

# Portfolio selection under independent possibilistic information <sup>☆</sup>

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## Abstract

This paper deals with a portfolio selection problem with independently estimated possibilistic return rates. Under such a circumstance, a distributive investment has been regarded as a good solution in the traditional portfolio theory. However, the conventional possibilistic approach yields a concentrated investment solution. Considering the reason why a distributive investment is advocated, a new approach to the possibilistic portfolio selection is proposed. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

The Markowitz model [7] is famous for portfolio selection. In this model, an expected return rate of a bond is treated as a random variable. Stochastic programming is applied so that the solution is obtained by minimizing the variance of the total expected return rate under the constraint that the mean of the total expected return rate is equal to a predetermined value. This model yields a distributive investment solution unless the expected return rates are completely positively dependent on each other. In the traditional portfolio theory, a distributive investment has been regarded as a good policy to reduce the risk.

Possibilistic programming is a known similar approach to stochastic programming. Therefore, an application of possibilistic programming to portfolio selection can be expected. In possibilistic programming approaches, the expected return rates are not treated as random variables but as possibilistic variables. Applying possibilistic programming may have a two-fold advantage [2]: (1) the knowledge of the expert can be easily introduced to the estimation of the return rates and (2) the reduced problem is more tractable than that of the stochastic programming approach. However, most of the previously proposed possibilistic programming approaches treat independent possibilistic variables and thus the application to the portfolio selection do not yield a distributive investment. This contradicts with the traditional portfolio theory.

In this paper, considering how a model yields a distributive investment solution, a new possibilistic programming approach to the portfolio selection is proposed. This approach is based on a regret which the

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decision maker may undertake. First, the previous possibilistic programming approaches are reviewed and their solutions to the portfolio selection problems are discussed. It is shown that concentrated investment solutions are obtained by those approaches. A minimax regret approach to the possibilistic portfolio selection is proposed. It is shown that a distributive investment solution is obtained by this approach, following the problem reduction. Some examples are given in order to compare the solutions obtained by the previous and proposed approaches.

## 2. Possibilistic portfolio selection problem

When return rates of all bonds are known exactly, a portfolio selection problem can be formulated as

$$\begin{aligned} & \text{maximize} && \mathbf{c}^T \mathbf{x}, \\ & \text{subject to} && \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0, \end{aligned} \quad (1)$$

where  $\mathbf{c} = (c_1, c_2, \dots, c_n)^T$ ,  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  and  $\mathbf{e} = (1, 1, \dots, 1)^T$ . The component  $c_i$  represents the return rate for the  $i$ th bond and  $x_i$  shows the investment rate to the  $i$ th bond. Thus the problem is maximizing the total return rate. An optimal solution can be obtained easily as shown in the following theorem.

**Theorem 1.** *An optimal solution to Problem (1) is a concentrated investment on a bond which has the maximum return rate. Namely, a solution  $x_{i^*} = 1$ ,  $x_i = 0$ ,  $\forall i \neq i^*$  where  $c_{i^*} \geq c_i$ ,  $i = 1, 2, \dots, n$  is an optimal solution to Problem (1).*

**Proof.** Trivial.  $\square$

However, in the real setting, one can seldom obtain the return rate without any uncertainty. Thus, the decision makers should make their decisions under uncertainty. Such an uncertainty has been treated as a random variable so far. The problem has been formulated as the following stochastic programming problem:

$$\begin{aligned} & \text{maximize} && \mathcal{C}^T \mathbf{x}, \\ & \text{subject to} && \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0, \end{aligned} \quad (2)$$

where  $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n)^T$  is a random variable vector obeying a multivariate probability distribution which has a mean vector  $\mathbf{m}$  and a covariance matrix  $V$ . To such a problem, Markowitz [7] proposed the following model to determine the investment rates  $\mathbf{x}$ :

$$\begin{aligned} & \text{minimize} && \mathbf{x}^T V \mathbf{x}, \\ & \text{subject to} && \mathbf{m}^T \mathbf{x} \geq z_0, \\ & && \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0, \end{aligned} \quad (3)$$

where  $z_0$  is a predetermined desirable expected return rate. In the original Markowitz model, the first inequality constraint  $\mathbf{m}^T \mathbf{x} \geq z_0$  is replaced with an equality constraint  $\mathbf{m}^T \mathbf{x} = z_0$ . This change enables the model to generate a suitable solution for an underestimated expected return rate  $z_0$ .

To formulate Problem (3), we need to estimate the probability distribution, strictly speaking, a mean vector  $\mathbf{m}$  and a covariance matrix  $V$ . This is not an easy task when the number of bonds is big. Therefore, several simplified methods, sacrificing generality, have been proposed. Moreover, it is difficult to reflect the unquantifiable factors such as the knowledge of experts, the trend of public opinion, and so on. Furthermore, even though the probability distribution can be estimated, there is no guarantee that the return rates truly obey it.

