ST 501: Homework 2

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1 Chapter 1 problems:

1. Question 1.46:

   Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed. If it lands heads up, a ball is drawn from urn A; otherwise, a ball is drawn from urn B.

   a. What is the probability that a red ball is drawn?

   Answer : Here, P(A) and P(B) represent the probability of choosing either urn A or B by flipping a coin respectively. The probability that the a red ball is drawn given that it is in url A is represented by $P(Red\ Ball|A)$, likewise for url B, $P(Red\ Ball|B)$.

$$P(Red\ ball\ is\ drawn) = P(Red\ ball\ is\ drawn\ from\ A) + P(Red\ ball\ is\ drawn\ from\ B) \tag{1}$$

$$\rightarrow P(A)P(Red\ Ball|A) + P(B)P(Red\ Ball|B) = \frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{2}{7} = \frac{31}{70} \approx 0.44\ or\ 44\% \tag{2}$$

   b. If a red ball is drawn, what is the probability that the coin landed heads up?.

   Using Bayer's Theorem, we get

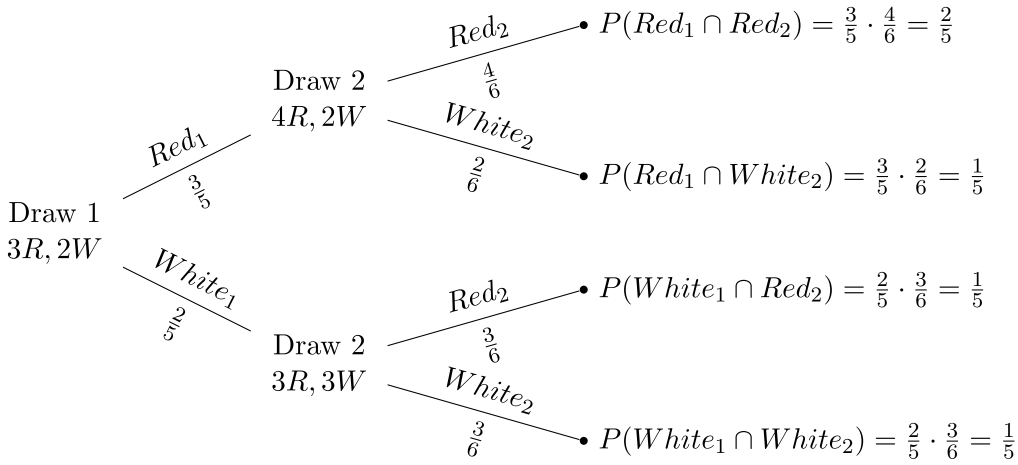
$$P(A|Red\ Ball) = \frac{P(A)P(Red\ Ball|A)}{P(Red\ Ball)} = \frac{\frac{1}{2} \times \frac{3}{5}}{\frac{31}{70}} = \frac{21}{31} \approx 0.68\ or\ 68\% \tag{3}$$

2. Question 1.48:

   An urn contains three red and two white balls. A ball is drawn, and then it and another ball of the same color are placed back in the urn. Finally, a second ball is drawn.

   a. What is the probability that the second ball drawn is white?

   Answer: The easiest way to approach this problem is to first draw a probability tree.



Looking at the probability three, we can see that the probability that a second ball drawn is white is equal to $P(Red_1 \cap White_2) + P(White_1 \cap White_2) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}$

b. If the second ball drawn is white, what is the probability that the first ball drawn was red?

Answer: Again, looking at the decision three we can see that if the second ball drawn is white, the probability that the first ball drawn was red is $\frac{P(White_1 \cap White_2)}{P(Red_1 \cap White_2) + P(White_1 \cap White_2)} = \frac{1/5}{2/5} = \frac{1}{2}$

