

## 历届试题选 (四)

一、已知函数  $f(x) = \arctan x + \sin x$ , 求  $f^{(11)}(0)$ . (2016—2017)

解: 令  $g(x) = \arctan x$ , 则  $g'(x) = \frac{1}{1+x^2}$ , 即  $(1+x^2)g'(x) = 1$ .

两边求  $n$  阶导数, 即  $[(1+x^2)g'(x)]^{(n)} = 0$ .

由莱布尼茨公式,

$$g^{(n+1)}(x)(1+x^2) + ng^{(n)}(x)(1+x^2)' + \frac{n \cdot (n-1)}{2!} g^{(n-1)}(x)(1+x^2)'' = 0,$$

即 
$$g^{(n+1)}(x)(1+x^2) + ng^{(n)}(x) \cdot 2x + \frac{n(n-1)}{2!} g^{(n-1)}(x) \cdot 2 = 0.$$

令  $x=0$ ,  $g^{(n+1)}(0) = -n(n-1)g^{(n-1)}(0)$ .

所以,  $g^{(11)}(0) = -10 \cdot 9 g^{(9)}(0) = 10 \cdot 9 \cdot 8 \cdot 7 g^{(7)}(0) = \cdots = -10! g'(0)$ .

因为  $g'(0) = 1$ , 所以,  $g^{(11)}(0) = -10!$ .

故 
$$f^{(11)}(x) = g^{(11)}(x) + (\sin x)^{(11)} = g^{(11)}(x) + \sin(x + \frac{11}{2}\pi).$$

令  $x=0$ , 得 
$$f^{(11)}(0) = g^{(11)}(0) + \sin \frac{11}{2}\pi = -10! - 1.$$

二、已知  $y = x^2 \cos^2 x + \frac{1}{1+x}$ , 求  $y^{(n)}(0)$  ( $n \geq 3$ ). (2017—2018)

解:  $y = x^2 \cos^2 x + \frac{1}{1+x} = x^2 \cdot \frac{1+\cos 2x}{2} + \frac{1}{1+x} = \frac{x^2}{2} + \frac{1}{2} x^2 \cos 2x + \frac{1}{1+x}.$

记  $g(x) = x^2 \cos 2x$ , 由莱布尼茨公式, 得

$$g^{(n)}(x) = (\cos 2x)^{(n)} \cdot x^2 + n(\cos 2x)^{(n-1)} \cdot (x^2)' + \frac{n(n-1)}{2} (\cos 2x)^{(n-2)} \cdot (x^2)''$$

$$g^{(n)}(x) = 2 \cos(2x + \frac{n}{2}\pi) \cdot x^2 + n \cdot 2^{n-1} \cos(2x + \frac{n-1}{2}\pi) \cdot 2x + \frac{n(n-1)}{2} 2^{n-2} \cos(2x + \frac{n-2}{2}\pi) \cdot 2.$$

令  $x=0$ , 得

$$g^{(n)}(0) = -2^{n-2} n(n-1) \cos \frac{n}{2}\pi.$$

注意到,  $(\frac{1}{1+x})^{(n)} = \frac{(-1)^n n!}{(1+x)^{n+1}}$ , 故

$$y^{(n)}(0) = g^{(n)}(0) + \left( \frac{(-1)^n n!}{(1+x)^{n+1}} \right) \Big|_{x=0} = -2^{n-2} n(n-1) \cos \frac{n\pi}{2} + (-1)^n n!.$$

三、设函数  $f(x) = x \ln(1-x^2)$ , 求  $f^{(11)}(0)$ . (2019—2020)

解: 记  $g(x) = \ln(1-x^2) = \ln(1+x) + \ln(1-x)$ , 则  $g'(x) = \frac{1}{1+x} + \frac{1}{x-1}$ .