A possibility distribution is known as an alternative to a probability distribution, where though the uncertain values are estimated substantially, the unquantifiable factors such as the knowledge of experts can be reflected easily. Thus, it may be worthwhile to introduce the possibility distribution to the portfolio selection problem. In this paper, we discuss about the possibilistic portfolio selection.

Treating each return rate as a possibilistic variable  $\gamma_i$  restricted by a possibility distribution  $\pi_{C_i}$ , we have the following possibilistic portfolio selection problem:

$$\begin{aligned} & \text{maximize} && \boldsymbol{\gamma}^T \mathbf{x}, \\ & \text{subject to} && \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0, \end{aligned} \quad (4)$$

where  $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \dots, \gamma_n)^T$  is a possibilistic variable vector restricted by a multivariate possibility distribution  $\pi_C$ . Since most approaches assume the independence among possibilistic variables in the possibilistic programming, we also assume the independence in order to show the incompatibility with

the traditional discussion in the portfolio theory. However, the proposed approach can be useful with some changes even when this assumption is removed. Under this assumption the multivariate possibility distribution  $\pi_C$  can be expressed as

$$\pi_C(\mathbf{c}) = \min_{i=1,2,\dots,n} \pi_{C_i}(c_i). \quad (5)$$

For the sake of simplicity, we assume that  $\pi_C$  is upper semi-continuous, in other words, each  $\pi_{C_i}$  is upper semi-continuous. Moreover, we assume

$$\lim_{c_i \rightarrow -\infty} \pi_{C_i}(c_i) = \lim_{c_i \rightarrow +\infty} \pi_{C_i}(c_i) = 0, \quad i = 1, 2, \dots, n. \quad (6)$$

### 3. Solutions by the conventional possibilistic programming approaches

Various approaches have been proposed to a possibilistic programming problem. In this section, we apply major possibilistic programming approaches [3, 4] to the possibilistic portfolio selection problem (4). We assume that the decision maker's attitude is uncertainty (risk) averse. From this point of view, a proper model is introduced in each approach.

#### 3.1. Fractile approach

Given appropriate level  $h^0 \in (0, 1]$ , Problem (4) is formulated so as to maximize the fractile  $z$  under a constraint that a necessity measure of the event that the objective function value is not less than  $z$  is greater than or equal to  $h^0$ :

$$\begin{aligned} &\text{maximize} && z, \\ &\text{subject to} && N_C(\{\mathbf{c} \mid \mathbf{c}^T \mathbf{x} \geq z\}) \geq h^0, \\ &&& \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0, \end{aligned} \quad (7)$$

where  $N_C$  is a necessity measure under a possibility distribution  $\pi_C$  and defined as follows for a set  $D$ :

$$N_C(D) = \begin{cases} \inf_{\mathbf{c} \notin D} 1 - \pi_C(\mathbf{c}) & \text{if } D \text{ is not a universal set,} \\ 1 & \text{if } D \text{ is a universal set.} \end{cases} \quad (8)$$

Thus,  $N_C(\{\mathbf{c} \mid \mathbf{c}^T \mathbf{x} \geq z\})$  shows a necessity degree to what extent the objective function value is not less than  $z$  and represented as

$$N_C(\{\mathbf{c} \mid \mathbf{c}^T \mathbf{x} \geq z\}) = \inf_{\mathbf{c}^T \mathbf{x} < z} 1 - \pi_C(\mathbf{c}). \quad (9)$$

Under the possibilistic independence (5), Problem (7) is reduced to the following linear programming problem [4]:

$$\begin{aligned} &\text{maximize} && \mathbf{c}^L(1 - h^0)^T \mathbf{x}, \\ &\text{subject to} && \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0, \end{aligned} \quad (10)$$

where  $\mathbf{c}^L(\cdot) = (c_1^L(\cdot), c_2^L(\cdot), \dots, c_n^L(\cdot))^T$  and

$$c_i^L(h) = \inf\{q \mid \pi_{C_i}(q) > h\}. \quad (11)$$

#### 3.2. Modality optimization approach

Given a target value  $z^0$ , Problem (4) is formulated so as to maximize a necessity measure of the event that the objective function value is not less than  $z^0$ :

$$\begin{aligned} &\text{maximize} && N_C(\{\mathbf{c} \mid \mathbf{c}^T \mathbf{x} \geq z^0\}), \\ &\text{subject to} && \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0. \end{aligned} \quad (12)$$

This problem is reduced to the following nonlinear programming problem [4]:

$$\begin{aligned} &\text{maximize} && h, \\ &\text{subject to} && \mathbf{c}^L(1 - h)^T \mathbf{x} \geq z^0, \\ &&& \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0. \end{aligned} \quad (13)$$

