(GE120)
DISCRETE STRUCTURES

LECTURE # 6

Types of Function

By,
Dr. Mehwish Manzur

Layout of Today's Lecture

- Examples of Functions
- Types of Functions

Examples of Functions

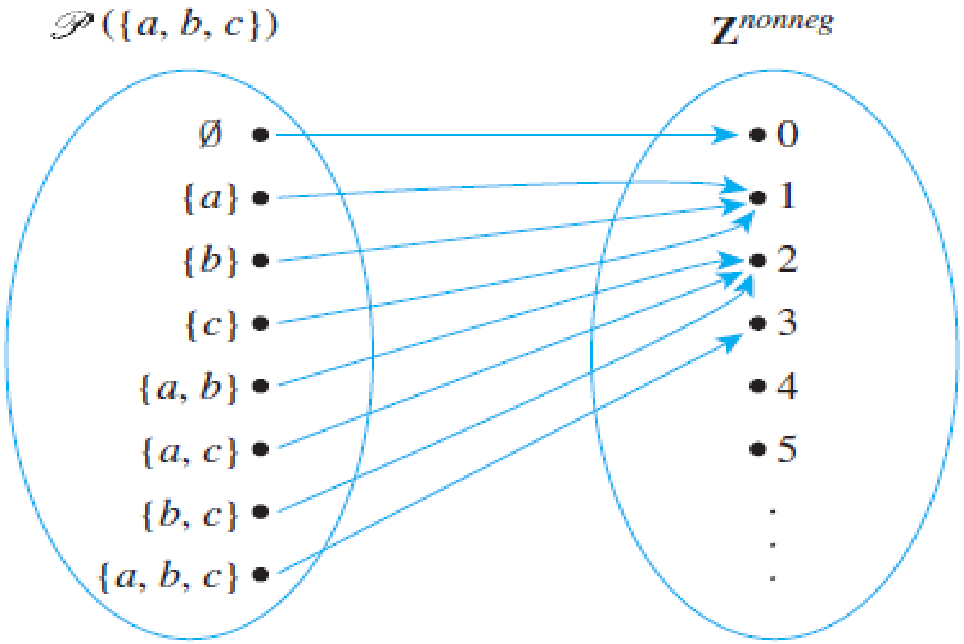
A Function defined on a power Set

P(A) denotes the set of all subsets of the set *A*. Define a function

$F: P(\{a, b, c\}) \rightarrow \mathbf{Z}^{nonneg}$ as follows: For each $X \in P(\{a, b, c\})$,

$F(X)$ = the number of elements in X.

Draw an arrow diagram for *F*.



Example:

Functions Defined on a Cartesian Product

Define functions $M : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $R : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows: For each ordered pair (a, b) of integers,

$$M(a, b) = ab \quad \text{and} \quad R(a, b) = (-a, b).$$

Then M is the multiplication function that sends each pair of real numbers to the product of the two, and R is the reflection function that sends each point in the plane that corresponds to a pair of real numbers to the mirror image of the point across the vertical axis. Find the following:

a. $M(-1, -1)$
b. $M\left(\frac{1}{2}, \frac{1}{2}\right)$
c. $M(\sqrt{2}, \sqrt{2})$
d. $R(2, 5)$
e. $R(-2, 5)$
f. $R(3, -4)$

Solution

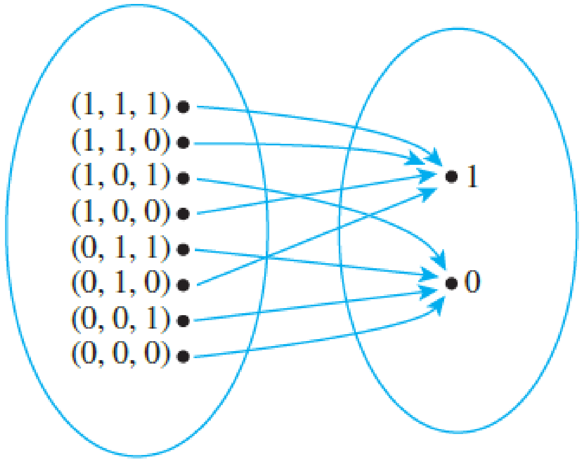
a. $(-1)(-1) = 1$
b. $(1/2)(1/2) = 1/4$
c. $\sqrt{2} \cdot \sqrt{2} = 2$
d. $(-2, 5)$
e. $(-(-2), 5) = (2, 5)$
f. $(-3, -4)$

Boolean Functions

An **(*n*-place) Boolean function** *f* is a function whose domain is the set of all ordered *n*-tuples of 0's and 1's and whose co-domain is the set {0, 1}.

