Problem Set 2

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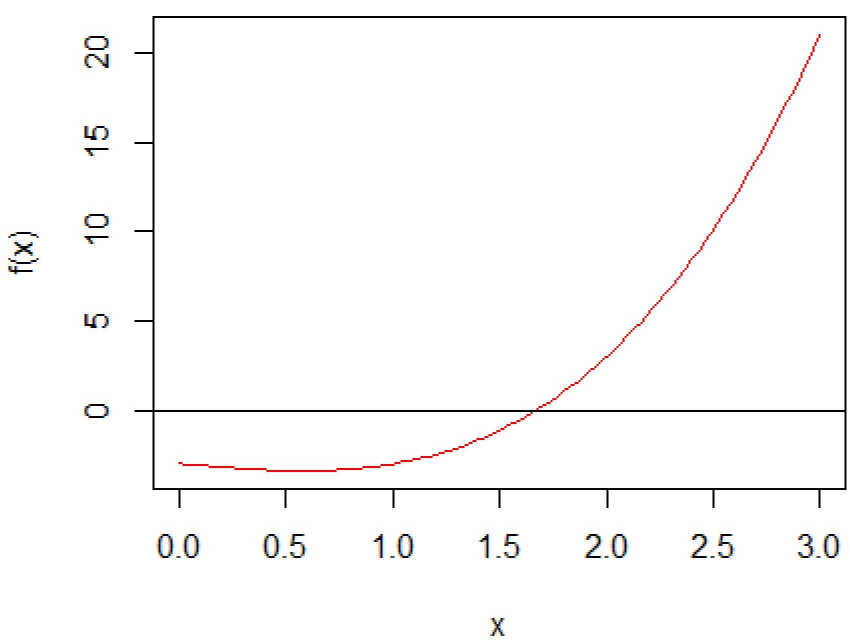
Problem Set 2

1. Use the bisection method with a hand calculator or computer to find the real root of $x^3 - x - 3 = 0$. Use an error tolerance of $\epsilon = 0.0001$. Graph the function $f(x) = x^3 - x - 3$ and label the root.

- **Step 1**: The graph below is generated using R, and illustrates the function $f(x) = x^3 - x - 3$



- **Step 2**: Based on the graph, we start with $[a, b] = [1.5, 2.0]$
- **Step 3**: To determine the no. of iterations, let $|a - c_n| = \epsilon = 0.0001$, then solve for n.

$$|a - c_n| \leq [\frac{1}{2}]^n (b - a)$$

$$0.0001 \le [\frac{1}{2}]^n (2.0 - 1.5)$$

$$0.0001 \le [\frac{1}{2}]^n (0.5)$$

$$\frac{0.0001}{0.5} \le [\frac{1}{2}]^n$$

$$0.0002 \le [\frac{1}{2}]^n$$

$$\ln 0.0002 \le n \ \ln [\frac{1}{2}]$$

$$\frac{\ln 0.0002}{\ln \frac{1}{2}} \le n$$

$$\frac{-8.517193191416237426654733697279 3}{-0.6931471805599453094172321214581 8} \le n$$

$$12.28771238 \le n$$

$$n \ge 13$$

- **Step 4**: The following table is generated starting with $Bisect(f(x) = x^3 - x - 3, 1.5, 2.0, c_1, \epsilon = 0.0001)$.

n	a	b	c	f(b)	f(c)	f(b)f(c)	|b - c| ≤ 0.0001?
1	1.5	2.0	1.75	3.0	0.609375	1.828125	No
2	1.5	1.75	1.625	0.609375	-0.333984375	-0.203521728515625	No
3	1.625	1.75	1.6875	0.609375	0.117919921875	0.0718574523925781	No
4	1.625	1.6875	1.65625	0.117919921875	-0.112884521484375	-0.0133113339543343	No
5	1.65625	1.6875	1.671875	0.117919921875	0.00129318237304687	0.000152491964399815	No
6	1.65625	1.671875	1.6640625	0.00129318237304687	-0.0561003684997559	-0.0000725480076653184	No
7	1.6640625	1.671875	1.66796875	0.00129318237304687	-0.0274799466133118	-0.0000355365825726039	No
8	1.66796875	1.671875	1.669921875	0.00129318237304687	-0.0131124928593636	-0.000016956844632432	No
9	1.669921875	1.671875	1.6708984375	0.00129318237304687	-0.00591443572193384	-0.00000764844402212361	No
10	1.6708984375	1.671875	1.67138671875	0.00129318237304687	-0.00231182214338332	-0.00000298960764544276	No
11	1.67138671875	1.671875	1.671630859375	0.00129318237304687	-0.00050961879605893	-0.000000659030044036779	No
12	1.671630859375	1.671875	1.6717529296875	0.00129318237304687	0.000391707055314328	0.000000506548659330586	No
13	1.671630859375	1.6717529296875	1.67169189453125	0.000391707055314328	-0.0000589745529850916	-0.0000000231007484882 69	Yes
14	1.67169189453125	1.6717529296875	**1.67172241210937**	0.000391707055314328	0.000166361580426155	0.0000000651650047861 67	Yes

- **To conclude**, the real root of the function $f(x) = x^3 - x - 3$ is 1.67172241210937.

