Inverted Pendulum

Kishan Patel

December 11, 2024

1 Introduction

2 Lagrange Mechanics

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = F_i \text{ with } \mathcal{L} = T - V \tag{1}$$

2.1 Finding the Lagrangian

$$\begin{aligned} T &= T_{\text{transitional}} + T_{\text{rotational}} \\ &= T_c + T_w + T_{w_r} \\ &= \frac{1}{2}m_c v_c^2 + \frac{1}{2}m_w v_w^2 + \frac{1}{2}I_w\omega^2 \end{aligned} \tag{2}$$

The velocity vectors are in reference to the center of gravity, I will assume that the density of each component is constant. Hence, the rod's center of gravity is at $\frac{1}{2}l$ and the weight's, at the end of the rod, center of mass is at l.

$$\begin{aligned} v_c &= \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} \\ v_w &= \begin{bmatrix} \dot{x} + l\dot{\theta}\cos\theta \\ -l\dot{\theta}\sin\theta \end{bmatrix} \end{aligned} \tag{3}$$

Velocity squared is $v^2 = v \cdot v$

$$\begin{aligned} v_c^2 &= \dot{x}^2 \\ v_w^2 &= \dot{x}^2 + 2l\dot{x}\dot{\theta}cos\theta + l^2\dot{\theta}^2\cos^2\theta + l^2\dot{\theta}^2\sin^2\theta \\ &= \dot{x}^2 + 2l\dot{x}\dot{\theta}cos\theta + l^2\dot{\theta}^2 \end{aligned} \tag{4}$$

Now, simplify kinetic energy: T

$$\begin{aligned} T &= \frac{1}{2}m_c\dot{x}^2 \\ &+ \frac{1}{2}m_w\dot{x}^2 + m_w l\dot{x}\dot{\theta}cos\theta + \frac{1}{2}m_w l^2\dot{\theta}^2 \\ &+ \frac{1}{2}I_w\dot{\theta}^2 \\ T &= \frac{1}{2}\dot{x}^2[m_c + m_w] + l\dot{x}\dot{\theta}\cos\theta m_w + \frac{1}{2}l^2\dot{\theta}^2[m_w + \frac{I_w}{l^2}] \end{aligned} \tag{5}$$

Now, will solve for the potential energy: V.

$$\begin{aligned} V &= V_w \\ &= m_w g l(1 - \cos\theta) \end{aligned} \tag{6}$$

Now, we can define the Lagrangian and solve for the mechanics with $q_1 = x$, and $q_2 = \theta$.

$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\dot{x}^2[m_c + m_w] + \frac{1}{2}l^2\dot{\theta}^2[m_w + \frac{I_w}{l^2}] + l\dot{x}\dot{\theta}\cos(\theta)m_w - m_w gl(1-\cos(\theta)) \\
&= \frac{1}{2}\dot{x}^2[m_c + m_w] + l^2\dot{\theta}^2 m_w + l\dot{x}\dot{\theta}\cos(\theta)m_w + m_w gl\cos(\theta) - m_w gl
\end{aligned}
\tag{7}
$$

$q_1 = x$:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial \mathcal{L}}{\partial x} + F_x \\
\frac{d}{dt}(\dot{x}[m_c + m_w] + l\, m_w\, \dot{\theta}\, \cos(\theta)) &= F_x \\
\ddot{x}[m_c + m_w] + lm_w\ddot{\theta}\cos(\theta) - lm_w\dot{\theta}^2\sin(\theta) &= F_x
\end{aligned}
\tag{8}
$$

$q_1 = \theta$:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \frac{\partial \mathcal{L}}{\partial \theta} \\
\frac{d}{dt}(2\, m_w\, \dot{\theta}\, l^2 + m_w\, \dot{x}\, \cos(\theta)\, l) &= -g\, l\, m_w\, \sin(\theta) - l\, m_w\, \dot{\theta}\, \dot{x}\, \sin(\theta) \\
2m_w\ddot{\theta}l^2 + m_w\ddot{x}\cos(\theta)l - m_w\dot{x}\dot{\theta}\sin(\theta)l &= -g\, l\, m_w\, \sin(\theta) - l\, m_w\, \dot{\theta}\, \dot{x}\, \sin(\theta) \\
2m_w\ddot{\theta}l^2 + m_w\ddot{x}\cos(\theta)l &= -glm_w\sin(\theta)
\end{aligned}
\tag{9}
$$

