Kishal Tandel **April 14, 2025**

Quantum Tunneling

Abstract

We present a comprehensive analytical treatment of quantum tunneling through a one-dimensional potential barrier, focusing on the behavior of the transmission coefficient $T(\eta)$ as a function of the dimensionless energy parameter $\eta = \frac{E}{V_0}$. By deriving expressions for all three regimes – $E < V_0$, $E = V_0$, and $E > V_0$ – and proving the continuity of $T(\eta)$ at $\eta = 1$ using L'Hôpital's Rule, we highlight an often-overlooked but pedagogically valuable aspect of the standard quantum mechanics curriculum. We supplement our results with numerical plot using Python to illustrate key features of tunneling behavior across energy regimes.

Contents

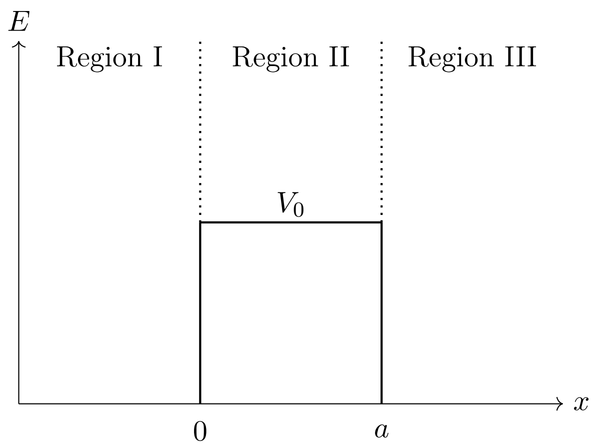
1. Introduction

Quantum tunneling is typically introduced in undergraduate quantum mechanics through standard barrier problems. However, the continuity of the transmission coefficient function at the threshold energy $E = V_0$ is often overlooked or treated qualitatively. This paper presents a detailed, fully analytical proof of this continuity, offering a pedagogical enhancement to the standard treatment found in most textbooks.

Consider the potential $V(x)$ defined by:

$$V(x) = \begin{cases} V_0 \quad (> 0) & 0 \leq x \leq a \\ 0 & \text{otherwise} \end{cases}$$

The given potential could be sketched as follows:



Potential Barrier

Here we have divided the space into three regions with Region II containing the potential barrier of height V_0 and width a. We will consider a quantum particle to be moving towards positive x direction in Region I and solve one dimensional time independent Schrödinger Equation for the following three mutually exclusive and exhaustive cases:

1. $0 < E < V_0$
2. $E = V_0$
3. $E > V_0$

where E is the particle's energy.

This classic quantum mechanical problem is typically introduced in standard texts such as Griffiths [1], but often without explicit treatment of continuity at the energy threshold.

While the standard tunneling problem is often presented via the transfer matrix method or continuity conditions [2], this paper offers an explicit derivation of the transmission and reflection coefficients for all energy regimes using a direct wavefunction matching approach.

2. Solving Schrödinger Equation

The time independent Schrödinger Equation is given as:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x)} \tag{1}$$

A detailed discussion of the Schrödinger equation and its solutions for piecewise constant potentials can be found in Shankar [3].

2.1 Case I: $\mathbf{0 < E < V_0}$

In Region I and Region III, (1) reduces to the following equation:

$$\frac{d^2}{dx^2}\psi(x) = -k^2\psi(x), \qquad \text{where } k = \frac{\sqrt{2mE}}{\hbar} \tag{2}$$

The solution to (2) is oscillatory and is given as:

$$\text{Region I:} \quad \psi_1(x) = Ae^{ikx} + Be^{-ikx} \tag{3}$$

$$\text{Region III:} \quad \psi_3(x) = Ce^{ikx} + De^{-ikx} \tag{4}$$

In (4), De^{-ikx} represents a particle reflected in Region III and hence travelling along negative x direction. Since there is no potential barrier in Region III, a particle having transmitted through the potential barrier in Region II and travelling along positive x direction in Region III cannot be reflected back. Hence $D = 0$. Therefore (4) reduces to the following equation:

$$\text{Region III:} \quad \psi_3(x) = Ce^{ikx} \tag{5}$$

In Region II, (1) reduces to the following equation:

$$\frac{d^2}{dx^2}\psi(x) = \kappa^2\psi(x), \qquad \text{where } \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \tag{6}$$

The solution to (6) is exponential and is given as:

$$\text{Region II:} \quad \psi_2(x) = Fe^{\kappa x} + Ge^{-\kappa x} \tag{7}$$

Using continuity of $\psi(x)$ at $x = 0$ and at $x = a$, we have:

$$\psi_1(0) = \psi_2(0) \qquad \text{and} \qquad \psi_2(a) = \psi_3(a) \tag{8}$$

$$\Rightarrow A + B = F + G \qquad \text{and} \qquad Fe^{\kappa a} + Ge^{-\kappa a} = Ce^{ika} \tag{9}$$

Using continuity of $\frac{d}{dx}\psi(x)$ at $x = 0$ and at $x = a$, we have:

$$\left.\frac{d}{dx}\psi_1(x)\right|_{x=0} = \left.\frac{d}{dx}\psi_2(x)\right|_{x=0} \qquad \text{and} \qquad \left.\frac{d}{dx}\psi_2(x)\right|_{x=a} = \left.\frac{d}{dx}\psi_3(x)\right|_{x=a} \tag{10}$$

$$\Rightarrow ikA - ikB = \kappa F - \kappa G \qquad \text{and} \qquad \kappa Fe^{\kappa a} - \kappa Ge^{-\kappa a} = ikCe^{ika} \tag{11}$$

Since we have five constants A, B, C, F, G and only four equations given by (9) and (11), the system is underdetermined.

