PH2102 Problem Set 6

Q 1) Starting from the rtelativistic velocity transformation rules show that under a special Lorentz transformation, we have

$$\frac{\gamma'_u}{\gamma\gamma_u} = 1 - \frac{u_x v}{c^2}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, $\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$ and $\gamma'_u = \frac{1}{\sqrt{1-\frac{u'^2}{c^2}}}$.

Q 2) a) Bob has won the grand prize in the quasar lottery and one of the first things he did after getting his hands on the money is - buy a 5m long stretch limo. Unfortunately, he had forgotten that his garage, meant to accommodate his Maruti 500 is only 3m long. Having studied PH2102 rather well, Bob formulated a fool proof plan to get his 5m long car to fit in his 3m garage. He asks his friend Alice, to wait just beside the door of the garage. By driving his car really fast he can ensure that Alice sees his car to be 3m long. This means that she can slam the door of the grage shut at the instant when the tail of Bob's car crosses the front door - at the same instant that the front end of the car touches the back wall of the garage. (what happens to Bob, and his car, after that is something that we will not worry about).

Find out how fast Bob was driving.

b) Just as Bob got near the garage, a thought struck him! The car may be contracted to Alice - but to him it is still 5m long. Instead, it is the garage that is moving rapidly towards him that has contracted. Its too late to do anything now - Bob has the sinking feeling that he has messed up - there is no way in which the car will fit into the garage!

So - does the car fit into the garage or not (is Alice right, or is Bob?)? Help the two friends out with a relativistic explanation! *Hint : think about what the phrase "the car fits in the garage" means - and note that the notion of simultaneity plays a role here.*

Q 3)

In this story Alice and Bob are twins. On their 20th birthday, the more adventurous of them, Alice, leaves on an interstellar trip, while Bob prefers to stay safely back home here on earth. Alice's ship travels out at $0.8c$ to a star system 4 light-years away, and immediately turns back to return to the earth. When Alice reunites with her twin, Bob has aged 10 years ($\frac{4 \text{ lt-yrs}}{0.8c} \times 2$) - but Alice's clock has run more slowly $\left(\gamma = \frac{5}{3}\right)$ and she has aged only six. This is the basis of the famous (or infamous) twin paradox. The fact that the two twins are now of different ages is not the paradox - though! This may seem strange to us because it does not conform to our expectations.However, just as Bob observes Alice's clock to run slow, Alice, too observes Bob's clocks to run slow! The paradox is that if Alice is treated as the observer - the same physics apparently says that it is she, rather than Bob, who should be older! The fact that it is indeed Alice who is younger when the two meet seems to run foul of the basic postulate of relativity - that the laws of physics are the same for all inertial observers!

There is a very quick way to resolve this paradox. This is to argue that while Bob is an inertial observer[a] Alice is not. While she moves uniformly with respect to Bob for almost all of the motion, she does have to turn around (in almot zero time - and thus with near infinite acceleration) and this renders her non-inertial. So, the fact that the conclusions are different for the two observers can simply be explained away by saying that only one of the two is inertial - only Bob's conclusion about Alice aging more slowly is the correct one.

While the above is a perfectly correct resolution of the twin paradox - just denying the paradox does seem a bit like cheating. In this problem we will explore this issue a bit more.

[a]in so far as an observer fixed to the earth can be treated as (approximately) inertial.

Let's say that the twins make a pact before Alice sets out on her journey. On their birthday every year (as measured by their respective clocks) Alice and Bob sends out a light signal to their sibling.

a) How many signals does Bob send out till they meet again? When does Alice receive these signals?

b) How many signals does Alice send Bob? At what times does Bob receive them?

c) In the light of your results above discuss the twin paradox.

Q 4) In the class we showed that two properties that the Lorentz transformation matrix L (with components $\Lambda^{\mu}{}_{\nu}$) obeys are

$$\det L = \pm 1, \qquad \left|\Lambda^0{}_0\right| \geq 1$$

a) Check that the natrix L for the special Lorentz transformation obeys these.

b) The Lorentz transformations are linear transformations that leave the interval $\eta_{\mu\nu}x^{\mu}x^{\nu}$ invariant. A transformation that connect spacetime measurements of one inertial observer to those of another must leave the interval invariant, and hence must be a Lorentz transformation. However, all Lorentz transformations can not connect the measurements of two inertial observers (note that both time reversal, $t \mapsto -t$, $\vec{r} \mapsto \vec{r}$, and parity, $t \mapsto t$, $\vec{r} \mapsto -\vec{r}$, leave the interval invariant and are hence Lorentz transformations). Using the fact that when you are considering transformations from one inertial observer to another, you can think of many (indeed an infinite number) of intervening inertial observers, each with very very close velocities, show that such transformations must obey

$$\det L = +1, \qquad \Lambda^0{}_0 \geq 1$$

This is a topological argument - like the one that helped us to decide that for rotations in three dimensions we must have $\det R = +1$