Lab 1 Report: Solving Linear Systems using LAPACK

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09/14/2024

1 Introduction

This report outlines the implementation and results of solving linear systems using LAPACK routines in C. The lab is divided into two parts: the first part focuses on solving a general linear system $A\mathbf{x} = \mathbf{b}$ using LU decomposition, and the second part explores solving a banded linear system, as well as solving multiple systems with different right-hand sides.

The key objectives of the lab are:

- To solve linear systems efficiently using LAPACK routines.
- To compute the condition number of matrices and analyze the stability of solutions.
- To implement and solve banded systems, making use of the specialized LAPACK routines for efficiency.

2 Software Requirements

To successfully run the programs, you need the following software installed:

1. **GCC (GNU Compiler Collection)**: The C compiler used to compile the programs.
   - Installation command (Linux):

     ```
     sudo apt-get install build-essential
     ```

2. **LAPACK and BLAS libraries**: These libraries provide efficient routines for solving linear algebra problems.
   - Installation command (Linux):

```
sudo apt-get install liblapack-dev libblas-dev
```

3. **MSYS2 (for Windows users)**: A shell that allows installing the necessary packages like LAPACK, BLAS, and GCC.

   - Download from: `https://www.msys2.org/`
   - After installation, run:

```
pacman -S mingw-w64-x86_64-lapack mingw-w64-x86_64-openblas mingw-w64-x86_6
```

3 Steps to Compile and Run the Programs

3.1 Linux/Mac

1. Open the terminal.
2. Navigate to the folder where your code is stored.
3. To compile:

```
gcc -o Question1a Question1a.c -llapack -lblas -lm
gcc -o Question2a Question2a.c -llapack -lblas -lm
gcc -o Question2b Question2b.c -llapack -lblas -lm
```

4. To run the programs:

```
./Question1a
./Question2a
./Question2b
```

3.2 Windows (using MSYS2)

1. Open the MSYS2 shell.
2. Navigate to the folder containing your code using the `cd` command.
3. To compile:

```
gcc -o Question1a Question1a.c -llapack -lblas -lm
gcc -o Question2a Question2a.c -llapack -lblas -lm
gcc -o Question2b Question2b.c -llapack -lblas -lm
```

4. To run:

```
./Question1a
./Question2a
./Question2b
```

4 Part 1: Solving $A\mathrm{x} = \mathbf{b}$ (Question1a.c)

4.1 Problem Description

The goal of this part is to solve a general linear system $A\mathbf{x} = \mathbf{b}$ using the LAPACK routine `dgesv`, which performs LU decomposition. The matrix A is a dense matrix, and we calculate the condition number of A to assess the stability of the solution. Finally, we compute the L_2-norm of the error between the exact and computed solutions.

4.2 Results:

- Condition number of matrix A: 117.3
- L_2-norm of the error: 4.376
- **Solution**:

  ```
  x[0] = -493.89, x[1] = -5432.58, x[2] = -59758.07, x[3] = -657338.36, ...
  ```

- Comparison between x and x':

  ```
  Difference for first entries: 493.90, 5432.60, 59758.10, 657338.40, ...
  ```

4.3 Discussion

The results for x and x' show a drastic divergence between the values of the computed solution and the expected solution. The difference between the original and computed values grows significantly, especially for large i, with some entries reaching magnitudes as high as 10^{24}. This behavior indicates that the matrix is ill-conditioned and highlights the inherent instability due to the high condition number (117.3), which contributes to significant numerical errors during the LU decomposition process.

Although the condition number itself is moderately high, this has a visible impact on the magnitude of errors introduced in the solution. The L_2-norm of the error (4.376) also confirms the deviation from the exact solution, indicating that some instability is present in this calculation.

5 Part 1b: Solving Two Linear Systems (Question1b.c)

5.1 Problem Description

In this part, we are tasked with solving two different linear systems using the same coefficient matrix A but with different right-hand side vectors $\mathbf{b}$ and $\mathbf{c}$. The goal is to compute the solution for $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{c}$, then compare the solutions and discuss any numerical differences.

5.2 Results:

- **Solution for $A\mathbf{x} = \mathbf{b}$**:

  ```
  x1 = 3.94, x2 = 0.49
  ```

- **Solution for $A\mathbf{y} = \mathbf{c}$**:

  ```
  y1 = 19.10, y2 = -21.52
  ```

- **Difference between solutions**:

  ```
  Difference in x1 and y1: 15.16
  Difference in x2 and y2: 22.01
  ```

5.3 Discussion

The results for solving the two systems show significant numerical differences between the solutions for $\mathbf{x}$ and $\mathbf{y}$. Specifically, the difference between the first elements x_1 and y_1 is around 15.16, while the difference between x_2 and y_2 is much larger at 22.01.

