Problem Set 1

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2. Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function and graph using R with the point/s identified. $f(x) = x^3 - 4x^2 - 2x - 5$ on $[-10, 10]$.

Solution

Since $f(x)$ is a polynomial, it is both *continuous* and $differrentiable$.

- **Step 1**: Get the derivative of $f(x)$

$$f(x) = x^3 - 4x^2 - 2x - 5$$
$$f'(x) = 3x^{3-1} - (2)4x^{2-1} - (1)2x^{1-1}$$
$$f'(x) = 3x^2 - 8x - 2$$

- **Step 2**: Use the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$
  - wherein:
    * a and b are based from the given interval, $[-10, 10]$
    * $f(a) = f(-10) = (-10)^3 - 4(-10)^2 - 2(-10) - 5 = -1385$
    * $f(b) = f(10) = (10)^3 - 4(10)^2 - 2(10) - 5 = 575$
    * $f'(c) = 3c^2 - 8c - 2$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$3x^2 - 8x - 2 = \frac{575 - (-1385)}{10 - (-10)}$$
$$3x^2 - 8x - 2 = \frac{1960}{20}$$
$$3x^2 - 8x - 2 = 98$$
$$3x^2 - 8x - 100 = 0$$

* **Step 3**: Since the result $3x^2 - 8x - 100 = 0$ is quadratic (format: $ax^2 + bx + c = 0$), use the quadratic formula to find the values of c.

$$c = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$c = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-100)}}{2(3)}$$
$$c = \frac{8 \pm \sqrt{64 + 1200}}{6}$$

$$c = \frac{8 \pm \sqrt{1264}}{6}$$

* **To conclude**, c has two values that satisfy the conclusions of the Mean Value Theorem:

$$c = \frac{8 + \sqrt{1264}}{6} \approx 7.258796$$

$$c = \frac{8 - \sqrt{1264}}{6} \approx -4.5921296$$

3. Find the point c that satisfies the Mean Value Theorem for integrals on the interval $[-10, 10]$. The function is $f(x) = 2e^x$.

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

$$\int_{-10}^{10} (2e^x)dx = (2e^\xi)(10 - (-10))$$

$$2\int_{-10}^{10} e^x dx = (2e^\xi)(20)$$

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$$2\left(e^{10} - e^{-10}\right) = (e^\xi)(2)(20)$$

$$2\left(e^{10} - \frac{1}{e^{10}}\right) = 40e^\xi$$

What is c?

4. Consider the function $f(x) = \cos(x/2)$.

a. Find the fourth Taylor polynomial for f at $x = \pi$

Derivatives:

$$f(x) = cos(x/2) = f(\pi) = cos(\pi/2) = 0$$

$$f'(x) = -\frac{sin(x/2)}{2} = f(\pi) = -\frac{sin(\pi/2)}{2} = -\frac{1}{2}$$

$$f''(x) = -\frac{cos(x/2)}{4} = f(\pi) = -\frac{cos(\pi/2)}{4} = 0$$

$$f'''(x) = \frac{sin(x/2)}{8} = f(\pi) = \frac{sin(\pi/2)}{8} = \frac{1}{8}$$

$$f''''(x) = \frac{cos(x/2)}{16} = f(\pi) = \frac{cos(\pi/2)}{16} = 0$$

Getting the 4th Taylor Polynomial:

$$p_4 = 0 + \frac{f(\pi)(x-\pi)}{1!} + \frac{f''(\pi)(x-\pi)^2}{2!} + \frac{f'''(\pi)(x-\pi)^3}{3!} + \frac{f''''(\pi)(x-\pi)^4}{4!}$$

$$p_4 = 0 + \frac{-\frac{1}{2}(x-\pi)}{1} + \frac{0(x-\pi)^2}{2} + \frac{\frac{1}{8}(x-\pi)^3}{6} + \frac{0(x-\pi)^4}{24}$$

$$p_4 = \frac{-\frac{1}{2}(x-\pi)}{1} + \frac{\frac{1}{8}(x-\pi)^3}{6}$$

$$p_4 = -\frac{-x+\pi}{2} + \frac{(x-\pi)^3}{48}$$

$$p_4 = \frac{-x+\pi}{2} + \frac{(x-\pi)^2}{24}$$ 3

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b. Use the fourth Taylor polynomial to approximate $\cos(x/4)$

c. Use the fourth Taylor polynomial to bound the error.

