Arithmetic Derivative

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1 Problem 484

Find the following sum:

$$\sum_{k=2}^{5 \cdot 10^{15}} gcd(k, k') \tag{1.1}$$

From the bounds of gcd, our sum has to between $N = 5 \cdot 10^{15}$ and $\frac{N(N+1)}{2}$.

2 Arithmetic Derivative

The arithmetic derivative is defined as follows:

$$\begin{aligned} p' &= 1 \text{ for any prime p} \\ (ab)' &= a'b + ab' \end{aligned} \tag{2.1}$$

2.1 Arithmetic Derivative of 1

The derivative of one can be found by looking at the base case of p', take the following $(1 \cdot p)' = 1' \cdot p + p' \cdot 1 = 1$, $(1')$ must be 0.

2.2 Arithmetic Derivative of Powers of p

$$(p^n)' = np^{n-1} \tag{2.2}$$

To prove this take the base case to be $p' = 1$, and assume formula above. Now using proof by induction, show that p^{n+1} has the same form.

$$\begin{aligned} (p^{n+1})' &= (pp^n)' \\ &= p'p^n + p(p^n)' \\ &= p^n + npp^{n-1} \\ &= (n+1)p^{n+1} \end{aligned} \tag{2.3}$$

2.3 Arithmetic Derivative of a Natural Number

Let $k \in \mathbb{N}$ have a prime factorization $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ where p_i is some prime and $\alpha_i \in \mathbb{N}$ is its corresponding power.

$$k' = \sum_{i=1}^{n} \alpha_i \frac{k}{p_i} \tag{2.4}$$

To prove this take the base case $(p_i^{\alpha_i})' = \alpha_i p_i^{\alpha_i - 1}$, and assume the formula above. Now using proof by induction, show that $L' = (kp_{n+1}^{\alpha_{n+1}})'$ has the same form.

$$\begin{aligned} (kp_{n+1}^{\alpha_{n+1}})' &= (p_{n+1}^{\alpha_{n+1}})'k + p_{n+1}^{\alpha_{n+1}}k' \\ &= \alpha_{n+1}p_{n+1}^{\alpha_{n+1}-1}k + p_{n+1}^{\alpha_{n+1}} \sum_{i=1}^{n} \alpha_i \frac{k}{p_i} \\ &= \alpha_{n+1}\frac{L}{p_{n+1}} + \sum_{i=1}^{n} \alpha_i \frac{L}{p_i} \\ &= \sum_{i=1}^{n+1} \alpha_i \frac{kp_{n+1}^{\alpha_{n+1}}}{p_i} \end{aligned} \tag{2.5}$$

It can be seen that k' has a factor of $p_1^{\alpha_1-1}\cdots p_n^{\alpha_n-1}$ in each part of its sum.

$$k' = p_1^{\alpha_1-1}\cdots p_n^{\alpha_n-1}\sum_{i=1}^{n}\alpha_i\frac{p_1\cdots p_n}{p_i} \tag{2.6}$$

2.4 The GCD of k and its Arithmetic Derivative

Using the closed form formula for k' we can say that the $p_1^{\alpha_1-1}\cdots p_n^{\alpha_n-1} \leq gcd(k,k')$. To remove the inequality, $\sum_{i=1}^{n}\alpha_i\frac{p_1\cdots p_n}{p_i}$ needs to be analyzed. If the sum has a factor of p_i then we can update $p_i^{\alpha_i-1}$ to $p_i^{\alpha_i}$ in our $gcd(k,k')$ calculation. So when does $p_i \mid \sum_{i=1}^{n}\alpha_i\frac{p_1\cdots p_n}{p_i}$, for all the terms $j\neq i$ there is exactly one p_i in them. Then when would $p_i|\alpha_i\frac{p_1\cdots p_n}{p_i}$? There will be at least one factor of p_i when $p_i \mid \alpha_i$. So when $\alpha_i = p_i * j\ \forall j\in\mathbb{N}$ we get an extra factor of p_i in our $gcd(k,k')$. We will introduce a new function $g(k) = \gcd(k,k')$ to help with readability.

$$g(k) = \gcd(k,k') = \prod_{i=1}^{n}\begin{cases} p_i^{\alpha_i-1}, & \text{if } p_i \nmid \alpha_i \\ p_i^{\alpha_i}, & \text{if } p_i \mid \alpha_i \end{cases} \tag{2.7}$$

3 Properties of g(k)

The function $g(k)$ is a multiplicative function when partitioned by the $p_i^{\alpha_i}$.

$$g(p_1^{\alpha_1}\cdots p_n^{\alpha_n}) = g(p_1^{\alpha_1})\cdots g(p_n^{\alpha_n}) \tag{3.1}$$

The $\sum g(k)$ can be factored if all the arguments share the same factor of p_i and α_i

$$\begin{aligned} &g(p_1^{\alpha_1}\cdots p_n^{\alpha_n}\cdot q_1^{\beta_1}\cdots q_m^{\beta_m}) + g(p_1^{\alpha_1}\cdots p_n^{\alpha_n}\cdot r_1^{\gamma_1}\cdots r_o^{\gamma_o}) \\ &= g(p_1^{\alpha_1}\cdots p_n^{\alpha_n})[g(q_1^{\beta_1}\cdots q_m^{\beta_m}) + g(r_1^{\gamma_1}\cdots r_o^{\gamma_o})] \end{aligned} \tag{3.2}$$

4 Creating Group G(p)

To find $\sum g(k)$ we need k's prime factorization which is expensive. So we can try to build all the k's using the primes which removes the need for prime factorization.