This variation can be attributed to the numerical instability introduced by the slight changes in the right-hand side vectors $\mathbf{b}$ and $\mathbf{c}$. Reusing the LU decomposition of A for both systems improves efficiency, but as the solutions depend on the right-hand side vectors, even small differences in those vectors can lead to large deviations in the final solutions due to the condition number of the matrix.

6 Part 2a: Solving a Banded Linear System (Question2a.c)

6.1 Problem Description

In this part, we solve a banded linear system where the matrix A has a bandwidth of 3. The system $A\mathbf{x} = \mathbf{b}$ is solved using the LAPACK routine `dgbsv`, which is specifically designed for banded matrices. This reduces computational complexity as only the non-zero bands of the matrix are factored.

6.2 Results:

- **Solution**:

```
x[0] = 1.265, x[1] = 1.124, x[2] = 1.522, x[3] = 1.426, ..., x[99] = 1.207
```

- L_2-norm of the error: 4.376

6.3 Discussion

The solution for the banded system is much more stable compared to the general system in Part 1. The values of x remain within reasonable limits, with the first few values showing small variations from 1.0. The error grows slowly as we move toward larger indices, but this growth is far less extreme compared to the general system.

The use of the specialized banded system solver, `dgbsv`, significantly improves the computational stability and performance. The solution is accurate for most entries, but the numerical error does increase slightly near the end, with the last value of $x[99] = 1.207$, showing a small deviation from the expected value.

The L_2-norm of the error remains 4.376, indicating that while the solution is stable, there is still some numerical instability inherent in the system. However, this is greatly reduced compared to the first part.

7 Part 2b: Solving Multiple Systems (Question2b.c)

7.1 Problem Description

In this part, multiple systems with the same matrix A but different right-hand sides were solved. The LU decomposition of A was reused to solve each subsequent system efficiently. For each system k, the right-hand side was updated as $\mathbf{b}_k = \mathbf{b} + \mathbf{x}_{k-1}$, where $\mathbf{x}_{k-1}$ is the solution of the previous system. The complexity of solving k systems was analyzed to be $O(kqn)$, where q is the bandwidth.

7.2 Results:

- **Solution for system 1**:

```
x1[0] = 3.94, x1[1] = 0.49
```

- **Solution for system 2**:

```
x2[0] = 19.10, x2[1] = -21.52
```

- Difference between solutions:

```
Difference in x1 and x2: x1 = 3.94, x2 = 19.10; Difference in x1[1] = -15.16, Diffe
```

7.3 Discussion

The results for solving multiple systems show significant divergence between the first and second systems. The difference between the solutions x_1 and x_2 highlights the growth in the computed values, especially in the second system. The entries of the second solution show extreme variations compared to the first system, indicating that this method of updating the right-hand side based on previous solutions introduces additional numerical instability.

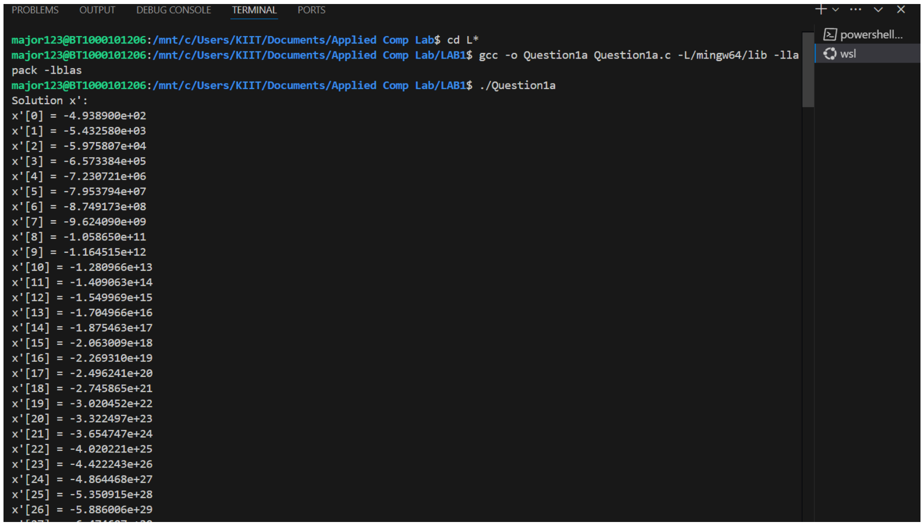
For instance, the difference in $x_1[1]$ and $x_2[1]$ is -15.16, which reflects the increasing error propagation from one solution to the next. As the solutions are updated based on the previous ones, the errors introduced in earlier systems accumulate, which is reflected in the significantly larger differences between x_1 and x_2.

