PH2102 # Assignment 9 **Instructor:ADG**

Mechanics II

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Q.1

Using *Work-Energy Theorem*, we have:

$$m\gamma(u)c^2 - m\gamma(0)c^2 = \int_{\vec{x}=\vec{0}}^{\vec{x}=\vec{x}} \vec{F}.\vec{dx} \tag{1}$$

where $\gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ and $\gamma(0) = 1$.

Since force $\vec{F}$ is uniform, we have:

$$\int_{\vec{x}=\vec{0}}^{\vec{x}=\vec{x}} \vec{F}.\vec{dx} = Fx \tag{2}$$

Substituting these in (1), we get:

$$mc^2\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right) = Fx \quad \Rightarrow \frac{mc^2}{F}\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} - 1\right) = x \tag{3}$$

Let $\xi = \frac{mc^2}{F}$. Then (3) becomes:

$$\frac{\xi}{\sqrt{1-\frac{u^2}{c^2}}} = x + \xi \tag{4}$$

Squaring both sides of (4), we get:

$$\xi^2 = (x+\xi)^2\left(1 - \frac{u^2}{c^2}\right) \quad \Rightarrow \xi^2 c^2 = (x+\xi)^2(c^2 - u^2) \tag{5}$$

$$\Rightarrow (x+\xi)^2 u^2 = (x+\xi)^2 c^2 - \xi^2 c^2 \quad \Rightarrow u^2 = \frac{x^2 + \xi^2 + 2x\xi - \xi^2}{(x+\xi)^2}c^2 \tag{6}$$

$$\boxed{\Rightarrow u = \frac{\sqrt{x(x+2\xi)}}{x+\xi}c} \tag{7}$$

Since $u = \frac{dx}{dt}$, we have from (7):

$$\frac{dx}{dt} = \frac{\sqrt{x(x+2\xi)}}{x+\xi}c \quad \Rightarrow \frac{x+\xi}{\sqrt{x(x+2\xi)}}\,dx = c\,dt \tag{8}$$

Integrating (8), we get:

$$\int_{x=0}^{x=x} \frac{x+\xi}{\sqrt{x(x+2\xi)}}\,dx = \int_{t=0}^{t=t} c\,dt \tag{9}$$

Let

$$\sqrt{x^2+2x\xi} = a \quad \Rightarrow x^2+2x\xi = a^2 \tag{10}$$

$$\Rightarrow 2(x+\xi)\,dx = 2a\,da \quad \Rightarrow (x+\xi)\,dx = a\,da \tag{11}$$

As x $\to$ 0, a $\to$ 0. As x $\to$ x, a $\to$ $\sqrt{x^2+2x\xi}$. Substituting these in (9), we get:

$$\int_{a=0}^{a=\sqrt{x^2+2x\xi}} da = c\int_{t=0}^{t=t} dt \quad \Rightarrow \sqrt{x^2+2x\xi} = ct \tag{12}$$

Squaring both sides of (12), we get:

$$x^2+2x\xi-c^2t^2 = 0 \tag{13}$$

$$\boxed{\Rightarrow x = -\xi \pm \sqrt{\xi^2+c^2t^2}} \tag{14}$$

All relativistic formulas approach classical formula in the limit of non-relativistic velocities. We would check upon this fact to find out which of the two equations given by (14) is the correct one.

Using *Work Energy Theorem* for non-relativistic velocities, we get:

$$\frac{1}{2}mu^2 = \int_{\vec{x}=\vec{0}}^{\vec{x}=\vec{x}} \vec{F}.\vec{dx} = Fx \tag{15}$$

$$\Rightarrow u^2 = \frac{2Fx}{m} \tag{16}$$

Since $u = \frac{dx}{dt}$, we have from (16):

$$\left(\frac{dx}{dt}\right)^2 = \frac{2Fx}{m} \quad \Rightarrow \frac{dx}{\sqrt{x}} = \sqrt{\frac{2F}{m}}t \tag{17}$$

Integrating (17), we get:

$$\int_{x=0}^{x=x} \frac{dx}{\sqrt{x}} = \int_{t=0}^{t=t} \sqrt{\frac{2F}{m}}t \quad \Rightarrow 2\sqrt{x} = \sqrt{\frac{2F}{m}}t \tag{18}$$

$$\boxed{\Rightarrow x = \frac{F}{2m}t^2} \tag{19}$$

The classical expression for displacement of a particle as a function of time under the action of a constant force is given by (19). The relativistic expression should match the classical expression under non-relativistic limit. We can rewrite (14) as:

$$x = \xi\left(\pm\sqrt{1+\frac{c^2t^2}{\xi^2}} - 1\right) \tag{20}$$

Since $\left|\frac{ct}{\xi}\right| < 1$, we can taylor expand (20) as:

$$x = \xi\left(\pm\left(1 + \frac{c^2t^2}{2\xi^2} + \cdots\right) - 1\right) \tag{21}$$