$$\text{则 } g^{(n)}(x) = \left( \frac{1}{1+x} \right)^{(n-1)} + \left( \frac{1}{x-1} \right)^{(n-1)} = \frac{(-1)^{n-1} (n-1)!}{(1+x)^n} + \frac{(-1)^{n-1} (n-1)!}{(x-1)^n}.$$

$$\text{所以, } g^{(n)}(0) = (-1)^{n-1} (n-1)! - (n-1)!.$$

由莱布尼茨公式, 得

$$f^{(n)}(x) = g^{(n)}(x) \cdot x + n g^{(n-1)}(x) \cdot 1,$$

$$\text{令 } x=0, \quad f^{(n)}(0) = n g^{(n-1)}(0) = [(-1)^n - 1] n(n-2)!.$$

$$\text{故 } f^{(11)}(0) = [(-1)^{11} - 1] \cdot 11 \cdot 9! = -\frac{11!}{5}.$$

四、设  $f(x) = (x^2 + x + 1) \cos^2 \frac{x}{2}$ , 求  $f^{(20)}(0)$ . (2020—2021)

$$\text{解: } f(x) = (x^2 + x + 1) \cos^2 \frac{x}{2} = \frac{1}{2} (x^2 + x + 1) (1 + \cos x).$$

由莱布尼茨公式,

$$\begin{aligned} f^{(n)}(x) &= (1 + \cos x)^{(n)} \cdot \frac{1}{2} (x^2 + x + 1) + n(1 + \cos x)^{(n-1)} \cdot \frac{1}{2} (x^2 + x + 1)' \\ &\quad + \frac{n(n-1)}{2} (1 + \cos x)^{(n-2)} \cdot \frac{1}{2} (x^2 + x + 1)'' \\ &= \cos(x + \frac{n\pi}{2}) \cdot \frac{1}{2} (x^2 + x + 1) + n \cos(x + \frac{(n-1)\pi}{2}) \cdot \frac{1}{2} (2x + 1) \\ &\quad + \frac{n(n-1)}{2} \cos(x + \frac{(n-2)\pi}{2}) \cdot 1 \end{aligned}$$

$$\text{故 } f^{(n)}(0) = \frac{1}{2} \cos \frac{n\pi}{2} + \frac{n}{2} \cos \frac{(n-1)\pi}{2} + \frac{n(n-1)}{2} \cos \frac{(n-2)\pi}{2}.$$

$$\begin{aligned} \text{因此, } f^{(10)}(0) &= \frac{1}{2} \cos 5\pi + 5 \cos \frac{9\pi}{2} + \frac{10 \cdot 9}{2} \cos 4\pi \\ &= \frac{1}{2} + \frac{9 \cdot 0}{2} = \frac{8!}{2}. \end{aligned}$$

五、设函数  $f(x) = (x^2 + x + 1) \cos 2x$ , 求  $f^{(8)}(0)$ . (2021—2022)

解: 由莱布尼茨公式, 得

$$f^{(n)}(x) = (x^2 + x + 1)(\cos 2x)^{(n)} + n(x^2 + x + 1)'(\cos 2x)^{(n-1)}$$

$$+ \frac{n(n-1)}{2}(x^2 + x + 1)''(\cos 2x)^{(n-2)}$$

$$= (x^2 + x + 1) \cdot 2^n \cos(2x + \frac{n\pi}{2}) + n(2x + 1) \cdot 2^{n-1} \cos(2x + \frac{(n-1)\pi}{2})$$

$$+ \frac{n(n-1)}{2} \cdot 2 \cdot 2^{n-2} \cos(2x + \frac{(n-2)\pi}{2}).$$

故  $f^{(n)}(0) = 2^n \cos \frac{n\pi}{2} + n \cdot 2^{n-1} \cos \frac{(n-1)\pi}{2} + n(n-1) \cdot 2^{n-2} \cos \frac{(n-2)\pi}{2}.$

因此,

$$f^{(8)}(0) = 2^8 \cos 4\pi + 8 \cdot 2^7 \cos \frac{7\pi}{2} + 56 \cdot 2^6 \cos 3\pi = 2^8 - 7 \cdot 2^9 = -13 \cdot 2^8 = -3328.$$

