PH2102 **Assignment 7** **Instructor:ADG**

Mechanics II

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Q. 1

Let S be the frame in which the particle's velocity and acceleration is $\vec{u}$ and $\vec{a}$ respectively. Let S' be the instantaneously co-moving frame in which the particle's acceleration is $\vec{a}_0$. Let $a_\parallel$ and $a_\perp$ be the acceleration components along the particle's velocity $\vec{u}$ and perpendicular to the particle's velocity $\vec{u}$ respectively w.r.t an observer in frame S. Similarly let $a_{\parallel,0}$ and $a_{\perp,0}$ be the components of acceleration $\vec{a}_0$ along the particle's velocity $\vec{u}$ and perpendicular to the particle's velocity $\vec{u}$ respectively w.r.t an observer in frame S'. The angle between vectors $\vec{u}$ and $\vec{a}$ is θ.

The acceleration components in the two frames are related by the following equations:

$$a'_x = \frac{a_x}{[\gamma(1-\frac{u_x v}{c^2})]^3} \quad , \quad a'_y = \frac{a_y}{[\gamma(1-\frac{u_x v}{c^2})]^2} + \frac{u_y v a_x}{c^2\gamma^2(1-\frac{u_x v}{c^2})^3}$$

Since S' is an ICF and we are taking acceleration components along the direction of particle's velocity and perpendicular to particle's velocity in S frame, we have, $a'_x = a_{\parallel,0}$, $a_x = a_\parallel = acos\theta$, $\gamma = \gamma_u$, $u_x = v = u$, $a'_y = a_{\perp,0}$, $a_y = a_\perp = asin\theta$, $u_y = 0$. Substituting these in the above equations, we get:

$$a_{\parallel,0} = \frac{acos\theta}{[\gamma_u(1-\frac{u^2}{c^2})]^3} \quad , \quad a_{\perp,0} = \frac{asin\theta}{[\gamma_u(1-\frac{u^2}{c^2})]^2}$$

where $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$. On Simplifying, we get:

$$a_{\parallel,0} = \gamma_u^3 acos\theta \quad , \quad a_{\perp,0} = \gamma_u^2 asin\theta$$

The magnitude of $\vec{a}_0$ is given by:

$$a_0 = \sqrt{(a_{\parallel,0})^2 + (a_{\perp,0})^2} = \gamma_u^2\sqrt{\gamma_u^2 cos^2\theta + sin^2\theta}\, a = \gamma_u^3\sqrt{cos^2\theta + (1-\beta_u^2)sin^2\theta}\, a$$

where $\beta_u = \frac{u}{c}$. Since $cos^2\theta + sin^2\theta = 1$, the above equation could be simplified as:

$$a_0 = \gamma_u^3\sqrt{cos^2\theta + sin^2\theta - \beta_u^2 sin^2\theta}\, a = \gamma_u^3\sqrt{1-\beta_u^2 sin^2\theta}\, a$$

Hence Proved.

Q. 2

Let $\tilde{X}$ and $\tilde{X}'$ be the space-time coordinates for observer S and S' respectively. Given that $\tilde{X} \to \tilde{X}' = L\tilde{X}$ where Lorentz Transformation matrix L is given as:

$$L = \frac{1}{64}\begin{pmatrix} 125 & -48 & -60 & 75 \\ -75 & 80 & 36 & -45 \\ -60 & 0 & 80 & -36 \\ 48 & 0 & 0 & 80 \end{pmatrix}$$

With respect to observer S, the 4-covector A_μ is given as:

$$A_\mu = \begin{pmatrix} 5 \\ 1 \\ -2 \\ 0 \end{pmatrix}$$

Let the 4-covector w.r.t. observer S' be A'_μ. The 4-vector in a particular frame is related to the 4-covector in the same frame by $A^\mu = \eta A_\mu$ where Minkowski Metric η is given as:

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Therefore, w.r.t. observer S, the 4-vector A^μ is given by:

$$A^\mu = \begin{pmatrix} 5 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

Since 4-vectors transforms the same way as coordinates do, we have $A^\mu \to A'^\mu = LA^\mu$, where A'^μ is the 4-vector w.r.t. observer S'.

$$\implies A'^\mu = \frac{1}{64}\begin{pmatrix} 125 & -48 & -60 & 75 \\ -75 & 80 & 36 & -45 \\ -60 & 0 & 80 & -36 \\ 48 & 0 & 0 & 80 \end{pmatrix} A^\mu = \frac{1}{64}\begin{pmatrix} 125 & -48 & -60 & 75 \\ -75 & 80 & 36 & -45 \\ -60 & 0 & 80 & -36 \\ 48 & 0 & 0 & 80 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

