Hochschule
Bonn-Rhein-Sieg
University of Applied Sciences

b-it Bonn-Aachen International Center for Information Technology

Institute for AI and Autonomous Systems

Calculus

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[Extension to slides created by Michal & Musharraf]

Introduction

What is Calculus?

A branch of mathematics which deals with "*study of continuous change*" of functions or sequences

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Major Branches

- Differential Calculus
- Integral Calculus

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General Applications

- Optimisation
- To Solve Differential Equations
- Function Approximations
- Length, Area, Volume, Center of Mass, Moment of Inertia Calculations

Applications in Robotics

- **Kinematics**
  - Modelling mobile robot (eg: differential drive) and environment
  - Transform between different spaces

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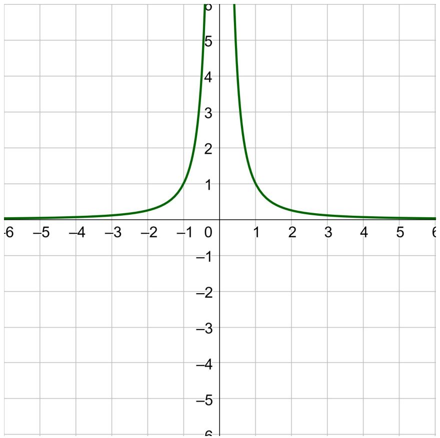
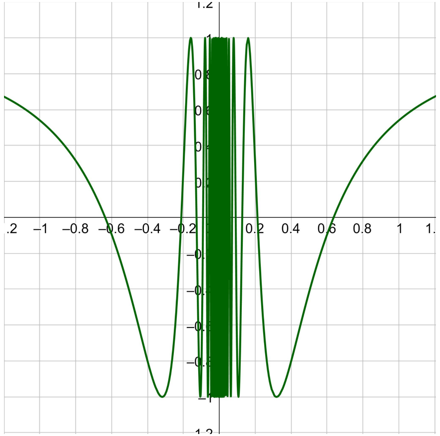
- Logic
  - Situational calculus

Limits and Continuity

Functions vs Relations

- Functions
  Eg: $y = x^2+4x$
- Relations
  Eg: $x^2+y^2 = 25$





Limits and Continuity

Functions vs Relations

- Functions have one unique output for a given input
  Eg: $y = x^2+4x$
- Relations can have more than one output
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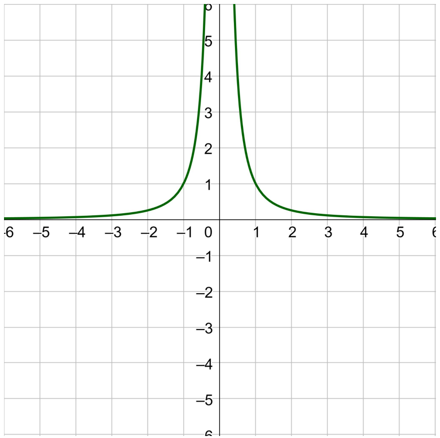

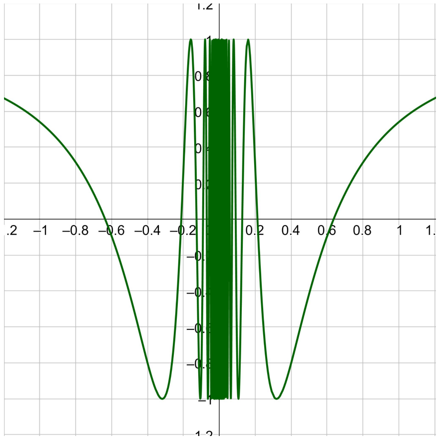

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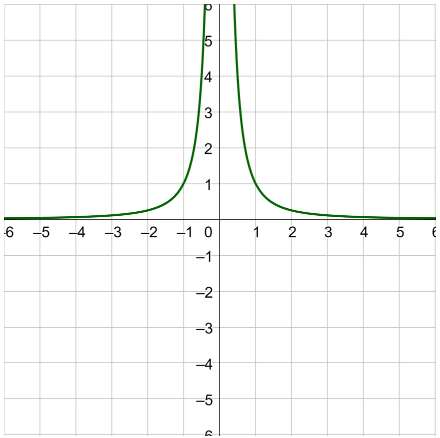

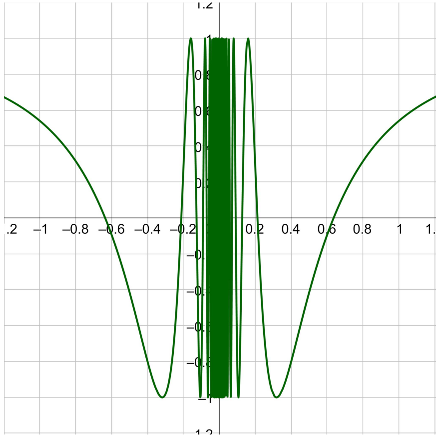

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Limit

- Unique value of a function when its input approaches a particular number from both sides

  $\lim_{x \to c} f(x) = L$

  Eg: $f(x) = \frac{x^2-1}{x-1}$.

