# Basel Problem, Riemann Function of $\zeta(2)$ Rigorous Non-Differential Proof

Start from $z^n - a^n = (z-a)\prod_{k=1}^{\frac{n-1}{2}}\left(z^2 - 2az\cos\frac{2k\pi}{n} + a^2\right)$

Let $z = 1 + \frac{x}{N}, \quad a = 1 - \frac{x}{N}, \quad N = n,$ we have:

$$\begin{aligned}
\left(1+\frac{x}{N}\right)^N - \left(1-\frac{x}{N}\right)^N &= \left[\left(1+\frac{x}{N}\right) - \left(1-\frac{x}{N}\right)\right]\prod_{k=1}^{\frac{N-1}{2}}\left[\left(1+\frac{x}{N}\right)^2 - 2\left(1+\frac{x}{N}\right)\left(1-\frac{x}{N}\right)\cos\left(\frac{2\pi k}{N}\right) + \left(1-\frac{x}{N}\right)^2\right] \\
&= \frac{2x}{N}\prod_{k=1}^{\frac{N-1}{2}}\left[2 + \frac{2x^2}{N^2} - 2\left(1-\frac{x^2}{N^2}\right)\cos\left(\frac{2\pi k}{N}\right)\right] \\
&= \frac{2x}{N}\prod_{k=1}^{\frac{N-1}{2}}\left[2 + \frac{2x^2}{N^2} - 2\cos\left(\frac{2\pi k}{N}\right) + \frac{2x^2}{N^2}\cos\left(\frac{2\pi k}{N}\right)\right] \\
&= \frac{4x}{N}\prod_{k=1}^{\frac{N-1}{2}}\left(\left(1-\cos\left(\frac{2\pi k}{N}\right)\right) + \left(1+\cos\left(\frac{2\pi k}{N}\right)\right)\frac{x^2}{N^2}\right) \\
&= \frac{4x}{N}\prod_{k=1}^{\frac{N-1}{2}}\left\{\left[1-\cos\left(\frac{2\pi k}{N}\right)\right]\left[1 + \frac{1+\cos\left(\frac{2\pi k}{N}\right)}{1-\cos\left(\frac{2\pi k}{N}\right)}\frac{x^2}{N^2}\right]\right\}
\end{aligned}$$

Let $\frac{4}{N}\prod_{k=1}^{\frac{N-1}{2}}\left(1-\cos\left(\frac{2\pi k}{N}\right)\right)$ be $C(N)$, we have:

$$\left(1+\frac{x}{N}\right)^N - \left(1-\frac{x}{N}\right)^N = C(N)x\prod_{k=1}^{\frac{N-1}{2}}\left(1 + \frac{1+\cos\left(\frac{2\pi k}{N}\right)}{1-\cos\left(\frac{2\pi k}{N}\right)}\frac{x^2}{N^2}\right)$$

According to the Binomial theorem:

$$\left(1+\frac{x}{N}\right)^N = \sum_{k=0}^{N} C_N^k\frac{x^k}{N^k} \qquad \left(1-\frac{x}{N}\right)^N = \sum_{k=0}^{N}(-1)^k C_N^k\frac{x^k}{N^k}$$

Subtract the two equations and extract the first-order term, we get the first-order term in the LHS of the original equation:

$$C_N^1\frac{x}{N} - (-1)C_N^1\frac{x}{N} = 2C_N^1\frac{x}{N} = 2x.$$

Since the coefficient of first-order term in RHS is $C(N)$, $C(N) = 2$, and we have:

$$\left(1+\frac{x}{N}\right)^N - \left(1-\frac{x}{N}\right)^N = 2x\prod_{k=1}^{\frac{N-1}{2}}\left(1 + \frac{1+\cos\left(\frac{2\pi k}{N}\right)}{1-\cos\left(\frac{2\pi k}{N}\right)}\frac{x^2}{N^2}\right)$$

Let $\theta = \dfrac{2k\pi}{N}$, we get $\cos(\theta) = 1 - \dfrac{\theta^2}{2} + \mathrm{O}(\theta^3)$, put it back to the equation, we have:

$$\left(1+\frac{x}{N}\right)^N - \left(1-\frac{x}{N}\right)^N = 2x\prod_{k=1}^{\frac{N-1}{2}}\left[1+\frac{1+\cos\left(\frac{2\pi k}{N}\right)}{1-\cos\left(\frac{2\pi k}{N}\right)}\frac{x^2}{N^2}\right] = 2x\prod_{k=1}^{\frac{N-1}{2}}\left\{1+\frac{1+\left[1-\frac{\theta^2}{2}+\mathrm{O}\left(\theta^3\right)\right]}{1-\left[1-\frac{\theta^2}{2}+\mathrm{O}\left(\theta^3\right)\right]}\frac{x^2}{N^2}\right\}$$

$$= 2x\prod_{k=1}^{\frac{N-1}{2}}\left[1+\frac{2-\frac{\theta^2}{2}+\mathrm{O}\left(\theta^3\right)}{\frac{\theta^2}{2}+\mathrm{O}\left(\theta^3\right)}\frac{x^2}{N^2}\right] = 2x\prod_{k=1}^{\frac{N-1}{2}}\left(1+\frac{\left(4-\theta^2+\mathrm{O}\left(\theta^3\right)\right)x^2}{\left(\theta^2+\mathrm{O}\left(\theta^3\right)\right)N^2}\right)$$

$$= 2x\prod_{k=1}^{\frac{N-1}{2}}\left(1+\frac{\left(4-\left(\frac{2k\pi}{N}\right)^2+\mathrm{O}\left(\left(\frac{2k\pi}{N}\right)^3\right)\right)x^2}{\left(\left(\frac{2k\pi}{N}\right)^2+\mathrm{O}\left(\left(\frac{2k\pi}{N}\right)^3\right)\right)N^2}\right)$$

$$= 2x\prod_{k=1}^{\frac{N-1}{2}}\left(1+\frac{\left(4-\left(\frac{2k\pi}{N}\right)^2+\mathrm{O}\left(\left(\frac{2k\pi}{N}\right)^3\right)\right)x^2}{(2k\pi)^2+\mathrm{O}\left(\frac{(2k\pi)^3}{N}\right)}\right)$$

(O in the equation is the time complexity Big O notation)

As $N$ approach to infinity, the LHS of the equation turns to $\lim\limits_{N\to\infty}\left(1+\dfrac{x}{N}\right)^N - \left(1-\dfrac{x}{N}\right)^N = e^x - e^{-x}$

We have: $e^x - e^{-x} = 2x\displaystyle\prod_{k=1}^{\infty}\left(1+\frac{(4+o(1))x^2}{(2k\pi)^2+o(1)}\right) = 2x\prod_{k=1}^{\infty}\left(1+\frac{(1+o(1))x^2}{k^2\pi^2+o(1)}\right) = 2x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right)$

Expand the Series representations of the LHS: $e^x - e^{-x} = -\displaystyle\sum_{k=0}^{\infty}\frac{(-x)^k - x^k}{k!}$

Compare the third-order term on the both side of $-\displaystyle\sum_{k=0}^{\infty}\frac{(-x)^k - x^k}{k!} = 2x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right)$, we have:

$$\frac{2x^3}{3!} = x^3\sum_{k=1}^{\infty}\frac{2}{k^2\pi^2} \qquad \Rightarrow \qquad \frac{\pi^2}{6} = \sum_{k=1}^{\infty}\frac{1}{k^2}$$

We conclude that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$.