历届期末试题选解（定积分的计算）

定积分计算小结：

一、定积分的换元积分法：

换元积分公式：

（1）第一换元公式：$\int_a^b f(\varphi(x))\varphi'(x)\mathrm{d}x = \int_\alpha^\beta f(u)\mathrm{d}u$，其中

$$u=\varphi(x)，\quad \varphi'(x)\mathrm{d}x=\mathrm{d}u，\quad \alpha=\varphi(a)，\quad \beta=\varphi(b). \text{（例 7，例 9，例 11）}$$

注：第一换元法又称凑微分法，不一定要换元，如果没有换元，就不必换限. 例如，

$$\int_0^{\frac{\pi}{2}} \mathrm{e}^{\sin x}\cos x\mathrm{d}x = \int_0^{\frac{\pi}{2}} \mathrm{e}^{\sin x}\mathrm{d}\sin x = \mathrm{e}^{\sin x}\Big|_0^{\frac{\pi}{2}} = \mathrm{e}-1.$$

（2）第二换元公式：$\int_a^b f(x)\mathrm{d}x = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t$，其中

$$x=\varphi(t)，\quad \mathrm{d}x=\varphi'(t)\mathrm{d}t，\quad a=\varphi(\alpha)，\quad b=\varphi(\beta). \text{（例 8）}$$

（3）常用的变量替换：

被积函数含有 $\sqrt{a^2-x^2}$，则可令 $x=a\sin t$；（见例 2，例 4）

被积函数含有 $\sqrt{a^2+x^2}$，则可令 $x=a\tan t$；

被积函数含有 $\sqrt{x^2-a^2}$，则可令 $x=a\sec t$；

被积函数含有 $\sqrt[m]{ax+b}$，则可令 $t=\sqrt[m]{ax+b}$；（见例 1，例 6，例 14）

被积函数含有 $\sqrt[m]{\dfrac{ax+b}{cx+d}}$，则可令 $t=\sqrt[m]{\dfrac{ax+b}{cx+d}}$.

对于三角函数有理式，可采用万能代换 $t=\tan\dfrac{x}{2}$，万能代换用到的公式：

$$\sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2},\ \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t. \quad \text{（见例 4）}$$

二、分部积分法：

分部积分公式：$\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\mathrm{d}x$. （例 3，例 12，例 15）

三、常用的结论：

(1) 根据定积分的几何意义，$\int_{-a}^{a}\sqrt{a^2-x^2}\,\mathrm{d}x$ 表示半圆 $y=\sqrt{a^2-x^2}$ 与 x 轴围成的半圆面积，即

$$\int_{-a}^{a}\sqrt{a^2-x^2}\,\mathrm{d}x=\frac{\pi}{2}a^2.\quad(\text{例2，例5})$$

(2) $\int_{-a}^{a}f(x)\mathrm{d}x=\frac{1}{2}\int_{-a}^{a}[f(x)+f(-x)]\mathrm{d}x$，特别地，如果 $f(x)$ 是奇函数，则 $\int_{-a}^{a}f(x)\mathrm{d}x=0$；如果 $f(x)$

是偶函数，则 $\int_{-a}^{a}f(x)\mathrm{d}x=2\int_{0}^{a}f(x)\mathrm{d}x$．(结论见书上 P249 例5) (例2，例4，例5，例7，例10，例

11，例13)

(3) $\int_{0}^{\pi}xf(\sin x)\mathrm{d}x=\frac{\pi}{2}\int_{0}^{\pi}f(\sin x)\mathrm{d}x$；(结论见书上 P249 例6) (例3，例10，例15)

(4) $\int_{0}^{\pi}f(\sin x)\mathrm{d}x=2\int_{0}^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x$，(例10)

证明：$\int_{0}^{\pi}f(\sin x)\mathrm{d}x=\int_{0}^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}f(\sin x)\mathrm{d}x$．

