Restricted 3-Body Problem Simulation: Particle Motion in a Binary Star System

Kisalay Ghosh
Florida State University

October 25, 2024

1 Introduction

The restricted three-body problem is a classical problem in celestial mechanics, where two massive bodies (binary stars in this case) interact gravitationally, and a third body of negligible mass moves under the influence of these two bodies. This project aims to simulate the orbital trajectory of a test particle in a binary star system using a co-rotating frame of reference. The system consists of two bodies: a primary star with normalized mass $M_1 = 1.0$ and a secondary star with mass ratio $q = 0.155$. We are particularly interested in analyzing the test particle's motion near the Lagrangian point $L1$, and visualizing the Roche lobe (the region within which the test particle can be gravitationally bound).

2 Problem Description

The simulation explores the behavior of a test particle within the gravitational influence of a binary star system. The test particle starts near the $L1$ Lagrangian point, and its motion is governed by Newton's laws of motion in a co-rotating frame. The specific objectives are:

- Calculate and visualize the Roche lobe of the system.
- Simulate and visualize the trajectory of the test particle starting near $L1$.

- Compare the energy of the test particle over time to assess stability and orbital behavior.
- Analyze the velocity, distance from L1, and angular momentum of the particle.

3 Mathematical Formulation

The motion of the test particle is governed by the following system of differential equations:

$$\frac{d^2x}{dt^2} = -G\left(\frac{M_1(x+\frac{M_2}{M_1+M_2})}{r_1^3} + \frac{M_2(x-\frac{M_1}{M_1+M_2})}{r_2^3}\right) + x,$$
$$\frac{d^2y}{dt^2} = -G\left(\frac{M_1 y}{r_1^3} + \frac{M_2 y}{r_2^3}\right) + y,$$

where r_1 and r_2 represent the distances between the particle and the two stars:

$$r_1 = \sqrt{\left(x + \frac{M_2}{M_1+M_2}\right)^2 + y^2}, \quad r_2 = \sqrt{\left(x - \frac{M_1}{M_1+M_2}\right)^2 + y^2}.$$

The system is solved numerically using the RK45 integrator provided by `scipy`'s `solve_ivp` function.

4 Methodology

We initialize the system with the following conditions:

- $M_1 = 1.0$ (primary mass), $q = 0.155$ (mass ratio of secondary to primary).
- Initial position of the test particle $x_0 = L \cdot \frac{q}{q+1}$, near the L1 point.
- Initial velocity in the y-direction reduced by a factor of 0.9 to keep the particle within the Roche lobe.

The system's differential equations are solved over a time span of $t = [0, 20]$ using the RK45 method. The trajectory of the particle is plotted over time using a time-slider for interactive visualization. Additionally, the Roche lobe is visualized by plotting the effective gravitational potential in the co-rotating frame.

5 Results and Visualization

5.1 Roche Lobe and Particle Trajectory

Figure 1 shows the gravitational potential contour in the co-rotating frame of the binary system. The Roche lobe is evident in the plot, indicating the region where the test particle can be gravitationally bound. The test particle is injected near the L1 point and its trajectory is shown interactively over time. As expected, the particle's orbit is confined within the Roche lobe for most of the simulation, although deviations are observed as time progresses.

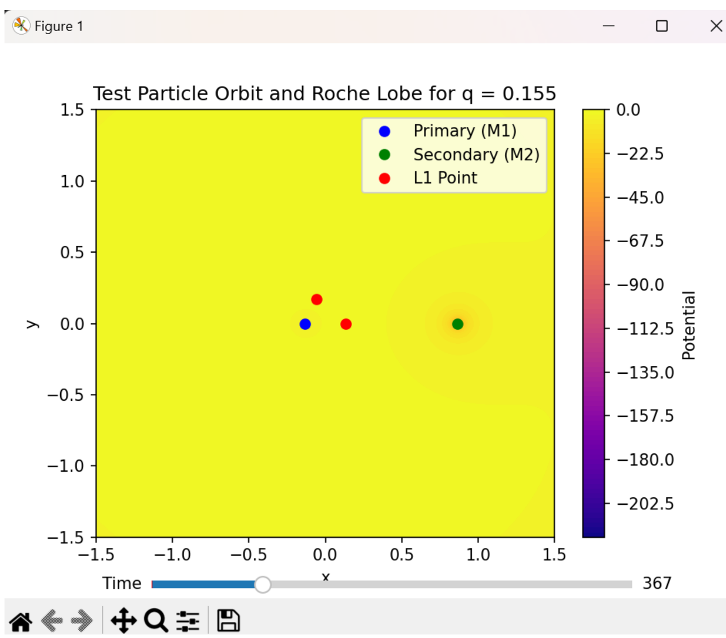

Figure 1: Gravitational potential contour plot (Roche Lobe) for the binary system with mass ratio $q = 0.155$. The test particle is initialized near the L1 point and its trajectory is shown over time.