Now note that Ae^{ikx} represents a particle (wave) travelling along positive x direction in Region I, Ce^{ikx} represents a particle (wave) travelling along positive x direction in Region III after having transmitted through the potential barrier and Be^{-ikx} represents a particle (wave) travelling along negative x direction after getting reflected from the wall of the potential barrier at $x = 0$. Accordingly we can write:

$$\psi_1(x) = Ae^{ikx} + Be^{-ikx} = \psi_{1,\text{right}} + \psi_{1,\text{left}} \tag{12}$$

$$\psi_3(x) = Ce^{ikx} = \psi_{3,\text{right}} \tag{13}$$

2.1.1 Finding Transmission and Reflection coefficients

Probability of finding a particle travelling right in Region I:

$$P_{1,\text{right}} = \psi^*_{1,\text{right}}\psi_{1,\text{right}} = (A^*e^{-ikx})(Ae^{ikx}) = A^*A \tag{14}$$

Probability of finding a particle travelling left in Region I:

$$P_{1,\text{left}} = \psi^*_{1,\text{left}}\psi_{1,\text{left}} = (B^*e^{ikx})(Be^{-ikx}) = B^*B \tag{15}$$

Probability of finding a particle travelling right in Region III:

$$P_{3,\text{right}} = \psi^*_{3,\text{right}}\psi_{3,\text{right}} = (C^*e^{-ikx})(Ce^{ikx}) = C^*C \tag{16}$$

The transmission coefficient T and reflection coefficient R are defined as follows:

$$T = \frac{P_{3,\text{right}}}{P_{1,\text{right}}} \qquad and \qquad R = \frac{P_{1,\text{left}}}{P_{1,\text{right}}} \tag{17}$$

Using (14), (15) and (16), (17) could be written as:

$$T = \frac{C^*C}{A^*A} = t^*t \quad \text{(say)} \qquad and \qquad R = \frac{B^*B}{A^*A} = r^*r \quad \text{(say)} \tag{18}$$

Now for notational simplicity, let $\dfrac{B}{A} = r$, $\dfrac{F}{A} = f$, $\dfrac{G}{A} = g$ and $\dfrac{C}{A} = t$. We have already found the four sets of equations given by (9) and (11) using boundary conditions and continuity of wavefunction and its first derivative. On dividing each of these four equations by A and using the notations as mentioned above, we get:

$$1 + r = f + g \tag{19}$$

$$ik - ikr = \kappa f - \kappa g \tag{20}$$

$$fe^{\kappa a} + ge^{-\kappa a} = te^{ika} \tag{21}$$

$$\kappa fe^{\kappa a} - \kappa ge^{-\kappa a} = ikte^{ika} \tag{22}$$

Multiplying (21) by κ and adding it to (22), we get:

$$2\kappa f e^{\kappa a} = ikte^{ika} + \kappa te^{ika} \tag{23}$$

Dividing (23) by $2\kappa e^{\kappa a}$ and then simplifying, we get:

$$f = \frac{te^{(ik-\kappa)a}(ik+\kappa)}{2\kappa} \tag{24}$$

Multiplying (21) by κ and then subtracting it from (22), we get:

$$-2\kappa g e^{-\kappa a} = ikte^{ika} - \kappa te^{ika} \tag{25}$$

Dividing (25) by $-2\kappa e^{-\kappa a}$ and then simplifying, we get:

$$g = \frac{te^{(ik+\kappa)a}(ik-\kappa)}{-2\kappa} \tag{26}$$

Multiplying (19) by ik and then adding it to (20) we get:

$$2ik = (ik+\kappa)f + (ik-\kappa)g \tag{27}$$

Using (24), (26) in (27) and then simplifying, we get:

$$2ik = [e^{-\kappa a}(ik+\kappa)^2 - e^{\kappa a}(ik-\kappa)^2]\frac{te^{ika}}{2\kappa} \tag{28}$$

Simplifying (28) further, we get:

$$2ik = [(-k^2+\kappa^2)(e^{-\kappa a} - e^{\kappa a}) + (2ik\kappa)(e^{-\kappa a} + e^{\kappa a})]\frac{te^{ika}}{2\kappa} \tag{29}$$

The hyperbolic $\sinh(x)$ and $\cosh(x)$ are given as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad and \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \tag{30}$$

Using (30), we can write (29) as:

$$2ik = [-(-k^2+\kappa^2)2\sinh(\kappa a) + (2ik\kappa)2\cosh(\kappa a)]\frac{te^{ika}}{2\kappa} \tag{31}$$