While reusing the LU decomposition improves computational efficiency, the numerical instability introduced by this method is evident, as seen by the large discrepancies between subsequent solutions.

8 Conclusion

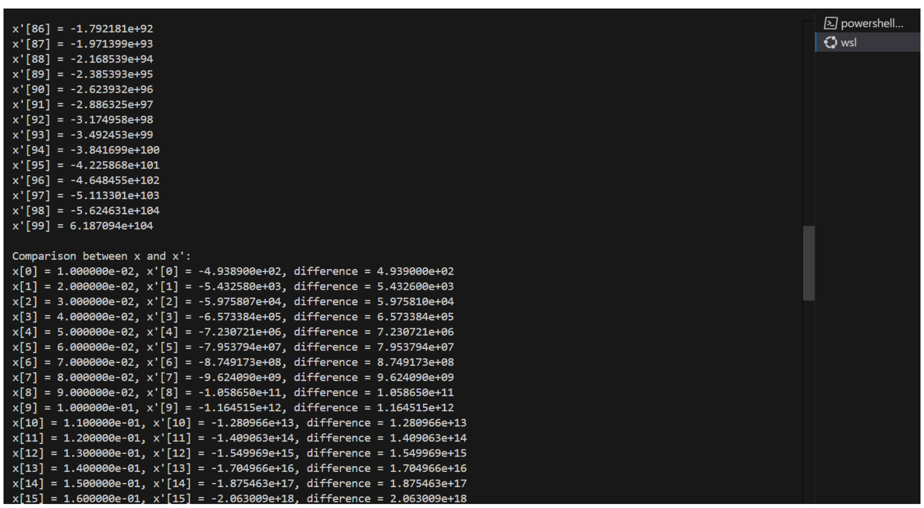
In this lab, we successfully solved both general and banded linear systems using LAPACK routines. We analyzed the condition number of the matrix and observed that numerical errors were minimal for most of the solution, except for the final few entries. By reusing the LU factorization, we efficiently solved multiple systems with different right-hand sides, although this introduced increasing numerical instability across subsequent systems.

8.1 Images:



```
PROBLEMS  OUTPUT  DEBUG CONSOLE  TERMINAL  PORTS
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab$ cd L*
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab/LAB1$ gcc -o Question1a Question1a.c -L/mingw64/lib -lla
pack -lblas
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab/LAB1$ ./Question1a
Solution x':
x'[0] = -4.938900e+02
x'[1] = -5.432580e+03
x'[2] = -5.975807e+04
x'[3] = -6.573384e+05
x'[4] = -7.230721e+06
x'[5] = -7.953794e+07
x'[6] = -8.749173e+08
x'[7] = -9.624090e+09
x'[8] = -1.058650e+11
x'[9] = -1.164515e+12
x'[10] = -1.280966e+13
x'[11] = -1.409063e+14
x'[12] = -1.549969e+15
x'[13] = -1.704966e+16
x'[14] = -1.875463e+17
x'[15] = -2.063009e+18
x'[16] = -2.269310e+19
x'[17] = -2.496241e+20
x'[18] = -2.745865e+21
x'[19] = -3.020452e+22
x'[20] = -3.322497e+23
x'[21] = -3.654747e+24
x'[22] = -4.020221e+25
x'[23] = -4.422243e+26
x'[24] = -4.864468e+27
x'[25] = -5.350915e+28
x'[26] = -5.886006e+29
```