#### 3.3. Spread minimization approach

Define a representative vector  $\hat{\mathbf{c}}$  of the possibility distribution  $\pi_C$ . Given  $h^0 \in (0, 1]$  and  $z^0$ , the width of the  $h^0$ -level set  $[C^T \mathbf{x}]_{h^0} = \{y \mid \pi_{C^T \mathbf{x}}(y) \geq h^0\}$  of the possibility distribution  $\pi_{C^T \mathbf{x}}$  on the objective function value can be minimized under the constraint  $\hat{\mathbf{c}}^T \mathbf{x} \geq z^0$ , where  $\pi_{C^T \mathbf{x}}$  is calculated by the extension principle [1], i.e.,

$$\pi_{C^T \mathbf{x}}(y) = \sup_{\mathbf{c}^T \mathbf{x} = y} \pi_C(\mathbf{c}). \quad (14)$$

Problem (4) is formulated as

$$\begin{aligned} & \text{minimize} && w, \\ & \text{subject to} && \max_{y^R, y^L \in [C^T x]_{h^0}} (y^R - y^L) \leq w, \\ & && \hat{c}^T x \geq z^0, \\ & && e^T x = 1, \quad x \geq 0, \end{aligned} \quad (15)$$

and reduced to the following linear programming problem [4]:

$$\begin{aligned} & \text{minimize} && (c_R(h^0)^T - c_L(h^0)^T)x, \\ & \text{subject to} && \hat{c}^T x \geq z^0, \\ & && e^T x = 1, \quad x \geq 0, \end{aligned} \quad (16)$$

where  $c_L(\cdot) = (c_{1L}(\cdot), c_{2L}(\cdot), \dots, c_{nL}(\cdot))^T$ ,  $c_R(\cdot) = (c_{1R}(\cdot), c_{2R}(\cdot), \dots, c_{nR}(\cdot))^T$  and

$$\begin{aligned} c_{iL}(h) &= \inf\{q \mid \pi_{C_i}(q) \geq h\}, \\ c_{iR}(h) &= \sup\{q \mid \pi_{C_i}(q) \geq h\}. \end{aligned} \quad (17)$$

Since the variance of a probability distribution corresponds to the spread of a possibility distribution, this model is a counterpart of the Markowitz model.

We have the following theorem.

**Theorem 2.** *A concentrated investment solution such that  $x_i = 1$  for some  $i \in \{1, 2, \dots, n\}$  is an optimal solution to Problem (10). The same assertion is valid for Problem (13). A semi-concentrated investment solution such that  $x_i + x_j = 1$  for some  $i, j \in \{1, 2, \dots, n\}$  is an optimal solution to Problem (16).*

**Proof.** The first and third assertions of this theorem are obvious when taking into account the number of constraints excluding non-negativity constraints of the problem. Indeed, Problem (10) is a linear programming problem with one constraint and Problem (16) is a linear programming problem with two constraints. Since a linear programming problem can be solved by the simplex method, Problems (10) and (16) have one and two basic variables, respectively. Hence concentrated and semi-concentrated investment solutions are optimal to Problems (10) and (16), respectively. Now, we prove the second assertion. Let  $(\hat{x}, \hat{h})$  be an

optimal solution to Problem (13). Consider an optimal solution  $x^*$  to a linear programming problem,

$$\begin{aligned} & \text{maximize} && c^L(1 - \hat{h})^T x, \\ & \text{subject to} && e^T x = 1, \quad x \geq 0. \end{aligned} \quad (18)$$

The solution  $(x^*, \hat{h})$  is also an optimal solution to Problem (13). By the same way of the proof of the first and second assertions, an optimal solution to Problem (18) is a concentrated investment solution.  $\square$

Theorem 2 shows that if each of Problems (10), (13) and (16) has a unique solution, no distributive investment solution is obtained. Moreover, when those problems are solved by the simplex method, the conventional possibilistic programming approaches to the portfolio selection do not yield a solution compatible with the traditional portfolio theory.

#### 4. A new possibilistic programming approach

Let us discuss why a distributive investment solution under independent return rate assumption is preferred by a decision maker who has an uncertainty (risk) averse attitude. We can observe at least the following two reasons:

- (a) *Property of a measure.* Consider the event that the total return rate is not less than a certain value. When the measure of the event under a distributive investment solution is greater than that under a concentrated investment solution, the distributive investment solution should be preferable. In other words, the uncertainty is decreased by distribution to many bonds.
- (b) *The worst regret criterion.* Suppose that the decision maker has invested his money in a bond according to a concentrated investment solution. If the return rate of another bond becomes better than that of the invested bond as a result, the decision maker may feel a regret. At the decision making stage, we cannot know the return rate determined in the future. Thus, any concentrated investment solution may bring a regret to the decision maker. In this sense, if the decision maker is interested in minimizing the worst regret which may be undertaken, a distributive investment solution must be preferable.

