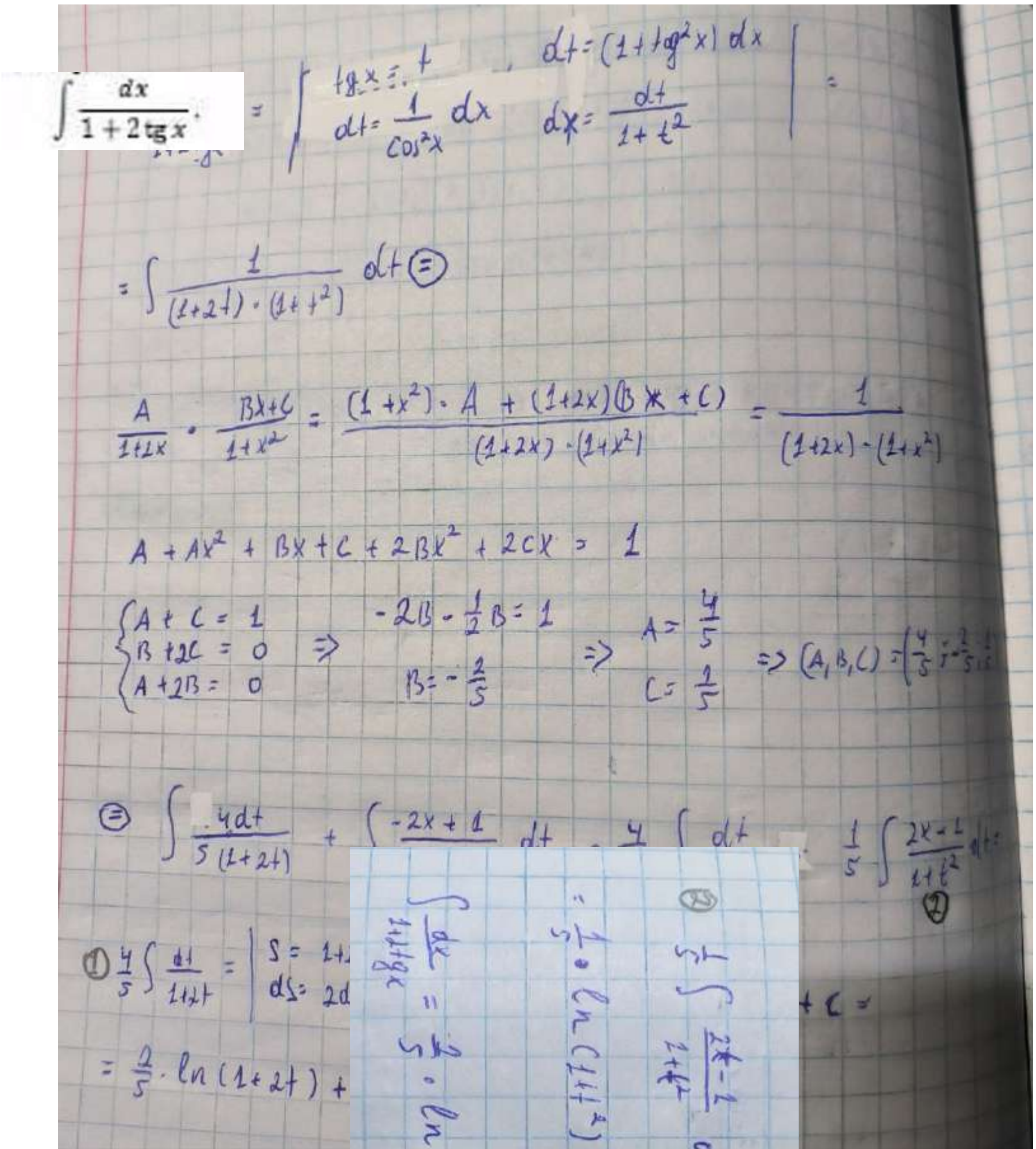
Знайти інтеграли.