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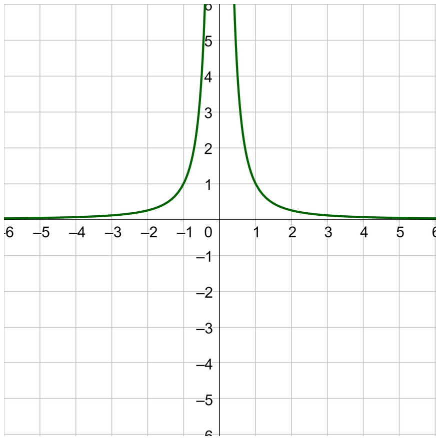
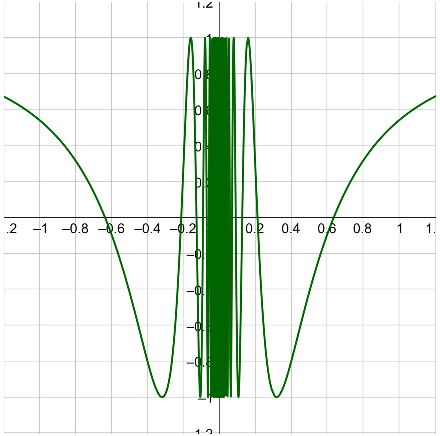
Continuity

