Probability Distributions:

Probability distributions are mathematical functions that describe the likelihood or probability of different outcomes of a random variable. There are two main types of probability distributions: discrete and continuous.

1. Discrete Probability Distributions:

Discrete probability distributions are used when the random variable can take only distinct, separate values, typically integers. Some common discrete probability distributions are:

a. Binomial Distribution:

The binomial distribution models the number of successes in a fixed number of independent and identical trials, where each trial has only two possible outcomes: success or failure. The probability mass function (PMF) of a binomial random variable X, which represents the number of successes in n trials, with the probability of success in each trial being p, is given by:

P(X = x) = (n choose x) * p^x * (1-p)^(n-x), for x = 0, 1, 2, ..., n

Example: Suppose you flip a fair coin 10 times. Let X be the number of heads obtained. Since the coin is fair, the probability of getting a head on any single toss is 0.5. The number of heads obtained follows a binomial distribution with n = 10 and p = 0.5.

b. Poisson Distribution:

The Poisson distribution is used to model the number of occurrences of an event in a fixed interval of time or space, given that the events occur independently and at a constant average rate. The PMF of a Poisson random variable X, which represents the number of occurrences of an event in a fixed interval, with an average rate of λ occurrences per interval, is given by:

P(X = x) = (e^(-λ) * λ^x) / x!, for x = 0, 1, 2, ...

Example: Suppose the average number of customers arriving at a bank per hour is 5. Let X be the number of customers arriving in a given hour. The number of customer arrivals follows a Poisson distribution with λ = 5.

2. Continuous Probability Distributions:

Continuous probability distributions are used when the random variable can take any value within a specific range or interval. Some common continuous probability distributions are:

a. Normal Distribution:

The normal distribution is a bell-shaped, continuous probability distribution that is widely used in many fields, including statistics, natural sciences, and social sciences. The probability density function (PDF) of a normal random variable X with mean μ and standard deviation σ is given by:

$$f(x) = (1 / (\sigma * \text{sqrt}(2\pi))) * e^{(-(x-\mu)^2 / (2\sigma^2))}, \text{ for } -\infty < x < \infty$$

Example: Suppose the heights of adult men in a certain population are normally distributed with a mean of 175 cm and a standard deviation of 6 cm. Let X be the height of a randomly selected adult man from this population. The height follows a normal distribution with $\mu = 175$ and $\sigma = 6$.

b. Uniform Distribution:

The uniform distribution is a continuous probability distribution where all values within a specified range are equally likely. The PDF of a uniform random variable X with a minimum value of a and a maximum value of b is given by:

$$f(x) = 1 / (b - a), \text{ for } a \leq x \leq b$$

$$= 0, \text{ otherwise}$$

Example: Suppose a fair die is rolled, and let X be the outcome. The outcome follows a uniform distribution with a minimum value of 1 and a maximum value of 6.

Expectation and Mean:

The expectation, also known as the mean or expected value, is a central measure of a probability distribution. It represents the long-run average value of a random variable if the experiment or process is repeated many times.

For a discrete random variable X, the mean or expected value, denoted as E(X), is calculated as:

$$E(X) = \Sigma x * P(X = x)$$

where the summation is taken over all possible values of X.

For a continuous random variable X with a probability density function f(x), the mean or expected value is calculated as:

$$E(X) = \int x * f(x)\, dx$$

where the integral is taken over the entire range of possible values of X.

Example: Consider the binomial distribution with n = 5 and p = 0.3. Let X be the number of successes. The mean or expected value of X is:

$$E(X) = \Sigma x * P(X = x)$$

= 0 * (5 choose 0) * (0.3)^0 * (0.7)^5 + 1 * (5 choose 1) * (0.3)^1 * (0.7)^4 + ... + 5 * (5 choose 5) * (0.3)^5 * (0.7)^0

= 1.5

Mean Property:

The mean or expected value has a useful property called the linearity property or the mean property. This property states that for any two random variables X and Y, and any constants a and b, we have:

E(aX + bY) = a * E(X) + b * E(Y)

This property is particularly useful when working with linear combinations of random variables, as it allows us to calculate the expected value of the linear combination by simply combining the expected values of the individual random variables.

Example: Let X and Y be two independent random variables, with E(X) = 2 and E(Y) = 3. Consider the random variable Z = 2X - 3Y.

Using the mean property, we can find the expected value of Z as:

E(Z) = E(2X - 3Y)

= 2 * E(X) - 3 * E(Y)

= 2 * 2 - 3 * 3

$= 4 - 9$

$= -5$

This example demonstrates how the mean property can be used to calculate the expected value of a linear combination of random variables, without having to find the complete probability distribution of the linear combination.

In summary,

probability distributions describe the likelihood of different outcomes of a random variable, while the expectation or mean represents the long-run average value of the random variable. The mean property allows us to calculate the expected value of linear combinations of random variables, making it a valuable tool in probability theory and its applications.