PH2102 # Assignment 8 **Instructor:ADG**

Mechanics II

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Q.1 Part(a)

The invariant space-time interval ds^2 is given by:

$$\boxed{ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}$$

For a light-like (null) trajectory, the space-time interval must satisfy $ds^2 = 0$. In the x^0-x^1 plane, the null condition is:

$$(dx^0)^2 - (dx^1)^2 = 0 \quad \Rightarrow \quad dx^0 = \pm dx^1$$

This equation represents the path of light, where $x^0 = x^1$ or $x^0 = -x^1$, corresponding to the light cone's axes in x^0-x^1 plane. We thus define *light cone coordinates* in the following way, to simplify the analysis of light-like trajectories.

$$x^+ = x^0 + x^1 \quad \text{and} \quad x^- = x^0 - x^1$$

In these coordinates, light moving along $x^1 = x^0$ corresponds to $x^- = 0$, and light moving along $x^1 = -x^0$ corresponds to $x^+ = 0$. We are thus aligning the coordinates axes along the light-like directions, making calculations involving light propagation easier. These coordinates are named *light cone coordinates* because they are aligned with the light cone structure of null paths.

Part(b)

The space-time invariant in terms of standard coordinates (x^0, x^1, x^2, x^3) can be written as

$$\boxed{ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2}$$

Now, using the transformations for light cone coordinates:

$$x^+ = x^0 + x^1, \quad x^- = x^0 - x^1$$

Taking differential of the above Equation, we get:

$$dx^+ = dx^0 + dx^1, \quad dx^- = dx^0 - dx^1$$

Since $dx^+ dx^- = (dx^0)^2 - (dx^1)^2$, therefore the space-time invariant in terms of *light cone coordinates* (x^+, x^-, x^2, x^3) can be written as :

$$\boxed{ds^2 = dx^+ dx^- - (dx^2)^2 - (dx^3)^2}$$

To find the metric η in *light cone coordinates*, we can use the following identity:

$$\boxed{\eta_{\mu\nu}dx^\mu dx^\nu = ds^2 = dx^+dx^- - (dx^2)^2 - (dx^3)^2}$$

where $dx^\mu = (dx^+, dx^-, dx^2, dx^3)$ and $\mu, \nu \in \{+, -, 2, 3\}$. On solving the above Equation we get the following equations in *light cone coordinates*:

$$(\eta_{+-} + \eta_{-+})dx^+dx^- = dx^+dx^- \quad \Rightarrow \quad \eta_{+-} + \eta_{-+} = 1$$

$$\eta_{22}(dx^2)^2 = -(dx^2)^2 \quad \Rightarrow \quad \eta_{22} = -1$$

$$\eta_{33}(dx^3)^2 = -(dx^3)^2 \quad \Rightarrow \quad \eta_{33} = -1$$

$$\eta_{\mu\nu} = 0 \quad \forall \quad \eta_{\mu\nu} \notin \{\eta_{+-}, \eta_{-+}, \eta_{22}, \eta_{33}\}$$

Therefore the metric η in *light cone coordinates* can be written as:

$$\boxed{\eta = \begin{pmatrix} 0 & r & 0 & 0 \\ (1-r) & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \text{where } r \in \mathbb{R}}$$

Part(c)

The special Lorentz transformation (boost along x^1) is given by:

$$x'^0 = \gamma\left(x^0 - \beta x^1\right), \quad x'^1 = \gamma\left(x^1 - \beta x^0\right), \quad x'^2 = x^2, \quad x'^3 = x^3 \tag{1}$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

The light cone coordinates are given as:

$$x^+ = x^0 + x^1, \quad x^- = x^0 - x^1 \tag{2}$$

where the other two coordinates x^2 and x^3 remain unchanged. Now, let's compute the Lorentz transformations in terms of x^+ and x^-.