Input			Output
P	***Q***	***R***	***S***
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

(a)



(b)

FIGURE Two Representations of a Boolean Function

Examples of Functions

A Boolean Function

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows:

For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f\,(x_1, x_2, x_3) = (x_1 + x_2 + x_3)\ mod\ 2.$$

Describe f using an input/output table.

$$f\,(1, 1, 1) = (1 + 1 + 1)\ mod\ 2 = 3\ mod\ 2 = 1$$

$$f\,(1, 1, 0) = (1 + 1 + 0)\ mod\ 2 = 2\ mod\ 2 = 0$$

Examples of Functions

A Boolean Function

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)\ mod\ 2$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

Checking Whether a Function Is Well Defined

It can sometimes happen that what appears to be a function defined by a rule is not really a function at all. To give an example, suppose we wrote, "Define a function $f : \mathbf{R} \to \mathbf{R}$ by specifying that for all real numbers x,

$f(x)$ is the real number y such that $x^2 + y^2 = 1$.

There are two distinct reasons why this description does not define a function. For almost all values of x,

1. there is no y that satisfies the given equation
2. there are two different values of y that satisfy the equation. For instance, when $x = 2$, there is no real number y such that $x^2 + y^2 = 1$, and when $x = 0$, both $y = -1$ and $y = 1$ satisfy the equation $x^2 + y^2 = 1$.

In general, we say that a "function" is **not well defined** if it fails to satisfy at least one of the requirements for being a function.

Example:

Checking Whether a Function Is Well Defined

A Function That Is Not Well Defined

Recall that **Q** represents the set of all rational numbers. Suppose you read that a function $f: \mathbf{Q} \rightarrow \mathbf{Z}$ is to be defined by the formula

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

That is, the integer associated by f to the number $\frac{m}{n}$ is m. Is f well defined? Why?

Solution The function f is not well defined. The reason is that fractions have more than one representation as quotients of integers. For instance, $\frac{1}{2} = \frac{3}{6}$. Now if f were a function,

since $\frac{1}{2} = \frac{3}{6}$.

$$f\left(\frac{1}{2}\right) \neq \left(\frac{3}{6}\right).$$

This contradiction shows that f is not well defined and, therefore, is not a function.

Sum/difference of Functions

Let F: $\mathbf{R} \rightarrow \mathbf{R}$ and G: $\mathbf{R} \rightarrow \mathbf{R}$ be functions. Define new functions $F + G$: $\mathbf{R} \rightarrow \mathbf{R}$: For all $x \in \mathbf{R}$,

$$(F + G)(x) = F(x) + G(x)$$

F and G must have same Domains and Codomains.

Equality of Functions

Theorem:

If $F: X \rightarrow Y$ and $G: X \rightarrow Y$ are functions,

Then $F = G$

if, and only if, $F(x) = G(x)$ for all $x \in X$.

Example

Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For every x in J_3,

$$f(x) = (x^2 + x + 1)\ mod\ 3 \quad \text{and} \quad g(x) = (x+2)^2\ mod\ 3.$$

Does $f = g$?

Solution

x	x^2+x+1	$f(x) = (x^2+x+1)\ mod\ 3$	$(x+2)^2$	$g(x) = (x+2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
1	3	$3\ mod\ 3 = 0$	9	$9\ mod\ 3 = 0$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$

Functions Acting on Sets

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and

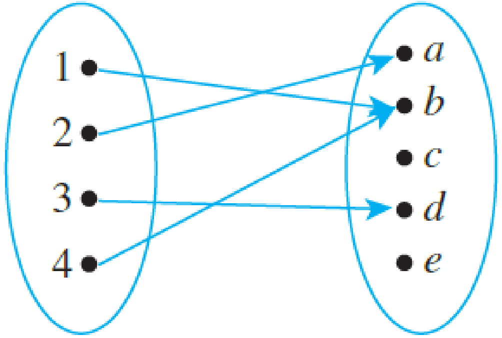
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Functions Acting on Sets

Example:

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F : X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Solution

$$F(A) = \{b\} \quad F(X) = \{a, b, d\} \quad F^{-1}(C) = \{1, 2, 4\} \quad F^{-1}(D) = \varnothing$$

Types of Functions

One-to-One Functions

A function *f* is said to be ***one-to-one***, or an ***injunction***, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of *f*. A function is said to be *injective* if it is one-to-one. $f(a)$ is the image of a in the range.

Or If every element of domain has unique image in Range.