令 $x=\pi-t$，则 $\int_{\frac{\pi}{2}}^{\pi}f(\sin x)\mathrm{d}x=\int_{\frac{\pi}{2}}^{0}f(\sin(\pi-t))(-\mathrm{d}t)=\int_{0}^{\frac{\pi}{2}}f(\sin t)\mathrm{d}t=\int_{0}^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x$．

于是，$\int_{0}^{\pi}f(\sin x)\mathrm{d}x=2\int_{0}^{\frac{\pi}{2}}f(\sin x)\mathrm{d}x.$

(5) $\int_{0}^{\frac{\pi}{2}}\sin^n x\mathrm{d}x=\int_{0}^{\frac{\pi}{2}}\cos^n x\mathrm{d}x=\begin{cases}\dfrac{n-1}{n}\dfrac{n-3}{n-2}\cdots\dfrac{2}{3},&n\text{为奇数时，}\\[2mm]\dfrac{n-1}{n}\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\dfrac{\pi}{2},&n\text{为偶数时．}\end{cases}$ (结论见 P253 例12) (例10)

(6) $\int\dfrac{1}{\sqrt{x^2+a^2}}\mathrm{d}x=\ln(x+\sqrt{x^2+a^2})+C$．(结论见 P202 例22)．(例11)

四、历届试题

例1. 求定积分 $\int_{0}^{4}\dfrac{\mathrm{d}x}{1+\sqrt{x}}$；(2016-2017)

解： 令 $t=\sqrt{x}$，则 $x=t^2$，$\mathrm{d}x=(t^2)'\mathrm{d}t=2t\mathrm{d}t$．

$x=0$ 时，$t=0$；$x=4$ 时，$t=2$．于是，

$$\int_{0}^{4}\dfrac{\mathrm{d}x}{1+\sqrt{x}}=\int_{0}^{2}\dfrac{1}{1+t}\cdot2t\mathrm{d}t=2\int_{0}^{2}(1-\dfrac{1}{1+t})\mathrm{d}t$$

$$=2(t-\ln(1+t))\Big|_{0}^{2}=4-2\ln3.$$

例 2. 求定积分 $\int_{-3}^{3}[\sqrt{9-x^2}+x\ln(1+x^2)]\mathrm{d}x$；(2016-2017)

解： 注意到 $f(x)=x\ln(1+x^2)$ 为奇函数，故 $\int_{-3}^{3}x\ln(1+x^2)\mathrm{d}x=0$.

所以，$\int_{-3}^{3}[\sqrt{9-x^2}+x\ln(1+x^2)]\mathrm{d}x=\int_{-3}^{3}\sqrt{9-x^2}\mathrm{d}x$.

令 $x=3\sin t,-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$，$\mathrm{d}x=(3\sin t)'\mathrm{d}t=3\cos t\mathrm{d}t$.

$x=-3$ 时，$t=-\dfrac{\pi}{2}$；$x=3$ 时，$t=\dfrac{\pi}{2}$.

于是，$\qquad\int_{-3}^{3}\sqrt{9-x^2}\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{9-9\sin^2 t}\cdot 3\cos t\mathrm{d}t$

$$=9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^2 t\mathrm{d}t=9\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\mathrm{d}t$$

$$=9(\frac{1}{2}t+\frac{1}{4}\sin 2t)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\frac{9}{2}\pi.$$

故 $\qquad\int_{-3}^{3}[\sqrt{9-x^2}+x\ln(1+x^2)]\mathrm{d}x=\dfrac{9}{2}\pi.$

例 3. 求定积分 $\int_{0}^{\pi}x\sqrt{\cos^2 x-\cos^4 x}\mathrm{d}x$；(2016-2017)

解一： $\int_{0}^{\pi}x\sqrt{\cos^2 x-\cos^4 x}\mathrm{d}x=\int_{0}^{\pi}x\sqrt{\cos^2 x(1-\cos^2 x)}\mathrm{d}x$

$$=\int_{0}^{\pi}x|\cos x|\sin x\mathrm{d}x$$

$$=\int_{0}^{\frac{\pi}{2}}x\cos x\sin x\mathrm{d}x-\int_{\frac{\pi}{2}}^{\pi}x\cos x\sin x\mathrm{d}x.$$