1. Просто знайти:



$$\int \frac{dx}{1+2\operatorname{tg} x} = \left| \begin{array}{ll} \operatorname{tg} x = t & dt = (1+\operatorname{tg}^2 x)\,dx \\ dt = \frac{1}{\cos^2 x}\,dx & dx = \frac{dt}{1+t^2} \end{array} \right| =$$

$$= \int \frac{1}{(1+2t)\cdot(1+t^2)}\,dt \;(=)$$

$$\frac{A}{1+2x} + \frac{Bx+C}{1+x^2} = \frac{(1+x^2)\cdot A + (1+2x)(Bx+C)}{(1+2x)\cdot(1+x^2)} = \frac{1}{(1+2x)\cdot(1+x^2)}$$

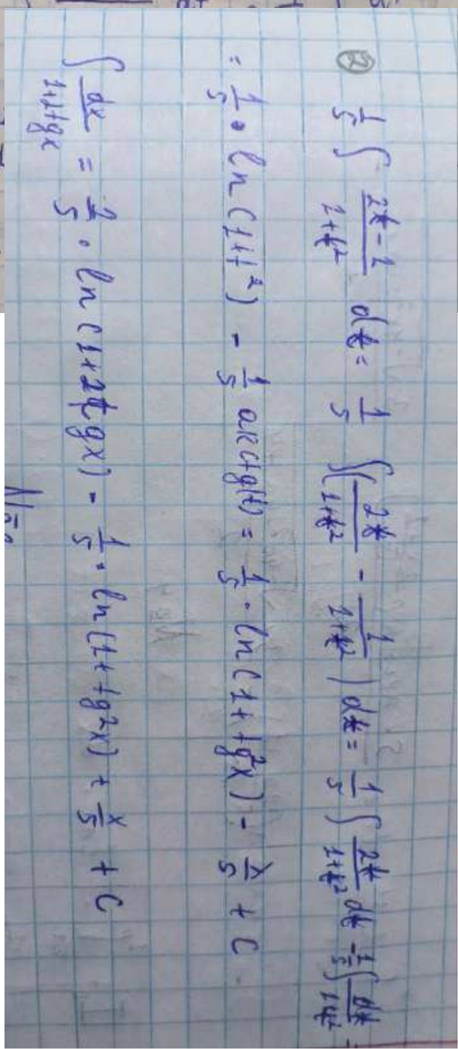
$$A + Ax^2 + Bx + C + 2Bx^2 + 2Cx = 1$$

$$\begin{cases} A + C = 1 \\ B + 2C = 0 \\ A + 2B = 0 \end{cases} \Rightarrow \begin{array}{l} -2B - \frac{1}{2}B = 1 \\ B = -\frac{2}{5} \end{array} \Rightarrow \begin{array}{l} A = \frac{4}{5} \\ C = \frac{1}{5} \end{array} \Rightarrow (A, B, C) = \left(\frac{4}{5}; -\frac{2}{5}; \frac{1}{5}\right)$$

$$(=) \int \frac{4\,dt}{5(1+2t)} + \int \frac{-2x+1}{\ldots}\,dt = \frac{4}{5}\int \frac{dt}{\ldots} - \frac{1}{5}\int \frac{2x-1}{1+t^2}\,dt \;(2)$$

$$(1)\; \frac{4}{5}\int \frac{dt}{1+2t} = \left| \begin{array}{l} s = 1+2t \\ ds = 2dt \end{array} \right| \ldots + C =$$

$$= \frac{2}{5}\cdot \ln(1+2t) + \ldots$$



$$(2)\; \frac{1}{5}\int \frac{2t-1}{1+t^2}\,dt = \frac{1}{5}\int \left(\frac{2t}{1+t^2} - \frac{1}{1+t^2}\right) dt = \frac{1}{5}\int \frac{2t}{1+t^2}\,dt - \frac{1}{5}\int \frac{dt}{1+t^2}$$

$$= \frac{1}{5}\cdot \ln(1+t^2) - \frac{1}{5}\operatorname{arctg}(t) = \frac{1}{5}\cdot \ln(1+\operatorname{tg}^2 x) - \frac{x}{5} + C$$

$$\int \frac{dx}{1+2\operatorname{tg} x} = \frac{2}{5}\cdot \ln(1+2\operatorname{tg} x) - \frac{1}{5}\cdot \ln(1+\operatorname{tg}^2 x) + \frac{x}{5} + C$$

