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Decision Support

A goal programming method for obtaining interval weights from an interval comparison matrix ☆

Ying-Ming Wang [a,b,*], Taha M.S. Elhag [a]

[a] *Project Management Division, School of Mechanical, Aerospace and Civil Engineering, The University of Manchester, P.O. Box 88, Manchester M60 1QD, UK*

[b] *School of Public Administration, Fuzhou University, Fuzhou 350002, PR China*

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Abstract

Crisp comparison matrices lead to crisp weight vectors being generated. Accordingly, an interval comparison matrix should give an interval weight estimate. In this paper, a goal programming (GP) method is proposed to obtain interval weights from an interval comparison matrix, which can be either consistent or inconsistent. The interval weights are assumed to be normalized and can be derived from a GP model at a time. The proposed GP method is also applicable to crisp comparison matrices. Comparisons with an interval regression analysis method are also made. Three numerical examples including a multiple criteria decision-making (MCDM) problem with a hierarchical structure are examined to show the potential applications of the proposed GP method.

Keywords: Analytic hierarchy process; Multiple criteria decision making; Interval comparison matrix; Interval regression analysis; Normalization

1. Introduction

Analytic hierarchy process (AHP), as a multiple criteria decision making (MCDM) tool and a weight estimation technique, has been extensively applied in many areas such as selection, evaluation, planning and development, decision making, forecasting, and so on [20]. The conventional AHP requires exact judgments

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* Corresponding author. Address: Project Management Division, School of Mechanical, Aerospace and Civil Engineering, The University of Manchester, P.O. Box 88, Manchester M60 1QD, UK. Tel.: +44 161 2005974; fax: +44 161 2004646.

E-mail addresses: msymwang@hotmail.com, Yingming.Wang@Manchester.ac.uk (Y.-M. Wang).

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and crisp comparison matrices. However, due to the complexity and uncertainty involved in real world decision problems, it is sometimes unrealistic or impossible to acquire exact judgments. It is more natural or easier to provide interval judgments for part or all of the judgments in a pairwise comparison matrix. A number of techniques have been developed to deal with interval comparison matrices.

Saaty and Vargas [14] put forward interval judgments as a way to model subjective uncertainty and use a Monte Carlo simulation approach to find out weights from interval comparison matrices. They also point out difficulties in using this approach. Arbel [1,2] interprets interval judgments as linear constraints on weights and formulates the weight estimation problem as a linear programming (LP) model. Kress [10] finds the infeasibility of Arbel's method in solving inconsistent interval comparison matrices. Salo and Hämäläinen [15,16] extend Arbel's approach to hierarchical structures. Their approach searches for the maximums and minimums for all interval weights and incorporates the resultant intervals into further synthesis of global interval weights. Arbel and Vargas [3,4] formulate a hierarchical problem as a nonlinear programming (NLP) model in which all local weights in a hierarchy are included as decision variables.

Moreno-Jimenez [13] studies the probability distribution of possible rankings of the alternatives in an interval comparison matrix of size $n = 2$ or $n = 3$. Islam et al. [8] suggest a lexicographic goal programming (LGP) method to find a crisp set of weights for an inconsistent interval comparison matrix and explore its properties and advantages as a weight estimation technique. Haines [7] proposes a statistical approach to extract preferences from interval comparison matrices. Two specific distributions on a feasible region are examined and the mean of the distributions is used as a basis for assessment and ranking. Mikhailov [11,12] proposes a fuzzy programming approach (FPP) method and its variant, to derive crisp weights from interval comparison matrices. Sugihara et al. [17] bring forward an interval regression analysis method, which involves the solution of lower and upper approximation models. The lower approximation model captures the interval weights included by interval judgments, whereas the upper approximation model captures the interval weights that involve interval judgments.

Wang et al. [18] propose a two-stage logarithmic goal programming (TLGP) method to generate weights from interval comparison matrices, which can be either consistent or inconsistent. The first stage is devised to minimize the inconsistency of interval comparison matrices and the second stage is developed to generate weights under the condition of minimal inconsistency. The weights are assumed to be multiplicative rather than additive. Wang et al. [19] also design a method of consistency test to check whether an interval comparison matrix is consistent or not. For consistent interval comparison matrices, Arbel's linear programming (LP) model is used to derive interval weights and for inconsistent interval comparison matrices, an eigenvector method-based nonlinear programming (NLP) approach is developed to generate interval weights that can meet pre-determined consistency requirements.

