Problem Set 2

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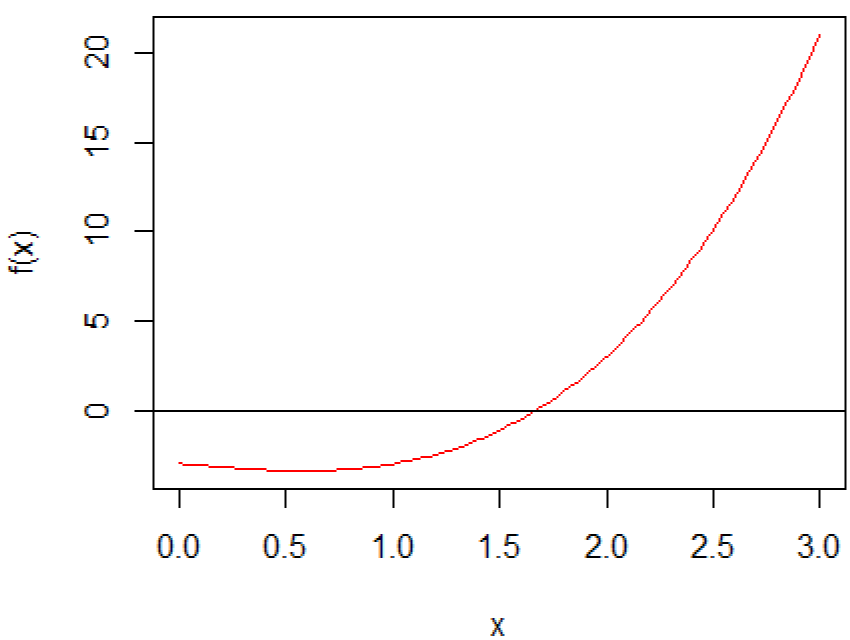
1. Use the bisection method with a hand calculator or computer to find the real root of $x^3 - x - 3 = 0$. Use an error tolerance of $\epsilon = 0.0001$. Graph the function $f(x) = x^3 - x - 3$ and label the root.

- **Step 1**: The graph below is generated using R, and illustrates the function $f(x) = x^3 - x - 3$



- **Step 2**: Based on the graph, we start with $[a, b] = [1.5, 2.0]$
- **Step 3**: To determine the no. of iterations, let $|a - c_n| = \epsilon = 0.0001$, then solve for n.

$$|a - c_n| \leq [\frac{1}{2}]^n (b - a)$$

$$0.0001 \leq [\frac{1}{2}]^n(2.0 - 1.5)$$

$$0.0001 \leq [\frac{1}{2}]^n(0.5)$$

$$\frac{0.0001}{0.5} \leq [\frac{1}{2}]^n$$

$$0.0002 \leq [\frac{1}{2}]^n$$

$$\ln 0.0002 \leq n \ \ln\ [\frac{1}{2}]$$

$$\frac{\ln 0.0002}{\ln \frac{1}{2}} \leq n$$

$$\frac{-8.517193191416237426654733697279 3}{-0.69314718055994530941723212145818} \leq n$$

$$12.28771238 \leq n$$

$$n \geq 13$$

- **Step 4**: The following table is generated starting with $Bisect(f(x) = x^3 - x - 3, 1.5, 2.0, c_1, \epsilon = 0.0001)$.

n	a	b	c	f(b)	f(c)	f(b)f(c)	|b - c| ≤ 0.0001?
1	1.5	2.0	1.75	3.0	0.609375	1.828125	No
2	1.5	1.75	1.625	0.609375	-0.333984375	-0.203521728515625	No
3	1.625	1.75	1.6875	0.609375	0.117919921875	0.0718574523925781	No
4	1.625	1.6875	1.65625	0.117919921875	-0.112884521484375	-0.0133113339543343	No
5	1.65625	1.6875	1.671875	0.117919921875	0.00129318237304687	0.000152491964399815	No
6	1.65625	1.671875	1.6640625	0.00129318237304687	-0.0561003684997559	-0.0000725480076653184	No
7	1.6640625	1.671875	1.66796875	0.00129318237304687	-0.0274799466133118	-0.0000355365825726039	No
8	1.66796875	1.671875	1.669921875	0.00129318237304687	-0.0131124928593636	-0.000016956844632432	No
9	1.669921875	1.671875	1.6708984375	0.00129318237304687	-0.00591443572193384	-0.00000764844402212361	No
10	1.6708984375	1.671875	1.67138671875	0.00129318237304687	-0.00231182214338332	-0.00000298960764544276	No
11	1.67138671875	1.671875	1.671630859375	0.00129318237304687	-0.000509618796058 93	-0.000000659030044036779	No
12	1.671630859375	1.671875	1.6717529296875	0.00129318237304687	0.000391707055314328	0.000000506548659330586	No
13	1.671630859375	1.6717529296875	1.67169189453125	0.000391707055314328	-0.0000589745529850916	-0.0000000231007484882 69	Yes
14	1.67169189453125	1.6717529296875	**1.67172241210937**	0.000391707055314328	0.000166361580426155	0.0000000651650047861 67	Yes

