STABILITY CONDITION ON A SINGULAR SURFACE AND ITS RESOLUTION

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Abstract. Let X be a surface with an ADE-singularity and let $\widetilde{X}$ be its crepant resolution. In this paper, we show that there exists a Bridgeland stability condition σ_X on $\mathrm{D}^b(X)$ and a weak stability condition $\sigma_{\widetilde{X}}$ on the derived category of the desingularisation $\mathrm{D}^b(\widetilde{X})$, such that pushforward of $\sigma_{\widetilde{X}}$-semistable objects are σ_X-semistable

We first construct Bridgeland stability conditions on $\mathrm{D}^b(\widetilde{X})$ associated to the contraction $\widetilde{X} \longrightarrow X$, generalizing the results of Tramel and Xia in [TX22], Then we deform it to a weak stability condition $\sigma_{\widetilde{X}}$ and show that it descends to $\mathrm{D}^b(X)$, producing the stability condition σ_X.

1. Introduction

Bridgeland stability conditions have been constructed on curves, surfaces, and a number of smooth threefolds, including all Fano threefolds (see, for example, [Bri08, BMMS12, Li19]). Recently, Langer constructed stability conditions on normal surfaces [Lan24].

In this article, we construct a stability condition on a singular surface with an ADE singularity which is compatible with a weak stability condition on resolution.

Our main result is the following:

Theorem 1.1. *Let X be a surface with an ADE singularity and $\pi\colon \widetilde{X} \longrightarrow X$ be its crepant resolution.*

Then there exist a stability condition $\sigma_X = (Z_X, \mathcal{A})$ on $\mathrm{D}^b(X)$, and a weak stability condition $\sigma_{\widetilde{X}} = (Z_{\widetilde{X}}, \mathcal{B}^0)$ such that they are related as follows.

1. $Z_{\widetilde{X}} = Z_X \circ \pi_*$
2. *$\pi_*\colon \mathcal{B}^0 \longrightarrow \mathcal{A}$ is an exact functor and for any $Z_{\widetilde{X}}$-semistable object E, its pushforward π_*E is Z_X-semistable.*
3. *In particular, π_* induces a morphism between the moduli spaces $\mathcal{M}_{\sigma_{\widetilde{X}}}(v) \longrightarrow \mathcal{M}_{\sigma_X}(\pi_* v)$.*

It is quite natural that we first study the derived category of the resolution $\widetilde{X}$. The construction will be realized by deforming the central charge of a stability condition on $\mathrm{D}^b(\widetilde{X})$ to a weak stability condition $\sigma_{\widetilde{X}}$, and then the construction of σ_X will be based on that of $\sigma_{\widetilde{X}}$.

By generalizing [TX22, Sect. 5,6], we will construct $\sigma_{\widetilde{X}}$ as the limit of stability conditions σ_ϵ, which are associated to the contractions $\pi\colon \widetilde{X} \longrightarrow X$, in the sense that the skyscraper sheaf $\mathcal{O}_x$ is strictly semistable for x in the exceptional locus Π, and is stable for any $x \in \widetilde{X} \setminus \Pi$.

In order to do so, we need to construct a heart of bounded t-structure, using the technique of torsion pair and tilting in the sense of [HRS96]. Combining arguments in [Tod13] and in [TX22, Sect. 3], we have a double-tilted heart $\mathcal{B}^0$ on $\mathrm{D}^b(\widetilde{X})$, with two different descriptions, which are useful for constructing the central charges on this heart and the support property then after. In [Tod13, Sect. 3], there is an assumption that $C^2 = -1$. This is used to prove a Bogomolov-Gieseker type result and to construct a Bridgeland stability condition, but our argument didn't use this assumption.

In summary, we have the following result:

Theorem 1.2. *Let X be a surface with an ADE singularity and $\pi\colon \widetilde{X} \longrightarrow X$ be its crepant resolution. Then there exists a path $\sigma_\epsilon = (Z_\epsilon, \mathcal{B}^0)$ of Bridgeland stability conditions on $\mathrm{D}^b(\widetilde{X})$ with the same heart, such that the end point of the path σ_ϵ is the weak stability condition $\sigma_{\widetilde{X}} = (Z_{\widetilde{X}}, \mathcal{B}^0)$.*

Finally, we prove that the end point $\sigma_{\widetilde{X}}$ descends to a pre-stability condition σ_X on $\mathrm{D}^b(X)$, viewed as the quotient of $\mathrm{D}^b(\widetilde{X})$, and show that there exists a compatability between $\sigma_{\widetilde{X}}$ and σ_X given by the pushforward $\mathbf{R}\pi_*$, which also gives a surjection between the set of semistable objects.

Theorem 1.3. *For any class $v \in K_{\mathrm{num}}(X)$, there exists a class $\widetilde{v} \in K_{\mathrm{num}}(\widetilde{X})$ such that there is a surjective map $\pi_*\colon \mathcal{M}_{\sigma_{\widetilde{X}}}(\widetilde{v}) \longrightarrow \mathcal{M}_{\sigma_X}(v)$.*

Furthermore, we use the compatibility with π_* to naturally induces a quadratic form on the quotient lattice $K_{\mathrm{num}}(X) \simeq K_{\mathrm{num}}(\widetilde{X})/\ker \pi_*$ which provides the support property of σ_X.

1.1. **Related works.** There are some recent related works. In [LR22], the authors constructed stability conditions on the derived category of stacks associated to surfaces with an ADE singularity.

Also, Vilches [Vil24] consider stability conditions associated to more general birational contractions of surfaces. In particular, for an ADE configuration, he also recovers the stability conditions σ_ϵ on the smooth surface $\widetilde{X}$.

By [BPPW22], the stability manifold $\mathrm{Stab}(\widetilde{X})$ admits a partial compactification in the space $\mathrm{Hom}(\Lambda, \mathbb{C}) \times \mathrm{Slice}(\widetilde{X})$ by massless semistable objects. Then, in terms of [BPPW22], $\sigma_{\widetilde{X}}$ can be thought of as a lax stability condition lying in that stratum of $\partial\,\mathrm{Stab}(\widetilde{X})$.

In [Bol23], Bolognese gives another local compatification of the stability manifold $\mathrm{Stab}(\widetilde{X})$. The stability condition σ_X we constructed can also be view as a generalized stability condition on $\widetilde{X}$ in Bolognese's sense.

Our result possibly provides a model of what could happen on higher dimensional singular varieties. One of the application can be to study the derived category of a smoothing of 1-nodal Fano threefolds and its Kuznetsov component (cf. [KS24]) as we need fiberwise stability conditions, using the theory of stability conditions in families developed in [BLM+21].

On the other hand, one may also want to consider the same family with the singular fiber replaced with its resolution. We want to relate the notion of (semi)stability on $\mathrm{D^b}(X)$ and that on its resolution $\mathrm{D}^b(\widetilde{X})$, and then our result may be applied.

1.2. **Plan of the paper.** This paper is organized as follows. Section 2 contains some preliminaries of stability conditions and the theory of tilting.

In Section 3, we first construct the heart by double tilting of coherent sheaves. We will moreover show that two double tilted hearts actually coincide and study the heart $\mathcal{B}^0$. Particularly, we are interested in some simple objects which will appear in the Jordan-Hölder filtrations.

In Section 4 and 5, we respectively define the central charge and the quadratic form for support property on $\mathrm{D}^b(\widetilde{X})$. Also, we verify the Harder-Narasimhan property on $\mathcal{B}^0$ with repsect to $Z_{\pi^*H,\beta,z}$.

In Section 6, we deform the central charge Z_ϵ and get a path $\sigma_\epsilon = (Z_\epsilon, \mathcal{B}^0)$ of stability conditions. We will show that the end point $\sigma_{\widetilde{X}} = (Z_{\widetilde{X}}, \mathcal{B}^0)$ is a weak stability condition; we also find a tilted heart $\mathrm{Coh}^\beta_H(X) \subseteq \mathrm{D}^b(X)$ which forms a Bridegland stability condition together with Z_X. Moreover, $\sigma_X = (Z_X, \mathrm{Coh}^\beta_H(X))$ is compatible with $\sigma_{\widetilde{X}}$ in the sense of Theorem 1.1 and 1.3.

In Section 7, we conclude that the induced Bridgeland stability condition on the singular surface also admits the support property, whose quadratic form is also induced by that of the end point $\sigma_{\widetilde{X}}$.

List of Notations

$Z_H := -H^{n-1} \cdot \mathrm{ch}_1 + iH^n \mathrm{ch}_0$, the slope stability with respect to a nef divisor H; page 5

$\mathrm{Coh}^\beta_H(X)$, the heart obtained by tilting $\mathrm{Coh}(X)$ with respect to μ_H at the slope β; page 5

$\mathcal{T}_H^{>0} := \{T \in \mathrm{Coh}(X) \mid \mu_H(S) > 0 \text{ for all } T \twoheadrightarrow S\}$, the torsion part with respect to μ_H; page 5

$\mathcal{F}_H^{\leq 0} := \{F \in \mathrm{Coh}(X) \mid \mu_H(G) \leq 0 \text{ for all } 0 \neq G \hookrightarrow F\}$, the torsion-free part with respect to μ_H; page 5

$\pi\colon \widetilde{X} \longrightarrow X$, the crepant resolution of a surface X with an ADE singularity x_0; page 5

C_i, $i = 1, \dots, n$, the exceptional (-2)-curves of π; page 5

$\Pi := \pi^{-1}(x_0) = \bigcup_i C_i$, the scheme-theoretic preimage of the singular point x_0; page 5

${}^{-1}\mathrm{Per}(\widetilde{X}/X)$, the category of perverse coherent sheaves on $\widetilde{X}$ over X; page 5

$\mathcal{T}_P := \{T \in \mathrm{Coh}(\widetilde{X}) \mid \mathbf{R}^1\pi_* T = 0, \mathrm{Hom}(T, E) = 0 \text{ for any coherent sheaf } E \text{ with } \mathbf{R}\pi_* E = 0\}$, the torsion part defining ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$; page 5

$\mathcal{F}_P := \{F \in \mathrm{Coh}(\widetilde{X}) \mid \mathbf{R}^0\pi_* F = 0\}$, the torsion-free part defining ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$; page 6

$\mu_{\pi^*H}(E)$, the slope of E with respect to π^*H on $\mathrm{Coh}(X)$ and ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$; page 6

$\mathcal{T}^{>0}_{\pi^*H,P} := \{T \in {}^{-1}\mathrm{Per}(\widetilde{X}/X) \mid \mu_{\pi^*H}(S) > 0 \text{ for all } T \twoheadrightarrow S\}$, the torsion part on ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$ defining $\mathcal{B}^0$; page 6

$\mathcal{F}^{\leq 0}_{\pi^*H,P} := \{F \in {}^{-1}\mathrm{Per}(\widetilde{X}/X) \mid \mu_{\pi^*H}(G) \leq 0 \text{ for all } 0 \neq G \hookrightarrow F\}$, the torsion-free part on ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$ defining $\mathcal{B}^0$; page 6

$\mathcal{F}_0 := \mathcal{F}_P$, viewed as a subcategory of $\mathrm{Coh}^0_{\pi^*H}(\widetilde{X})$; page 6

$\mathcal{T}_0 := \{T \in \mathrm{Coh}^0(\widetilde{X}) \mid \mathrm{Hom}(T, \mathcal{F}_0) = 0\}$, the left orthogonal complement of $\mathcal{F}_0$ in $\mathrm{Coh}^0_{\pi^*H}(\widetilde{X})$; page 6

$H^i_{\mathcal{A}}$, the i-th cohomology with respect to $\mathcal{A}$. If $\mathcal{A} = \mathrm{Coh}(\widetilde{X})$, we omit the subscript $\mathcal{A}$; if $\mathcal{A} = {}^{-1}\mathrm{Per}(\widetilde{X}/X)$, we write H^i_P; page 7

$\mathcal{B}^0 = \mathcal{B}^0_{\pi^*H,-1} = \mathcal{B}^0_{\pi^*H,P}$, the heart $\langle \mathcal{F}^{\leq 0}_{\pi^*H,P}[1], \mathcal{T}^{>0}_{\pi^*H,P}\rangle = \langle \mathcal{F}_0[1], \mathcal{T}_0\rangle$; page 9

$\mathcal{T}_{\mathcal{B}^0} := \mathrm{Coh}(\widetilde{X}) \cap \mathcal{B}^0$, the torsion part on $\mathrm{Coh}(\widetilde{X})$ defining $\mathcal{B}^0$; page 9

$\mathcal{F}_{\mathcal{B}^0} := \mathrm{Coh}(\widetilde{X}) \cap \mathcal{B}^0[-1]$, the torsion-free part on $\mathrm{Coh}(\widetilde{X})$ defining $\mathcal{B}^0$; page 9

$Z_{\pi^*H,\beta,z} := -\mathrm{ch}_2 + \beta \cdot \mathrm{ch}_1 + z\,\mathrm{ch}_0 + i(\pi^*H) \cdot \mathrm{ch}_1$, the stability function on $\mathcal{B}^0$; page 12

$Q_{A,B} := \Delta + A(\Im Z_{\pi^*H,\beta,z})^2 + B(\Re Z_{\pi^*H,\beta,z})^2$, the quadratic form on $K_{\mathrm{num}}(\widetilde{X})$ for the support property of $(Z_{\pi^*H,\beta,z}, \mathcal{B}^0)$; page 16

$\sigma_\epsilon := (Z_{\pi^*H,\epsilon\beta,z}, \mathcal{B}^0)$, Bridgeland stability conditions on $\widetilde{X}$ obtained from deformation; page 21

$\mathrm{pr}\colon K_{\mathrm{num}}(\widetilde{X}) \longrightarrow K_{\mathrm{num}}(\widetilde{X})/\ker \pi_*$, the projection between the numerical K-groups; page 21

$\mathbf{L}^i_{\mathcal{A}}\pi^* := H^i_{\mathcal{A}} \circ \mathbf{L}\pi^*$, if $\mathcal{A} = \mathrm{Coh}(\widetilde{X})$, we omit the subscript $\mathcal{A}$; if $\mathcal{A} = {}^{-1}\mathrm{Per}(\widetilde{X}/X)$, we write $\mathbf{L}^i_P\pi^*$; page 22

$Z_{\widetilde{X}} := Z_{\pi^*H,0,z}$, the weak stability function at the end point of the path in $\mathrm{Stab}(\widetilde{X})$; page 23

$\sigma_{\widetilde{X}} := (Z_{\widetilde{X}}, \mathcal{B}^0)$, the weak stability condition lying on the boundary of $\mathrm{Stab}(\widetilde{X})$; page 23

Z_X, the stability function on $\mathrm{Coh}^0_H(X)$ with the compatibility $Z_X \circ \pi_* = Z_{\widetilde{X}}$; page 24

$\sigma_X := (Z_X, \mathrm{Coh}^0_H(X))$, the induced Bridgeland stability condition on X; page 25

$Q_{\widetilde{X}} := \Delta + A(\Im Z_{\widetilde{X}})^2$, the quadratic form on $K_{\mathrm{num}}(\widetilde{X})/\ker\pi_*$ for the support property of $\sigma_{\widetilde{X}}$; page 28

2. Preliminaries

In this section, we review the notions of weak and Bridgeland stability conditions and bounded t-structures.

Let $\mathcal{D}$ be a triangulated category.

Definition 2.1. Let $\mathcal{A}$ be an abelian category. A group homomorphism $Z\colon K(\mathcal{A}) \longrightarrow \mathbb{C}$ is said to be a weak stability function (resp. stability function) if for any $E \in \mathcal{A}$, we have $\Im Z(E) \geq 0$ with $\Im Z(E) = 0 \Rightarrow \Re Z(E) \leq 0$ (resp. $\Re Z(E) < 0$)

Definition 2.2. Fix a finite rank lattice Λ and a surjective group homomorphism $v\colon K(\mathcal{D}) \longrightarrow \Lambda$. A weak stability condition (resp. Bridgeland stability condition) on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{A})$ consisting of a group homomorphism (called the central charge of σ) $Z\colon \Lambda \longrightarrow \mathbb{C}$ and a heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{D}$, such that the following conditions hold:

(a) The composition $Z \circ v\colon K(\mathcal{A}) = K(\mathcal{D}) \longrightarrow \Lambda \longrightarrow \mathbb{C}$ is a weak stability function (resp. stability function) on $\mathcal{A}$. This gives a notion of slope: for any $E \in \mathcal{A}$, we set

$$\mu_\sigma(E) = \mu_Z(E) := \begin{cases} \frac{-\Re Z(E)}{\Im Z(E)}, & \text{if } \Im Z(E) > 0, \\ +\infty, & \text{if } \Im Z(E) = 0, \end{cases} \tag{1}$$

We say that an object $E \in \mathcal{A}$ is σ-semistable if for every nonzero subobject $F \subset E$, we have $\mu_\sigma(F) \leq \mu_\sigma(E)$.

(b) Any object $E \in \mathcal{A}$ admits a Harder-Narasimhan filtration in σ-semistable ones; for any $E \in \mathcal{A}$, there is a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$, such that the quotients $A_i := E_i/E_{i-1}$ are σ-semistable (called the HN factors) and we have an inequality of their slopes $\mu_\sigma(A_1) > \cdots > \mu_\sigma(A_n)$.

(c) (Support property) There exists a quadratic form Q on $\Lambda \otimes \mathbb{R}$ such that $Q|_{\ker Z}$ is negative definite, and $Q(v(E)) \geq 0$, for all σ-semistable $E \in \mathcal{A}$.

Any pair $\sigma = (Z, \mathcal{A})$ satisfying the conditions (a) and (b) is said to be a pre-stability condition.

We now briefly review the theory of torsion pairs and tilting introduced in [HRS96].

Definition 2.3. Given an abelian category $\mathcal{A}$. A torsion pair in $\mathcal{A}$ is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories satisfying the following conditions:

(1) $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$

(2) For every $A \in \mathcal{A}$, there exists a short exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

If we have a torsion pair in an abelian category, we can obtain a tilted heart. More precisely, we have the following result.

Theorem 2.4. [HRS96] *Let $\mathcal{A}$ be the heart of bounded t-structure in $\mathcal{D}$ with a torsion pair $(\mathcal{T}, \mathcal{F})$. The category*

$$\mathcal{A}^\sharp := \{E \in \mathcal{D} \mid H^i(E) = 0 \text{ if } i \neq 0, -1, H^0(E) \in \mathcal{T}, H^{-1}(E) \in \mathcal{F}\}$$

is the heart of a bounded t-structure on $\mathcal{D}$.

In particular, given a notion of stability and a slope, we can construct a torsion pair by the existence of Harder-Narasimhan filtration.

Proposition 2.5. [HRS96] *Given a weak stability condition $\sigma = (Z, \mathcal{A})$, and a real number $\mu \in \mathbb{R}$, we obtain the following two subcategories of $\mathcal{A}$:*

$$\mathcal{T}_\sigma^{>\mu} := \{E \in \mathcal{A} \mid \text{All HN factors } F \text{ of } E \text{ have slopes } \mu_\sigma(F) > \mu\}$$
$$\mathcal{F}_\sigma^{\leq\mu} := \{E \in \mathcal{A} \mid \text{All HN factors } F \text{ of } E \text{ have slopes } \mu_\sigma(F) \leq \mu\}$$

Then the pair $(\mathcal{T}_\sigma^{>\mu}, \mathcal{F}_\sigma^{\leq\mu})$ gives a torsion pair.

