Algorithm and Data Structure Analysis (ADSA)

Lecture 6: Order Statistics

Review: Sorting Algorithms Comparison Sorts

- Comparison sorts: O(n lg n) at best (decision tree with n! leaves→O(n lg n) height)
- Counting sort: O(n+k) =O(n) for n inputs in the range 1..k (k=O(n))
- Radix sort: O(*dn+dk*)=O(n) for *n* numbers on d digits that range from 1..*k* (constant d, k=O(n))
- Bucket sort:
  - Use n buckets (linked lists) to divide interval [0,1) (range k=O(n)) into subintervals of size 1/n (k/n)
  - Uniform input distribution → O(1) bucket size→expected total time O(n)

Order Statistics

- The *i*th *order statistic* in a set of *n* elements is the *i*th smallest element
- The *minimum* is thus the 1st order statistic
- The *maximum* is (duh) the *n*th order statistic
- The *median* is the *n*/2 order statistic
  – If *n* is even, there are 2 medians
- *How can we calculate order statistics?*
- *What is the running time?*

Order Statistics

- *How many comparisons are needed to find the minimum element in a set? The maximum?*

```
MINIMUM(A, n)
  min = A[1]
  for i = 2 to n
      if min > A[i]
          min = A[i]
  return min
```

There is a lower bound of n-1 comparisons for this problem

Order Statistics

- *How many comparisons are needed to find the minimum element in a set? The maximum?*
- *Can we simultaneously find the minimum and maximum with less than twice the cost?*

Order Statistics

- *How many comparisons are needed to find the minimum element in a set? The maximum?*
- *Can we simultaneously find the minimum and maximum with less than twice the cost?*
- Yes:
  - Walk through elements by pairs
    - Compare each element in pair to the other
    - Compare the largest to maximum, smallest to minimum
  - Total cost: 3 comparisons per 2 elements = $O(3n/2)$

Finding Order Statistics: The Selection Problem

- A more interesting problem is *selection*: finding the *i*th smallest element of a set

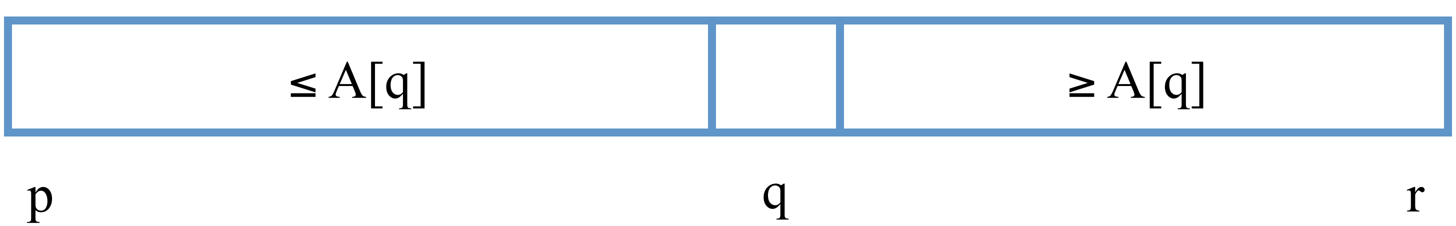
Finding Order Statistics: The Selection Problem

- A more interesting problem is *selection*: finding the *i*th smallest element of a set
- We will show:
  - A practical randomized algorithm with O(n) expected running time
  - A cool algorithm of theoretical interest only with O(n) worst-case running time

Randomized Selection

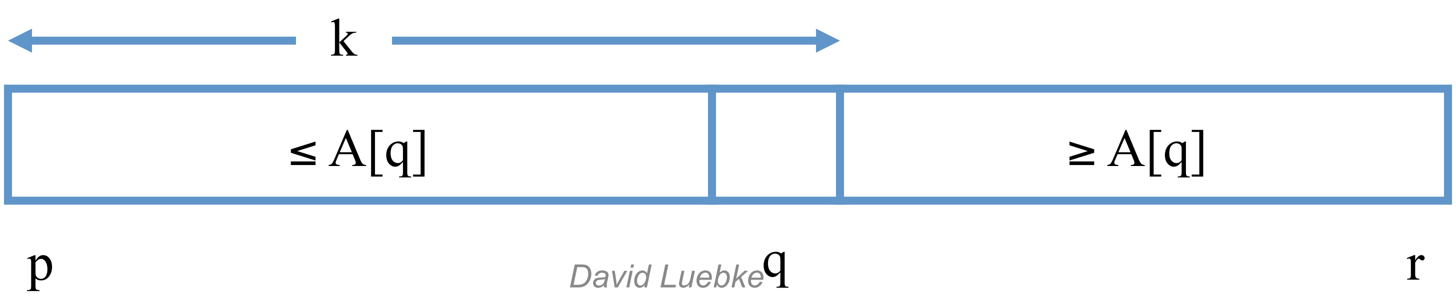
- Key idea: use partition() from quicksort
  - But, only need to examine one subarray
  - This savings shows up in running time: O(n)
- We will again use a slightly different partition than the book:

  q = RandomizedPartition(A, p, r)



