

《微积分 I-1》历届期末试题解答 (一)

一、原函数与不定积分的概念

如果已知 $\varphi(x)$ 是 $f(x)$ 的原函数, 那么, $f(x) = \varphi'(x)$. 解题时, 可以先求出 $f(x)$ 再求解, 也可以直接用分部积分公式.

例 1. 设 $e^x \sin x$ 为 $f(x)$ 的一个原函数, 求 $\int e^{-x} f(x) dx$. (2017-2018)

解一: $f(x) = (e^x \sin x)' = e^x (\sin x + \cos x)$,

则 $\int e^{-x} f(x) dx = \int (\cos x + \sin x) dx = \sin x - \cos x + C$.

解二: $\int e^{-x} f(x) dx = \int e^{-x} (e^x \sin x)' dx = e^{-x} \cdot e^x \sin x - \int (e^{-x})' e^x \sin x dx$
 $= \sin x + \int \sin x dx = \sin x - \cos x + C$.

例 2. 设 $f(x)$ 的一个原函数为 $\frac{\cos(\ln x)}{x}$, 试求 $\int x^2 \cdot f(x) dx$. (2019-2020)

解一: 因为 $f(x)$ 的一个原函数为 $\frac{\cos(\ln x)}{x}$, 即 $f(x) = (\frac{\cos(\ln x)}{x})'$.

故 $\int x^2 \cdot f(x) dx = \int x^2 \cdot [\frac{\cos(\ln x)}{x}]' dx$
 $= x^2 \frac{\cos(\ln x)}{x} - \int 2x \cdot \frac{\cos(\ln x)}{x} dx$
 $= x \cos(\ln x) - 2 \int \cos(\ln x) dx$.

因为 $\int \cos(\ln x) dx = \int (x)' \cos(\ln x) dx$
 $= x \cos(\ln x) - \int x (\cos(\ln x))' dx$
 $= x \cos(\ln x) + \int \sin(\ln x) dx$
 $= x \cos(\ln x) + \int (x)' \sin(\ln x) dx$
 $= x \cos(\ln x) + x \sin(\ln x) - \int \cos(\ln x) dx$,

故 $\int \cos(\ln x) dx = \frac{x}{2} [\cos(\ln x) + \sin(\ln x)] + C$.

因此, $\int x^2 \cdot f(x) dx = x \cos(\ln x) - 2 \cdot \frac{x}{2} [\cos(\ln x) + \sin(\ln x)] + C$
 $= -x \sin(\ln x) + C$.

解二: $f(x) = \left[\frac{\cos(\ln x)}{x} \right]' = \frac{x \cdot (-\sin(\ln x)) \cdot \frac{1}{x} - \cos(\ln x) \cdot 1}{x^2} = -\frac{\sin(\ln x) + \cos(\ln x)}{x^2}.$

于是,
$$\begin{aligned} \int x^2 \cdot f(x) dx &= -\int (\cos \ln x + \sin \ln x) dx \\ &= -\int \cos \ln x dx - \int \sin \ln x dx \\ &= -\int \cos \ln x dx - \int (x)' \sin \ln x dx \\ &= -\int \cos \ln x dx - [x \sin(\ln x) - \int x \cos \ln x \cdot \frac{1}{x} dx] \\ &= -\int \cos \ln x dx - x \sin(\ln x) + \int \cos \ln x dx \\ &= -x \sin(\ln x) + C. \end{aligned}$$

二、不定积分的凑微分法

如果 $f(x)$ 的原函数为 $F(x)$, 则

$$\int f(\varphi(x))\varphi'(x)dx = \int f(u)du = F(u) + C = F(\varphi(x)) + C.$$

例 3. $\int \sec^4 x dx$. (2016-2017)

解:
$$\begin{aligned} \int \sec^4 x dx &= \int \sec^2 x \sec^2 x dx = \int (1 + \tan^2 x) d \tan x \\ &= \tan x + \frac{1}{3} \tan^3 x + C. \end{aligned}$$

注: 一般地, 当 m 为正整数, n 为非负整数时. 令 $\tan x = u$

$$\begin{aligned} \int \tan^n x \sec^{2m} x dx &= \int \tan^n x \sec^{2m-2} x \sec^2 x dx \\ &= \int \tan^n x (1 + \tan^2 x)^{m-1} \sec^2 x dx \\ &= \int \tan^n x (1 + \tan^2 x)^{m-1} d \tan x \\ &= \int u^n (1 + u^2)^{m-1} du = \dots. \end{aligned}$$