Under non-relativistic limit, terms of order higher than $\frac{c^2t^2}{\xi^2}$ are negligible. Hence (21) becomes:

$$x = \frac{c^2t^2}{2\xi} \quad \text{or} \quad x = -2\xi - \frac{c^2t^2}{2\xi} \tag{22}$$

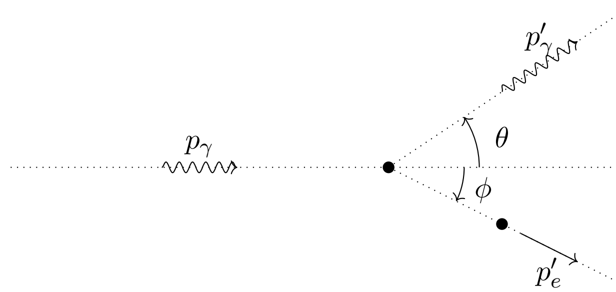
Substituting $\xi = \frac{mc^2}{F}$ in (22), we get:

$$x = \frac{F}{2m}t^2 \quad \text{or} \quad x = -\left(\frac{2mc^2}{F} + \frac{F}{2m}t^2\right) \tag{23}$$

Clearly only the first expression in (23) matches the classical formula. Hence we only take positive sign in (20). The relativistic expression for displacement of a particle as function of time under the action of a constant force is therefore given by:

$$\boxed{x = \xi\left(\sqrt{1 + \frac{c^2t^2}{\xi^2}} - 1\right)} \tag{24}$$

Q.2



From *Enegry Conservation*, we have:

$$m_ec^2 + p_\gamma c = p'_\gamma c + E'_e \tag{25}$$

From *Momentum Conservation*, we have:

$$p_\gamma = p'_\gamma cos\theta + p'_e cos\phi \tag{26}$$

$$0 = p'_\gamma sin\theta - p'_e sin\phi \tag{27}$$

Now (26) and (27) can be rearranged as:

$$p'_e cos\phi = p_\gamma - p'_\gamma cos\theta \tag{28}$$

$$p'_e sin\phi = p'_\gamma sin\theta \tag{29}$$

Squaring (28) and (29) and then adding, we get:

$${p'_e}^2(cos^2\phi + sin^2\phi) = {p'_\gamma}^2(cos^2\theta + sin^2\theta) + p_\gamma^2 - 2p_\gamma p'_\gamma cos\theta \tag{30}$$

$$\Rightarrow {p'_e}^2 = {p'_\gamma}^2 + p_\gamma^2 - 2p_\gamma p'_\gamma cos\theta \tag{31}$$

Rearranging (25), we get:

$$E'_e = m_e c^2 + (p_\gamma - p'_\gamma)c \tag{32}$$

Squaring (32), we get:

$$ {E'_e}^2 = m_e^2 c^4 + (p_\gamma - p'_\gamma)^2 c^2 + 2m_e c^3 (p_\gamma - p'_\gamma) \tag{33}$$

From *Relativistic Energy-Momentum* relation, we have:

$$ {E'_e}^2 = {p'_e}^2 c^2 + m_e^2 c^4 \tag{34}$$

Using (34) in (33), we get:

$$ {p'_e}^2 = (p_\gamma - p'_\gamma)^2 + 2m_e c(p_\gamma - p'_\gamma) \tag{35}$$

Using (31) in (35), we get:

$$ {p'_\gamma}^2 + p_\gamma^2 - 2p_\gamma p'_\gamma cos\theta = p_\gamma^2 + {p'_\gamma}^2 - 2p_\gamma p'_\gamma + 2m_e c(p_\gamma - p'_\gamma) \tag{36}$$

$$\Rightarrow p_\gamma p'_\gamma(1-cos\theta) = m_e c(p_\gamma - p'_\gamma) \quad \Rightarrow \frac{1}{p'_\gamma} - \frac{1}{p_\gamma} = \frac{1}{m_e c}(1-cos\theta) \quad \Rightarrow \frac{h}{p'_\gamma} - \frac{h}{p_\gamma} = \frac{h}{m_e c}(1-cos\theta) \tag{37}$$