Markowitz model [7] and the other stochastic programming approaches yield a distributive investment solution because of (a). Indeed, we have

$$\text{Prob}(\lambda X_1 + (1 - \lambda)X_2 \geq k) > \text{Prob}(X_i \geq k), \quad \forall \lambda \in (0, 1), \quad i = 1, 2, \quad (19)$$

when independent random variables  $X_1$  and  $X_2$  obey the same marginal normal (probability) distribution. Moreover, we have

$$\text{Var}(\lambda X_1 + (1 - \lambda)X_2) < \text{Var}(X_i), \quad \forall \lambda \in (0, 1), \quad i = 1, 2, \quad (20)$$

where  $\text{Var}(X)$  is the variance, i.e., an uncertainty criterion, of a random variable  $X$ .

In possibilistic programming approaches, we could not obtain a distributive investment solution since possibility and necessity measures do not have the property mentioned in (a). For possibility and necessity measures, we have

$$\begin{aligned} \text{Pos}(\lambda X_1 + (1 - \lambda)X_2 \geq k) &= \text{Pos}(X_i \geq k), \\ \forall \lambda \in [0, 1], \quad i &= 1, 2, \\ \text{Nes}(\lambda X_1 + (1 - \lambda)X_2 \geq k) &= \text{Nes}(X_i \geq k), \\ \forall \lambda \in [0, 1], \quad i &= 1, 2, \end{aligned} \quad (21)$$

where  $X_1$  and  $X_2$  are mutually independent possibilistic variables restricted by the same marginal possibility distribution. Moreover we have

$$\text{Spd}(\lambda X_1 + (1 - \lambda)X_2) = \text{Spd}(X_i), \quad \forall \lambda \in [0, 1], \quad i = 1, 2, \quad (22)$$

where  $\text{Spd}(X)$  is the spread of the possibility distribution  $\pi_X$  restricts a possibilistic variable  $X$ , i.e.,

$$\text{Spd}(X) = \sup\{r \mid \pi_X(r) > 0\} - \inf\{r \mid \pi_X(r) > 0\}. \quad (23)$$

Hence, the conventional possibilistic programming approaches fail to yield a distributive investment solution without introducing the concept of the regret. In what follows, a regret is introduced into the possibilistic portfolio selection problem (4).

Suppose that a decision maker is informed about the determined return rates  $\mathbf{c}$  after he/she has invested

his/her money in bonds according to a feasible solution  $\mathbf{x}$  to (4), he/she will have a regret  $r(\mathbf{x}; \mathbf{c})$  which can be quantified as

$$r(\mathbf{x}; \mathbf{c}) = \max_{\substack{\mathbf{y} \\ \mathbf{e}^T \mathbf{y} = 1, \mathbf{y} \geq 0}} F(\mathbf{c}^T \mathbf{y}, \mathbf{c}^T \mathbf{x}), \quad (24)$$

where  $F: D_1 \times D_2 \rightarrow \mathbf{R}$  ( $D_1, D_2 \subseteq \mathbf{R}$ ) is a continuous function such that  $F(\cdot, r)$  is strictly increasing and  $F(r, \cdot)$  is strictly decreasing. Those properties of the function  $F$  reflect the fact that the right-hand side of Eq. (24) evaluates the regret  $r(\mathbf{x}; \mathbf{c})$ . Since  $F$  is continuous, so  $r(\cdot; \cdot)$  is.

At the decision making stage, the decision maker cannot know the return rate  $\mathbf{c}$  determined in the future but a possibility distribution  $\pi_C(\mathbf{c})$ . By the extension principle [1], a possibility distribution  $\pi_{R(\mathbf{x})}$  on regrets can be defined as

$$\pi_{R(\mathbf{x})}(r) = \sup_{\substack{\mathbf{c} \\ r = r(\mathbf{x}; \mathbf{c})}} \pi_C(\mathbf{c}). \quad (25)$$

We regard the possibilistic portfolio selection problem (1) as a problem of minimizing a regret  $R(\mathbf{x})$  with a possibility distribution  $\pi_{R(\mathbf{x})}$ , i.e.,

$$\begin{aligned} \text{minimize} \quad & R(\mathbf{x}), \\ \text{subject to} \quad & \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0. \end{aligned} \quad (26)$$

Since  $R(\mathbf{x})$  is a possibilistic variable restricted by a possibility distribution  $\pi_{R(\mathbf{x})}$ , (26) is also a possibilistic programming problem. We apply the fractile model to Problem (26) so that, given  $h^0$ , Problem (26) is formulated as

$$\begin{aligned} \text{minimize} \quad & z, \\ \text{subject to} \quad & N_{R(\mathbf{x})}(\{r \mid r \leq z\}) \geq h^0, \\ & \mathbf{e}^T \mathbf{x} = 1, \quad \mathbf{x} \geq 0. \end{aligned} \quad (27)$$