Example 2.6. Let X be a projective variety of dimension n and H be a nef divisor. Consider the lattice Λ_H generated by vectors of the form $(H^n \operatorname{ch}_0(E), H^{n-1}\operatorname{ch}_1(E))$ and the natural surjection $v_H \colon K(X) \longrightarrow \Lambda_H$. The pair $(Z_H, \operatorname{Coh}(X))$ given by

$$Z_H := -H^{n-1} \cdot \operatorname{ch}_1 + iH^n \operatorname{ch}_0$$

defines a weak stability condition on $\mathrm{D}^b(X)$ with respect to Λ_H; the quadratic form Q can be chosen to be 0.

This is called the slope stability and we write $\mu_H := \mu_{Z_H}$ for its slope function.

The most common example of heart constructed by tilting is the following:

Definition 2.7. We write $\operatorname{Coh}^\beta_H(X) \subseteq \mathrm{D}^{\mathrm{b}}(\mathrm{X})$ for the tilt of coherent sheaves on a projective variety X with respect to the slope stability Z_H at the slope $\beta \in \mathbb{R}$. The torsion pair defining this tilt is denoted by

$$\mathcal{T}_H^{>0} := \{T \in \operatorname{Coh}(X) \mid \mu_H(S) > 0 \text{ for all } T \twoheadrightarrow S\}$$
$$\mathcal{F}_H^{\leq 0} := \{F \in \operatorname{Coh}(X) \mid \mu_H(G) \leq 0 \text{ for all } 0 \neq G \hookrightarrow F\}$$

3. Construction of hearts

Let X be a singular surface with an ADE singularity x_0 and $\pi \colon \widetilde{X} \longrightarrow X$ be its resolution. Denote the exceptional (-2)-curves by C_i, $i = 1, \ldots, n$, and let $\Pi := \pi^{-1}(x_0) = \bigcup_i C_i$ be the exceptional locus. We now start to construct our Bridgeland stability conditions on $\widetilde{X}$. In this section, we first focus on the heart of a bounded t-structures, and the construction is a generalization of [TX22, Sect. 3, 4].

We will construct a heart $\mathcal{B}^0$ of bounded t-structure by tilting the category ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ of perverse coherent sheaves. It is shown to coincides with a tilt of the heart $\operatorname{Coh}^0(\widetilde{X})$ (see Proposition 2.5).

Let H be an ample divisor on X, and we can then define the slope stability μ_{π^*H} on $\operatorname{Coh}(\widetilde{X})$ with respect to the nef divisor π^*H. We first recall from [VdB04] that ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ is a tilt of $\operatorname{Coh}(\widetilde{X})$ with respect to the torsion pair $(\mathcal{T}_P, \mathcal{F}_P)$, where

$$\mathcal{T}_P := \{T \in \operatorname{Coh}(\widetilde{X}) \mid \mathbf{R}^1\pi_*T = 0, \operatorname{Hom}(T, E) = 0 \text{ for any coherent sheaf } E \text{ with } \mathbf{R}\pi_*E = 0\}$$

$$\mathcal{F}_P := \{F \in \mathrm{Coh}(\widetilde{X}) \mid \mathbf{R}^0\pi_*F = 0\}$$

We then want to tilt the perverse coherent sheaves once more. By abuse of notation, we define the slope stability with respect to π^*H as

$$\mu_{\pi^*H}(E) := \begin{cases} \frac{\mathrm{ch}_1(E)\cdot\pi^*H}{\mathrm{ch}_0(E)}, & \text{when } \mathrm{ch}_0(E) \neq 0 \\ \infty, & \text{when } \mathrm{ch}_0(E) = 0 \end{cases}$$

We need to check that ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$ admits Harder-Narasimhan property with respect to μ_{π^*H}.

Lemma 3.1. *(cf.* [Tod13, Lemma 3.6]*) Any object in* ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$ *admits a Harder-Narasimhan filtration with respect to* μ_{π^*H}*-stability.*

Proof. First, by [VdB04, Prop. 3.2.7, 3.3.1], we see that the category of perverse coherent sheaves is equivalent to a category $\mathrm{Coh}(\mathcal{M})$ of $\mathcal{M}$-modules for a certain sheaf of algebra $\mathcal{M}$ on X, and in particular is Noetherian.

We will use the analogue of [Bri07, Prop. 2.4] for weak stability conditions. Assume the contrary that there exists an infinite sequence $\cdots \subsetneq E_i \subsetneq \cdots \subsetneq E_2 \subsetneq E_1$ with $\mu_{\pi^*H}(E_{i+1}) > \mu_{\pi^*H}(E_i/E_{i+1})$ for all i. Since every perverse coherent sheaf has non-negative ch_0, we may assume $\mathrm{ch}_0(E_i)$ is constant. But then $\mu_{\pi^*H}(E_i/E_{i+1})) = \infty$, which is a contradiction. $\square$

The Harder-Narasimhan property guarantees that the pair of subcategories defined below is a torsion pair on ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$:

$$\mathcal{T}^{>0}_{\pi^*H,P} := \{T \in {}^{-1}\mathrm{Per}(\widetilde{X}/X) \mid \mu_{\pi^*H}(S) > 0 \text{ for all } T \twoheadrightarrow S\}$$

$$\mathcal{F}^{\leq 0}_{\pi^*H,P} := \{F \in {}^{-1}\mathrm{Per}(\widetilde{X}/X) \mid \mu_{\pi^*H}(G) \leq 0 \text{ for all } 0 \neq G \hookrightarrow F\}$$

Therefore, we obtain a double tilted category $\mathcal{B}^0_{\pi^*H,P} = \langle\, \mathcal{F}^{\leq 0}_{\pi^*H,P}[1], \mathcal{T}^{>0}_{\pi^*H,P}\,\rangle$.

On the other hand, note that every sheaf in $\mathcal{F}_P$ is a torsion sheaf and hence lies in $\mathrm{Coh}^0(\widetilde{X}) := \mathrm{Coh}^0_{\pi^*H}(\widetilde{X})$ (see Definition 2.7). Note that $\mathcal{F}_P = \{F \in \mathrm{Coh}(\widetilde{X}) \,|\, \mathbf{R}^0\pi_*F = 0\}$ is also contained in $\mathrm{Coh}^0(\widetilde{X})$. We can then consider the category $\mathcal{F}_0 := \mathcal{F}_P$ and its left orthogonal complement $\mathcal{T}_0 := \{T \in \mathrm{Coh}^0(\widetilde{X}) \mid \mathrm{Hom}(T, \mathcal{F}_0) = 0\}$.

Lemma 3.2. *The pair* $(\mathcal{T}_0, \mathcal{F}_0)$ *is a torsion pair on* $\mathrm{Coh}^0(\widetilde{X})$*.*

Proof. The semiorthogonality is clear. Now consider an object $E \in \mathrm{Coh}^0(\widetilde{X})$. It fits into the distinguished triangle $H^{-1}(E)[1] \longrightarrow E \longrightarrow H^0(E)$.

As $(\mathcal{T}_P, \mathcal{F}_P)$ is a torsion pair on $\mathrm{Coh}(\widetilde{X})$ we obtain a short exact sequence $0 \longrightarrow T_0 \longrightarrow H^0(E) \longrightarrow F_0 \longrightarrow 0$ with $T_0 \in \mathcal{T}_P$ and $F_0 \in \mathcal{F}_P$. Note that although this triangle arises from a short exact sequence in $\mathrm{Coh}(\widetilde{X})$, it also gives short exact sequence in $\mathrm{Coh}^0(\widetilde{X})$. This can be seen as follows:

It suffices to show that $T_0 \in \mathcal{T}^{>0}_{\pi^*H}$ as $\mathcal{T}^{>0}_{\pi^*H} \subseteq \mathrm{Coh}^0(\widetilde{X})$. Assume the contrary that there is a subsheaf $A \subseteq T_0$ such that the quotient T_0/A has negative slope. Then the octahedron axiom gives a short exact sequence $0 \longrightarrow T_0/A \longrightarrow H^0(E)/A \longrightarrow F_0 \longrightarrow 0$. As $\mathbf{R}^0\pi_*F_0 = 0$, F_0 does not contribute to the slope of $H^0(E)/A$, we see that $\mu_{\pi^*H}(H^0(E)/A) = \mu_{\pi^*H}(T_0/A) < 0$, which is a contradiction as $H^0(E) \in \mathcal{T}^{>0}_{\pi^*H}$.

In particular, $H^0(E) \longrightarrow F_0$ is surjective in $\mathrm{Coh}^0(\widetilde{X})$. By the octahedron axiom we have a triangle $C \longrightarrow E \longrightarrow F_0$. As $E \longrightarrow H^0(E) \longrightarrow F_0$ is a surjection in $\mathrm{Coh}^0(\widetilde{X})$, we see that C lies in $\mathrm{Coh}^0(\widetilde{X})$. Finally, it is trivial that $\mathrm{Hom}(C, \mathcal{F}_0) = 0$ and so $C \in \mathcal{T}_0$. $\square$

Denote the tilted heart of $\mathrm{Coh}^0(\widetilde{X})$ with respect to this pair by $\mathcal{B}^0_{\pi^*H,-1}$, and denote the i-th cohomology with respect to a heart $\mathcal{A}$ by $H^i_{\mathcal{A}}$. For simplicity, we write H^i for $H^i_{\mathrm{Coh}(\widetilde{X})}$ and H^i_P for $H^i_{^{-1}\mathrm{Per}(\widetilde{X}/X)}$.

We then compare two double tilted hearts above using the following lemma.

Lemma 3.3. *Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T},\mathcal{F})$. Assume that on the tilted category $\mathcal{A}^\sharp = \langle \mathcal{F}[1], \mathcal{T}\rangle$ we also have another torsion pair $(\mathcal{T}^\sharp, \mathcal{F}^\sharp)$ with $\mathcal{F}^\sharp \subseteq \mathcal{A}$.*

Then the double tilted category $(\mathcal{A}^\sharp)^\sharp$ is the tilt of $\mathcal{A}$ with respect to the torsion pair $(\mathcal{T}\cap\mathcal{T}^\sharp, [\mathcal{F},\mathcal{F}^\sharp])$, where the bracket $[\ ]$ denotes the extension closure.

Proof. It is clear that $(\mathcal{A}^\sharp)^\sharp \subseteq \langle \mathcal{A}, \mathcal{A}[1]\rangle$. By [Pol07, Lemma 1.1.2] the heart $(\mathcal{A}^\sharp)^\sharp$ is a tilt of $\mathcal{A}$, and moreover the torsion part of this tilting is $(\mathcal{A}^\sharp)^\sharp \cap \mathcal{A} = \mathcal{T}^\sharp \cap \mathcal{A} = \mathcal{T}^\sharp \cap \mathcal{T}$.

It remains the show that $(\mathcal{T}\cap\mathcal{T}^\sharp)^\perp = [\mathcal{F},\mathcal{F}^\sharp]$. The inclusion $(\mathcal{T}\cap\mathcal{T}^\sharp)^\perp \supseteq [\mathcal{F},\mathcal{F}^\sharp]$ is clear.

To show that $(\mathcal{T}\cap\mathcal{T}^\sharp)^\perp \subseteq [\mathcal{F},\mathcal{F}^\sharp]$, we consider an object $E \in (\mathcal{T}\cap\mathcal{T}^\sharp)^\perp = (\mathcal{A}^\sharp)^\sharp[-1]\cap\mathcal{A}$. As $E[1] \in (\mathcal{A}^\sharp)^\sharp$, there is a distinguished triangle

$$H^{-1}_{\mathcal{A}^\sharp}(E[1])[1] \longrightarrow E[1] \longrightarrow H^0_{\mathcal{A}^\sharp}(E[1]).$$

Now we look at the long exact cohomology sequence with respect to the heart $\mathcal{A}$. It turns out to be the following exact sequence:

$$0 \longrightarrow H^0_{\mathcal{A}}(H^{-1}_{\mathcal{A}^\sharp}(E[1])) \longrightarrow E \longrightarrow H^{-1}_{\mathcal{A}}(H^0_{\mathcal{A}^\sharp}(E[1])) \longrightarrow 0.$$

It is clear that we have $H^0_{\mathcal{A}}(H^{-1}_{\mathcal{A}^\sharp}(E[1])) = H^{-1}_{\mathcal{A}^\sharp}(E[1]) \in \mathcal{F}^\sharp$ and $H^{-1}_{\mathcal{A}}(H^0_{\mathcal{A}^\sharp}(E[1])) \in \mathcal{F}$ and the assertion follows. ☐

Proposition 3.4. *These two hearts coincide, that is, $\mathcal{B}^0_{\pi^*H,P} = \mathcal{B}^0_{\pi^*H,-1}$.*

Proof. Note first that $\mathcal{F}^{\leq 0}_{\pi^*H,P} \subset \mathrm{Coh}(\widetilde{X})$. Indeed, for $F \in \mathcal{F}^{\leq 0}_{\pi^*H,P} \subseteq {}^{-1}\,\mathrm{Per}(\widetilde{X}/X)$, we consider the distinguished triangle $H^{-1}(F)[1] \longrightarrow F \longrightarrow H^0(F)$. As $H^{-1}(F) \in \mathcal{F}_P$, it is either 0 or a torsion coherent sheaf and hence has slope $+\infty$, but as $F \in \mathcal{F}^{\leq 0}_{\pi^*H,P}$ there cannot be such a torsion subobject. We then see that $H^{-1}(F) = 0$, and hence the assumptions of Lemma 3.3 are all satisfied.

Applying Lemma 3.3 twice, we see that both $\mathcal{B}^0_{\pi^*H,P}$ and $\mathcal{B}^0_{\pi^*H,-1}$ are obtained from $\mathrm{Coh}(\widetilde{X})$ by a single tilt. As a torsion pair is determined by its torsion part, it suffices to prove that the torsion parts of $\mathcal{B}^0_{\pi^*H,P}$ and $\mathcal{B}^0_{\pi^*H,-1}$ as tilts of $\mathrm{Coh}(\widetilde{X})$ are the same, that is, $\mathcal{B}^0_{\pi^*H,P} \cap \mathrm{Coh}(\widetilde{X}) = \mathcal{B}^0_{\pi^*H,-1} \cap \mathrm{Coh}(\widetilde{X})$.

Given $E \in \mathcal{B}^0_{\pi^*H,P} \cap \mathrm{Coh}(\widetilde{X})$, we first consider the exact triangle $F[1] \longrightarrow E \longrightarrow T$ with $F \in \mathcal{F}^{\leq 0}_{\pi^*H,P}$ and $T \in \mathcal{T}^{>0}_{\pi^*H,P}$ given by the definition of $\mathcal{B}^0_{\pi^*H,P}$. Taking the long exact cohomology sequence with respect to $\mathrm{Coh}(\widetilde{X})$,we see that $E = H^0(E) = H^0(T) = T$ and $F = 0$.

By Lemma 3.3, we know that $\mathcal{B}^0_{\pi^*H,-1} \cap \mathrm{Coh}(\widetilde{X}) = \mathcal{T}_0 \cap \mathcal{T}^0_{\pi_*H}$. It is clear that $E \in \mathcal{T}_0$ and we claim that $E \in \mathcal{T}^0_{\pi_*H}$, that is, there does not exist any quotient of E in $\mathrm{Coh}(\widetilde{X})$ with non-positive slope. Assume the contrary that there is a quotient $f\colon E \twoheadrightarrow Q$ in $\mathrm{Coh}(\widetilde{X})$ with $\mu_{\pi^*H}(Q) \leq 0$. Note that the torsion part $\mathcal{T}_P$ is closed under quotient so $Q \in \mathcal{T}_P \subseteq {}^{-1}\,\mathrm{Per}(\widetilde{X}/X)$.

Let $K := \ker f$ and consider the long exact cohomology sequence of $K \longrightarrow E \longrightarrow Q$ with respect to ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$. This gives us an exact sequence

$$0 \longrightarrow H_P^0(K) \longrightarrow E \longrightarrow Q \longrightarrow H_P^1(K) \longrightarrow 0.$$

As $H_P^1(K) \in \mathcal{F}_P$, it has no contribution to the slope, we have $\mu_{\pi^*H}(\operatorname{Im} H_P^0(f)) = \mu_{\pi^*H}(Q) \leq 0$, which is a contradiction as T cannot have a quotient in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ with non-positive slope.

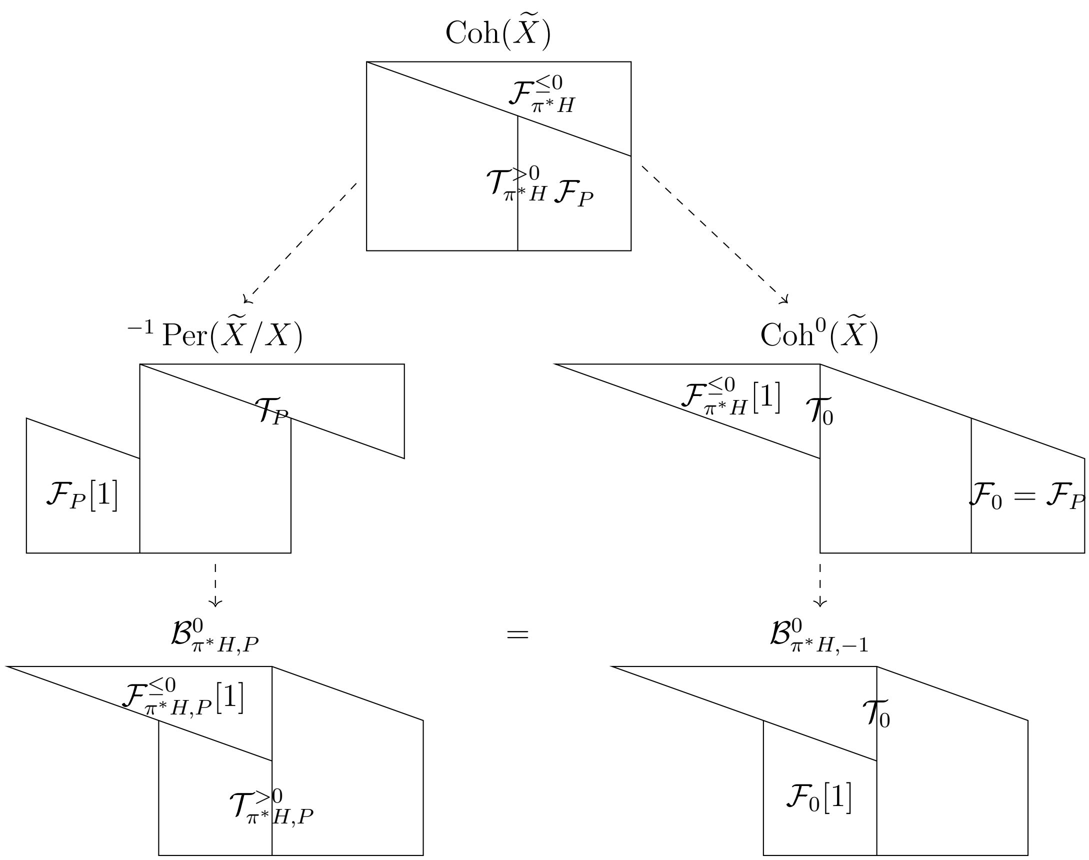
Conversely, given $E \in \mathcal{B}^0_{\pi^*H,-1} \cap \operatorname{Coh}(\widetilde{X})$, we consider similarly the exact triangle $F[1] \longrightarrow E \longrightarrow T$ with $F \in \mathcal{F}^{\leq 0}_{\pi^*H}$ and $T \in \mathcal{T}^{>0}_{\pi^*H}$ given by the definition of $\mathcal{B}^0_{\pi^*H,-1}$. Taking the long exact cohomology sequence with respect to $\operatorname{Coh}(\widetilde{X})$ tells us again that $E = T \in$ and $F = 0$. It then suffices to show that there does not exist a quotient of E in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ with non-positive slope.