Here is the Latex code that I used to generate the probability tree.

```
\usepackage[latin1]{inputenc}
\usepackage{tikz}
\usetikzlibrary{trees}

% Set the overall layout of the tree
\tikzstyle{level 1}=[level distance=3.5cm, sibling distance=3.5cm]
\tikzstyle{level 2}=[level distance=3.5cm, sibling distance=2cm]

% Define styles for bags and leafs
\tikzstyle{bag} = [text width=4em, text centered]
\tikzstyle{end} = [circle, minimum width=3pt,fill, inner sep=0pt]

% The sloped option gives rotated edge labels. Personally
% I find sloped labels a bit difficult to read. Remove the sloped options
% to get horizontal labels.
\begin{tikzpicture}[grow=right, sloped]
\node[bag] {Draw 1 $3R, 2W$}
    child {
        node[bag] {Draw 2 $3R, 3W$}
            child {
                node[end, label=right:
                    {$P(White_1\cap White_2)=\frac{2}{5}\cdot\frac{3}{6}=\frac{1}{5}$
                        ↪ }] {} % 4th
                edge from parent
                node[above] {$White_2$} % 2nd draw (Bottom Branch)
                node[below] {$\frac{3}{6}$}
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        }
        child {
            node[end, label=right:
                {$P(White_1\cap Red_2)=\frac{2}{5}\cdot\frac{3}{6}=\frac{1}{5}$}]
                    ↪ {} % 3rd
            edge from parent
            node[above] {$Red_2$} %Second Draw ( 2nd from the bottom branch)
            node[below] {$\frac{3}{6}$}
        }
        edge from parent
        node[above] {$White_1$} % First Draw (Bottom branch)
        node[below] {$\frac{2}{5}$}
    }
    child {
        node[bag] {Draw 2 $4R, 2W$}
        child {
            node[end, label=right:
                {$P(Red_1\cap White_2)=\frac{3}{5}\cdot\frac{2}{6}=\frac{1}{5}$}]
                    ↪ {} % 2nd
            edge from parent
            node[above] {$White_2$} % Second Draw (2nd brach from the top)
            node[below] {$\frac{2}{6}$}
        }
        child {
            node[end, label=right:
                {$P(Red_1\cap Red_2)=\frac{3}{5}\cdot\frac{4}{6}=\frac{2}{5}$}]
                    ↪ {} % 1st
            edge from parent
            node[above] {$Red_2$} % Second Draw (Top branch)
            node[below] {$\frac{4}{6}$}
        }
        edge from parent
        node[above] {$Red_1$} % First Draw (Top branch)
        node[below] {$\frac{3}{5}$}
    };
\end{tikzpicture}
```

3. Question 1.50:

   Two dice are rolled, and the sum of the face values is six. What is the probability that at least one of the dice came up a three?

   Hint: We can use the sample space that we made for problem 1.2 from the previous homework and highlight all elements where the sum of the face values is six.

$$\mathbf{\Omega} = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}. \tag{4}$$

Answer : We can see that there are five elements where the the sum of the faces is six and that there is only one element where at least one of the dice came up a three. Therefore, the probability of an event A, where at least one of the dice came up a three is: $P(A) = \frac{1}{5}$.

4. Question 1.59:

   A box has three coins. One has two heads, one has two tails, and the other is a fair coin with one head and one tail. A coin is chosen at random, is flipped, and comes up heads.

   a. What is the probability that the coin chosen is the two-headed coin?

   Answer : The easier approach to this problem would be to again draw a probability tree. However this time we can use Bayer's Rule. Let's define the event of getting head as H, the event of getting tails as T, the event of getting a two head coin as C_{hh}, the event of getting a two tails coin as C_{tt} and the event of getting a normal coin as C_n. The probability that the coin chosen is the two-headed coin is:

$$P(C_{hh}|H) = \frac{P(H|C_{hh}) \times P(C_{hh})}{P(H|C_{hh}) \times P(C_{hh}) + P(H|C_{tt}) \times P(C_{tt}) + P(H|C_n) \times P(C_n)} \tag{5}$$

$$P(C_{hh}|H) = \frac{1 \times 1/3}{1 \times 1/3 + 0 \times 1/3 + 1/2 \times 1/3} = \frac{2}{3} \tag{6}$$

   b. What is the probability that if it is thrown another time it will come up heads?