As is well known, crisp comparison matrices lead to crisp weight vectors to be generated. It is more natural and logical that an interval comparison matrix should give an interval weight estimate rather than an exact point estimate. Our literature review shows that among the methods mentioned above, some of them are only applicable to consistent interval comparison matrices, some of them can only derive a crisp set of weights, which is a point estimate, from an inconsistent interval comparison matrix. Although Monte Carlo simulation and the TLGP method can be used to generate interval weights from both consistent and inconsistent interval comparison matrices, they are still somewhat complicated. The former needs hundreds or thousands of simulations, whereas the latter requires the solution of $(2n + 1)$ LP models. Sugihara et al.'s interval regression analysis method is the only one that can derive an interval weight vector from an interval comparison matrix at a time simply by solving one model. Illuminated by this method, we develop in this paper an alternate goal programming (GP) method to obtain normalized interval weights from an interval comparison matrix, which may be either consistent or inconsistent. The proposed GP method shows some advantages and is also applicable to crisp comparison matrices.

The rest of the paper is organized as follows. Section 2 develops the GP method, which is compared with the interval regression analysis in Section 3. The synthesis problem of interval weights is investigated in Section 4. Three numerical examples including a MCDM problem with a hierarchical structure are examined in Section 5 using the GP method to show its potential applications and validity. The paper is concluded in Section 6.

2. Interval comparison matrix and a goal programming method

Suppose a decision maker (DM) provides interval judgments instead of precise judgments for a pairwise comparison matrix. For example, it could be judged that criterion i is between l_{ij} and u_{ij} times as important as criterion j with l_{ij} and u_{ij} being non-negative real numbers and $l_{ij} \leqslant u_{ij}$. Then, an interval comparison matrix can be expressed as

$$A = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \cdots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \cdots & 1 \end{bmatrix}, \tag{1}$$

where $l_{ij} = 1/u_{ji}$ and $u_{ij} = 1/l_{ji}$ for all $i, j = 1, \ldots, n$; $i \neq j$. The above interval comparison matrix can be split into two crisp nonnegative matrices:

$$A_{\mathrm{L}} = \begin{bmatrix} 1 & l_{12} & \cdots & l_{1n} \\ l_{21} & 1 & \cdots & l_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ l_{n1} & l_{n2} & \cdots & 1 \end{bmatrix} \quad \text{and} \quad A_{\mathrm{U}} = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1n} \\ u_{21} & 1 & \cdots & u_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ u_{n1} & u_{n2} & \cdots & 1 \end{bmatrix}, \tag{2}$$

where $A_{\mathrm{L}} \leqslant A \leqslant A_{\mathrm{U}}$. Note that A_{L} and A_{U} are no longer reciprocal matrices.

For the interval comparison matrix A, there should exist a normalized interval weight vector, $W = ([w_1^{\mathrm{L}}, w_1^{\mathrm{U}}], \ldots, [w_n^{\mathrm{L}}, w_n^{\mathrm{U}}])^{\mathrm{T}}$, which is close to A in the sense that $a_{ij} = [l_{ij}, u_{ij}] \approx [w_i^{\mathrm{L}}, w_i^{\mathrm{U}}]/[w_j^{\mathrm{L}}, w_j^{\mathrm{U}}]$ for all $i,j = 1, \ldots, n$; $i \neq j$. According to [17], the interval weight vector W is said to be normalized if and only if

$$\sum_i w_i^{\mathrm{U}} - \max_j (w_j^{\mathrm{U}} - w_j^{\mathrm{L}}) \geqslant 1, \tag{3}$$

$$\sum_i w_i^{\mathrm{L}} + \max_j (w_j^{\mathrm{U}} - w_j^{\mathrm{L}}) \leqslant 1, \tag{4}$$

which can be equivalently rewritten as

$$w_i^{\mathrm{L}} + \sum_{j=1, j \neq i}^{n} w_j^{\mathrm{U}} \geqslant 1, \quad i = 1, \ldots, n, \tag{5}$$

$$w_i^{\mathrm{U}} + \sum_{j=1, j \neq i}^{n} w_j^{\mathrm{L}} \leqslant 1, \quad i = 1, \ldots, n. \tag{6}$$