其中 $\qquad\int_{0}^{\frac{\pi}{2}}x\cos x\sin x\mathrm{d}x=\dfrac{1}{2}\int_{0}^{\frac{\pi}{2}}x\sin 2x\mathrm{d}x$

$$=\frac{1}{2}x(-\frac{1}{2}\cos 2x)\Big|_{0}^{\frac{\pi}{2}}-\frac{1}{2}\int_{0}^{\frac{\pi}{2}}(-\frac{1}{2}\cos 2x)\mathrm{d}x\qquad\text{（分部积分）}$$

$$=\frac{\pi}{8}+\frac{1}{4}\int_{0}^{\frac{\pi}{2}}\cos 2x\mathrm{d}x=\frac{\pi}{8}+\frac{1}{4}\cdot\frac{1}{2}\sin 2x\Big|_{0}^{\frac{\pi}{2}}=\frac{\pi}{8}.$$

$$\int_{\frac{\pi}{2}}^{\pi}x\cos x\sin x\mathrm{d}x=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}x\sin 2x\mathrm{d}x$$

$$=\frac{1}{2}x(-\frac{1}{2}\cos 2x)\Big|_{\frac{\pi}{2}}^{\pi}-\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(-\frac{1}{2}\cos 2x)\mathrm{d}x$$

$$= -\frac{\pi}{4} - \frac{\pi}{8} + \frac{1}{4}\int_{\frac{\pi}{2}}^{\pi}\cos 2x\mathrm{d}x = -\frac{3\pi}{8} + \frac{1}{4}\cdot\frac{1}{2}\sin 2x\Big|_{\frac{\pi}{2}}^{\pi} = -\frac{3\pi}{8}.$$

故 $$\int_0^{\pi} x\sqrt{\cos^2 x - \cos^4 x}\mathrm{d}x = \frac{\pi}{8} - (-\frac{3\pi}{8}) = \frac{\pi}{2}.$$

解二： 注意到，$\sqrt{\cos^2 x - \cos^4 x} = \sqrt{\cos^2 x(1-\cos^2 x)} = \sqrt{(1-\sin^2 x)\sin^2 x}$ 是 $\sin x$ 的函数，于是，

$$\int_0^{\pi} x\sqrt{\cos^2 x - \cos^4 x}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi}\sqrt{\cos^2 x - \cos^4 x}\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\pi}|\cos x|\sin x\mathrm{d}x$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\cos x\sin x\mathrm{d}x - \frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}\cos x\sin x\mathrm{d}x$$

$$= \frac{\pi}{4}\int_0^{\frac{\pi}{2}}\sin 2x\mathrm{d}x - \frac{\pi}{4}\int_{\frac{\pi}{2}}^{\pi}\sin 2x\mathrm{d}x$$

$$= \frac{\pi}{4}(-\frac{1}{2}\cos 2x)\Big|_0^{\frac{\pi}{2}} - \frac{\pi}{4}(-\frac{1}{2}\cos 2x)\Big|_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

例 4. 求定积分 $\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\frac{1}{(1+x)\sqrt{1-x^2}}\mathrm{d}x$；（2017-2018）

解一： 令 $x = \sin t$，则 $\mathrm{d}x = \cos t\mathrm{d}t$。$x = -\frac{1}{\sqrt{2}}$ 时，$t = -\frac{\pi}{4}$；$x = \frac{1}{\sqrt{2}}$ 时，$t = \frac{\pi}{4}$。