例 4. 求不定积分 $\int \frac{dx}{x(1+\ln x)}$. (2018-2019)

解:
$$\int \frac{dx}{x(1+\ln x)} = \int \frac{1}{1+\ln x} d(1+\ln x) = \ln|1+\ln x| + C.$$

注:
$$\int \frac{f'(x)}{f(x)} dx = \int \frac{1}{f(x)} df(x) = \ln|f(x)| + C.$$

例 5. 求不定积分 $\int \frac{x^2}{1-x^6} dx$. (2020-2021)

解: $\int \frac{x^2}{1-x^6} dx = \frac{1}{3} \int \frac{1}{1-(x^3)^2} \cdot 3x^2 dx = \frac{1}{3} \int \frac{1}{1-(x^3)^2} \cdot dx^3.$

令 $u = x^3$, 则

$$\begin{aligned}\int \frac{x^2}{1-x^6} dx &= \frac{1}{3} \int \frac{1}{1-u^2} \cdot du = \frac{1}{6} \int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du \\&= \frac{1}{6} [\ln|1+u| - \ln|1-u|] + C = \frac{1}{6} \ln \left| \frac{u+1}{u-1} \right| + C \\&= \frac{1}{6} \ln \left| \frac{x^3+1}{x^3-1} \right| + C.\end{aligned}$$

例 6. $\int \frac{x \ln(1+x^2)}{1+x^2} dx$. (2021-2022)

解:
$$\begin{aligned}\int \frac{x \ln(1+x^2)}{1+x^2} dx &= \frac{1}{2} \int \frac{\ln(1+x^2)}{1+x^2} d(1+x^2) \\&= \frac{1}{2} \int \ln(1+x^2) d \ln(1+x^2) = \frac{1}{4} \ln^2(1+x^2) + C.\end{aligned}$$

三、不定积分的第二换元法

$\int f(x) dx = \int f(\varphi(t)) d\varphi(t) = \int f(\varphi(t)) \varphi'(t) dt \Big|_{t=\varphi^{-1}(x)}$, 其中 $t = \varphi^{-1}(x)$ 为 $x = \varphi(t)$ 的反函数.

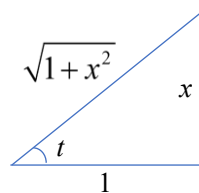
常见的代换:

- (1) 被积函数含有 $\sqrt{a^2 + x^2}$, 故可作变换 $x = a \tan t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$).
- (2) 被积函数含有 $\sqrt{a^2 - x^2}$, 故可作变换 $x = a \sin t$ ($-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$).
- (3) 被积函数含有 $\sqrt{x^2 - a^2}$, 故可作变换 $x = a \sec t$ ($0 < t < \frac{\pi}{2}$ 或 $\frac{\pi}{2} < t < \pi$).
- (4) 被积函数含有指数函数, 可作指数代换 $t = a^x$ 或 $t = e^x$.

例 7. $\int \frac{dx}{x^2 \sqrt{1+x^2}}$. (2016-2017)

解: 令 $x = \tan t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$), 则

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{1+x^2}} &= \int \frac{\sec^2 t}{\tan^2 t \sqrt{1+\tan^2 t}} dt = \int \frac{\cos t}{\sin^2 t} dt \\ &= \int \frac{1}{\sin^2 t} d\sin t = -\frac{1}{\sin t} + C \\ &= -\frac{\sqrt{1+x^2}}{x} + C.\end{aligned}$$

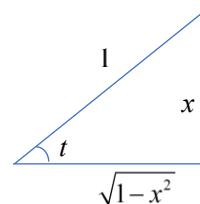


例 8. $\int \frac{1-2x}{\sqrt{1-x^2}} dx$. (2017-2018)

解一：
$$\begin{aligned}\int \frac{1-2x}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} (-2x) dx \\ &= \int \frac{1}{\sqrt{1-x^2}} dx + \int \frac{1}{\sqrt{1-x^2}} d(1-x^2) \\ &= \arcsin x + 2\sqrt{1-x^2} + C.\end{aligned}$$

