В-3

№ 1

1. $L = \langle -x^2-2x+3,\ 3x^2+4x-1,\ 2x^2+2x+2 \rangle$

$$\begin{pmatrix} -1 & -2 & 3 \\ 3 & 4 & -1 \\ 2 & 2 & 2 \end{pmatrix} \underset{2p+4p}{\sim} \begin{pmatrix} -1 & -2 & 3 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \underset{3p-2p}{\sim} \begin{pmatrix} -1 & -2 & 3 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{2p+1p}{\sim}$$

$$\begin{pmatrix} -1 & -2 & 3 \\ 0 & -2 & 8 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow L = \langle -x^2-2x+3,\ 3x^2+4x-1 \rangle \Rightarrow \dim\langle L \rangle = 2$$

$f(x) = x^2+3$

Нехай $\exists\ \alpha, \beta \in R$:

$\alpha(-x^2-2x+3) + \beta(3x^2+4x-1) = x^2+3$

$$\left(\begin{array}{cc|c} -1 & 3 & 1 \\ -2 & 4 & 0 \\ 3 & -1 & 3 \end{array}\right) \underset{3p+3\cdot 1p}{\overset{2p-2\cdot 1p}{\sim}} \left(\begin{array}{cc|c} -1 & 3 & 3 \\ 0 & -2 & -2 \\ 0 & 8 & 6 \end{array}\right) \underset{3p+4\cdot 2p}{\sim}$$

$$\left(\begin{array}{cc|c} -1 & 3 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{array}\right) \Rightarrow \beta = -1 \text{ і } \beta = 0$$ – протиріччя, а отже $f(x) \notin$ лінійній оболонці

$$T_{e\to B} = \begin{pmatrix} 0 & 1 & \\ 0 & 0 & 1 \end{pmatrix} \begin{vmatrix} 3 & -2 & 5 \\ 1 & 4 & -2 \end{pmatrix} \Rightarrow T_{e\to B} = \begin{pmatrix} 3 & -2 & 5 \\ 1 & 4 & -2 \end{pmatrix}$$

№2

$U = \{a_1, a_2, a_3\}$, $V = \{b_1, b_2, b_3\}$

$a_1 = (1, 1, -1, -1)$, $a_2 = (3, -1, 1, -2)$, $a_3 = (2, -2, 2, -1)$

$b_1 = (2, 1, 2, -3)$, $b_2 = (1, 2, 3, -3)$, $b_3 = (1, -1, -1, 0)$

1.

$$B_u: \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & -1 & 1 & -2 \\ 2 & -2 & 2 & -1 \end{pmatrix} \underset{3p - 2\cdot 1p.}{\overset{2p - 3\cdot 1p}{\sim}} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -4 & 4 & 1 \\ 0 & -4 & 4 & 1 \end{pmatrix} \Rightarrow$$

$\Rightarrow \dim U = 2$, $B_u = \{a_1, a_2\}$

$$B_V = \begin{pmatrix} 2 & 1 & 2 & -3 \\ 1 & 2 & 3 & -3 \\ 1 & -1 & -1 & 0 \end{pmatrix} \begin{matrix} 1p-2\cdot 3p \\ \sim \\ 2p-3p. \end{matrix} \begin{pmatrix} 0 & 3 & 4 & -3 \\ 0 & 3 & 4 & -3 \\ 1 & -1 & -1 & 0 \end{pmatrix} \Rightarrow$$

$\Rightarrow \dim V = 2$, $B_V = \{ b_2, b_3 \}$

2.

$$B_{V+U} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & -1 & 1 & -2 \\ 1 & 2 & 3 & -3 \\ 1 & -1 & -1 & 0 \end{pmatrix} \begin{matrix} 2p-3\cdot 1p. \\ \sim \\ 3p-1p \\ 4p-1p \end{matrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -4 & 4 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & -2 & 0 & 1 \end{pmatrix} \begin{matrix} 2p+ \\ \sim \\ 3p+1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 20 & -7 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 8 & -3 \end{pmatrix} \begin{matrix} 2p-2{,}5\cdot 4p. \\ \sim \end{matrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 8 & -3 \end{pmatrix} \Rightarrow$$