- **To conclude**, the real root of the function $f(x) = x^3 - x - 3$ is 1.67172241210937.

2. The function $f(x) = -3x^3 + 2e^{\frac{x^2}{2}} - 1$ has values of zero near $x = -0.5$ and $x = 0.5$.

a. What is the derivative of f?

b. If you begin Newton's method at $x = 0$, which root is reached? How many iterations to achieve an error less than 10^{-5}?

c. Begin Newton's method at another starting point to get the other zero.

3. Use the function from no.2 and find the root using the secant method where $x_0 = 0$ and $x_1 = 1$. Use an error tolerance of $\epsilon = 0.001$.

4. Consider the system

$$10.2x + 2.4y - 4.5z = 14.067,$$

$$-2.3x - 7.7y + 11.1z = -0.996,$$

$$-5.5x - 3.2x + 0.9z = -12.645.$$

a. Present the augmented matrix of the system.

$$\left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ -2.3 & -7.7 & 11.1 & -0.996 \\ -5.5 & -3.2 & 0.9 & -12.645 \end{array}\right)$$

b. Solve the system using $A_x = LU_x = L_y = b$ and round the final answer to 4 decimal digits.

c. Find the residual vector if the correct solution is $x = 1.4531001$, $y = -1.5891949$, $z = -0.2748947$.

- **Step 1**: The following are given values of the variables b, A, and $\bar{x}$:

$$A = \begin{pmatrix} 10.2 & 2.4 & -4.5 \\ -2.3 & -7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix}$$

$$b = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 1.4531001 \\ -1.5891949 \\ -0.2748947 \end{pmatrix}$$

- **Step 2**: Use the equation $r = b - A\bar{x}$ to calculate the residual error:

$$r = b - A\bar{x}$$

$$r = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix} - \begin{pmatrix} 10.2 & 2.4 & -4.5 \\ -2.3 & -7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix} \begin{pmatrix} 1.4531001 \\ -1.5891949 \\ -0.2748947 \end{pmatrix}$$

$$r = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix} - \begin{pmatrix} 10.2(1.4531001) & + & 2.4(-1.5891949) & - & 4.5(-0.2748947) \\ -2.3(1.4531001) & + & -7.7(-1.5891949) & + & 11.1(-0.2748947) \\ -5.5(1.4531001) & + & -3.2(-1.5891949) & + & 0.9(-0.2748947) \end{pmatrix}$$

$$r = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix} - \begin{pmatrix} 12.24457941 \\ 5.84333933 \\ -3.1540321 \end{pmatrix}$$

$$r = \begin{pmatrix} 14.067 - 12.24457941 \\ -0.996 - 5.84333933 \\ -12.645 + 3.1540321 \end{pmatrix}$$

$$r = \begin{pmatrix} 1.82242059 \\ -6.83933933 \\ -9.4909679 \end{pmatrix}$$

- **To conclude**, the residual vector of the solution $x = 1.4531001, y = -1.5891949, z = -0.2748947$ is $r = (1.82242059, -6.83933933, -9.4909679)^T$.

5. Compute the Frobenius norm, maximum column sum, and maximum row sum of the matrix:

$$\begin{pmatrix} 10.2 & 2.4 & 4.5 \\ -2.3 & 7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix}$$

6. Solve the system of equations given in no. 4, starting with the initial vector of $[0, 0, 0]$:

a. Solve using the Jacobi method with 2-digit precision.

b. Solve using Gauss-Seidel method with 2-digit precision.

c. Solve for e if the true solution is $x = (1.5, 0.33, 0.45)^T$.