六、求函数  $y = \sqrt[3]{\frac{(x+1)(x+2)}{(1+x^2)(2+x^2)}}$  在  $x=0$  处的导数  $\left. \frac{dy}{dx} \right|_{x=0}$ . (2017—2018)

解:  $\ln y = \frac{1}{3} [\ln(x+1) + \ln(x+2) - \ln(1+x^2) - \ln(2+x^2)],$

两边求导, 得

$$\frac{1}{y} y' = \frac{1}{3} \left[ \frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{1+x^2} - \frac{2x}{2+x^2} \right],$$

即  $y' = \frac{1}{3} \sqrt[3]{\frac{(x+1)(x+2)}{(1+x^2)(2+x^2)}} \left[ \frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{1+x^2} - \frac{2x}{2+x^2} \right].$

故  $\left. \frac{dy}{dx} \right|_{x=0} = \frac{1}{3} \left( 1 + \frac{1}{2} \right) = \frac{1}{2}.$

七、求函数  $y = \sqrt[6]{\frac{x^2-1}{(x+2)(x+4)}}$  在  $x=2$  处的微分  $dy|_{x=2}$ . (2019—2020)

解:  $\ln y = \frac{1}{6} [\ln(x^2-1) - \ln(x+2) - \ln(x+4)],$

两边微分, 得

$$\frac{1}{y} dy = \frac{1}{6} \left[ \frac{2x}{x^2-1} - \frac{1}{x+2} - \frac{1}{x+4} \right] dx,$$

即  $dy = \frac{1}{6} \sqrt[6]{\frac{x^2-1}{(x+2)(x+4)}} \left[ \frac{2x}{x^2-1} - \frac{1}{x+2} - \frac{1}{x+4} \right] dx.$

令  $x=2$ , 得  $dy = \frac{1}{6}\sqrt{\frac{3}{24}}[\frac{4}{3} - \frac{1}{4} - \frac{1}{6}]dx = \frac{11}{144\sqrt{2}}dx$ .

八、设  $y = x \cdot \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$ , 则  $\left. \frac{d^2y}{dx^2} \right|_{x=0} = \underline{\hspace{2cm}}$ . (2021—2022)

解: 记  $f(x) = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$ , 则  $y = xf(x)$ .

于是,  $\frac{dy}{dx} = f(x) + xf'(x)$ ,  $\frac{d^2y}{dx^2} = f'(x) + f'(x) + xf''(x) = 2f'(x) + xf''(x)$ .

故  $\left. \frac{d^2y}{dx^2} \right|_{x=0} = 2f'(0)$ .

因为  $\ln|f(x)| = \frac{1}{2}[\ln|x-1| + \ln|x-2| - \ln|x-3| - \ln|x-4|]$ , 两边求导, 得

$$\frac{1}{f(x)} f'(x) = \frac{1}{2} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} \right],$$

即  $f'(x) = \frac{1}{2} \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}} \left[ \frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} \right]$ .

于是,  $f'(0) = \frac{1}{2} \sqrt{\frac{1}{6}} \left[ -1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right] = -\frac{11\sqrt{6}}{144}$ .

故  $\left. \frac{d^2y}{dx^2} \right|_{x=0} = 2f'(0) = -\frac{11}{72}\sqrt{6}$ .

九、求函数  $y = \arctan \frac{1-x^2}{1+x^2}$  的微分  $dy$  和  $dy|_{x=1}$ . (2017—2018)

解:  $dy = \frac{1}{1 + \left(\frac{1-x^2}{1+x^2}\right)^2} d\left(\frac{1-x^2}{1+x^2}\right) = \frac{1}{1 + \left(\frac{1-x^2}{1+x^2}\right)^2} \cdot \frac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} dx$

$$= -\frac{2x}{1+x^4} dx.$$

于是,  $dy|_{x=1} = -dx$ .