- For all values of input to a function, it must satisfy,

  $\lim_{x\to c} f(x) = f(c)$





Characteristics of Data

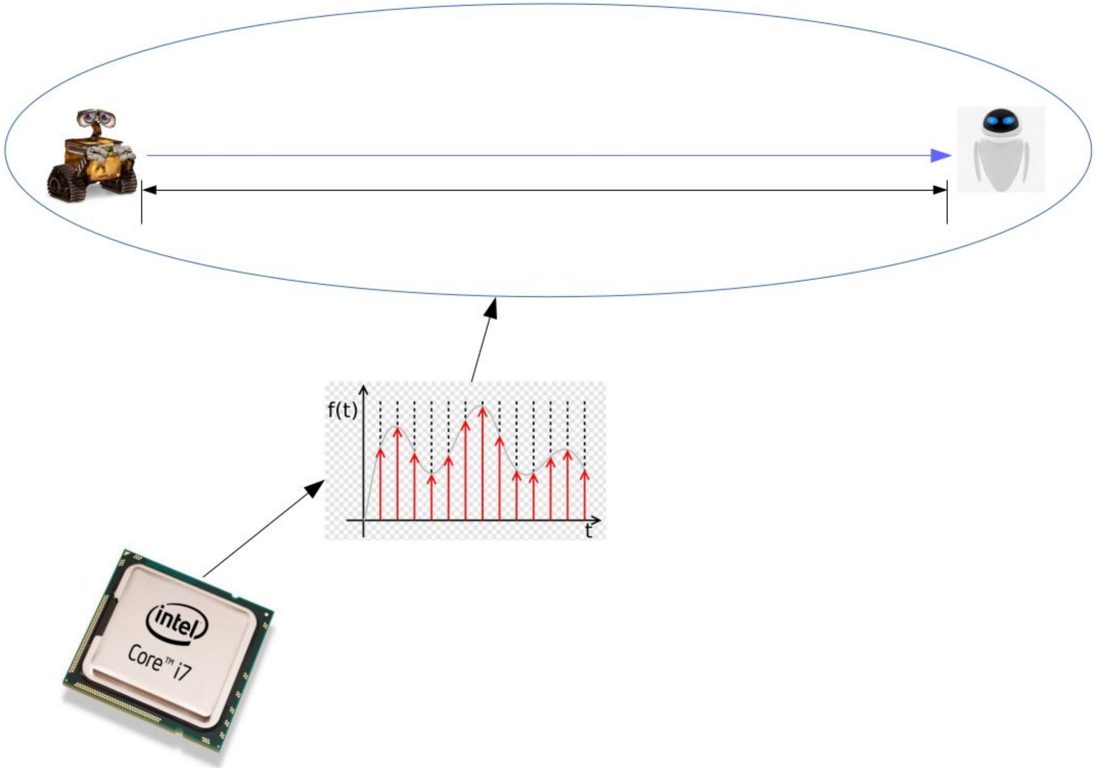
Analog

- continuous

Digital

- discrete

Differential Calculus

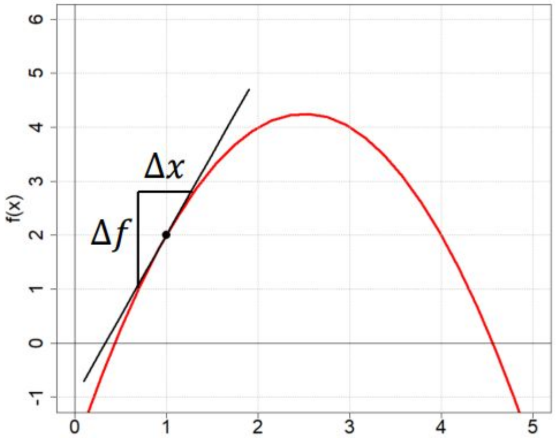


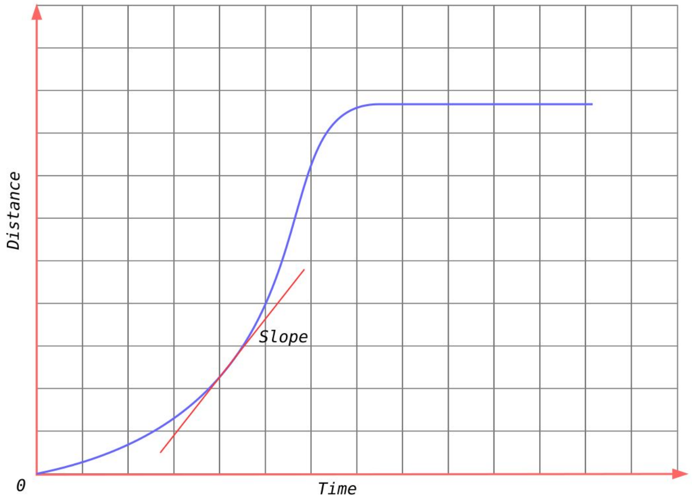
Differential Calculus

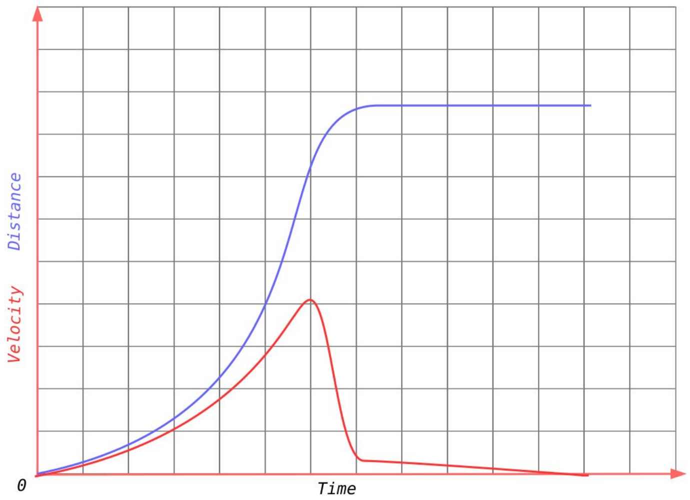
Derivative of a Function

- Rate of change
- In a 2D system, derivatives = slope = tan(θ)