Now we are in position to transform Problem (27) to a linear programming problem. From Eqs. (8) and (25), we have

$$\begin{aligned} N_{R(\mathbf{x})}(\{r \mid r \leq z\}) &= \inf_{r > z} (1 - \pi_{R(\mathbf{x})}(r)) \\ &= \inf_{r > z} \left( 1 - \sup_{\substack{\mathbf{c} \\ r = r(\mathbf{x}; \mathbf{c})}} \pi_C(\mathbf{c}) \right) \\ &= \inf_{\substack{\mathbf{c} \\ r(\mathbf{x}; \mathbf{c}) > z}} (1 - \pi_C(\mathbf{c})). \end{aligned} \quad (28)$$

Thus,  $N_{R(x)}(\{r \mid r \leq z\}) \geq h^0$  is equivalent to

$$\pi_C(\mathbf{c}) > 1 - h^0 \text{ implies } r(\mathbf{x}; \mathbf{c}) \leq z. \quad (29)$$

By the continuity of  $r(\mathbf{x}; \mathbf{c})$  with respect to  $\mathbf{c}$ , Eq. (29) becomes equivalent to

$$\sup_{\mathbf{c} \in (C)_{1-h^0}} r(\mathbf{x}; \mathbf{c}) \leq z, \quad (30)$$

where  $(C)_{1-h^0}$  is a  $(1 - h^0)$ -level set, i.e.,

$$(C)_{1-h^0} = \{\mathbf{c} \mid \pi_C(\mathbf{c}) > 1 - h^0\}. \quad (31)$$

A closure of the  $(1 - h^0)$ -level set,  $\text{cl}(C)_{1-h^0}$ , can be expressed as

$$\begin{aligned} \text{cl}(C)_{1-h^0} &= \{\mathbf{c} = (c_1, c_2, \dots, c_n) \mid c_i^L(1 - h^0) \\ &\leq c_i \leq c_i^R(1 - h^0), i = 1, 2, \dots, n\}, \end{aligned} \quad (32)$$

where  $c_i^L(\cdot)$  is a function defined by Eq. (11) and  $c_i^R(\cdot)$  is defined by

$$c_i^R(h) = \sup\{q \mid \pi_{C_i}(q) > h\}. \quad (33)$$

By the continuity of  $r(\mathbf{x}; \mathbf{c})$  with respect to  $\mathbf{c}$ , the supremum, ‘sup’, and the  $(1 - h^0)$ -level set,  $(C)_{1-h^0}$ , can be replaced with the maximum, ‘max’, and the closure,  $\text{cl}(C)_{1-h^0}$  in Eq. (30), respectively. Hence, we have

$$N_{R(x)}(\{r \mid r \leq z\}) \geq h^0 \Leftrightarrow \max_{\mathbf{c} \in \text{cl}(C)_{1-h^0}} r(\mathbf{x}; \mathbf{c}) \leq z. \quad (34)$$

Introducing Eqs. (34) and (24) into Eq. (27), we have

$$\begin{aligned} &\text{minimize } z, \\ &\text{subject to } \max_{\substack{\mathbf{c}, \mathbf{y} \\ \mathbf{c} \in \text{cl}(C)_{1-h^0} \\ \mathbf{e}^T \mathbf{y} = 1, \mathbf{y} \geq 0}} F(\mathbf{c}^T \mathbf{y}, \mathbf{c}^T \mathbf{x}) \leq z, \\ &\mathbf{e}^T \mathbf{x} = 1, \mathbf{x} \geq 0. \end{aligned} \quad (35)$$

Since  $\text{cl}(C)_{1-h^0} = [c_i^L(1 - h^0), c_i^R(1 - h^0)]$ , Problem (35) is a minimax problem with linear constraints. From the assumption of  $F$  and Theorem 1, we have

$$\max_{\substack{\mathbf{c}, \mathbf{y} \\ \mathbf{c} \in \text{cl}(C)_{1-h^0} \\ \mathbf{e}^T \mathbf{y} = 1, \mathbf{y} \geq 0}} F(\mathbf{c}^T \mathbf{y}, \mathbf{c}^T \mathbf{x})$$

$$\begin{aligned} &= \max_{\mathbf{c} \in \text{cl}(C)_{1-h^0}} F\left(\max_{\mathbf{e}^T \mathbf{y} = 1, \mathbf{y} \geq 0} \mathbf{c}^T \mathbf{y}, \mathbf{c}^T \mathbf{x}\right) \\ &= \max_{i=1,2,\dots,n} \max_{\mathbf{c} \in \text{cl}(C)_{1-h^0}} F(c_i, \mathbf{c}^T \mathbf{x}), \end{aligned} \quad (36)$$

where  $\mathbf{e}_i$  is a unit vector whose  $i$ th component is one. Hence, Problem (35) is reduced to

$$\begin{aligned} &\text{minimize } z, \\ &\text{subject to } \max_{\mathbf{c} \in \text{cl}(C)_{1-h^0}} F(c_i, \mathbf{c}^T \mathbf{x}) \leq z, i = 1, 2, \dots, n, \\ &\mathbf{e}^T \mathbf{x} = 1, \mathbf{x} \geq 0. \end{aligned} \quad (37)$$

