ST 501: Homework 4

Kirill Tsarapkin

1 Chapter 2 problems:

1. Question 2.45:

   Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = .1$.

   a. Find the probability that the lifetime is less than 10.

$$\text{The random variable is: } X \sim \frac{1}{10}e^{-\frac{x}{10}} \text{ for } x \geq 0. \tag{1}$$

$$P(X < 10) = \int_0^{10} \frac{1}{10}e^{-\frac{x}{10}}\, dx = \frac{e-1}{e} \approx 0.632 \tag{2}$$

   b. Find the probability that the lifetime is between 5 and 15.

$$P(5 < X < 15) = \int_5^{15} \frac{1}{10}e^{-\frac{x}{10}}\, dx = \frac{e-1}{e^{3/2}} \approx 0.383 \tag{3}$$

   c. Find t such that the probability that the lifetime is greater than t is .01.

$$P(t < X) = \frac{1}{100} = \int_t^{\infty} \frac{1}{10}e^{-\frac{x}{10}}\, dx = e^{\frac{-t}{10}} \to e^{\frac{-t}{10}} = \frac{1}{100} \to t = 20\, ln(10) \approx 46.05 \tag{4}$$

2 Chapter 4 problems:

2. Question 4.2:

   If X is a discrete uniform random variable-that is, P(X = k) = 1/n for k = 1, 2, . . . , n. Find E(X) and Var(X)

   Hint: Look up the sum of the first n numbers and the sum of the first n^2 numbers.

$$E(X) = \sum_{k=1}^{n} k\frac{1}{n} = \frac{1+2+.....+n}{n} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2} \tag{5}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \sum_{k=1}^{n} \frac{k^2}{n} - [\sum_{k=1}^{n} \frac{k}{n}]^2 \tag{6}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \sum_{k=1}^{n} \frac{k^2}{n} - [\sum_{k=1}^{n} \frac{k}{n}]^2 = \frac{1^1 + 2^2 + ... + n^2}{n} - [\frac{n+1}{2}]^2 \tag{7}$$

$$Var(X) = \frac{\frac{n(n+1)(2n+1)}{6}}{n} - [\frac{n+1}{2}]^2 = \frac{(n+1)(2n+1)}{6} - [\frac{n+1}{2}]^2 = \frac{n^2-1}{12} \tag{8}$$

3. Question 4.4:

   Let X have the cdf $F(x) = 1 - x^{-\alpha}$, $x \geq 1$.

   Note: As $x \geq 1$, the expectation of X exists as long as $E(X) < \infty$. There will be some values of α where this is not the case! (Similarly for V ar(X).)

   a. Find E(X) for those values of α for which E(X) exists.

$$E(X) = \int_1^\infty x f(x) dx = \int_1^\infty x \frac{dF(x)}{dx} dx = \int_1^\infty x \alpha x^{-\alpha-1} dx = \int_1^\infty \alpha x^{-\alpha} dx \tag{9}$$

$$E(x) = \frac{\alpha}{\alpha - 1} \text{ for } \alpha > 1 \tag{10}$$

   b. Find Var(X) for those values of α for which it exists.

$$Var(X) = E(X^2) - [E(X)]^2 = \int_1^\infty x^2 \frac{\alpha}{x^{\alpha+1}} dx - [\int_1^\infty x \frac{\alpha}{x^{\alpha+1}} dx]^2 = \tag{11}$$

$$= \frac{\alpha}{\alpha-2} - [\frac{\alpha}{\alpha-1}]^2 = \frac{\alpha}{(\alpha-2)(\alpha-1)^2} \; for \; \alpha > 2 \tag{12}$$

4. Question 4.14:

   Let X be a continuous random variable with the density function $f(x) = 2x$, $0 \leq x \leq 1$

   a. Find E(X).

$$E(X) = \int_0^1 x f(x) dx = \int_0^1 2x^2 \, dx = \frac{2}{3} \tag{13}$$

   b. Find $E(X^2)$ and Var(X).