Assume the contrary that there is a quotient $f\colon E \twoheadrightarrow Q$ in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ with $\mu_{\pi^*H}(Q) \leq 0$. Let $K := \ker f$ and consider the long exact cohomology sequence of $K \longrightarrow E \longrightarrow Q$ with respect to $\operatorname{Coh}(\widetilde{X})$. This gives us an exact sequence

$$0 \longrightarrow H^{-1}(Q) \longrightarrow K \longrightarrow E \longrightarrow H^0(Q) \longrightarrow 0.$$

Let Z be the central charge of slope stability on $\operatorname{Coh}(\widetilde{X})$. Then $Z(H^0(Q)) = Z(E) - [Z(K) - Z(H^{-1}(Q))] = Z(E) - Z(K)$, and hence $\mu_{\pi^*H}(H^0(Q)) = \mu_{\pi^*H}(Q) \leq 0$, which is a contradiction as E cannot have such a quotient in $\operatorname{Coh}(\widetilde{X})$. $\square$

The result above can be understood via the picture below.



From now on we write $\mathcal{B}^0 = \mathcal{B}^0_{\pi^*H,-1} = \mathcal{B}^0_{\pi^*H,P}$ for this heart, and $\mathcal{T}_{\mathcal{B}^0} := \mathrm{Coh}(\widetilde{X}) \cap \mathcal{B}^0$ and $\mathcal{F}_{\mathcal{B}^0} := \mathrm{Coh}(\widetilde{X}) \cap \mathcal{B}^0[-1]$ for its torsion part and torsion-free part respectively.

The same argument in Lemma 3.4 proves the following fact:

Lemma 3.5. *$\mathcal{F}^{\leq \alpha}_{\pi^*H,P} \subseteq \mathcal{F}^{\leq \alpha}_{\pi^*H}$ for every $\alpha \in \mathbb{R}$.*

Proof. Let $F \in \mathcal{F}^{\leq 0}_{\pi^*H,P}$. If $H^{-1}(F) \neq 0$, then $H^{-1}(F) \subseteq F$ would be a subobject with $\mu_{\pi^*H}(H^{-1}(F)) = +\infty$, and therefore $F = H^0(F)$ is a coherent sheaf.

Assume the contrary that there is a subsheaf $G \subseteq F$ with $\mu_{\pi^*H}(G) > \alpha$. Consider the following exact sequence:

$$0 \longrightarrow H^0_P(G) \longrightarrow F \longrightarrow F/G \longrightarrow H^1_P(G) \longrightarrow 0.$$

Let $Q := \ker(F/G \longrightarrow H^1_P(G)$. Then since $H^1_P(G) \in \mathcal{F}_P$ has no contribution to the slope, there is a subobject $H^0_P(G)$ of F in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ with $\mu_{\pi^*H}(H^0_P(G)) = \mu_{\pi^*H}(G) > \alpha$, which is a contradiction. □

Remark 3.6. The inclusion in Lemma 3.5 above is strict. For example, let $\alpha = 0$ and let $\mathcal{I}_x$ be the ideal sheaf of a point $x \in C_1$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{I}_x \arrow[r, hook] \arrow[d, dotted] & \mathcal{O}_{\widetilde{X}} \arrow[r, two heads] \arrow[d, two heads] & \mathcal{O}_x \\
\mathcal{O}_{C_1}(-1) \arrow[r, hook] & \mathcal{O}_{C_1} \arrow[ur, two heads] &
\end{tikzcd}$$

Then, the sheaf $\mathcal{I}_x$ admits a nonzero morphism to $\mathcal{O}_{C_1}(-1)$ so $\mathcal{I}_x \notin \mathcal{T}_P$. Therefore, $\mathcal{I}_x \notin {}^{-1}\operatorname{Per}(\widetilde{X}/X)$ and hence can not lie in $\mathcal{F}^0_{\pi^*H,P}$.

In the remaining of this section, we will classify the simple objects in $\mathcal{B}^0$ supporting on the exceptional locus Π.

We will need the following lemmas about the pushforward on the numerical K-group. For the numerical K-group of the singular surface X, we defined $K_{\mathrm{num}}(X)$ by the quotient of $K(\mathrm{D^b(X)})$ by the right kernel of the Euler characteristic $\chi \colon K(\mathrm{D^{perf}(X)}) \times \mathrm{K(D^b(X))} \longrightarrow \mathbb{Z}$.

Lemma 3.7. *The kernel of the pushforward $\pi_* \colon K_{\mathrm{num}}(\widetilde{X}) \longrightarrow K_{\mathrm{num}}(X)$ is exactly generated by $\mathcal{O}_{C_i}(-1)$ for $i = 1, \ldots, n$.*

Proof. We have $\operatorname{rank} K_{\mathrm{num}}(X) = \rho(X) + 2$ and $\operatorname{rank} K_{\mathrm{num}}(\widetilde{X}) = \rho(\widetilde{X}) + 2$. As $\pi_*\mathcal{O}_{C_i} = 0$, we know that the kernel of π_* contains the classes $[\mathcal{O}_{C_i}(-1)]$ for all i. Moreover, the pushforward π_* on the numerical Grothendieck group $K_{\mathrm{num}}(\widetilde{X})$ has kernel of rank n.

This can be seen as follows. We know that the difference of ranks of Neron-Severi group $\rho(\widetilde{X}) - \rho(X)$ is just n, exactly given by the exceptional curves $C_1, \ldots, C_n$.

Therefore, we just need to note that π_* is surjective on the numerical K-group; this holds as any sheaf supported on x_0 is equivalent to $m\mathcal{O}_{x_0}$ for some m in $K_{\mathrm{num}}(X)$. □

Remark 3.8. One may also note that by projection formula, $\mathbf{R}\pi_*$ is essentially surjective on $\mathrm{D}_{qco}(\widetilde{X})$. This together with the formula $\mathbf{R}\pi_*\mathbf{L}\pi^* = \mathrm{id}_{\mathrm{D_{qco}}(X)}$ shows that the functor $\mathbf{L}\pi^* \colon \mathrm{D_{qco}}(X) \longrightarrow \mathrm{D}_{qco}(\widetilde{X})$ is fully faithful and induces a semiorthogonal decomposition $\mathrm{D}_{qco}(\widetilde{X}) = \langle \ker \mathbf{R}\pi_*, \mathrm{D}_{qco}(X) \rangle$.

Lemma 3.9. [Bri02, Lemma 3.2] *For $E \in \mathrm{D}(\widetilde{\mathrm{X}})$, $E \in {}^{-1}\operatorname{Per}(\widetilde{X}/X)$ if and only if E satisfies the conditions below.*

(i) $H^i(E) = 0$ *for* $i \neq 0, -1$.
(ii) $\mathbf{R}^1\pi_* H^0(E) = 0$ *and* $\mathbf{R}^0\pi_* H^{-1}(E) = 0$.
(iii) $\operatorname{Hom}_X(H^0(E), F) = 0$ *for any sheaf* F *on* $\widetilde{X}$ *with* $\mathbf{R}\pi_*(F) = 0$.

The above result by Bridgeland implies that $\mathbf{R}\pi_*$ induces an exact functor of abelian categories $\mathbf{R}\pi_* \colon {}^{-1}\operatorname{Per}(\widetilde{X}/X) \longrightarrow \operatorname{Coh}(X)$. Moreover, as the derived pushforward $\mathbf{R}\pi_* \colon \mathrm{D}^b(\widetilde{X}) \longrightarrow \mathrm{D}^b(X)$ is an exact triangulated functor, the assertion is equivalent to the t-exactness, i.e $\mathbf{R}\pi_*({}^{-1}\operatorname{Per}(\widetilde{X}/X)) \subseteq \operatorname{Coh}(X)$. Recall that given $E \in {}^{-1}\operatorname{Per}(\widetilde{X}/X)$, to show $\mathbf{R}\pi_*(E) \in \operatorname{Coh}(X)$, it suffices to show that $\mathbf{R}\pi_*(H^0(E))$ and $\mathbf{R}\pi_*(H^{-1}(E)[1])$ are both in $\operatorname{Coh}(X)$.

Remark 3.10. Lemma 3.9 implies that if G is a coherent sheaf on X, $\mathbf{L}^0\pi^*G := H^0_{\operatorname{Coh}(\widetilde{X})}(\mathbf{L}\pi^*G)$ is a perverse coherent sheaf. This can be seen by Lemma 3.9: conditions (i),(ii) are clear, and for (iii), let F be a sheaf with $\mathbf{R}\pi_*F = 0$. Then $\operatorname{Hom}(\mathbf{L}^0\pi^*G, F) = \operatorname{Hom}(\mathbf{L}\pi^*G, F) = \operatorname{Hom}(G, \mathbf{R}\pi_*F) = 0$; here the first equality holds because $\operatorname{Hom}(H^{-i}(\mathbf{L}\pi^*G)[i], F)$ are negative Ext and hence are 0. This particularly implies that the sheaf $\mathcal{O}_\Pi$ is a perverse coherent sheaf.

We also need the following lemma about simple objects in atilted category.

Lemma 3.11. *Let E be a simple object in an abelian category $\mathcal{A}$. Assume that $\mathcal{A}$ admits a torsion pair $(\mathcal{T}, \mathcal{F})$.*

(i) *If $E \in \mathcal{T}$, so $E \in \mathcal{A}^\sharp = \langle \mathcal{F}[1], \mathcal{T} \rangle$, and if there is no short exact sequence of the form*
$$0 \longrightarrow F \longrightarrow T \longrightarrow E \longrightarrow 0$$
with $0 \neq F \in \mathcal{F}$ and $T \in \mathcal{T}$, then E is a simple object in $\mathcal{A}^\sharp$.
(ii) *If $E \in \mathcal{F}$, so $E[1] \in \mathcal{A}^\sharp = \langle \mathcal{F}[1], \mathcal{T} \rangle$, and if there is no short exact sequence of the form*
$$0 \longrightarrow E \longrightarrow F \longrightarrow T \longrightarrow 0$$
with $F \in \mathcal{F}$ and $0 \neq T \in \mathcal{T}$, then $E[1]$ is a simple object in $\mathcal{A}^\sharp$.

Proof. We will prove (i) as (ii) is similar. Assume that there is a short exact sequence $0 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 0$ in $\mathcal{A}^\sharp$. The long exact cohomology sequence with respect to the heart $\mathcal{A}$ is of the form
$$0 \longrightarrow H^{-1}_{\mathcal{A}}(Q) \longrightarrow K \xrightarrow{f} E \longrightarrow H^0_{\mathcal{A}}(Q) \longrightarrow 0$$
.

As E is simple in $\mathcal{A}$, either $\operatorname{Im} f = 0$ or $= E$. In the former case, we have $H^{-1}_{\mathcal{A}}(Q) = K$, which means that they are both 0 as $K \in \mathcal{T}$ and $H^{-1}_{\mathcal{A}}(Q) \in \mathcal{F}$. In the latter case, $H^0_{\mathcal{A}}(Q) = 0$ and we obtain the short exact sequence
$$0 \longrightarrow H^{-1}_{\mathcal{A}}(Q) \longrightarrow K \longrightarrow E \longrightarrow 0.$$
□

Our claim is the following:

Proposition 3.12. *The only simple objects in $\mathcal{B}^0$ supported on the exceptional locus Π are $\mathcal{O}_{C_i}(-1)[1]$ and $\mathcal{O}_\Pi$.*

We separate the proof of this proposition into two parts.

Lemma 3.13. *$\mathcal{O}_{C_i}(-1)[1]$ and $\mathcal{O}_\Pi$ are simple objects in $\mathcal{B}^0$.*

Proof. For the first assertion, we consider a short exact sequence $0 \longrightarrow K \longrightarrow \mathcal{O}_{C_i}(-1)[1] \longrightarrow Q \longrightarrow 0$ in $\mathcal{B}^0$. As $\mathcal{O}_{C_i}(-1)$ is in $\ker \mathbf{R}\pi_*$ and $\mathbf{R}\pi_*$ is t-exact, we see that $\mathbf{R}\pi_* K = \mathbf{R}\pi_* Q = 0$.

Observe that for any $F \in \mathcal{B}^0$ with $\mathbf{R}\pi_* F = 0$, we always have that $H^0(F) = 0$. This is because $\mathbf{R}\pi_* H^0(F) = 0$, but as $H^0(F)$ is in the torsion part $\mathcal{T}_0$, it can only be 0. We then see by the long exact cohomology sequence that $H^{-1}(K) = K[-1]$ is a subsheaf of $\mathcal{O}_{C_i}(-1)$. But as K is also in the kernel of $\mathbf{R}\pi_*$, K is either 0 or exactly $\mathcal{O}_{C_i}(-1)[1]$.

For the second assertion, we will apply Lemma 3.11(i) to the object $\mathcal{O}_\Pi \in {}^{-1}\operatorname{Per}(\widetilde{X}/X)$, viewing $\mathcal{B}^0 = \mathcal{B}^0_{\pi^*H,P} = \langle \mathcal{F}^{\leq 0}_{\pi^*H,P}[1], \mathcal{T}^{>0}_{\pi^*H,P} \rangle$. We first need to verify that $\mathcal{O}_\Pi$ is simple in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$. We consider a short exact sequence $0 \longrightarrow K \longrightarrow \mathcal{O}_\Pi \longrightarrow Q \longrightarrow 0$ in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$. As $\mathbf{R}\pi_*\mathcal{O}_\Pi = \mathcal{O}_{x_0}$ is simple in $\operatorname{Coh}(X)$, by the t-exactness of $\mathbf{R}\pi_*$ we know that either $\mathbf{R}\pi_* K = 0, \mathbf{R}\pi_* Q = \mathcal{O}_{x_0}$ or $\mathbf{R}\pi_* Q = 0, \mathbf{R}\pi_* K = \mathcal{O}_{x_0}$.

By the long exact cohomology sequence we have $H^{-1}(K) = 0$, so K is a coherent sheaf and hence $K \in {}^{-1}\operatorname{Per}(\widetilde{X}/X) \cap \operatorname{Coh}(\widetilde{X}) = \mathcal{T}_P$. However, in the former case, $\mathbf{R}\pi_* K = 0$ implies that $K \in \mathcal{F}_P$, then K can only be 0.

In the latter case, we assume that Q is nonzero. Since $\mathbf{R}\pi_* Q = 0$, the t-exactness tells us that $\mathbf{R}\pi_* H^0(Q) = 0$, we know that it belongs to both $\mathcal{F}_P$ and $\mathcal{T}_P$ and hence it must be 0. Now the long exact cohomology sequence gives the short exact sequence $0 \longrightarrow H^{-1}(Q) \longrightarrow H^0(K) \longrightarrow \mathcal{O}_\Pi \longrightarrow 0$ in $\operatorname{Coh}(\widetilde{X})$. Consider the truncation distinguished triangle $\tau^{\leq -1}\mathbf{L}\pi^*\mathcal{O}_{x_0} \longrightarrow \mathbf{L}\pi^*\mathcal{O}_{x_0} \longrightarrow \tau^{\geq 0}\mathbf{L}\pi^*\mathcal{O}_{x_0} = \mathcal{O}_\Pi$ and apply the functor $\operatorname{Hom}(\bullet, H^{-1}(Q)[1])$.

By adjunction we see that $\operatorname{Hom}(\mathbf{L}\pi^*\mathcal{O}_{x_0}, H^{-1}(Q)[1]) = 0$ since $\mathbf{R}\pi_* H^{-1}(Q) = 0$. It is also clear that $\operatorname{Hom}(\tau^{\leq -1}\mathbf{L}\pi^*\mathcal{O}_{x_0}, H^{-1}(Q)) = 0$; thus we have $\operatorname{Ext}^1(\mathcal{O}_\Pi, H^{-1}(Q)) = 0$, which means that $K = Q[-1] \oplus \mathcal{O}_\Pi$ but this is not in the heart ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$, a contradiction.

It remains to check that there can not be a short exact sequence of the form

$$0 \longrightarrow F \longrightarrow T \longrightarrow \mathcal{O}_\Pi \longrightarrow 0,$$

with $F \in \mathcal{F}^{\leq 0}_{\pi^*H,P}$ and $T \in \mathcal{T}^{>0}_{\pi^*H,P}$. This is trivial as $\mathcal{O}_\Pi$ has no contribution to the slope function μ_{π^*H} on perverse coherent sheaves. $\square$

Lemma 3.14. *$\mathcal{O}_{C_i}(-1)[1]$ and $\mathcal{O}_\Pi$ are the only simple objects in $\mathcal{B}^0$ supported on the exceptional locus Π.*

Proof. Given a simple object E in $\mathcal{B}^0$ with $\operatorname{supp} E = \Pi$, we know that E must be equal to either $H^0(E)$ or $H^{-1}(E)[1]$.

We first deal with the case that $E = H^0(E)$. Note that $\mathbf{R}\pi_* E$ can not be 0 otherwise $E = 0$, so there exists a nonzero morphism $\mathcal{O}_{x_0} \longrightarrow \mathbf{R}\pi_* E$, which induces by adjunction a nonzero morphism $\mathbf{L}\pi^*\mathcal{O}_{x_0} \longrightarrow E$. Then, applying the functor $\operatorname{Hom}(\bullet, E)$ to the distinguished triangle $\tau^{\leq -1}\mathbf{L}\pi^*\mathcal{O}_{x_0} \longrightarrow \mathbf{L}\pi^*\mathcal{O}_{x_0} \longrightarrow \mathcal{O}_\Pi$, we get a nonzero morphism $\mathcal{O}_\Pi \longrightarrow E$. As they are both simple, this must be an isomorphism.

Now for the case that $E = H^{-1}(E)[1]$, the assertion follows directly from the lemma below. $\square$

Lemma 3.15. *Given a coherent sheaf F with $\mathbf{R}\pi_* F = 0$, then F is an extension of the sheaves $\mathcal{O}_{C_i}(-1)$.*

Proof. Note first that $\mathrm{ch}_1(F) = a_1C_1 + \cdots a_nC_n$ with all $a_i \geq 0$. By induction on $\sum_i a_i$, it is enough to show that either $\mathrm{Hom}(\mathcal{O}_{C_i}(-1), F)$ or $\mathrm{Hom}(F, \mathcal{O}_{C_i}(-1))$ is nonzero for some i, as $\mathcal{O}_{C_i}(-1)$ are simple in the category $\ker \mathbf{R}\pi_* \cap \mathrm{Coh}(\widetilde{X})$.

