第四次小测

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1. 证明拉普拉斯算子 $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ 的旋转不变性。

原坐标（x,y）和旋转后坐标（x',y'）的关系如下：

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

答：依照题目可以推导出以下关系：

$$\frac{\partial x}{\partial x'} = \cos\theta \qquad \frac{\partial x}{\partial y'} = -\sin\theta$$

$$\frac{\partial y}{\partial y'} = \cos\theta \qquad \frac{\partial y}{\partial x'} = \sin\ \theta$$

$f(x, y)$对于x'的偏导数可以写作：

$$\begin{aligned}\frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x'} \\ &= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\end{aligned}$$

$f(x, y)$对于y'的偏导数可以写作：

$$\begin{aligned}\frac{\partial f}{\partial y'} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y'} \\ &= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta\end{aligned}$$

再求二次偏导数：

$$\begin{aligned}\frac{\partial^2 f}{\partial x'^2} &= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x'}\right) \\ &= \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) \\ &= \cos\ \theta\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\right) + \sin\ \theta\frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial y}\right) \\ &= \cos\ \theta\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial x'}\right) + \sin\ \theta\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial x'} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial x'}\right) \\ &= \cos\ \theta\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\sin\theta\right) + \sin\ \theta\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\cos\theta + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\sin\theta\right) \\ &= \cos^2\ \theta\frac{\partial^2 f}{\partial x^2} + 2\sin\ \theta\cos\ \theta\frac{\partial f}{\partial y\,\partial x} + \sin^2\ \theta\frac{\partial^2 f}{\partial y^2}\end{aligned}$$

$$\frac{\partial^2 f}{\partial y'^2} = \frac{\partial}{\partial y'}(\frac{\partial f}{\partial y'})$$
$$= \frac{\partial}{\partial y'}(-\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta)$$
$$= -\sin\theta\frac{\partial}{\partial y'}(\frac{\partial f}{\partial x}) + \cos\theta\frac{\partial}{\partial y'}(\frac{\partial f}{\partial y})$$
$$= -\sin\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\frac{\partial y}{\partial y'}) + \cos\theta(\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\frac{\partial x}{\partial y'} + \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\frac{\partial y}{\partial y'})$$
$$= -\sin\theta(-\frac{\partial}{\partial x}(\frac{\partial f}{\partial x})\sin\theta + \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})\cos\theta) + \cos\theta(-\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})\sin\theta + \frac{\partial}{\partial y}(\frac{\partial f}{\partial y})\cos\theta)$$
$$= \sin^2\theta\frac{\partial^2 f}{\partial x^2} - 2\sin\theta\cos\theta\frac{\partial f}{\partial y\,\partial x} + \cos^2\theta\frac{\partial^2 f}{\partial y^2}$$

两者相加即可证明拉普拉斯算子的旋转不变性：

$$\frac{\partial^2 f}{\partial x'^2} + \frac{\partial^2 f}{\partial y'^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

2. 证明傅里叶变换的旋转性：$f(r,\theta+\theta_0)$的傅里叶变换为$F(\omega,\theta+\theta_0)$。

提示：$cos(a-b) = \cos a\cdot\cos b + \sin a\cdot\sin b$

答： 由极坐标变化，我们可以得到：

$$f(x,y) \leftrightarrow f_r(r,\theta) = f(\mathrm{r}\cos\theta, \mathrm{r}\sin\theta)$$
$$F(u,v) \leftrightarrow F_\rho(\rho,\varphi) = F(\rho\cos\varphi, \rho\sin\varphi)$$

则原来的$f(x,y) \leftrightarrow F(u,v)$变成$f_r(r,\theta) \leftrightarrow F_\rho(\rho,\varphi)$，又因为$dxdy = \mathrm{rdrd}\theta$，

由$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+yv)}dxdy$，可得：

$$F[f_r(r,\theta)] = F_\rho(\rho,\varphi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_r(r,\theta)e^{-j2\pi(\rho\cos\varphi r\cos\theta+\rho\sin\varphi r\sin\theta)}rdrd\theta$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_r(r,\theta)e^{-j2\pi\rho r(\cos\varphi\cos\theta+\sin\varphi\sin\theta)}rdrd\theta$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_r(r,\theta)e^{-j2\pi\rho r\cos(\varphi-\theta)}rdrd\theta$$

则有 $F[f_r(r,\theta+\theta_0)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_r(r,\theta+\theta_0)e^{-j2\pi\rho r\cos(\varphi-\theta)}rdrd\theta$

令$\theta+\theta_0 = \theta'$，则可得：

$$上式 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_r(r,\theta')e^{-j2\pi\rho r\cos[\varphi-(\theta'-\theta_0)]}rdrd\theta'$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_r(r,\theta')e^{-j2\pi\rho r\cos[(\varphi+\theta_0)-\theta']}rdrd\theta'$$

与上面得到的结果对比可以发现$F[f(r,\theta+\theta_0)]$=$F(\omega,\theta+\theta_0)$