

Taylor's Theorem :-

Let $f : [a, b] \rightarrow R$, $f, f', f'', \dots, f^{(n-1)}$ be continuous on $[a, b]$ and suppose $f^{(n)}$ exists on (a, b) . Then there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \dots + \frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{(n-1)} + \frac{f^{(n)}(c)}{n!}(b-a)^n$$

Proof :- Define

$$F(x) = f(b) - f(x) - f'(x)(b-x) - \frac{f''(x)}{2!}(b-x)^2 - \dots - \frac{f^{(n-1)}(x)}{(n-1)!}(b-x)^{(n-1)}$$

We will show that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$ for some $c \in (a, b)$, which will prove the theorem.

Note that

$$F'(x) = -\frac{f^{(n)}(x)}{(n-1)!}(b-x)^{(n-1)} \quad \text{Eq (1.)}$$

Define $g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^n F(a)$. It is easy to check that $g(a) = g(b) = 0$ and

hence by Rolle's theorem there exists some $c \in (a, b)$ such that

$$g'(c) = F'(c) + \frac{n(b-c)^{(n-1)}}{(b-a)^n} F(a) = 0 \dots \dots \dots \text{Eq (2.)}$$

From (1) and (2) we obtain that $\frac{f^{(n)}(c)}{(n-1)!}(b-c)^{(n-1)} = \frac{n(b-c)^{(n-1)}}{(b-a)^n} F(a)$. This implies that $F(a) = \frac{(b-a)^n}{n!} f^{(n)}(c)$. This proves the theorem.