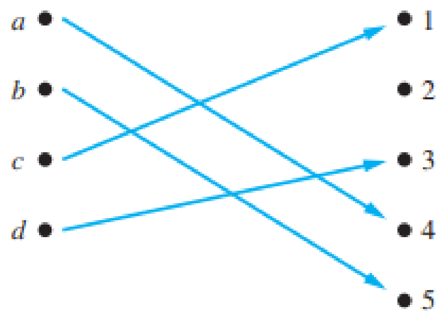


FIGURE 3 **A One-to-One Function.**

Remark: We can express that f is one-to-one using quantifiers as

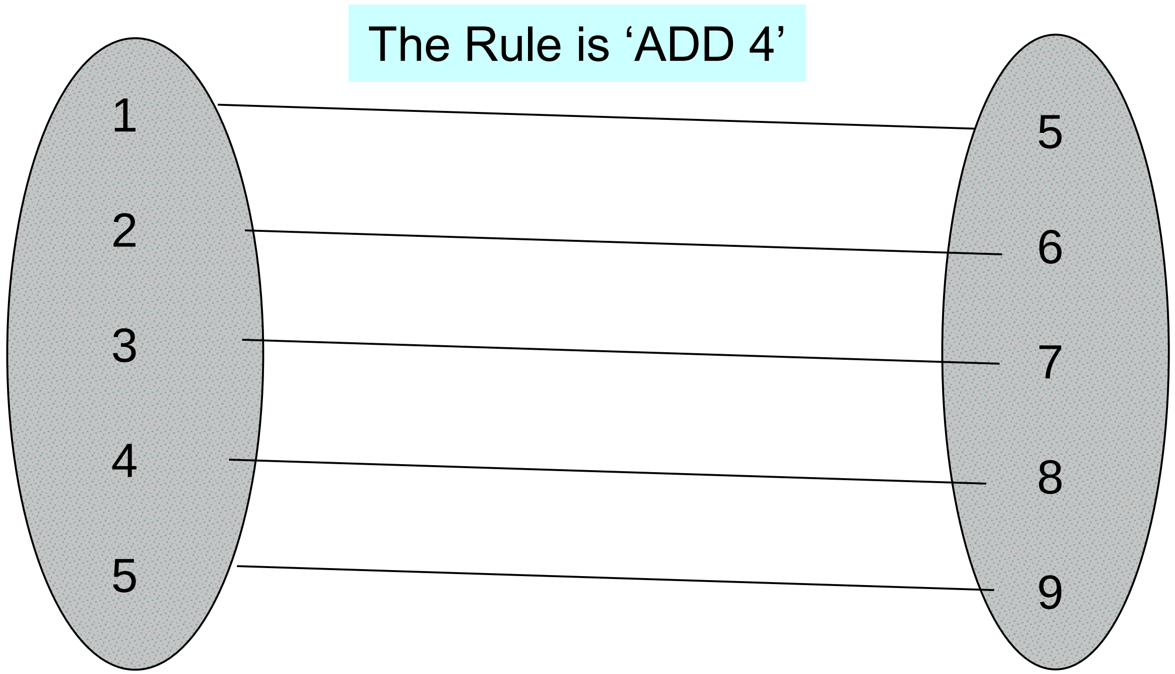
$$\forall a\, \forall b\, (f(a) = f(b) \rightarrow a = b)$$

or equivalently,

$$\forall a\, \forall b\, (a \neq b \rightarrow f(a) \neq f(b)),$$

where the universe of discourse is the domain of the function.

