Algorithm and Data Structure Analysis (ADSA)

Minimum Spanning Trees

Properties of MSTs

An MST of a given graph G can be constructed by greedy algorithms.

Crucial properties:

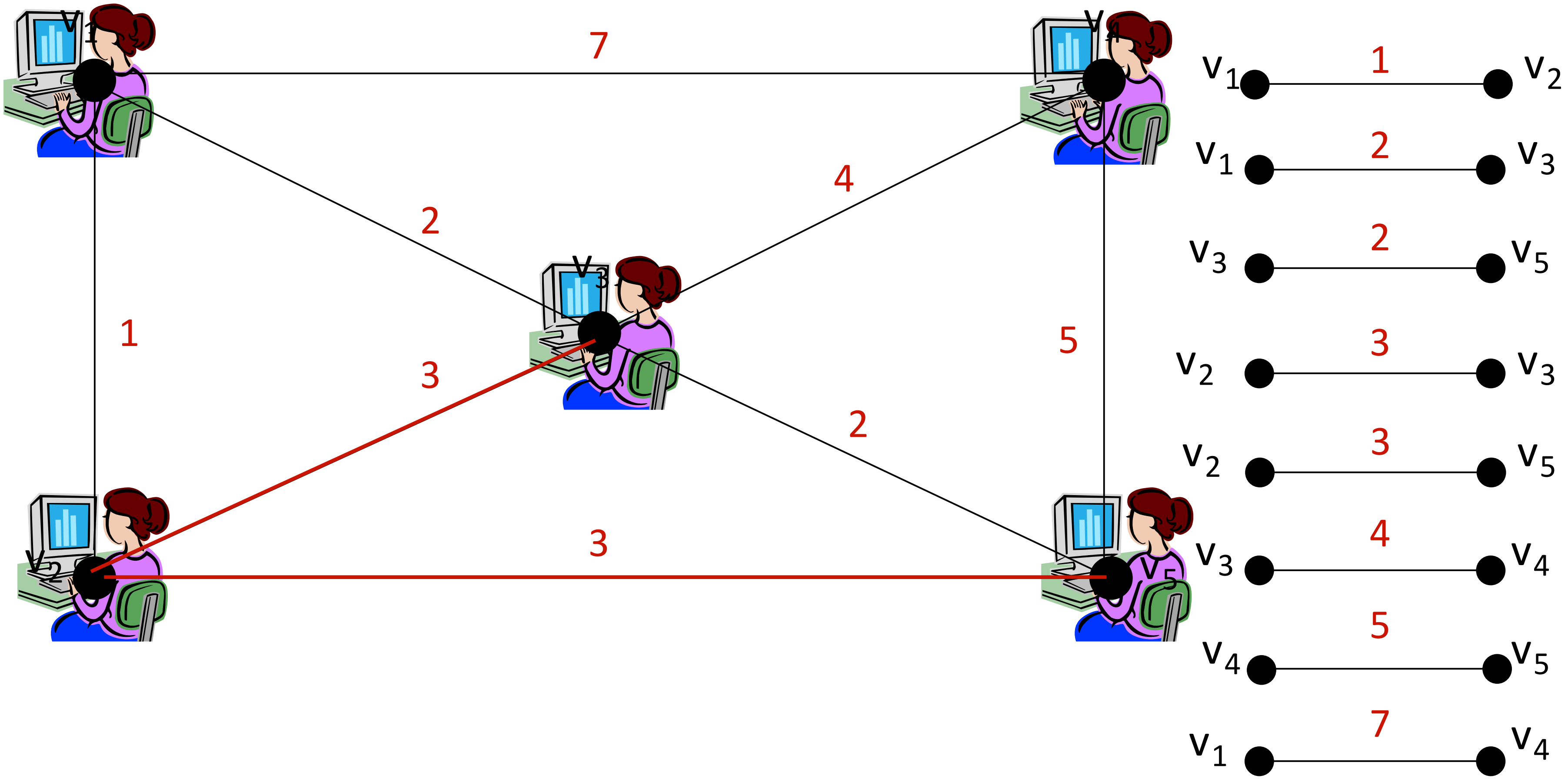
- Cut property (Let e be an edge of minimum cost in a cut C. Then there is an MST that contains e)
- Cycle property (an edge of maximal cost in any cycle does not need to be considered for computing an MST)

Description of Kruskal's algorithm:

- Sort in the edges of the graph with respect to increasing weights.
- Introduce the edges in ascending order that do not create cycles.