Assume the contrary that for all i, we have $\mathrm{Hom}(\mathcal{O}_{C_i}(-1), F) = 0 = \mathrm{Hom}(F, \mathcal{O}_{C_i}(-1))$. By adjunction formula we can easily compute that the restrictions of canonical divisor $K_{\widetilde{X}}$ on the curves C_i are trivial for all i. Therefore, by Serre duality we have that $\mathrm{Ext}^2(F, \mathcal{O}_{C_i}(-1)) = 0$.

Hence, for every i, $\chi(F, \mathcal{O}_{C_i}(-1)) \leq 0$, which by Hirzebruch-Riemann-Roch implies that $\mathrm{ch}_1(F) \cdot C_i \geq 0$. However, since the intersection matrix of the exceptional curves is negative definite (see, for example, [Rei97]), we have

$$0 > \mathrm{ch}_1(F)^2 = \sum_i a_i(\mathrm{ch}_1(F) \cdot C_i) \geq 0,$$

which is a contradiction. $\square$

Then, Lemma 3.13 and 3.14 combine to prove Proposition 3.12.

4. Central charges

In this section we will show that the central charge $Z_{\pi^*H,\beta,z}$ is a stability function on $\mathcal{B}^0$ and that it satisfies the Harder-Narasimhan property under some suitable assumptions. This extends the results of [TX22] to resolutions of ADE singularities and fixes some gaps in the original proof.

Let $z \in \mathbb{R}$ and $\beta \in \mathrm{NS}_{\mathbb{R}}(\widetilde{X})$ be a class with $(\pi^*H) \cdot \beta = 0$. On $\mathrm{D}^b(\widetilde{X})$ we define a function

$$Z_{\pi^*H,\beta,z} := -\mathrm{ch}_2 + \beta \cdot \mathrm{ch}_1 + z\,\mathrm{ch}_0 + i(\pi^*H) \cdot \mathrm{ch}_1$$

We first prove that it is a stability function on $\mathcal{B}^0$. When x_0 is an A_1 singularity, this is a special case of [TX22, Lemma 5.2].

Lemma 4.1. *The function $Z_{\pi^*H,\beta,z}$ is a stability function on $\mathcal{B}^0$, when z and β are chosen such that $\beta \cdot C_i > 0$ for all i, $\beta \cdot \mathrm{ch}_1(\mathcal{O}_\Pi) < 1$, and $z > -\frac{\beta^2}{2}$.*

Proof. Given $E \in \mathcal{B}^0$, we need to prove that $\Im Z_{\pi^*H,\beta,z}(E) \geq 0$ and that if $\Im Z_{\pi^*H,\beta,z}(E) = 0$ then $\Re Z_{\pi^*H,\beta,z}(E) < 0$.

Such E fits into a distinguished triangle $F[1] \longrightarrow E \longrightarrow T$ for some $F \in \mathcal{F}_0$ and $T \in \mathcal{T}_0$. As $(\pi^*H) \cdot C_i = 0$ for all i and F is supported on these exceptional curves, we have $\Im Z_{\pi^*H,\beta,z}(E) = \Im Z_{\pi^*H,\beta,z}(T) = \Im Z_{\pi^*H,\beta,z}(H^0(T)) - \Im Z_{\pi^*H,\beta,z}(H^{-1}(T))$. By the construction of $\mathrm{Coh}^0(\widetilde{X})$, we know that $\Im Z_{\pi^*H,\beta,z}(H^0(T)) \geq 0$ and that $\Im Z_{\pi^*H,\beta,z}(H^{-1}(T)) \leq 0$.

To prove the second assertion, we assume that $\Im Z_{\pi^*H,\beta,z}(E) = 0$. Notice that this is equivalent to $\Im Z_{\pi^*H,\beta,z}(F) = \Im Z_{\pi^*H,\beta,z}(H^{-1}(T) = \Im Z_{\pi^*H,\beta,z}(H^0(T)) = 0$.

Therefore, it suffices to prove that in $\mathcal{B}^0$, if $A[1] \in \mathcal{F}_0[1], B \in \mathcal{T}^{>0}_{\pi^*H}, C[1] \in \mathcal{F}^{\leq 0}_{\pi^*H}[1]$, with $\Im Z_{\pi^*H,\beta,z}(A[1]) = \Im Z_{\pi^*H,\beta,z}(B) = \Im Z_{\pi^*H,\beta,z}(C[1]) = 0$, we have $\Re Z_{\pi^*H,\beta,z}(A[1]), \Re Z_{\pi^*H,\beta,z}(B)$, and $\Re Z_{\pi^*H,\beta,z}(C[1])$ are all negative.

For $A \in \mathcal{F}_0 = \mathcal{F}_P$, we have $\mathrm{ch}_0(A) = 0$, so we only need to show that

$$-\mathrm{ch}_2(A[1]) + \beta \cdot \mathrm{ch}_1(A[1]) < 0.$$

As $\mathbf{R}^0\pi_*A = 0$, we have $\mathbf{R}\pi_*A = \mathbf{R}^1\pi_*A[-1]$ and hence $[\mathbf{R}\pi_*A] = -n[x_0]$ for some $n \geq 0$. Then since x_0 is an ADE singularity, C_i are K-trivial and the relative Todd class is 0. Now by Grothendieck-Riemann-Roch we have $-n = \mathrm{ch}_2(\mathbf{R}\pi_*A) = \pi_*\mathrm{ch}_2(A)$. Since π_*

has no effect on ch_2 on surfaces we see that $-\mathrm{ch}_2(A[1]) = \mathrm{ch}_2(A) = -n \leq 0$. Also, as $\mathrm{ch}_1(A[1]) = -\sum_i (a_i C_i)$ for some $a_i \geq 0$ and $\beta \cdot C_i > 0$ for all i, we in summary see that $\Re Z_{\pi^* H,\beta,z}(A) > 0$.

For $B \in \mathcal{T}^{>0}_{\pi^* H}$ with $\Im Z_{\pi^* H,\beta,z}(B) = 0$, we have $\mathrm{ch}_0(B) = 0$, and hence B is supported either on points or on Π. If B is supported on points then it is automatically that $\Re Z_{\pi^* H,\beta,z}(B) < 0$; therefore we may assume that B is supported on Π and write $\mathrm{ch}_1(B) = \sum_i (b_i C_i)$ for some $b_i \geq 0$. Our assumption is now $\sum_i m_i(\beta \cdot C_i) < 1$, where m_i is the multiplicities of C_i in the scheme-theoretic preimage Π. Then $-\mathrm{ch}_2(B) + \beta \cdot \mathrm{ch}_1(B) = -\mathrm{ch}_2(B) + \sum_i (b_i \beta \cdot C_i) = -\mathrm{ch}_2(B) + \sum_i \frac{b_i}{m_i} m_i(\beta \cdot C_i) < 0$ as $\mathrm{ch}_2(B)$ is the length of $\mathbf{R}^0\pi_* B$ and hence $\geq \max_i \{\frac{b_i}{m_i}\}$.

For $C[1] \in \mathcal{F}^0_{\pi_* H}[1]$ with $\Im Z_{\pi^* H,\beta,z}(C) = 0$, the sheaf C must be slope semistable. We can therefore use Bogomolov-Gieseker inequality and Hodge index theorem to see that $z > -\frac{\beta^2}{2}$ implies $\Re Z_{\pi^* H,\beta,z}(C[1]) < 0$ (cf. [TX22, Lemma 5.2]). Indeed,

$$\begin{aligned}
\Re Z_{\pi^* H,\beta,z}(C) &= z\,\mathrm{ch}_0(C) + \beta \cdot \mathrm{ch}_1(C) - \mathrm{ch}_2(C) \\
&\geq \mathrm{ch}_0(C)\{z + \frac{\beta \cdot \mathrm{ch}_1(C)}{\mathrm{ch}_0(C)} - \frac{\mathrm{ch}_1^2(C)}{\mathrm{ch}_0^2(C)}\}(\text{Bogomolov-Giesker}) \\
&= \mathrm{ch}_0(C)\{z - \frac{(\mathrm{ch}_1(C) - \mathrm{ch}_0(C)\beta)^2}{2\,\mathrm{ch}_0^2(C)} + \frac{\beta^2}{2}\}.
\end{aligned}$$

We know that $(\mathrm{ch}_1(C) - \mathrm{ch}_0(C)\beta)^2 \leq 0$ by Hodge index theorem, and that $z + \frac{\beta^2}{2} > 0$, so $\Re Z_{\pi^* H,\beta,z}(C[1]) < 0$, which completes the proof. $\square$

Remark 4.2. In the constructions, we may instead choose to tilt $\mathrm{Coh}(\widetilde{X})$ and ${}^{-1}\mathrm{Per}(\widetilde{X}/X)$ at any slope $\alpha \in \mathbb{R}$. The same argument in Section 3 will give us a double tilted heart $\mathcal{B}^\alpha$.

To define a central charge on this heart, we may choose

$$Z^\alpha_{\pi^* H,\beta,z}(E) := -\mathrm{ch}_2(E) + \beta \cdot \mathrm{ch}_1(E) + z\,\mathrm{ch}_0(E) + i[(\pi^* H) \cdot \mathrm{ch}_1(E) - \alpha\,\mathrm{ch}_0(E)].$$

The same proof works to show that it is a stability function if the assumption $z > -\frac{\beta^2}{2}$ is replaced with $z - \frac{\alpha^2}{2(\pi^* H)^2} > -\frac{\beta^2}{2}$.

Before proving the Harder-Narasimhan property for the pair, we give the following lemma:

Lemma 4.3. *If T is a 0-dimensional torsion sheaf on X, then $\mathbb{D}(T)$ is contained in $\mathcal{D}^{\geq 2}(X)$.*

Proof. T must be an extension of skyscraper sheaves. If x is a smooth point on X, then $\mathbb{D}(\mathcal{O}_x) = \mathcal{O}_x[-2]$ so its clear that $\mathbb{D}(\mathcal{O}_x) \in \mathcal{D}^{\geq 2}(X)$.

It remains to show that for the singular point x_0, $\mathbb{D}(\mathcal{O}_{x_0}) \in \mathcal{D}^{\geq 2}(X)$. It is automatic that $\mathbb{D}(\mathcal{O}_{x_0}) \in \mathcal{D}^{\geq 0}(X)$, and $H^0(\mathbb{D}(\mathcal{O}_{x_0})) = \mathcal{H}om(\mathcal{O}_{x_0}, \mathcal{O}_X) = 0$ as $\mathcal{O}_{x_0}$ is torsion.

Assume the contrary that $H^1(\mathbb{D}(\mathcal{O}_{x_0}))$ is nonzero. Then as pushforward onto the open set $X \setminus x_0$ commutes with $\mathbb{D}$, we see that $H^1(\mathbb{D}(\mathcal{O}_{x_0}))$ is supported on x_0, and hence there exists a nonzero morphism $\mathcal{O}_{x_0} \longrightarrow H^1(\mathbb{D}(\mathcal{O}_{x_0}))$.

On the other hand, consider a complete intersection Z of Cartier divisors D_1, D_2 with $x_0 \in Z$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D_1 - D_2) \longrightarrow \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0.$$

This implies that $\mathbb{D}(\mathcal{O}_Z)$ is the complex $(\mathcal{O}_X \longrightarrow \mathcal{O}_X(D_1) \oplus \mathcal{O}_X(D_2) \longrightarrow \mathcal{O}_X(D_1 + D_2))$, which is quasi-isomorphic to $\mathcal{O}_Z(D_1 + D_2)[-2] \simeq \mathcal{O}_Z[-2]$. Now we look at the composition of morphisms

$$\mathbb{D}(\mathcal{O}_Z)[1] = \mathcal{O}_Z[-1] \twoheadrightarrow \mathcal{O}_{x_0}[-1] \longrightarrow H^1(\mathbb{D}(\mathcal{O}_{x_0}))[-1] \longrightarrow \mathbb{D}(\mathcal{O}_{x_0}).$$

This gives us that $\mathrm{Hom}(\mathbb{D}(\mathcal{O}_Z)[1], \mathbb{D}(\mathcal{O}_{x_0})) \neq 0$. However, by duality, we also see that $\mathrm{Hom}(\mathbb{D}(\mathcal{O}_Z)[1], \mathbb{D}(\mathcal{O}_{x_0})) = \mathrm{Hom}(\mathcal{O}_{x_0}, \mathcal{O}_Z[-1]) = 0$, which is a contradiction. □

We now turn to the Harder-Narasimhan property.

Lemma 4.4. *Given $E \in \mathcal{B}^0$, any sequence of inclusions*

$$0 = A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_j \hookrightarrow \cdots \hookrightarrow E$$

in $\mathcal{B}^0$ with $\Im Z_{\pi^*H,\beta,z}(A_j) = 0$ for all j terminates.

Proof. For each j, consider the distinguished triangles

$$A_j \longrightarrow A_{j+1} \longrightarrow B_{j+1} \tag{2}$$

$$A_j \longrightarrow E \longrightarrow E_j. \tag{3}$$

The octahedron axiom gives the following exact triangle for each j:

$$B_j \longrightarrow E_{j-1} \longrightarrow E_j. \tag{4}$$

The induced long exact cohomology sequence of (2) with respect to $\mathrm{Coh}(\widetilde{X})$ gives a sequence of coherent sheaves

$$0 = H^{-1}(A_0) \hookrightarrow H^{-1}(A_1) \hookrightarrow \cdots \hookrightarrow H^{-1}(A_j) \hookrightarrow \cdots \hookrightarrow H^{-1}(E).$$

As $\mathrm{Coh}(\widetilde{X})$ is Noetherian, $H^{-1}(A_j)$ is constant for j sufficiently large, say $H^{-1}(A_{N_1}) = H^{-1}(A_{N_1+1}) = \cdots$ for some N_1.

After omitting the first N_1 terms, we may quotient all objects in the sequence by the common subobject $H^{-1}(A_{N_1})[1]$ and assume that all $A_j = H^0(A_j)$ are sheaves. Moreover, as $\Im Z_{\pi^*H,\beta,z}(A_j) = 0$, we know A_j is torsion for each j.

Considering again the long exact cohomology sequence of (2), this tells us that for each j, the sheaf $H^{-1}(B_j)$ is torsion. Furthermore, since $\mathrm{Coh}(\widetilde{X})$ is Noetherian, it remains to show that $H^0(B_j) = 0$ for sufficiently large j.

By Lemma 3.5, $\mathcal{F}^{\leq 0}_{\pi^*H,P} \subseteq \mathcal{F}^{\leq 0}_{\pi^*H}$, and hence there is a torsion-free subsheaf

$$H^{-1}_P(B_j) = H^0(H^{-1}_P(B_j)) \subseteq H^{-1}(B_j)$$

which implies that $H^{-1}(B_j) = H^{-1}(H^0_P(B_j)) \in \mathcal{F}_P$ for each j. Particularly, since $H^{-1}(B_j) \in \mathcal{F}_P$ and $H^0(B_j) \in \mathcal{T}_P$, we obtain that B_j is a perverse coherent sheaf for each j.

On the other hand, the long exact sequences of (3) and (4) yield a sequence of surjections in the Noetherian category $\mathrm{Coh}(\widetilde{X})$:

$$H^0(E) \twoheadrightarrow H^0(E_1) \twoheadrightarrow H^0(E_2) \twoheadrightarrow \cdots$$

Therefore, $H^0(E_{N_2}) = H^0(E_{N_2+1}) = \cdots$ for some N_2.

We then replace the triangle (3) by $A_j/A_{N_2} \longrightarrow E/A_{N_2} = E_{N_2} \longrightarrow E_j$ and hence we can assume that $H^0(E) = H^0(E_j)$ is constant. Therefore, the long exact sequence of (3) becomes $0 \longrightarrow H^{-1}(E) \longrightarrow H^{-1}(E_j) \longrightarrow A_j \longrightarrow 0$.

Consider the following commutative diagram:

$$\begin{tikzcd}
 & H^{-1}(E)[1] \arrow[r] \arrow[d] & H^{-1}(E_j)[1] \arrow[d] \\
A_j \arrow[ur, dotted] \arrow[r] \arrow[d] \arrow[dr, "0"] & E \arrow[r] \arrow[d] & E_j \arrow[d] \\
0 \arrow[r] & H^0(E) \arrow[r, equal] & H^0(E_j)
\end{tikzcd}$$

This implies that we may then replace the triangle (3) by $A_j \longrightarrow H^{-1}(E)[1] \longrightarrow H^{-1}(E_j)[1]$ to assume furthermore that E and E_j are shifts of sheaves, say $E = \mathcal{E}[1]$ and $E_j = \mathcal{E}_j[1]$ for some sheaves $\mathcal{E}, \mathcal{E}_j$.

Consider the following distinguished triangle:

$$\mathcal{E}_{j-1} \longrightarrow \mathcal{E}_j \longrightarrow B_j \tag{5}$$

As B_j is in ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$, we have $\mathbf{R}^1\pi_* B_j = 0$, and therefore $\mathbf{R}^1\pi_*\mathcal{E}_j \twoheadrightarrow \mathbf{R}^1\pi_*\mathcal{E}_{j+1}$. As π is an isomorphism on $\widetilde{X} \setminus \Pi$, we know that if F is a coherent sheaf, then $\mathbf{R}^1\pi_* F$ is supported on the singular point x_0. We may thus assume that $\mathbf{R}^1\pi_*\mathcal{E}_j$ is constant for j sufficiently large. Applying $\mathbf{R}\pi_*$ on (5) and taking the long exact sequence with respect to $\operatorname{Coh}(\widetilde{X})$ gives

$$0 \longrightarrow \mathbf{R}^0\pi_*\mathcal{E}_{j-1} \longrightarrow \mathbf{R}^0\pi_*\mathcal{E}_j \longrightarrow \mathbf{R}^0\pi_* B_j \longrightarrow 0. \tag{6}$$

As $H^0(B_j)$ lies in $\mathcal{T}^0_{\pi_* H}$ and satisfies $\Im Z_{\pi_* H,\beta}(B_j) = 0$, it must be supported on the union of points and Π. This implies that for each j, the sheaf $\mathbf{R}^0\pi_* B_j = \mathbf{R}^0\pi_* H^0(B_j)$ is a 0-dimensional torsion sheaf on X.

Moreover, for each j, the sheaf $\mathbf{R}^0\pi_*\mathcal{E}_j$ is torsion-free. Indeed, assume the contrary. As $\mathcal{E}_j|_{\widetilde{X}\setminus\Pi}$ is torsion-free, there must exist a nonzero morphis $\mathcal{O}_{x_0} \longrightarrow \mathbf{R}^0\pi_*\mathcal{E}_j$. The underived adjointness gives a nonzero morphism $\mathcal{O}_\Pi \longrightarrow \mathcal{E}_j$ which is a contradiction, since $\mathcal{O}_\pi \in \mathcal{T}_{\mathcal{B}^0}$ and $\mathcal{E}_j \in \mathcal{F}_{\mathcal{B}^0}$. This means that for each j, $\mathbf{R}^0\pi_*\mathcal{E}_j$ is a subsheaf of $(\mathbf{R}^0\pi_*\mathcal{E}_j)^{\vee\vee}$ and the quotient $(\mathbf{R}^0\pi_*\mathcal{E}_j)^{\vee\vee})/\mathbf{R}^0\pi_*\mathcal{E}_j$ is supported on points.