One-to-One Functions

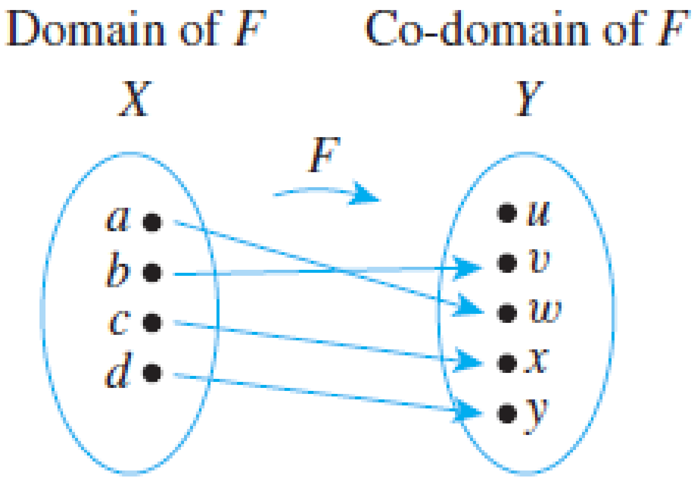


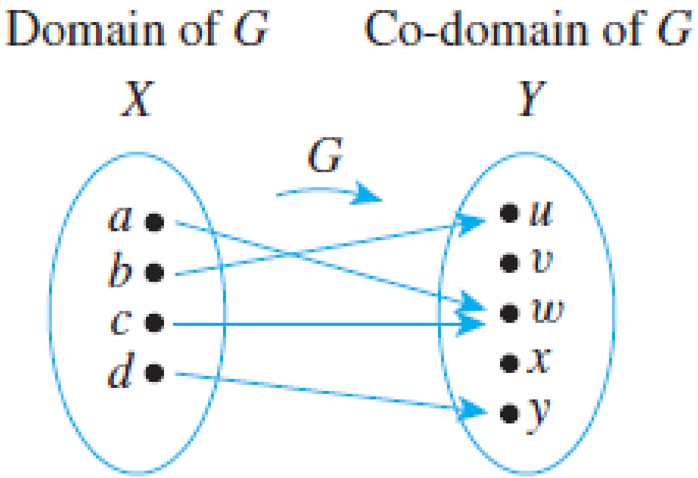
Dom (R) = {1, 2, 3, 4, 5}

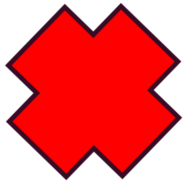
Codomain(R)={5, 6, 7, 8, 9,10}

One-to-One Functions

Identifying One-to-One functions defined on sets







EXAMPLE Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

Solution:

The function f is one-to-one because f takes on different values at the four elements of its domain.

EXAMPLE Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.

Solution: The function $f(x) = x^2$ is not one-to-one because, for instance,

$f(1) = f(-1) = 1$,

but $1 \neq -1$.

Note that the function $f(x) = x^2$ with its domain restricted to **Z**+ is one-to-one.

One-to-One Functions on Infinite Sets

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

To show that f is *not* one-to-one, you will ordinarily

find elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

One-to-One Functions on Infinite Sets

Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all} \quad x \in \mathbf{R}$$

and

$$g(n) = n^2 \quad \text{for all} \quad n \in \mathbf{Z}.$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

One-to-One Functions on Infinite Sets

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for all real numbers x, then f is one-to-one.

Proof:

Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that $x_1 = x_2$.]* By definition of f,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

which is what was to be shown.

One-to-One Functions on Infinite Sets

If the function $g: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $g(n) = n^2$, for all $n \in \mathbf{Z}$, then g is not one-to-one.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \quad \text{and also}$$
$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $\quad g(n_1) = g(n_2) \quad \text{but} \quad n_1 \neq n_2,$

and so g is not one-to-one.

Types of Functions

Onto Functions

A function f from A to B is called **onto**, or a **surjection**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

Remark: A function f is onto if $\forall y \, \exists x \, (f(x) = y)$,

where the domain for x is the domain of the function and the domain for y is the codomain of the function

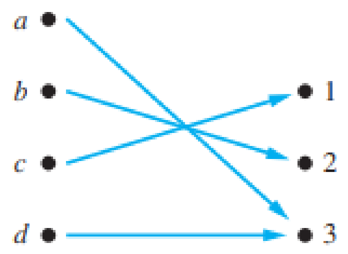
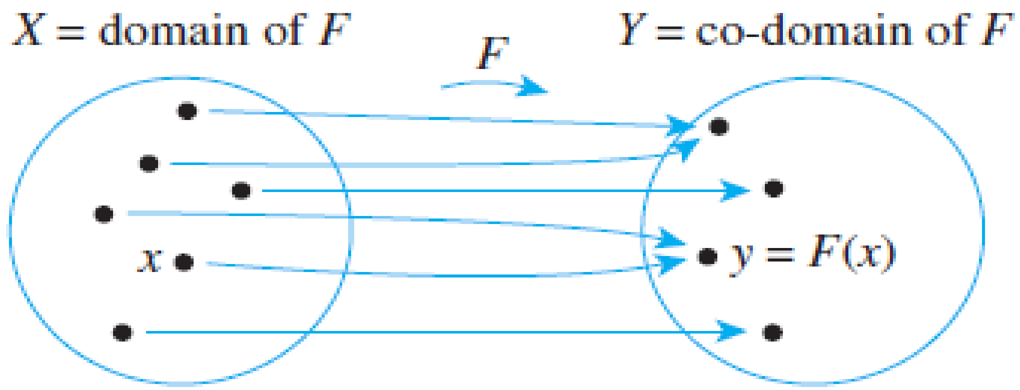


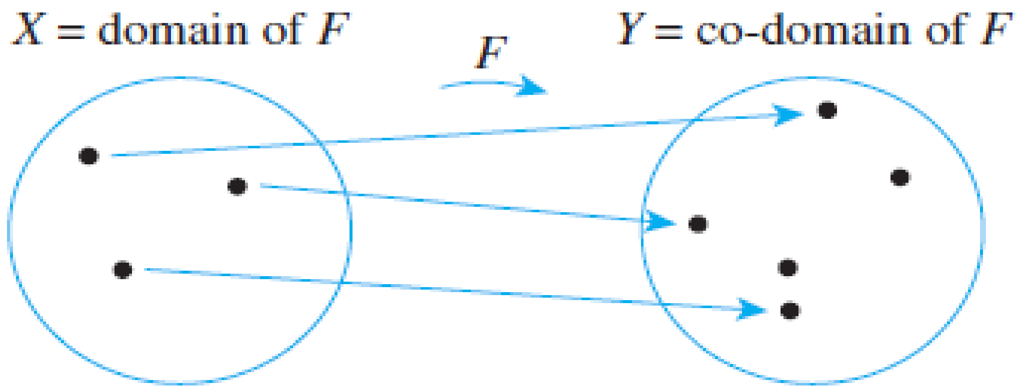
FIGURE 4 An Onto Function.