1. Sort the edges with respect to increasing weight

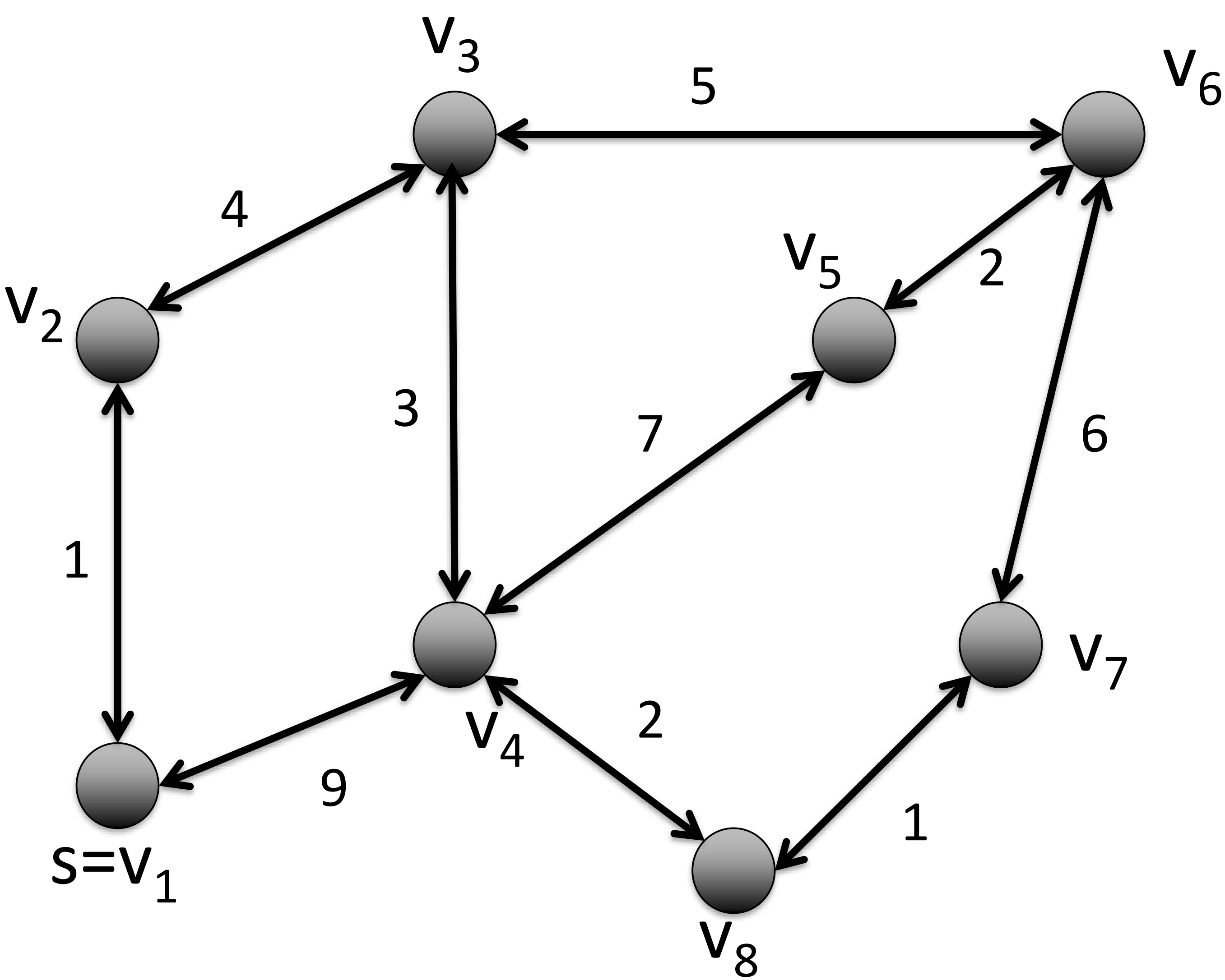
2. Introduce edges in ascending order that do not create cycles