Randomized Selection

```
RandomizedSelect(A, p, r, i)
    if (p == r) then return A[p];
    q = RandomizedPartition(A, p, r)
    k = q - p + 1;
    if (i == k) then return A[q];    // not in book
    if (i < k) then
        return RandomizedSelect(A, p, q-1, i);
    else
        return RandomizedSelect(A, q+1, r, i-k);
```



Randomized Selection

- Analyzing **`RandomizedSelect()`**
  - Worst case: partition always 0:n-1

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    $$= O(n^2)$$

    - No better than sorting!

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    $T(n) = T(9n/10) + O(n)$

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    $T(n) = T(9n/10) + O(n)$

    $= O(n)$ (Master Theorem, case 3)

    - Better than sorting!
    - *What if this had been a 99:1 split?*

Randomized Selection

- Average case
  - For upper bound, assume *i*th element always falls in larger side of partition:

$$T(n) \le \frac{1}{n}\sum_{k=0}^{n-1} T(\max(k, n-k-1)) + \Theta(n)$$

$$\le \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n)$$

What happened here?

  - Let's show that T(*n*) = O(*n*) by substitution

Randomized Selection

- Assume T(n) ≤ cn for sufficiently large c:

$$
\begin{aligned}
T(n) &\le \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n) && \textit{The recurrence we started with} \\
&\le \frac{2}{n}\sum_{k=n/2}^{n-1} ck + \Theta(n) && \textit{Substitute } T(n) \le cn \textit{ for } T(k) \\
&= \frac{2c}{n}\left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k\right) + \Theta(n) && \textit{“Split” the recurrence} \\
&= \frac{2c}{n}\left(\frac{1}{2}(n-1)n - \frac{1}{2}\left(\frac{n}{2}-1\right)\frac{n}{2}\right) + \Theta(n) && \textit{Expand arithmetic series} \\
&= c(n-1) - \frac{c}{2}\left(\frac{n}{2}-1\right) + \Theta(n) && \textit{Multiply it out}
\end{aligned}
$$

Randomized Selection

- Assume T(*n*) ≤ *cn* for sufficiently large *c*:

$$
\begin{aligned}
T(n) &\le c(n-1) - \frac{c}{2}\left(\frac{n}{2}-1\right) + \Theta(n) && \textit{The recurrence so far} \\
&= cn - c - \frac{cn}{4} + \frac{c}{2} + \Theta(n) && \textit{Multiply it out} \\
&= cn - \frac{cn}{4} - \frac{c}{2} + \Theta(n) && \textit{Subtract c/2} \\
&= cn - \left(\frac{cn}{4} + \frac{c}{2} - \Theta(n)\right) && \textit{Rearrange the arithmetic} \\
&\le cn \quad \text{(if c is big enough)} && \textit{What we set out to prove}
\end{aligned}
$$

Worst-Case Linear-Time Selection

- Randomized algorithm works well in practice
- What follows is a worst-case linear time algorithm, really of theoretical interest only
- Basic idea:
  - Generate a good partitioning element
  - Call this element *x*