故 $$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\frac{1}{(1+x)\sqrt{1-x^2}}\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{(1+\sin t)\cos t}\cos t\mathrm{d}t = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{1+\sin t}\mathrm{d}t$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1-\sin t}{1-\sin^2 t}\mathrm{d}t = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1-\sin t}{\cos^2 t}\mathrm{d}t$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\cos^2 t}\mathrm{d}t + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{-\sin t}{\cos^2 t}\mathrm{d}t.$$

因为 $\frac{-\sin t}{\cos^2 t}$ 为奇函数，则 $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{-\sin t}{\cos^2 t}\mathrm{d}t = 0$。

故 $$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}}\frac{1}{(1+x)\sqrt{1-x^2}}\mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{\cos^2 t}\mathrm{d}t = \tan t\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2.$$

解二： 令 $x = \sin t$，则 $dx = \cos t\, dt$．$x = -\dfrac{1}{\sqrt{2}}$ 时，$t = -\dfrac{\pi}{4}$；$x = \dfrac{1}{\sqrt{2}}$ 时，$t = \dfrac{\pi}{4}$．

故
$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{(1+x)\sqrt{1-x^2}}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(1+\sin t)\cos t}\cos t\, dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t}\, dt .$$

利用==万能代换== $t = \tan\dfrac{x}{2}$，则

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t}\, dt = \int_{-\tan\frac{\pi}{8}}^{\tan\frac{\pi}{8}} \frac{1}{1+\dfrac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\, dt$$

$$= 2\int_{-\tan\frac{\pi}{8}}^{\tan\frac{\pi}{8}} \frac{1}{(t+1)^2}\, dt$$

$$= -\frac{2}{t+1}\Bigg|_{-\tan\frac{\pi}{8}}^{\tan\frac{\pi}{8}}$$

$$= -\frac{2}{\tan\dfrac{\pi}{8}+1} + \frac{2}{-\tan\dfrac{\pi}{8}+1}$$

$$= \frac{4\tan\dfrac{\pi}{8}}{1-\tan^2\dfrac{\pi}{8}} = 2\tan\frac{\pi}{4} = 2 .$$

解三： 令 $x = \sin t$，则 $dx = \cos t\, dt$．$x = -\dfrac{1}{\sqrt{2}}$ 时，$t = -\dfrac{\pi}{4}$；$x = \dfrac{1}{\sqrt{2}}$ 时，$t = \dfrac{\pi}{4}$．

故
$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{(1+x)\sqrt{1-x^2}}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(1+\sin t)\cos t}\cos t\, dt = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t}\, dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\sin^2\dfrac{t}{2} + \cos^2\dfrac{t}{2} + 2\sin\dfrac{t}{2}\cos\dfrac{t}{2}}\, dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\left(\sin\dfrac{t}{2} + \cos\dfrac{t}{2}\right)^2}\, dt$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2\left(\dfrac{1}{\sqrt{2}}\sin\dfrac{t}{2} + \dfrac{1}{\sqrt{2}}\cos\dfrac{t}{2}\right)^2}\, dt$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2(\frac{\pi}{4} - \frac{t}{2})} \mathrm{d}t$$

$$= -\tan(\frac{\pi}{4} - \frac{t}{2}) \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = -\tan\frac{\pi}{8} + \tan\frac{3\pi}{8}$$

$$= -\tan(\frac{\pi}{4} - \frac{\pi}{8}) + \tan(\frac{\pi}{8} + \frac{\pi}{4})$$

$$= -\frac{1 - \tan\frac{\pi}{8}}{1 + \tan\frac{\pi}{8}} + \frac{1 + \tan\frac{\pi}{8}}{1 - \tan\frac{\pi}{8}} = \frac{2 \cdot 2\tan\frac{\pi}{8}}{1 - \tan^2\frac{\pi}{8}} = 2\tan\frac{\pi}{4} = 2.$$

解四： 令 $x = \sin t$，则 $\mathrm{d}x = \cos t\, \mathrm{d}t$．$x = -\frac{1}{\sqrt{2}}$ 时，$t = -\frac{\pi}{4}$；$x = \frac{1}{\sqrt{2}}$ 时，$t = \frac{\pi}{4}$．