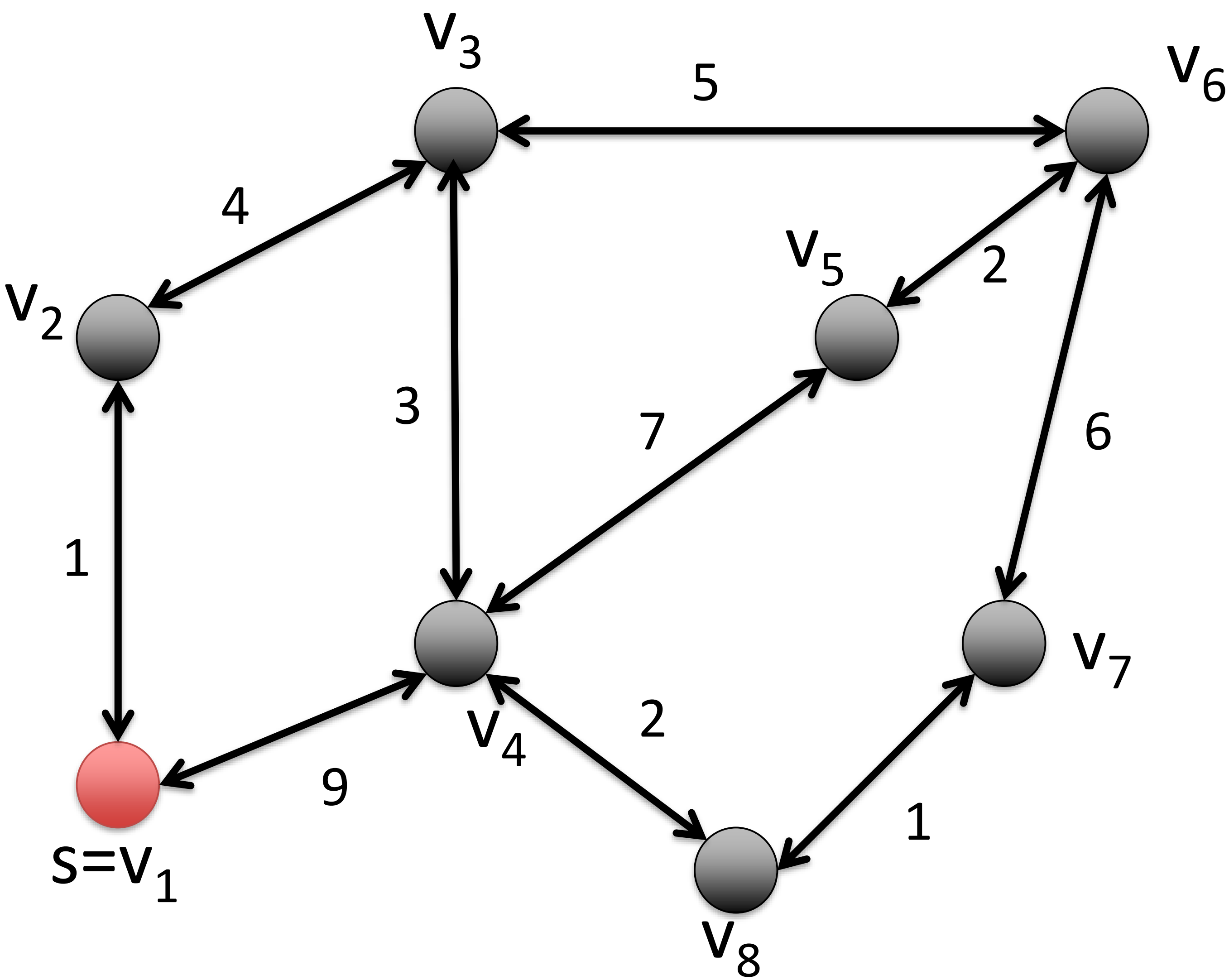
Jarnik-Prim Algorithm

- Similar to Dijkstra's algorithm for the single-source shortest path problem.
- Start with an arbitrary node s of V.
- Let S be the set of already connected nodes.
- In the beginning S={s} holds.
- Insert in each iteration an edge of minimal cost that connects a node u of S to a node v not contained in S (it's an edge of minimal cost in this cut).
- Add v to S and continue until all nodes are contained in S.