Onto Functions on Sets



Each element y in Y equals $F(x)$ for at least one x in X.

A Function That Is Onto

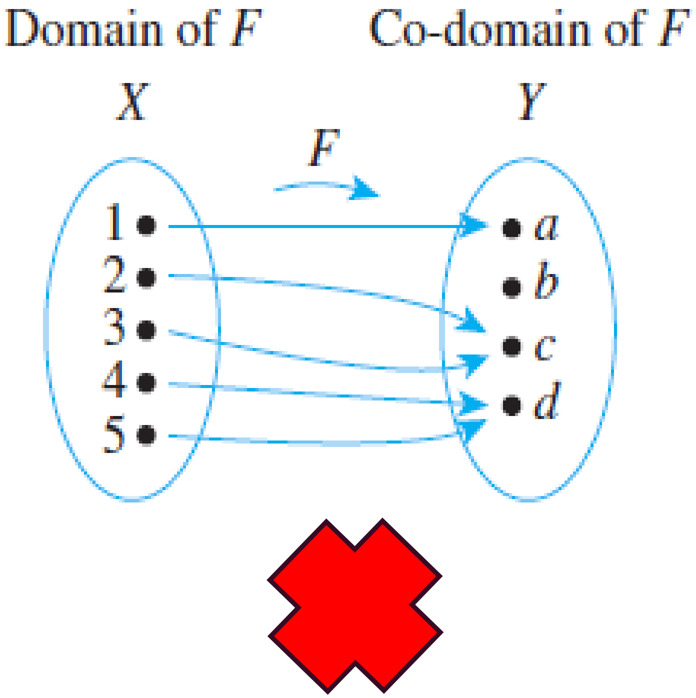


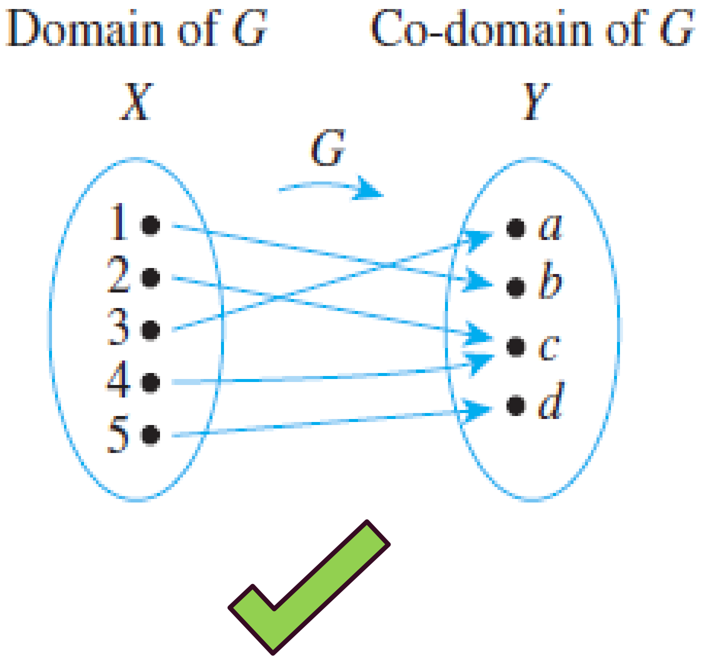
At least one element in Y does not equal $F(x)$ for any x in X.

A Function That Is Not Onto

Onto Functions on Sets

Identifying Onto Functions





- F is not onto because b ≠ F(x) for any x in X.
- G is onto because each element of Y equals G(x) for some x in X:
  a = G(3), b = G(1), c = G(2) = G(4), and d = G(5)

EXAMPLE Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

Solution:

Because all three elements of the codomain are images of elements in the domain, we see that f is onto.

EXAMPLE Is the function $f(x) = x^2$ from the set of integers to the set of integers onto?

Solution:

The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

Onto Functions on Infinite Sets

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus to prove F is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that y is any element of Y

and **show** that there is an element X of X with $F(x) = y$.

To prove F is *not* onto, you will usually

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

Onto Functions on Infinite Sets

Define $f: \mathbf{R} \to \mathbf{R}$

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

and

Is f onto? Prove or give a counterexample.

Onto Functions on Infinite Sets

To prove that *f* is onto, you must prove

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

1. There exists real number x such that y = f(x)?
2. Does f really send x to y?

Onto Functions on Infinite Sets

If $f: \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for all real numbers x, then f is onto.

