Taylor's Theorem :-

Let $f : [a,b] \to R,\ f, f', f'', \ldots\ldots., f^{(n-1)}$ be continuous on $[a,b]$ and suppose $f^{(n)}$exits on (a,b). Then there exits $c \in (a,b)$ such that

$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots.. + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{(n-1)} + \frac{f^{(n)}(c)}{n!}(b-a)^n$.

Proof :- Define

$F(x) = f(b) - f(x) - f'(x)(b-x) - \frac{f''(x)}{2!}(b-x)^2 - \ldots\ldots. - \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{(n-1)}$.

We will show that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$ for some $c \in (a,b)$, which will prove the theorem.

Note that

$$F'(x) = -\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{(n-1)} \cdots\cdots\cdots Eq(1.)$$

Define $g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^n F(a)$. It is easy to check that $g(a) = g(b) = 0$ and hence by Rolle's theorem there exits some $c \in (a,b)$such that

$$g'(c) = F'(c) + \frac{n(b-c)^{(n-1)}}{(b-a)^n} F(a) = 0 \cdots\cdots\cdots Eq(2.)$$

From (1) and (2) we obtain that $\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{(n-1)} = \frac{n(b-c)^{(n-1)}}{(b-a)^n} F(a)$ .This implies that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$. This proves the theorem.