As is known, if the interval comparison matrix A is the precise comparison about the interval weight vector W, namely, $a_{ij} = [l_{ij}, u_{ij}] \equiv [w_i^{\mathrm{L}}, w_i^{\mathrm{U}}]/[w_j^{\mathrm{L}}, w_j^{\mathrm{U}}]$, then A must be able to be written as

$$A = \begin{bmatrix} 1 & \dfrac{[w_1^{\mathrm{L}}, w_1^{\mathrm{U}}]}{[w_2^{\mathrm{L}}, w_2^{\mathrm{U}}]} & \cdots & \dfrac{[w_1^{\mathrm{L}}, w_1^{\mathrm{U}}]}{[w_n^{\mathrm{L}}, w_n^{\mathrm{U}}]} \\ \dfrac{[w_2^{\mathrm{L}}, w_2^{\mathrm{U}}]}{[w_1^{\mathrm{L}}, w_1^{\mathrm{U}}]} & 1 & \cdots & \dfrac{[w_2^{\mathrm{L}}, w_2^{\mathrm{U}}]}{[w_n^{\mathrm{L}}, w_n^{\mathrm{U}}]} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{[w_n^{\mathrm{L}}, w_n^{\mathrm{U}}]}{[w_1^{\mathrm{L}}, w_1^{\mathrm{U}}]} & \dfrac{[w_n^{\mathrm{L}}, w_n^{\mathrm{U}}]}{[w_2^{\mathrm{L}}, w_2^{\mathrm{U}}]} & \cdots & 1 \end{bmatrix}. \tag{7}$$

According to the division operation rule on interval numbers, i.e. $[b_{\mathrm{L}}, b_{\mathrm{U}}]/[d_{\mathrm{L}}, d_{\mathrm{U}}] = [b_{\mathrm{L}}/d_{\mathrm{U}}, b_{\mathrm{U}}/d_{\mathrm{L}}]$, where $[b_{\mathrm{L}}, b_{\mathrm{U}}]$ and $[d_{\mathrm{L}}, d_{\mathrm{U}}]$ are two positive interval numbers, the interval comparison matrix A defined by (7) can be further rewritten as

$$A = \begin{bmatrix} 1 & \left[\frac{w_1^L}{w_2^U}, \frac{w_1^U}{w_2^L}\right] & \cdots & \left[\frac{w_1^L}{w_n^U}, \frac{w_1^U}{w_n^L}\right] \\ \left[\frac{w_2^L}{w_1^U}, \frac{w_2^U}{w_1^L}\right] & 1 & \cdots & \left[\frac{w_2^L}{w_n^U}, \frac{w_2^U}{w_n^L}\right] \\ \vdots & \vdots & \cdots & \vdots \\ \left[\frac{w_n^L}{w_1^U}, \frac{w_n^U}{w_1^L}\right] & \left[\frac{w_n^L}{w_2^U}, \frac{w_n^U}{w_2^L}\right] & \cdots & 1 \end{bmatrix}, \tag{8}$$

which can be split into the following two crisp nonnegative matrices:

$$A_L = \begin{bmatrix} 1 & \frac{w_1^L}{w_2^U} & \cdots & \frac{w_1^L}{w_n^U} \\ \frac{w_2^L}{w_1^U} & 1 & \cdots & \frac{w_2^L}{w_n^U} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{w_n^L}{w_1^U} & \frac{w_n^L}{w_2^U} & \cdots & 1 \end{bmatrix} \quad \text{and} \quad A_U = \begin{bmatrix} 1 & \frac{w_1^U}{w_2^L} & \cdots & \frac{w_1^U}{w_n^L} \\ \frac{w_2^U}{w_1^L} & 1 & \cdots & \frac{w_2^U}{w_n^L} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{w_n^U}{w_1^L} & \frac{w_n^U}{w_2^L} & \cdots & 1 \end{bmatrix}. \tag{9}$$

It is easy to prove that

$$A_L W_U = W_U + (n-1)W_L, \tag{10}$$

$$A_U W_L = W_L + (n-1)W_U, \tag{11}$$

where $W_L = (w_1^L, \ldots, w_n^L)^T$ and $W_U = (w_1^U, \ldots, w_n^U)^T$. Eqs. (10) and (11) are important links between the lower and upper bounds of the interval weight vector W.