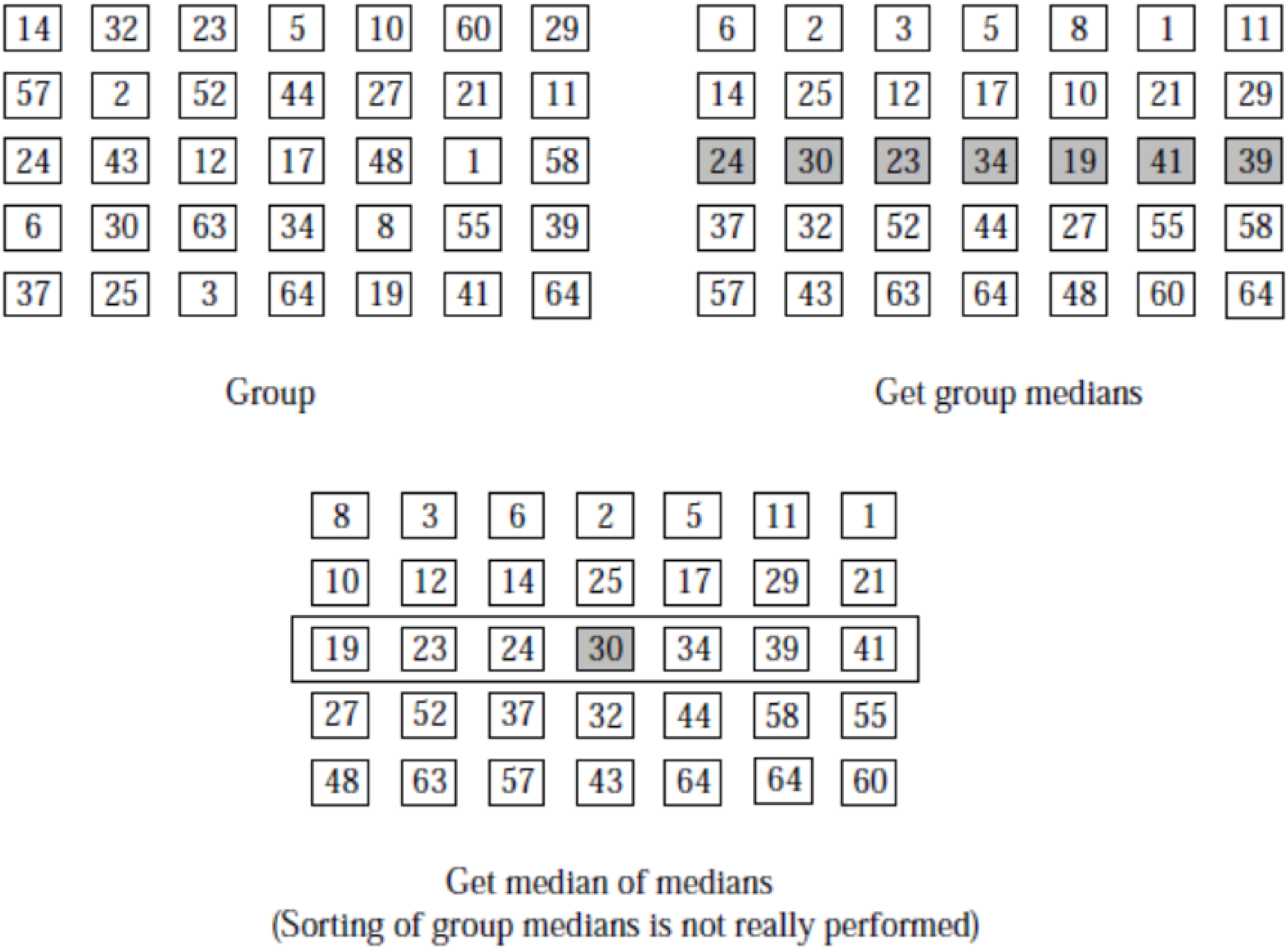

Figure 8: Choosing the Pivot. 30 is the final pivot.