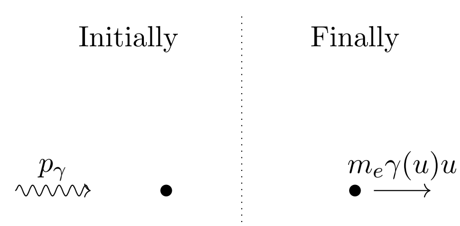
Using $\lambda = \dfrac{h}{p_\gamma}$ in (37), we have:

$$\boxed{\lambda' - \lambda = \frac{h}{m_e c}(1 - cos\theta)} \tag{38}$$

Q.3 Proof by Contradiction

Let us suppose that a free electron can absorb a photon incident on it i.e.,

$$\gamma + e \rightarrow e \tag{39}$$



We have from *Energy Conservation*:

$$E_\gamma + m_e c^2 = m_e \gamma(u) c^2 \tag{40}$$

$$\Rightarrow E_\gamma = m_e(\gamma(u) - 1)c^2 \tag{41}$$

We have from *Momentum Conservation*:

$$p_\gamma = m_e \gamma(u) u \tag{42}$$

Since $E_\gamma = p_\gamma c$, we have from (41) and (42):

$$m_e(\gamma(u) - 1)c^2 = m_e \gamma(u) u c \quad \Rightarrow u = \left(1 - \frac{1}{\gamma(u)}\right) c \quad \Rightarrow 1 - \frac{u}{c} = \sqrt{1 - \frac{u^2}{c^2}} \tag{43}$$

On Squaring (43), we get:

$$1+\frac{u^2}{c^2}-2\frac{u}{c}=1-\frac{u^2}{c^2} \quad \Rightarrow \frac{u^2}{c^2}=\frac{u}{c} \quad \Rightarrow u=c \tag{44}$$

which is a contradiction as material particles cannot move with speed greater than or equal to c. Hence phenomenon described by (39) does not happen in nature.

Q.4

Note: $1\,\mathbf{u} \approx 931.5\,\mathbf{MeV}/c^2$

1. Alpha Decay: $^{220}\mathbf{Fr} \to {}^{216}\mathbf{At} + \alpha$

The Kinetic Energy of the α particle is given by:

$$\boxed{T_\alpha = \frac{(m_{\text{Fr}}-m_\alpha)^2 - m_{\text{At}}^2}{2m_{\text{Fr}}}c^2} \tag{45}$$

where:

- $m_{\text{Fr}} = 220.012326\,\text{u}$,
- $m_{\text{At}} = 216.002422\,\text{u}$,
- $m_\alpha = 4.001506\,\text{u}$.

Substituting these values in (45), we get:

$$\boxed{T_\alpha \approx 7.68\,\text{MeV}} \tag{46}$$

2. Beta Decay: $^{220}\mathbf{Fr} \to {}^{220}\mathbf{Ra} + e^- + \nu_e$

The maximum Kinetic Energy T_e^{max} of the β particle (when neutrino is not emitted) is given by:

$$\boxed{T_e^{max} = \frac{(m_{\text{Fr}}-m_e)^2 - m_{\text{Ra}}^2}{2m_{\text{Fr}}}c^2} \tag{47}$$

where:

- $m_{\text{Fr}} = 220.012326\,\text{u}$,
- $m_{\text{Ra}} = 220.011027\,\text{u}$,
- $m_e = 0.000548\,\text{u}$.

Substituting these values (47), we get:

$$\boxed{T_e^{max} \approx 0.7\,\text{MeV}} \tag{48}$$

Q. 5

The 4-momenta of the two particles w.r.t Alice are given as:

$$P_1^{\mu\,A} = m_1\gamma(u_1)(c, \vec{u}_1), \quad P_2^{\mu\,A} = m_2\gamma(u_2)(c, \vec{u}_2) \tag{49}$$

The 4-momenta of the two particles w.r.t Bob are given as:

$$P_1^{\mu\,B} = \left(\frac{E_1}{c}, m_1\gamma(u_{1,B})\vec{u}_{1,B}\right), \quad P_2^{\mu\,B} = (m_2c, \vec{0}), \tag{50}$$

where $\vec{u}_{1,B}$ is velocity of particle-1 and E_1 is the total energy of particle-1 w.r.t Bob. Since the scalar product of the 4-momenta of the two particles is an invariant, we have:

$$P_1^{\mu\,B}P_{2\,\mu}^{B} = P_1^{\mu\,A}P_{2\,\mu}^{A} \tag{51}$$

$$\Rightarrow m_2E_1 = m_1m_2\gamma(u_1)\gamma(u_2)(c^2 - \vec{u}_1\cdot\vec{u}_2) \tag{52}$$

$$\boxed{\Rightarrow E_1 = m_1\gamma(u_1)\gamma(u_2)(c^2 - \vec{u}_1\cdot\vec{u}_2)} \tag{53}$$