解二： 令 $x = \sin t$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$, 则

$$\begin{aligned}\int \frac{1-2x}{\sqrt{1-x^2}} dx &= \int \frac{1-2\sin t}{\cos t} \cos t dt = \int (1-2\sin t) dt = t + 2\cos t + C \\ &= \arcsin x + 2\sqrt{1-x^2} + C.\end{aligned}$$

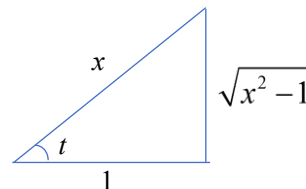


注： 本题既可以凑微分，也可以应第二换元法。

例 9. 求不定积分 $\int \frac{dx}{x^2\sqrt{x^2-1}}$. (2018-2019, 2021-2022)

解： 当 $x > 1$ 时，令 $x = \sec t$ ($0 < t < \frac{\pi}{2}$)，则 $dx = \sec t \tan t dt$.

$$\begin{aligned}\text{于是, } \int \frac{dx}{x^2\sqrt{x^2-1}} &= \int \frac{1}{\sec^2 t \sqrt{\sec^2 t - 1}} \sec t \tan t dt \\ &= \int \frac{1}{\sec^2 t \tan t} \sec t \tan t dt \\ &= \int \cos t dt = \sin t + C = \frac{\sqrt{x^2-1}}{x} + C.\end{aligned}$$



当 $x < -1$ 时，令 $u = -x$ ，则 $u > 1$ 于是，

$$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{1}{(-u)^2 \sqrt{(-u)^2-1}} d(-u) = -\int \frac{1}{u^2 \sqrt{u^2-1}} du$$

$$= -\frac{\sqrt{u^2-1}}{u} + C = \frac{\sqrt{x^2-1}}{x} + C.$$

故 $\int \frac{dx}{x^2\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} + C.$

例 10. $\int \frac{dx}{e^x(1+e^x)}.$ (2019-2020)

解: 令 $u = e^x$, 则 $x = \ln u$, $dx = \frac{1}{u} du$.

于是, $\int \frac{dx}{e^x(1+e^x)} = \int \frac{1}{u(1+u)} \frac{1}{u} du$

$$= \int \frac{1+u-u}{u^2(1+u)} du = \int \frac{1}{u^2} du - \int \frac{1}{u(1+u)} du$$

$$= -\frac{1}{u} - \int \frac{1+u-u}{u(1+u)} du = -\frac{1}{u} - \int \frac{1}{u} du + \int \frac{1}{1+u} du$$

$$= -\frac{1}{u} - \ln u + \ln(1+u) + C$$

$$= -e^{-x} - x + \ln(1+e^x) + C.$$

注: 对于带有指数函数的被积函数, 可以考虑作指数代换.

例 11. $\int \frac{dx}{(1+2x^2)\sqrt{1+x^2}}.$ (2019-2020)

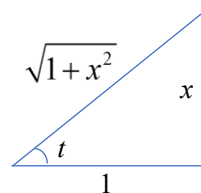
解: 令 $x = \tan t$ ($-\frac{\pi}{2} < t < \frac{\pi}{2}$), 则 $dx = (\tan t)' dt = \sec^2 t dt$, 于是,

$$\int \frac{dx}{(1+2x^2)\sqrt{1+x^2}} = \int \frac{1}{(1+2\tan^2 t)\sqrt{1+\tan^2 t}} \sec^2 t dt$$

$$= \int \frac{1}{(1+2\tan^2 t)\cos t} dt = \int \frac{\cos t}{\cos^2 t + 2\sin^2 t} dt$$

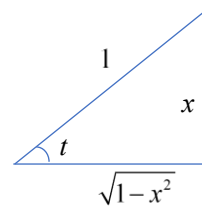
$$= \int \frac{1}{1+\sin^2 t} d\sin t = \arctan(\sin t) + C$$

$$= \arctan \frac{x}{\sqrt{1+x^2}} + C.$$



例 12. 求不定积分 $\int \frac{\sqrt{1-x^2}}{x^2} dx$. (2020-2021)

解一: 令 $x = \sin t, t \in (-\frac{\pi}{2}, 0) \cup (-\frac{\pi}{2}, \frac{\pi}{2})$, 则



$$\begin{aligned}\int \frac{\sqrt{1-x^2}}{x^2} dx &= \int \frac{\sqrt{1-\sin^2 t}}{\sin^2 t} \cos t dt \\&= \int \frac{\cos^2 t}{\sin^2 t} dt = \int \frac{1-\sin^2 t}{\sin^2 t} dt = \int \frac{1}{\sin^2 t} dt - \int dt \\&= -\cot t - t + C = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C.\end{aligned}$$