Note that since it is special Lorentz Transformation (boost along x^1), $x'^2 = x^2$ and $x'^3 = x^3$ in light cone coordinates. Using Eqn(1) and Eqn(2), we can write x'^+ and x'^- as:

$$x'^+ = x'^0 + x'^1 = \gamma\left(1-\beta\right)x^0 + \gamma\left(1-\beta\right)x^1 = \gamma\left(1-\beta\right)\left(x^0 + x^1\right) = \gamma\left(1-\beta\right)x^+$$

$$x'^- = x'^0 - x'^1 = \gamma\left(1+\beta\right)x^0 - \gamma\left(1+\beta\right)x^1 = \gamma\left(1+\beta\right)\left(x^0 - x^1\right) = \gamma\left(1+\beta\right)x^-$$

Thus, in matrix form, the special Lorentz transformation (boost along x^1) in light cone coordinates is given by:

$$\boxed{\begin{pmatrix} x'^+ \\ x'^- \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \gamma(1-\beta) & 0 & 0 & 0 \\ 0 & \gamma(1+\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^+ \\ x^- \\ x^2 \\ x^3 \end{pmatrix}}$$

Thus the special Lorentz Transformation matrix (boost along x^1) in light cone coordinates is given as:

$$\boxed{\begin{pmatrix} \gamma(1-\beta) & 0 & 0 & 0 \\ 0 & \gamma(1+\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

Part(d)

In *light cone coordinates*, a general 4-vector will be of the form $A^\mu = (A^+, A^-, A^2, A^3)$. The corresponding 4-covector will be of the form $A_\mu = (A_+, A_-, A_2, A_3)$. The 4-covector is related to the 4-vector through the metric η as:

$$\boxed{A_\mu = \eta A^\mu}$$

$$\Rightarrow \begin{pmatrix} A_+ \\ A_- \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} 0 & r & 0 & 0 \\ (1-r) & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A^+ \\ A^- \\ A^2 \\ A^3 \end{pmatrix}, \qquad \text{where } r \in \mathbb{R}$$

Thus in *light cone coordinates*, the 4-covector in terms of the components of 4-vector can be written as:

$$\boxed{A_\mu = \left(rA^-, (1-r)A^+, -A^2, -A^3\right), \qquad \text{where } r \in \mathbb{R}}$$

Q.2 Part(a)

We begin with:

$$T = \int_{\vec{v}=\vec{0}}^{\vec{v}=\vec{u}} \frac{d}{dt}\left(m\gamma(v)\vec{v}\right) \cdot d\vec{r} = \int_{\vec{v}=\vec{0}}^{\vec{v}=\vec{u}} d\left(m\gamma(v)\vec{v}\right) \cdot \frac{d\vec{r}}{dt} = \int_{\vec{v}=\vec{0}}^{\vec{v}=\vec{u}} d\left(m\gamma(v)\vec{v}\right) \cdot \vec{v} \tag{3}$$

Note that m is the rest mass. Relativistic mass m_r is given as $m_r = m\gamma(v)$. Therefore taking differential of the term $m\gamma(v)\vec{v}$, we get:

$$d\left(m\gamma(v)\vec{v}\right) = d\left(m_r\vec{v}\right) = m_r\, d\vec{v} + \vec{v}\, dm_r \tag{4}$$

Substituting Eqn(4) in Eqn(3), we get:

$$T = \int_{\vec{v}=\vec{0}}^{\vec{v}=\vec{u}} \left(m_r\, d\vec{v} + \vec{v}\, dm_r\right) \cdot \vec{v} = \int_{\vec{v}=\vec{0}}^{\vec{v}=\vec{u}} \left(m_r\, \vec{v} \cdot d\vec{v} + v^2\, dm_r\right) \tag{5}$$

Now we know that:

$$m_r = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{6}$$

$$\Rightarrow m_r^2 c^2 - m_r^2 v^2 = m^2 c^2 \tag{7}$$

Taking differential of both sides in Eqn(7), we get:

$$2m_r\, c^2\, dm_r - 2m_r^2\, \vec{v}\cdot d\vec{v} - v^2\, 2m_r\, dm_r = 0 \tag{8}$$