4.1 Brute Force

The brute force method of solving our sum would require calculating the prime factorization of each k.

$$O(\sum g(k)) \propto \sum_{k=1}^{N}\sqrt{k} \approx \int_1^N \sqrt{x}dx \approx \frac{2}{3}N^{\frac{3}{2}} \tag{4.1}$$

So we must find a procedure that has a significantly lower time complexity. So we must calculate $g(k)$ in terms of some other $g(l)$ which saves the number of computations needed.

4.2 Motivation

If we have computed $g(p_1^{\alpha_1}\cdots p_n^{\alpha_n})$ then $g(q^\alpha\cdot p_1^{\alpha_1}\cdots p_n^{\alpha_n}) = g(q^\alpha)g(p_1^{\alpha_1}\cdots p_n^{\alpha_n})$. So create groups G such that we can calculate all the numbers k that have a factor of q^α.

4.3 Building Group G(p)

Let $p^{\alpha_n} \mid k \in G(p)$. Now, does $G(p_i) \cap G(p_j) = \emptyset$ when $i \neq j$? No. There needs to be one more condition. We will require that $q \nmid k \in G(p)$ where $q < p$.

$$G(p) = \{k \quad | \quad p^\alpha \mid k \wedge q \nmid k \wedge 1 < k \leq N \quad q < p, \alpha \in \mathbb{N}\} \tag{4.2}$$

The union of all our groups $G(p)$ is all the values of k we need for $\sum g(k)$.

$$\sum g(k) = \sum_{p_i} \sum_{k \in G(p_i)} g(k) \tag{4.3}$$

Why is $G(p)$ useful? There are $G(p) = \{p\}$; the p's that have this property conform to the following:

$$G(p) = \{p\} \quad \text{iff} \quad p^2 \nleq N \tag{4.4}$$

The smallest element in $G(p)$ is p, the second smallest is p^2 since $p^2 < p \cdot p_{+1}$ where p_{+1} is the prime immediately after p. For our N we get $P^* = 70,710,707$.

$$\begin{aligned} \sum_{p=P^*}^{\pi(N)} G(p) &= \pi(N) - \pi(P^*) + 1 \\ &\approx \frac{N}{\log N} - \frac{P^*}{\log P^*} + 1 \\ &\approx 138,319,418,975,671 - 3,912,265 + 1 \\ &\approx 138,319,415,063,407 \end{aligned} \tag{4.5}$$

Since $\pi(^1P^*) \approx 3,912,265$ ...?. However, there is a $G(p) = \{p, p^2\}$ (show that there is only one p that allows for this).

$$\begin{aligned} G(p) = \{p, p^2\} &\quad \text{iff} \quad p \cdot p_{+1} \nleq N \\ p = {}^1P^* &= 70,710,677 \\ \sum_{k \in G(^1P^*)} g(k) &= 1 + p \end{aligned} \tag{4.6}$$

4.4 Categorizing G(p) into 13 Meta-Groups

A meta group of n is a group of $G(p)$ such that $\mu(G(p)) = n$. The measure $\mu(G)$ measures the most amount of primes a value in G has. For example, $\mu(p) = 1$, $\mu(2 \cdot 3) = 2$, $\mu(2^2 \cdot 3 \cdot 5 \cdot 7) = 4$, etc... So for the $G(2)$ the product of the first 13 primes is less then N, which is also the most amount of primes in a number in G, hence $\mu(G(2)) = 13$.

$$\mu(G(p)) = \underset{k \in G(p)}{\text{argmax}} \quad \mu(k) \tag{4.7}$$

For the problem at hand we can precompute when our measure changes; see table below. The constant's ${}^nP^*$ convey that all primes below it have a measure of at least n. The can be formalized by $n < \mu(G(p))$ for $\forall p < {}^nP^*$. Combining all the relations one can derive the closed form solution to $\mu(G(p))$. The difficulty of computing the sum $\sum_{k \in G(p)} g(k)$ is related to the measure $\mu(G(p))$.

$$\begin{aligned} \mu(G(p)) &= 13 \qquad \text{iff} \quad p = 2 \\ \mu(G(p)) &= n \qquad \text{iff} \quad {}^{n-1}P^* < p \leq {}^nP^* \\ \mu(G(p)) &= 1 \qquad \text{iff} \quad {}^1P^* \leq p \end{aligned} \tag{4.8}$$

μ	${}^{n}P^{*}$
13	${}^{13}P^{*}$=2
12	${}^{12}P^{*}$=5
11	${}^{11}P^{*}$=11
10	${}^{10}P^{*}$=19
9	${}^{9}P^{*}$=37
8	${}^{8}P^{*}$=73
7	${}^{7}P^{*}$=157
6	${}^{6}P^{*}$=397
5	${}^{5}P^{*}$=1,361
4	${}^{4}P^{*}$=8,387
3	${}^{3}P^{*}$=170,957
2	${}^{2}P^{*}$=70,710,649
1	${}^{1}P^{*}$=70,710,677

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