Leibniz notation: $y = x^2;\ \dfrac{dy}{dx} = 2x$







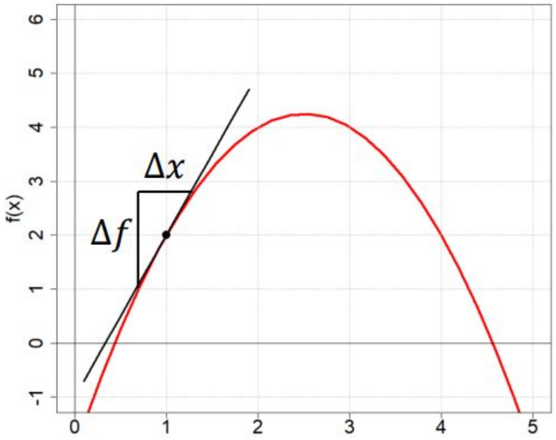
Differential Calculus

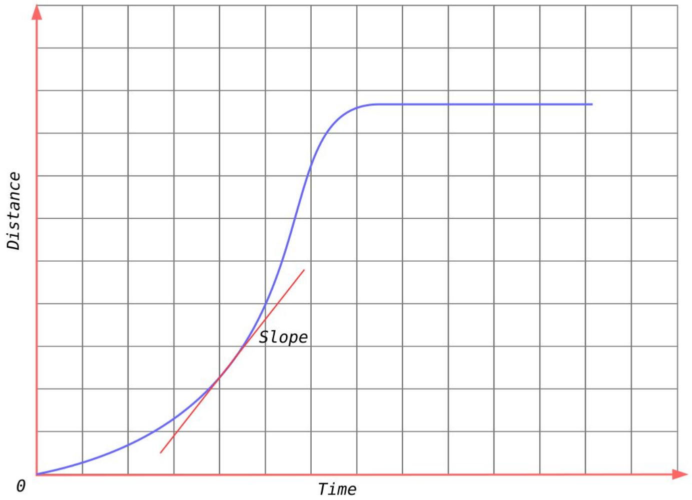
Derivative of a Function

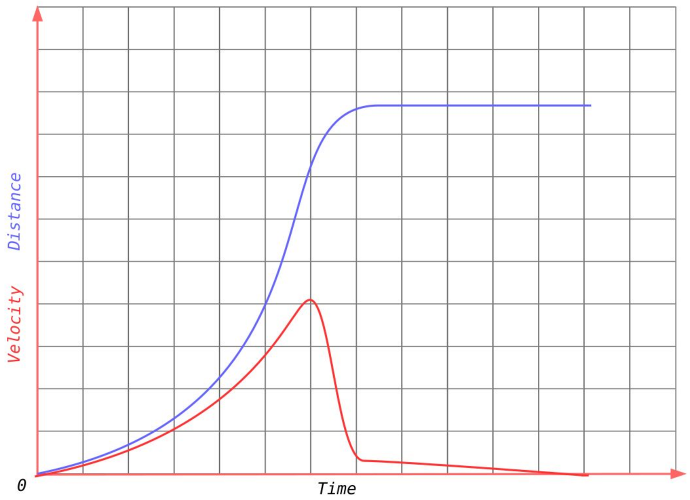
- Rate of change
- In a 2D system, derivatives = slope = tan(θ)
- *Another view:*

  The line itself is the best linear approximation

Leibniz notation: $y=x^2; \quad \frac{dy}{dx}=2x$







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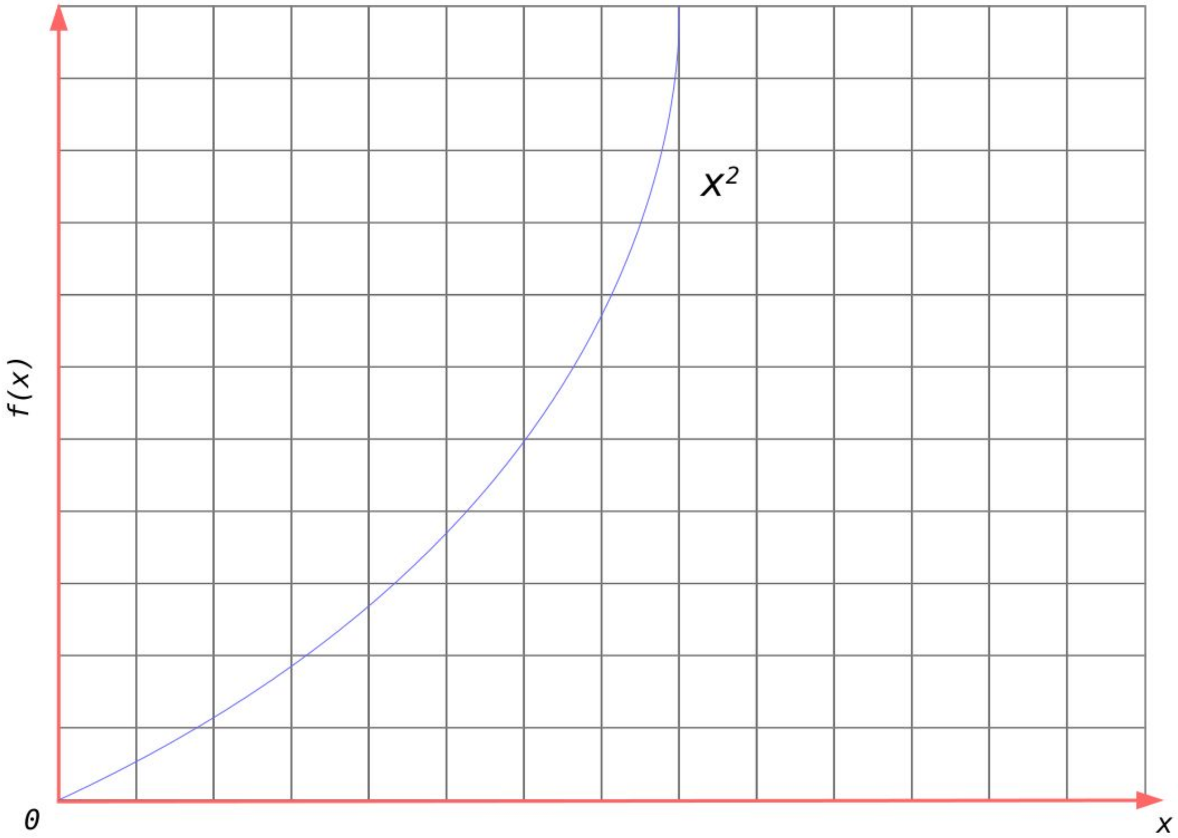
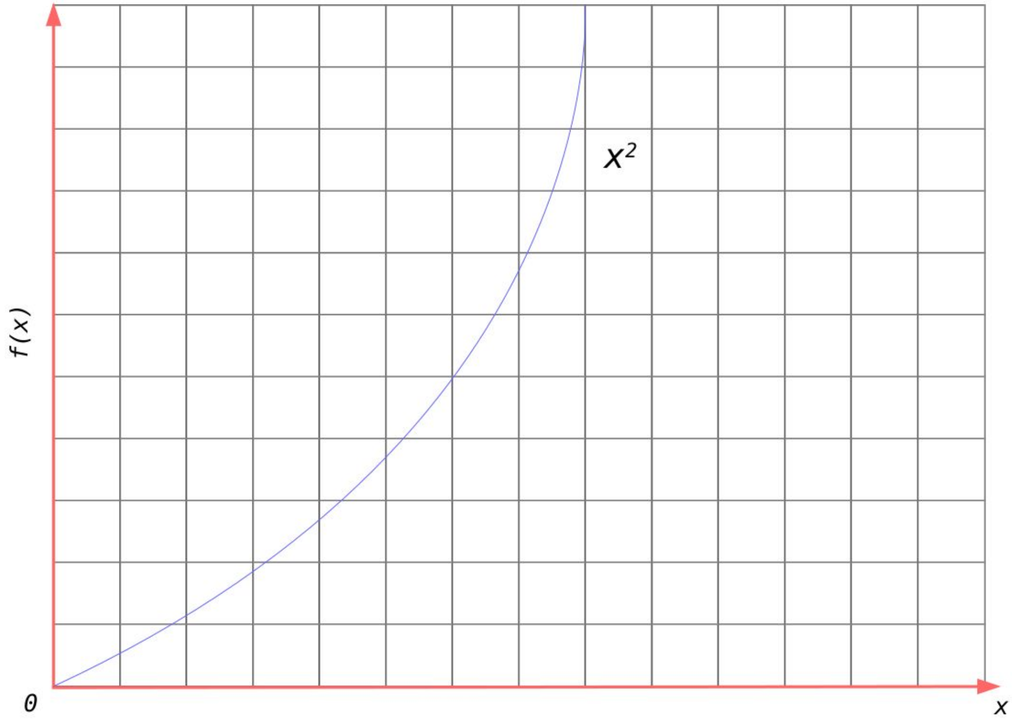