解二: $\int \frac{\sqrt{1-x^2}}{x^2} dx = \int (-\frac{1}{x})' \sqrt{1-x^2} dx$

$$\begin{aligned}&= -\frac{\sqrt{1-x^2}}{x} - \int (-\frac{1}{x})(\sqrt{1-x^2})' dx \\&= -\frac{\sqrt{1-x^2}}{x} - \int (-\frac{1}{x}) \frac{-2x}{2\sqrt{1-x^2}} dx \\&= -\frac{\sqrt{1-x^2}}{x} - \int \frac{1}{\sqrt{1-x^2}} dx \\&= -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C.\end{aligned}$$

注: 本题的解法是第二换元法和分部积分法.

四、不定积分的分部积分法

分部积分法公式: $\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$.

u 的选择: (1) 反三角函数和对数函数; (2) 幂函数; (3) 指数函数和三角函数.

例题: 见例 1, 例 2, 例 12, 例 13, 例 15.

例 13. 求不定积分 $\int x \arctan x dx$. (2017-2018, 2020-2021)

解: 分部积分法

$$\begin{aligned}\int x \arctan x dx &= \int [\frac{1}{2}(x^2+1)]' \arctan x dx \\&= \frac{1}{2}(x^2+1) \arctan x - \int \frac{1}{2}(x^2+1)(\arctan x)' dx\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(x^2+1)\arctan x - \int \frac{1}{2}(x^2+1) \frac{1}{1+x^2} dx \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2} \int dx \\
&= \frac{1}{2}(x^2+1)\arctan x - \frac{1}{2}x + C.
\end{aligned}$$

注：被积函数有对数或反三角函数，通常用分部积分。

五、不定积分的一题多解

例 14. 求不定积分 $\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} dx$. (2017-2018)

解一： $3\sin x + 4\cos x = A(2\sin x + \cos x) + B(2\sin x + \cos x)'$

$$= A(2\sin x + \cos x) + B(2\cos x - \sin x)$$

$$= (2A - B)\sin x + (A + 2B)\cos x.$$

$$\text{令 } \begin{cases} 2A - B = 3 \\ A + 2B = 4 \end{cases} \Rightarrow \begin{cases} A = 2 \\ B = 1 \end{cases}, \text{ 故}$$

$$\begin{aligned}
\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} dx &= \int \frac{2(2\sin x + \cos x) + (2\sin x + \cos x)'}{2\sin x + \cos x} dx \\
&= \int 2dx + \int \frac{(2\sin x + \cos x)'}{2\sin x + \cos x} dx \\
&= 2x + \ln|2\sin x + \cos x| + C.
\end{aligned}$$

注：(1) 对于不定积分 $\int \frac{a_1 \sin x + b_1 \cos x}{a_2 \sin x + b_2 \cos x} dx$ ，可以将分子，分解成

$$a_1 \sin x + b_1 \cos x = A(a_2 \sin x + b_2 \cos x) + B(a_2 \sin x + b_2 \cos x)'$$

$$\begin{aligned}
\text{则 } \int \frac{a_1 \sin x + b_1 \cos x}{a_2 \sin x + b_2 \cos x} dx &= \int \left(A + B \frac{(a_2 \sin x + b_2 \cos x)'}{a_2 \sin x + b_2 \cos x} \right) dx \\
&= Ax + B \ln|a_2 \sin x + b_2 \cos x| + C.
\end{aligned}$$

$$(2) \int \frac{f'(x)}{f(x)} dx = \int \frac{1}{f(x)} df(x) = \ln|f(x)| + C.$$

解二：(万能代换)：令 $\tan \frac{x}{2} = t$ ，则

$$\int \frac{3\sin x + 4\cos x}{2\sin x + \cos x} dx = 4 \int \frac{2 + 3t - 2t^2}{(1+t^2)(1+4t-t^2)} dt$$

$$\begin{aligned}
&= \int \left(\frac{-2t+4}{1+t^2} + \frac{4-2t}{1+4t-t^2} \right) dt \\
&= -\int \frac{2t}{1+t^2} dt + 4 \int \frac{1}{1+t^2} dt + \int \frac{4-2t}{1+4t-t^2} dt \\
&= -\ln(1+t^2) + 4 \arctan t + \ln|1+4t-t^2| + C \\
&= 4 \arctan t + \ln \frac{|1+4t-t^2|}{1+t^2} + C \\
&= 4 \cdot \frac{x}{2} + \ln \frac{\left| 1 + 4 \tan \frac{x}{2} - \tan^2 \frac{x}{2} \right|}{1 + \tan^2 \frac{x}{2}} + C \\
&= 2x + \ln \left| \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + 2 \cdot \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right| + C \\
&= 2x + \ln |\cos x + 2 \sin x| + C.
\end{aligned}$$