Since, the pendulum weight is being modeled as point mass $\frac{I_w}{l^2} = m_w$. sub in $\ddot{x}$ into $\ddot{\theta}$. Now, sub $\ddot{\theta}$ back into $\ddot{x}$ Using Matlab syms to simplify even more.

$$
\begin{aligned}
\ddot{x} &= \frac{2\, l\, m_w\, \sin(\theta)\, \dot{\theta}^2 + 2\, F_x + g\, m_w\, \cos(\theta)\, \sin(\theta)}{-m_w \cos(\theta)^2 + 2\, m_c + 2\, m_w} \\
\ddot{\theta} &= -\frac{l\, m_w\, \cos(\theta)\, \sin(\theta)\, \dot{\theta}^2 + F_x\, \cos(\theta) + g\, m_c\, \sin(\theta) + g\, m_w\, \sin(\theta)}{l\left(-m_w\cos(\theta)^2 + 2\, m_c + 2\, m_w\right)}
\end{aligned}
\tag{10}
$$

To make the system more accurate, add friction terms $-\delta_c\dot{x}$ and $-\delta_p\dot{\theta}$ to $\ddot{x}$ and $\ddot{\theta}$ respectively.

3 State Space Formulation

$$
\text{Let } \boldsymbol{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \text{, and } F_x = u \tag{11}
$$

$$
\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{2\, l\, m_w\, \sin(x_3)\, {x_4}^2 + 2\, u + g\, m_w\, \cos(x_3)\, \sin(x_3)}{-m_w\cos(x_3)^2 + 2\, m_c + 2\, m_w} \\ x_4 \\ -\frac{l\, m_w\, \cos(x_3)\, \sin(x_3)\, {x_4}^2 + u\, \cos(x_3) + g\, m_c\, \sin(x_3) + g\, m_w\, \sin(x_3)}{l\left(-m_w\cos(x_3)^2 + 2\, m_c + 2\, m_w\right)} \end{bmatrix} = f(x,u) \tag{12}
$$

3.1 linearization about fixed point

Now, find states to linearized at $f(x,u) = 0$

$$
\begin{aligned}
f(x,u) &= 0 \\
f([x_1,0,x_3,0]^T,0) &= \begin{bmatrix} 0 \\ \frac{g\, m_w\ \cos(x_3)\ \sin(x_3)}{-m_w \cos(x_3)^2+2\, m_c+2\, m_w} \\ 0 \\ -\frac{g\, m_c\ \sin(x_3)+g\, m_w\ \sin(x_3)}{l\left(-m_w \cos(x_3)^2+2\, m_c+2\, m_w\right)} \end{bmatrix}
\end{aligned} \tag{13}
$$

$f(x,u) = 0$ when $x_1 \in R$, $x_2 = 0$, $x_3 = \pi n\ \forall n \in Z$, $x_4 = 0$, and $u = 0$.

$$
\bar{x} = \begin{bmatrix} x_1 \\ 0 \\ \pi n = 0 \\ 0 \end{bmatrix} \tag{14}
$$

Now, we are ready for the state space equation $\dot{x} = Ax + Bu$. To find the matrices A and B we will take the Jacobian of $f(x,u)$. For the model, we will want x_3 to be 0 as that is the upright position of the pendulum.

$$
\begin{aligned}
\dot{x} &= \partial_x f\Big|_{[x1,0,0,0]^T,0}(x-\bar{x}) + \partial_u f\Big|_{[x1,0,0,0]^T,0}(u-0) \\
\dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\delta & \frac{g\, m_w}{2\, m_c+m_w} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\delta & -\frac{g\, m_c+g\, m_w}{l\,(2\, m_c+m_w)} & 0 \end{bmatrix}(x-\bar{x}) + \begin{bmatrix} 0 \\ \frac{2}{2\, m_c+m_w} \\ 0 \\ -\frac{1}{l\,(2\, m_c+m_w)} \end{bmatrix} u \\
y &= Cx
\end{aligned} \tag{15}
$$

Now, we have our state space equation and measurement equation. The δ term is friction of the cart moving horizontally.

3.2 Controllability

The rank$(\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}) = 4$; therefore, we have full controllability of the system with our control $F_x = u$.