We consider the dual triangle of (6). By Lemma 4.3, $H^0(\mathbb{D}(\mathbf{R}^0\pi_* B_j)) = H^1(\mathbb{D}(\mathbf{R}^0\pi_* B_j)) = 0$, then $(\mathbf{R}^0\pi_*\mathcal{E}_j)^{\vee\vee}$ is constant. We can thus use the double dual argument (cf. [Bri08, Proposition 7.1]) as following to say that $\mathbf{R}^0\pi_* B_j = 0$ for j sufficiently large. Consider the commutative diagram:

$$\begin{tikzcd}
\mathbf{R}^0\pi_*\mathcal{E}_{j-1} \arrow[r] \arrow[d, hook] & \mathbf{R}^0\pi_*\mathcal{E}_j \arrow[r] \arrow[d, hook] & \mathbf{R}^0\pi_* B_j \arrow[d] \\
(\mathbf{R}^0\pi_*\mathcal{E}_{j-1})^{\vee\vee} \arrow[r, equal] \arrow[d, two heads] & (\mathbf{R}^0\pi_*\mathcal{E}_j)^{\vee\vee} \arrow[r] \arrow[d, two heads] & 0 \arrow[d] \\
(\mathbf{R}^0\pi_*\mathcal{E}_{j-1})^{\vee\vee}/\mathbf{R}^0\pi_*\mathcal{E}_{j-1} \arrow[r, two heads, "\phi"] & (\mathbf{R}^0\pi_*\mathcal{E}_{j-1})^{\vee\vee}/\mathbf{R}^0\pi_*\mathcal{E}_j \arrow[r] & 0
\end{tikzcd}$$

The snake lemma gives us a short exact sequence

$$0 \longrightarrow \mathbf{R}^0\pi_*B_j \longrightarrow (\mathbf{R}^0\pi_*\mathcal{E}_{j-1})^{\vee\vee}/\mathbf{R}^0\pi_*\mathcal{E}_{j-1} \longrightarrow (\mathbf{R}^0\pi_*\mathcal{E}_j)^{\vee\vee}/\mathbf{R}^0\pi_*\mathcal{E}_j \longrightarrow 0.$$

Therefore, by looking at the length of $(\mathbf{R}^0\pi_*\mathcal{E}_j)^{\vee\vee}/\mathbf{R}^0\pi_*\mathcal{E}_j$, we see that $\mathbf{R}^0\pi_*B_j = 0$ for j sufficiently large.

By construction of $\mathcal{B}^0$, the sheaf $H^0(B_j) \in \mathcal{T}_P$, that is, it cannot have nonzero morphisms to any object F with $\mathbf{R}\pi_*F = 0$. When j is sufficiently large, since $\mathbf{R}\pi_*H^0(B_j) = \mathbf{R}\pi_*B_j = \mathbf{R}^0\pi_*B_j = 0$, $H^0(B_j)$ can only be 0 . □

Corollary 4.5. *The pair $(Z_{\pi^*H,\beta,z}, \mathcal{B}^0)$ satisfies the Harder-Narasimhan property.*

Proof. By Lemma 4.4 and as $\mathrm{Im}(\Im Z_{\pi^*H,\beta,z})$ is discrete, the assumptions of [BM11, Proposition B.2] are satisfied. □

In summary, we have proved the following theorem.

Theorem 4.6. *The pair $(Z_{\pi^*H,\beta,z}, \mathcal{B}^0)$ forms a pre-stability condition when z and β is chosen so that $\beta \cdot C_i > 0$ for all i , $\beta \cdot \mathrm{ch}_1(\mathcal{O}_\Pi) < 1$, and $z > -\frac{\beta^2}{2}$.*

5. Support property on the resolution

The aim of this section is to find a quadratic form Q such that $(Z_{\pi^*H,\beta,z}, \mathcal{B}^0)$ satisties the support property with respect to Q (see Definition 2.2(c)). We follow the basic approach of [TX22, Sect. 6], but we address some gaps and generalize it to the case of an ADE configuration.

We start with the following:

Definition 5.1. (i) We define on the numerical K-group $K_{\mathrm{num}}(\widetilde{X})$ the quadratic form

$$Q_{A,B} := \Delta + A(\Im Z_{\pi^*H,\beta,z})^2 + B(\Re Z_{\pi^*H,\beta,z})^2 \tag{7}$$

where $\Delta := \mathrm{ch}_1^2 - 2\,\mathrm{ch}_2\,\mathrm{ch}_0$ is the discriminant.

(ii) For every $s \geq 1$, we consider the following function on $\mathrm{D}^b(\widetilde{X})$:

$$Z_{\pi^*H,\beta,sz} = -\mathrm{ch}_2 + \beta \cdot \mathrm{ch}_1 + sz\,\mathrm{ch}_0 + i(\pi^*H) \cdot \mathrm{ch}_1 .$$

From now on we let $z > 0$. Then it is clear that the pair $(Z_{\pi^*H,\beta,sz}, \mathcal{B}^0)$ is a pre-stability condition with the same assumptions as in Lemma 4.1. In this section, we will show that for appropriate choices of A and B, $(Z_{\pi^*H,\beta,z}, \mathcal{B}^0)$ satiesfies the support property with respect to $Q_{A,B}$.

We may first verify the first assertion. Let $\widetilde{\Lambda} := K_{\mathrm{num}}(\widetilde{X})$.

Lemma 5.2. *The quadratic form $Q_{A,B}$ is negative definite on* $\ker Z_{\pi^*H,\beta,sz} \subseteq \widetilde{\Lambda} \otimes \mathbb{R}$, *for every $A, B \geq 0$ and $s \geq 1$.*

Proof. Given a class $x = (r, c, d)$ in $\widetilde{\Lambda} \otimes \mathbb{R}$, assume that $Z_{\pi^*H,\beta,sz}(x) = 0$ for some $s \geq 1$. Note first that the quadratic form $\Delta(x) = \mathrm{ch}_1(x)^2 - 2\,\mathrm{ch}_2(x)\,\mathrm{ch}_0(x) = c^2 - 2rd$ can also be rewritten using the twisted Chern characters $\mathrm{ch}^\beta(x) = \mathrm{ch}(x) \cdot e^{-\beta}$ as $\mathrm{ch}_1^\beta(x)^2 - 2\,\mathrm{ch}_2^\beta(x)\,\mathrm{ch}_0^\beta(x)$.

As we chose β such that $(\pi^*H) \cdot \beta = 0$, we see from $\Im Z_{\pi^*H,\beta,sz}(x) = 0$ that $\mathrm{ch}_1^\beta(x) = \mathrm{ch}_1(x) - \beta\,\mathrm{ch}_0(x) = c - r\beta$ is contained in $(\pi^*H)^\perp$, so we may apply Hodge index theorem to see that $\mathrm{ch}_1^\beta(x)^2 \leq 0$. Since we chose $z > \max\{-\frac{\beta^2}{2}, 0\}$, it can be seen that

$$Q_{A,B}(x) = \Delta(x) = \mathrm{ch}_1^\beta(x)^2 - 2\,\mathrm{ch}_2^\beta(x)\,\mathrm{ch}_0^\beta(x) \leq -2(sz + \frac{\beta^2}{2})r^2 \leq 0.$$

□

Definition 5.3. Define the set $\mathcal{D} := \{E \in \mathcal{B}^0 \,|\, E \text{ is } Z_{\pi^*H,\beta,sz}\text{-semistable for } s \text{ sufficiently large}\}$.

We first prove a crucial lemma in several steps.

Lemma 5.4. *There are constants $A, B \geq 0$ such that $Q_{A,B}(E) \geq 0$ for every $Z_{\pi^*H,\beta,z}$-semistable object $E \in \mathcal{D}$ with $\Im Z_{\pi^*H,\beta,z}(E) > 0$.*

To prove this, we may want to classify the objects in $\mathcal{D}$, particularly those with the imaginary parts of their central charges being positive.

Lemma 5.5. *(cf.* [TX22, Lemma 6.5]*) Given E in the set $\mathcal{D}$, then E must be one of the following forms:*

1. *$E = H^0(E)$ is a (possibly torsion) slope semistable sheaf in $\mathcal{T}^{>0}_{\pi^*H}$;*
2. *$H^0(E)$ is either 0 or a sheaf supported on the union of points and Π, and $H^{-1}(E)$ fits into a short exact sequence*

$$0 \longrightarrow G \longrightarrow H^{-1}(E) \longrightarrow F \longrightarrow 0$$

where F is a slope semistable sheaf in $\mathcal{F}^{\leq 0}_{\pi^*H}$, and $\mathbf{R}^0\pi_*G = 0$. Moreover, G must be 0 unless $(\pi^*H) \cdot \mathrm{ch}_1(F) = 0$.

Proof. Given $E \in \mathcal{D}$, we first decompose E as $G[1] \longrightarrow E \longrightarrow T$ with $G \in \mathcal{F}_0$, and with $T \in \mathcal{T}_0$ fitting into an exact triangle $F[1] \longrightarrow T \longrightarrow S$, where $F \in \mathcal{F}^{\leq 0}_{\pi^*H}$ and $S \in \mathcal{T}^{>0}_{\pi^*H}$.

$$\begin{tikzcd} & & F[1] \arrow[d] \\ G[1] \arrow[r] & E \arrow[r] & T \arrow[d] \\ & & S \end{tikzcd}$$

First consider the case $\mathrm{ch}_0(E) > 0$. In this case, as s tends to ∞, the phase $\phi_{\pi^*H,\beta,sz}(E)$ tends to 0. However, as G is supported on the exceptional curves, $\phi_{\pi^*H,\beta,sz}(G[1]) = 1$ for every s. As E is $Z_{\pi^*H,\beta,sz}$-semistable for large s, this implies that $G = 0$ and $E = T$.

We know that $\mathrm{ch}_0(F[1]) < 0$ and hence $\phi_{\pi^*H,\beta,sz}(F[1]) \to 1$ as s grows large. Since $\mathrm{Hom}(F[1], R) = \mathrm{Ext}^{-1}(F, R) = 0$ for any sheaf $R \in \mathcal{F}_0$, F lies in $\mathcal{T}^0$. Moreover, $S = H^0(H^0_{\mathrm{Coh}^0(\widetilde{X})}(E))$, we see that $F[1]$ is a subobject of $T = E$ in $\mathcal{B}^0$, so $F[1] = 0$ similarly and $E = S$ is a sheaf in $\mathcal{T}^{>0}_{\pi^*H}$ with positive rank.

It remains to verify that it is slope semistable with respect to μ_{π^*H}. We proceed by considering the Harder-Narasimhan filtration of S with respect to μ_{π^*H}, that is a sequence of inclusions:

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{m-1} \subseteq S_m = S$$

with the quotients factors S_i/S_{i-1} being μ_{π^*H}-semistable and the slopes are decreasing.

S_{m-1} is in $\mathrm{Coh}^0_{\pi^*H}(\widetilde{X})$ and hence it fits into the distinguished triangle $S'_{m-1} \longrightarrow S_{m-1} \longrightarrow F'$ with $S'_{m-1} \in \mathcal{T}_0$ and $F' \in \mathcal{F}_0$. This particularly means that $\mathbf{R}\pi_*F' = 0$ and the slope of S'_{m-1} is the same as that of S_{m-1}.

The long exact cohomology sequence tells us that S'_{m-1} is also a sheaf, Now, consider the composition $S'_{m-1} \hookrightarrow S$ and let $Q := S/S'_{m-1}$. It is a sheaf of slope $\mu_{\pi^*H}(Q) > 0$ since $S \in \mathcal{T}^{>0}_{\pi^*H}$. Moreover, $\mathrm{Hom}(Q, \mathcal{F}_0) = \mathrm{Hom}(S, \mathcal{F}_0) = 0$. These together imply that $S'_{m-1} \in \mathcal{T}_0$ and $Q \in \mathcal{B}^0$. Therefore S'_{m-1} is a subobject of S in $\mathcal{B}^0$.

But when s is sufficiently large, we have $\phi_{\pi^*H,\beta,sz}(S'_{m-1}) > \phi_{\pi^*H,\beta,sz}(S)$ since $\mu_{\pi^*H}(S'_{m-1}) > \mu_{\pi^*H}(S)$, which leads to a contradiction as $S = E$ is $Z_{\pi^*H,\beta,sz}$-semistable. Thus, we see that S itself must be slope semistable.

Next, consider the case $\mathrm{ch}_0(E) < 0$. Then $\phi_{\pi^*H,\beta,sz}(E) \to 1$ as $s \to \infty$. Since S is a quotient of E in $\mathcal{B}^0$, we must also have $\phi_{\pi^*H,\beta,sz}(S) \to 1$ as $s \to \infty$. This is possible only if $\mathrm{ch}_0(S) = 0$ and $(\pi^*H) \cdot \mathrm{ch}_1(S) = 0$, which means that S is a torsion sheaf supported on the union of finitely many points and the exceptional locus Π.

If $(\pi^*H) \cdot \mathrm{ch}_1(F) < 0$ then $\Im Z_{\pi^*H,\beta,sz}(E) > 0$, and so $G = 0$ by semistability of E as s large. In this case, we can use the Harder-Narasimhan filtrations as above to show that F is slope semistable.

If $(\pi^*H) \cdot \mathrm{ch}_1(F) = 0$, then as every subobject of F is contained in $\mathcal{F}^{\leq 0}_{\pi^*H}$, there can not exist a subobject with slope greater than $0 = \mu_{\pi^*H}(F)$, that is, F is slope semistable.

Finally, assume that $\mathrm{ch}_0(E) = 0$. If $T = 0$, then $E = G[1]$. If $T \neq 0$, then either $(\pi^*H) \cdot \mathrm{ch}_1(E) > 0$ or $(\pi^*H) \cdot \mathrm{ch}_1(E) = 0$.

In the former case, the phase $\phi_{\pi^*H,\beta,sz}(E)$ is between 0 and 1 for every s, and hence $G = 0$ and $F = 0$ as before. Then E is a torsion sheaf supported on a curve $C' \nsubseteq \Pi$, and therefore $E = H^0(E)$ is a slope semistable sheaf with slope $\mu_{\pi^*H}(E) = \infty > 0$. In the latter case, we have $F = 0$ and $S = H^0(E)$ is supported on the union of points and Π. □

Lemma 5.6. *There is a positive constant A_0, depending only on H, such that for any sheaf E supported on a curve C' not containing any of exceptional curves C_i, we have* $\mathrm{ch}_1(E)^2 + A_0[(\pi^*H) \cdot \mathrm{ch}_1(E)]^2 \geq 0$.

Proof. We first claim that there exist some $b_i > 0$ such that $(\sum_i b_i C_i) \cdot C_j < 0$ for all j.

The exceptional curves C_i span a negative definite convex cone $\sigma \subseteq \mathrm{NS}(\widetilde{X}) \otimes \mathbb{R}$ (again, see [Rei97, Theorem A.7]) and it suffices to prove that $\sigma \cap -\sigma^\vee$ is non-empty. Assume the contrary. There must be a linear function which is positive on σ and negative on $-\sigma^\vee$, which can be represented by pairing with a vector v.

But then as v is postive on $\sigma^\vee$, we know that v lies in $(\sigma^\vee)^\vee = \sigma$, which implies that $(v, v) => 0$. This is a contradiction to the negative definiteness and hence our claim is proved.

By [Rei97, Exercise 4.16], for sufficiently small $\epsilon > 0$, the class $H' := \pi^*H - \epsilon(\sum_i b_i C_i)$ is ample. Now let C' be the support of E with multiplicities and write $C' = \alpha + a_1C_1 + \cdots a_nC_n$ with $\alpha \cdot C_j = 0$ for all j. As $C' \cdot H' > 0$, we obtain that

$$(\pi^*H) \cdot \alpha > \epsilon(\sum_i b_i C_i) \cdot (\sum_i a_i C_i) = \sum_i \epsilon b_i (C_i \cdot C') \geq 0.$$

By the openness of the ample cone on X, there is a constant $N > 0$ depending only on H such that $(\pi^*H)^2\alpha^2 + N(\pi^*H \cdot \alpha)^2 \geq 0$. It suffices to show that $(\pi^*H \cdot \alpha)^2 \geq M(\sum_i a_i C_i)^2$ for some constant $M < 0$ depending only on H. We can then choose $A_0 := \frac{N}{(\pi^*H)^2} - \frac{1}{M} > 0$.

As $C_1, \cdots, C_n$ are exceptional curves of an ADE singularity, by [Car05, Proposition 10.18(iii)], every entry of the inverse of the intersection matrix is negative. This implies that if $(\sum_i a_i C_i) \cdot C_j \geq 0$ for all $j = 1, \cdots, n$, then all $a_i \leq 0$. Consider the unit sphere S (with respect to the metric given by D^2) and the convex cone σ generated by all the curves C_i in $\mathrm{NS}_{\mathbb{R}}(\widetilde{X})$. Then $D \mapsto [D \cdot (\sum_i b_i C_i)]^2$ is a continuous positive function for D in the compact set $\sigma \cap S$, and hence there is a minimum K. Moreover, this number K must be positive as any D in σ satisfies $D \cdot (\sum_i b_i C_i) > 0$.

We set $D := (-\sum_i a_i C_i)$ and then $[D \cdot (\sum_i b_i C_i)]^2 \geq -K D^2$. Now

$$(\pi^* H \cdot \alpha)^2 > [\epsilon(\sum_i b_i C_i) \cdot (\sum_i a_i C_i)]^2 \geq -\epsilon^2 K [(\sum_i a_i C_i)^2].$$

Choose $M := -\epsilon^2 K$ and the assertion follows. ☐

Lemma 5.7. *Let $E \in \mathcal{D}$ be a $Z_{\pi^* H, \beta, z}$-semistable object with $\mathrm{ch}_0(E) < 0$ and $\Im Z_{\pi^* H, \beta, z}(E) > 0$. Then $\Delta(E) \geq 0$.*

Proof. By Lemma 5.5, such E must fit into an exact triangle $F[1] \longrightarrow E \longrightarrow S$, with S a torsion sheaf support on the union of points and Π, and with F a slope semistable sheaf with $\mu_{\pi^* H}(F) \geq 0$.

Firstly, we can decompose S into $S_1 \oplus S_2$ where $\operatorname{supp} S_1 \subseteq \Pi$ and $\operatorname{supp} S_2$ is 0-dimensional, then by Hirzebruch-Riemann-Roch, we obtain

$$\begin{aligned}
\Delta(E) &= \Delta(F) + \mathrm{ch}_1(S_1 \oplus S_2)^2 - 2\,\mathrm{ch}_0(F[1])\,\mathrm{ch}_2(S_1 \oplus S_2) + 2\,\mathrm{ch}_1(F[1])\,\mathrm{ch}_1(S_1 \oplus S_2) \\
&= \Delta(F) + \mathrm{ch}_1(S_1)^2 - 2\,\mathrm{ch}_0(F[1])\,\mathrm{ch}_2(S_1) + 2\,\mathrm{ch}_1(F[1])\,\mathrm{ch}_1(S_1) - 2\,\mathrm{ch}_0(F[1])\,\mathrm{ch}_2(S_2) \\
&= \Delta(F) - \chi(S_1, S_1) - 2\chi(S_1, F[1]) + 2\,\mathrm{ch}_0(F)\,\mathrm{ch}_2(S_2).
\end{aligned}$$

As $\mathrm{ch}_0(F)\,\mathrm{ch}_2(S_2)$ are is non-negative, we may assume that $S = S_1$ is supported on the exceptional locus Π. By adjunction formula, we note that the canonical bundle $\omega_{\widetilde{X}}$ is trivial on the exceptional locus, and hence by Serre duality we have $\dim \mathrm{Hom}_{\widetilde{X}}(S, S) = \dim \mathrm{Ext}^2_{\widetilde{X}}(S, S)$. Therefore, we have $\chi_{\widetilde{X}}(S, S) \leq 2 \dim \mathrm{Hom}_{\widetilde{X}}(S, S)$.