故

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{(1+x)\sqrt{1-x^2}} \mathrm{d}x = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{(1+\sin t)\cos t} \cos t\, \mathrm{d}t = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin t} \mathrm{d}t.$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{1}{1+\sin t} + \frac{1}{1-\sin t} \right) \mathrm{d}t$$

$$= \frac{1}{2} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2}{1-\sin^2 t} \mathrm{d}t = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos^2 t} \mathrm{d}t = \tan t \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = 2.$$

例 5. 计算 $\int_{-1}^{1} (1 + x + \sqrt{1-x^2})^2 \mathrm{d}x$；（2017—2018）

解：
$$\int_{-1}^{1} (1 + x + \sqrt{1-x^2})^2 \mathrm{d}x = \int_{-1}^{1} [(1 + \sqrt{1-x^2})^2 + x^2 + 2x\sqrt{1-x^2}] \mathrm{d}x$$

$$= \int_{-1}^{1} [1 + 2\sqrt{1-x^2} + 1 - x^2 + x^2] \mathrm{d}x + 2\int_{-1}^{1} x\sqrt{1-x^2} \mathrm{d}x$$

$$= 2\int_{-1}^{1} (1 + \sqrt{1-x^2}) \mathrm{d}x + 2\int_{-1}^{1} x\sqrt{1-x^2} \mathrm{d}x.$$

注意到，$x\sqrt{1-x^2}$ 是奇函数，故 $\int_{-1}^{1} x\sqrt{1-x^2} \mathrm{d}x = 0$．

而
$$\int_{-1}^{1} (1 + \sqrt{1-x^2}) \mathrm{d}x = \int_{-1}^{1} \mathrm{d}x + \int_{-1}^{1} \sqrt{1-x^2} \mathrm{d}x = 2 + \frac{1}{2}\pi \cdot 1^2 = 2 + \frac{\pi}{2}.$$

于是，$\int_{-1}^{1} (1 + x + \sqrt{1-x^2})^2 \mathrm{d}x = 4 + \pi$．

注：（1）若 $f(x)$ 是奇函数，则 $\int_{-a}^{a} f(x)\mathrm{d}x = 0$；（2）$\int_{-a}^{a} \sqrt{a^2 - x^2} \mathrm{d}x = \frac{1}{2}\pi a^2$（半圆的面积）．

例 6. 计算 $\int_{-3}^{1} \frac{x}{1 + \sqrt{1-x}} \mathrm{d}x$；（2018—2019）

解一： $\displaystyle\int_{-3}^{1}\frac{x}{1+\sqrt{1-x}}\mathrm{d}x=\int_{-3}^{1}\frac{x(1-\sqrt{1-x})}{1^2-(\sqrt{1-x})^2}\mathrm{d}x=\int_{-3}^{1}(1-\sqrt{1-x})\mathrm{d}x$

$$=[x+\frac{2}{3}\sqrt{(1-x)^3}\,]\Big|_{-3}^{1}=1-(-3+\frac{2}{3}\cdot 8)=-\frac{4}{3}.$$

解二： 令 $\sqrt{1-x}=t$ ，则 $x=1-t^2$ ， $\mathrm{d}x=-2t\mathrm{d}t$.

$$\int_{-3}^{1}\frac{x}{1+\sqrt{1-x}}\mathrm{d}x=\int_{2}^{0}\frac{1-t^2}{1+t}(-2t)\mathrm{d}t$$

$$=2\int_{0}^{2}(t-t^2)\mathrm{d}t$$

$$=2(\frac{t^2}{2}-\frac{t^3}{3})\Big|_{0}^{2}=2\cdot(2-\frac{8}{3})=-\frac{4}{3}.$$

注： 换元后，一定要注意换限.