Proof:

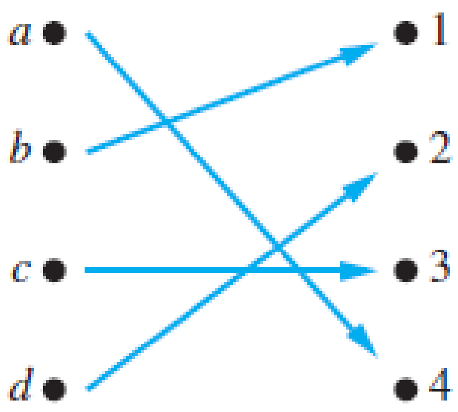
Let $y \in \mathbf{R}$. *[We must show that $\exists x$ in $\mathbf{R}$ such that $f(x) = y$.]* Let $x = (y+1)/4$. Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$
\begin{aligned}
f(x) &= f\left(\frac{y+1}{4}\right) && \text{by substitution} \\
&= 4 \cdot \left(\frac{y+1}{4}\right) - 1 && \text{by definition of } f \\
&= (y+1) - 1 = y && \text{by basic algebra.}
\end{aligned}
$$

[This is what was to be shown.]

One-to-One Correspondence

The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective.

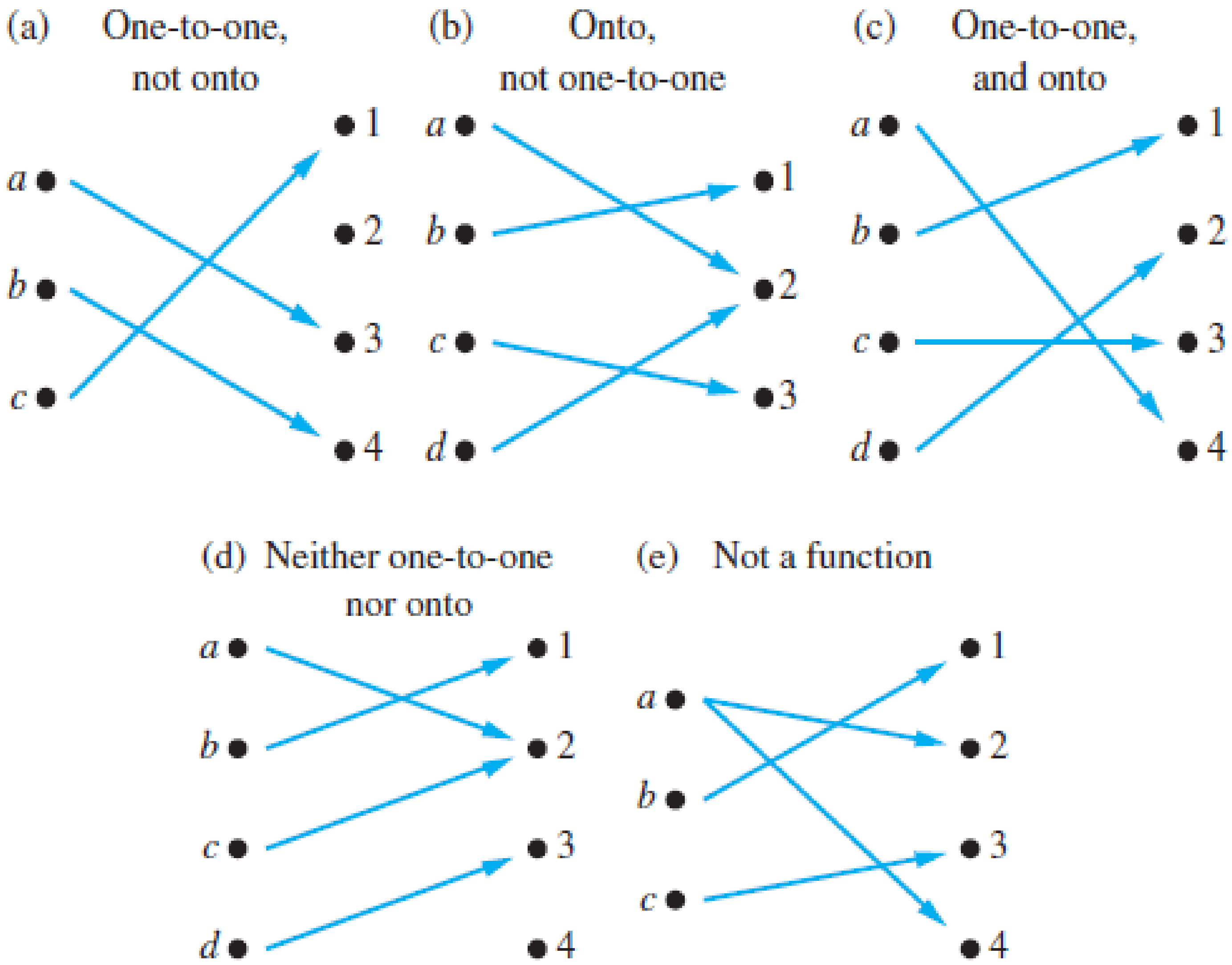


Example:

Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ with $f(a) = 4$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f a bijection?