Differential Calculus

How to handle discrete functions?



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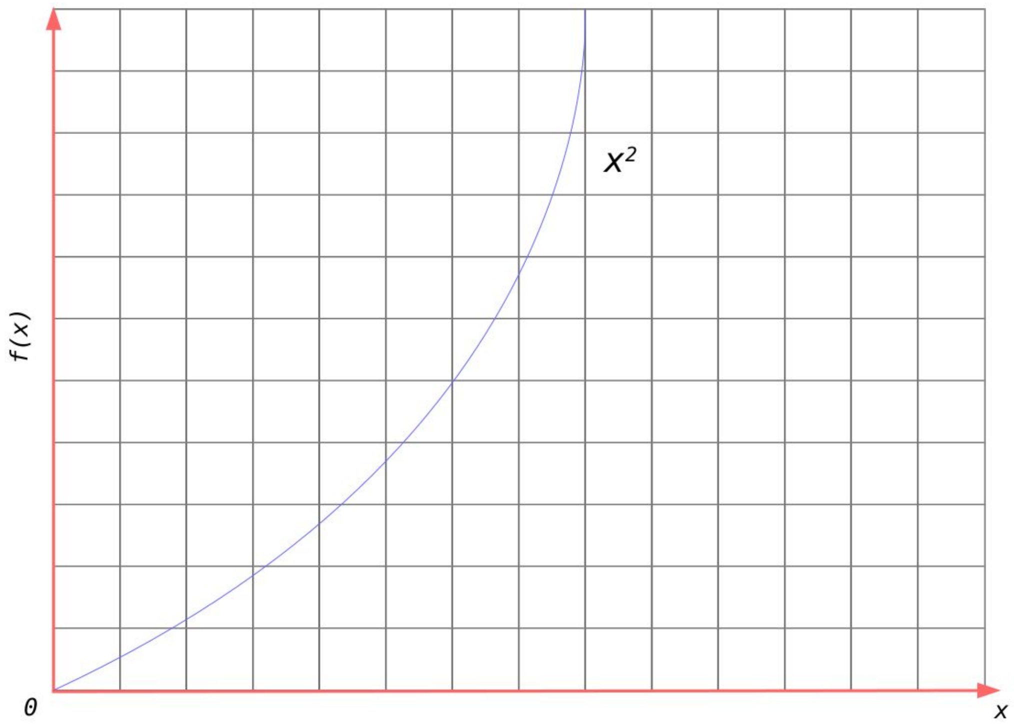
Approximate!