十、设方程  $e^{x-y} = y-1$  确定了隐函数  $y = y(x)$ , 求此隐函数在点  $(2, 2)$  处的一阶导数和二阶导数. (2019—2020)

解：方程两边对  $x$  求导，得  $e^{x-y}(x-y)' = y'$ ，即  $e^{x-y}(1-y') = y'$ ，解得  $y' = \frac{e^{x-y}}{1+e^{x-y}}$ 。

$$\begin{aligned}\text{于是, } y'' &= \left( \frac{e^{x-y}}{1+e^{x-y}} \right)' = \frac{(e^{x-y})' \cdot (1+e^{x-y}) - e^{x-y} \cdot (1+e^{x-y})'}{(1+e^{x-y})^2} \\ &= \frac{e^{x-y} \cdot (1-y') \cdot (1+e^{x-y}) - e^{x-y} \cdot e^{x-y} \cdot (1-y')}{(1+e^{x-y})^2} \\ &= \frac{e^{x-y} \cdot (1-y')}{(1+e^{x-y})^2} \\ &= \frac{e^{x-y} \cdot \left(1 - \frac{e^{x-y}}{1+e^{x-y}}\right)}{(1+e^{x-y})^2} = \frac{e^{x-y}}{(1+e^{x-y})^3}.\end{aligned}$$

$$\text{故 } y'\Big|_{\substack{x=2 \\ y=2}} = \frac{e^{2-2}}{1+e^{2-2}} = \frac{1}{2}, \quad y''\Big|_{\substack{x=2 \\ y=2}} = \frac{e^{2-2}}{(1+e^{2-2})^3} = \frac{1}{8}.$$

十一、求由方程  $y = \tan(x+y)$  所确定的隐函数  $y = y(x)$  的导数  $y'(x)$  和  $y''(x)$ . (2017—2018)

解：方程  $y = \tan(x+y)$  两边对  $x$  求导，得

$$y' = \sec^2(x+y) \cdot (x+y)' = \sec^2(x+y)(1+y'),$$

移项后，得  $y' = -\csc^2(x+y)$ 。

于是， $y'' = -2\csc(x+y) \cdot (-\csc(x+y)\cot(x+y))(1+y')$

$$= 2\csc^2(x+y)\cot(x+y)(1-\csc^2(x+y))$$

$$= -2\csc^2(x+y)\cot^3(x+y))$$

$$= -\frac{2\cos^3(x+y)}{\sin^5(x+y)}.$$

十二、设方程  $\ln(x^2 + y^2) = 2\arctan \frac{y}{x}$  确定了隐函数  $y = y(x)$ ，求此隐函数在点  $(1,0)$  处的

一阶导数和二阶导数. (2018—2019)

解：方程  $\ln(x^2 + y^2) = 2\arctan \frac{y}{x}$  两边对  $x$  求导，得

$$\frac{1}{x^2 + y^2} \cdot (2x + 2y \cdot y') = 2 \cdot \frac{1}{1 + (\frac{y}{x})^2} \cdot \frac{xy' - y}{x^2},$$

整理得,  $y' = \frac{x+y}{x-y}$ , 于是,  $y'|_{\substack{x=1 \\ y=0}} = \frac{x+y}{x-y} \Big|_{\substack{x=1 \\ y=0}} = 1.$

由  $y' = \frac{x+y}{x-y}$ , 得

$$\begin{aligned} y'' &= \frac{(1+y')(x-y) - (x+y)(1-y')}{(x-y)^2} \\ &= \frac{-2y + 2xy'}{(x-y)^2} = \frac{-2y + 2x \cdot \frac{x+y}{x-y}}{(x-y)^2} = \frac{2y^2 + 2x^2}{(x-y)^3}. \end{aligned}$$

故  $y''|_{\substack{x=1 \\ y=0}} = \frac{2y^2 + 2x^2}{(x-y)^3} \Big|_{\substack{x=1 \\ y=0}} = 2.$

十三、设方程  $2^{xy} = x^2 + y$  确定了函数  $y = y(x)$ , 求  $dy|_{x=0}$ . (2020—2021)