$\Rightarrow \dim(V+U) = 4 \Rightarrow R^4 = V+U$

$\dim(U+V) = \dim V + \dim U - \dim(U \cap V) \Rightarrow \dim(U\cap V) = 0$, а

отже $R^4 = V \oplus U$

3. $x = (0, 2, 0 -1)$

$\alpha_1 a_1 + \alpha_2 a_2 + \beta_1 \cdot b_2 + \beta_2 \cdot b_3 = x$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 2 \\ -1 & 1 & 3 & -1 & 0 \\ -1 & -2 & -3 & 0 & -1 \end{array}\right) \begin{array}{l} \\ 2p-1p \\ 3p+1p \\ 4p+1p \end{array} \sim \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -4 & 1 & -2 & 2 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 \end{array}\right) \begin{array}{l} \\ \\ :4 \\ \end{array} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -4 & 1 & -2 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 \end{array}\right) \begin{array}{l} \\ 2p+4\cdot 3p \\ \\ 4p-3p \end{array} \sim \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 5 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & -1 \end{array}\right) \begin{array}{l} 1p-3\cdot 3p \\ 2p+2\cdot 4p \\ \\ \\ \end{array} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & -1 \end{array}\right) \begin{array}{l} 1p+2p \\ \\ \\ 4p-3\cdot 2p \end{array} \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} 1p-4p \\ \cdot(-1) \\ \\ \\ \end{array} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \alpha_1 = 1, \ \beta_2 = -1$$

$x = a_1 - b_3$

$P_{R_{u \parallel v}} x = a_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ $\quad P_{R_{v \parallel u}} = -b_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$

№ 3

1) $\varphi(x) \to A \cdot x, \quad A = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

1. $\varphi(x+y) = \varphi(x) + \varphi(y)$; $x, y \in M_2(R)$

$\varphi(x+y) = A \cdot (x+y) = Ax + Ay = \varphi(x) + \varphi(y)$ $+$

2. $\varphi(\lambda x) = \lambda \varphi(x)$

$\varphi(\lambda x) = A \cdot (\lambda x) = A \cdot \lambda \cdot x = \lambda \cdot Ax = \lambda \cdot \varphi(x)$ $+$ $\Rightarrow$

отже $\varphi(x)$ - лінійний оператор

2) $E = (E_1, E_2, E_3, E_4) =$

$E \quad (1 \quad 0) \quad \quad (0 \quad 1) \quad \quad (0 \quad 0)$

№ 3

1) $\varphi(x) \to x \cdot A$, $A = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

1. $\varphi(x+y) = \varphi(x) + \varphi(y)$; $x, y \in M_2(R)$

$\varphi(x+y) = (x+y) \cdot A = x \cdot A + y \cdot A = \varphi(x) + \varphi(y)$

$\varphi(\lambda x) = \lambda(\varphi(x)) = (\lambda \cdot x) A = \lambda x \cdot A = \lambda \cdot \varphi(x)$ $\Rightarrow$ $\varphi(x)$ - лінійний оператор

2) $E = (E_1, E_2, E_3, E_4)$

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\varphi(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

$\varphi(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 0 & 0 \end{pmatrix}$

$\varphi(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$

$\varphi(E_4) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & -1 \end{pmatrix}$

$$M = \begin{pmatrix} -2 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

3. Знайдемо $\ker \varphi$

$$x \cdot A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

$$\begin{cases} -2x_1 + 3x_2 = 0 \\ x_1 - x_2 = 0 \\ -2x_3 + 3x_4 = 0 \\ x_3 - x_4 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3 = x_4 = 0.$$

Отже $\ker \varphi = \left\{ \lambda \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \lambda \in \mathbb{R} \right\} \Rightarrow \operatorname{def} \varphi = 0$

$\operatorname{rank}(\varphi) = \operatorname{rank}(\operatorname{im}) \Rightarrow$ макс. к-ть лін. незалежних векторів в стандартному базисі.

$$M = \begin{pmatrix} -2 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix} \Rightarrow \operatorname{rank} M = 4 \Rightarrow \operatorname{rank}(\operatorname{im}) = 4 \Rightarrow \operatorname{rank}(\varphi) = 4$$

В-дь:

2) $M = \begin{pmatrix} -2 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

3) $\operatorname{rank}(\ker \varphi) = 0$

$\operatorname{rank}(\varphi) = 4$