В-1

№1

$$B = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

$$A = \begin{pmatrix} 13 & 12 \\ 1 & -5 \end{pmatrix}$$

1)

$$\alpha_1 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ \alpha_2 + \alpha_3 + \alpha_4 = 0 \\ \alpha_3 + \alpha_4 = 0 \\ \alpha_4 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

ОТЖЕ СИСТЕМА МАТРИЦЬ ЛІНІЙНО НЕЗАЛЕЖНА І УТВОРЮЮТЬ БАЗИС.

$$A = \alpha_1 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -5 \\ \alpha_2 + \alpha_3 + \alpha_4 = 1 \\ \alpha_3 + \alpha_4 = 12 \\ \alpha_4 = 13 \end{cases} \Rightarrow \begin{array}{l} \alpha_4 = 13 \\ \alpha_3 = -1 \\ \alpha_2 = -11 \\ \alpha_1 = -6 \end{array}$$

Отже координати $A = (-6, -11, -1, 13)$ в базисі $B$.

2) $E\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right)$,

$$E_M = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \quad B_M = \begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{pmatrix}$$

$$T_{EB} = \left(\begin{array}{cccc|cccc}0&0&0&1&1&0&0&0\\0&0&1&1&0&1&0&0\\0&1&1&1&0&0&1&0\\1&1&1&1&0&0&0&1\end{array}\right) = $$

$$\left(\begin{array}{cccc|cccc}1&1&1&1&0&0&0&1\\0&1&1&1&0&0&1&0\\0&0&1&1&0&1&0&0\\0&0&0&1&1&0&0&0\end{array}\right) \sim$$

$$\Rightarrow T_{E\to B} = \begin{pmatrix}0&0&-1&1\\0&-1&1&0\\-1&1&0&0\\1&0&0&0\end{pmatrix}$$

Отже кординати $A = (-6, -11, -1, 13)$ в базисі $B$

2) $E\left\{\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}\right\}$

$$E_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B_M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$T_{e-B} = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \Rightarrow$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

№ 2

$U = \{a_1, a_2, a_3\}$, $V = \{b_1, b_2, b_3\}$

$a_1 = (1, -2, 3)$, $a_2 = (2, -1, 4)$, $a_3 = (1, 1, 0)$

$b_1 = (-2, 1, 3)$, $b_2 = (-1, 0, 2)$, $b_3 = (1, 1, -3)$

1) $B_u = \begin{pmatrix} 1 & 2 & 1 \\ -2 & -1 & 1 \\ 3 & 4 & 0 \end{pmatrix} \underset{3p-3\cdot 1p.}{\overset{2p+2\cdot 1p.}{\sim}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & -2 & -3 \end{pmatrix} \underset{3p+\frac{2}{3}2p}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \Rightarrow$

$\Rightarrow \dim U = 3$, $B_u = (a_1, a_2, a_3)$

$B_v = \begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & 2 & -3 \end{pmatrix} \overset{1p+3\cdot 2p}{\sim} \begin{pmatrix} 1 & -1 & 4 \\ 1 & 0 & 1 \\ 3 & 2 & -3 \end{pmatrix} \underset{3p-3\cdot 1p.}{\overset{2p-1p.}{\sim}} \begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -3 \\ 0 & 5 & -15 \end{pmatrix} \underset{3p-5\cdot 2p}{\sim}$

$\begin{pmatrix} 1 & -1 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim V = 2$, $B_v = (b_1, b_2)$

2)

$$B_{u+v} = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 1 & 1 & 0 \\ -2 & 1 & 3 \\ -1 & 0 & 2 \\ 1 & 1 & -3 \end{pmatrix} \begin{array}{l} \\ 2p-2\cdot 1p. \\ 3p-1p. \\ 4p+2\cdot 1p \\ 5p+1p. \\ 6p-1p. \end{array} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 3 & 3 \\ 0 & -3 & 11 \\ 0 & -2 & 5 \\ 0 & 3 & -6 \end{pmatrix} \begin{array}{l} \\ \\ 3p-2p \\ 4p+2p. \\ 5p+\frac{2}{3}2p \\ 6p-2p \end{array} \sim \begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \\ 0 & 0 & 9 \\ 0 & 0 & \frac{11}{3} \\ 0 & 0 & -8 \end{pmatrix} \begin{array}{l} \\ \\ \\ 4p-\frac{9}{5}3p. \\ 5p-\frac{11}{5}3p. \\ 6p+\frac{8}{5}3p. \end{array}$$

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim(U+V) = 3, \quad B_{u+v} = \{a_1, a_2, a_3\}$$

$x = (1, 4, 1)$

$\alpha_1 \cdot a_1 + \alpha_2 \cdot a_2 + \alpha_3 \cdot a_3 = x$

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 1 \\ -2\alpha_1 + \alpha_2 + \alpha_3 = 4 \\ 3\alpha_1 + 4\alpha_2 = 1 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -2 & -1 & 1 & 4 \\ 3 & 4 & 0 & 1 \end{array}\right) \begin{array}{l} \\ 2p+2\cdot 1p \\ \sim \\ 3p-3\cdot 1p. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 3 & 3 & 6 \\ 0 & -2 & -3 & -2 \end{array}\right) :3 \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -3 & -2 \end{array}\right) \begin{array}{l} \sim \\ 3p+2\cdot 2p. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 2 \end{array}\right) \cdot(-1) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{array}\right) \begin{array}{l} 1p-3p \\ \sim \\ 2p-3p. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right) \overset{1p-2\cdot 2p}{\sim} \left(\begin{array}{ccc|c} 1 & & & -5 \\ & 1 & & 4 \\ & & 1 & -2 \end{array}\right)$$

$\alpha_1 = -5,\ \alpha_2 = 4,\ \alpha_3 = -2 \Rightarrow x_c \in B_{U+V}$

3)