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

h – small, positive, fixed epsilon



Differential Calculus - Rules (single variable)

Addition Rule ⟶ $\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$

Product Rule ⟶ $\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)$

Power Rule ⟶ $\frac{d}{dx}[x^n] = n\cdot x^{n-1}$

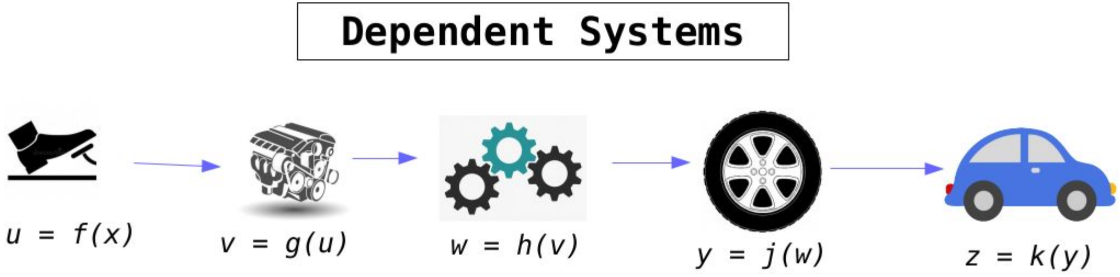
Quotient Rule ⟶ $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot\frac{d}{dx}f(x) - f(x)\cdot\frac{d}{dx}g(x)}{(g(x))^2}$

Derivative of Constant ⟶ $\frac{d}{dx}[c] = 0$

Chain Rule ⟶ $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

Other Rules: For logarithmic and exponential functions

Differential Calculus - Chain Rule



$$\frac{dz}{dx} = \frac{dz}{dy} * \frac{dy}{dw} * \frac{dw}{dv} * \frac{dv}{du} * \frac{du}{dx}$$

Differential Calculus - Multivariable Functions

How to handle derivatives of multivariable functions?

Partial Derivatives: Taking derivative of each variable and holding others as constants

- Notations: $\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$

$$f_x = \frac{\partial f}{\partial x} \qquad f_y = \frac{\partial f}{\partial y}$$

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- Application: *Jacobians*

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \frac{\partial F_m}{\partial x_2} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

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Cross Partial Derivatives

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

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Total Derivatives: if there is a function f(x,y), then the total derivative is represented as sum f partial derivatives of each variable times the derivative of that variable

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In this example, assuming function ‘f’ is not directly dependent on ‘t’ and ‘t’ is independent variable, the total derivative can be represented as,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

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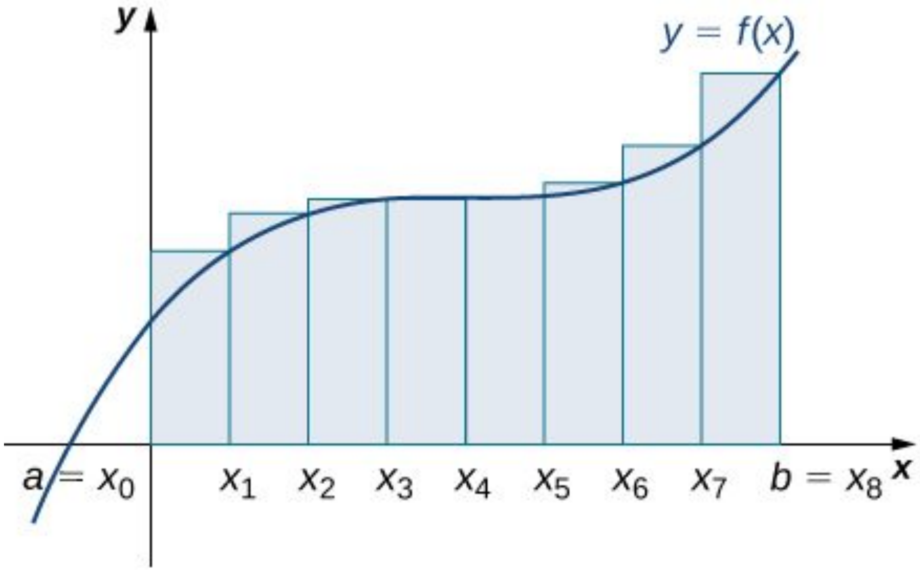
Taylor Series: Approximations of functions. At x=0, this series is also termed as Maclaurin series.

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Integral Calculus

Integration: it is analogous to summation. It is also termed as antiderivative



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- *Indefinite integrals:* bounds are undefined
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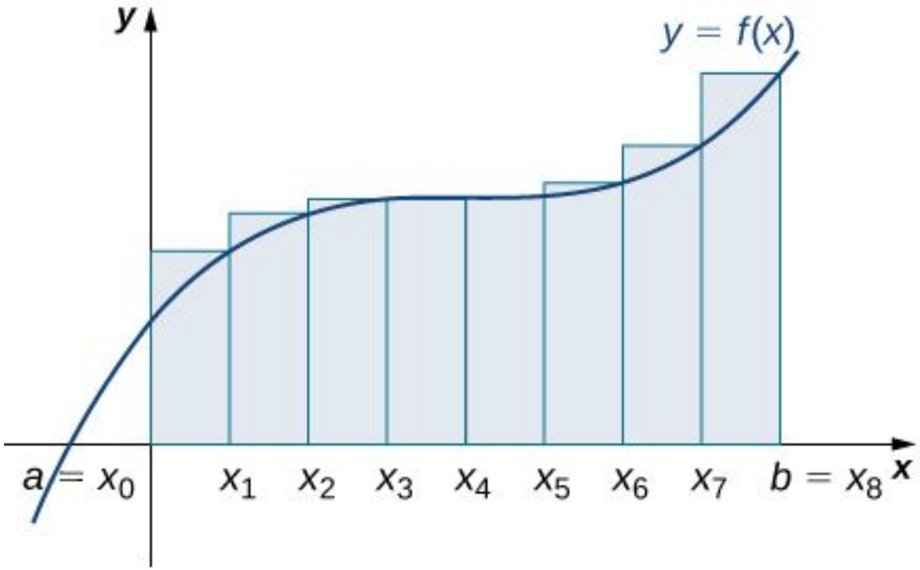