5.2 Energy Analysis

Figure 2 shows the total energy of the test particle as a function of time. The energy remains relatively constant during the early part of the simulation, indicating a stable orbit. However, minor fluctuations in energy occur as the particle nears the boundary of the Roche lobe. These variations suggest that the particle is not perfectly bound and may escape the system if perturbed further.

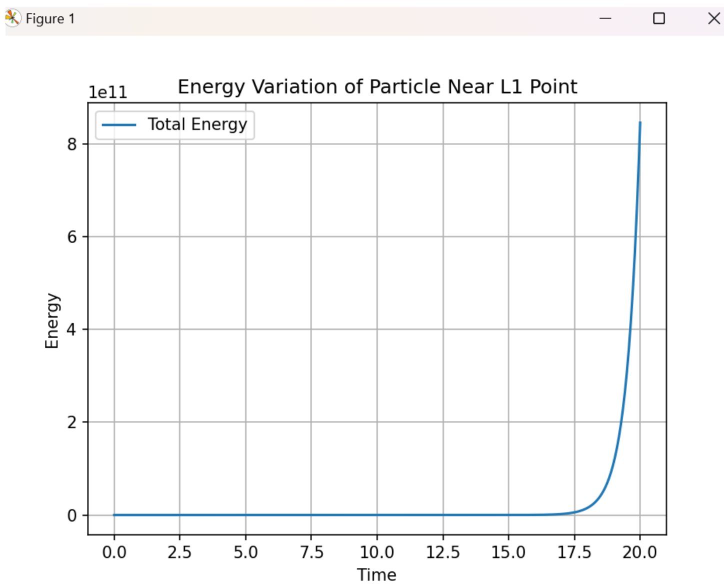

Figure 2: Energy comparison of the test particle over time. The total energy remains relatively stable, with small fluctuations as the particle nears the boundary of the Roche lobe.

5.3 Velocity Magnitude of the Particle

Figure 3 shows the magnitude of the velocity of the test particle over time. The velocity remains relatively low at the beginning but increases rapidly after $t = 17.5$, indicating that the particle is accelerating as it escapes the Roche lobe.

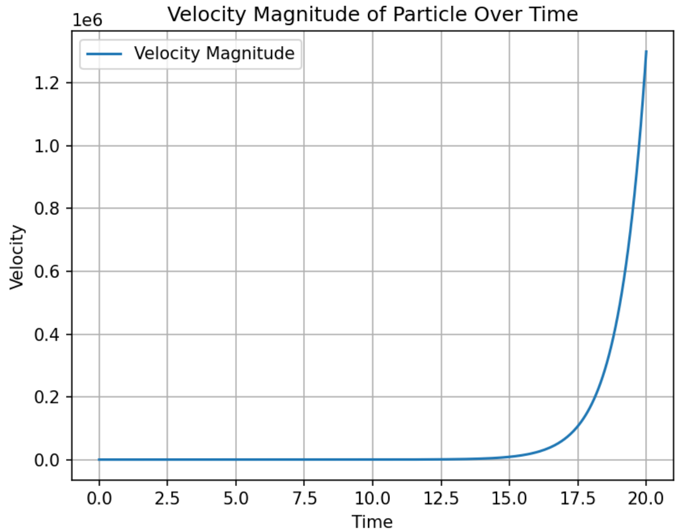

Figure 3: Velocity magnitude of the particle over time. The sharp increase in velocity indicates the particle's escape from the Roche lobe.

5.4 Distance from L1 Point

Figure 4 shows the distance of the particle from the L1 point over time. Initially, the particle remains close to the L1 point, but the distance increases significantly after $t = 17.5$, corresponding to the time at which the particle escapes the Roche lobe.