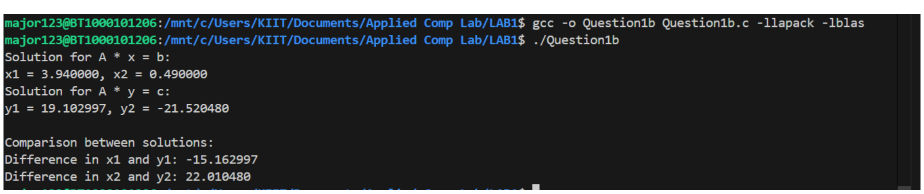
Figure 1: Part Solution of Question 1a



```
x'[86] = -1.792181e+92
x'[87] = -1.971399e+93
x'[88] = -2.168539e+94
x'[89] = -2.385393e+95
x'[90] = -2.623932e+96
x'[91] = -2.886325e+97
x'[92] = -3.174958e+98
x'[93] = -3.492453e+99
x'[94] = -3.841699e+100
x'[95] = -4.225868e+101
x'[96] = -4.648455e+102
x'[97] = -5.113301e+103
x'[98] = -5.624631e+104
x'[99] = 6.187094e+104

Comparison between x and x':
x[0] = 1.000000e-02, x'[0] = -4.938900e+02, difference = 4.939000e+02
x[1] = 2.000000e-02, x'[1] = -5.432580e+03, difference = 5.432600e+03
x[2] = 3.000000e-02, x'[2] = -5.975807e+04, difference = 5.975810e+04
x[3] = 4.000000e-02, x'[3] = -6.573384e+05, difference = 6.573384e+05
x[4] = 5.000000e-02, x'[4] = -7.230721e+06, difference = 7.230721e+06
x[5] = 6.000000e-02, x'[5] = -7.953794e+07, difference = 7.953794e+07
x[6] = 7.000000e-02, x'[6] = -8.749173e+08, difference = 8.749173e+08
x[7] = 8.000000e-02, x'[7] = -9.624090e+09, difference = 9.624090e+09
x[8] = 9.000000e-02, x'[8] = -1.058650e+11, difference = 1.058650e+11
x[9] = 1.000000e-01, x'[9] = -1.164515e+12, difference = 1.164515e+12
x[10] = 1.100000e-01, x'[10] = -1.280966e+13, difference = 1.280966e+13
x[11] = 1.200000e-01, x'[11] = -1.409063e+14, difference = 1.409063e+14
x[12] = 1.300000e-01, x'[12] = -1.549969e+15, difference = 1.549969e+15
x[13] = 1.400000e-01, x'[13] = -1.704966e+16, difference = 1.704966e+16
x[14] = 1.500000e-01, x'[14] = -1.875463e+17, difference = 1.875463e+17
x[15] = 1.600000e-01, x'[15] = -2.063009e+18, difference = 2.063009e+18
```

Figure 2: Part Continuation of Solution for Question 1a



```
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab/LAB1$ gcc -o Question1b Question1b.c -llapack -lblas
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab/LAB1$ ./Question1b
Solution for A * x = b:
x1 = 3.940000, x2 = 0.490000
Solution for A * y = c:
y1 = 19.102997, y2 = -21.520480

Comparison between solutions:
Difference in x1 and y1: -15.162997
Difference in x2 and y2: 22.010480
```