解: 方程  $2^{xy} = x^2 + y$  两边微分, 得

$$2^{xy} \ln 2 \cdot d(xy) = 2xdx + dy$$

即  $2^{xy} \ln 2 \cdot (ydx + xdy) = 2xdx + dy.$

将  $x=0, y=1$  代入, 得  $\ln 2 \cdot dx = dy.$

故  $dy|_{x=0} = 2.$

十四、设方程  $y - x - \frac{1}{2} \sin y = 0$  确定了隐函数  $y = y(x)$ , 求此隐函数的一阶导数和二阶导数. (2021—2022)

解: 方程  $y - x - \frac{1}{2} \sin y = 0$  两边求导数, 得

$$\frac{dy}{dx} - 1 - \frac{1}{2} \cos y \frac{dy}{dx} = 0,$$

所以,  $\frac{dy}{dx} = \frac{2}{2 - \cos y}.$

继续求导, 得

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{2}{(2-\cos y)^2} (2-\cos y)' \\
&= -\frac{2}{(2-\cos y)^2} \sin y \cdot y' \\
&= -\frac{2}{(2-\cos y)^2} \sin y \cdot \frac{2}{2-\cos y} \\
&= -\frac{4 \sin y}{(2-\cos y)^3}.
\end{aligned}$$

十五、求星形线  $\begin{cases} x = a \cos^3 \theta \\ y = a \sin^3 \theta \end{cases}$  在  $\theta = \frac{\pi}{4}$  处的二阶导数  $\frac{d^2 y}{dx^2}$  的值. (2016—2017)

解: 
$$\frac{dy}{dx} = \frac{(a \sin^3 \theta)'}{(a \cos^3 \theta)'} = \frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta (-\sin \theta)} = -\tan \theta;$$

$$\frac{d^2 y}{dx^2} = \frac{(-\tan \theta)'}{(a \cos^3 \theta)'} = \frac{-\sec^2 \theta}{3a \cos^2 \theta (-\sin \theta)} = \frac{1}{3a \cos^4 \theta \sin \theta}.$$

故 
$$\left. \frac{d^2 y}{dx^2} \right|_{\theta=\frac{\pi}{4}} = \left. \frac{1}{3a \cos^4 \theta \sin \theta} \right|_{\theta=\frac{\pi}{4}} = \frac{1}{3a \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{2}}} = \frac{4\sqrt{2}}{3a}.$$

十六、求函数  $\begin{cases} x = \frac{t}{1+t^2} \\ y = \frac{t^2}{1+t^2} \end{cases}$  在  $t=2$  所对应点处的切线方程和法线方程. (2017—2018)

解: 
$$\frac{dy}{dx} = \frac{(\frac{t^2}{1+t^2})'}{(\frac{t}{1+t^2})'} = \frac{\frac{2t(1+t^2)-t^2 \cdot 2t}{(1+t^2)^2}}{\frac{(1+t^2)-t \cdot 2t}{(1+t^2)^2}} = \frac{2t}{1-t^2},$$

令  $t=2$ , 可得对应点  $(\frac{2}{5}, \frac{4}{5})$  处的切线斜率为  $k = \left. \frac{dy}{dx} \right|_{t=2} = \left. \frac{2t}{1-t^2} \right|_{t=2} = -\frac{4}{3}.$

故所求的切线方程为  $y - \frac{4}{5} = -\frac{4}{3}(x - \frac{2}{5})$ , 即  $y = -\frac{4}{3}x + \frac{4}{3};$

法线方程为  $y - \frac{4}{5} = \frac{3}{4}(x - \frac{2}{5})$ , 即  $y = \frac{3}{4}x + \frac{1}{2}.$

十七、计算由摆线的参数方程  $\begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$  ( $0 < t < 2\pi$ ) 所确定的函数  $y = y(x)$  的一阶

导数和二阶导数. (2018—2019)

解: 
$$\frac{dy}{dx} = \frac{(1 - \cos t)'}{(t - \sin t)'} = \frac{\sin t}{1 - \cos t},$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{\left(\frac{\sin t}{1 - \cos t}\right)'}{(t - \sin t)'} = \frac{\cos t(1 - \cos t) - \sin t \cdot \sin t}{(1 - \cos t)^3} \\ &= \frac{\cos t - 1}{(1 - \cos t)^3} = -\frac{1}{(1 - \cos t)^2}.\end{aligned}$$