$$\int_a^x \frac{1-3t}{\sqrt{1-t-t^2}}\,dt.$$

$$= \int_a^x \frac{1-3t}{\sqrt{\frac{5}{4}-(t+\frac{1}{2})^2}}\,dt =$$

① $\int \frac{1-3t}{\sqrt{\frac{5}{4}-(t+\frac{1}{2})^2}}\,dt = \left| \begin{matrix} u = t+\frac{1}{2}, & t = u - \frac{1}{2} \\ du = dt & \end{matrix} \right| = \int \frac{1-3\cdot(u-\frac{1}{2})}{\sqrt{\frac{5}{4}-u^2}}\,du =$

$= \frac{5}{2}\int \frac{du}{\sqrt{\frac{5}{4}-u^2}} - 3\int \frac{u}{\sqrt{\frac{5}{4}-u^2}}\,du = \frac{5}{4}\cdot \arcsin\frac{2t+1}{\sqrt{5}} + \frac{3\sqrt{-4t^2-4t+4}}{2} + C$

① $\frac{5}{4}\int \frac{du}{\sqrt{\frac{5}{4}-u^2}} = \left| \boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}} \right| = \frac{5}{4}\cdot \arcsin\frac{2u}{\sqrt{5}} + C$

② $3\int \frac{u\cdot du}{\dots}$ ... $\sqrt{\ } + C =$

$$\int \arcsin\sqrt{x}\,dx.$$

$= -3\sqrt{\frac{5}{4}-u \dots}$

$$\int x\cos^2 x\,dx.$$

$\int \cancel{x}\cdot\cos^2 x\,dx = \int x\cdot\left(\frac{1}{2}+\frac{\cos 2x}{2}\right)dx =$

$= \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cdot\cos 2x\,dx \;\boxed{=}\; \frac{x^2}{4} + \frac{x\cdot\sin 2x}{4} + \frac{\cos 2x}{8} + C$

① $\int x\,dx = \frac{x^2}{2} + C$

② $\int x\cdot\cos 2x\cdot dx = \left| \begin{matrix} u = x & dv = \cos 2x \\ du = dx & v = \frac{\sin 2x}{2} \end{matrix} \right| = \frac{x\cdot\sin 2x}{2} - \frac{1}{2}\int \sin 2x\,dx$

$= \frac{x\cdot\sin 2x}{2} \dots \cos 2x + C$

$$\int \arcsin\sqrt{x}\,dx = \left| \begin{matrix} t = \sqrt{x} \\ dt = \frac{dx}{2\sqrt{x}} \end{matrix} \right| = 2\int t \cdot \arcsin t\,dt =$$

$$= \left| \begin{matrix} u = \arcsin t & dv = t \\ du = \frac{dt}{\sqrt{1-t^2}} & v = \frac{t^2}{2} \end{matrix} \right| = t^2 \cdot \arcsin t - \int \frac{t^2}{\sqrt{1-t^2}}\,dt \; \boxed{=}$$

1

1 $$\int \frac{t^2}{\sqrt{1-t^2}}\,dt \; =$$

$$\left(\int \frac{t^2}{\sqrt{1-t^2}} \right)' = \left((At+B) \cdot \sqrt{1-t^2} + C \cdot \int \frac{dt}{\sqrt{1-t^2}} \right)'$$

$$A \cdot \sqrt{1-t^2} - \frac{\cancel{2}t \cdot (At+B)}{\cancel{2} \cdot \sqrt{1-t^2}} + \frac{C}{\sqrt{1-t^2}} = \frac{t^2}{\sqrt{1-t^2}}$$

$$A - At^2 - At^2 - Bt + C = t^2 \Rightarrow \begin{cases} -2A = 1 \\ -B = 0 \\ A + C = 0 \end{cases} \Rightarrow \begin{matrix} A = -\frac{1}{2} \\ B = 0 \\ C = \frac{1}{2} \end{matrix}$$

$$= -\frac{t \cdot \sqrt{1-t^2}}{2} + \frac{1}{2} \underbrace{\int \frac{dt}{\sqrt{1-t^2}}}_{\arcsin t} = -\frac{t \cdot \sqrt{1-t^2}}{2} + \frac{\arcsin t}{2} + C$$