Let $Y = X^2$, then $X = \sqrt{Y}$ and so we get

$$CDF \rightarrow F(Y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = \int_0^{\sqrt{y}} 2x \, dx = y \; , [0,1] \tag{14}$$

$$PDF \rightarrow f(y) = \frac{dF(Y)}{dy} = \frac{d}{dy}(y) = 1 \; , [0,1] \tag{15}$$

$$E(X^2) = E(Y) = \int_0^1 y f(y) \, dy = \int_0^1 y \, dy = \frac{1}{2} \tag{16}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{1}{2} - [\frac{2}{3}]^2 = \frac{1}{18} \tag{17}$$

5. Question 4.16:

Suppose that $E(X) = \mu$ and $Var(X) = \sigma^2$. Let $Z = (X - \mu)/\sigma$. Show that E(Z) = 0 and Var(Z) = 1.

Note: This transformation is a linear function of X!

$$E(Z) = \int_{-\infty}^{\infty} Zf(x)dx = \int_{-\infty}^{\infty} (\frac{X-\mu}{\sigma})f(x)\,dx = \frac{1}{\sigma}[\int_{-\infty}^{\infty} Xf(x)dx - \mu \int_{-\infty}^{\infty} f(x)dx] \tag{18}$$

$$E(Z) = \frac{1}{\sigma}[E(X) - \mu E(X)] = \frac{1}{\sigma}[\mu - \mu] = 0 \tag{19}$$

Or we can use linearity of the expectation and pull out the constants to obtain

$$E(Z) = E(\frac{X-\mu}{\sigma}) = \frac{E(X)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0 \tag{20}$$

Since from the result above E(Z)=0, $Var(Z) = E(Z^2) - [E(Z)]^2 = E(Z^2)$

$$Var(Z) = E(Z^2) = E[(\frac{X-\mu}{\sigma})^2] = \frac{1}{\sigma^2}E[\mu^2 - 2\mu X + X^2] = \frac{1}{\sigma^2}[\mu^2 - 2\mu^2 + E(X^2)] = \frac{1}{\sigma^2}[E(X^2) - \mu^2] \tag{21}$$

$$\text{Where } E(X^2) = Var(X) + [E(X)]^2 = \sigma^2 + \mu^2 \tag{22}$$

$$\text{Therefore, we get } Var(Z) = \frac{1}{\sigma^2}[E(X^2) - \mu^2] = \frac{1}{\sigma^2}[\sigma^2 + \mu^2 - \mu^2] = 1 \tag{23}$$

6. Question 4.30:

Find $E[1/(X + 1)]$, where X is a Poisson random variable.

Note : Recall the Poisson PMF from the notes. You'll need to use kernel matching or the MacLauren series of e^ω to simplify the summation.

The expected value of a Posson random variable is

$$E(X) = \sum_{k=0}^{\infty} \frac{k\lambda^k}{k!} e^{-\lambda} \tag{24}$$

Therefore, we get

$$E(\frac{1}{X+1}) = \sum_{k=0}^{\infty} \frac{1}{k+1} \frac{\lambda^k}{k!} e^{-\lambda} \tag{25}$$

Now since (k+1)k! = (k+1)! , we can rewrite the above equation as

$$E(\frac{1}{X+1}) = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} \tag{26}$$

Multiply equation (10) by $\frac{\lambda}{\lambda}$

$$E(\frac{1}{X+1}) = e^{-\lambda} \frac{\lambda}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+1)!} = e^{-\lambda} \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{k+1}}{(k+1)!} = \frac{e^{-\lambda}}{\lambda} (\frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + ...) \tag{27}$$

We know that the Maclaurin Series for function e^x is $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2} + \frac{x^3}{3!}$.... Therefore we can simplify the above equation to get the expectation value

$$E(\frac{1}{X+1}) = \frac{e^{-\lambda}}{\lambda} (e^{\lambda} - 1) = \frac{1 - e^{-\lambda}}{\lambda} \tag{28}$$