2. The function $f(x) = -3x^3 + 2e^{\frac{x^2}{2}} - 1$ has values of zero near $x = -0.5$ and $x = 0.5$.

a. What is the derivative of f?

$$f(x) = -9x^2 + 2xe^{\frac{x^2}{2}}$$

b. If you begin Newton's method at $x = 0$, which root is reached? How many iterations to achieve an error less than 10^{-5}?

Using x = 0, the root is not reached, instead it approaches negative infinity. To work around this, we used x = 0.2 instead. The root reached is 0.85679 with a tolerance of 0.000001. It takes nine iterations to reach the root.

x1	-1.33953440932477
x2	-0.527931301157682
x3	0.75290290670584
x4	0.852258759028248
x5	0.855919335612942
x6	0.856625728386131
x7	0.856765756971528
x8	0.856793656325713
x9	0.856799220585234

c. Begin Newton's method at another starting point to get the other zero.

Using x = 0.5. The root reached is 0.85680 with a tolerance of 0.000001. It takes ten iterations to reach the root.

x1	1.02945423563182
x10	0.856801232870126
x11	0.856800751254564
x2	0.917056576862918
x3	0.873593043199066
x4	0.860910010453705
x5	0.857761807490951
x6	0.857022787711302
x7	0.85685181945609
x8	0.856812403795683
x9	0.85680332407157

3. Use the function from no.2 and find the root using the secant method where $x_0 = 0$ and $x_1 = 1$. Use an error tolerance of $\epsilon = 0.001$.

4. Consider the system

$$10.2x + 2.4y - 4.5z = 14.067,$$

$$-2.3x - 7.7y + 11.1z = -0.996,$$

$$-5.5x - 3.2x + 0.9z = -12.645.$$

a. Present the augmented matrix of the system.

$$\left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ -2.3 & -7.7 & 11.1 & -0.996 \\ -5.5 & -3.2 & 0.9 & -12.645 \end{array}\right)$$

b. Solve the system using $A_x = LU_x = L_y = b$ and round the final answer to 4 decimal digits.

- **Step 1**: Determine the LU matrices first.

$$A = LU$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ -2.3 & -7.7 & 11.1 & -0.996 \\ -5.5 & -3.2 & 0.9 & -12.645 \end{array}\right)$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

$$U = \left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ -2.3 & -7.7 & 11.1 & -0.996 \\ -5.5 & -3.2 & 0.9 & -12.645 \end{array}\right)$$

- Row operation: $0.22549019607843137254901960784314 R_1 + R_2 \rightarrow R_2$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.22549019607843137254901960784314 & 1 & 0 \\ * & * & 1 \end{pmatrix}$$

$$U = \left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ 0 & -7.15882352941 & 10.0852941 & 2.1759705882352941 \\ -5.5 & -3.2 & 0.9 & -12.645 \end{array}\right)$$

- Row operation: $0.53921568627450980392156862745098 R_1 + R_3 \rightarrow R_3$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.22549019607843137254901960784314 & 1 & 0 \\ 0.53921568627450980392156862745098 & * & 1 \end{pmatrix}$$

$$U = \left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ 0 & -7.15882352941 & 10.0852941 & 2.1759705882352941 \\ 0 & -1.90588235294117647 & -1.5264705882352941 & -5.05985294117647 \end{array}\right)$$

- Row operation: $-0.26622843056696795398520953163516 R_2 + R_3 \rightarrow R_3$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.22549019607843137254901960784314 & 1 & 0 \\ 0.53921568627450980392156862745098 & -0.26622843056696795398520953163516 & 1 \end{pmatrix}$$

$$U = \left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ 0 & -7.15882352941 & 10.0852941 & 2.1759705882352941 \\ 0 & 0 & -4.211462612982744 & 4.4805477065107 \end{array}\right)$$

- **Step 2**: Use matrix L in the equation $L_y = b$ to find y via forward substitution.

$$L_y = b$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0.22549019607843137254901960784314 & 1 & 0 \\ 0.53921568627450980392156862745098 & -0.26622843056696795398520953163516 & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, b = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix}$$

- Solving for y_1

$$y_1 = 14.067$$

- Solving for y_2

$$0.22549019607843137254901960784314 y_1 + y_2 = -0.996$$

$$y_2 = -4.1679705882352941176470588235295$$

- Solving for y_3

$$0.53921568627450980392156862745098 y_1 + 0.26622843056696795398520953163516 y_2 + y_3 = -12.645$$

$$y_3 = -19.120514790468364831552999178307$$

- Therefore:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14.067 \\ -4.1679705882352941176470588235295 \\ -19.120514790468364831552999178307 \end{pmatrix}$$