If we apply the functor $\mathrm{Hom}_{\widetilde{X}}(S, \cdot)$ to the triangle $F[1] \longrightarrow E \longrightarrow S$, we obtain an exact sequence

$$\mathrm{Hom}_{\widetilde{X}}(S, E) \longrightarrow \mathrm{Hom}_{\widetilde{X}}(S, S) \longrightarrow \mathrm{Ext}^1_{\widetilde{X}}(S, F[1])$$

By semistability we see that $\mathrm{Hom}_{\widetilde{X}}(S, E) = 0$ and similarly $\mathrm{Hom}_{\widetilde{X}}(S, F[1]) = 0$ as the phase of S is 1 and those of E and $F[1]$ are less than 1. Since $\mathrm{Ext}^2_{\widetilde{X}}(S, F[1]) = \mathrm{Ext}^3_{\widetilde{X}}(S, F) = 0$, this immediately implies that $\dim \mathrm{Hom}_{\widetilde{X}}(S, S) \leq \dim \mathrm{Ext}^1_{\widetilde{X}}(S, F[1]) = -\chi(S, F[1])$.

In summary, we have the inequality $\chi(S, S) \leq 2 \dim \mathrm{Hom}_{\widetilde{X}}(S, S) \leq -2\chi(S, F[1])$, that is, $-\chi(S, S) - 2\chi(S, F[1]) \geq 0$. As $\Delta(F) \geq 0$ by Bogomolov-Gieseker, this completes the proof. ☐

Proof of Lemma 5.4. Let $E \in \mathcal{D}$ be an $Z_{\pi^* H, \beta, z}$-semistable object with $\Im Z_{\pi^* H, \beta, z}(E) > 0$.

If $\mathrm{ch}_0(E) > 0$, then by Lemma 5.5, E can only be a slope semistable torsion-free sheaf. In this case, we apply the Bogomolov-Gieseker inequality to see that $Q_{A,B}(E) \geq \Delta(E) \geq 0$ for any $A, B \geq 0$.

If $\mathrm{ch}_0(E) = 0$, by Lemma 5.5, E must be a torsion sheaf supported on a curve $C' \nsubseteq \Pi$. Moreover, as E is $Z_{\pi^*H,\beta,z}$-semistable, the support C' cannot contain C_i for any i as otherwise $E|_{C_i}$ would be a $Z_{\pi^*H,\beta,z}$-destabilizing subsheaf. Then Lemma 5.6 implies that $Q_{A_0,B}(E) \geq 0$ for every such torsion sheaf E and any positive B.

Finally, if $\mathrm{ch}_0(E) < 0$, by Lemma 5.7, $Q_{A,B}(E) \geq \Delta(E) \geq 0$ for any $A, B \geq 0$. □

Lemma 5.8. *There exists a constant $B_0 > 0$ such that $Q_{A,B}(E) \geq 0$ for all $A \geq 0, B \geq B_0$ and for any $Z_{\pi^*H,\beta,z}$-stable object $E \in \mathcal{B}^0$ with $\Im Z_{\pi^*H,\beta,z}(E) = 0$.*

Proof. We consider the same decomposition as in Lemma 5.5:

$$\begin{tikzcd} & & F[1] \arrow[d] \\ G[1] \arrow[r] & E \arrow[r] & T \arrow[d] \\ & & S \end{tikzcd}$$

where $G \in \mathcal{F}_0$, $F \in \mathcal{F}^{\leq 0}_{\pi^*H}$ and $S \in \mathcal{T}^{>0}_{\pi^*H}$. As E is stable, it must be equal to $G[1]$, to $F[1]$, or to S.

If $E = F[1]$, then as F is a torsion-free slope-stable sheaf, $Q_{A,B}(E) \geq \Delta(E) = \Delta(F) \geq 0$ for any $A, B > 0$ by the Bogomolov-Gieseker inequality. If $E = G[1]$, then as E is stable, E can only be $\mathcal{O}_{C_i}(-1)[1]$ for some i by Proposition 3.12. As there are only finitely many choices, we can find some $B_1 > 0$ such that $Q_{A,B}(\mathcal{O}_{C_i}(-1)[1]) \geq 0$ for all i and $B \geq B_1$.

Finally assume that $E = S$. If S is supported on a point then $\Delta(E) = \Delta(S) = 0$, so we may assume that S is supported on Π, and then by Proposition 3.12, S can only be $\mathcal{O}_\Pi$. We can similarly find a desired constant $B_2 > 0$ such that $Q_{A,B_2}(\mathcal{O}_\Pi) \geq 0$ and set $B_0 := \max\{B_1, B_2\}$. □

Combining all above, we can finally prove our main result in this section.

Theorem 5.9. *The pair $\sigma = (Z_{\pi^*H,\beta,z}, \mathcal{B}^0)$ is a Bridgeland stability condition, with respect to the lattice $K_{\mathrm{num}}(\widetilde{X})$, and it satisfies the support property with respect to the lattice $\widetilde{\Lambda}$ with the quadratic form (7) for A, B sufficiently large.*

Proof. Let A_0, B_0 be the constants given by Lemma 5.6 and 5.8. It remains to prove that $Q_{A_0,B_0}(E) \geq 0$ for any $Z_{\pi^*H,\beta,z}$-semistable object $E \in \mathcal{B}^0$.

As the image of $\Im Z_{\pi^*H,\beta,z}(E)$ is discrete, we prove by induction on $\Im Z_{\pi^*H,\beta,z}(E)$. If $\Im Z_{\pi^*H,\beta,z}(E) = 0$, then the assertion follows from Lemma 5.8. If $\Im Z_{\pi^*H,\beta,z}(E) > 0$ is minimal, then as any possible destabilizing subobjects must have smaller imaginary part, E must be in $\mathcal{D}$ and hence the assertion follows from Lemma 5.4.

Now let E be an object in the set $\{F \in \mathcal{B}^0 \mid F \text{ is } Z_{\pi^*H,\beta,z}\text{-semistable and } Q_{A_0,B_0}(F) < 0\}$ with the smallest imaginary part. Then by Lemma 5.4, $E \notin \mathcal{D}$. The induction hypothesis gives the support property for objects with smaller imaginary part, and hence by wall-crossing, there is some $s > 1$ such that E is strictly $Z_{\pi^*H,\beta,sz}$-semistable (cf. [BMS16, Theorem 3.5] and [Bri08, Sect. 9]).

We can then consider the Jordan-Hölder factors $E_1, \ldots, E_m$ of E, which have smaller imaginary parts. By induction hypothesis we have $Q_{A,B}(E_i) \geq 0$ for every i.

Now we may apply [BMS16, Lemma A.6]. More precisely, since all of the Jordan-Hölder factors E_i have the same phase, i.e. they lie on the same ray of the image of $Z_{\pi^*H,\beta,sz}$, we can find constants $a_{ij} > 0$ such that $[E_i] - a_{ij}[E_j]$ is in the kernel of $Z_{\pi^*H,\beta,z}$ and hence $Q_{A,B}([E_i] - a_{ij}[E_j]) \leq 0$.

Then linear algebra tells us that $Q_{A_0,B_0}(m[E_i] + n[E_j]) \geq 0$ for any positive n, m, and hence $Q_{A_0,B_0}(E) \geq 0$, which is a contradiction. Therefore, $Q_{A_0,B_0}(E) \geq 0$ for every $Z_{\pi^*H,\beta,z}$-semistable object E, and the assertion follows. $\square$

6. Deformation and compatibility of pre-stability conditions

In this section, we will see that by deforming along a path σ_ϵ of stability conditions on $\widetilde{X}$, the end point will induce a pre-stability condition σ_X on the singular surface X, which is compatible with the pushforward $\mathbf{R}\pi_*$.

More precisely, we will give a path Z_ϵ in the space of central charges, parametrized by ϵ, which lifts to a path of Bridgeland stability conditions $\sigma_\epsilon := (Z_{\pi^*H,\epsilon\beta,z}, \mathcal{B}^0)$ in $\operatorname{Stab}(\mathrm{D}^b(\widetilde{X}))$, but the end point, denoted by $\sigma_{\widetilde{X}} = (Z_{\widetilde{X}}, \mathcal{B}^0)$, is only a weak pre-stability condition, which will be compatible with a stability condition σ_X on $\mathrm{D}^b(X)$ in the following sense.

The path is chosen so that at its end point $\sigma_{\widetilde{X}}$, the central charge of $\mathcal{O}_{C_i}(-1)$ is 0. This means that there is a function $Z_X \colon K_{\mathrm{num}}(X) \longrightarrow \mathbb{C}$ with $Z_{\widetilde{X}} = Z_X \circ \pi_*$, where $\pi_* \colon K_{\mathrm{num}}(\widetilde{X}) \longrightarrow K_{\mathrm{num}}(X)$ is the pushforward on the numerical K-groups.

Let v_π be the composition of $v \colon K(\mathrm{D}^b(\widetilde{X})) \longrightarrow \widetilde{\Lambda}$ and let $\Lambda := K_{\mathrm{num}}(X)$. We consider the projection map of the numerical K-group $\mathrm{pr}\colon K_{\mathrm{num}}(\widetilde{X}) \longrightarrow K_{\mathrm{num}}(\widetilde{X})/\ker\pi_*$. Our results in this section will give the following commutative diagram:

$$\begin{tikzcd}
K(\widetilde{X}) = K(\mathcal{B}^0) \arrow[r, "v"] \arrow[dr, "v_\pi"] \arrow[dd, "\pi_*"] & \widetilde{\Lambda} = K_{\mathrm{num}}(\widetilde{X}) \arrow[d, "\mathrm{pr}"] \arrow[dr, "Z_{\widetilde{X}}"] & \\
 & \widetilde{\Lambda}/\ker\pi_* \arrow[r] \arrow[d] & \mathbb{C} \\
K(X) = K(\mathcal{A}) \arrow[r, "v'"] & \Lambda = K_{\mathrm{num}}(X) \arrow[ur, "Z_X"] &
\end{tikzcd}$$

Our goal is to moreover obtain a result like Toda's that the pushforward gives an exact functor between the hearts $\mathcal{B}^0$ and $\mathcal{A}$. The plan is to realize the hearts as tilts of perverse coherent sheaves and coherent sheaves respectively, and then show that there are some relations between the torsion pairs defined them.

We will first construct $\mathcal{A}$ as the tilted heart $\mathrm{Coh}^\alpha_H(X)$ of the category of coherent sheaves. Similarly as Lemma 3.9, to prove the desired exactness, we need to find a real number α such that $\pi_*(\mathcal{B}^0) \subset \mathrm{Coh}^\alpha_H(X)$.

Our proof will rely on the following lemma, the proof of which is obvious.

Lemma 6.1. *Let $\mathcal{D}_1, \mathcal{D}_2$ be triangulated categories. Given two abelian categories $\mathcal{A}_1 \subseteq \mathcal{D}_1$ and $\mathcal{A}_2 \subseteq \mathcal{D}_2$, assume that there are torsion pairs $\mathcal{A}_i = \langle \mathcal{T}_i, \mathcal{F}_i \rangle$ for $i = 1, 2$. If a triangulated functor $G \colon \mathcal{D}_1 \longrightarrow \mathcal{D}_2$ sends $\mathcal{A}_1, \mathcal{T}_1$, and $\mathcal{F}_1$ to $\mathcal{A}_2, \mathcal{T}_2$, and $\mathcal{F}_2$ respectively, then $G(\mathcal{A}_1^\sharp) \subseteq \mathcal{A}_2^\sharp$.*

We now make the following optimistic claim: If $E \in {}^{-1}\operatorname{Per}(\widetilde{X}/X)$ is slope semistable, then π_*E will be a slope semistable sheaf (with respect to H) of the same slope. If this is the case, then Lemma 6.1 applies and we can take $\alpha = 0$. However, to prove this we can not use the adjunction argument as the derived pullback $\mathbf{L}\pi^*$ is not t-exact. For example, $\mathbf{L}\pi^*\mathcal{O}_{x_0}$ is even a non-bounded complex. This also results in that we can not prove that the derived pullback $\mathbf{L}\pi^*$ is exact by Lemma 6.1.

Let $\mathbf{L}^i_{\mathcal{A}}\pi^* := H^i_{\mathcal{A}} \circ \mathbf{L}\pi^*$ denote the i-th pullback with respect to a heart $\mathcal{A}$ of a bounded t-structure.

The following is the most crucial result in this section.

Proposition 6.2. *Let $(Z_{\mathcal{A}}, \mathcal{A})$ and $(Z_{\mathcal{B}}, \mathcal{B})$ be weak stability conditions on $\mathrm{D}^b(\widetilde{X})$ and $\mathrm{D}^b(X)$ respectively. Assume that*

(1) *$\mathbf{R}\pi_*$ is t-exact with respect to the t-structures associated with $\mathcal{A}$ and $\mathcal{B}$,*
(2) *$Z_{\mathcal{A}} = Z_{\mathcal{B}} \circ \pi_*$.*

Then given a $Z_{\mathcal{A}}$-semistable object $E \in \mathcal{A}$, the object $\mathbf{R}\pi_*E \in \mathcal{B}$ is $Z_{\mathcal{B}}$-semistable.

Proof. Let $\mu_{\mathcal{A}}$ and $\mu_{\mathcal{B}}$ be the slope with respect to the stability functions $Z_{\mathcal{A}}$ and $Z_{\mathcal{B}}$ respectively. Assume the contrary that $\mathbf{R}\pi_*E$ is $Z_{\mathcal{B}}$-unstable. Then either there exists a quotient $\mathbf{R}\pi_*E \twoheadrightarrow G$ with $\mu_{\mathcal{B}}(G) < \mu_{\mathcal{B}}(\mathbf{R}\pi_*E)$, or there exists a nonzero subobject $K \subseteq \mathbf{R}\pi_*E$ with $Z_{\mathcal{B}}(K) = 0$.

In the first case, as $\mathbf{R}\pi_*$ is t-exact with respect to the t-structures given by $\mathcal{A}$ and $\mathcal{B}$, its left adjoint $\mathbf{L}_{\mathcal{A}}\pi^*$ is right t-exact. In particular, $\mathbf{L}^0_{\mathcal{A}}\pi^* \colon \mathcal{B} \longrightarrow \mathcal{A}$ is right exact. Therefore, the natural map $f \colon \mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_*E \longrightarrow \mathbf{L}^0_{\mathcal{A}}\pi^*G$ is a surjection in $\mathcal{A}$. We claim that this is a $Z_{\mathcal{A}}$-destabilizing quotient of $\mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_*E$, that is $\mu_{\mathcal{A}}(\mathbf{L}^0_{\mathcal{A}}\pi^*G) < \mu_{\mathcal{A}}(\mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_*E)$.

To show this, we will prove that $\mathbf{L}^0_{\mathcal{A}}\pi^*$ preserves the slope, i.e. $\mu_{\mathcal{B}}(M) = \mu_{\mathcal{A}}(\mathbf{L}^0_{\mathcal{A}}\pi^*M)$ for any object $M \in \mathcal{B}$. Indeed, by condition (1) and the equality $\mathbf{R}\pi_*\mathbf{L}\pi^* = \mathrm{id}$, we see that $\tau^{<0}_{\mathcal{A}}(\mathbf{L}\pi^*M)$ is in the kernel of $\mathbf{R}\pi_*$:

$$\mathbf{R}\pi_*(\tau^{<0}_{\mathcal{A}}(\mathbf{L}\pi^*M)) = \tau^{<0}_{\mathcal{B}}(\mathbf{R}\pi_*\mathbf{L}\pi^*M) = \tau^{<0}_{\mathcal{B}}(M) = 0$$

Therefore $M = \mathbf{R}\pi_*\mathbf{L}\pi^*M = \mathbf{R}\pi_*\mathbf{L}^0_{\mathcal{A}}\pi^*M$, so $\mu_{\mathcal{B}}(M) = \mu_{\mathcal{B}}(\mathbf{R}\pi_*\mathbf{L}^0_{\mathcal{A}}\pi^*M) = \mu_{\mathcal{A}}(\mathbf{L}^0_{\mathcal{B}}\pi^*M)$ since the condition (2) implies that $\mathbf{R}\pi_*$ also preserve the slope.

Next, we consider the map $g \colon \mathbf{L}\pi^*\mathbf{R}\pi_*E \longrightarrow E$ given by adjunction. Applying $H^0_{\mathcal{A}}$, we get a morphism $H^0_{\mathcal{A}}(g) \colon \mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_*E \longrightarrow E$. By the definition, we know that $\mathbf{R}\pi_*(g) = \mathrm{id}_{\mathbf{R}\pi_*E}$, and thus the pushforward of $\mathrm{cone}(g)$ is 0.

Moreover, by condition (1), $\mathbf{R}\pi_*$ commutes with the cohomology functors $H^i_{\mathcal{A}}$ with respect to the heart $\mathcal{A}$, which implies that every cohomology of $\mathrm{cone}(g)$ with respect to $\mathcal{A}$ lies in the kernel of $\mathbf{R}\pi_*$. Particularly, by condition (2) we see $Z_{\mathcal{A}}(\ker H^0_{\mathcal{A}}(g)) = Z_{\mathcal{A}}(\mathrm{cok}\, H^0_{\mathcal{A}}(g)) = 0$,

Consider the quotient $f \colon \mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_*E \longrightarrow \mathbf{L}^0_{\mathcal{A}}\pi^*G$. By the nine lemma, we may consider the following commutative diagram in $\mathcal{A}$:

$$\begin{tikzcd}
T \arrow[r, hook] \arrow[d, hook] & \ker(f) \arrow[r, two heads] \arrow[d] & \ker(f)/T \arrow[dr, hook] \arrow[dd, hook] & & \\
\ker(H^0_{\mathcal{A}}(g)) \arrow[r, hook] \arrow[d, two heads] & \mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_*E \arrow[rr, "H^0_{\mathcal{A}}(g)"] \arrow[d, two heads, "f"] \arrow[dr, two heads] & & E \arrow[r, two heads] & \mathrm{cok}(H^0_{\mathcal{A}}(g)) \\
\ker(H^0_{\mathcal{A}}(g))/T \arrow[r, hook] & \mathbf{L}^0_{\mathcal{A}}\pi^*G \arrow[dr, two heads] & \mathrm{Im}(H^0_{\mathcal{A}}(g)) \arrow[ur, hook] \arrow[d, two heads] & & \\
& & \mathbf{L}^0_{\mathcal{A}}\pi^*G/(\ker(H^0_{\mathcal{A}}(g))/T) & &
\end{tikzcd}$$

Then, we see that $Z_{\mathcal{A}}(T) = 0$ and $\mu_{Z_{\mathcal{A}}}(\ker(f)/T) = \mu_{Z_{\mathcal{A}}}(\ker(f)) > \mu_{Z_{\mathcal{A}}}(\mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_* E) = \mu_{Z_{\mathcal{A}}}(E)$. Hence, $\ker(f)/T$ is a destabilizing subobject of E, which is a contradiction.

