

Arithmetic Derivative

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1 Problem 484

Find the following sum:

$$\sum_{k=2}^{5 \cdot 10^{15}} \gcd(k, k') \quad (1.1)$$

From the bounds of \gcd , our sum has to between $N = 5 \cdot 10^{15}$ and $\frac{N(N+1)}{2}$.

2 Arithmetic Derivative

The arithmetic derivative is defined as follows:

$$\begin{aligned} p' &= 1 \text{ for any prime } p \\ (ab)' &= a'b + ab' \end{aligned} \quad (2.1)$$

2.1 Arithmetic Derivative of 1

The derivative of one can be found by looking at the base case of p' , take the following $(1 \cdot p)' = 1' \cdot p + p' \cdot 1 = 1$, $(1')$ must be 0.

2.2 Arithmetic Derivative of Powers of p

$$(p^n)' = np^{n-1} \quad (2.2)$$

To prove this take the base case to be $p' = 1$, and assume formula above. Now using proof by induction, show that p^{n+1} has the same form.

$$\begin{aligned} (p^{n+1})' &= (pp^n)' \\ &= p'p^n + p(p^n)' \\ &= p^n + np^{n-1} \\ &= (n+1)p^{n+1} \end{aligned} \quad (2.3)$$

2.3 Arithmetic Derivative of a Natural Number

Let $k \in \mathbb{N}$ have a prime factorization $p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ where p_i is some prime and $\alpha_i \in \mathbb{N}$ is its corresponding power.

$$k' = \sum_{i=1}^n \alpha_i \frac{k}{p_i} \quad (2.4)$$

To prove this take the base case $(p_i^{\alpha_i})' = \alpha_i p_i^{\alpha_i-1}$, and assume the formula above. Now using proof by induction, show that $L' = (kp_{n+1}^{\alpha_{n+1}})'$ has the same form.

$$\begin{aligned} (kp_{n+1}^{\alpha_{n+1}})' &= (p_{n+1}^{\alpha_{n+1}})'k + p_{n+1}^{\alpha_{n+1}}k' \\ &= \alpha_{n+1} p_{n+1}^{\alpha_{n+1}-1} k + p_{n+1}^{\alpha_{n+1}} \sum_{i=1}^n \alpha_i \frac{k}{p_i} \\ &= \alpha_{n+1} \frac{L}{p_{n+1}} + \sum_{i=1}^n \alpha_i \frac{L}{p_i} \\ &= \sum_{i=1}^{n+1} \alpha_i \frac{kp_{n+1}^{\alpha_{n+1}}}{p_i} \end{aligned} \quad (2.5)$$

It can be seen that k' has a factor of $p_1^{\alpha_1-1} \cdots p_n^{\alpha_n-1}$ in each part of its sum.

$$k' = p_1^{\alpha_1-1} \cdots p_n^{\alpha_n-1} \sum_{i=1}^n \alpha_i \frac{p_1 \cdots p_n}{p_i} \quad (2.6)$$

2.4 The GCD of k and its Arithmetic Derivative

Using the closed form formula for k' we can say that the $p_1^{\alpha_1-1} \cdots p_n^{\alpha_n-1} \leq \gcd(k, k')$. To remove the inequality, $\sum_{i=1}^n \alpha_i \frac{p_1 \cdots p_n}{p_i}$ needs to be analyzed. If the sum has a factor of p_i then we can update $p_i^{\alpha_i-1}$ to $p_i^{\alpha_i}$ in our $\gcd(k, k')$ calculation. So when does $p_i \mid \sum_{i=1}^n \alpha_i \frac{p_1 \cdots p_n}{p_i}$, for all the terms $j \neq i$ there is exactly one p_i in them. Then when would $p_i \mid \alpha_i \frac{p_1 \cdots p_n}{p_i}$? There will be at least one factor of p_i when $p_i \mid \alpha_i$. So when $\alpha_i = p_i * j \forall j \in \mathbb{N}$ we get an extra factor of p_i in our $\gcd(k, k')$. We will introduce a new function $g(k) = \gcd(k, k')$ to help with readability.