Solution:

The function f is one-to-one and onto. It is one-to-one because no two values in the domain are assigned the same function value. It is onto because all four elements of the codomain are images of elements in the domain. Hence, f is a bijection.


(a) One-to-one, not onto
(b) Onto, not one-to-one
(c) One-to-one, and onto
(d) Neither one-to-one nor onto
(e) Not a function
a
b
c
d
1
2
3
4

Inverse Functions

Let f be a one-to-one correspondence from the set A to the set B. The ***inverse function*** of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

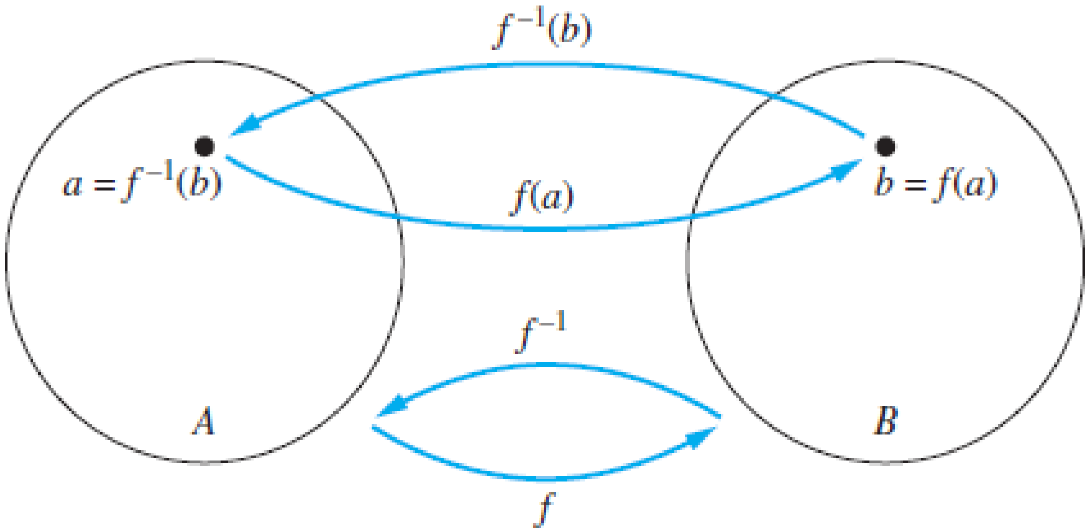


FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

Theorem

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.

EXAMPLE Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

Solution:

The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$.

EXAMPLE Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?

Solution: The function f has an inverse because it is a one-to-one correspondence. To reverse the correspondence, suppose that y is the image of x, so that $y = x + 1$. Then $x = y - 1$. This means that $y - 1$ is the unique element of $\mathbf{Z}$ that is sent to y by f. Consequently, $f^{-1}(y) = y - 1$.

EXAMPLE

Let f be the function from **R** to **R** with $f\ (x)\ =\ x^2$. Is f invertible?

Solution:

Because $f\ (-2) = f\ (2) = 4$, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.

(Note we can also show that f is not invertible because it is not onto.)

Finding an Inverse Function

The function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula

$f(x) = 4x - 1$, for all real numbers x

Solution For any *[particular but arbitrarily chosen]* y in $\mathbf{R}$, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But

$$\begin{aligned} & f(x) = y \\ \Leftrightarrow \quad & 4x - 1 = y && \text{by definition of } f \\ \Leftrightarrow \quad & x = \frac{y+1}{4} && \text{by algebra.} \end{aligned}$$

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

Some Important Functions

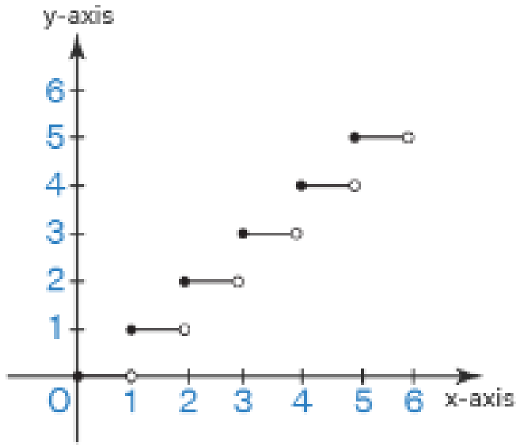
Floor Function:

- It is a function that takes an input as a real number and gives an output that is an integral value less than or equal to the input real number.
- The floor function gives the greatest integer output which is lesser than or equal to a given number.

The floor function is denoted by floor(x) or $\lfloor x \rfloor$.