例 7. 计算 $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\sin x}{1+\cos x}\mathrm{d}x$ ； （2018—2019）

解一： $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\sin x}{1+\cos x}\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+\cos x}\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x}{1+\cos x}\mathrm{d}x$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2\cos^2\frac{x}{2}}\mathrm{d}x-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+\cos x}\mathrm{d}(1+\cos x)$$

$$=\tan\frac{x}{2}\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}-\ln(1+\cos x)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=2.$$

解二： $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\sin x}{1+\cos x}\mathrm{d}x=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+\cos x}\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x}{1+\cos x}\mathrm{d}x$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+\cos x}\mathrm{d}x+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x}{1+\cos x}\mathrm{d}x.$$

注意到， $\dfrac{\sin x}{1+\cos x}$ 是奇函数，故 $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x}{1+\cos x}\mathrm{d}x=0$.

又 $\dfrac{1}{1+\cos x}$ 为偶函数，则

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{1+\cos x}\mathrm{d}x=2\int_{0}^{\frac{\pi}{2}}\frac{1}{1+\cos x}\mathrm{d}x$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2\cos^2 \frac{x}{2}} \mathrm{d}x = 2\tan \frac{x}{2}\Big|_0^{\frac{\pi}{2}} = 2.$$

例 8. 计算 $\int_{-1}^0 \frac{x^2}{(x+2)^3} \mathrm{d}x$；（2019—2020）

解： 令 $t = x + 2$，则

$$\int_{-1}^0 \frac{x^2}{(x+2)^3} \mathrm{d}x = \int_1^2 \frac{(t-2)^2}{t^3} \mathrm{d}t$$

$$= \int_1^2 (\frac{1}{t} - \frac{4}{t^2} + \frac{4}{t^3}) \mathrm{d}t$$

$$= (\ln t + \frac{4}{t} - \frac{2}{t^2})\Big|_1^2$$

$$= (\ln t + \frac{4}{t} - \frac{2}{t^2})\Big|_1^2$$

$$= \ln 2 + 2 - \frac{1}{2} - (0 + 4 - 2) = \ln 2 - \frac{1}{2}.$$

例 9. 计算 $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \mathrm{d}x$；（2019—2020）

解一： $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{(1+\sin x)(1-\sin x)} \mathrm{d}x$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x}{\cos^2 x} \mathrm{d}x = \int_0^{\frac{\pi}{4}} (\sec^2 x - \sec x \tan x) \mathrm{d}x$$

$$= (\tan x - \sec x)\Big|_0^{\frac{\pi}{4}} = 1 - \sqrt{2} - (-1) = 2 - \sqrt{2}.$$

解二： $\int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} \mathrm{d}x = \int_0^{\frac{\pi}{4}} \frac{1}{(\sin \frac{x}{2} + \cos \frac{x}{2})^2} \mathrm{d}x$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{(\tan \frac{x}{2} + 1)^2} \frac{1}{\cos^2 \frac{x}{2}} \mathrm{d}x$$

$$= 2\int_0^{\frac{\pi}{4}} \frac{1}{(\tan \frac{x}{2} + 1)^2} \mathrm{d}(\tan \frac{x}{2} + 1)$$

$$= -\frac{2}{\tan\frac{x}{2}+1}\bigg|_0^{\frac{\pi}{4}} = -\frac{2}{\tan\frac{\pi}{8}+1} - (-2) = 2 - \sqrt{2}.$$

注： $\tan\frac{\pi}{4} = \frac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = 1 \Rightarrow \tan^2\frac{\pi}{8} + 2\tan^2\frac{\pi}{8} + 1 = 2 \Rightarrow \tan\frac{\pi}{8} = \sqrt{2} - 1.$

例 10. 计算 $\int_{-\pi}^{\pi}(x+x^2)\sin^3 x\mathrm{d}x$；(2019—2020)

解： 因为 $f(x) = x^2\sin^3 x$ 为奇函数，所以，$\int_{-\pi}^{\pi}x^2\sin^3 x\mathrm{d}x = 0.$