We then turn to the second case. Consider the composition

$$\mathbf{L}^0_{\mathcal{A}}\pi^* K \longrightarrow \mathbf{L}^0_{\mathcal{A}}\pi^*\mathbf{R}\pi_* E \longrightarrow E.$$

The composition is nonzero as its pushforward is the inclusion $K \hookrightarrow \mathbf{R}\pi_* E$. The image of this morphism gives a $Z_{\mathcal{A}}$-destabilizing subobject of E, which is also a contradiction. □

Remark 6.3. In the above proof, the equality $M = \mathbf{R}\pi_*\mathbf{L}\pi^* M = \mathbf{R}\pi_*\mathbf{L}^0_{\mathcal{A}}\pi^* M$ follows only from the condition (1), and it implies particularly that the functor $\mathbf{R}\pi_* \colon \mathcal{A} \longrightarrow \mathcal{B}$ is essentially surjective. In the case where $\mathcal{A} = {}^{-1}\operatorname{Per}(\widetilde{X}/X)$ and $\mathcal{B} = \operatorname{Coh}(X)$, this was also proven in [BKS18, Theorem 2.14].

We fix the following notations of slope stability.

- Let $Z_{\pi^* H}$ denote the central charge of slope stability with respect to $\pi^* H$ on ${}^{-1}\operatorname{Per}(\widetilde{X}/X)$ given by $-(\pi^* H)\cdot \operatorname{ch}_1 + i \operatorname{ch}_0$.
- Let Z_H denote the central charge of slope stability with respect to H on $\operatorname{Coh}(X)$ given by $-H \cdot \operatorname{ch}_1 + i\operatorname{ch}_0$.

Corollary 6.4. *If $E \in {}^{-1}\operatorname{Per}(\widetilde{X}/X)$ is Z_{π^*H}-semistable, then $\mathbf{R}\pi_* E$ is a Z_H-semistable coherent sheaf.*

Proof. We will apply Proposition 6.2. The condition (1) follows from Lemma 3.9.

For the condition (2), by Grothendieck-Riemann-Roch, for any $E \in {}^{-1}\operatorname{Per}(\widetilde{X}/X)$, we have $\operatorname{ch}_0(E) = \operatorname{ch}_0(\mathbf{R}\pi_* E)$ and $(\pi^* H)\cdot \operatorname{ch}_1(E) = H \cdot \operatorname{ch}_1(\mathbf{R}\pi_* E)$. Therefore, $Z_H(\mathbf{R}\pi_* E) = Z_{\pi^* H}(E)$. □

Combining Lemma 6.1 and Corollary 6.4 shows the following result.

Proposition 6.5. *The pushforward functor $\mathbf{R}\pi_* \colon \mathcal{B}^0 \longrightarrow \operatorname{Coh}^0_H(X)$ is exact.*

As we are going to show that the stability conditions on $\widetilde{X}$ and on X are compatible via π_*, the essential surjectivity will be useful.

Proposition 6.6. *The pushforward functor $\mathbf{R}\pi_* \colon \mathcal{B}^0 \longrightarrow \operatorname{Coh}^0_H(X)$ is essentially surjective.*

Proof. Similarly as in Remark 6.3, by the t-exactness of $\mathbf{R}\pi_* \colon \mathcal{B}^0 \longrightarrow \operatorname{Coh}^0_H(X)$, we see that for any $E \in \operatorname{Coh}^0_H(X)$, $\mathbf{R}\pi_* \circ H^0_{\mathcal{B}^0} \circ \mathbf{L}\pi^* E = E$. □

We are going to construct the pre-stability condition on $\widetilde{X}$ and then show that it descends to the singular surface X. By Section 4, we know that the pair $(Z_{\pi^* H,\beta,z}, \mathcal{B}^0)$ gives a Bridgeland stability condition with support property for suitable choices of β and z. Note that $Z_{\pi^* H,\beta,z}([\mathcal{O}_{C_i}(-1)]) = 1 + \frac{C_i^2}{2} + \beta\cdot C_i = \beta \cdot C_i$. Consider the functions $Z_{\pi^* H,\epsilon\beta}$. If we assume that $z > \max\{0, -\frac{\beta^2}{2}\}$, then for every $0 < \epsilon << 1$, the pair $(Z_{\pi^* H,\epsilon\beta}, \mathcal{B}^0)$ is a stability condition by Section 4 and 5.

For simplicity we denote $Z_{\widetilde{X}} := Z_{\pi^* H,0,z}$. Note that at the end point $\epsilon = 0$, we have $Z_{\pi^* H,0}([\mathcal{O}_{C_i}(-1)]) = 0$ for all i, and hence the pair $\sigma_{\widetilde{X}} := (Z_{\widetilde{X}}, \mathcal{B}^0)$ can not be a stability condition. However, we claim that it is a weak pre-stability condition.

Proposition 6.7. *The pair $\sigma_{\widetilde{X}} = (Z_{\widetilde{X}}, \mathcal{B}^0)$ forms a weak pre-stability condition.*

Proof. Let E be an object in $\mathcal{B}^0$. We note that $Z_{\widetilde{X}}$ is a weak stability function on $\mathcal{B}^0$ as follows. First, $\Im Z_{\widetilde{X}}(E) = \Im Z_{\pi^* H,\beta_\epsilon}(E) \geq 0$. If $\Im Z_{\widetilde{X}}(E) = \Im Z_{\pi^* H,\beta_\epsilon}(E) = 0$, then $\Re Z_{\pi^* H,\beta_\epsilon}(E) < 0$ for all ϵ, and by continuity we see that $\Re Z_{\widetilde{X}}(E) \leq 0$.

For the Harder-Narasimhan property, one may first note that [BM11, Prop. B.2] can also be extended to the case of weak stability condition, as all of them can be deduced from the analogue [Bri07, Prop. 2.4]; the proof of this version is exactly the same as that of Bridgeland stability conditions. Then since $\Im Z_{\widetilde{X}} = \Im Z_{\pi^* H,\beta,z}$, Lemma 4.4 implies that $(Z_{\widetilde{X}}, \mathcal{B}^0)$ admits Harder-Narasimhan property. $\square$

By Lemma 3.7, $Z_{\widetilde{X}} \colon K_{\mathrm{num}}(\widetilde{X}) \longrightarrow \mathbb{C}$ factors through $\pi_* \colon K_{\mathrm{num}}(\widetilde{X}) \longrightarrow K_{\mathrm{num}}(X)$, i.e. there is a function Z_X such that $Z_{\widetilde{X}} = Z_X \circ \pi_*$.

Proposition 6.8. *The function Z_X gives a stability function on $\mathrm{Coh}^0_H(X)$.*

Proof. Given $E \in \mathrm{Coh}^0_H(X)$, by Proposition 6.6 we may write $E = \mathbf{R}\pi_* \widetilde{E}$ for some $\widetilde{E} \in \mathcal{B}^0$. Recall that by the definition of $\mathcal{B}^0$, we may obtain the following diagram, with $G \in \mathcal{F}_0$ and $T \in \mathcal{T}_0$.

$$\begin{tikzcd} & & F[1] \arrow[d] \\ G[1] \arrow[r] & \widetilde{E} \arrow[r] & T \arrow[d] \\ & & S \end{tikzcd}$$

Since $Z_{\widetilde{X}}$ is a weak stability function and $Z_X(E) = Z_{\widetilde{X}}(\widetilde{E})$, it now suffices to show that if $Z_{\widetilde{X}}(\widetilde{E}) = 0$ then $\mathbf{R}\pi_* \widetilde{E} = 0$. Note that $Z_{\widetilde{X}}(\widetilde{E}) = 0$ is equivalent to that $Z_{\widetilde{X}}(G[1]) = Z_{\widetilde{X}}(S) = Z_{\widetilde{X}}(F[1]) = 0$.

First, as G is supported on the exceptional locus Π, $\mathbf{R}\pi_*(G[1])$ is supported on the singular point of X; as we also know $\mathbf{R}\pi_*(G[1]) \in \mathrm{Coh}^0_H(X)$ is a sheaf of finite length, we easily see that $Z_{\widetilde{X}}(G[1]) = 0$ implies that $\mathbf{R}\pi_*(G[1]) = 0$. Secondly, assume that $Z_{\widetilde{X}}(S) = 0$ and that S is nonzero. It can be seen that S can only be supported on the union of points and Π. Therefore, $\mathbf{R}\pi_*(S)$ must be supported on points on X and hence $\Re Z_{\widetilde{X}}(S) = \Re Z_X(\mathbf{R}\pi_* S) < 0$, a contradiction.

We then turn to $F = H^{-1}(T) \in \mathcal{F}^{\leq 0}_{\pi^* H}$. Assume that $Z_{\widetilde{X}}(F) = 0$ and that F is nonzero. It has slope $\mu_{\pi^* H}(F) := \frac{(\pi^* H)\cdot \mathrm{ch}_1(F)}{\mathrm{ch}_0(F)} \leq 0$ and $\Im Z_{\widetilde{X}}(F) = (\pi^* H) \cdot \mathrm{ch}_1(F) = 0$, so it has positive rank and the slope is exactly 0 (otherwise the slope should be $+\infty$).

Note that F is of maximal possible slope in the category $\mathcal{F}^{\leq 0}_{\pi^* H}$, so it must be $\mu_{\pi^* H}$-semistable and torsion-free. Bogomolov-Gieseker inequality then implies that $\mathrm{ch}_1^2(F) \geq 2\,\mathrm{ch}_0(F)\,\mathrm{ch}_2(F)$. On the other hand, as $(\pi^* H) \cdot \mathrm{ch}_1(F) = 0$, Hodge index theorem gives that $(\mathrm{ch}_1(F))^2 \leq 0$. Now we obtain the inequality

$$0 = \Re Z_{\widetilde{X}}(F) = z\,\mathrm{ch}_0(F) - \mathrm{ch}_2(F) \geq \mathrm{ch}_0(F)(z - \frac{(\mathrm{ch}_1(F))^2}{2\,\mathrm{ch}_0^2(F)}) > 0.$$

which is absurd.

In summary, if $Z_{\widetilde{X}}(\widetilde{E}) = 0$, then $\widetilde{E} = G[1], T = 0$ and $\mathbf{R}\pi_* \widetilde{E} = \mathbf{R}\pi_* G[1] = 0$ as claimed. $\square$

Proposition 6.9. *The pair* $\sigma_X := (Z_X, \mathrm{Coh}^0_H(X))$ *is a pre-stability condition.*

Proof. It remains to prove the Harder-Narasimhan property. Again, by the essential surjectivity we may write $E = \pi_* G$ for some $G \in \mathcal{B}^0$. Then G admits a Harder-Narasimhan filtration $0 = G_0 \subset G_1 \subset \cdots \subset G_{m-1} \subset G_m = G$ with the quotients G_i/G_{i-1} $Z_{\widetilde{X}}-$semistable and the slopes (with respect to $Z_{\widetilde{X}}$) are decreasing. This induces a filtration of $0 = E_0 \subset E_1 \subset \cdots \subset E_{m-1} \subset E_m = E$, where $E_i := \pi_* G_i$. Note that the slopes with respect to Z_X are still decreasing. To check that this is the desired Harder-Narasimhan filtration, we still need the Z_X-semistability of the quotients. As π_* is exact, the quotient $E_i/E_{i-1} = \pi_* G_i/\pi_* G_{i-1}$ is isomorphic to $\pi_*(G_i/G_{i-1})$, so it suffices to show the following result. □

Corollary 6.10. *The pushforward of an* $Z_{\widetilde{X}}$*-semistable object in* $\mathcal{B}^0$ *is* Z_X*-semistable.*

Proof. We can apply Proposition 6.2 directly. The condition (1) follows from Proposition 6.5 and the condition (2) is just the definition. □

At the end of this section, we are going to prove the surjectivity of the morphism between the set of semistable objects. We start with the following lemma:

Lemma 6.11. *If* $E \in \mathrm{Coh}^0_H(X)$ *is an* Z_X*-semistable object, then there exists a* $Z_{\widetilde{X}}$*-semistable object* $\widetilde{E} \in \mathcal{B}^0$ *with* $\mathbf{R}\pi_* \widetilde{E} = E$.

Proof. By Proposition 6.6, we can first find an object $G \in \mathcal{B}^0$ with $\mathbf{R}\pi_* G = E$. If G is $Z_{\widetilde{X}}$-semistable then the assertion is proved.

We then assume that G is $Z_{\widetilde{X}}$-unstable and consider the Harder-Narasimhan filtration of G with respect to $\sigma_{\widetilde{X}}$. That is, a sequence of inclusions

$$0 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G$$

with $G_i/G_{i-1} Z_{\widetilde{X}}$-semistable and $\mu_{Z_{\widetilde{X}}}(G_1) > \cdots > \mu_{Z_{\widetilde{X}}}(G_n/G_{n-1})$.

As E is Z_X-semistable, $\mathbf{R}\pi_*(G_i/G_{i-1}) = 0$ except for one i. But if $\mathbf{R}\pi_*(G_i/G_{i-1}) = 0$, then $\mu_{Z_{\widetilde{X}}}(G_i/G_{i-1}) = +\infty$. Therefore, the filtration can only be a two-step filtration $0 \subseteq G_1 \subseteq G$, and now we pick $\widetilde{E} := G/G_1$. □

To prove Theorem 1.3, it thus remains to show that we can choose $\widetilde{E}$ to have a fixed Chern character $\widetilde{v}$.

Lemma 6.12. *Consider a linear combination* $\sum_i a_i C_i$ *with* $a_i \in \mathbb{R}$*. If some of the* a_i *are negative, then there exists* k *such that* $a_k < 0$ *and* $C_k \cdot \sum_i a_i C_i > 0$.

Proof. Let $a_{j_1}, a_{j_2}, \ldots, a_{j_m}$ be the negative coefficients, then we have to show that there exists some j_k with $C_{j_k} \cdot \sum_i a_i C_i > 0$.

For each k, we have

$$\begin{aligned} C_{j_k} \cdot \sum_i a_i C_i &= \sum_i a_i C_{j_k} \cdot C_i \\ &= \Big(\sum_{i \neq j_1, \ldots, j_m} a_i C_{j_k} \cdot C_i\Big) + \sum_{l=1}^{m} a_{j_l} C_{j_k} \cdot C_{j_l} \\ &\geq \sum_{l=1}^{m} a_{j_l} C_{j_k} \cdot C_{j_l} \end{aligned}$$

We now consider the subdiagram generated by $C_{j_i}, \ldots C_{j_m}$, whose intersection matrix $B = (b_{kl})$ is just $(C_{j_k} \cdot C_{j_l})$. As a subdiagram of Dynkin diagram of ADE type is a disjoint union of Dynkin diagrams of ADE type, the matrix B is a direct sum of intersection matrices for exceptional curves of some ADE singularities.

Hence, by [Car05, Proposition 10.18(iii)], if $\sum_{l=1}^{m} a_{j_l} C_{j_k} \cdot C_{j_l} \leq 0$ for all $k = 1, \ldots m$, then $a_{j_l} \geq 0$ for all l, which is a contradiction. This means that there exists some k such that $C_{j_k} \cdot \sum_i a_i C_i > 0$. □

We will need the following result about extension by massless object.

Lemma 6.13. *Let $\sigma = (Z, \mathcal{A})$ be a weak stability condition. Consider a short exact sequence in the heart $\mathcal{A}$:*

$$0 \longrightarrow A \longrightarrow B \longrightarrow Q \longrightarrow 0.$$

Assume that $Z(Q) = 0$.

(i) *If B is Z-semistable, then A is Z-semistable.*
(ii) *If A is Z-semistable, Q is simple and $\mathrm{Hom}_{\mathcal{A}}(Q, B) = 0$, then B is Z-semistable.*

Proof. For (i), if B is Z-semistable, then for any nonzero subobject $E \subseteq A$, it is also a subobject of B, and hence $\mu_Z(E) \leq \mu_Z(B) = \mu_Z(A)$.

For (ii), we assume that A is Z-semistable. Let $E \subseteq B$ be a nonzero subobject and let f be the composition $E \hookrightarrow B \longrightarrow Q$.

By our assumptions, f can not be injective. Then we have a short exact sequence

$$0 \longrightarrow \ker f \longrightarrow A \longrightarrow B/E \longrightarrow 0,$$

and hence $\mu_Z(E) = \mu_Z(\ker f) \leq \mu_Z(A) = \mu_Z(B)$. □

Let $\mathcal{M}_\sigma(v)$ be the set of σ-semistable objects with Chern character v.

Lemma 6.14. *Given a class $v \in \Lambda$, there exists a class $\widetilde{v} \in \widetilde{\Lambda}$ such that for any $E \in \mathcal{M}_{\sigma_X}(v)$, there exists an object $\widetilde{E} \in \mathcal{M}_{\sigma_{\widetilde{X}}}(\widetilde{v})$ such that $\mathbf{R}\pi_* \widetilde{E} = E$.*

Proof. Let $\widetilde{v_0}$ be such that $\pi_* \widetilde{v_0} = v$. Then there is a unique class $\widetilde{v} \in \widetilde{v_0} + \ker \pi_*$ such that its projection to the kernel can be written as $\mathrm{pr}_{\ker \pi_*}(\widetilde{v}) = \sum_i a_i C_i$ with $0 \leq a_i < 1$ for all i.

It then suffices to show that there exists a $Z_{\widetilde{X}}$-semistable object $\widetilde{E}$ of class $\widetilde{v}$ such that $\mathbf{R}\pi_* \widetilde{E} = E$. By Lemma 6.11, there exists a $Z_{\widetilde{X}}$-semistable object $E' \in \mathcal{B}^0$ such that $\mathbf{R}\pi_* E' =$

E. We write $\mathrm{pr}_{\ker \pi_*}(\mathrm{ch}(E')) = \sum_i b_i C_i$. Note that the b_i might be rational numbers, not necessarily integers.

By Hirzebruch-Riemann-Roch, $\chi(\mathcal{O}_{C_i}(-1)[1], E') = C_i \cdot \mathrm{ch}_1(E')$ for any i. Note that by semistability and Serre duality $\mathrm{Hom}(\mathcal{O}_{C_i}(-1)[1], E'[j]) = 0$ unless $j = 1, 2$, and hence

$$\chi(\mathcal{O}_{C_i}(-1)[1], E') = \dim \mathrm{Hom}(E', \mathcal{O}_{C_i}(-1)[1]) - \dim \mathrm{Ext}^1(\mathcal{O}_{C_i}(-1)[1], E).$$

Now if $\mathrm{ch}(E') \neq \widetilde{v}$, then for some j, we either have $b_j \geq 1$ or some $b_j < 0$. Assuming that there is some $b_j \geq 1$, we see by Lemma 6.12 that there exists k such that $b_k > 0$ and $0 > C_k \cdot \mathrm{ch}_1(E') = \chi(\mathcal{O}_{C_k}(-1)[1], E')$.

This implies that $\dim \mathrm{Ext}^1(\mathcal{O}_{C_k}(-1)[1], E') > 0$ and we can consider a nontrivial extension $0 \longrightarrow E' \longrightarrow F \longrightarrow \mathcal{O}_{C_k}(-1)[1] \longrightarrow 0$ in $\mathcal{B}^0$.