- An example of floor function is

$$\lfloor 2.3 \rfloor = 2, \text{ and } \lfloor -3.4 \rfloor = -4.$$

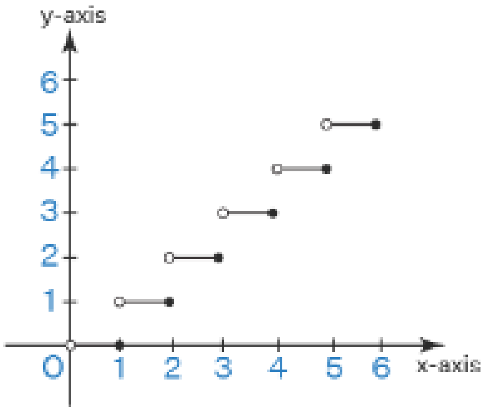


Floor Function

Ceiling Function:

- It is a function that takes an input as a real number and gives an output that is an integral value **greater than or equal to the input real number**. The ceiling function gives the least integer output which is greater than or equal to the given number.
- The ceiling function is denoted by ceil(x) or $\lceil x \rceil$.
- An example of a ceiling function is

$$\lceil 4.7 \rceil = 5, \text{ or } \lceil -2.8 \rceil = -2.$$



Ceiling Function

Examples

Compute $\lfloor x \rfloor$ *and* $\lceil x \rceil$ *for each of the following values of x:*

a. 25/4
b. *0.999*
c. *−2.01*

Solution:

a. 25/4 = *6.25 and 6 < 6.25 < 7;* hence $\lfloor 25/4 \rfloor = 6$ and $\lceil 25/4 \rceil = 7$

b. 0 < *0.999 < 1;* hence $\lfloor 0.999 \rfloor = 0$ and $\lceil 0.999 \rceil = 1$

c. −3 < −*2.01* < −*2;* hence $\lfloor -2.01 \rfloor = -3$ and $\lceil -2.01 \rceil = -2$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$

Example

The 1,370 students at a college are given the opportunity to take buses to an out-of-town game. Each bus holds a maximum of 40 passengers.

a. For reasons of economy, the athletic director will send only full buses. What is the maximum number of buses the athletic director will send?

b. If the athletic director is willing to send one partially filled bus, how many buses will be needed to allow all the students to take the trip?

$$a).\ \lfloor 1370/40 \rfloor = \lfloor 34.25 \rfloor = 34$$

$$\text{b).}\ \lceil 1370/40 \rceil = \lceil 34.25 \rceil = 35$$

General Values of Floor

If k is an integer, what are $\lfloor \mathrm{k} \rfloor$ and $\lfloor k+1/2 \rfloor$? Why?

Solution:

Suppose k is an integer. Then

$\lfloor k \rfloor = k$ becuase k is an integer and $k \le k < k+1$

and

$\left\lfloor \mathrm{k}+\dfrac{1}{2} \right\rfloor = k$ becuase k is an integer and $k \le k + \frac{1}{2} < k+1$

Cont…

Is the following statement true or false? For all real Numbers *x* and *y*, $\lfloor x+y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$

Solution: The statement is false, take x = y = 1/2, then

$$\lfloor x+y \rfloor = \left\lfloor \frac{1}{2} + \frac{1}{2} \right\rfloor = \lfloor 1 \rfloor = 1$$

where as

$$\lfloor x \rfloor + \lfloor y \rfloor = \left\lfloor \frac{1}{2} \right\rfloor + \left\lfloor \frac{1}{2} \right\rfloor = 0 + 0 = 0$$

Hence

$$\lfloor x+y \rfloor \neq \lfloor x \rfloor + \lfloor y \rfloor$$

Cont….

Theorem: For any integer n

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}; & \text{if n is even} \\ \frac{n-1}{2}; & \text{if n is odd} \end{cases}$$

Proof: Suppose *n is a integer.Then* either *n is odd or n is even.*

Case 1: in this case n = 2k+1 for some integers k.

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k+1}{2} \right\rfloor = \left\lfloor \frac{2k}{2} + \frac{1}{2} \right\rfloor$$

$$\left\lfloor k + \frac{1}{2} \right\rfloor = k$$

because k is an integer and $k \le k + 1/2 < k + 1$. But since

$$n = 2k+1 \Rightarrow n-1 = 2k \Rightarrow k = \frac{n-1}{2}$$

Cont…

Since both the left-hand and right-hand sides equal *k, they are equal to each* other. That is,

$$\left\lfloor \frac{n}{2} \right\rfloor = k$$

Case 2: In this case, *n* = *2k for some integer k.*

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \lfloor k \rfloor = k$$

Since k=n/2 by the definition of even number. So

$$\left\lfloor \frac{n}{2} \right\rfloor = k = \frac{n}{2}$$


Thank you!