Solving (31) for t, we get:

$$t = \frac{4ik\kappa e^{-ika}}{(k^2-\kappa^2)2\sinh(\kappa a) + (2ik\kappa)2\cosh(\kappa a)} \tag{32}$$

The complex conjugate of t can be written as:

$$t^* = \frac{-4ik\kappa e^{ika}}{(k^2-\kappa^2)2\sinh(\kappa a) - (2ik\kappa)2\cosh(\kappa a)} \tag{33}$$

Using (18), we have:

$$T = t^*t = \frac{-4ik\kappa e^{ika}}{(k^2-\kappa^2)2\sinh(\kappa a) - (2ik\kappa)2\cosh(\kappa a)} \times \frac{4ik\kappa e^{-ika}}{(k^2-\kappa^2)2\sinh(\kappa a) + (2ik\kappa)2\cosh(\kappa a)} \tag{34}$$

$$\Rightarrow T = \frac{16k^2\kappa^2}{4(k^2-\kappa^2)^2\sinh^2(\kappa a) + 16k^2\kappa^2\cosh^2(\kappa a)} \tag{35}$$

$$\Rightarrow T = \frac{1}{\dfrac{(k^2-\kappa^2)^2}{4k^2\kappa^2}\sinh^2(\kappa a) + \cosh^2(\kappa a)} \tag{36}$$

Using the identity $\cosh^2(x) - \sinh^2(x) = 1$ in (36), it can be written as:

$$T = \frac{1}{1 + \left[\dfrac{(k^2-\kappa^2)^2}{4k^2\kappa^2} + 1\right]\sinh^2(\kappa a)} \tag{37}$$

Now we will first simplify the denominator of (37) as follows:

$$\frac{(k^2-\kappa^2)^2}{4k^2\kappa^2} + 1 = \frac{k^2}{4\kappa^2} + \frac{\kappa^2}{4k^2} + \frac{1}{2} \tag{38}$$

Using the expression for k and κ from (2) and (6) respectively, we can write (38) as:

$$\frac{(k^2-\kappa^2)^2}{4k^2\kappa^2} + 1 = \frac{V_0^2}{4E(V_0-E)} \tag{39}$$

Substituting (39) in (37) and letting $\eta = \dfrac{E}{V_0}$, we can write:

$$\boxed{T = \left[1 + \frac{\sinh^2(\kappa a)}{4\eta(1-\eta)}\right]^{-1}}, \qquad \eta \in (0,1) \tag{40}$$

The particle could either be transmitted through the barrier or be reflected from the barrier. Hence we must have:

$$T + R = 1 \qquad \Rightarrow \qquad R = T - 1 \tag{41}$$

Using expression for κ from (6) and the notation of η, we can rewrite κ as:

$$\boxed{\kappa = \frac{\sqrt{2mV_0(1-\eta)}}{\hbar}} \tag{42}$$

2.2 Case II: $\mathbf{E = V_0}$

In this case the solution in Region I and Region III will remain same as given by (3) and (5) respectively, i.e,

$$\text{Region I:} \quad \psi_1(x) = Ae^{ikx} + Be^{-ikx}, \qquad k = \frac{\sqrt{2mE}}{\hbar} \tag{43}$$

$$\text{Region III:} \quad \psi_3(x) = Ce^{ikx}, \qquad k = \frac{\sqrt{2mE}}{\hbar} \tag{44}$$

In Region II, (1) reduces to the following equation:

$$\frac{d^2}{dx^2}\psi_2(x) = 0 \tag{45}$$

The solution to (45) is linear and is given by:

$$\psi_2(x) = P + Qx \tag{46}$$

Using continuity of $\psi(x)$ at $x = 0$ and at $x = a$, we have:

$$\psi_1(0) = \psi_2(0) \qquad \text{and} \qquad \psi_2(a) = \psi_3(a) \tag{47}$$

$$\Rightarrow A + B = P \qquad \text{and} \qquad P + Qa = Ce^{ika} \tag{48}$$

Using continuity of $\dfrac{d}{dx}\psi(x)$ at $x = 0$ and at $x = a$, we have:

$$\frac{d}{dx}\psi_1(x)\bigg|_{x=0} = \frac{d}{dx}\psi_2(x)\bigg|_{x=0} \qquad \text{and} \qquad \frac{d}{dx}\psi_2(x)\bigg|_{x=a} = \frac{d}{dx}\psi_3(x)\bigg|_{x=a} \tag{49}$$

$$\Rightarrow ikA - ikB = Q \qquad \text{and} \qquad Q = ikCe^{ika} \tag{50}$$

Since we have five constants A, B, C, P, Q and only four equations given by (48) and (50), the system is underdetermined.

