PH2102 **Assignment 6** **Instructor:ADG**

Mechanics II

Kishal Bipin Tandel, 23MS115 **October 14, 2024**

Q1. The relativistic velocity transformation is given by:

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

Squaring both sides, we get:

$$u'^2 = \frac{(u - v)^2}{\left(1 - \frac{uv}{c^2}\right)^2} = \frac{u^2 - 2uv + v^2}{\left(1 - \frac{uv}{c^2}\right)^2} \tag{1}$$

Now γ'_u is given by:

$$\gamma'_u = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} \tag{2}$$

Substituting Eqn(1) in Eqn(2), we get:

$$\gamma'_u = \frac{1}{\sqrt{1 - \frac{1}{c^2} \cdot \frac{u^2 - 2uv + v^2}{\left(1 - \frac{uv}{c^2}\right)^2}}} \tag{3}$$

Simplifying Eqn(3), we get:

$$\gamma'_u = \frac{1}{\sqrt{\frac{\left(1 - \frac{uv}{c^2}\right)^2 - \frac{1}{c^2}(u^2 - 2uv + v^2)}{\left(1 - \frac{uv}{c^2}\right)^2}}} \tag{4}$$

Now, expanding $\left(1 - \frac{uv}{c^2}\right)^2$:

$$\left(1 - \frac{uv}{c^2}\right)^2 = 1 - \frac{2uv}{c^2} + \frac{u^2 v^2}{c^4} \tag{5}$$

Substituting Eqn(5) in Eqn(4) and then simplifying, we get:

$$\gamma'_u = \frac{1 - \frac{uv}{c^2}}{\sqrt{1 - \frac{2uv}{c^2} + \frac{u^2v^2}{c^4} - \frac{u^2}{c^2} + \frac{2uv}{c^2} - \frac{v^2}{c^2}}} = \frac{1 - \frac{uv}{c^2}}{\sqrt{1 - \frac{u^2}{c^2} - \frac{v^2}{c^2} + \frac{u^2v^2}{c^4}}} \tag{6}$$

We also know that:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Therefore the product $\gamma\gamma_u$ is:

$$\gamma\gamma_u = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \cdot \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}-\frac{u^2}{c^2}+\frac{u^2v^2}{c^4}}} \tag{7}$$

Dividing Eqn(6) by Eqn(7), we get:

$$\frac{\gamma_u'}{\gamma\gamma_u} = 1 - \frac{uv}{c^2} \tag{8}$$

Hence proved.

Q2. **a)** From Alice's perspective, Bob's car of proper length $L_0 = 5\,m$ will appear to be of contracted length $L = 3\,m$. Using Lorentz contraction formula to find Bob's vecocity v, we get:

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow 3 = 5\sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \frac{3}{5} = \sqrt{1-\frac{v^2}{c^2}}$$

$$\Rightarrow \left(\frac{3}{5}\right)^2 = 1-\frac{v^2}{c^2}$$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow v = \frac{4}{5}c = 0.8c$$

Thus, Bob is driving at $v = 0.8c$.

b) From Alice's perspective, the car would have contracted and so would fit in the garage when the car is moving with $0.8c$. From Bob's perspective the garage was moving at $0.8c$ and so the garage must have contracted to a length less than $3m$ and so the car would not fit inside.

This problem arises due to relativity of simultaneity according to which, two events which are simultaneous to one inertial observer are, in general, not simultaneous to another inertial observer.The two events here are:

1. Front end of Bob's car is inside the garage.
2. Rear end of Bob's car is inside the garage.

Therefore, even though from Alice's perspective, the car fits in the garage meaning, these events were simultaneous, from Bob's perspective, the car did not fit in the garage meaning, these events were not simultaneous. So this is just a difference of perspective.

But in reality, the car could either be completely inside the garage or not. To solve this problem we consider the situation when the car stops. Alice slammed the door of the garage when the car was completely in according to her.But as the car comes to rest,it must stretch out to its rest length in Alice's frame. This is because the notion of rigidity loses its meaning in relativity, for when the car changes its speed (the brief period of acceleration when the car just starts to move or when it just comes to a halt), different parts of the car do not, in general, accelerate simultaneously.In this way, the car stretches or shrinks to reach the length appropriate to its new velocity in an observer's frame.

So when Alice slammed the door when the car was completely inside the garage according to her and when the car came to rest, either of the following two situations could happen:

1. The car fits inside the garage but is damaged.
2. The car broke through the walls of the garage and so does not fit inside the garage.

In either of these cases, Bob is at loss due to lack of his understanding of the concepts of relativity in detail.

Q3.

From Bob's reference frame:

i) Alice's speed $v = 0.8c$,
ii) The distance to the star system L_0 (proper length)= 4 lt-years.
Therefore,

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-0.64}} = \frac{5}{3}$$

So, time taken t by Alice for reaching the star system with respect to Bob,

$$t = \frac{L_0}{v} = \frac{4}{0.8} = 5 \text{ years}$$

Therefore time taken by Alice for the round way trip w.r.t. Bob is $2 \times t = 10$ years.

