В-2

№ 1

$B\{-2x^2+3x-1,\ x^2-2x+4,\ -x^2+5x-2\}$

$f(x)=x^2+2x+4$

1) $\lambda_1\cdot(-2x^2+3x-1)+\lambda_2(x^2-2x+4)+\lambda_3(-x^2+5x-2)=$

$=x^2(-2\lambda_1+\lambda_2-\lambda_3)+x(3\lambda_1-2\lambda_2+5\lambda_3)-\lambda_1+4\lambda_2-2\lambda_3=0$

$$\begin{cases}-2\lambda_1+\lambda_2-\lambda_3=0\\ 3\lambda_1-2\lambda_2+5\lambda_3=0\\ -\lambda_1+4\lambda_2-2\lambda_3=0\end{cases}\Rightarrow\begin{pmatrix}-2&1&-1\\3&-2&5\\-1&4&-2\end{pmatrix}\begin{matrix}1р-2\cdot3р.\\ \sim\\ 2р+3\cdot3р.\end{matrix}$$

$$\begin{pmatrix}0&-7&5\\0&10&-1\\-1&4&-2\end{pmatrix}\begin{matrix}1р+3\cdot2р\\ \sim\\ 3р-2\cdot2р\end{matrix}\begin{pmatrix}0&-23&0\\0&10&-1\\-1&-16&0\end{pmatrix}\begin{matrix}:-23\\ \sim\end{matrix}\begin{pmatrix}0&1&0\\0&10&-1\\-1&-16&0\end{pmatrix}\begin{matrix}2р-10\cdot1р\\ \sim\\ 3р+16\cdot1р.\end{matrix}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\Rightarrow\lambda_1=\lambda_2=\lambda_3=0,$$ а отже система векторів лінійно незалежна і утворює базис

$$f(x) = \lambda_1 \cdot (-2x^2+3x+1) + \lambda_2 \cdot (x^2-2x+4) + \lambda_3 \cdot (-x^2+5x-2) \Rightarrow$$

$$\Rightarrow$$

$$\begin{cases} -2\lambda_1 + \lambda_2 - \lambda_3 = 1 \\ 3\lambda_2 + 2\lambda_2 + 5\lambda_3 = 2 \\ -\lambda_1 + 4\lambda_2 - 2\lambda_3 = 4 \end{cases}$$

$$\left(\begin{array}{ccc|c} -2 & 1 & -1 & 1 \\ 3 & -2 & 5 & 2 \\ -1 & 4 & -2 & 4 \end{array}\right) \underset{3р+3\cdot 3р}{\overset{1р-2\cdot 3р}{\sim}} \left(\begin{array}{ccc|c} 0 & -7 & 3 & -7 \\ 0 & 10 & -1 & 14 \\ -1 & 4 & -2 & 4 \end{array}\right) \underset{2р-2\cdot 2р}{\overset{1р+3\cdot 2р}{\sim}}$$

$$\left(\begin{array}{ccc|c} 0 & 23 & 0 & 35 \\ 0 & 10 & -1 & 14 \\ -1 & -16 & 0 & -24 \end{array}\right) \begin{array}{l} :23 \\ \\ \cdot(-1) \end{array} \quad \left(\begin{array}{ccc|c} 0 & 1 & 0 & \frac{35}{23} \\ 0 & 10 & -1 & 14 \\ 1 & 16 & 0 & 24 \end{array}\right) \underset{3р-16\cdot 1р}{\overset{2р-10\cdot 1р}{\sim}}$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & \frac{35}{23} \\ 0 & 0 & -1 & -\frac{28}{3} \\ 1 & 0 & 0 & -\frac{8}{23} \end{array}\right) \Rightarrow \lambda_1 = -\frac{8}{23}, \ \lambda_3 = \frac{28}{23}, \ \lambda_2 = \frac{35}{23}$$

Отже координати $f(x) = \left(-\frac{8}{23}, \frac{35}{23}, \frac{28}{23}\right)$ в базисі B

2) $E(1, x, x^2)$

$$E_M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B_M = \begin{pmatrix} -2 & 1 & -1 \\ 3 & -2 & 5 \\ 1 & 4 & -2 \end{pmatrix}$$

$$T_{e \to B} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & 3 & -2 & 5 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right) \Rightarrow T_{e \to B} = \begin{pmatrix} -2 & 1 & -1 \\ 3 & -2 & 5 \\ 1 & 4 & -2 \end{pmatrix}$$

№2

$U = \{a_1, a_2, a_3\}$, $V = \{b_1, b_2, b_3\}$

$a_1 = (1, 1, -1, -1)$, $a_2 = (3, -1, 1, -2)$, $a_3 = (2, -2, 2, -1)$

$b_1 = (2, 1, 2, -3)$, $b_2 = (1, 2, 3, -3)$, $b_3 = (1, -1, -1, 0)$

1.

$$B_U: \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & -1 & 1 & -2 \\ 2 & -2 & 2 & -1 \end{pmatrix} \underset{3p - 2 \cdot 1p.}{\overset{2p - 3 \cdot 1p}{\sim}} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -4 & 4 & 1 \\ 0 & -4 & 4 & 1 \end{pmatrix} \Rightarrow$$