2.2.1 Finding Transmission and Reflection coefficients

Note that (12),(13), (14), (15), (16), (17) and (18) still holds in this case.
Now for notational simplicity, let $\dfrac{B}{A} = r$, $\dfrac{P}{A} = p$, $\dfrac{Q}{A} = q$ and $\dfrac{C}{A} = t$. We have already found the four sets of equations given by (48) and (50) using boundary conditions and continuity of wavefunction and its first derivative. On dividing each of these four equations by A and using the notations as mentioned above, we get:

$$1 + r = p \tag{51}$$

$$ik - ikr = q \tag{52}$$

$$p + qa = te^{ika} \tag{53}$$

$$q = ikte^{ika} \tag{54}$$

Multiplying (54) by a and subtracting it from (53), we get:

$$p = (1 - ika)te^{ika} \tag{55}$$

Multiplying (51) by ik and adding it to (52), we get:

$$2ik = pik + q \tag{56}$$

Using (54) and (55) in (56), we get:

$$2ik = (2 - ika)ikte^{ika} \tag{57}$$

Solving (57) for t, we get:

$$t = \frac{2e^{-ika}}{2 - ika} \tag{58}$$

The complex conjugate of t can be written as:

$$t^* = \frac{2e^{ika}}{2 + ika} \tag{59}$$

Using (18), we have:

$$T = t^* t = \frac{2e^{ika}}{2 + ika} \times \frac{2e^{-ika}}{2 - ika} = \frac{4}{4 + k^2 a^2} \tag{60}$$

$$\Rightarrow \boxed{T = \left[1 + \left(\frac{ka}{2}\right)^2\right]^{-1}}, \qquad k = \frac{\sqrt{2mE}}{\hbar} = \frac{\sqrt{2mV_0}}{\hbar} \tag{61}$$

Note that $\eta = \dfrac{E}{V_0} = 1$ in this case.

The particle could either be transmitted through the barrier or be reflected from the barrier. Hence we must have:

$$T + R = 1 \qquad \Rightarrow \qquad R = T - 1 \tag{62}$$

2.3 Case III: $\mathbf{E > V_0}$

In this case the solution in Region I and Region III will remain same as given by (3) and (5) respectively, i.e,

$$\text{Region I:} \quad \psi_1(x) = Ae^{ikx} + Be^{-ikx}, \qquad k = \frac{\sqrt{2mE}}{\hbar} \tag{63}$$

$$\text{Region III:} \quad \psi_3(x) = Ce^{ikx}, \qquad k = \frac{\sqrt{2mE}}{\hbar} \tag{64}$$

In Region II, (1) reduces to the following equation:

$$\frac{d^2}{dx^2}\psi_2(x) = -\alpha^2 \psi_2(x), \qquad \text{where } \alpha = \frac{\sqrt{2m(E - V_0)}}{\hbar} \tag{65}$$

The solution to (65) is oscillatory and is given by:

$$\psi_2(x) = Me^{i\alpha x} + Ne^{-i\alpha x} \tag{66}$$

Using continuity of $\psi(x)$ at $x = 0$ and at $x = a$, we have:

$$\psi_1(0) = \psi_2(0) \qquad \text{and} \qquad \psi_2(a) = \psi_3(a) \tag{67}$$

$$\Rightarrow A + B = M + N \qquad \text{and} \qquad Me^{i\alpha a} + Ne^{-i\alpha a} = Ce^{ika} \tag{68}$$

Using continuity of $\frac{d}{dx}\psi(x)$ at $x = 0$ and at $x = a$, we have:

$$\frac{d}{dx}\psi_1(x)\bigg|_{x=0} = \frac{d}{dx}\psi_2(x)\bigg|_{x=0} \qquad \text{and} \qquad \frac{d}{dx}\psi_2(x)\bigg|_{x=a} = \frac{d}{dx}\psi_3(x)\bigg|_{x=a} \tag{69}$$

$$\Rightarrow kA - kB = \alpha M - \alpha N \qquad \text{and} \qquad \alpha Me^{i\alpha a} - \alpha Ge^{-i\alpha a} = kCe^{ika} \tag{70}$$

Since we have five constants A, B, C, M, N and only four equations given by (68) and (70), the system is underdetermined.

2.3.1 Finding Transmission and Reflection coefficients

Note that (12),(13), (14), (15), (16), (17) and (18) still holds in this case.

Now for notational simplicity, let $\frac{B}{A} = r$, $\frac{M}{A} = m$, $\frac{N}{A} = n$ and $\frac{C}{A} = t$. We have already found the four sets of equations given by (68) and (70) using boundary conditions and continuity of wavefunction and its first derivative. On dividing each of these four equations by A and using the notations as mentioned above, we get:

$$1 + r = m + n \tag{71}$$

$$k - kr = \alpha m - \alpha n \tag{72}$$

$$me^{i\alpha a} + ne^{-i\alpha a} = te^{ika} \tag{73}$$

$$\alpha me^{i\alpha a} - \alpha ne^{-i\alpha a} = kte^{ika} \tag{74}$$

Multiplying (73) by $i\alpha$ and adding it to (74), we get:

$$2\alpha me^{i\alpha a} = kte^{ika} + \alpha te^{ika} \tag{75}$$

Dividing (75) by $2\alpha e^{i\alpha a}$ and then simplifying, we get:

$$m = \frac{te^{(ik - i\alpha)a}(k + \alpha)}{2\alpha} \tag{76}$$

Multiplying (73) by α and then subtracting it from (74), we get:

$$-2\alpha ne^{-i\alpha a} = kte^{ika} - \alpha te^{ika} \tag{77}$$

Dividing (77) by $-2\alpha e^{-i\alpha a}$ and then simplifying, we get:

$$n = \frac{te^{(ik + i\alpha)a}(k - \alpha)}{-2\alpha} \tag{78}$$

Multiplying (71) by k and then adding it to (72) we get:

$$2k = (k+\alpha)m + (k-\alpha)n \tag{79}$$

Using (76), (78) in (79) and then simplifying, we get:

$$2k = [e^{-i\alpha a}(k+\alpha)^2 - e^{i\alpha a}(k-\alpha)^2]\frac{te^{ika}}{2\alpha} \tag{80}$$