If $\mathrm{Hom}(\mathcal{O}_{C_i}(-1)[1], F)$ is nonzero, then since F is a nontrivial extension and $\mathcal{O}_{C_i}(-1)[1]$ is simple, the composition map $(\mathcal{O}_{C_i}(-1)[1] \longrightarrow F \longrightarrow \mathcal{O}_{C_i}(-1)[1])$ must be 0. Hence, it factors to give a nonzero morphism $\mathcal{O}_{C_i}(-1)[1] \longrightarrow E'$, which is a contradiction. Therefore, the conditions of Lemma 6.13 are satisfied and so F is $Z_{\widetilde{X}}$-semistable.

Replacing E' by F, we obtain an $Z_{\widetilde{X}}$-semistable object F such that $\mathbf{R}\pi_* F = E$ and $\mathrm{pr}_{\ker \pi_*}(\mathrm{ch}(F)) = (b_j - 1)C_j + \sum_{i \neq j} b_i C_i$. Repeating this process if necessary, we reach a $Z_{\widetilde{X}}$-semistable object E'' such that $\mathbf{R}\pi_* E'' = E$ and $\mathrm{pr}_{\ker \pi_*}(\mathrm{ch}(F)) = \sum_i b'_i C_i$ with $b'_i \leq 0$ for all i.

Now, if there exists any j with $b'_j < 0$, then Lemma 6.12 implies that there exists some l such that $b'_l < 0$ and $\chi(\mathcal{O}_{C_l}(-1)[1], E'') > 0$.

Then $\dim \mathrm{Hom}(E'', \mathcal{O}_{C_l}(-1)[1])$ must be positive, and hence there exists a nonzero morphism $f\colon E'' \longrightarrow \mathcal{O}_{C_l}(-1)[1]$ and $\mathrm{pr}_{\ker \pi_*}(\mathrm{ch}(\ker f)) = (b'_l + 1)C_l + \sum_{i \neq l} b_i C_i$. Note that by Lemma 6.13(i), $\ker f$ is still $Z_{\widetilde{X}}$-semistable.

Replacing E'' by $\ker f$ and repeating this process if necessary, we obtain a subobject $\widetilde{E} \subseteq E''$ so that $\mathrm{pr}_{\ker \pi_*}(\mathrm{ch}(\widetilde{E})) = \sum_i a_i C_i$ with $0 \leq a_i < 1$ for all i, that is $\widetilde{E}$ is of class $\widetilde{v}$. □

Combining the lemmas above, we can prove the following result.

Theorem 6.15. *For any class $v \in \Lambda$, there exists a class $\widetilde{v} \in \widetilde{\Lambda}$ such that there is a surjective map $\pi_*\colon \mathcal{M}_{\sigma_{\widetilde{X}}}(\widetilde{v}) \longrightarrow \mathcal{M}_{\sigma_X}(v)$.*

One may note that by Theorem 6.15, if $\mathcal{M}_{\sigma_{\widetilde{X}}}(\widetilde{v})$ satisfies boundedness (in the sense of [BLM+21, Definition 9.4]), then so does $\mathcal{M}_{\sigma_X}(v)$.

7. Support property on the singular surface

In this final section, we prove the support property of the stability condition $\sigma_X = (Z_X, \mathcal{B}^0)$. As above, we want to take advantage of the essential surjectivity of $\mathbf{R}\pi_*$. We claim that if $\sigma_{\widetilde{X}}$ admits the support property with respect to the lattice $\widetilde{\Lambda}/\ker \pi_*$, then we can obtain an induced quadratic form for the support property on σ_X.

Note that the Chern character of $\mathrm{pr}\, E$ is the same as the Chern character of $\mathbf{R}\pi_* E$ on X. Since $\widetilde{\Lambda}/\ker \pi_* \simeq \Lambda$, a quadratic form $Q_{\widetilde{X}}$ on $\widetilde{\Lambda}/\ker \pi_*$ can be identified with a quadratic form Q_X on Λ. By abuse of notations, we write $Z_{\widetilde{X}}$ for the map $\widetilde{\Lambda}/\ker\pi_* \longrightarrow \mathbb{C}$ that $Z_{\widetilde{X}}$ factors through.

Now given a σ_X-semistable object F in $\mathrm{Coh}^0_H(X)$, we write $F = \mathbf{R}\pi_* G$ for some $\sigma_{\widetilde{X}}$-semistable G by Lemma 6.11, and then $Q_X(F) = Q_{\widetilde{X}}(G) \geq 0$. On the other hand, by the compatibility $Z_X \circ \pi_* = Z_{\widetilde{X}}$, if $F \in \ker Z_X$ and $F = \mathbf{R}\pi_* G$, then G must be in $\ker Z_{\widetilde{X}}$ and thus $Q_X(F) = Q_{\widetilde{X}}(G) < 0$.

We are going to try to extend the argument in Section 5 to construct a quadratic form on $(\widetilde{\Lambda}/\ker\pi_*) \otimes \mathbb{R} \simeq \Lambda \otimes \mathbb{R}$. First, we note that on the singular surface X we can define the intersection products and Chern classes (see, for example, [Lan23]).

By the openness of the ample cone on X, there is a constant $A > 0$, depending only on the ample divisor H, such that $C^2 + A(H \cdot C)^2 \geq 0$ for any curve $C \subseteq X$. Now we define a quadratic form on $\widetilde{\Lambda}/\ker\pi_*$ by

$$Q_{\widetilde{X}} := \Delta + A(\Im Z_{\widetilde{X}})^2 \tag{8}$$

Similarly, we need to verify the followings two statements:

(i) $Q_{\widetilde{X}}$ is negative definite on $\ker Z_{\widetilde{X}}$.

(ii) For any $E \in \mathcal{B}^0$ which is $Z_{\widetilde{X}}$-semistable, we have $Q_{\widetilde{X}}(v_\pi(E)) \geq 0$.

By abuse of notation, for $E \in \mathcal{B}^0$, we write $Q_{\widetilde{X}}(E) := Q_{\widetilde{X}}(v_\pi(E)) = \Delta(\mathrm{pr}\, E) + A(\Im Z_{\widetilde{X}}(E))^2$.

For any $s \geq 1$, define the function $Z_{\widetilde{X},s}\colon \widetilde{\Lambda}/\ker\pi_* \longrightarrow \mathbb{C}$ by $Z_{\widetilde{X},s}(E) := Z_{\widetilde{X}}(E) + (s-1)z\,\mathrm{ch}_0(E)$ and the pair $\sigma_{\widetilde{X},s} := (Z_{\widetilde{X},s}, \mathcal{B}^0)$. With the assumptions of Lemma 4.1 and $z > 0$, $\sigma_{\widetilde{X},s}$ also gives a weak stability condition for each s and one can easily verify this as in Proposition 6.7.

We can immediately prove the condition (i) by showing the following analogue of Lemma 5.2:

Lemma 7.1. *$Q_{\widetilde{X}}$ is negative definite on $\ker Z_{\widetilde{X},s} \subseteq (\widetilde{\Lambda}/\ker\pi_*) \otimes \mathbb{R}$ for all $s \geq 1$.*

Proof. Given a nonzero class x in $(\widetilde{\Lambda}/\ker\pi_*) \otimes \mathbb{R}$ and assume that $Z_{\widetilde{X},s}(x) = 0$ for some $s \geq 1$. This implies that

$$-\mathrm{ch}_2(x) + sz\,\mathrm{ch}_0(x) = 0 \text{ and } (\pi^*\mathrm{H}) \cdot \mathrm{ch}_1(\mathrm{x}) = 0. \tag{9}$$

As $\mathrm{ch}_2(x) = sz \cdot \mathrm{ch}_0(x)$, we see that $2\,\mathrm{ch}_2(x)\,\mathrm{ch}_0(x) \geq 0$. It then suffices to show that $(\mathrm{ch}_1(x))^2 < 0$. By the equations (9), we see that $\mathrm{ch}_1(x)$ is orthogonal to π^*H. Note that although π^*H is only nef, $(\pi^*H)^2 > 0$ as H is ample on X, and hence by Hodge index theorem the assertion is proved. □

As for the condition (ii), similar to in Section 5, we can define the set

$$\mathcal{D}_0 := \{E \in \mathcal{B}^0 \mid E \text{ is } Z_{\widetilde{X},s}\text{-semistable for sufficiently large } s\}$$

and classify objects in $\mathcal{D}_0$ similarly. The proof of the following lemma is verbatim the same as Lemma 5.5.

Lemma 7.2. *Given E in the set $\mathcal{D}_0$, then E must be one of the following forms:*

1. *$E = H^0(E)$ is a (possibly torsion) slope semistable sheaf in $\mathcal{T}^{\geq 0}_{\pi^*H}$;*
2. *$H^0(E)$ is either 0 or a sheaf supported on the union of points and Π, and $H^{-1}(E)$ fits into a short exact sequence*

$$0 \longrightarrow G \longrightarrow H^{-1}(E) \longrightarrow F \longrightarrow 0$$

where F is a slope semistable sheaf in $\mathcal{F}^{\le 0}_{\pi^*H}$, and $\mathbf{R}^0\pi_*G = 0$. Moreover, G must be 0 unless $(\pi^*H)\cdot \mathrm{ch}_1(F) = 0$.

We deal with the $Z_{\widetilde{X}}$-semistable objects in $\mathcal{D}_0$ with positive imaginary part and the semistable objects in $\mathcal{B}^0$ with $\Im Z_{\widetilde{X}}(E) = 0$ respectively.

Lemma 7.3. *If E is an object in $\mathcal{D}_0$, with $\Im Z_{\widetilde{X}}(E) > 0$, then $Q_{\widetilde{X}}(E) \ge 0$.*

Proof. Given $E \in \mathcal{D}_0$ with $\Im Z_{\widetilde{X}}(E) > 0$, we divide the proof into three cases by Lemma 7.2 as in the proof of Lemma 5.4.

If $\mathrm{ch}_0(E) > 0$, then E is a slope semistable torsion-free sheaf. Note that X is $\mathbb{Q}$-Cartier. Therefore, we can still define $\pi^*\colon\ \mathrm{NS}(X) \longrightarrow \mathrm{NS}(\widetilde{X})$ and $\pi^*\pi_*\colon\ \mathrm{NS}(X) \longrightarrow \mathrm{NS}(X)$ is the orthogonal projection with kernel generated by the curves $[C_i]$, and the kernel is negative definite with respect to the intersection pairing.

Recall that $K_{\mathrm{num}}(X) \simeq \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$ and $\pi^*\pi_*$ preserves ch_0 and ch_2, so we similarly obtain that $\pi^*\pi_* K_{\mathrm{num}}(X) \longrightarrow K_{\mathrm{num}}(X)$ is a orthogonal projection with kernel generated by the classes $[\mathcal{O}_{C_i}(-1)]$; furthermore, the kernel must be negative definite with respect to the quadratic form $\Delta = \mathrm{ch}_1^2 - 2\,\mathrm{ch}_0\,\mathrm{ch}_2$.

Now we see that $\Delta(E) = \Delta(\pi^*\pi_*E) + \Delta(E - \pi^*\pi_*E) = \Delta(\pi_*E) + \Delta(E - \pi^*\pi_*E)$. As $(E - \pi^*\pi_*E) \in \ker \pi_*$, the last term is negative, and hence $\Delta(\mathrm{pr}\, E) = \Delta(\pi_*E) > \Delta(E)$. The Bogomolov-Gieseker inequality on $\widetilde{X}$ says that $\Delta(E) \ge 0$, and hence we have $Q_{\widetilde{X}}(E) \ge \Delta(\mathrm{pr}\, E) \ge \Delta(E) \ge 0$.

If $\mathrm{ch}_0(E) = 0$, then E can only be a torsion sheaf supported on a curve $\widetilde{C} \not\subseteq \Pi$ with $C_i \not\subseteq \widetilde{C}$ for all i, then $\mathbf{R}\pi_*E$ is a torsion sheaf supported on a curve C, and hence $Q_{\widetilde{X}}(E) \ge C^2 + A(H\cdot C)^2 \ge 0$ by our choice of A.

If $\mathrm{ch}_0(E) < 0$, then E will fit in a distinguished triangle $F[1] \longrightarrow E \longrightarrow S$ where S is a torsion sheaf supported on a union of points and Π, and F is a slope semistable sheaf with slope $\mu_{\pi^*H}(F) < 0$.

Note that $\mathbf{R}\pi_*S$ is a sheaf supported on points, and hence $\mathrm{ch}_0(\mathrm{pr}\, S) = 0, \mathrm{ch}_1(\mathrm{pr}\, S) = 0$, and $\mathrm{ch}_2(\mathrm{pr}\, S) \ge 0$. Then we may use the relation $\mathrm{ch}(\mathrm{pr}\, E) = \mathrm{ch}(\mathrm{pr}\, F[1]) + \mathrm{ch}(\mathrm{pr}\, S)$ to compute the discriminant $\Delta(\mathrm{pr}\, E)$:

$$\begin{aligned}
\Delta(\mathrm{pr}\, E) &= \mathrm{ch}_1^2(\mathrm{pr}\, E) - 2\,\mathrm{ch}_0(\mathrm{pr}\, E)\,\mathrm{ch}_2(\mathrm{pr}\, E)\\
&= [\mathrm{ch}_1(\mathrm{pr}\, S) - \mathrm{ch}_1(\mathrm{pr}\, F)]^2 - 2[\mathrm{ch}_0(\mathrm{pr}\, S) - \mathrm{ch}_0(\mathrm{pr}\, F)][\mathrm{ch}_2(\mathrm{pr}\, S) - \mathrm{ch}_2(\mathrm{pr}\, F)]\\
&= \mathrm{ch}_1^2(\mathrm{pr}\, F) + 2\,\mathrm{ch}_0(\mathrm{pr}\, F)\,\mathrm{ch}_2(\mathrm{pr}\, S) - 2\,\mathrm{ch}_0(\mathrm{pr}\, F)\,\mathrm{ch}_2(\mathrm{pr}\, F)\\
&\ge \Delta(\mathrm{pr}\, F)
\end{aligned}$$

Then as F is slope semistable torsion-free, we have $\mathrm{ch}_0(F) \ge 0$, and the computation in the first case says that $\Delta(\mathrm{pr}\, F) \ge 0$. To sum up, we have $Q_{\widetilde{X}}(E) \ge \Delta(\mathrm{pr}\, E) \ge \Delta(\mathrm{pr}\, F) \ge 0$, which proves the assertion. □

Remark 7.4. The computation above can also be used to deduce the Bogomolov-Gieseker inequality on X. More precisely, given a slope semistable torsion-free sheaf G on X, then we claim that $\Delta_X(G) := \mathrm{ch}_1^2(G) - 2\,\mathrm{ch}_0(G)\,\mathrm{ch}_2(G) \ge 0$.

We start with considering a perverse coherent sheaf E such that $G = \mathbf{R}\pi_*E$. E can fit in a triangle $F[1] \longrightarrow E \longrightarrow T$ where $F = H^{-1}(E)$ and $T = H^0(E)$.

First, we note that E must be a slope semistable perverse coherent sheaf (that is, the converse of Corollary 6.4 holds), otherwise the pushforward of a destabilizing subobject of

E gives a destabilizing subobject of G. Then, since $F \in \mathcal{F}_0 = \{A \in \mathrm{Coh}(\widetilde{X}) \mid \mathbf{R}^0\pi_* A = 0\}$, $F[1]$ does not affect the slope of E and hence T is a slope semistable coherent sheaf.

We want to apply the Bogomolov-Gieseker on $\widetilde{X}$ to T so we still need to verify that T is also torsion-free. By considering the pushforward of the triangle $F[1] \longrightarrow E \longrightarrow T$, we see that $\mathbf{R}\pi_* F[1] = 0$; otherwise it will be a torsion subsheaf of G, and thus $G = \mathbf{R}\pi_* T$.

Now T must be a torsion-free coherent sheaf or G cannot be torsion-free. Bogomolov-Gieseker then applies on T, showing that $\Delta_{\widetilde{X}}(T) \geq 0$. By the computation in the above lemma, we obtain that $\Delta_X(G) \geq \Delta_{\widetilde{X}}(T) \geq 0$, which proves the Bogomolov-Gieseker inequality on the singular surface X.

Lemma 7.5. *If $E \in \mathcal{B}^0$ is a $Z_{\widetilde{X}}$-stable object with $\Im Z_{\widetilde{X}}(E) = 0$, then $Q_{\widetilde{X}}(E) \geq \Delta(\mathrm{pr}\, E) \geq 0$.*

Proof. Consider $E \in \mathcal{B}^0$ with E being $Z_{\widetilde{X}}$-stable and $\Im Z_{\widetilde{X}}(E) = 0$. For such E, we again take advantage of the decomposition:

$$\begin{tikzcd} & & F[1] \arrow[d] \\ G[1] \arrow[r] & E \arrow[r] & T \arrow[d] \\ & & S \end{tikzcd}$$

where E is equal to one of $G[1], F[1]$, and S.

If $E = F[1]$, then $Q_{\widetilde{X}}(E) \geq \Delta(\mathrm{pr}\, F) \geq \Delta(F) \geq 0$ as in Lemma 7.3. If $E = G[1]$, then E must be one of $\mathcal{O}_{C_i}(-1)[1]$, so $Q_{\widetilde{X}}(E) \geq \Delta(\mathrm{pr}\, E) = 0$. If $E = S$, then $\mathbf{R}\pi_* E$ can only be supported on points, and hence $Q_{\widetilde{X}}(E) \geq \Delta(\mathrm{pr}\, E) = \Delta_X(\mathbf{R}\pi_* E) = 0$ □

We can now conclude the proof of (ii) and obtain the support property of $\sigma_{\widetilde{X}}$.

Theorem 7.6. *The stability condition $\sigma_{\widetilde{X}} = (Z_{\widetilde{X}}, \mathcal{B}^0)$ satisfies the support property, with respect to the lattice $\widetilde{\Lambda}/\ker \pi_*$ and the quadratic form $Q_{\widetilde{X}}$. In particular, this induces a stability condition $\sigma_X = (Z_X, \mathrm{Coh}^0_H(X))$ and $\mathrm{Stab}(X) \neq \varnothing$.*

Proof. With Lemma 7.1, 7.3 and 7.5, the proof is almost exactly the same as Theorem 5.9. The only thing to notice is that every $Z_{\widetilde{X}}$-semistable object E with $\Im Z_{\widetilde{X}}(E) = 0$ admits a Jordan-Hölder filtration so we can pass to the $Z_{\widetilde{X}}$-stable objects similarly. This is because $\sigma_{\widetilde{X}}$ and σ_ϵ have the same heart $\mathcal{B}^0$ and $\mathcal{P}(1)_{\sigma_{\widetilde{X}}} = \mathcal{P}(1)_{\sigma_\epsilon}$.

We now see that $Q_{\widetilde{X}}$ is the desired quadratic form on the lattice $\widetilde{\Lambda}/\ker \pi_* \simeq \Lambda$ which gives the support property of $\sigma_{\widetilde{X}}$, and this induces the support property of $\sigma_X = (Z_X, \mathrm{Coh}^0_H(X))$. □

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