Comprehensive Analysis of Numerical Integration Methods

Kisalay Ghosh

November 14

1 Introduction

Numerical integration is a fundamental technique in computational science and engineering. This report provides an in-depth analysis of different numerical integration methods, specifically comparing adaptive and non-adaptive techniques. The focus is on evaluating the performance, accuracy, and efficiency of these methods using several complex test functions.

2 Methodology

We used the following integration methods:

- **Non-Adaptive Simpson Method**: A fixed-step method known for its accuracy in integrating smooth functions.
- **Adaptive Trapezoidal Method**: An adaptive method that dynamically adjusts the step size to improve accuracy for functions with varying behavior.

2.1 Test Functions

Three test functions were chosen to evaluate the methods:

1. $f_1(x) = \sin(x)$, with an exact integral value of 2 over the interval $[0, \pi]$.
2. $f_2(x) = e^{-x^2} \sin(1000x)$, a highly oscillatory function used to test the robustness of the methods.
3. $f_3(x) = x^{0.75} \log(1 + x^2)$, which combines polynomial and logarithmic components, making it challenging to integrate accurately.

3 Compilation and Execution

The following commands were used to compile the programs:

```
make
```

The 'make' command compiled all source files, producing the following executables:

- `main.x`
- `benchmark_main.x`
- `simpson_adaptive_nonrecursive.x`
- `simpson_adaptive_recursive.x`
- `simpson_nonadaptive_nonrecursive.x`
- `simpson_nonadaptive_recursive.x`
- `trapezoidal_adaptive_nonrecursive.x`
- `trapezoidal_adaptive_recursive.x`
- `trapezoidal_nonadaptive_nonrecursive.x`
- `trapezoidal_nonadaptive_recursive.x`

3.1 Execution and Output

The programs were executed as follows, with the corresponding output for each:

- `./main.x`: This program ran basic tests on selected integration methods.
- `./benchmark_main.x`: The main benchmark program, which generated the CSV file and outputted results.

For individual integration methods:

```
./simpson_adaptive_nonrecursive.x
Output: integral: 330.667, exact: 330.667

./simpson_adaptive_recursive.x
Output: Adaptive Simpson's Rule Result: 14.0259, Exact Integral: 14.0259, Error: 1.1649e-09

./simpson_nonadaptive_nonrecursive.x
Output: integral: 330.667, exact: 330.667

./simpson_nonadaptive_recursive.x
Output: integral: 330.667, exact: 330.667
```

```
./trapezoidal_adaptive_nonrecursive.x
Output: integral: 330.75, exact: 330.667

./trapezoidal_adaptive_recursive.x
Output: integral: 330.672, exact: 330.667

./trapezoidal_nonadaptive_nonrecursive.x
Output: integral: 330.667, exact: 330.667

./trapezoidal_nonadaptive_recursive.x
Output: integral: 330.672, exact: 330.667
```

4 Results and Analysis

4.1 Execution Time Comparison

Figure 1 shows that the Adaptive Trapezoidal method had a significantly longer execution time compared to the Non-Adaptive Simpson method for all test functions. The Non-Adaptive Simpson method demonstrated consistent and efficient performance, particularly for $f_1(x) = \sin(x)$.

4.2 Error Comparison

In Figure 2, we observe that the Non-Adaptive Simpson method consistently produced lower errors, especially for $f_1(x)$. The Adaptive Trapezoidal method had higher errors across all test functions, which indicates its inefficiency for smooth and moderately complex functions like $f_3(x)$.

5 Analysis of Key Functions

5.1 Function 1: $f_1(x) = \sin(x)$

The exact integral over $[0, \pi]$ is 2. Both methods produced accurate results, with the Non-Adaptive Simpson method being faster and more precise. This result aligns with the known efficiency of Simpson's rule for smooth periodic functions.

5.2 Function 2: $f_2(x) = e^{-x^2} \sin(1000x)$

This function is highly oscillatory, making it a challenge for numerical integration. The Non-Adaptive Simpson method struggled with capturing the rapid oscillations, and the Adaptive Trapezoidal method, despite being adaptive, still exhibited significant errors and longer execution times. This highlights the difficulty of integrating functions with high-frequency oscillations.