Simplifying (80) further, we get:

$$2k = [(k^2+\alpha^2)(e^{-i\alpha a} - e^{i\alpha a}) + (2k\alpha)(e^{-i\alpha a} + e^{i\alpha a})]\frac{te^{ika}}{2\alpha} \tag{81}$$

The hyperbolic $\sinh(x)$ and $\cosh(x)$ are given as:

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad and \qquad \cosh(x) = \frac{e^x + e^{-x}}{2} \tag{82}$$

Using (82), we can write (81) as:

$$2k = [-(-k^2+\alpha^2)2\sinh(i\alpha a) + (2k\alpha)2\cosh(i\alpha a)]\frac{te^{ika}}{2\alpha} \tag{83}$$

But $\sinh(ix) = i\sin(x)$ and $\cosh(ix) = \cos(x)$. Therefore we have from (83):

$$2k = [-i(-k^2+\alpha^2)2\sin(\alpha a) + (2k\alpha)2\cos(\alpha a)]\frac{te^{ika}}{2\alpha} \tag{84}$$

Solving (84) for t, we get:

$$t = \frac{4k\alpha e^{-ika}}{-i(k^2+\alpha^2)2\sin(\alpha a) + (2k\alpha)2\cos(\alpha a)} \tag{85}$$

The complex conjugate of t can be written as:

$$t^* = \frac{4k\alpha e^{ika}}{i(k^2+\alpha^2)2\sin(\kappa a) + (2k\alpha)2\cos(\kappa a)} \tag{86}$$

Using (18), we have:

$$T = t^*t = \frac{4k\alpha e^{ika}}{i(k^2+\alpha^2)2\sin(\kappa a) + (2k\alpha)2\cos(\kappa a)} \times \frac{4k\alpha e^{-ika}}{-i(k^2+\alpha^2)2\sin(\alpha a) + (2k\alpha)2\cos(\alpha a)} \tag{87}$$

$$\Rightarrow T = \frac{16k^2\alpha^2}{4(k^2+\alpha^2)^2\sin^2(\alpha a) + 16k^2\alpha^2\cos^2(\alpha a)} \tag{88}$$

$$\Rightarrow T = \frac{1}{\frac{(k^2+\alpha^2)^2}{4k^2\alpha^2}\sin^2(\alpha a) + \cos^2(\alpha a)} \tag{89}$$

Using the identity $\cos^2(x) + \sin^2(x) = 1$ in (89), it can be written as:

$$T = \frac{1}{1 + \left[\dfrac{(k^2 + \alpha^2)^2}{4k^2\alpha^2} - 1\right]\sin^2(\alpha a)} \tag{90}$$

Now we will first simplify the denominator of (90) as follows:

$$\frac{(k^2 + \alpha^2)^2}{4k^2\alpha^2} - 1 = \frac{k^2}{4\alpha^2} + \frac{\alpha^2}{4k^2} - \frac{1}{2} \tag{91}$$

Using the expression for k and α from (64) and (65) respectively, we can write (91) as:

$$\frac{(k^2 + \alpha^2)^2}{4k^2\alpha^2} - 1 = \frac{V_0^2}{4E(E - V_0)} \tag{92}$$

Substituting (92) in (90) and letting $\eta = \dfrac{E}{V_0}$, we can write:

$$\boxed{T = \left[1 + \frac{\sin^2(\alpha a)}{4\eta(\eta - 1)}\right]^{-1}}, \qquad \eta \in (1, \infty) \tag{93}$$

The particle could either be transmitted through the barrier or be reflected from the barrier. Hence we must have:

$$T + R = 1 \qquad \Rightarrow \qquad R = T - 1 \tag{94}$$

Using expression for α from (65) and the notation of η, we can rewrite α as:

$$\boxed{\alpha = \frac{\sqrt{2mV_0(\eta - 1)}}{\hbar}} \tag{95}$$

3. Continuity of Transmission coefficient function $\mathbf{T}(\eta)$ and Reflection coefficient function $\mathbf{R}(\eta)$

Continuity of the transmission coefficient at the classical threshold $\eta = 1$ is often discussed heuristically in introductory contexts [4]; here, we provide a rigorous analytic derivation using L'Hôpital's Rule. Since we have already found the Transmission and Reflection coefficients, we will now inspect their continuity.