Since $A'^\mu = \eta A'_\mu$, we have:

$$\eta A'_\mu = \frac{1}{64}\begin{pmatrix} 125 & -48 & -60 & 75 \\ -75 & 80 & 36 & -45 \\ -60 & 0 & 80 & -36 \\ 48 & 0 & 0 & 80 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

Now, pre-multiplying both side by η^{-1} and noting that $\eta^{-1} = \eta$, we have:

$$A'_\mu = \frac{1}{64}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 125 & -48 & -60 & 75 \\ -75 & 80 & 36 & -45 \\ -60 & 0 & 80 & -36 \\ 48 & 0 & 0 & 80 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\implies A'_\mu = \frac{1}{64}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\begin{pmatrix}553\\ -383\\ -140\\ 240\end{pmatrix}$$

$$\implies A'_\mu = \frac{1}{64}\begin{pmatrix}553\\ 383\\ 140\\ -240\end{pmatrix}$$

Hence the components of the 4-covector as measured by observer S' are given by A'_μ.

Q. 3 Part(a)

A Lorentz transformation matrix L obeys:

$$L^T\eta L = \eta \tag{1}$$

Since L_1 and L_2 are given to be Lorentz transformation matrices, we have:

$$L_1^T\eta L_1 = \eta \tag{2}$$

$$L_2^T\eta L_2 = \eta \tag{3}$$

Now we substitute L_1L_2 in place of L in L.H.S of Eqn(1) and compute it to check if it is a Lorentz transformation matrix. Therefore we have:

$$(L_1L_2)^T\eta(L_1L_2) = L_2^TL_1^T\eta L_1L_2 \tag{4}$$

Using Eqn(2), R.H.S of Eqn(4) can be written as:

$$(L_1L_2)^T\eta(L_1L_2) = L_2^T\eta L_2 \tag{5}$$

Using Eqn(3), R.H.S of Eqn(5) can be written as:

$$(L_1L_2)^T\eta(L_1L_2) = \eta \tag{6}$$

From Eqn(1), Eqn(2), Eqn(3) and Eqn(6), we conclude that if L_1 and L_2 are Lorentz Transformation matrices then L_1L_2 is also a Lorentz Transformation matrix.

Hence Proved.

Part(b)

Given that:

$$\det(L_1) = 1,\ \ \det(L_2) = 1 \tag{7}$$

Applying determinant on both sides of Eqn(6), we get:

$$\det((L_1L_2)^T\eta L_1L_2) = \det(\eta) \tag{8}$$

$$\implies \det((L_1L_2)^T)\det(\eta)\det(L_1L_2) = \det(\eta)$$

Since $\det((L_1L_2)^T) = \det(L_1L_2)$ and $\det(\eta) \neq 0$, we have from the above Equation:

$$(\det(L_1L_2))^2 = 1 \implies \det(L_1L_2) = \pm 1 \implies \det(L_1)\det(L_2) = \pm 1 \tag{9}$$

Using Eqn(7) in Eqn(9) we neglect negative value.

$$\implies \det(L_1)\det(L_2) = +1 \implies \det(L_1L_2) = +1$$

Hence proved.

Part(c)

We first prove some identities which we would be using for our proof. Let L be a general Lorentz Transformation matrix. Then we have:

$$\eta = (L)^T \eta (L) \tag{10}$$

Writing Eqn(10) in component form, we have:

$$\eta_{\mu\nu} = \sum_{\rho,\sigma} (L)^T_{\mu\rho} \eta_{\rho\sigma} (L)_{\sigma\nu} = \eta_{\rho\sigma} \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} \tag{11}$$

where we have used the fact that $(L)^T_{\mu\rho} = (L)_{\rho\mu}$ .Note that $\eta_{\rho\sigma} = 0$ for all $\rho \neq \sigma$ and $\rho, \sigma \in \{0, 1, 2, 3\}$. Taking the 00-th element of $\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu}$, we get:

$$1 = \eta_{00} = \eta_{\rho\sigma} \Lambda^{\rho}{}_{0} \Lambda^{\sigma}{}_{0} = (\Lambda^0{}_0)^2 - (\Lambda^1{}_1)^2 - (\Lambda^2{}_2)^2 - (\Lambda^3{}_3)^2$$
$$\implies (\Lambda^1{}_1)^2 + (\Lambda^2{}_2)^2 + (\Lambda^3{}_3)^2 = (\Lambda^0{}_0)^2 - 1 \tag{12}$$

We would be using Eqn(12) for our proof later.

Let us define $A = L_1 L_2$. Accordingly, we use notation such that $A^{\mu}{}_{\nu}$, $(L_1)^{\mu}{}_{\nu}$ and $(L_2)^{\mu}{}_{\nu}$ denotes the $\mu - th$ row,$\nu - th$ column element of A, L_1 and L_2 respectively where $\mu, \nu \in \{0, 1, 2, 3\}$.