$$\boxed{=} \; t^2 \cdot \arcsin t + \frac{t \cdot \sqrt{1-t^2}}{2} - \frac{\arcsin t}{2} = x \cdot \arcsin x + \frac{\sqrt{x} \cdot \sqrt{1-x}}{2} -$$

$$- \frac{\arcsin\sqrt{x}}{2} + C$$

$$\int_2^x \frac{t^2-1}{(t-2)(t^2+t+1)}\,dt.$$

$$\int \frac{t^2-1}{(t-2)\cdot(t^2+t+1)}\,dt = \frac{1}{7}\cdot\int\underbrace{\frac{4t+5}{t^2+t+1}}_{1}\,dt + \frac{3}{7}\int\underbrace{\frac{dt}{t-2}}_{2} \boxed{=}$$

$$\frac{t^2-1}{(t-2)\cdot(t^2+t+1)} = \frac{At+B}{t^2+t+1} + \frac{C}{t-2} = \frac{t^2\cdot(A+C)+t(-2A+B+C)-2B+C}{(t-2)\cdot(t^2+t+1)}$$

$$\begin{cases} A+C=1 \\ -2A+B+C=0 \\ -2B+C=-1 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 7 & 3 \end{array}\right)$$

$$C=\frac{3}{7},\quad B=\frac{5}{7},\quad A=\frac{4}{7}$$

1) $$\int\frac{4t+5}{t^2+t+1}\,dt = 2\int\underbrace{\frac{2t+1}{t^2+t+1}}_{1.1}\,dt + 3\int\frac{dt}{t^2+t+1} =$$

$$4t+5 = 2t+1+2t+1+3 = 2\cdot(2t+1)+3$$

1.1) $$\int\frac{2t+1}{t^2+t+1}\,dt = \left|\int\frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+C\right| = \ln|t^2+t+1|+C$$

1.2) $$\int\frac{dt}{t^2+t+1} = \int\frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}} = \left|\begin{array}{l} u=t+\frac{1}{2} \\ du=dt \end{array}\right| = \int\frac{dt}{u^2+\frac{3}{4}} =$$

$$= \left|\int\frac{dx}{a^2+x^2} = \frac{1}{a}\cdot\operatorname{arctg}\frac{x}{a}+C\right| = \frac{2}{\sqrt{3}}\cdot\operatorname{arctg}\frac{2}{\sqrt{3}}\cdot\left(t+\frac{1}{2}\right)+C =$$

$$= 2\sqrt{3}\cdot\operatorname{arctg}(2\sqrt{3}\cdot t+\sqrt{3})+C$$

$$= 2\ln|t^2+t+1| + 2\sqrt{3}\cdot\operatorname{arctg}(2\sqrt{3}t+\sqrt{3})+C$$

2) $$\int\frac{dt}{t-2} = \ln|t-2|$$

$$\boxed{=}\ \frac{1}{7}\left(2\ln|t^2+t+1| + 2\sqrt{3}\cdot\operatorname{arctg}(2\sqrt{3}\cdot t+\sqrt{3}) + 3\cdot\ln|t-2|+C\right.$$

$$\int_a^x \frac{3t^2-8t+7}{(t+1)(t^2-4t+4)}\,dt. = \int \frac{t+1}{t^2-4t+4}\,dt + 2\int\frac{dt}{t+1} \quad \boxed{=}$$

$$\int_a^x \frac{dt}{(t-1)^2(t+10)} = \frac{1}{121}\int\frac{-t+12}{t^2-2t+1}\,dt + \frac{1}{121}\int\frac{dt}{t+10} \quad \boxed{=}$$

① ②

$$\frac{At+B}{t^2-2t+1}+\frac{C}{t+10} = \frac{t^2\cdot(A+C)+t(10A+B-2C)+10B+C}{(t-1)^2\cdot(t+10)}$$

$$\begin{cases} A+C=0 \\ 10A+B+2C=0 \\ 10B+C=1 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 10 & 1 & -2 & 0 \\ 0 & 10 & 1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -12 & 0 \\ 0 & 0 & 121 & 1 \end{array}\right) \Rightarrow \begin{array}{l} C=1/121 \\ B=12/121 \\ A=-1/121 \end{array}$$