故 $\qquad \int_{-\pi}^{\pi}(x+x^2)\sin^3 x\mathrm{d}x = \int_{-\pi}^{\pi}x\sin^3 x\mathrm{d}x$

$$= 2\int_0^{\pi}x\sin^3 x\mathrm{d}x \quad \text{（因为 } x\sin^3 x \text{ 为偶函数）}$$

$$= 2\cdot\frac{\pi}{2}\int_0^{\pi}\sin^3 x\mathrm{d}x$$

$$= 2\cdot\pi\int_0^{\frac{\pi}{2}}\sin^3 x\mathrm{d}x = 2\pi\cdot\frac{2}{3} = \frac{4}{3}\pi.$$

例 11. 计算 $\int_{-2}^{2}\frac{x+1}{\sqrt{4x^2+9}}\mathrm{d}x$；(2020—2021)

解： $\int_{-2}^{2}\frac{x+1}{\sqrt{4x^2+9}}\mathrm{d}x = \int_{-2}^{2}\frac{x}{\sqrt{4x^2+9}}\mathrm{d}x + \int_{-2}^{2}\frac{1}{\sqrt{4x^2+9}}\mathrm{d}x.$

因为 $f(x) = \frac{x}{\sqrt{4x^2+9}}$ 是奇函数，所以，$\int_{-2}^{2}\frac{x}{\sqrt{4x^2+9}}\mathrm{d}x = 0.$

由于 $\frac{1}{\sqrt{4x^2+9}}$ 为偶函数，则

$$\int_{-2}^{2}\frac{x+1}{\sqrt{4x^2+9}}\mathrm{d}x = \int_{-2}^{2}\frac{1}{\sqrt{4x^2+9}}\mathrm{d}x = 2\int_0^{2}\frac{1}{\sqrt{4x^2+9}}\mathrm{d}x$$

$$= \int_0^{2}\frac{1}{\sqrt{(2x)^2+9}}\mathrm{d}(2x) = \ln\left(2x+\sqrt{4x^2+9}\right)\bigg|_0^2$$

$$= \ln 9 - \ln 3 = \ln 3.$$

例 12. 计算 $\int_0^{\frac{\pi}{2}}\mathrm{e}^x\sin^2 x\mathrm{d}x$；(2020—2021)

解： $\displaystyle\int_0^{\frac{\pi}{2}}\mathrm{e}^x\sin^2 x\mathrm{d}x=\int_0^{\frac{\pi}{2}}\mathrm{e}^x\frac{1-\cos 2x}{2}\mathrm{d}x=\frac{1}{2}\int_0^{\frac{\pi}{2}}\mathrm{e}^x\mathrm{d}x-\frac{1}{2}\int_0^{\frac{\pi}{2}}\mathrm{e}^x\cos 2x\mathrm{d}x$.

其中，$\qquad\displaystyle\int_0^{\frac{\pi}{2}}\mathrm{e}^x\mathrm{d}x=\mathrm{e}^x\Big|_0^{\frac{\pi}{2}}=\mathrm{e}^{\frac{\pi}{2}}-1$ ，

$$\int_0^{\frac{\pi}{2}}\mathrm{e}^x\cos 2x\mathrm{d}x=\mathrm{e}^x\cos 2x\Big|_0^{\frac{\pi}{2}}+2\int_0^{\frac{\pi}{2}}\mathrm{e}^x\sin 2x\mathrm{d}x\qquad\text{（分部积分）}$$

$$=-\mathrm{e}^{\frac{\pi}{2}}-1+2\int_0^{\frac{\pi}{2}}\mathrm{e}^x\sin 2x\mathrm{d}x$$

$$=-\mathrm{e}^{\frac{\pi}{2}}-1+2(\mathrm{e}^x\sin 2x\Big|_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\mathrm{e}^x\cos 2x\mathrm{d}x)\qquad\text{（分部积分）}$$

$$=-\mathrm{e}^{\frac{\pi}{2}}-1-4\int_0^{\frac{\pi}{2}}\mathrm{e}^x\cos 2x\mathrm{d}x\ ,$$