   Answer: Since the coin turned up heads in the first flip, $P(C_{tt}) = 0$. The probability that if the coin is thrown another time it will come up heads is :

$$P(C_{hh}) \times P(H|C_{hh}) + P(C_n) \times P(H|C_n) = \frac{2}{3} \times 1 + \frac{1}{3} \times \frac{1}{2} = \frac{5}{6} \tag{7}$$

   c. Answer part (a) again, supposing that the coin is thrown a second time and comes up heads again.

   Answer: First lets define a new event H_{hh}, where in the first toss the coin turns up heads and then in the second toss the coin turns up to be heads again. Since the coin turned up heads in the first flip, $P(C_{tt}) = 0$. Supposing that the coin is thrown a second time and comes up heads again, the probability that the coin chosen is the two-headed coin is :

$$P(C_{hh}|H_{hh}) = \frac{P(H_{hh}|C_{hh}) \times P(C_{hh})}{P(H_{hh}|C_{hh}) \times P(C_{hh}) + P(H_{hh}|C_{tt}) \times P(C_{tt}) + P(H_{hh}|C_n) \times P(C_n)} \tag{8}$$

$$P(C_{hh}|H_{hh}) = \frac{1 \times 4/6}{1 \times 4/6 + 0 + 1/2 \times 1/3} = \frac{4}{5} \tag{9}$$

5. Question 1.60:

   A factory runs three shifts. In a given day, 1% of the items produced by the first shift are defective, 2% of the second shiftâs items are defective, and 5% of the third shiftâs items are defective.

   If the shifts all have the same productivity, what percentage of the items produced in a day are defective?

   Answer : Let's define event D as the percent of the defective items. Then the percentage of the items produced in a day are defective is :

$$P(D|1st\ Shift)P(1st\ Shift) + P(D|2nd\ Shift)P(2nd\ Shift) + P(D|3rd\ Shift)P(3rd\ Shift) \tag{10}$$

$$\frac{1}{3} \times \frac{1}{100} + \frac{1}{3} \times \frac{2}{100} + \frac{1}{3} \times \frac{5}{100} = \frac{2}{75} \approx 2.67\% \tag{11}$$

   If an item is defective, what is the probability that it was produced by the third shift?

   Answer : The probability that a defective item was produced by the third shift is :

$$\frac{P(D|3rd\ Shift)P(3rd\ Shift)}{P(D|1st\ Shift)P(1st\ Shift) + P(D|2nd\ Shift)P(2nd\ Shift) + P(D|3rd\ Shift)P(3rd\ Shift)} \tag{12}$$

$$\frac{\frac{1}{3} \times \frac{5}{100}}{\frac{1}{3} \times \frac{1}{100} + \frac{1}{3} \times \frac{2}{100} + \frac{1}{3} \times \frac{5}{100}} = \frac{5}{8} \approx 62.50\% \tag{13}$$

6. Question 1.69:

   If A and B are disjoint, can they be independent?

   We know that if A and B are disjoint, then $P(A \cap B) = 0$, so $P(A \cup B) = P(A) + P(B)$ and when A and B are independent, $P(A \cap B) = P(A)(B)$.

   Answer: If A and B are disjoint, they can be independent if either $P(A) = 0$ *or* $P(B) = 0$. However if $P(A) > 0$ and $P(B) > 0$ then if A and B are disjoint, they can NOT be independent.

7. Question 1.70:

   If $A \subset B$, can A and B be independent?

   Answer: A and B can be independent, such that $P(A \cup B) = P(A)P(B)$ if A is the empty set, $P(A) = 0$ since the empty set is a subset of every set or if $B = \Omega$, $P(B) = 1$.

8. Question 1.74:

   What is the probability that the following system works if each unit fails independently with probability p (see Figure 1.5)?