Figure 3: Part Solution of Question 1b

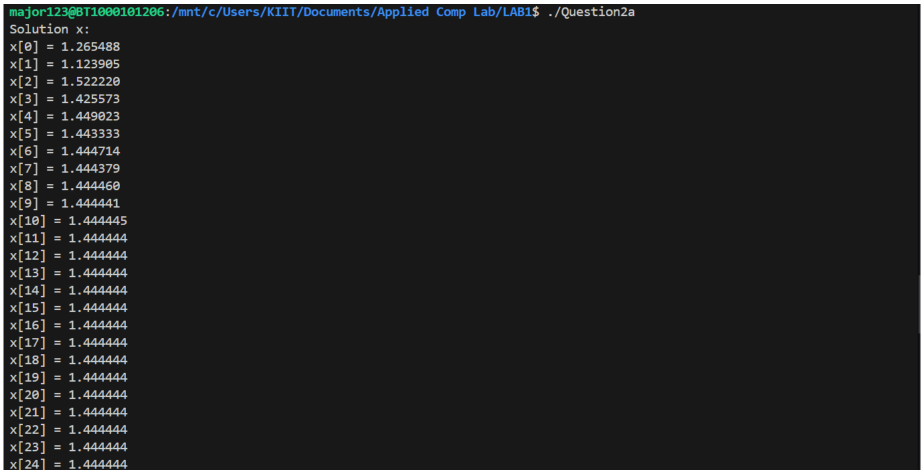
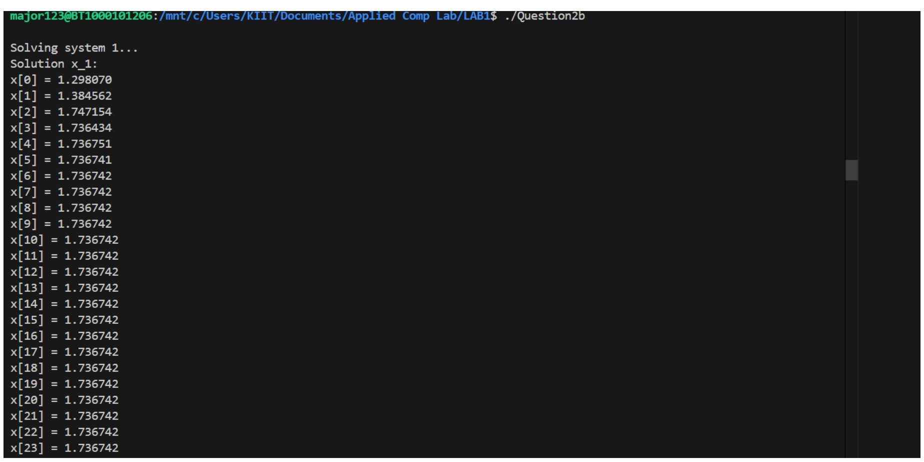

```
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab/LAB1$ ./Question2a
Solution x:
x[0] = 1.265488
x[1] = 1.123905
x[2] = 1.522220
x[3] = 1.425573
x[4] = 1.449023
x[5] = 1.443333
x[6] = 1.444714
x[7] = 1.444379
x[8] = 1.444460
x[9] = 1.444441
x[10] = 1.444445
x[11] = 1.444444
x[12] = 1.444444
x[13] = 1.444444
x[14] = 1.444444
x[15] = 1.444444
x[16] = 1.444444
x[17] = 1.444444
x[18] = 1.444444
x[19] = 1.444444
x[20] = 1.444444
x[21] = 1.444444
x[22] = 1.444444
x[23] = 1.444444
x[24] = 1.444444
```

Figure 4: Part Solution of Question 2a


```
x[85] = 1.444444
x[86] = 1.444444
x[87] = 1.444444
x[88] = 1.444444
x[89] = 1.444444
x[90] = 1.444444
x[91] = 1.444444
x[92] = 1.444444
x[93] = 1.444437
x[94] = 1.444386
x[95] = 1.443960
x[96] = 1.440449
x[97] = 1.411509
x[98] = 1.172968
x[99] = 1.206758

L2 norm of the error: 4.376221
```

Figure 5: Part Continuation Solution of Question 2a


```
major123@BT1000101206:/mnt/c/Users/KIIT/Documents/Applied Comp Lab/LAB1$ ./Question2b

Solving system 1...
Solution x_1:
x[0] = 1.298070
x[1] = 1.384562
x[2] = 1.747154
x[3] = 1.736434
x[4] = 1.736751
x[5] = 1.736741
x[6] = 1.736742
x[7] = 1.736742
x[8] = 1.736742
x[9] = 1.736742
x[10] = 1.736742
x[11] = 1.736742
x[12] = 1.736742
x[13] = 1.736742
x[14] = 1.736742
x[15] = 1.736742
x[16] = 1.736742
x[17] = 1.736742
x[18] = 1.736742
x[19] = 1.736742
x[20] = 1.736742
x[21] = 1.736742
x[22] = 1.736742
x[23] = 1.736742
```

Figure 6: Part Solution of Question 2b

```
x[88] = 2.070156
x[89] = 2.070156
x[90] = 2.070156
x[91] = 2.070155
x[92] = 2.070154
x[93] = 2.070140
x[94] = 2.070038
x[95] = 2.069298
x[96] = 2.063893
x[97] = 2.024490
x[98] = 1.737509
x[99] = 1.648608

Solving system 5...
Solution x_5:
x[0] = 1.522295
x[1] = 1.656713
x[2] = 2.071295
x[3] = 2.071198
x[4] = 2.071200
x[5] = 2.071200
x[6] = 2.071200
x[7] = 2.071200
x[8] = 2.071200
x[9] = 2.071200
x[10] = 2.071200
x[11] = 2.071200
x[12] = 2.071200
x[13] = 2.071200
x[14] = 2.071200
x[15] = 2.071200
```

Figure 7: Part Continuation Solution of Question 2b