$\Rightarrow \dim U = 2$, $B_U = \{a_1, a_2\}$

$$B_V = \begin{pmatrix} 2 & 1 & 2 & -3 \\ 1 & 2 & 3 & -3 \\ 1 & -1 & -1 & 0 \end{pmatrix} \underset{2p-3p}{\overset{1p-2\cdot 3p}{\sim}} \begin{pmatrix} 0 & 3 & 4 & -3 \\ 0 & 3 & 4 & -3 \\ 1 & -1 & -1 & 0 \end{pmatrix} \Rightarrow$$

$\Rightarrow \dim V = 2$, $B_V = \{b_2, b_3\}$

2.

$$B_{V+U} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 3 & -1 & 1 & -2 \\ 1 & 2 & 3 & -3 \\ 1 & -1 & -1 & 0 \end{pmatrix} \begin{matrix} 2p-3\cdot 1p \\ \sim \\ 3p-1p \\ 4p-1p \end{matrix} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -4 & 4 & 1 \\ 0 & 1 & 4 & -2 \\ 0 & -2 & 0 & 1 \end{pmatrix} \begin{matrix} 2p+ \\ \sim \\ 3p+ \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 20 & -7 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 8 & -3 \end{pmatrix} \overset{2p-2{,}5\cdot 4p}{\sim} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 8 & -3 \end{pmatrix} \Rightarrow$$

$\Rightarrow \dim(V+U) = 4 \Rightarrow R^4 = V+U$

$\dim(U+V) = \dim V + \dim U - \dim(U\cap V) \Rightarrow \dim(U \cap V) = 0$, a

отже $R^4 = V \oplus U$

3. $X = (0, 2, 0 -1)$

$\alpha_1 a_1 + \alpha_2 a_2 + \beta_1 \cdot b_2 + \beta_2 \cdot b_3 = \cancel{x}$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 1 & -1 & 2 & -1 & 2 \\ -1 & 1 & 3 & -1 & 0 \\ -1 & -2 & -3 & 0 & -1 \end{array}\right) \begin{array}{l} \\ 2p-1p \\ 3p+1p \\ 4p+1p \end{array} \sim \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -4 & 1 & -2 & 2 \\ 0 & 4 & 4 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 \end{array}\right) \begin{array}{l} \\ \\ :4 \\ \\ \end{array} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -4 & 1 & -2 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 \end{array}\right) \begin{array}{l} \\ 2p+4\cdot 3p \\ \\ 4p-3p \end{array} \sim \left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 5 & -2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & -1 \end{array}\right) \begin{array}{l} 1p-3\cdot 3p \\ 2p+2\cdot 4p \\ \\ \\ \end{array} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & -1 \end{array}\right) \begin{array}{l} 1p+2p \\ \\ \\ 4p-3\cdot 2p \end{array} \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} 1p-4p \\ \cdot(-1) \\ \\ \\ \end{array} \sim$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \alpha_1 = 1, \ \beta_2 = -1$$

$x = a_1 - b_3$

$Pr_{R_{uнv}} x = a_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ $\quad R_{R_{vнu}} = -b_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix}$

№ 3

1) $\varphi(X) \to A \cdot X, \quad A = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$

1. $\varphi(X+Y) = \varphi(X) + \varphi(Y); \; X, Y \in M_2(R)$

$\varphi(X+Y) = A \cdot (X+Y) = AX + AY = \varphi(X) + \varphi(Y)$ (+)

2. $\varphi(\alpha X) = \alpha \varphi(X)$

$\varphi(\alpha X) = A \cdot (\alpha X) = A \cdot \alpha \cdot X = \alpha \cdot AX = \alpha \cdot \varphi(X)$ (+) $\Rightarrow$

Отже $\varphi(X)$ - лінійний оператор

2) $E = (E_1, E_2, E_3, E_4) =$

$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\varphi(E_1) = A \cdot E_1 = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}$

$\varphi(E_2) = A \cdot E_2 = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 3 \end{pmatrix}$

$\varphi(E_3) = A \cdot E_3 = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$

$\varphi(E_4) = A \cdot E_4 = \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$

Отже $M = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix}$

3)

Знайдемо $\ker\varphi$:

$$A \cdot X = \Theta \Rightarrow \begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} -2x_1 + x_3 & -2x_2 + x_4 \\ 3x_1 - x_3 & 3x_2 - x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{cases} -2x_1 + x_3 = 0 \\ -2x_2 + x_4 = 0 \\ 3x_1 - x_3 = 0 \\ 3x_2 - x_4 = 0 \end{cases} \Rightarrow x_1 = x_2 = x_3 = x_4 = 0 \Rightarrow \ker(\varphi) = \left\{ \lambda \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \middle| \lambda \in \mathbb{R} \right\}$$

В-дь

2) $M = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix}$

3) базис ядра: $\lambda \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ або $\lambda \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$.