Let us define $T : (0, \infty) \to [0, 1]$ by

$$T(\eta) = \begin{cases} \left[1 + \dfrac{\sinh^2(\kappa a)}{4\eta(1-\eta)}\right]^{-1}, \qquad \kappa = \dfrac{\sqrt{2mV_0(1-\eta)}}{\hbar} & \forall \quad \eta \in (0,1) \\ \left[1 + \left(\dfrac{ka}{2}\right)^2\right]^{-1}, \qquad k = \dfrac{\sqrt{2mV_0}}{\hbar} & \text{if} \quad \eta = 1 \\ \left[1 + \dfrac{\sin^2(\alpha a)}{4\eta(\eta-1)}\right]^{-1}, \qquad \alpha = \dfrac{\sqrt{2mV_0(\eta-1)}}{\hbar} & \forall \quad \eta \in (1,\infty) \end{cases}$$

We note that 1, $\sinh^2(\kappa a)$ and $4\eta(1-\eta)$ are positive smooth functions when $\eta \in (0,1)$. Also note that 1 and $4\eta(\eta-1)$ are positive smooth functions and $\sin^2(\alpha a)$ is non-negative smooth function when $\eta \in (1,\infty)$. Hence $T(\eta)$ is continuous when $\eta \in (0,1) \cup (1,\infty)$. Now we will inspect the continuity of $T(\eta)$ at $\eta = 1$.

Note that $\dfrac{d}{d\eta}\kappa = \dfrac{-mV_0}{\hbar\sqrt{2mV_0(1-\eta)}}$ and $2\sinh(\kappa a)\cosh(\kappa a) = \sinh(2\kappa a)$. The limit of $T(\eta)$ as $\eta \to 1^-$ is given as:

$$\lim_{\eta\to1^-} T(\eta) = \frac{1}{1 + \lim\limits_{\eta\to1^-} \dfrac{\sinh^2(\kappa a)}{4\eta(1-\eta)}} \tag{96}$$

Since the limit in (96) gives an indeterminate form $(\frac{0}{0})$, we have using L'Hôpital's Rule:

$$\lim_{\eta\to1^-} \frac{\sinh^2(\kappa a)}{4\eta(1-\eta)} = \lim_{\eta\to1^-} \frac{2a\sinh(\kappa a)\cosh(\kappa a)\dfrac{d}{d\eta}\kappa}{4-8\eta} = \frac{mV_0 a}{4\hbar}\lim_{\eta\to1^-} \frac{\sinh(2\kappa a)}{\sqrt{2mV_0(1-\eta)}} \tag{97}$$

The limit in (97) again gives an indeterminate form $(\frac{0}{0})$. Using L'Hôpital's Rule again gives:

$$\lim_{\eta\to1^-} \frac{\sinh^2(\kappa a)}{4\eta(1-\eta)} = -\frac{mV_0 a}{4\hbar}\cdot\frac{4a}{2mV_0}\lim_{\eta\to1^-} \cosh(2\kappa a)\sqrt{2mV_0(1-\eta)}\frac{d}{d\eta}\kappa \tag{98}$$

$$\Rightarrow \lim_{\eta\to1^-} \frac{\sinh^2(\kappa a)}{4\eta(1-\eta)} = \frac{mV_0 a^2}{2\hbar^2}\lim_{\eta\to1^-} \cosh(2\kappa a) \tag{99}$$

Since $\kappa \to 0$ as $\eta \to 1^- \Rightarrow \cosh(2\kappa a) \to 1$ as $\eta \to 1^-$. Therefore we have from (99):

$$\lim_{\eta\to1^-} \frac{\sinh^2(\kappa a)}{4\eta(1-\eta)} = \frac{mV_0 a^2}{2\hbar^2} \tag{100}$$

Using (100) in (96), we have:

$$\boxed{\lim_{\eta\to 1^-} T(\eta) = \left[1 + \frac{mV_0a^2}{2\hbar^2}\right]^{-1}} \tag{101}$$

Now note that $\dfrac{d}{d\eta}\alpha = \dfrac{mV_0}{\hbar\sqrt{2mV_0(\eta-1)}}$ and $2\sin(\alpha a)\cosh(\alpha a) = \sin(2\alpha a)$. The limit of $T(\eta)$ as $\eta \to 1^+$ is given as:

$$\lim_{\eta\to 1^+} T(\eta) = \frac{1}{1 + \lim\limits_{\eta\to 1^+} \dfrac{\sin^2(\alpha a)}{4\eta(\eta-1)}} \tag{102}$$

Since the limit in (102) gives an indeterminate form $(\frac{0}{0})$, we have using L'Hôpital's Rule:

$$\lim_{\eta\to 1^+} \frac{\sin^2(\alpha a)}{4\eta(\eta-1)} = \lim_{\eta\to 1^+} \frac{2a\sin(\alpha a)\cos(\alpha a)\dfrac{d}{d\eta}\alpha}{8\eta-4} = \frac{mV_0a}{4\hbar}\lim_{\eta\to 1^+}\frac{\sin(2\alpha a)}{\sqrt{2mV_0(\eta-1)}} \tag{103}$$