3.3 Observability

$$
C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{16}
$$

To actualize the system, I will take the simplest case where the sensors will capture the positional state $x1 = x$ only. Now calculate the rank Observability matrix to see if the system is full state observable from just the position of the cart.

$$
O = \begin{bmatrix} C^T & A^TC^T & (A^T)^2C^T & (A^T)^3C^T \end{bmatrix} \tag{17}
$$

The rank of the Observability matrix is 4; therefore, we can estimate all the other states.

4 Controller with $C = I$

4.1 Proportional Control

For the proportional control let $u - \bar{u} = -K(x - \bar{x})$.

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
\dot{x} &= Ax - BKx \\
\dot{x} &= (A - BK)x
\end{aligned} \tag{18}
$$

Since the system is controllable, the matrix K can be placed so that $A - BK$ can have any eigenvalues. Now what eigenvalues should one choose? $\lambda_i \in \{z \in C : \text{Re}(z) < 0\}$. But, what is the optimal choice? For this, the Linear Quadratic Regulator (L.Q.R.) will be used. Define the cost J which is a quadratic function in x and u.

$$\begin{aligned} J &= \int_0^\infty x^T Q x + u^T R u \; dt \\ \dot{x} &= Ax + Bu \\ Q &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix} \\ R &= \begin{bmatrix} 1 \end{bmatrix} \end{aligned} \tag{19}$$

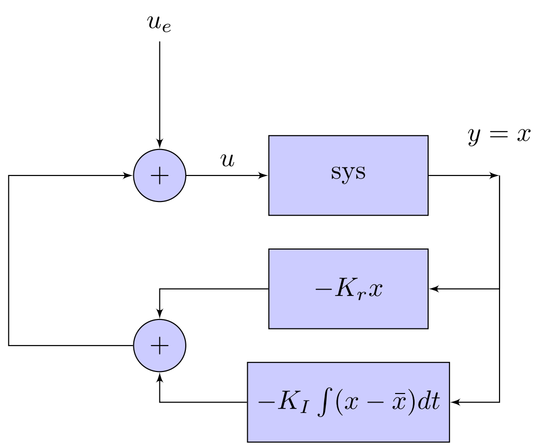
Now finding the minimum of the cost function is all done for us in Matlab. $K_r = lqr(A, B, Q, R)$. The interpretation of Q and R is that it defines the cost of controlling specific states.

4.2 Integral Control

For the proportional control let $u - \bar{u} = -K_I \int_0^t (x - \bar{x}) dt$.

$$\begin{aligned} w &= \int_0^t (x - \bar{x}) dt \\ \dot{w} &= x - \bar{x} \\ w(0) &= 0 \end{aligned} \tag{20}$$

To implement integral action set up a new state w which will update with the following discrete rule $w_n = w_{n-1} + \dot{w}_n dt$. The internal gain $K_I = [.1, .1, .1, .1]$.



5 Controller with $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

Since the controller has access to only the positional state, an estimator must be built to estimate other states.

5.1 Estimator

A system needs to build that will give us $\hat{x}$. The estimate $\hat{x}$ needs to approach the actual state x. So the error $e = x - \hat{x}$ needs to approach 0.

$$
\begin{aligned}
e &= x - \hat{x} \\
\dot{e} &= \dot{x} - \dot{\hat{x}} \\
\dot{e} &= Ax + Bu - (A\hat{x} + Bu + f) \\
\dot{e} &= A(x - \hat{x}) - L(y - \hat{y}) \\
\dot{e} &= (A - LC)e
\end{aligned} \tag{21}
$$

So if L can be placed such that $A - LC$ has negative eigenvalues, then the estimator will approach the actual state, that is, if the observability matrix has full rank.

$$
\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + Bu + L(y - \hat{y}) \\
\hat{y} &= C\hat{x}
\end{aligned} \tag{22}
$$

5.2 Kalman Filter

The matrix L needs to be chosen optimally to make the estimator. Let V_d be the covariance matrix for the disturbance in the system, and let L_n be the covariance matrix for the noise in the measurement.

$$
\begin{aligned}
\dot{x} &= Ax + Bu + w_d \\
y &= Cx + w_n
\end{aligned} \tag{23}
$$

$$
K_f = lqr(A', C', V_d, V_n)' \tag{24}
$$