We define :

$$\vec{l_1} = \begin{pmatrix} (L_1)^0{}_1 \\ (L_1)^0{}_2 \\ (L_1)^0{}_3 \end{pmatrix}, \quad \vec{l_2} = \begin{pmatrix} (L_2)^1{}_0 \\ (L_2)^2{}_0 \\ (L_2)^3{}_0 \end{pmatrix} \tag{13}$$

Therefore we have:

$$A^0{}_0 = (L_1)^0{}_0 (L_2)^0{}_0 + \vec{l_1}.\vec{l_2}$$
$$\implies A^0{}_0 \geq (L_1)^0{}_0 (L_2)^0{}_0 - |\vec{l_1}.\vec{l_2}| \tag{14}$$

Now using Eqn(12) we can calculate the norm of $\vec{l_1}$ and $\vec{l_2}$ as follows:

$$\|\vec{l_1}\| = \sqrt{((L_1)^1{}_1)^2 + ((L_1)^2{}_2)^2 + ((L_1)^3{}_3)^2} = \sqrt{((L_1)^0{}_0)^2 - 1} \tag{15}$$

$$\|\vec{l_2}\| = \sqrt{((L_2)^1{}_1)^2 + ((L_2)^2{}_2)^2 + ((L_2)^3{}_3)^2} = \sqrt{((L_2)^0{}_0)^2 - 1} \tag{16}$$

Now using Cauchy-Schwarz inequality, we have:

$$|\vec{l_1}.\vec{l_2}| \leq \|\vec{l_1}\| \|\vec{l_2}\| \tag{17}$$

Substituting Eqn(17) in Eqn(14), we get:

$$A^0{}_0 \geq (L_1)^0{}_0 (L_2)^0{}_0 - \|\vec{l_1}\| \|\vec{l_2}\| \tag{18}$$

Since it is given that $(L_1)^0{}_0 > 1$ and $(L_2)^0{}_0 > 1$, we use the inequality, $\sqrt{x^2 - 1} > x - 1 \ \forall x > 1$ in Eqn(15) and Eqn(16) to get:

$$\|\vec{l_1}\| > (L_1)^0{}_0 - 1 \tag{19}$$
$$\|\vec{l_2}\| > (L_2)^0{}_0 - 1 \tag{20}$$

Note that ${(L_1)^0}_0 - 1 > 0$ and ${(L_2)^0}_0 - 1 > 0$. Multiplying Eqn(19) and Eqn(20), we get:

$$\|\vec{l_1}\|\|\vec{l_2}\| > ({(L_1)^0}_0 - 1)({(L_2)^0}_0 - 1) \tag{21}$$

$$\implies \|\vec{l_1}\|\|\vec{l_2}\| > {(L_1)^0}_0{(L_2)^0}_0 - {(L_1)^0}_0 - {(L_2)^0}_0 + 1 \tag{22}$$

Substituting Eqn(22) in Eqn(18), we get:

$${A^0}_0 > {(L_1)^0}_0{(L_2)^0}_0 - [{(L_1)^0}_0{(L_2)^0}_0 - {(L_1)^0}_0 - {(L_2)^0}_0 + 1] \tag{23}$$

$$\implies {A^0}_0 > {(L_1)^0}_0 + {(L_2)^0}_0 - 1 \tag{24}$$

Since ${(L_1)^0}_0 > 1$ and ${(L_2)^0}_0 > 1$, Eqn(24) becomes:

$${A^0}_0 > 2 - 1$$

$$\implies {A^0}_0 > 1$$

Therefore if Lorentz transformation matrices L_1 and L_2 have 0-0 components greater than 1 then L_1L_2 also has 0-0 component greater than 1.

Hence proved.

Remark:

Here I prove one non-standard inequality which I simply stated while solving Q. 3 Part(c). The inequality was:

$$\sqrt{x^2 - 1} > x - 1 \quad \forall x > 1 \tag{25}$$

Proof(by contradiction):
Let us assume that Eqn(25) is not true. Then $\exists\ x > 1$ such that following holds:

$$\sqrt{x^2 - 1} \leq x - 1 \tag{26}$$

Since $x > 1$, both L.H.S and R.H.S of Eqn(26) are positive. On squaring Eqn(26), we get:

$$x^2 - 1 \leq x^2 + 1 - 2x$$

$$\implies -2 \leq -2x$$

$$\implies x \leq 1$$

which is a contradiction to our assumption that $\exists\ x > 1$ such that Eqn(26) holds. Hence our initial assumption was wrong and thus the inequality written as Eqn(25) holds.

Hence proved.