

# Type-C Stanley Symmetric Functions and Shifted Primed Tableaux

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## Goal

It is known that Stanley symmetric functions are Shur-positive, i.e. the coefficients of the Shur expansion are non-negative. Our goal here is to introduce a crystal structure on the set of unimodal factorizations that is isomorphic to the crystal structure of type-A crystal. In particular, we use Kraśkiewicz insertion [?, ?] to use the notion of Primed Tableaux and introduce crystal operators on those tableaux instead.

## Background

### Stanley Symmetric Functions

- Coxeter group of type  $C_n$  is defined to be generated by  $\{s_0, s_1, \dots, s_{n-1}\}$  subject to relations
  - $s_i^2 = 1$  for all  $i$ ,
  - $s_i s_j = s_j s_i$  provided  $|i - j| > 1$ ,
  - $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$  for all  $i > 0$ ,
  - $s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0$ .
- Each Coxeter group element  $w = s_{i_1} \dots s_{i_l}$  is represented by a word  $i_1 \dots i_l$  and many other equivalent words. Among those, the words of shortest length are called *reduced words*.
- A word  $i_1 \dots i_l$  is called unimodal if there exists an index  $\nu$  with  $i_1 > \dots > i_\nu < \dots < i_l$ . *Unimodal factorization* of a Coxeter group element  $w$  is a factorization of its reduced word into unimodal factors. Denote the set of unimodal factorizations of  $w$  as  $U(w)$ .
- For example, given  $w = s_2 s_1 s_2 s_0 s_1 s_0$ , some of the elements of  $U(w)$  are  $(212)(0)(10)$ ,  $(21)()(201)(0)$ ,  $(1)(2101)(0)$ ,  $()(12)(01)(01)$ .
- Given unimodal factorization  $\mathbf{A}$ , define its *weight*  $\text{wt}(\mathbf{A})$  to be the vector consisting of the number of elements in each factor, and  $\text{nz}(\mathbf{A})$  to be the number of non-empty factors.
- *Type-C Stanley symmetric function* is defined as

$$F_w^C(\mathbf{x}) = \sum_{\mathbf{A} \in U(w)} 2^{\text{nz}(\mathbf{A})} \mathbf{x}^{\text{wt}(\mathbf{A})}.$$

### Type-A crystals and Schur Functions

- Consider the set of words  $\mathcal{B}_n^h$  of length  $h$  in the alphabet  $\{1, 2, \dots, n+1\}$ , with no equivalency relations. We impose a crystal structure on  $\mathcal{B}_n^h$  by defining lowering operators  $f_i$  and raising operators  $e_i$  for  $1 \leq i \leq n$  and a weight function.
- The weight of  $\mathbf{b} \in \mathcal{B}_n^h$  is the tuple  $\text{wt}(\mathbf{b}) = (a_1, \dots, a_{n+1})$ , where  $a_i$  is the number of letters  $i$  in  $\mathbf{b}$ .
- Lowering operator  $f_i$  acts on  $\mathbf{b}$  by changing a particular letter  $i$  to  $i+1$  if such letter exists, and is not defined otherwise. Raising operator  $e_i$  acts as the inverse of  $f_i$ .
- Crystal operators induce digraph structure on  $\mathcal{B}_n^h$ , with the character of each connected component equal to a symmetric Schur polynomial  $s_\lambda(x_1, \dots, x_{n+1})$  with  $\lambda \vdash h$ .
- In the limit  $n \rightarrow \infty$  Shur polynomials become Schur functions  $s_\lambda(\mathbf{x})$ . Schur functions form an orthonormal basis to the vector space of symmetric functions with integer coefficients.

## Unimodal Tableaux and Shifted Primed Tableaux

A *shifted diagram*  $\mathcal{S}(\lambda)$  associated to a partition  $\lambda = (\lambda_1, \dots, \lambda_\ell)$  with  $\lambda_i > \lambda_{i+1}$  is the set of boxes in positions  $(i, j)$  satisfying  $1 \leq i \leq \ell$  and  $i \leq j \leq \lambda_i + i - 1$ .

### Unimodal Tableaux

- A *unimodal tableau*  $\mathbf{P}$  of shape  $\lambda$  associated to a Coxeter group element  $w$  of type  $C_n$  is a filling of a shifted diagram  $\mathcal{S}(\lambda)$  with letters from the alphabet  $\{0 < 1 < 2 < \dots < n-1\}$  such that
  - the rows of  $\mathbf{P}$ , denoted by  $P_1, \dots, P_\ell$ , are unimodal words,
  - $P_i$  is the longest unimodal subsequence in a concatenated word  $P_{i+1} P_i$ ,
  - the concatenated word  $P_\ell P_{\ell-1} \dots P_1$  a reduced type-C word that represents  $w$ .
- For example, unimodal tableau 

4	3	2	0	1
2	1	2		

 corresponds to  $w = s_2 s_1 s_2 s_4 s_3 s_2 s_0 s_1$ .

### Primed Tableaux

- A *primed tableau*  $\mathbf{T}$  of shape  $\lambda$  on  $n$  letters is a filling of  $\mathcal{S}(\lambda)$  with letters from the alphabet  $\{1' < 1 < 2' < 2 < \dots < m' < m\}$  such that:
  - The entries are weakly increasing along each column and each row of  $\mathbf{T}$ .
  - Each row contains at most one  $i'$  for every  $i = 1, \dots, m$ .
  - Each column contains at most one  $i$  for every  $i = 1, \dots, m$ .
- The weight of a primed tableau, denoted by  $\text{wt}(\mathbf{T})$ , is the vector with  $i$ -th coordinate equal to the total number of letters in  $\mathbf{T}$  that are either  $i$  or  $i'$ .
- For example, 

1	1	2'	3'	3
2'	2	3'		

 is a primed tableau of weight  $(2, 3, 3)$ .