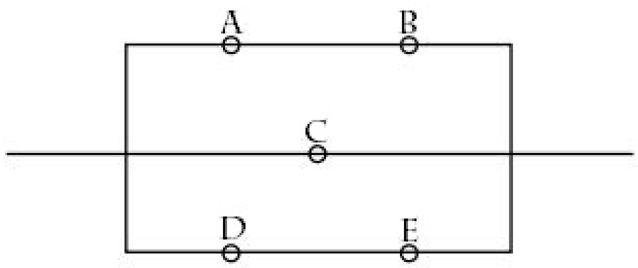

FIGURE 1.5

Answer: There are 5 units in the system, if each unit fails independently with probability p, then each unit works with probability 1-p. Since all of the units are independent, the probability that the system works is $P((A \cap B) \cup C \cup (D \cap E))$.

$$\begin{gathered} P(A \cap B) + P(C) + P(D \cap E) - P(C \cap (D \cap E)) - P((A \cap B) \cap C)) \\ +P((A \cap B) \cap C \cap (D \cap E)) - P((A \cap B) \cap (D \cap E)) \\ = (1 \text{ - p})^2 + (1-p) - (1-p)^3 + (1-p)^2 - (1-p)^3 + (1-p)^5 - (1-p)^4 \\ = \text{-p}^5 + 4p^4 - 4p^3 + 1 \end{gathered} \tag{14}$$

9. Question 1.77

A player throws darts at a target. On each trial, independently of the other trials, he hits the bull's-eye with probability .05. How many times should he throw so that his probability of hitting the bull's-eye at least once is .5?

Hint: Start with a complement rule to simplify things!

Answer: Let's define an event B where the player hits the bull's eye at least once with N number of throws, then the probability of event B is $P(B) = 1 - (1 - 0.05)^N$

$$P(B) = 1 - (0.95)^N = 0.5 \to 0.5 = (0.95)^N \to N = \frac{ln(0.5)}{ln(0.95)} \approx 13.51 \tag{15}$$

Therefore, the player should throw the dart 14 times so that his probability of hitting the bull's-eye at least once is 0.5.

2 Other problem :

10. Question 1

Two pennies, one with P(head)=u and one with P(Head)=w, are to be tossed together independently. Define

a. $p_0 = P(0\ heads\ occur)$

b. $p_1 = P(1\ head\ occurs)$

c. $p_2 = P(2\ heads\ occur)$

a) Let's define H_1 as the event of getting heads in the first coin, H_2 as the event of getting heads in the second coin, T_1 as the event of getting tails in the first coin and T_2 as the event of getting tails in the second coin.

$$P(0\ heads\ occur) = P(2\ tails\ occur) = P(T_1 \cap T_2) \tag{16}$$

$$Because\ two\ events\ are\ independent,\ P(T_1 \cap T_2) = P(T_1)P(T_2) \tag{17}$$

$$P(T_1)P(T_2) = (1-u)(1-w) = uw - u - w + 1 \tag{18}$$

b.

$$P(1\ head\ occurs) = P(H_1 \cap T_2) + P(T_1 \cap H_2) = P(H_1)P(T_2) + P(T_1)P(H_2) \tag{19}$$

$$P(1\ head\ occurs) = u(1-w) + (1-u)w = u - uw + w - wu \tag{20}$$

c.

$$P(2\ heads\ occur) = P(H_1)P(H_2) = uw \tag{21}$$

Can u and w be chosen such that $p_0 = p_1 = p_2$? Explain. Note: You will need to show whether or not a quadratic equation has a solution. Remember the quadratic formula and that you only have a solution if $b^2 - 4ac \geq 0$.

Answer: Set $P(0\ heads\ occur)) = P(1\ head\ occurs) = P(2\ heads\ occur)$. Alternatively, we can simply graph this in R and see if the lines intersect.

$$\begin{gathered} uw - u - w + 1 = u - uw + w - wu = uw \\ \text{First } u - uw + w - wu = uw \to u + w = 3uw \\ \text{Second } uw - u - w + 1 = uw \to u + w = 1 \\ \to 1 = 3uw \to \tfrac{1}{3} = uw \to u + w = 3uw = 3\tfrac{1}{3} \to u = 1 - w \\ \to \tfrac{1}{3} = (1-w)w = w - w^2 \to w^2 - w + \tfrac{1}{3} = 0 \\ w = \frac{1 \pm \sqrt{1 - \frac{4}{3}}}{2} = \frac{1 \pm \sqrt{-\frac{1}{3}}}{2} \end{gathered} \tag{22}$$

This solution only has complex solutions. Since there are no real solutions to $p_0 = p_1 = p_2$, we can conclude that there are no u and w such that $p_0 = p_1 = p_2$