How to use quantitative techniques to optimize portfolio construction? Supporting Material

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1 Portfolio Solution with Multiple Assets

We derive here the portfolio solution when we have more than two risky assets. We use the following notation:

- N is the total number of assets,
- $w = [w_1, w_2, ..., w_N]^\top$ is a $N \times 1$ vector containing portfolio allocations to the N risky assets,
- $E[r - r_f] = [E[r_1 - r_f], E[r_2 - r_f], ..., E[r_N - r_f]]^\top$ is a $N \times 1$ vector of expected returns in excess of the risk-free rate r_f,
- Σ is a $N \times N$ matrix containing all covariances between the N assets with element $\Sigma_{i,j} = Cov[r_i, r_j]$,
- $\top$ denotes the transpose operator; if w is a $N \times 1$ vector, then $w^\top$ is a $1 \times N$ vector.

Using this notation, we can write the portfolio expected return as

$$E[r_p] = r_f + w^\top E[r - r_f] \tag{1}$$

and the portfolio return variance as

$$Var[r_p] = w^\top \Sigma w. \tag{2}$$

Let's find the optimal allocation w^*. You want to maximize our tradeoff between portfolio expected return and variance, which is the preference metric we have chosen to use:

$$\max_w \quad E\left[r_p\right] - \frac{\text{Risk Aversion}}{2} Var\left[r_p\right]$$

which we can write as

$$\max_w \quad r_f + w^\top E[r - r_f] - \frac{\text{Risk Aversion}}{2} w^\top \Sigma w$$

using equations (1) and (2).

Remember from Calculus that to find the maximum we need to set the first derivative with respect to w equal to 0. Deriving our portfolio preference metric, we obtain the first order conditions:

$$E[r - r_f] - \text{Risk Aversion } \Sigma w = 0$$

from which we obtain the optimal allocation:

$$w^* = \frac{1}{\text{Risk Aversion}} \Sigma^{-1} E[r - r_f] \tag{3}$$

where Σ^{-1} is the inverse of the matrix Σ[1].

1.1 An example with two assets

Let's solve for the optimal allocation in the case in which we allocate over the next year between a risk-free asset, a stock index S, and long-term bond B. We use the following parameters:

- our risk aversion is 3,
- the risk-free asset pays $r_f = 3\%$,
- the stock index S has an expected excess return of $E[r_S - r_f] = 7\%$, a volatility of 20%, and hence a variance of $Var[r_S] = 20\%^2 = 0.04$,
- the long-term bond B has an expected excess return of $E[r_B - r_f] = 3\%$, a volatility of 10%, and hence a variance of $Var[r_B] = 10\%^2 = 0.01$,
- and the correlation between asset A and B is $Corr[r_S, r_B] = 0.5$.

The matrix Σ is in this case

$$\Sigma = \begin{bmatrix} Var[r_S] = 0.04 & Cov[r_S, r_B] = Vol[r_S]Vol[r_B]Corr[r_S, r_B] = 0.01 \\ Cov[r_B, r_S] = Vol[r_B]Vol[r_S]Corr[r_B, r_S] = 0.01 & Var[r_B] = 0.01 \end{bmatrix}$$

and the vector of expected return in excess of the risk-free rate is

$$E[r - r_f] = \begin{bmatrix} E[r_S - r_f] = 0.07 \\ E[r_B - r_f] = 0.03 \end{bmatrix}.$$

Using the formula for the inverse of a 2×2 matrix, we obtain

$$\Sigma^{-1} = \frac{1}{Var[r_S]Var[r_B] - Cov[r_S, r_B]^2} \begin{bmatrix} Var[r_B] & -Cov[r_S, r_B] \\ -Cov[r_B, r_S] & Var[r_S] \end{bmatrix}.$$

Then, using our formula for the optimal allocation in Equation (4), we obtain

$$\begin{aligned} w^* &= \frac{1}{\text{Risk Aversion}} \times \frac{1}{Var[r_S]Var[r_B] - Cov[r_S, r_B]^2} \begin{bmatrix} Var[r_B] & -Cov[r_S, r_B] \\ -Cov[r_B, r_S] & Var[r_S] \end{bmatrix} \\ &\quad \times \begin{bmatrix} E[r_S - r_f] \\ E[r_B - r_f] \end{bmatrix} \\ &= \frac{1}{\text{Risk Aversion}} \times \frac{1}{Var[r_S]Var[r_B] - Cov[r_S, r_B]^2} \begin{bmatrix} Var[r_B]E[r_S - r_f] - Cov[r_S, r_B]E[r_B - r_f] \\ Var[r_S]E[r_B - r_f] - Cov[r_B, r_S]E[r_S - r_f] \end{bmatrix}. \end{aligned}$$

[1]Deriving a second time confirms that this is a maximum.

Using the expression for correlation $Corr[r_S, r_B] = \frac{Cov[r_S, r_B]}{\sqrt{Var[r_S] \times Var[r_B]}}$, note that the expression $Var[r_S]Var[r_B] - Cov[r_S, r_B]^2$ in the denominator can also be expressed as $Var[r_S]Var[r_B]\left(1 - Corr[r_S, r_B]^2\right)$ as we did in the video.

Thefore, we obtain an optimal allocation to the stock index S of

$$w_S^* = \frac{0.01 \times 0.07 - 0.01 \times 0.03}{3 \times (0.04 \times 0.01 - 0.01^2)} = 44.44\%,$$

and an optimal allocation to the long-term bond B of

$$w_B^* = \frac{0.04 \times 0.03 - 0.01 \times 0.07}{3 \times (0.04 \times 0.01 - 0.01^2)} = 55.56\%.$$

Alternatively, using the correlation $Corr[r_S, r_B] = \frac{Cov[r_S, r_B]}{\sqrt{Var[r_S] \times Var[r_B]}} = \frac{0.01}{\sqrt{0.04 \times 0.01}} = 0.5$, we obtain an optimal allocation to the stock index S of

$$w_S^* = \frac{0.01 \times 0.07 - 0.01 \times 0.03}{3 \times (0.04 \times 0.01 \times (1 - 0.5^2))} = 44.44\%,$$

and an optimal allocation to the long-term bond B of

$$w_B^* = \frac{0.04 \times 0.03 - 0.01 \times 0.07}{3 \times (0.04 \times 0.01 \times (1 - 0.5^2))} = 55.56\%.$$

Finally, the allocation to the risk-free asset is $100\% - 44.44\% - 55.56\% = 0$.