Problem (37) can be solved by a relaxation procedure and nonlinear programming techniques. Moreover, we consider the case when  $F$  can be expressed as

$$F(r_1, r_2) = \varphi(f(r_1)r_2 + g(r_1)), \quad (38)$$

where  $\varphi: \mathbf{R} \rightarrow \mathbf{R}$  is strictly increasing,  $f: D_1 \rightarrow \mathbf{R}$  and  $g: D_2 \rightarrow \mathbf{R}$  satisfies

- (a)  $f(r) < 0, \forall r \in [L, R]$ ,
- (b)  $f'(r_1)r_2 + g'(r_1) > 0, \forall (r_1, r_2) \in [L, R] \times [L, R]$ ,
- (c) for all  $i \in \{1, 2, \dots, n\}$ ,

$$\inf_{\substack{\mathbf{c}, \mathbf{x} \\ \mathbf{c} \in (C)_0 \\ \mathbf{e}^T \mathbf{x} = 1, \mathbf{x} \geq 0}} (f'(c_i)\mathbf{c}^T \mathbf{x} + g'(c_i) + f(c_i)x_i) \geq 0.$$

$L$  and  $R$  are defined by

$$L = \min_{i=1,2,\dots,n} c_i^L(0), \quad R = \max_{i=1,2,\dots,n} c_i^R(0). \quad (39)$$

When  $F$  is represented by Eq. (38), Problem (37) can be written as

$$\begin{aligned} &\text{minimize } q, \\ &\text{subject to } \max_{\mathbf{c} \in \text{cl}(C)_{1-h^0}} f(c_i)\mathbf{c}^T \mathbf{x} + g(c_i) \leq q, \\ &i = 1, 2, \dots, n, \\ &\mathbf{e}^T \mathbf{x} = 1, \mathbf{x} \geq 0. \end{aligned} \quad (40)$$

Let  $Q = \{\varphi(r) \mid r \in \mathbf{R}\} \subseteq \mathbf{R}$  and  $\varphi^{-1}: Q \rightarrow \mathbf{R}$  be the inverse function of  $\varphi$ . We have

$$\begin{aligned} &\frac{\partial \varphi^{-1}(F(c_i, \mathbf{c}^T \mathbf{x}))}{\partial c_j} \\ &= \begin{cases} f(c_i)x_j \leq 0 & \text{if } i \neq j, \\ f'(c_i)\mathbf{c}^T \mathbf{x} + g'(c_i) + f(c_i)x_i \geq 0 & \text{if } i = j. \end{cases} \end{aligned} \quad (41)$$

Table 1  
Examples of  $f$ ,  $g$  and  $\varphi$

No.	Name	$f(r)$	$g(r)$	$\varphi(r)$
1	Usual minimax regret	$-1$	$r$	$r$
2	Minimax regret rate	$-\frac{1}{1+r}$	$\frac{r}{1+r}$	$r$
3	Linear combination of 1 and 2 ( $\lambda_1, \lambda_2 \geq 0$ )	$-\frac{\lambda_1(1+r) + \lambda_2}{1+r}$	$\frac{\lambda_1 r^2 + (\lambda_1 + \lambda_2)r}{1+r}$	$r$

Hence, Problem (40) can be reduced to the following linear programming problem:

minimize  $q$ ,

subject to

$$f(c_i^R(1-h^0)) \left( \sum_{\substack{j=1 \\ j \neq i}}^n c_j^L(1-h^0)x_j + c_i^R(1-h^0)x_i \right) \leq q - g(c_i^R(1-h^0)), \quad i = 1, 2, \dots, n,$$

$$e^T x = 1, \quad x \geq 0. \quad (42)$$

Problem (42) has  $(n+1)$  constraints excluding non-negativity constraints. Therefore, even if a simplex method is applied,  $(n+1)$  variables can take positive values. This means that Problem (42) can yield a distributive investment solution.