- **Step 3**: Use matrix U and y from Step 2 in the equation $U_x = y$ to find x via back substitution.

$$U = \left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ 0 & -7.15882352941 & 10.0852941 & 2.17597058823529 41 \\ 0 & 0 & -4.211462612982744 & 4.4805477065107 \end{array}\right)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, y = \begin{pmatrix} 14.067 \\ -4.1679705882352941176470588235295 \\ -19.120514790468364831552999178307 \end{pmatrix}$$

- Solving for x_3

$$-4.211462612982744 x_3 = -19.120514790468364831552999178307$$

$$x_3 = 4.5401126752319353800228277093271$$

- Solving for x_2

$$-7.1588235294117647...x_2 + 10.085294117647...x_3 = -4.1679705882352941 17647...$$
$$x_2 = 6.978289385115162866926161140214 8$$

- Solving for x_1

$$10.2x_1 + 2.4x_2 - 4.5x_3 = 14.067$$
$$x_1 = 1.7401580896340508166156801917114$$

- Therefore:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1.7401580896340508166156801917114 \\ 6.9782893851151628669261611402148 \\ 4.5401126752319353800228277093271 \end{pmatrix} \approx \begin{pmatrix} 1.7402 \\ 6.9783 \\ 4.5401 \end{pmatrix}$$

- **To conclude**, the solution of matrix A is $x = (1.7402, 6.9783, 4.5401)^T$.

c. Find the residual vector if the correct solution is $x = 1.4531001$, $y = -1.5891949$, $z = -0.2748947$.

- **Step 1**: The following are given values of the variables b, A, and $\bar{x}$:

$$A = \begin{pmatrix} 10.2 & 2.4 & -4.5 \\ -2.3 & -7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix}$$

$$b = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} 1.4531001 \\ -1.5891949 \\ -0.2748947 \end{pmatrix}$$

- **Step 2**: Use the equation $r = b - A\bar{x}$ to calculate the residual error:

$$r = b - A\bar{x}$$

$$r = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix} - \begin{pmatrix} 10.2 & 2.4 & -4.5 \\ -2.3 & -7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix} \begin{pmatrix} 1.4531001 \\ -1.5891949 \\ -0.2748947 \end{pmatrix}$$

$$r = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix} - \begin{pmatrix} 10.2(1.4531001) & + & 2.4(-1.5891949) & - & 4.5(-0.2748947) \\ -2.3(1.4531001) & + & -7.7(-1.5891949) & + & 11.1(-0.2748947) \\ -5.5(1.4531001) & + & -3.2(-1.5891949) & + & 0.9(-0.2748947) \end{pmatrix}$$

$$r = \begin{pmatrix} 14.067 \\ -0.996 \\ -12.645 \end{pmatrix} - \begin{pmatrix} 12.24457941 \\ 5.84333933 \\ -3.1540321 \end{pmatrix}$$

$$r = \begin{pmatrix} 14.067 - 12.24457941 \\ -0.996 - 5.84333933 \\ -12.645 + 3.1540321 \end{pmatrix}$$

$$r = \begin{pmatrix} 1.82242059 \\ -6.83933933 \\ -9.4909679 \end{pmatrix}$$

- **To conclude**, the residual vector of the solution $x = 1.4531001, y = -1.5891949, z = -0.2748947$ is $r = (1.82242059, -6.83933933, -9.4909679)^T$.

5. Compute the Frobenius norm, maximum column sum, and maximum row sum of the matrix:

$$\begin{pmatrix} 10.2 & 2.4 & 4.5 \\ -2.3 & 7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix}$$

The Frobenius norm is defined as the square root of the sum of the squares of all the matrix elements. Solving for the Frobenius norm of the given matrix:

$$||A||_f = \sqrt{10.2^2 + 2.4^2 + 4.5^2 + -2.3^2 + 7.7^2 + 11.1^2 + -5.5^2 + -3.2^2 + 0.9^2}$$
$$||A||_f = \sqrt{104.04 + 5.76 + 20.25 + 5.29 + 59.29 + 123.21 + 30.25 + 10.24 + 0.81}$$
$$||A||_f = \sqrt{359.14}$$
$$||A||_f = 18.9509894201$$

The maximum column sum, ||A||*1, of the matrix is the largest sum of each column.*

$$||A||_1 = 18$$

The maximum row sum, //A//∞, of the matrix is the largest sum of each row.

$$||A||_\infty = 21.1$$

6. Solve the system of equations given in no. 4, starting with the initial vector of $[0, 0, 0]$:

a. Solve using the Jacobi method with 2-digit precision.

b. Solve using Gauss-Seidel method with 2-digit precision.

c. Solve for e if the true solution is $x = (1.5, 0.33, 0.45)^T$.