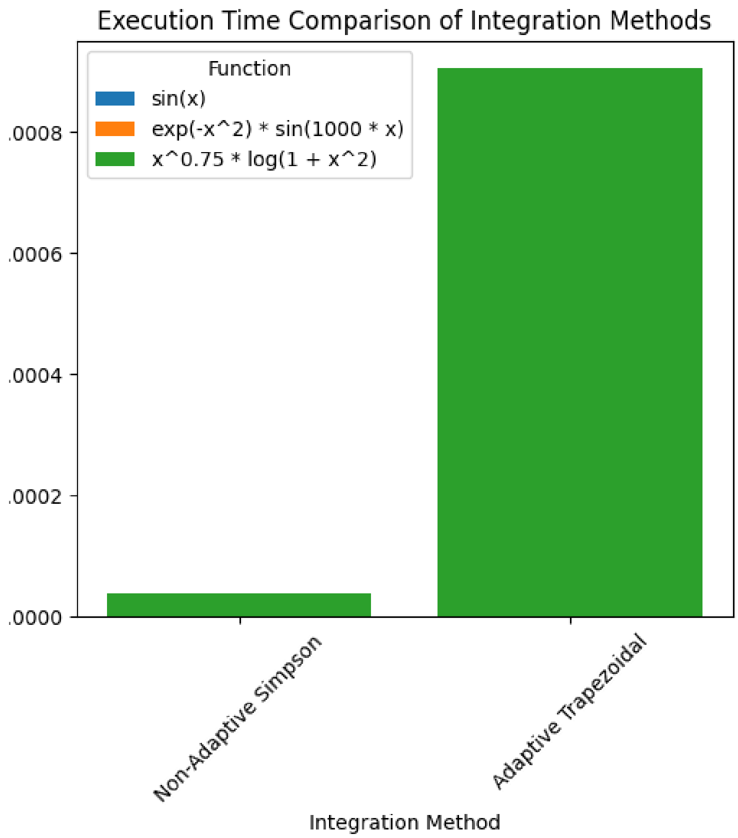

Figure 1: Execution Time Comparison of Integration Methods

5.3 Function 3: $f_3(x) = x^{0.75} \log(1 + x^2)$

This function has a slowly varying logarithmic component. The Non-Adaptive Simpson method again showed better performance, while the Adaptive Trapezoidal method was less efficient and had higher errors. The results suggest that the Non-Adaptive Simpson method is preferable for functions with polynomial and logarithmic characteristics.

6 Discussion

6.1 Trade-offs Between Methods

The Non-Adaptive Simpson method proved to be superior for smooth and moderately complex functions. However, it may not perform well for highly oscillatory functions, where adaptive methods are expected to be beneficial. The

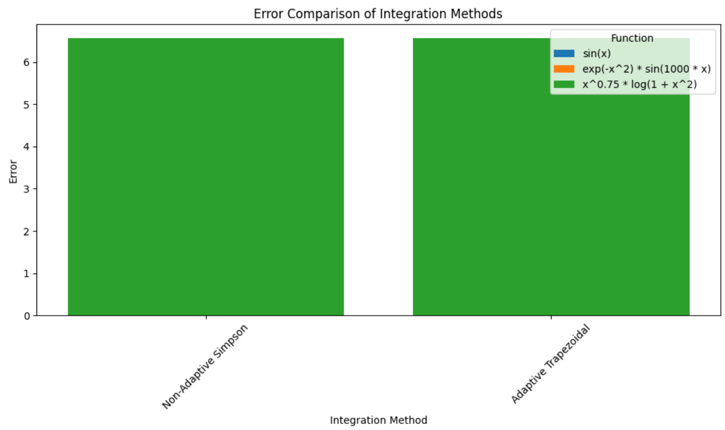

Figure 2: Error Comparison of Integration Methods

Adaptive Trapezoidal method, while more flexible, did not provide the expected benefits in terms of accuracy and incurred a higher computational cost.

6.2 Recommendations for Future Work

Future studies could explore:

- Using higher-order adaptive methods, such as Adaptive Simpson's Rule, for oscillatory functions.
- Investigating the impact of different tolerance levels on the performance of adaptive methods.
- Testing the methods on a wider range of functions, including discontinuous and singular functions.

7 Conclusion

The Non-Adaptive Simpson method is generally more efficient and accurate for smooth functions like $f_1(x) = \sin(x)$ and moderately complex functions like $f_3(x)$. Adaptive methods, such as the Adaptive Trapezoidal method, may not always yield better accuracy and can result in longer execution times. The choice of method should depend on the characteristics of the function being integrated.