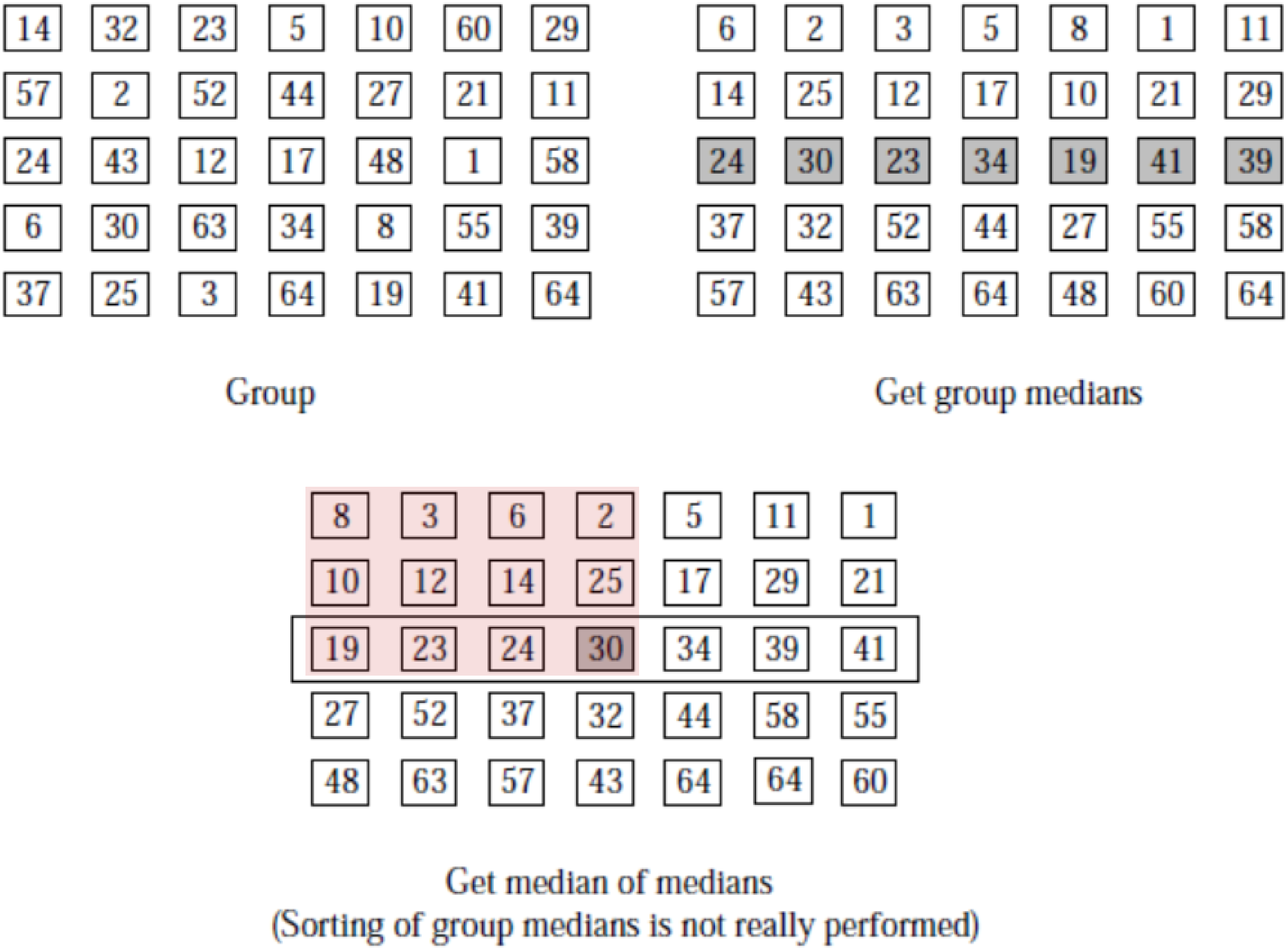

Figure 8: Choosing the Pivot. 30 is the final pivot.

Worst-Case Linear-Time Selection

- The algorithm in words:
  1. Divide *n* elements into groups of 5
  2. Find median of each group (*How? How long?*)
  3. Use Select() recursively to find median *x* of the $\lfloor n/5 \rfloor$ medians
  4. Partition the *n* elements around *x*. Let *k* = rank(*x*)
  5. **if** (i == k) **then** return x

     if (i < k) **then** use Select() recursively to find *i*th smallest element in first partition

     else (i > k) use Select() recursively to find (*i*-*k*)th smallest element in last partition

Worst-Case Linear-Time Selection

- (Sketch situation on the board)
- *How many of the 5-element medians are ≤ x?*
  - At least 1/2 of the medians = $\lfloor \lfloor n/5 \rfloor / 2 \rfloor = \lfloor n/10 \rfloor$
- *How many elements are ≤ x?*
  - At least $3 \lfloor n/10 \rfloor$ elements
- For large *n*, $3 \lfloor n/10 \rfloor \geq n/4$ *(How large?)*
- So at least $n/4$ elements $\leq x$
- Similarly: at least $n/4$ elements $\geq x$

Worst-Case Linear-Time Selection

- Thus after partitioning around *x*, step 5 will call Select() on at most 3*n*/4 elements
- The recurrence is therefore:

$$
\begin{aligned}
T(n) &\le T(\lfloor n/5 \rfloor) + T(3n/4) + \Theta(n) \\
&\le T(n/5) + T(3n/4) + \Theta(n) && \lfloor n/5 \rfloor \le n/5 \\
&\le cn/5 + 3cn/4 + \Theta(n) && \text{Substitute } T(n) = cn \\
&= 19cn/20 + \Theta(n) && \text{Combine fractions} \\
&= cn - (cn/20 - \Theta(n)) && \text{Express in desired form} \\
&\le cn \quad \text{if } c \text{ is big enough} && \text{What we set out to prove}
\end{aligned}
$$

Worst-Case Linear-Time Selection

- Intuitively:
  - Work at each level is a constant fraction (19/20) smaller
    - Geometric progression!
  - Thus the O(n) work at the root dominates

Linear-Time Median Selection

- Given a "black box" O(n) median algorithm, what can we do?
  - *i*th order statistic:
    - Find median *x*
    - Partition input around *x*
    - if ($i \le (n+1)/2$) recursively find *i*th element of first half
    - else find ($i - (n+1)/2$)th element in second half
    - $T(n) = T(n/2) + O(n) = O(n)$
  - *Can you think of an application to sorting?*

Linear-Time Median Selection

- Worst-case O(n lg n) quicksort
  - Find median *x* and partition around it
  - Recursively quicksort two halves
  - T(n) = 2T(n/2) + O(n) = O(n lg n)

The End