$\dim(U\cap V) = \dim U + \dim V - \dim(U+V) = 3+2-3 = 2$

Нехай $z \in U\cap V \Rightarrow z = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = \beta_1 b_1 + \beta_2 b_2$

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = -2\beta_1 - \beta_2 \\ -2\alpha_1 - \alpha_2 + \alpha_3 = \beta_1 \\ 3\alpha_1 + 4\alpha_2 = 3\beta_1 + 2\beta_2 \end{cases} \Rightarrow \begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 + 2\beta_1 + \beta_2 = 0 \\ -2\alpha_1 - \alpha_2 + \alpha_3 - \beta_1 = 0 \\ 3\alpha_1 + 4\alpha_2 - 3\beta_1 - 2\beta_2 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -1 & 1 & -1 & 0 \\ 3 & 4 & 0 & -3 & -2 \end{pmatrix} \begin{matrix} 2p+2\cdot 1p \\ \sim \\ 3p-3\cdot 1p \end{matrix} \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 3 & 3 & 3 & 2 \\ 0 & -2 & -3 & -9 & -5 \end{pmatrix} \underset{3p+\frac{2}{3}2p}{\sim}$$

$$\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 3 & 3 & 3 & 2 \\ 0 & 0 & -1 & -7 & -\frac{11}{3} \end{pmatrix} \begin{matrix} \\ :3 \\ :-1 \end{matrix} \sim \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & \frac{2}{3} \\ 0 & 0 & 1 & 7 & \frac{11}{3} \end{pmatrix} \begin{matrix} 1p-3p \\ \sim \\ 2p-3p \end{matrix}$$

$$\begin{pmatrix} 1 & 2 & 0 & -5 & -\frac{8}{3} \\ 0 & 1 & 0 & -6 & -3 \\ 0 & 0 & 1 & 7 & \frac{11}{3} \end{pmatrix} \overset{1p-2\cdot 2p}{\sim} \begin{pmatrix} 1 & 0 & 0 & 7 & \frac{10}{3} \\ 0 & 1 & 0 & -6 & -3 \\ 0 & 0 & 1 & 7 & \frac{11}{3} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \alpha_1 = -7\beta_1 - \frac{10}{3}\beta_2 \\ \alpha_2 = 6\beta_1 + 3\beta_2 \\ \alpha_3 = -7\beta_1 + \frac{11}{3}\beta_2 \end{cases}$$

| $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\beta_1$ | $\beta_2$ |
|---|---|---|---|---|
| $-7$ | $6$ | $-7$ | $1$ | $0$ |
| $-\frac{10}{3}$ | $3$ | $-\frac{11}{3}$ | $0$ | $1$ |

, ОТКЕ:

$$z_1 = -\frac{10}{3}a_1 + 3a_2 - \frac{11}{3}a_3 = b_2$$

$$z_2 = -7a_1 + 6a_2 - 7a_3 = \beta_1$$

$z_1, z_2$ - ВЕКТОРИ $U \cap V$

№2

$e = \{e_1, e_2\}$

1. $x_1 = e_1 - e_2$, $x_2 = e_1 + e_2$

$y_1 = e_1 + 2e_2$, $y_2 = -e_1 + 3e_2$, $z = -2e_1 + e_2 \Rightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad Y = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$$

$$A^T_{ee} = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 2 \\ 1 & 1 & -1 & 3 \end{array}\right) \underset{2p-1p}{\sim} \left(\begin{array}{cc|cc} 1 & -1 & 1 & 2 \\ 0 & 2 & -2 & 1 \end{array}\right) : 2$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & \frac{5}{2} \end{array}\right) \xrightarrow{1p-2p} \left(\begin{array}{cc|cc} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & -1 & \frac{1}{2} \end{array}\right) \Rightarrow A_\varphi = \begin{pmatrix} 0 & -1 \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix}$$

$$\varphi(z) = A \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -\frac{9}{2} \end{pmatrix}$$

2.

$$\operatorname{Im}\varphi = \{\varphi(x) \mid x \in V\} = \{A \cdot x \mid x \in V\}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -x_2 \\ \frac{5}{2}x_1 + \frac{1}{2}x_2 \end{pmatrix} \Rightarrow \operatorname{im}\varphi = \left\{ \begin{pmatrix} -x_2 \\ \frac{5}{2}x_1 + \frac{1}{2}x_2 \end{pmatrix} \,\middle|\, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V \right\}$$

$$\operatorname{def}(\varphi) = \dim(\ker \varphi)$$

$$\ker(\varphi) = \{ x \in V, \varphi(x) = \theta \}$$

$$A \cdot x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -1 \\ \frac{5}{2} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -\alpha_2 = 0 \\ \frac{5}{2}\alpha_1 + \frac{1}{2}\alpha_2 = 0 \end{cases} \Rightarrow \alpha_1 = \alpha_2 = 0 \Rightarrow x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \ker\varphi = \{(0,0)\}$$

$$\operatorname{def}\varphi = \dim(\ker\varphi) = 1$$

3. $B = \{y_1, y_2\}$

$$A_{eB} = T_{e \to B} \cdot A_{ee} \cdot T_{e \to B}^{-1}$$

$$T_{C \to B} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$$

$$T_{C \to B}^{-1} = \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right)_{2p - 2 \cdot 1p} \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 5 & -2 & 1 \end{array}\right)_{:5} \sim \left(\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{2}{5} & \frac{1}{5} \end{array}\right)^{1p + 2p}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & \frac{3}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{1}{5} \end{array}\right) \Rightarrow T_{C \to B}^{-1} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

$$A_{eB} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ \frac{2}{5} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -1 \\ \frac{3}{2} & 1 \end{pmatrix}$$