5. If $fl(x)$ is the machine approximated number of a real number x and ϵ is the corresponding relative error, then show that $fl(x) = (1 - \epsilon)x$.

We know that the machine-approximated number fl(x) is given by:

$$\text{fl}(x) = x(1 + \epsilon)$$

Since ϵ is the relative error, it's defined as:

$$\epsilon = \frac{\text{fl}(x) - x}{x}$$

Solving for fl(x):

$$\text{fl}(x) = x + \epsilon x$$

Now, we can rewrite this as:

$$\text{fl}(x) = x(1 + \epsilon)$$

$$\text{fl}(x) = (1 - (-\epsilon))x$$

$$\text{fl}(x) = (1 - \epsilon)x$$

Thus, we have shown that $\text{fl}(x) = (1 - \epsilon)x$.

6. For the following numbers x and their corresponding approximations x_A, find the number of significant digits in x_A with respect to x and find the relative error.

a. $x = 451.01$, $x_A = 451.023$

- **Step 1**: Using $\beta = 10$, determine the value of s:

$$\beta^s \leq |x|$$
$$10^s \leq |451.01|$$
$$10^2 \leq |451.01|$$
$$100 \leq 451.01$$

* Since $10^2 = 100 \leq 451.01$ **therefore**, $s = \boxed{2}$.

- **Step 2**: Determine the number of r significant digits:

$$|x - x_A| \leq \frac{1}{2}\beta^{s-r+1}$$
$$|451.01 - 451.023| \leq 10^{2-r+1}$$
$$|-0.013| \leq 10^{3-r}$$
$$|-0.013| \leq \frac{1}{2}10^{3-4}$$

$$0.013 \leq \frac{10^{-1}}{2}$$
$$0.013 \leq 0.05$$

* Since $r =$ 4 **therefore**, x_A has 4 significant digits with respect to x.

- **Step 3**: Determine the error value:

$$E(x_A) = x - x_A$$
$$E(x_A) = 451.01 - 451.023$$
$$E(x_A) = \text{-0.013}$$

- **Step 4**: Determine the relative error:

$$Er(x_A) = \frac{E(x_A)}{x}$$
$$Er(x_A) = \frac{-0.013}{451.01}$$
$$Er(x_A) = -0.000028824 = -2.8824\ x\ 10^{-5}$$

b. $x = -0.04518$, $x_A = -0.045113$

- **Step 1**: Using $\beta = 10$, determine the value of s

$$\beta^s \leq |x|$$
$$10^s \leq |-0.04518|$$
$$10^s \leq 0.04518$$
$$10^{-2} \leq 0.04518$$
$$0.01 \leq 0.04518$$

* Since $10^{-2} = 0.01 \leq 0.04518$ **therefore**, $s =$ -2.

- **Step 2**: Determine the number of r significant digits:

$$|x - x_A| \leq \frac{1}{2}\beta^{s-r+1}$$
$$|-0.04518 - (-0.045113)| \leq \frac{1}{2}10^{-2-r+1}$$
$$|-0.000067| \leq \frac{1}{2}10^{-1-r}$$
$$0.000067 \leq \frac{10^{-1-2}}{2}$$
$$0.000067 \leq \frac{10^{-3}}{2}$$
$$0.000067 \leq 0.0005$$

* Since $r =$ 2 **therefore**, x_A has 2 significant digits with respect to x.