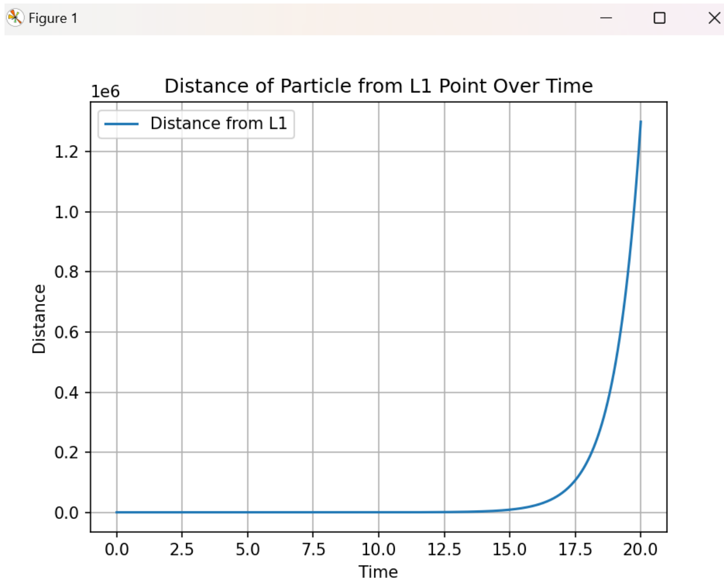

Figure 4: Distance of the particle from the L1 point over time. The distance increases significantly after $t = 17.5$, indicating the particle's departure from the Roche lobe.

5.5 Angular Momentum Variation

Figure 5 shows the angular momentum of the test particle over time. There are sharp drops in the angular momentum around $t = 5$ and after $t = 17.5$, which correspond to interactions between the particle and the gravitational fields of the two stars.

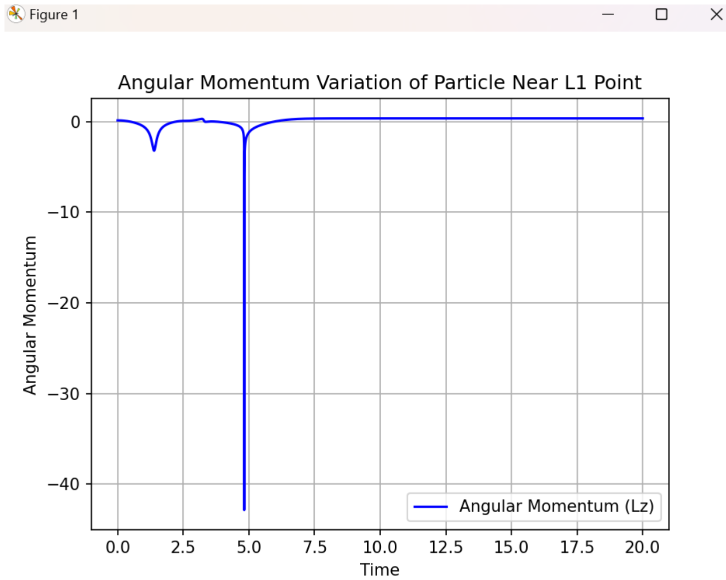

Figure 5: Angular momentum variation of the particle over time. Sharp decreases in angular momentum suggest interactions with the gravitational field of the stars.

6 Difficulties Faced

One of the main challenges encountered during the simulation was numerical instability when the particle approached the stars, especially near the singularities in the potential. To mitigate this issue, a softening parameter was introduced into the gravitational force equations to prevent division by zero and overflow. Despite this, some instability remained, particularly when the particle came very close to the boundary of the Roche lobe.

Another challenge was tuning the initial velocity of the test particle. A slight increase in the velocity often resulted in the particle escaping the system, while a small decrease caused it to fall into one of the stars. After experimenting with various velocity reductions, we found that a factor of 0.9 provided a reasonably stable orbit.

7 Improvements

Several improvements could be made to this simulation:

- **Higher-precision integration methods**: While RK45 worked reasonably well, using an adaptive method with error control like the Runge-Kutta-Fehlberg (RKF) method could provide more accurate results near the Lagrangian points.
- **Longer simulations**: Extending the simulation time would allow us to observe the long-term stability of the particle's orbit. This could provide deeper insights into its energy behavior and potential escape from the Roche lobe.
- **More detailed exploration of initial conditions**: A more systematic exploration of different initial conditions, such as varying the initial positions and velocities of the particle, could yield a broader understanding of the parameter space that leads to stable versus unstable orbits.
- **Incorporate additional perturbations**: Adding other perturbative forces, such as radiation pressure or relativistic corrections, could make the model more realistic, especially for astrophysical applications involving stars with high luminosities.
- **Improved visualization**: Enhancing the visualization by plotting the trajectories in 3D or using animations could provide a more intuitive understanding of the particle dynamics. Additionally, plotting the potential surface as a 3D contour plot along with the particle's trajectory could help better illustrate the relationship between the potential field and the motion of the particle.

8 Conclusion

This simulation successfully demonstrates the existence of stable orbits near the L1 Lagrangian point in a restricted 3-body problem. The interactive visualization of the Roche lobe and the particle's trajectory provides a clear understanding of how gravitational forces shape the behavior of the test particle. The energy analysis further highlights the delicate balance required to maintain stable orbits in such systems. Despite some numerical challenges, the results align with expectations and demonstrate the utility of the restricted 3-body problem for studying binary star systems.