From Alice's reference frame:

i) Bob's speed $v = 0.8c$, and hence $\gamma = \frac{5}{3}$
ii) The distance to the star system, $L = \frac{L_0}{\gamma} = 2.4$ lt-years.
So the time taken t_0 by the star system to reach her w.r.t her is,

$$t_0 = \frac{L}{v} = \frac{2.4}{0.8} = 3 \text{ years}$$

Therefore time taken by Bob for the round way trip w.r.t. Alice is $2 \times t_0 = 6$ years.

Since Bob and Alice are to send signals every year on their birthday w.r.t their times, Bob would have sent 10 signals and Alice would have sent 6 signals till they meet again.

To find the time, when Alice and Bob receive each other's signals, we will need to take into account the Doppler Effect.

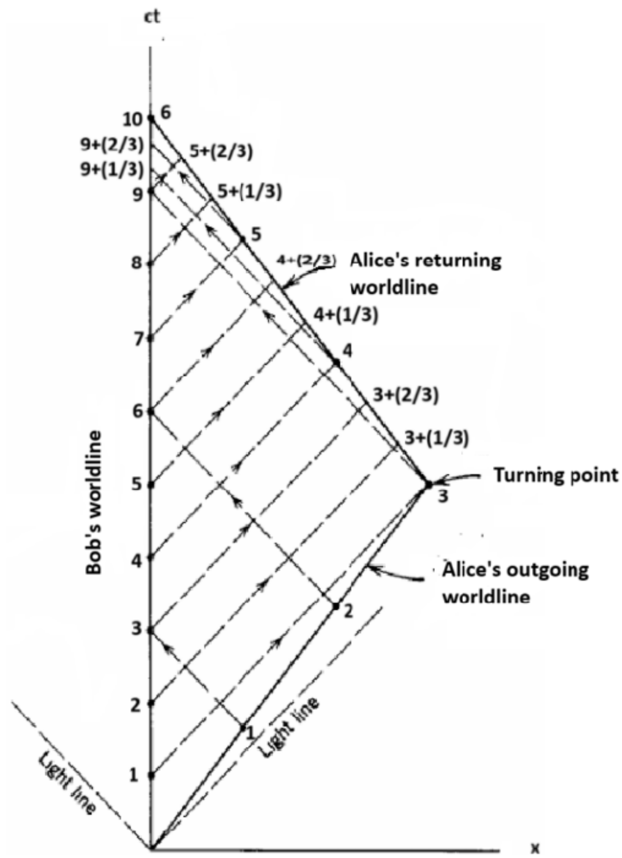

Figure 1: Space-Time Diagram of the Twin Paradox with Light Signals

The space-time diagram above shows the time when Alice and Bob receive each other's signals. Each interval between two successive dots on Alice's and Bob's worldline corresponds to one year passed in their respective frames. The time when one receives other's signals is also marked on the diagram.

Part (a)

We have already shown that Bob will send 10 signals till he meets Alice again which can also be seen from the diagram above. Now let us analyse the time when Alice receives these signals.

Outward Journey (Alice receding at $0.8c$)

In this case, Alice's Doppler factor w.r.t. Bob is given by:

$$K = \sqrt{\frac{c+v}{c-v}} = \sqrt{\frac{c+0.8c}{c-0.8c}} = 3$$

This means that Bob's annual signals reach Alice at gaps of three years according to her clock. So, during the outward journey which lasts for three years according to Alice, Alice receives only one signal from Bob. So, at the moment she receives Bob's first signal, Alice fires her rockets and turns back home.

Return Journey (Alice approaching at $0.8c$)

In this case, Alice's Doppler factor w.r.t Bob is given by: $\frac{1}{K} = \frac{1}{3}$ (since velocity for outward and inward journey is same)

Therefore, Bob's next signals reach Alice at gaps of $\frac{1}{3}$ year in her reference frame. Therefore, during the three years of her return journey, Alice receives nine signals from Bob, the final one getting to her just as she is entering the earth again.

Total Signals Received by Alice

The total number of signals Alice receives is:

$$1\,(\text{outgoing}) + 9\,(\text{returning}) = 10\,\text{signals}$$

Thus, Alice receives all 10 signals sent by Bob by the time they meet again. So they both agree that Bob has aged by 10 years during the round trip.

Part (b)

Alice sends a signal every year according to her clock. Since Alice experiences 6 years of proper time during her journey, she sends a total of 6 signals, which is also clear from the space-time diagram. Now let us analyse the time when Bob receives these signals.

Outward Journey (Alice receding at $0.8c$)

For this case Bob's Doppler Factor w.r.t. Alice is given by $\frac{1}{K} = \frac{1}{3}$.