Before describing numerical examples, we show some meaningful combinations of  $f$ ,  $g$  and  $h$  in Table 1. The first one is the usual minimax regret model [5] where the worst regret of  $x$  under a return rate vector  $c$  is defined by the difference between the optimal total return rate with respect to  $c$  and the obtained total return rate  $c^T x$ . The second one is a regret rate model which is equivalent with the achievement rate model [6]. In the regret rate model, the worst regret of  $x$  under a return rate vector  $c$  is defined by the ratio of the difference between the optimal total return rate with respect to  $c$  and the obtained total return rate  $c^T x$  to the optimal total income rate. Note that, because of the negativity of  $f(r)$ , for all  $r \in [L, R]$ , we cannot consider the ratio to the optimal total return rate. The third one is the non-negative linear combination of the first and second ones.

## 5. Numerical examples

In order to see what solution is obtained from Problem (42), 3 numerical examples are given in what follows. For comparison, not only the proposed approach but also a stochastic and the previous possibilistic programming approaches are applied to each example. In those examples, we have 5 bonds whose return rates are restricted by the following type of possibility distributions with a center  $c_i^c$  and a spread  $w_i$ :

$$\pi_{c_i}(q) = \exp\left(-\frac{(q - c_i^c)^2}{w_i}\right). \quad (43)$$

As the corresponding probability distribution  $p_{\mathcal{C}_i}$ , we consider the following normal distribution  $N(c_i^c, \sqrt{w_i}/2)$ :

$$p_{\mathcal{C}_i}(q) = \frac{1}{\sqrt{\pi w_i}} \exp\left(-\frac{(q - c_i^c)^2}{w_i}\right). \quad (44)$$

We assume the independence among return rates. The joint possibility distribution is given as in Eq. (5) and the joint probability distribution is given as

$$p_{\mathcal{C}}(c) = \prod_{i=1}^n p_{\mathcal{C}_i}(c_i). \quad (45)$$

Since  $p_{\mathcal{C}_i}$  is a normal distribution,  $p_{\mathcal{C}}$  becomes a multivariate normal distribution with a mean vector  $m = (c_1^c, c_2^c, \dots, c_n^c)$  and a covariance matrix

$$V = \begin{pmatrix} w_1/2 & 0 & 0 & \dots & 0 \\ 0 & w_2/2 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & w_n/2 \end{pmatrix}. \quad (46)$$

Table 2  
Solutions for Example 1

Approach	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
Markowitz	0.061225	0.081633	0.122449	0.244898	0.489796
Fractile	0	0	0	0	1
Modality	0	0	0	0	1
Spread	0	0	0	0	1
Regret	0.25	0.211325	0.146446	0	0.392229

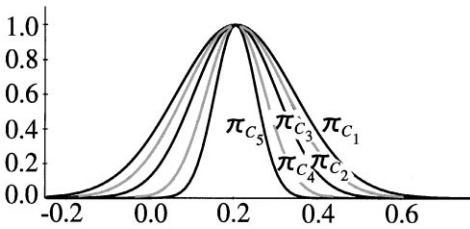


Fig. 1. Five bonds for Example 1.

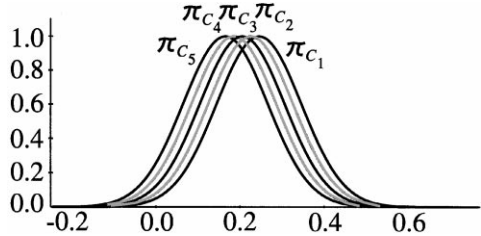


Fig. 2. Five bonds for Example 2.

In the following examples, we assume that functions  $f$ ,  $g$  and  $\varphi$  are defined by those in the first combination in Table 1, i.e., the usual minimax regret model.

**Example 1.** Let us consider five bonds whose return rates are depicted as possibility distributions in Fig. 1. All  $c_i^c$ 's are equal to 0.2 but  $w_i$  decreases as  $i$  increases:

$$w_1 = 0.04, \quad w_2 = 0.03, \quad w_3 = 0.02, \\ w_4 = 0.01 \quad \text{and} \quad w_5 = 0.005.$$

The optimal solutions to Problems (3), (10), (13), (16) and (42) are obtained as shown in Table 2 with setting  $z^0 = 0.18$  and  $h^0 = 0.9$ . Whereas the previous possibilistic programming approaches, i.e., the fractile approach, the modality optimization approach and the spread minimization approach, yield a concentrated investment on the 5th bond, the proposed possibilistic programming approach yields a distributive investment on the 1st, 2nd, 3rd and 5th bonds. However, the solution is rather different from the distributive investment solution of the Markowitz (stochastic programming) model (3). The reason is that a bond with a small variance gathers the investment rate around it since minimization of the variance which is not related to the original objective, maximization of the total return rate, is adopted by the Markowitz model.

This property of the Markowitz model will also be observed in Example 3.