故 $\displaystyle\int_0^{\frac{\pi}{2}}\mathrm{e}^x\cos 2x\mathrm{d}x=-\frac{1}{5}(\mathrm{e}^{\frac{\pi}{2}}+1).$

因此，$\displaystyle\int_0^{\frac{\pi}{2}}\mathrm{e}^x\sin^2 x\mathrm{d}x=\frac{1}{2}(\mathrm{e}^{\frac{\pi}{2}}-1)-\frac{1}{2}(-\frac{1}{5}(\mathrm{e}^{\frac{\pi}{2}}+1))=\frac{3}{5}\mathrm{e}^{\frac{\pi}{2}}-\frac{2}{5}.$

例 13. 计算 $\displaystyle\int_{-3}^{3}\frac{x^3\cos^2 x}{\sqrt{1+x^2+x^4}}\mathrm{d}x$ ；（2021—2022）

解： 因为 $\dfrac{x^3\cos^2 x}{\sqrt{1+x^2+x^4}}$ 为奇函数，则 $\displaystyle\int_{-3}^{3}\frac{x^3\cos^2 x}{\sqrt{1+x^2+x^4}}\mathrm{d}x=0.$

例 14. 计算 $\displaystyle\int_{-1}^{6}\frac{1}{1+\sqrt[3]{x+2}}\mathrm{d}x$ ；（2021—2022）

解：令 $t=\sqrt[3]{x+2}$ ，则 $x=t^3-2$ ，$\mathrm{d}x=(t^3-2)'\mathrm{d}t=3t^2\mathrm{d}t$.

$x=-1$ 时，$t=1$；$x=6$ 时，$t=\sqrt[3]{6+2}=2$.

于是，$\qquad\displaystyle\int_{-1}^{6}\frac{1}{1+\sqrt[3]{x+2}}\mathrm{d}x=\int_{1}^{2}\frac{1}{1+t}3t^2\mathrm{d}t$

$$=3\int_{1}^{2}\frac{t^2-1+1}{1+t}\mathrm{d}t$$

$$=3\int_{1}^{2}(t-1)\mathrm{d}t+3\int_{1}^{2}\frac{1}{1+t}\mathrm{d}t$$

$$=3\cdot\frac{1}{2}(t-1)^2\Big|_1^2+3\ln(1+t)\Big|_1^2=\frac{3}{2}+3\ln\frac{3}{2}.$$

例 15. 计算 $\displaystyle\int_0^{\pi}x\sin^2 x\mathrm{d}x$ ；（2021—2022）

解一:
$$\int_0^\pi x\sin^2 x\,\mathrm{d}x = \int_0^\pi x\cdot\frac{1-\cos 2x}{2}\,\mathrm{d}x$$
$$= \frac{1}{2}\int_0^\pi x\,\mathrm{d}x - \frac{1}{2}\int_0^\pi x\cos 2x\,\mathrm{d}x$$
$$= \frac{1}{4}x^2\Big|_0^\pi - \frac{1}{2}\left(x\cdot\frac{1}{2}\sin 2x\Big|_0^\pi - \frac{1}{2}\int_0^\pi \sin 2x\,\mathrm{d}x\right) \qquad (\text{分部积分})$$
$$= \frac{\pi^2}{4} + \frac{1}{4}\int_0^\pi \sin 2x\,\mathrm{d}x$$
$$= \frac{\pi^2}{4} + \frac{1}{4}\left(-\frac{1}{2}\cos 2x\right)\Big|_0^\pi = \frac{\pi^2}{4}.$$

解二:
$$\int_0^\pi x\sin^2 x\,\mathrm{d}x = \frac{\pi}{2}\int_0^\pi \sin^2 x\,\mathrm{d}x = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\,\mathrm{d}x = \pi\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}.$$