Chernoff Bound for Symmetric Independent Random Variables

Here is a Proof of special Chernoff Bound for symmetric independent random variables (each value assume 1 or -1 with equal probability p)

Let $X = \sum_{i=1}^{n} x_i$, where x_i is the symmetric independent random variable with

$$x_i = \begin{cases} 1, \\ -1, \end{cases} \qquad \text{with } \Pr\left(x_i = 1\right) = \Pr\left(x_i = -1\right) = p \le \frac{1}{2}$$

For any $a > 0$, we have:

$$\Pr(X \ge a) = \Pr(tX \ge ta) = \Pr\left(e^{tX} \ge e^{ta}\right) \le \frac{E\left(e^{tX}\right)}{e^{ta}} \qquad \text{by Markov Inequality}$$

According to the properties of Independent Random Variables,

$$E\left(e^{tX}\right) = \left(e^{t\sum_{i=1}^{n} x_i}\right) = E\left(\prod_{i=1}^{n} e^{tx_i}\right) = \prod_{i=1}^{n} E\left(e^{tx_i}\right)$$

$$E\left(e^{tx_i}\right) = e^t \Pr\left(x_i = 1\right) + e^{-t} \Pr\left(x_i = -1\right) = pe^t + pe^{-t}$$

By Taylor Series Expansion:

$$e^t = 1 + t + \frac{t^2}{2!} + \ldots + \frac{t^i}{i!} + \ldots = \sum_{i=0}^{\infty} \frac{t^i}{i!}$$

$$e^{-t} = 1 - t + \frac{t^2}{2!} + \ldots + \frac{(-1)^i t^i}{i!} + \ldots = \sum_{i=0}^{\infty} \frac{(-1)^i t^i}{i!}$$

Thus, we have:

$$E\left(e^{tx_i}\right) = pe^t + pe^{-t} = p\left(\sum_{i=0}^{\infty} \frac{t^i}{i!} + \sum_{i=0}^{\infty} \frac{(-1)^i t^i}{i!}\right)$$

$$= p\sum_{i=0}^{\infty} \frac{2t^{2i}}{(2i)!} = \sum_{i=0}^{\infty} \frac{2pt^{2i}}{(2i)!} = \sum_{i=0}^{\infty} \frac{\left(\frac{t^2}{2}\right)^i \times 2^i \times 2p}{(2i)!}$$

$$\le \sum_{i=0}^{\infty} \frac{\left(\frac{t^2}{2}\right)^i}{i!} = e^{\frac{t^2}{2}} \qquad \text{by Taylor Series Expansion}$$

Combine this inequality with the equation, we have:

$$E\left(e^{tX}\right) = \left(e^{t\sum_{i=1}^{n} x_i}\right) = E\left(\prod_{i=1}^{n} e^{tx_i}\right) = \prod_{i=1}^{n} E\left(e^{tx_i}\right) \le \prod_{i=1}^{n} e^{\frac{t^2}{2}} = e^{\frac{t^2 n}{2}}$$

and: $$\Pr(X \ge a) \le \frac{E\left(e^{tX}\right)}{e^{ta}} \le \frac{e^{\frac{t^2 n}{2}}}{e^{ta}} = e^{\frac{t^2 n}{2} - ta}$$

Setting $t = a/n$, we obtain: $\Pr(X \ge a) \le e^{-a^2/2n}$,

and similarly, we have: $\Pr(X \le -a) \le e^{-a^2/2n}$

Combing above two, we have: $\Pr\left(\left|X\right| \ge a\right) \le 2e^{-a^2/2n}$

This is the special case Chernoff Bound for symmetric independent random.