**Example 2.** Let us consider five bonds whose return rates are depicted as possibility distributions in Fig. 2. All  $w_i$ 's are equal to 0.02 but  $c_i^c$  decreases as  $i$  increases:

$$c_1^c = 0.24, \quad c_2^c = 0.22, \quad c_3^c = 0.2, \\ c_4^c = 0.18 \quad \text{and} \quad c_5^c = 0.16.$$

Table 3 shows the optimal solutions by Markowitz (stochastic programming), fractile, modality optimization, spread minimization and the usual minimax regret approaches with setting  $z^0 = 0.18$  and  $h^0 = 0.9$ . We can see that a distributive investment solution is not obtained by the previous possibilistic programming approaches, i.e., the fractile approach, the modality optimization approach and the spread minimization approach, but by the proposed minimax regret approach. Markowitz model solution has almost equal investment rate shares, whereas proposed minimax approach solution claims a decreasing share going from right to left in Fig. 2 which corresponds to the decrease of the return rates.



Table 3  
Solutions for Example 2

Approach	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
Markowitz	0.200003	0.200002	0.200002	0.199997	0.199996
Fractile	1	0	0	0	0
Modality	1	0	0	0	0
Spread	1	0	0	0	0
Regret	0.293198	0.246599	0.2	0.153401	0.106802

Table 4  
Solutions for Example 3

Approach	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$
Markowitz	0.192775	0.255173	0.232509	0.131889	0.187654
Fractile	0	1	0	0	0
Modality	1	0	0	0	0
Spread	0	0.42857	0	0.57143	0
Regret	0.408018	0.306634	0.252826	0.032522	0

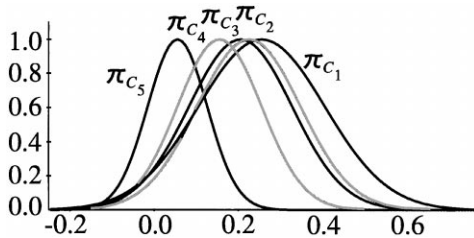


Fig. 3. Five bonds for Example 3.

**Example 3.** Let us consider five bonds whose return rates are depicted as possibility distributions in Fig. 3. The parameters  $c_i^c$ 's and  $w_i$ 's are defined as

$$c_1^c = 0.25, \quad c_2^c = 0.22, \quad c_3^c = 0.2,$$

$$c_4^c = 0.15, \quad c_5^c = 0.05,$$

$$w_1 = 0.0225, \quad w_2 = 0.015, \quad w_3 = 0.015,$$

$$w_4 = 0.01 \quad \text{and} \quad w_5 = 0.005.$$

The larger  $c_i^c$  is, the larger  $w_i$  is. The 5th bond with  $c_5^c = 0.05$  and  $w_5 = 0.005$ , intuitively speaking, seems to be inferior since it has the lowest return rate and is set apart from all the others on the critical edge close to zero. The optimal solutions by Markowitz, fractile, modality optimization, spread minimization and

minimax regret approaches are obtained as shown in Table 4 with setting  $z^0 = 0.18$  and  $h^0 = 0.9$ . By the proposed minimax regret approach, we got a distributive investment solution but not by the other possibilistic programming approaches. The solution obtained from the proposed minimax regret approach does not support investment in the 5th bond but the Markowitz model solution does. In this case, the solution to Problem (42) seems to be better than that to Problem (3) since it is following the return rate pattern.

## 6. Concluding remarks

In this paper, it has been revealed that a distributive investment solution cannot be obtained by the previous possibilistic programming approaches. A new possibilistic programming approach based on the worst regret has been proposed. Under the possibilistic independence among return rates, the problem can be reduced to a linear programming problem. It has been shown that a distributive investment solution can be obtained by the proposed approach. Three numerical examples have been given to demonstrate the advantages of the proposed approach over the traditional approaches.

## References

- [1] D. Dubois, H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980.
- [2] M. Inuiguchi, Stochastic programming problems versus fuzzy mathematical programming problems, *Jpn. J. Fuzzy Theory Systems* 4 (1992) 97–109.
- [3] M. Inuiguchi, H. Ichihashi, Y. Kume, Relationships between modality constrained programming problems and various fuzzy mathematical programming problems, *Fuzzy Sets and Systems* 49 (1992) 243–259.
- [4] M. Inuiguchi, H. Ichihashi, Y. Kume, Modality constrained programming problems: A unified approach to fuzzy mathematical programming problems in the setting of possibility theory, *Inform. Sci.* 67 (1993) 93–126.
- [5] M. Inuiguchi, M. Sakawa, Minimax regret solution to linear programming problems with an interval objective function, *Eur. J. Oper. Res.* 86 (1995) 526–536.
- [6] M. Inuiguchi, M. Sakawa, An achievement rate approach to linear programming problems with an interval objective function, *J. Oper. Res. Soc.* 48 (1997) 25–33.
- [7] H. Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, Wiley, New York, 1959.