① $$\int\frac{12-t}{t^2-2t+1}\,dt = \left|\begin{array}{l} t-1=x \\ dt=dx \end{array}\right| = \int\frac{11-x}{x^2}\,dx = 11\int\frac{dx}{x^2} - \int\frac{dx}{x} =$$

$$= 11\cdot -1x^{-1} - \ln|x| + C = \frac{-11}{t-1} - \ln|t-1| + C$$

② $$\int\frac{dt}{t+10} = \ln|t+10| + C$$

$$\boxed{=}\ \frac{1}{121}\left(\frac{-11}{t-1} - \ln|t-1| + \ln|t+10|\right) + C$$

$$\int_a^x \frac{\cos t}{\sin^2 t-1}\,dt. = \left|\begin{array}{l} \sin t = x \\ \cos t\,dt = dx \end{array}\right| = \int\frac{dx}{x^2-1} = \int\frac{dx}{(x-1)(x+1)} \quad \boxed{=}$$

$$\frac{A}{x-1}+\frac{B}{x+1} = \frac{x(A+B)+A-B}{x^2-1} \Rightarrow \begin{cases} A+B=0 \\ A-B=1 \end{cases} \Rightarrow \begin{array}{l} A=\frac{1}{2} \\ B=-\frac{1}{2} \end{array}$$

$$\boxed{=}\ \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{dx}{x+1} = \frac{1}{2}(\ln|x-1| + \ln|x+1|) = \frac{1}{2}\ln\left|\frac{\sin x-1}{\sin x+1}\right| + C$$

$$\int \frac{x^3+1}{x^3-x^2}\,dx.$$

$$\int dx + \int \frac{1+x^2}{x^3-x^2}\,dx \;\boxed{=}$$

$$\begin{array}{r|l} -\; x^3+1 & x^3-x^2 \\ x^3-x^2 & 1 \\ \hline 1+x^2 & \end{array}$$

$$\frac{Ax+B}{x^2} + \frac{C}{x-1} = \frac{x^2(A+C)+x(-A+B)-B}{x^3-x^2}$$

$$\begin{cases} A+C=1 \\ B-A=0 \\ -B=+1 \end{cases} \Rightarrow B=-1,\ A=-1,\ C=2$$

$$\boxed{=} \int dx + \int \ldots \frac{dx}{x^2} + 2\int \frac{dx}{x-1} =$$

$$= \ldots - \ln| \ldots$$

$$\int \frac{t^3}{t^2-4}\,dt = \left| \begin{array}{l} t^2-4=x,\ t^2=x+4 \\ 2t\,dt=dx,\ t\,dt=\frac{dx}{2} \end{array} \right| = \int \frac{t^2}{2x}\,dx = \int \frac{x+4}{2x}\,dx$$

$$= \int \frac{dx}{2} + \int \frac{2}{x}\,dx = \frac{1}{2}\cdot x + 2\cdot \ln|x| + C = \frac{t^2-4}{2} + 2\ln|t^2-4| + C$$

A БО

$$\int \frac{t^3}{t^2-4}\,dt = \left| \begin{array}{r|l} -\; t^3 & t^2-4 \\ t^3-4t & t \\ \hline 4t & \end{array} \right| = \int t\,dt + \int \frac{4t}{t^2-4}\,dt = \int t\,dt + 2\int \frac{2t}{t^2-4}\,dt$$

$$= \frac{t^2}{2} + 2\ln|t^2-4| + C$$

...невизначених коефіц...

$\int \frac{dx}{\ldots x+2)} = \frac{1}{3}\int (x-1)\,dx + \frac{1}{3}\int (x+2) = \frac{1}{3} \ln\left|\frac{x-1}{x+2}\right| + C$

$\int_a^x \frac{3t\,dt}{t^4+16}.$

$\frac{A}{x-1} + \frac{B}{x+2} = \frac{x(A+B) + 2A - B}{(x-1)(x+2)} \Rightarrow \begin{cases} A+B=0 \\ 2A-B=1 \end{cases} \Rightarrow \begin{matrix} A = -\frac{1}{3} \\ B = \frac{1}{3} \end{matrix}$