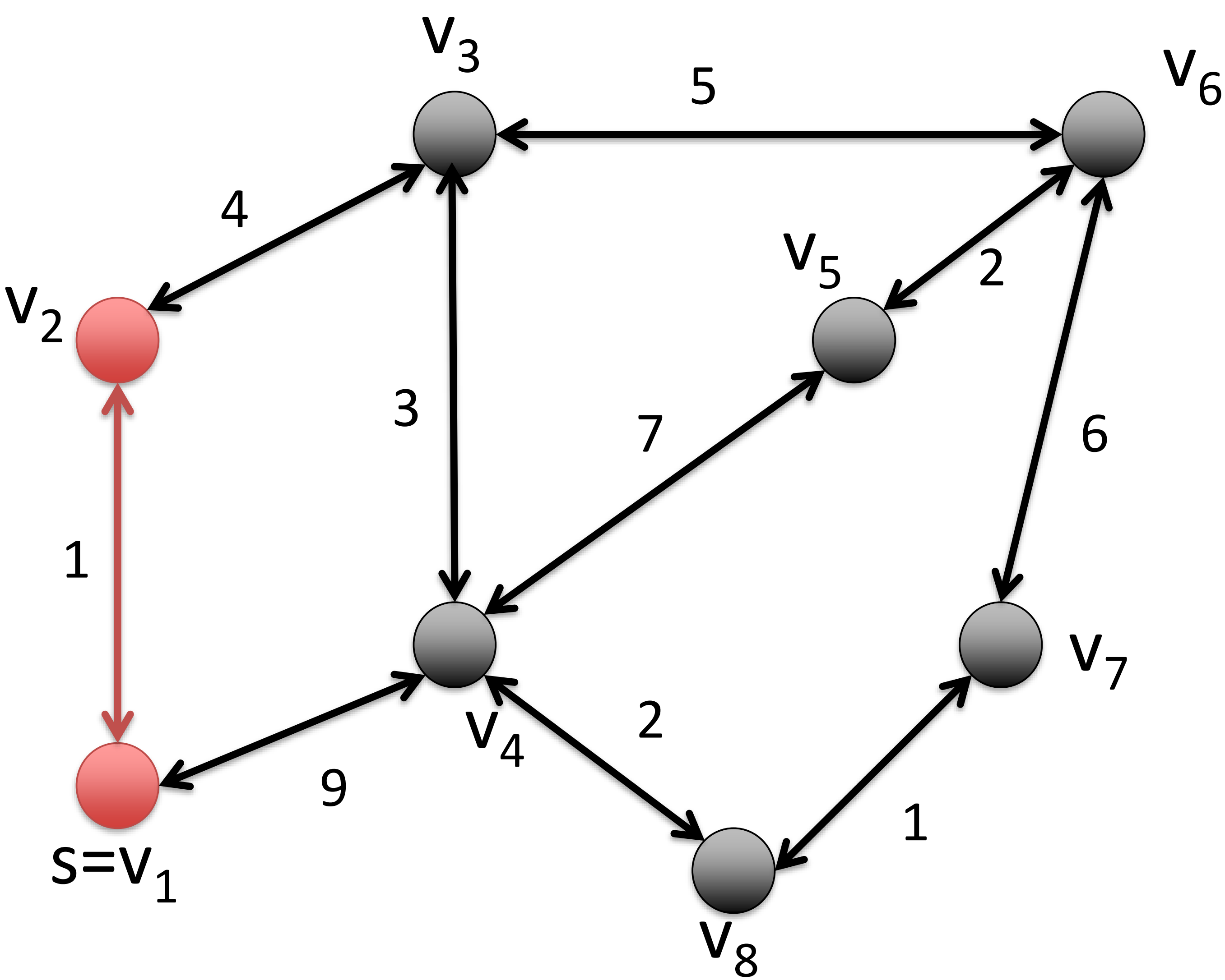
Example



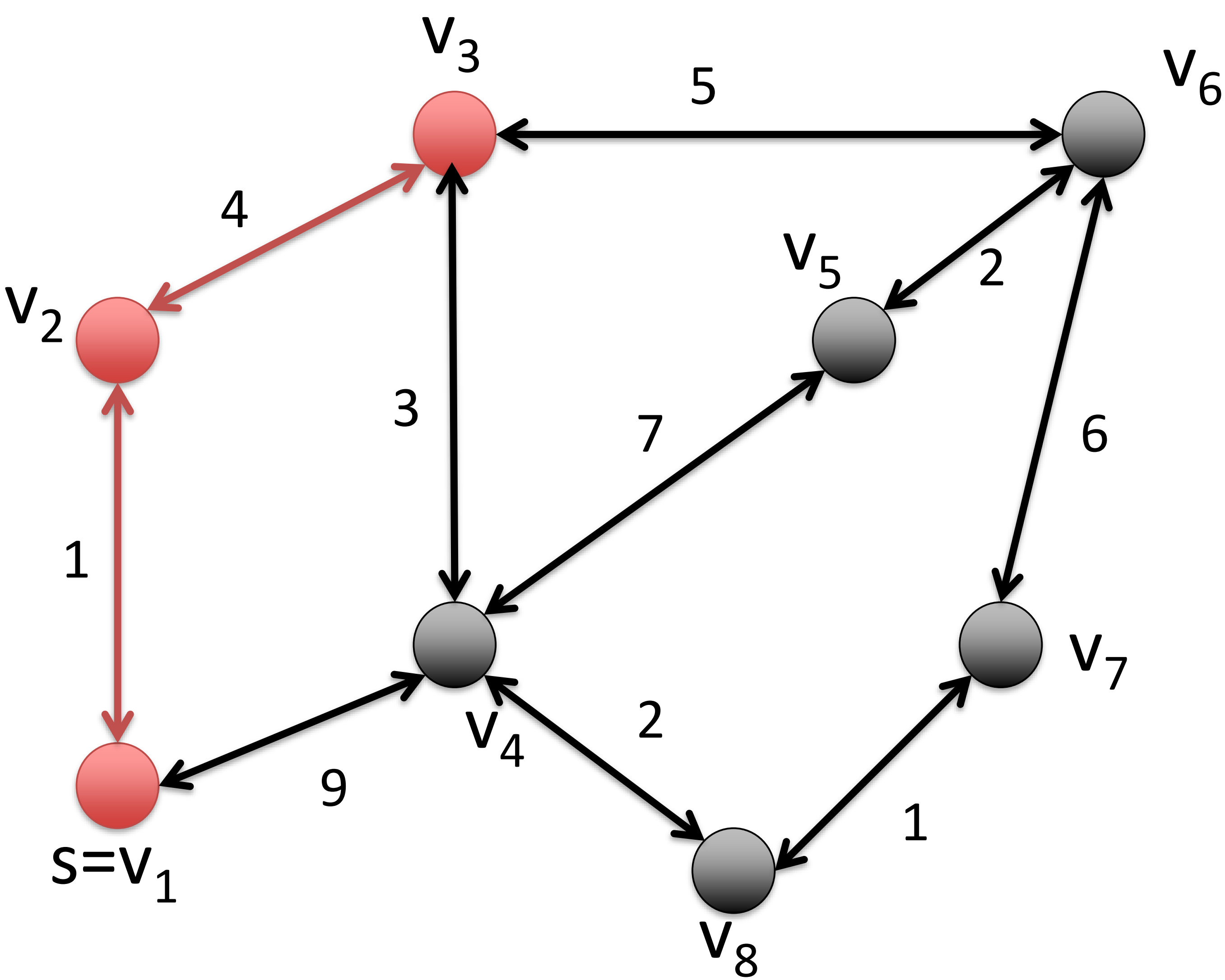
Example



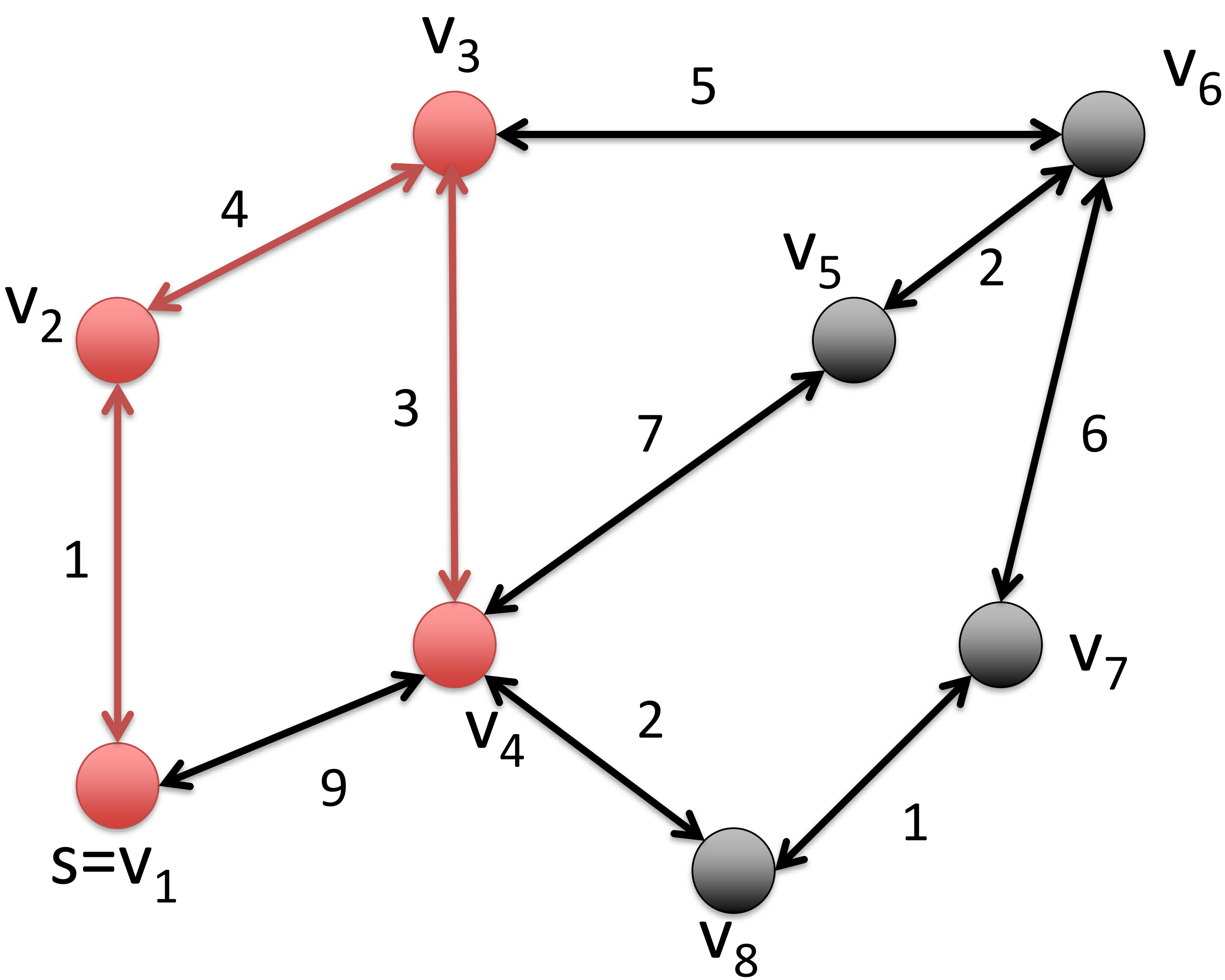
Example



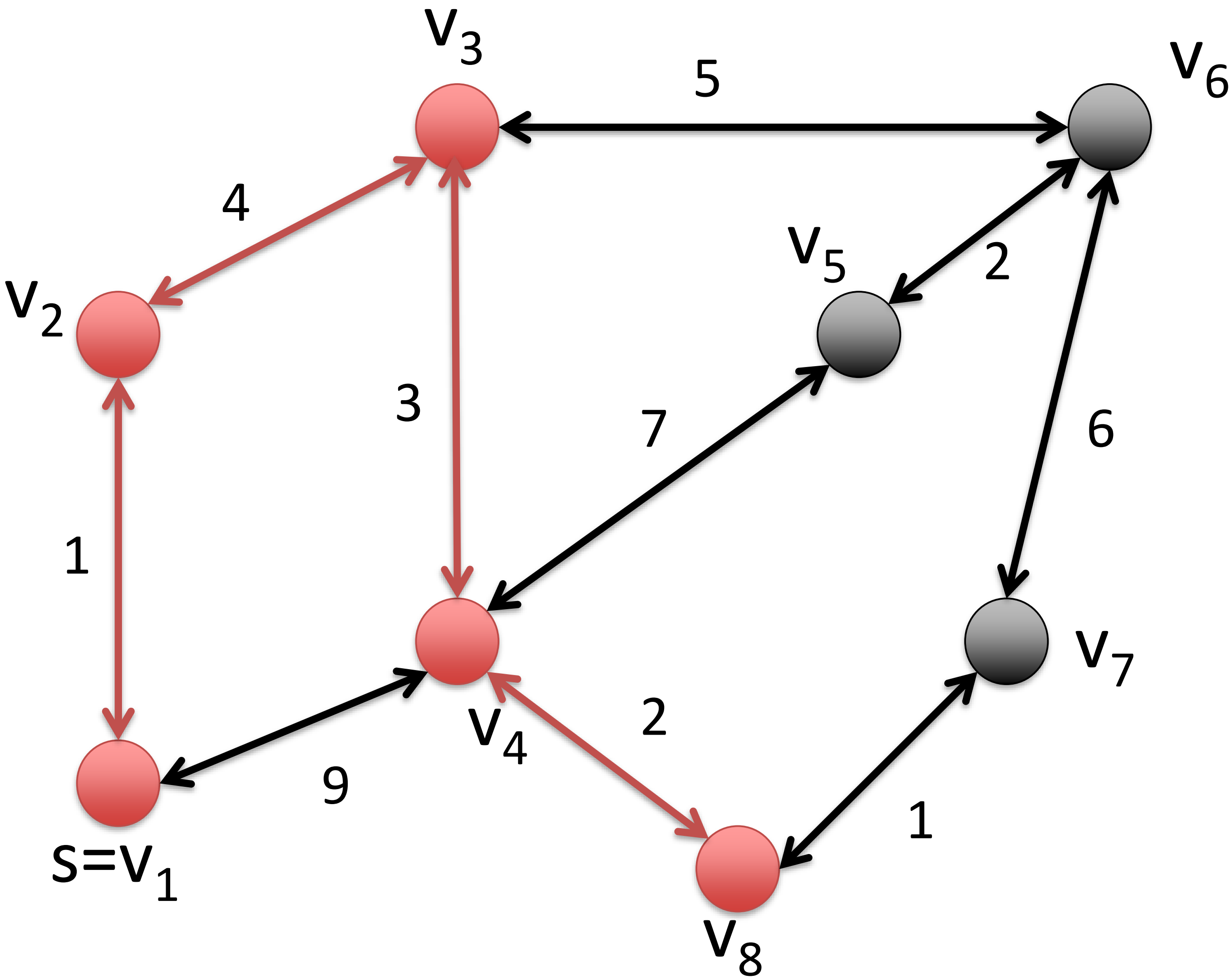
Example



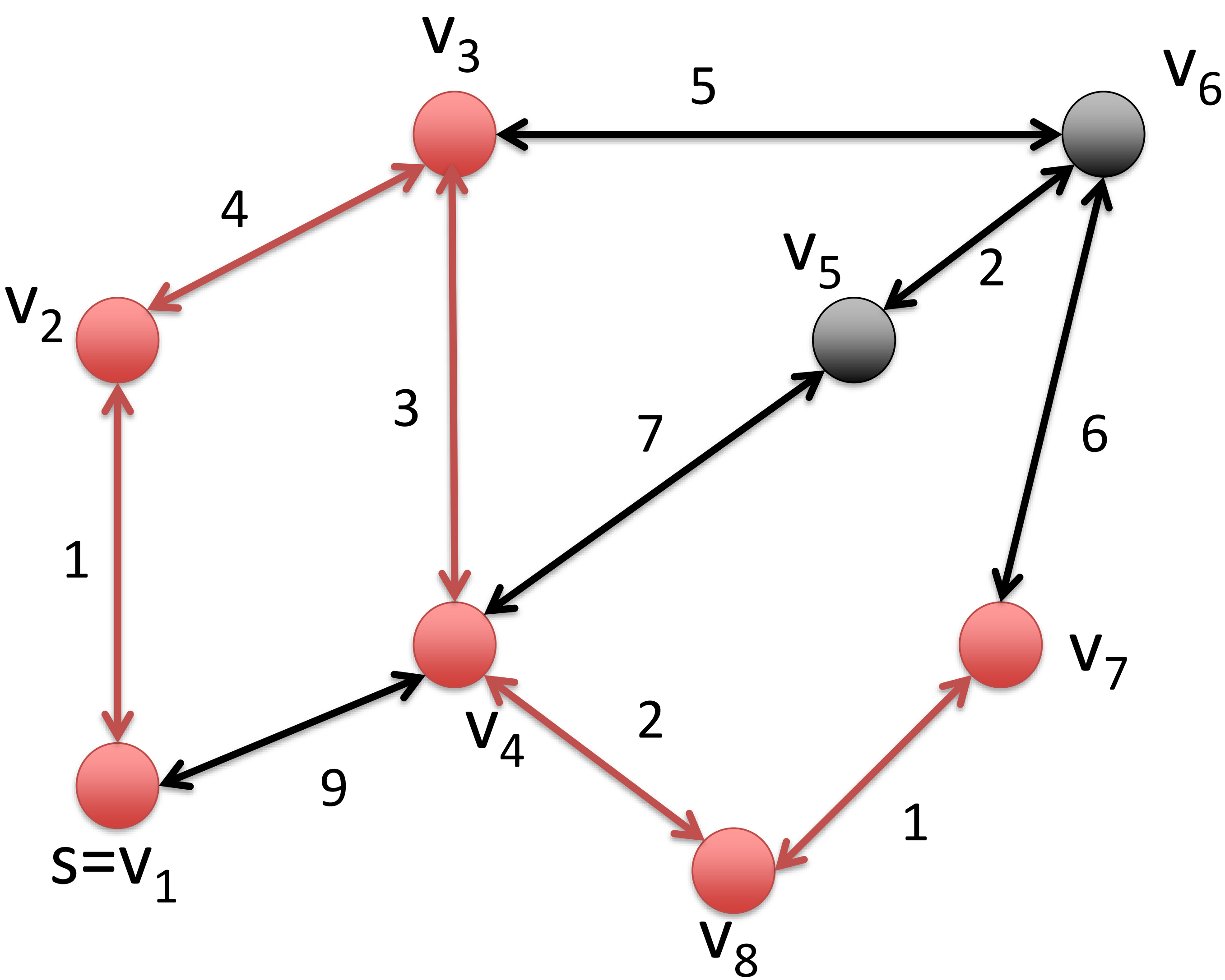
Example



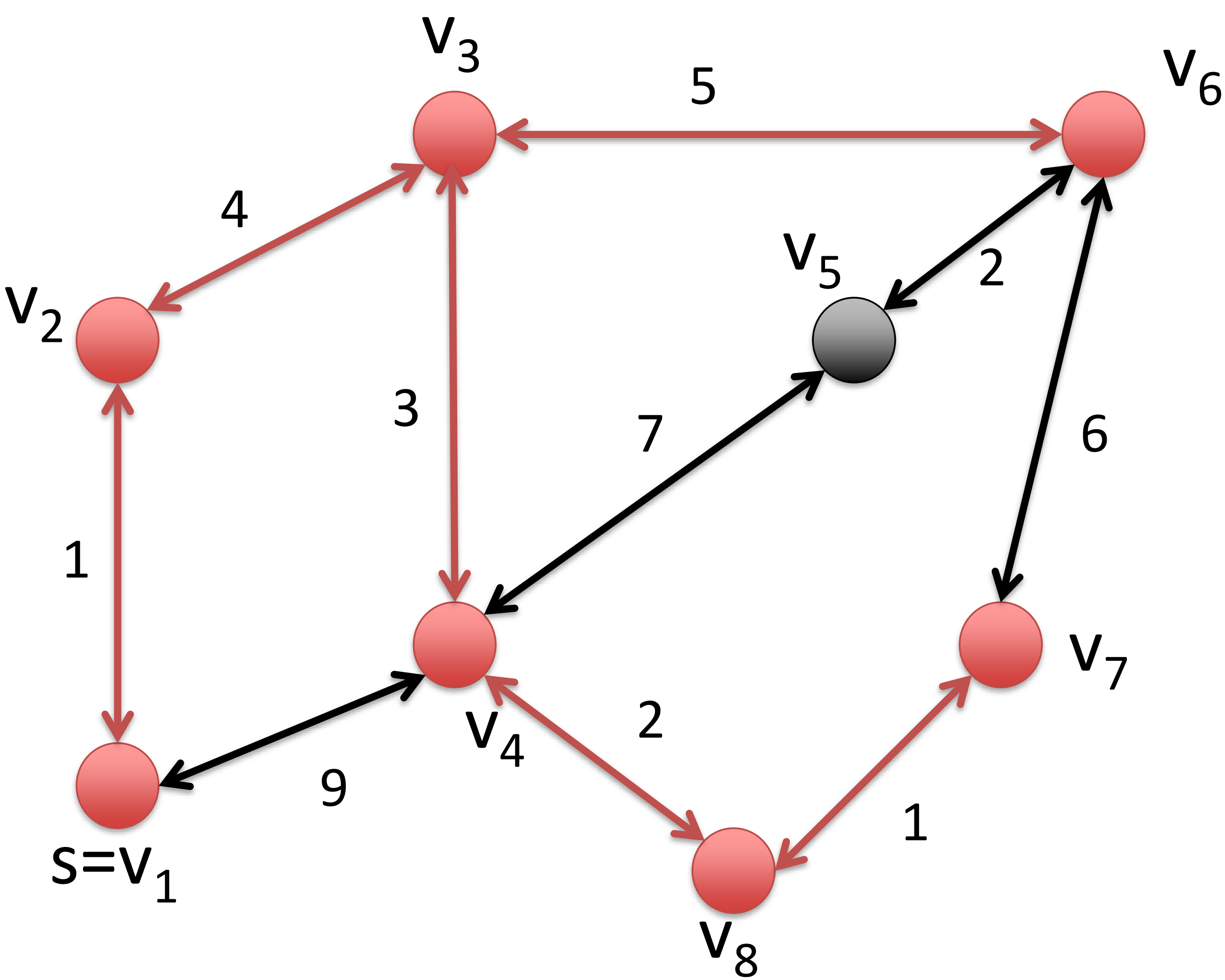
Example



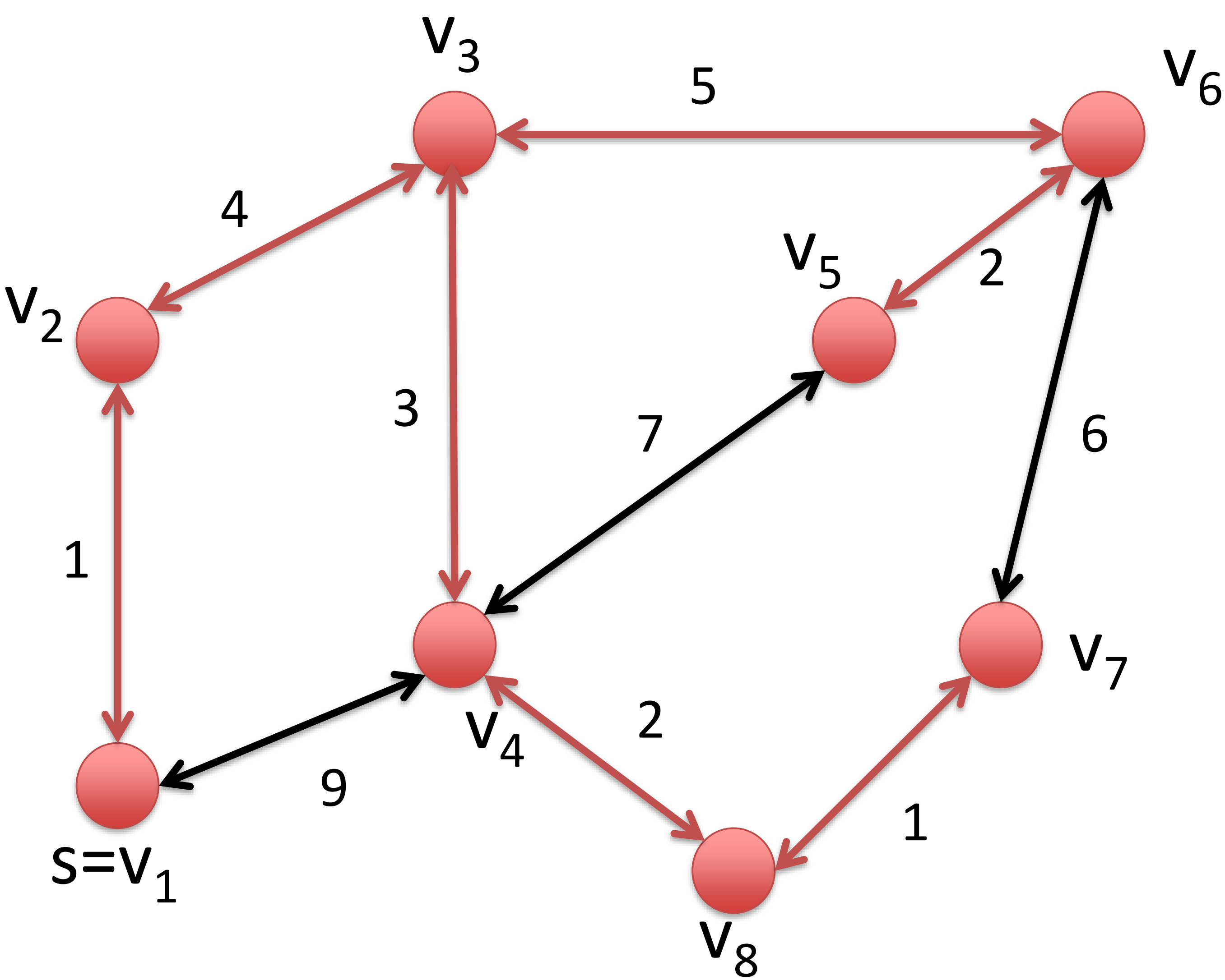
Example