The limit in (103) again gives an indeterminate form $(\frac{0}{0})$. Using L'Hôpital's Rule again gives:

$$\lim_{\eta\to 1^+} \frac{\sin^2(\alpha a)}{4\eta(\eta-1)} = \frac{mV_0a}{4\hbar}\cdot\frac{4a}{2mV_0}\lim_{\eta\to 1^+}\cos(2\alpha a)\sqrt{2mV_0(\eta-1)}\frac{d}{d\eta}\alpha \tag{104}$$

$$\Rightarrow \lim_{\eta\to 1^+} \frac{\sin^2(\alpha a)}{4\eta(\eta-1)} = \frac{mV_0a^2}{2\hbar^2}\lim_{\eta\to 1^+}\cos(2\alpha a) \tag{105}$$

Since $\alpha \to 0$ as $\eta \to 1^+ \Rightarrow \cos(2\alpha a) \to 1$ as $\eta \to 1^+$. Therefore we have from (105):

$$\lim_{\eta\to 1^+} \frac{\sin^2(\alpha a)}{4\eta(\eta-1)} = \frac{mV_0a^2}{2\hbar^2} \tag{106}$$

Using (106) in (102), we have:

$$\boxed{\lim_{\eta\to 1^+} T(\eta) = \left[1 + \frac{mV_0a^2}{2\hbar^2}\right]^{-1}} \tag{107}$$

At $\eta = 1$, we have:

$$\boxed{T(1) = \left[1 + \left(\frac{ka}{2}\right)^2\right]^{-1} = \left[1 + \frac{mV_0a^2}{2\hbar^2}\right]^{-1}} \tag{108}$$

From (101), (107) and (108), we have:

$$\boxed{\lim_{\eta\to 1^-} T(\eta) = \lim_{\eta\to 1^+} T(\eta) = T(1) = \left[1 + \frac{mV_0a^2}{2\hbar^2}\right]^{-1}} \tag{109}$$

Hence $T(\eta)$ is also continuous at $\eta = 1$. Therefore $T(\eta)$ is continuous on $(0,\infty)$.

Now define $R : (0,\infty) \to [0,1]$ by

$$R(\eta) = 1 - T(\eta) \qquad \forall \quad \eta \in (0,\infty) \tag{110}$$

Since both 1 and $T(\eta)$ are continuous on $(0,\infty) \Rightarrow R(\eta)$ is continuous on $(0,\infty)$.

4. Plotting Transmission and Reflection coefficients vs η using Python

Let us consider an electron moving along positive x direction in Region I. It will encounter a potential barrier of height 1 eV and of width 1 nm. Let us plot the values of Transmission and Reflection coefficients vs η. We will use the expressions for T, R, κ, k and α as determined in this article.

4.1 Python Code

```
import numpy as np
import matplotlib.pyplot as plt

# Constants
m = 9.11 * 10**( -31)                  # mass of electron (kg)
hbar = 1.055 * 10**( -34)              # reduced Planck's constant (Js)
V0 = 1 * 1.6 * 10**( -19)              # barrier height in Joules (1 eV)
a = 10**( -9)                          # barrier width (m)

# Tunneling region : eta <= 1
eta_tunnel = np.linspace(0,1,500)[1:-1]    # exclude 0 and 1
kappa = np.sqrt(2 * m * V0 * (1 - eta_tunnel ) )/hbar
sinh_term = np.sinh(kappa * a)**2
T_kappa = 1/(1+(sinh_term/(4 * eta_tunnel * (1 - eta_tunnel))))
R_kappa = 1 - T_kappa

# E=V0 : eta=1
eta=np.array([1])
k=np.sqrt(2 * m * V0)/hbar
T=np.array([1/(1+((k * a)/2)**2)])
R=1-T

# Over-barrier region : eta > 1
eta_above = np.linspace (1,7,3000)[1:]     # exclude 1
alpha = np.sqrt(2 * m * V0 * ( eta_above-1))/hbar
sin_term = np.sin(alpha * a)**2
T_alpha = 1/(1 +(sin_term/(4 * eta_above * ( eta_above - 1))))
R_alpha = 1 - T_alpha

# Plotting
plt.plot(eta_tunnel , T_kappa , color ='red', label =r'Transmission
   Coefficient $T$')
plt.plot(eta_tunnel , R_kappa , color ='blue', label =r'Reflection
   Coefficient $R$')
plt.plot(eta,T,'r,')
plt.plot(eta,R,'b,')
plt.plot(eta_above , T_alpha , color ='red')
plt.plot(eta_above , R_alpha , color ='blue')
plt.title('Quantum Tunneling')
plt.xlabel(r'$\eta = \dfrac{E}{ V_0}$')
plt.ylabel ('Coefficient Value')
plt.legend()
plt.grid()
plt.ylim(-0.05,1.05)
plt.xlim(0,7)
plt.show()
```