- **Step 3**: Determine the error value:

$$E(x_A) = x - x_A$$
$$E(x_A) = -0.04518 - (-0.045113)$$
$$E(x_A) = -0.000067$$

- **Step 4**: Determine the relative error:

$$Er(x_A) = \frac{E(x_A)}{x}$$
$$Er(x_A) = \frac{-0.000067}{-0.04518}$$
$$Er(x_A) = 0.00148$$

c. $x = 23.4604$, $x_A = 23.4213$

- **Step 1**: Using $\beta = 10$, determine the value of s

$$\beta^s \leq |x|$$
$$10^s \leq |23.4604|$$
$$10^s \leq 23.4604$$
$$10^1 \leq 23.4604$$
$$10 \leq 23.4604$$

* Since $10^1 = 1 \leq 23.4604$ **therefore**, $s = \boxed{1}$.

- **Step 2**: Determine the number of r significant digits:

$$|x - x_A| \leq \frac{1}{2}\beta^{s-r+1}$$
$$|23.4604 - 23.4213| \leq \frac{10^{1-r+1}}{2}$$
$$|0.0391| \leq \frac{10^{2-r}}{2}$$
$$0.0391 \leq \frac{10^{2-3}}{2}$$
$$0.0391 \leq \frac{10^{-1}}{2}$$
$$0.0391 \leq 0.05$$

* Since $r = \boxed{3}$ **therefore**, x_A has 3 significant digits with respect to x.

- **Step 3**: Determine the error value:

$$E(x_A) = x - x_A$$
$$E(x_A) = 23.4604 - 23.4213$$
$$E(x_A) = 0.0391$$

- **Step 4**: Determine the relative error:

$$Er(x_A) = \frac{E(x_A)}{x}$$
$$Er(x_A) = \frac{0.0391}{23.4604}$$
$$Er(x_A) = 0.0017$$

7. Find the condition number for the following functions

a. $f(x) = 2x^2$

$$f'(x) = 4x$$
$$Cond = \left|\frac{f'(c)}{f(c)}c\right|$$
$$Cond = \left|\frac{4x}{2x^2}x\right|$$
$$Cond = \left|\frac{4x^2}{2x^2}\right|$$
$$Cond = 2$$

* **To conclude**, the condition number of f is 2, therefore it is *$well-conditioned$*.

b. $f(x) = 2\pi^x$

$$f'(x) = 2\pi^x \; ln(\pi)$$
$$Cond = \left|\frac{f'(c)}{f(c)}c\right|$$
$$Cond = \left|\frac{2\pi^x \; ln(\pi)}{2\pi^x}x\right|$$
$$Cond = x \; ln(\pi)$$

* **To conclude**, the condition number of f is $x \; ln(\pi)$, therefore it is *$ill-conditioned$*.

c. $f(x) = 2b^x$

$$f'(x) = 2b^x \; ln(b)$$
$$Cond = \left|\frac{f'(c)}{f(c)}c\right|$$
$$Cond = \left|\frac{2b^x \; ln(b)}{2b^x}x\right|$$
$$Cond = x \; ln(b)$$

* **To conclude**, the condition number of f is $x \; ln(b)$, therefore it is *$ill-conditioned$*.

8. Determine if the following series converges or diverges. If it converges, determine its sum.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

- **Step 1**: Use the Ratio Test to determine the convergence of the given series.

$$L = \frac{a_{n+1}}{a_n}$$

$$L = \frac{\frac{1}{2^{1+1}}}{\frac{1}{2}}$$

$$L = \frac{\frac{1}{4}}{\frac{1}{2}}$$

$$L = \frac{2}{4} = \frac{1}{2}$$

* **To conclude**, since $L = \frac{1}{2} < 1$, the series converge absolutely, hence the series will *converge*.

- **Step 2**: Since the series converges, calculate the sum.

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{1(1-\frac{1}{2}^1)}{1-\frac{1}{2}}$$

$$S_n = \frac{1-\frac{1}{2}}{1-\frac{1}{2}}$$

$$S_n = \frac{\frac{1}{2}}{\frac{1}{2}}$$

$$S_n = \frac{2}{2} = 1$$