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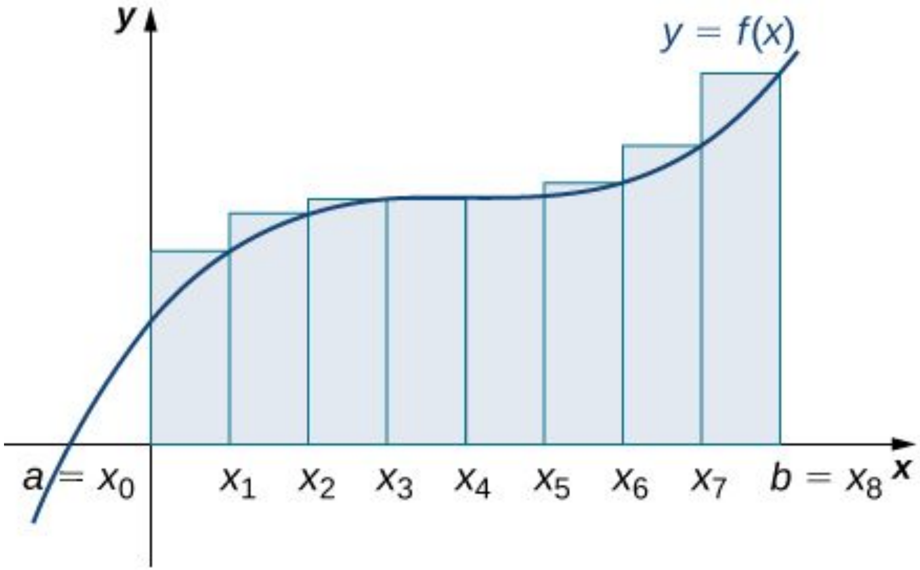
Example:

$y_1 = x^2 + 20$
$y_2 = x^2 + 100$

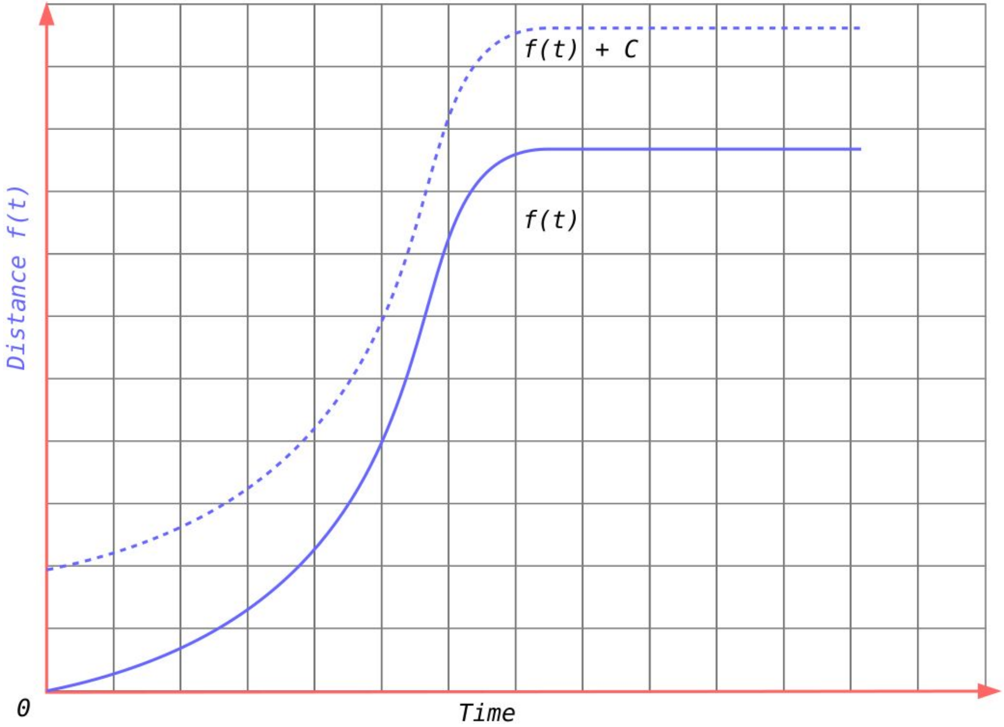
$y'_1 = y'_2 = 2x$

Thus,
$\int 2x \, dx = x^2 + c$



Integral Calculus

Bounds



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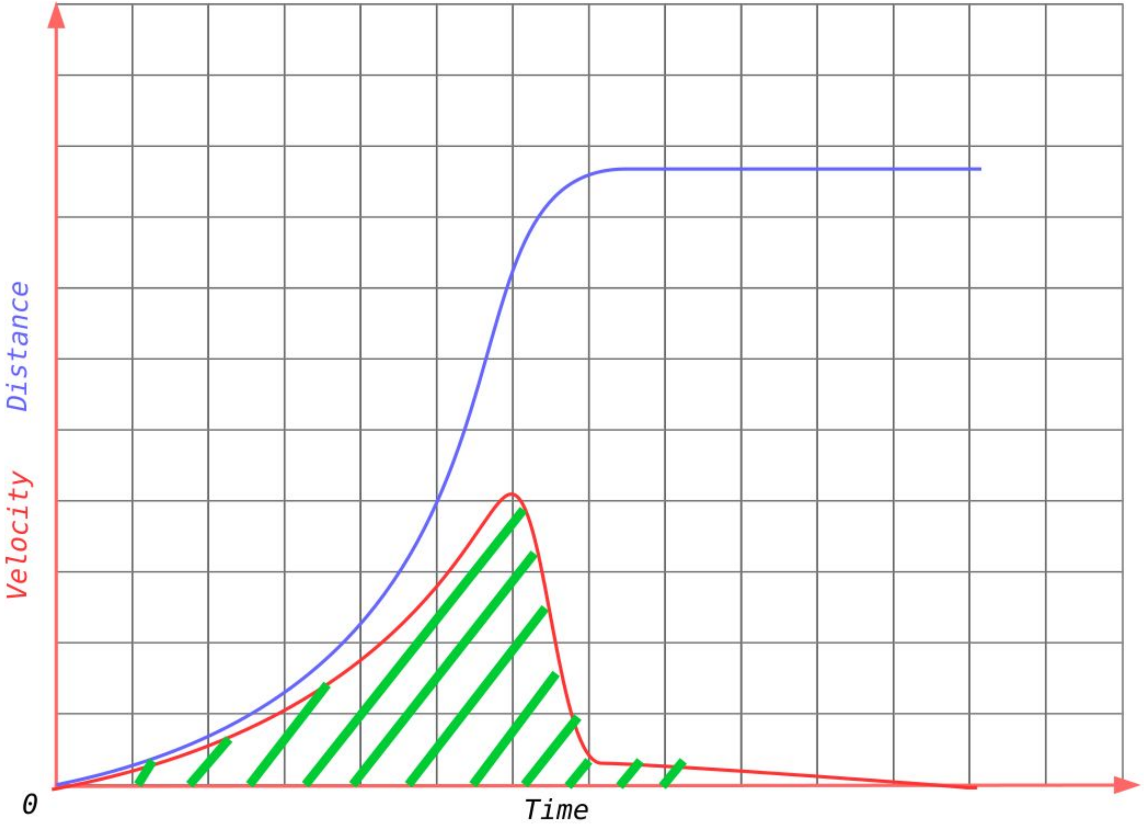

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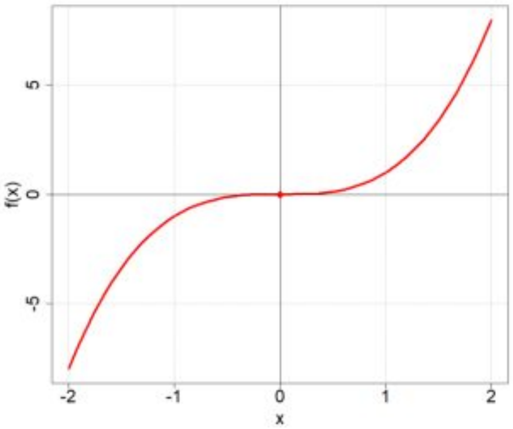
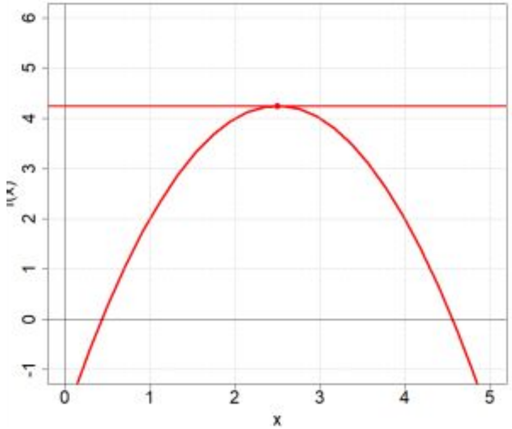
Optimisation

Inflection Point vs Critical Point





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Thank You!