Q.6

Part (a)

To prove the given inequality, we consider an ICF moving with particle-1. The 4-momenta in this ICF are given by:

$$P_1^{\mu} = (m_1c, \vec{0}), \quad P_2^{\mu} = (m_2\gamma(u)c, m_2\gamma(u)\vec{u}), \tag{54}$$

where $\vec{u}$ is velocity of particle-2 w.r.t the ICF moving with particle-1. The scalar invariant is given by:

$$P_1^{\mu}P_{2\,\mu} = m_1m_2\gamma(u)c^2 \tag{55}$$

Since $\gamma(u) \geq 1\,\forall\,u$, we have from (55):

$$\boxed{P_1^{\mu}P_{2\,\mu} \geq m_1m_2c^2 \quad \Rightarrow P_1^{\mu}P_{2\,\mu} > m_1m_2} \tag{56}$$

Part (b)

Given that there are n particles in a system with rest masses $m_1, \ldots, m_n$ and 4-momenta $P_1, \ldots, P_n$. The scalar invariant could be written as:

$$P^{\mu}P_{\mu} = \left(\sum_{i=1}^{n}\frac{E_i}{c}\right)^2 - \left(\sum_{i=1}^{n}\vec{p}_i\right)\cdot\left(\sum_{i=1}^{n}\vec{p}_i\right) \tag{57}$$

where E_i and $\vec{p}_i$ are the total energy and 3-momentum vector of the i^{th} particle.

To prove the given inequality, we can choose the ***Center Of Mass*** **(C.O.M.)** of the system as our frame of reference. In this frame, we have:

$$\sum_{i=1}^{n}\vec{p}_i = M\vec{v}_{C.O.M} = \vec{0}, \quad \text{where } M = \sum_{i=1}^{n}m_i \tag{58}$$

Using (58) in (57), we get:

$$P^\mu P_\mu = \left(\sum_{i=1}^{n} \frac{E_i}{c}\right)^2 \tag{59}$$

Since $E_i \geq m_i c^2$ for all velocities and for all particles, we have from (59):

$$P^\mu P_\mu \geq \left(\sum_{i=1}^{n} m_i c\right)^2 = \left(\sum_{i=1}^{n} m_i\right)^2 c^2 = M^2c^2 \tag{60}$$

$$\boxed{\Rightarrow P^\mu P_\mu \geq M^2c^2, \quad \text{where } M = \sum_{i=1}^{n} m_i} \tag{61}$$

The equality in (61) would be satisfied when all the particles are at rest w.r.t the C.O.M of the system.

Part (c)

Let T be the minimum kinetic energy of the projectile. We have:

$$E_{initial} = T + m_1c^2 + m_2c^2, \tag{62}$$

$$p_{initial} = p_1 \tag{63}$$

Using relativistic energy-momentum relation, we have:

$$p_1 = \sqrt{\frac{(T + m_1c^2)^2 - m_1^2c^4}{c^2}} = \sqrt{\frac{T^2 + 2Tm_1c^2}{c^2}} \tag{64}$$

From *Momentum Conservation* and from (63), we have:

$$p_{final} = p_{initial} = p_1 \tag{65}$$

Using relativistic energy-momentum relation, we have:

$$E_{final} = \sqrt{p_{final}^2c^2 + M^2c^4} \tag{66}$$

From *Energy Conservation*, we have:

$$E_{initial} = E_{final} \tag{67}$$

Using (62), (64), (65) and (66) in (67), we get:

$$T + m_1c^2 + m_2c^2 = \sqrt{p_{initial}^2c^2 + M^2c^4} \tag{68}$$

$$\Rightarrow T + m_1c^2 + m_2c^2 = \sqrt{T^2 + 2Tm_1c^2 + M^2c^4} \tag{69}$$

On squaring (69), we get:

$$T^2 + m_1^2c^4 + m_2^2c^4 + 2Tm_1c^2 + 2Tm_2c^2 + 2m_1m_2c^4 = T^2 + 2Tm_1c^2 + M^2c^4 \tag{70}$$

On simplifying (70), we get:

$$(m_1^2 + m_2^2 + 2m_1m_2)c^4 + 2Tm_2c^2 = M^2c^4 \tag{71}$$

$$\boxed{\Rightarrow T = \frac{M^2 - (m_1 + m_2)^2}{2m_2}c^2} \tag{72}$$

The minimum kinetic energy of the projectile is given by (72).