十八、求星形线  $\begin{cases} x = \cos^3 t \\ y = \sin^3 t \end{cases} (0 < t < 2\pi)$  在点  $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$  处的切线方程. (2019—2020)

解: 
$$\frac{dy}{dx} = \frac{(\sin^3 t)'}{(\cos^3 t)'} = \frac{3\sin^2 t \cos t}{3\cos^2 t (-\sin t)} = -\tan t.$$

因为点  $(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$  对应于  $t = \frac{\pi}{4}$ , 故所求切线斜率  $k = \left. \frac{dy}{dx} \right|_{t=\frac{\pi}{4}} = -1$ .

因此, 所求切线方程为  $y - \frac{\sqrt{2}}{4} = -(x - \frac{\sqrt{2}}{4})$ , 即  $y = -x + \frac{\sqrt{2}}{2}$ .

十九、 $y = y(x)$  由  $\begin{cases} x = 2t - 1 \\ te^y + y + 1 = 0 \end{cases}$  所确定, 求  $\left. \frac{dy}{dx} \right|_{x=-1}$  及  $\left. \frac{d^2y}{dx^2} \right|_{x=-1}$ . (2020—2021)

解: 方程  $te^y + y + 1 = 0$  两边对  $t$  求导, 得  $e^y + te^y \frac{dy}{dt} + \frac{dy}{dt} = 0$ .

于是, 
$$\frac{dy}{dt} = -\frac{e^y}{1 + te^y} = \frac{e^y}{y};$$

故 
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^y}{2y}.$$

因此, 
$$\frac{d^2y}{dx^2} = \frac{d}{dy} \left( \frac{e^y}{2y} \right) \cdot \frac{dy}{dx} = \frac{1}{2} \frac{ye^y - e^y}{y^2} \cdot \frac{e^y}{2y} = \frac{y-1}{4y^3} e^{2y}.$$

当  $x = -1$  时,  $t = 0$ , 则  $y = -1$ .

故 
$$\left. \frac{dy}{dx} \right|_{x=-1} = -\frac{e^{-1}}{2}, \quad \left. \frac{d^2y}{dx^2} \right|_{x=-1} = \frac{1}{2} e^{-2}.$$



二十、已知笛卡尔叶形线的参数方程为

$$\begin{cases} x = \frac{3at}{1+t^3} \\ y = \frac{3at^2}{1+t^3} \end{cases}, \text{ 其中 } a > 0 \text{ 为常数.}$$

求由此参数方程所确定的函数  $y = y(x)$  在  $t = 1$  处的一阶导数和二阶导数。

(2021—2022)

解: 
$$\frac{dy}{dx} = \frac{(\frac{3at^2}{1+t^3})'}{(\frac{3at}{1+t^3})'} = \frac{(\frac{t^2}{1+t^3})'}{(\frac{t}{1+t^3})'} = \frac{\frac{2t(1+t^3) - t^2 \cdot 3t^2}{(1+t^3)^2}}{\frac{(1+t^3) - t \cdot 3t^2}{(1+t^3)^2}} = \frac{2t - t^4}{1 - 2t^3}$$

$$\frac{d^2y}{dx^2} = \frac{(\frac{2t-t^4}{1-2t^3})'}{(\frac{3at}{1+t^3})'} = \frac{\frac{(2-4t^3)(1-2t^3) - (2t-t^4)(-6t^2)}{(1-2t^3)^2}}{3a \cdot \frac{(1+t^3) - t \cdot 3t^2}{(1+t^3)^2}}$$

$$= \frac{2(1+t^3)^4}{3a \cdot (1-2t^2)^3}.$$

故 
$$\left. \frac{dy}{dx} \right|_{t=1} = \frac{2-1}{1-2} = -1, \quad \left. \frac{d^2y}{dx^2} \right|_{t=1} = \frac{2^5}{-3a} = -\frac{32}{3a}.$$