注：三角函数有理式可以通过万能代换 $\tan \frac{x}{2} = t$ ，将积分转换成有理函数的积分，从而求得原积分。

万能代换的常用公式：

$$\tan \frac{x}{2} = t \Rightarrow x = 2 \arctan t, \sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2}{1+t^2} dt.$$

解三：令 $\tan x = t$ ，则 $x = \arctan t$ ，即 $dx = (\arctan t)' dt = \frac{1}{1+t^2} dt$ 。

$$\begin{aligned}
\int \frac{3 \sin x + 4 \cos x}{2 \sin x + \cos x} dx &= \int \frac{3 \tan x + 4}{2 \tan x + 1} dx \\
&= \int \frac{3t+4}{2t+1} \cdot \frac{1}{1+t^2} dt \\
&= \int \left(\frac{2}{2t+1} + \frac{-t+2}{1+t^2} \right) dt \\
&= \int \frac{2}{2t+1} dt - \int \frac{t}{1+t^2} dt + 2 \int \frac{1}{1+t^2} dt \\
&= \ln|2t+1| - \frac{1}{2} \ln(1+t^2) + 2 \arctan t + C \\
&= \ln \left| \frac{1+2t}{\sqrt{1+t^2}} \right| + 2 \arctan t + C
\end{aligned}$$

$$\begin{aligned}
 &= \ln \left| \frac{1+2\tan x}{\sec x} \right| + 2x + C \\
 &= \ln |\cos x + 2\sin x| + 2x + C.
 \end{aligned}$$

六、其他类型的题目

例 15. 设常数 a, b 满足 $\int \sqrt{x^2+4}dx = ax\sqrt{x^2+4} + b\ln(x+\sqrt{x^2+4}) + C$, 则 $a =$ _____, $b =$ _____. (2021-2022)

解一:

$$\begin{aligned}
 \int \sqrt{x^2+4}dx &= x\sqrt{x^2+4} - \int x(\sqrt{x^2+4})'dx \\
 &= x\sqrt{x^2+4} - \int x \frac{2x}{2\sqrt{x^2+4}}dx \\
 &= x\sqrt{x^2+4} - \int \frac{x^2+4-4}{\sqrt{x^2+4}}dx \\
 &= x\sqrt{x^2+4} - \int \sqrt{x^2+4}dx + 4\int \frac{1}{\sqrt{x^2+4}}dx
 \end{aligned}$$

故 $\int \sqrt{x^2+4}dx = \frac{x}{2}\sqrt{x^2+4} + 2\ln(x+\sqrt{x^2+4}) + C.$

故 $a = \frac{1}{2}, b = 2.$

解二: 对式子 $\int \sqrt{x^2+4}dx = ax\sqrt{x^2+4} + b\ln(x+\sqrt{x^2+4}) + C$ 两边求导, 得

$$\begin{aligned}
 \sqrt{x^2+4} &= a(x\sqrt{x^2+4})' + b[\ln(x+\sqrt{x^2+4})]' \\
 &= a\sqrt{x^2+4} + ax \frac{2x}{2\sqrt{x^2+4}} + b \frac{1}{x+\sqrt{x^2+4}} \left(1 + \frac{2x}{2\sqrt{x^2+4}}\right) \\
 &= a\sqrt{x^2+4} + \frac{ax^2}{\sqrt{x^2+4}} + \frac{b}{\sqrt{x^2+4}} \\
 &= \frac{2ax^2 + 4a + b}{\sqrt{x^2+4}},
 \end{aligned}$$

即 $x^2+4 = 2ax^2 + 4a + b.$

故 $a = \frac{1}{2}, b = 2.$

注: $\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln(x+\sqrt{x^2+a^2}) + C.$