Dividing both sides of Eqn(8) by $2m_r$, we get:

$$m_r\, \vec{v}\cdot d\vec{v} + v^2\;\; dm_r = c^2\, dm_r \tag{9}$$

Substituting Eqn(9) in Eqn(5), we get:

$$T = \int\limits_{\vec{v}=\vec{0}}^{\vec{v}=\vec{u}} c^2\, dm_r = \;\; c^2 \int\limits_{m_r=m}^{m_r=m\gamma(u)} dm_r = m\gamma(u)c^2 - mc^2 = m(\gamma(u)-1)c^2 \tag{10}$$

Hence the relativistic expression for kinetic energy for a point particle is given as:

$$\boxed{T = m(\gamma(u)-1)c^2} \tag{11}$$

Part(b)

We know that:

$$\gamma(u) = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} \tag{12}$$

When $u << c$, $\left|\frac{u^2}{c^2}\right| < 1$ and thus we can use binomial expansion and write Eqn(12) as:

$$\gamma(u) = 1 + \frac{u^2}{2c^2} + \frac{3u^4}{8c^4} + \cdots \tag{13}$$

Substituting Eqn(13) in Eqn(11), we get:

$$T = m\left(1 + \frac{u^2}{2c^2} + \frac{3u^4}{8c^4} + \cdots - 1\right)c^2 = m\left(\frac{u^2}{2c^2} + \frac{3u^4}{8c^4} + \cdots\right)c^2 \tag{14}$$

When $u << c$, we just need to consider the first term in Eqn(14) as other higher order terms would be negligibly small. Hence Eqn(14) reduces to:

$$\boxed{T = \frac{1}{2}mu^2} \tag{15}$$

Hence the relativistic kinetic energy expression for a point particle reduces to the familiar Newtonian formula when $u << c$.

The next order correction to the classical formula is given by $\frac{3mu^4}{8c^2}$ which is the second term in Eqn(14) for the relativistic kinetic energy expression.

Let T_N denote the kinetic energy measured using Newtonian formula and let T denote the kinetic energy measured using relativistic formula. Then percentage error δ incurred if we were to use Newtonian formula rather than relativistic formula is given as:

$$\delta = \frac{|T_N - T|}{T} \times 100\% = \frac{\left|\frac{1}{2}mu^2 - m(\gamma(u)-1)c^2\right|}{m(\gamma(u)-1)c^2} \times 100\% = \frac{\left|\frac{1}{2}u^2 - (\gamma(u)-1)c^2\right|}{(\gamma(u)-1)c^2} \times 100\%$$

Case(i): $u = 0.01c$

$$\delta = \frac{\left|\frac{1}{2}(0.01c)^2 - \left(\frac{1}{\sqrt{1-(0.01)^2}} - 1\right)c^2\right|}{\left(\frac{1}{\sqrt{1-(0.01)^2}} - 1\right)c^2} \times 100\% = \frac{\left|\frac{1}{2}(0.01)^2 - \left(\frac{1}{\sqrt{1-(0.01)^2}} - 1\right)\right|}{\left(\frac{1}{\sqrt{1-(0.01)^2}} - 1\right)} \times 100\% \approx 0.0075\%$$

Case(ii): $u = 0.1c$

$$\delta = \frac{\left|\frac{1}{2}(0.1c)^2 - \left(\frac{1}{\sqrt{1-(0.1)^2}} - 1\right)c^2\right|}{\left(\frac{1}{\sqrt{1-(0.1)^2}} - 1\right)c^2} \times 100\% = \frac{\left|\frac{1}{2}(0.1)^2 - \left(\frac{1}{\sqrt{1-(0.1)^2}} - 1\right)\right|}{\left(\frac{1}{\sqrt{1-(0.1)^2}} - 1\right)} \times 100\% \approx 0.75\%$$

Case(iii): $u = 0.5c$

$$\delta = \frac{\left|\frac{1}{2}(0.5c)^2 - \left(\frac{1}{\sqrt{1-(0.5)^2}} - 1\right)c^2\right|}{\left(\frac{1}{\sqrt{1-(0.5)^2}} - 1\right)c^2} \times 100\% = \frac{\left|\frac{1}{2}(0.5)^2 - \left(\frac{1}{\sqrt{1-(0.5)^2}} - 1\right)\right|}{\left(\frac{1}{\sqrt{1-(0.5)^2}} - 1\right)} \times 100\% \approx 19.20\%$$