## Kraśkiewicz insertion

The Kraśkiewicz insertion gives a bijection

$$\text{KR}: U^\pm(w) \rightarrow \bigcup_{\lambda} [\mathcal{UT}_w(\lambda) \times \mathcal{PT}(\lambda)],$$

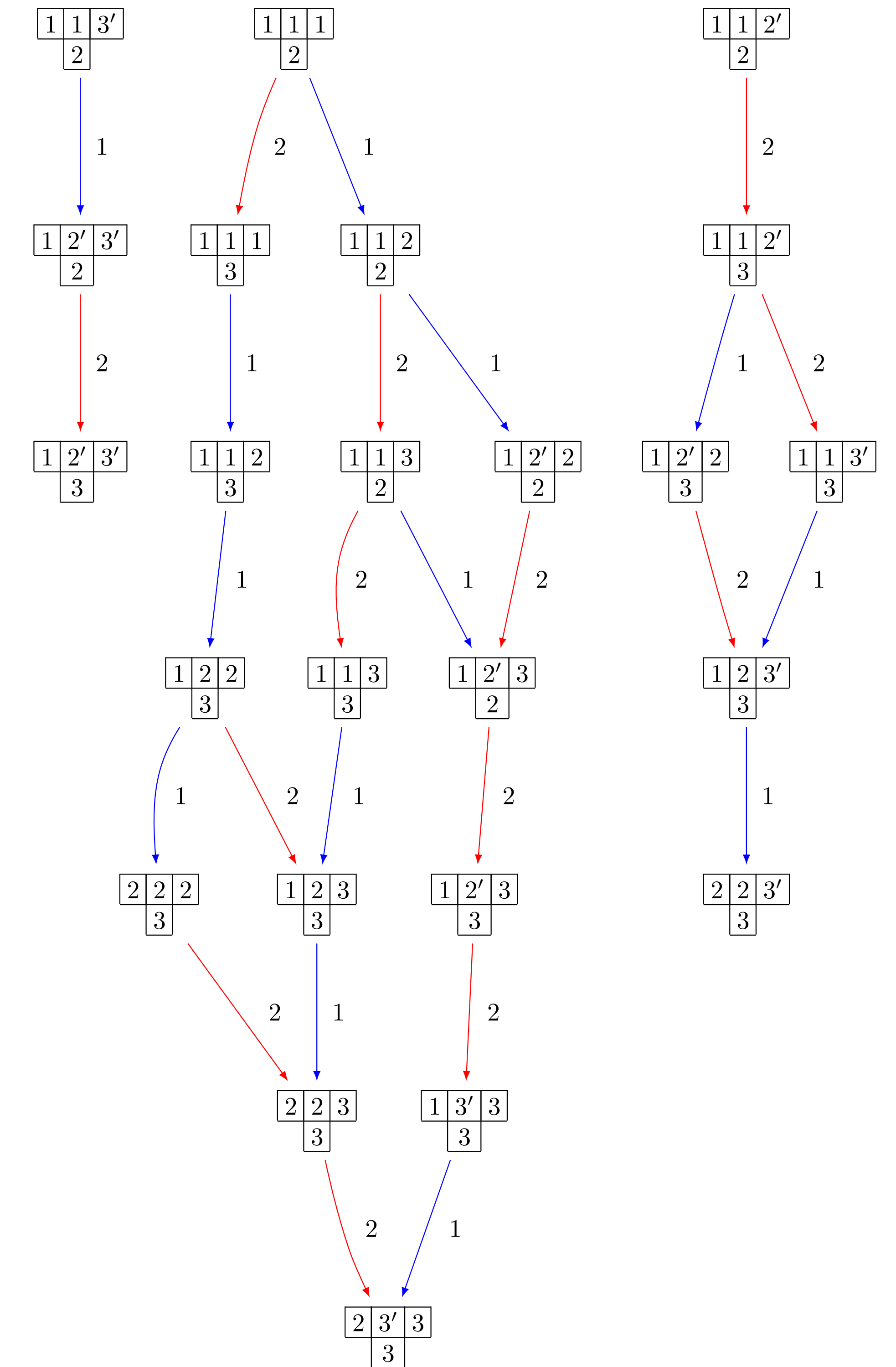
where  $U^\pm(w)$  is the set of all unimodal factorizations of  $w$  with a sign assigned to each non-zero factor,  $\mathcal{UT}_w(\lambda)$  is the set of all unimodal tableaux of shape  $\lambda$  associated with  $w$ , and  $\mathcal{PT}(\lambda)$  is the set of all primed tableaux of shape  $\lambda$ . Moreover, the weight of a unimodal factorization is equal to the weight of a primed tableau in the image.

## Lowering operator $f_i$ on Primed Tableau

We can introduce a crystal structure on the set  $\mathcal{PT}(\lambda)$  instead and induce the structure on  $U^\pm(w)$  via the Kraśkiewicz insertion. It is convenient to define the crystal operators on the subset of  $\mathcal{PT}(\lambda)$  with no primed elements on the diagonal, and extend the action on the whole set afterwards.

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## Crystal



## Bibliography

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