4.2 Plot

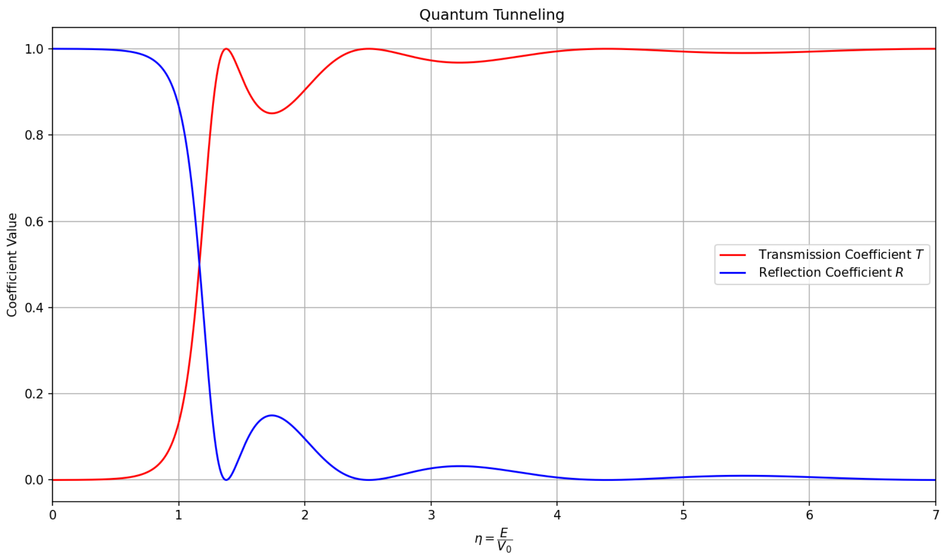


Figure 1: Quantum tunneling: Transmission and Reflection Coefficients vs η

4.3 Inference from the Plot

The smooth transition of transmission and reflection coefficients across energy regimes, also explored through wave packet dynamics in [5], is reflected in our plot of $T(\eta)$ and $R(\eta)$.
The graph depicting the transmission coefficient $T(\eta)$ and reflection coefficient $R(\eta)$ as functions of the dimensionless parameter $\eta = \frac{E}{V_0}$ reveals the nuanced behavior of quantum particles interacting with a potential barrier.

For $\eta < 1$, we observe that $T(\eta)$ remains strictly non-zero as $\eta \to 1^-$, indicating that particles with energy less than the barrier height still possess a finite probability of tunneling through the barrier. This behavior arises from the continuity of the wavefunction and its derivative across the potential barrier. As $\eta \to 0$, $T(\eta) \to 0$, and $R(\eta) \to 1$, which agrees with intuitive expectations of nearly total reflection at very low energies.

At $\eta = 1$, a critical threshold, the function $T(\eta)$ is shown to be continuous, as confirmed by analytical limits using L'Hôpital's Rule. This reflects the smooth transition between classically forbidden and allowed energy regimes in quantum mechanics.

For $\eta > 1$, the particle energy exceeds the potential of the barrier. In this regime, $T(\eta)$ exhibits oscillatory behavior due to constructive and destructive interference between forward and backward traveling waves. As $\eta \to \infty$, $T(\eta) \to 1$ and $R(\eta) \to 0$, aligning with the classical limit of full transmission at high energies.

5. Comparison with Classical Mechanics

For $E < V_0$ ($\eta < 1$), classical mechanics predicts total reflection, as the particle cannot overcome the potential barrier, implying $T = 0$ and $R = 1$. However, in quantum mechanics, the wave-like nature of particles allows for non-zero penetration into the classically forbidden region, governed by the exponential solution to the Schrödinger equation in the barrier region (Region II). The continuity of the wavefunction and its derivative enforces boundary conditions that yield a non-zero transmission probability, a distinctly quantum phenomenon known as tunneling.

At $E = V_0$ ($\eta = 1$), classical mechanics predicts total reflection, since the particle has zero kinetic energy at the barrier and cannot proceed. In quantum mechanics, however, the continuity of the wavefunction and its derivative still allows for a non-zero transmission probability. Although the wavefunction is linear in the barrier region (Region II) at this energy, the boundary conditions ensure that partial transmission occurs even at this critical point.

For $E > V_0$ ($\eta > 1$), classical mechanics predicts total transmission. Quantum mechanics, in contrast, introduces partial reflection due to abrupt changes in the potential in a region. The wavefunction in the barrier region (Region II) undergoes interference due to its complex exponential form, resulting in oscillatory behavior of both T and R. These oscillations encode the resonant and anti-resonant conditions determined by the phase relation between waves, a concept entirely absent in classical particle dynamics.

Thus, the quantum description not only accommodates classically forbidden phenomena (tunneling), but also provides richer structure in classically allowed regimes, revealing the fundamentally wave-like nature of microscopic systems.

References

[1] D. J. Griffiths, *Introduction to Quantum Mechanics*, 3rd ed. (Pearson Education, 2005).

[2] K. Ghosh, "Quantum tunneling: A pedagogical treatment using the transfer matrix method," Am. J. Phys. **77**, 661–666 (2009).

[3] R. Shankar, *Principles of Quantum Mechanics*, 2nd ed. (Springer, 2008).

[4] D. Falk, "Tunneling and the uncertainty principle: A heuristic view," Am. J. Phys. **57**, 1111–1113 (1989).

[5] M. A. Doncheski and R. W. Robinett, "Wave packet construction in one-dimensional quantum scattering problems via the stationary phase approximation," Am. J. Phys. **72**, 982–988 (2004).