$$g(k) = \gcd(k, k') = \prod_{i=1}^n \begin{cases} p_i^{\alpha_i-1}, & \text{if } p_i \nmid \alpha_i \\ p_i^{\alpha_i}, & \text{if } p_i \mid \alpha_i \end{cases} \quad (2.7)$$

3 Properties of g(k)

The function $g(k)$ is a multiplicative function when partitioned by the $p_i^{\alpha_i}$.

$$g(p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = g(p_1^{\alpha_1}) \cdots g(p_n^{\alpha_n}) \quad (3.1)$$

The $\sum g(k)$ can be factored if all the arguments share the same factor of p_i and α_i

$$\begin{aligned} & g(p_1^{\alpha_1} \cdots p_n^{\alpha_n} \cdot q_1^{\beta_1} \cdots q_m^{\beta_m}) + g(p_1^{\alpha_1} \cdots p_n^{\alpha_n} \cdot r_1^{\gamma_1} \cdots r_o^{\gamma_o}) \\ &= g(p_1^{\alpha_1} \cdots p_n^{\alpha_n})[g(q_1^{\beta_1} \cdots q_m^{\beta_m}) + g(r_1^{\gamma_1} \cdots r_o^{\gamma_o})] \end{aligned} \quad (3.2)$$

4 Creating Group G(p)

To find $\sum g(k)$ we need k 's prime factorization which is expensive. So we can try to build all the k 's using the primes which removes the need for prime factorization.

4.1 Brute Force

The brute force method of solving our sum would require calculating the prime factorization of each k .

$$O(\sum g(k)) \propto \sum_{k=1}^N \sqrt{k} \approx \int_1^N \sqrt{x} dx \approx \frac{2}{3} N^{\frac{3}{2}} \quad (4.1)$$

So we must find a procedure that has a significantly lower time complexity. So we must calculate $g(k)$ in terms of some other $g(l)$ which saves the number of computations needed.

4.2 Motivation

If we have computed $g(p_1^{\alpha_1} \cdots p_n^{\alpha_n})$ then $g(q^\alpha \cdot p_1^{\alpha_1} \cdots p_n^{\alpha_n}) = g(q^\alpha)g(p_1^{\alpha_1} \cdots p_n^{\alpha_n})$. So create groups G such that we can calculate all the numbers k that have a factor of q^α .

4.3 Building Group G(p)

Let $p^{\alpha_n} \mid k \in G(p)$. Now, does $G(p_i) \cap G(p_j) = \emptyset$ when $i \neq j$? No. There needs to be one more condition. We will require that $q \nmid k \in G(p)$ where $q < p$.

$$G(p) = \{k \mid p^\alpha \mid k \wedge q \nmid k \wedge 1 < k \leq N \quad q < p, \alpha \in \mathbb{N}\} \quad (4.2)$$

The union of all our groups $G(p)$ is all the values of k we need for $\sum g(k)$.

$$\sum g(k) = \sum_{p_i} \sum_{k \in G(p_i)} g(k) \quad (4.3)$$

Why is $G(p)$ useful? There are $G(p) = \{p\}$; the p's that have this property conform to the following:

$$G(p) = \{p\} \quad \text{iff} \quad p^2 \not\leq N \quad (4.4)$$

The smallest element in $G(p)$ is p, the second smallest is p^2 since $p^2 < p \cdot p_{next}$. For our N we get ${}^{12}P^* = 70, 710, 707$.

$$\begin{aligned} \sum_{p={}^{12}P^*}^{\pi(N)} G(p) &= \pi(N) - \pi({}^{12}P^*) + 1 \\ &\approx \frac{N}{\log N} - \frac{{}^{12}P^*}{\log {}^{12}P^*} + 1 \\ &\approx 138, 319, 418, 975, 671 - 3, 912, 265 + 1 \\ &\approx 138, 319, 415, 063, 407 \end{aligned} \quad (4.5)$$

Since $\pi({}^{12}P^*) \approx 3, 912, 265$, the remaining groups contain all the numbers that are only composed of the first 3, 912, 265 prime numbers.