$\int \int_a^x \cos^5 t\,dt. = \int \frac{1}{2}\,dx + \int \frac{\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

$\int x \cos x\,dx.$ $\left| \begin{matrix} u = x & dv = \cos x \end{matrix} \right| = \ldots \int \sin x\,dx$

$= \cos x + x \ldots$

$\int \frac{3t}{t^4+16}\,dt = \left| \begin{matrix} t = x^2 \\ dt = 2dx \cdot x \end{matrix} \right| = \frac{3}{2}\int \frac{dx}{x^2+16} = \frac{3}{2} \cdot \mathrm{arctg}\,\frac{x}{4} + C =$

$= \frac{3}{2} \cdot \mathrm{arctg}\left(\frac{t^2}{4}\right) + C$

$\int \cos^5 t\,dt = \int \cos t \cdot (1 - \sin^2 t)^2\,dt = \left| \begin{matrix} \sin t = x \\ \cos t\,dt = dx \end{matrix} \right| = \int (1 - x^2)^2\,dx =$

$= \int dx - 2\int x^2\,dx + \int x^4\,dx = x - \frac{2x^3}{3} + \frac{x^5}{5} + C =$

$= \sin t - \frac{2}{3}\sin^3 t + \frac{\sin^5 t}{5} + C$

1.1 Підстановка Чебешова:

$\int \frac{dt}{t^{11}\cdot\sqrt{1+t^4}} = \int t^{-11}\cdot(1 \ldots$

$= \int (x^2-1)^{\frac{11}{4}} \cdot \frac{(x^2-1)^{\frac12}}{x} \cdot (- \ldots$

$= -\frac12 \int (x^2-1)^2\, dx = -\frac12 (\int x \ldots$

$= -\frac12 \cdot \left(\frac{x^5}{5} - \frac23 x^3 + x + C \right)$

$1 + t^4 = x^2 \cdot t^4 \Rightarrow x = (t^{-4} \ldots$

$= \ldots - \frac{1}{10} \cdot (t^{-4}+1)^{\frac52} + \ldots$

$\int \frac{t^{-11}}{\sqrt{1+t^4}} dt.$

$\int x\sqrt{x^2+2}\,dx. =$

$= \frac12 \int \sqrt{t-2} \cdot \sqrt{t} \cdot \frac{dt}{\sqrt{t-2}} =$

$= \frac12 \int \sqrt{t}\, dt = \frac12 \cdot \frac23 \cdot t^{\frac32} + C$

$= \frac{(x^2+2)^{\frac32}}{3} + C$

$m = 1, \quad n = 2, \quad p = \frac12$

1) $p \notin \mathbb{Z}$,

2) $\frac{1+1}{2} = 1 \in \mathbb{Z}$

$x^2 + 2 = t$

$x = \sqrt{t-2}$

$dx = \frac{dt}{2\sqrt{t-2}}$

$\sqrt{x^2+2} = \sqrt{t}$

$(1+t^2)^{-\frac12} dt = \int$

$\frac{(x^2-1)^{-\frac12}}{x} \cdot \left(-1x(x^2-1)^{-\frac32}\right)) dx =$

$\ldots 1) dx = -\frac{x^3}{3} + x + C =$

$1 + t^2 \Rightarrow x = (t^{-2}+1)^{\frac12}$

$\ldots +1)^{\frac32} + (t^{-2}+1)^{\frac12} + C$

$m = -4, \quad n = 2, \quad p = -\frac12$

1) $p = -\frac12 \notin \mathbb{Z}$

2) $\frac{-4+1}{2} = -1{,}5 \notin \mathbb{Z}$

3) $\frac{-4+1}{2} - \frac12 = -2 \in \mathbb{Z}$

$(1+t^2) = x^2 t^2 \Rightarrow t^2 = \frac{1}{x^2-1}$

$t = (x^2-1)^{-\frac12}$

$dt = -\frac12 (x^2-1)^{-\frac32} \cdot 2x\, dx$

$(1+t^2)^{-\frac12} = \left(1 + \frac{1}{x^2-1}\right)^{-\frac12} =$

$= \frac{(x^2-1)^{-\frac12}(-1)}{x}, \quad x > 0$