Example



Example



Jarnik-Prim Algorithm Implementation

Function $jpMST : Set$ **of** $Edge$
  $d = \langle\infty, \ldots, \infty\rangle : NodeArray[1..n]$ **of** $\mathbb{R} \cup \{\infty\}$ // $d[v]$ is the distance of v from the tree
  $parent : NodeArray$ **of** $NodeId$ // parent$[v]$ is shortest edge between S and v
  $Q : NodePQ$ // uses $d[\cdot]$ as priority
  $Q.insert(s)$ for some arbitrary $s \in V$
  while $Q \neq \emptyset$ **do**
    $u := Q.deleteMin$
    $d[u] := 0$ // $d[u] = 0$ encodes $u \in S$
    foreach $edge\ e = (u, v) \in E$ **do**
      if $c(e) < d[v]$ **then** // $c(e) < d[v]$ implies $d[v] > 0$ and hence $v \notin S$
        $d[v] := c(e)$
        $parent[v] := u$
        if $v \in Q$ **then** $Q.decreaseKey(v)$ **else** $Q.insert(v)$
    invariant $\forall v \in Q : d[v] = \min\{c((u,v)) : (u,v) \in E \wedge u \in S\}$
  return $\{(v, parent[v]) : v \in V \setminus \{s\}\}$

Fig 11.3 Mehlhorn/Sanders

Runtime

- We can carry over the analysis for Dijkstra’s algorithm.
- Crucial again is the implementation of the priority queue.
- Overall runtime is $O(m + n \log n)$ when using Fibonacci heaps for the implementation of the priority queue.

Comparison

- Jarnik-Prim Algorithm can be implemented in time $O(m+n \log n)$
- Kruskal's Algorithm can be implemented in time $O(m \log m)$

Jarnik-Prim Algorithm is more efficient for dense graphs, i. e. where $m = \Theta(n^2)$ holds.