Even though Alice is moving away from Bob towards right, her light signals sent to Bob are moving to the left, and this means that Bob will see them at a gap that is thrice as big than the one that Alice sends them out at. Thus, Alice sends out a total of three signals during her outward journey, and the third of these reaches Bob after nine years (according to his clock).

Return Journey (Alice approaching at $0.8c$)

For this case Bob's Doppler Factor w.r.t Alice is given by: K = 3.

This means that Bob is going to receive Alice's yearly signals at gaps of $\frac{1}{3}$ years in his reference frame (since Alice is approaching Bob in this case). Alice sends out three signals while coming back. Bob receives all of them crammed up in the last one year of his frame of reference.

Total Signals Received by Bob

The total number of signals received by Bob is:

$$3\,(\text{Alice outgoing}) + 3\,(\text{Alice returning}) = 6\,\text{signals}$$

Thus, Bob receives all 6 signals sent by Alice by the time they meet again. So they both agree that Alice has aged by 6 years during the round trip.

Part (c)

There is no disagreement about the signals: Alice sends six and Bob receives six; Bob sends ten and Alice receives ten. Everything works out, each seeing the correct Doppler shift of the other's clock and each agree to the number of signals that the other sent. The different total times recorded by the twins corresponds to the fact that Bob sees Alice recede for nine years and return in one year, although Alice both receded for three of her years and returned for three of her years. Bob's records will show that he received signals at a slow rate for nine years and at a rapid rate for one year. Alice's records will show that she received signals at a slow rate for three years and at a rapid rate for another three years. The essential asymmetry is thereby revealed by a Doppler effect analysis. When Alice and Bob compare records, they will agree that Bob's clock recorded ten years and Alice's recorded only six. Ten years have passed for Bob during Alice's six-year round trip.

Q4. Part (a)

The Special Lorentz transformation matrix (for relative velocity only along x-direction) is given by:

$$L = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}.$$

Expanding along the fourth row, we get:

$$\det L = \det \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \det \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Simplifying:

$$\det L = \gamma^2 - \gamma^2\beta^2 = \gamma^2(1-\beta^2).$$

Using the fact that $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we find:

$$\det L = \frac{1}{1-\beta^2} \times (1-\beta^2) = 1.$$

Thus, we conclude that $\det L = +1$ for this type of Special Lorentz Transformation, satisfying the first condition.

Now, from the Lorentz matrix, we observe that:

$$\Lambda^0{}_0 = \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

Since $\beta \in [0, 1)$, we know that $1 - \beta^2 \in (0, 1]$. Therefore, $\gamma \geq 1$.

Thus, we conclude that:

$$\left|\Lambda^0{}_0\right| = \gamma \geq 1,$$

which satisfies the second condition.

Part (b)

The Lorentz transformations are linear transformations that preserve the spacetime interval:

$$\eta_{\mu\nu}x^\mu x^\nu$$

A transformation matrix L that preserves this interval must obey:

$$L^T \eta L = \eta$$

The determinant of such transformations must be $\det L = \pm 1$, as the Lorentz transformation group includes both proper (with $\det L = +1$) and improper (with $\det L = -1$) transformations.
While improper Lorentz transformations like parity ($t \to t, \mathbf{r} \to -\mathbf{r}$) and time reversal ($t \to -t, \mathbf{r} \to \mathbf{r}$) preserve the interval, they are discrete transformations and are not continuously connected to the identity transformation. These improper transformations have $\det L = -1$.

When considering a Lorentz transformation between two inertial frames with velocities that differ by an infinitesimal amount, the transformation is continuous and can be connected to identity transformation in a continuous way. If the determinant of a transformation is either $+1$ or -1, and we perform a continuous transformation, the determinant must remain where it started (since ± 1 are discrete values).

Thus, if the transformation starts as a proper Lorentz transformation (with $\det L = +1$), then any continuous Lorentz transformation must also satisfy $\det L = +1$.

Therefore, for any continuous Lorentz transformations between inertial frames where velocities differ by an infinitesimal amount, we must have:

$$\det L = +1.$$

Any Lorentz transformation that connects different inertial observers can be reached by continuously changing the relative velocity v from 0 to some finite value. This defines a continuous path in the space of Lorentz transformations starting from the identity matrix. This means that, starting from the identity, we can smoothly vary the matrix components without any discontinuities to reach any other proper Lorentz transformation.

Because ${\Lambda^0}_0$ starts at 1 (for the identity transformation) and is a continuous function of the velocity, ${\Lambda^0}_0$ cannot suddenly become ≤ -1 unless there is a discontinuity in the transformation, which contradicts the notion of continuous transformations between inertial observers.

For continuous Lorentz transformations (which can be continuously connected to the identity transformation) between inertial frames, we must have ${\Lambda^0}_0 \geq 1$ since ${\Lambda^0}_0 \leq -1$ would imply a discontinuous transformation like time reversal.

Thus, for continuous Lorentz transformations between inertial observers, we conclude that:

$$\det L = +1, \quad {\Lambda^0}_0 \geq 1.$$