed tableaux of shape λ . Moreover, the weight of a unimodal factorization is equal to the weight of a primed tableau in the image.

Lowering operator f_i on Primed Tableau

We can introduce a crystal structure on the set $\mathcal{PT}(\lambda)$ instead and induce the structure on $U^\pm(w)$ via the Kraśkiewicz insertion. In particular, we obtain the Schur expansion of the characteristic function of $\mathcal{PT}^\pm(\lambda)$, also known as Q -Schur function, and get the expansion of type-C Stanley symmetric function by summing over all elements of $\mathcal{UT}_w(\lambda)$.

It is convenient to define the crystal operators on the subset of $\mathcal{PT}^\pm(\lambda)$ with no primed elements on the diagonal (denoted by $\mathcal{PT}(\lambda)$), and extend the action on the whole set afterwards. Consider a primed tableau \mathbf{T} .

- Construct the *reading word* $\text{rw}(\mathbf{T})$ as follows:
 - ① List all primed letters in the tableau, column by column, in decreasing order within each column, moving from the rightmost column to the left, and with all the primes removed.
 - ② Then list all unprimed elements, row by row, in increasing order within each row, moving from the bottommost row to the top.
- Apply a bracketing rule for letters i and $i + 1$ in $\text{rw}(\mathbf{T})$ to find an i that operator f_i would act on. If no such i exist, the operator f_i is not defined for \mathbf{T} .
- As an example, crystal structure for $\mathcal{PT}((3, 1))$ is shown below.



Schur expansion of the characteristic function of $\mathcal{PT}((3, 1))$, also known as P -Schur function, is

$$P_{(3,1)} = s_{(2,1,1)} + s_{(3,1)} + s_{(2,2)}.$$

Characteristic function of $\mathcal{PT}^\pm((3, 1))$ is

$$Q_{(3,1)} = 4P_{(3,1)}.$$

And, finally, taking long Coxeter group element

$$w = s_0 s_1 s_0 s_1,$$

characteristic function of $U^\pm(w)$

$F_w^C = 4s_{(2,1,1)} + 4s_{(3,1)} + 4s_{(2,2)}$, since there is only one unimodal tableau corresponding to w , namely

$$\begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline & 0 & \\ \hline \end{array}.$$

In general, the highest weight elements of $\mathcal{PT}(\lambda)$ are exactly the ones with Yamanouchi reading words, and Schur expansion of F_w^C can be found similarly.