Due to the presence of subjectivity and uncertainty, the DM's subjective judgments cannot be 100% exact. Therefore, Eqs. (10) and (11) may not hold precisely. Based on such an analysis, we introduce the following deviation vectors:

$$E = (A_L - I)W_U - (n-1)W_L, \tag{12}$$

$$\Gamma = (A_U - I)W_L - (n-1)W_U, \tag{13}$$

where $E = (\varepsilon_1, \ldots, \varepsilon_n)^T$, $\Gamma = (\gamma_1, \ldots, \gamma_n)^T$ and I is a n by n unit matrix whose elements on the leading diagonal are 1, and all the other elements are 0. It is most desirable that the absolute values of deviation variables should be kept as small as possible, which leads to the following optimization model to be constructed:

$$\text{Minimize} \quad J = \sum_{i=1}^{n} (|\varepsilon_i| + |\gamma_i|)$$

$$\text{s.t.} \quad \begin{cases} (A_L - I)W_U - (n-1)W_L - E = 0, \\ (A_U - I)W_L - (n-1)W_U - \Gamma = 0, \\ w_i^L + \sum_{j=1, j\neq i}^{n} w_j^U \geqslant 1, \quad i = 1, \ldots, n, \\ w_i^U + \sum_{j=1, j\neq i}^{n} w_j^L \leqslant 1, \quad i = 1, \ldots, n, \\ W_U - W_L \geqslant 0, \\ W_L, W_U \geqslant 0, \end{cases} \tag{14}$$

where the first two constraints are Eqs. (12) and (13), the middle two constraints are the normalization constraints on the interval weight vector W, and the last two constraints are those on the lower and upper bounds of W and its nonnegativity.

Let

$$\varepsilon_i^+ = \frac{\varepsilon_i + |\varepsilon_i|}{2} \quad \text{and} \quad \varepsilon_i^- = \frac{-\varepsilon_i + |\varepsilon_i|}{2}, \quad i = 1, \ldots, n, \tag{15}$$

$$\gamma_i^+ = \frac{\gamma_i + |\gamma_i|}{2} \quad \text{and} \quad \gamma_i^- = \frac{-\gamma_i + |\gamma_i|}{2}, \quad i = 1, \ldots, n. \tag{16}$$

Then $E^+ = (\varepsilon_i^+, \ldots, \varepsilon_n^+)^{\mathrm{T}} \geqslant 0$, $E^- = (\varepsilon_i^-, \ldots, \varepsilon_n^-)^{\mathrm{T}} \geqslant 0$, $\Gamma^+ = (\gamma_i^+, \ldots, \gamma_n^+)^{\mathrm{T}} \geqslant 0$ and $\Gamma^- = (\gamma_i^-, \ldots, \gamma_n^-)^{\mathrm{T}} \geqslant 0$. Based on ε_i^+ and ε_i^-, ε_i and $|\varepsilon_i|$ can be expressed as

$$\varepsilon_i = \varepsilon_i^+ - \varepsilon_i^-, \quad i = 1, \ldots, n, \tag{17}$$

$$|\varepsilon_i| = \varepsilon_i^+ + \varepsilon_i^-, \quad i = 1, \ldots, n, \tag{18}$$

where $\varepsilon_i^+ \cdot \varepsilon_i^- = 0$ for $i = 1$ to n. As such, γ_i and $|\gamma_i|$ can be expressed as

$$\gamma_i = \gamma_i^+ - \gamma_i^-, \quad i = 1, \ldots, n, \tag{19}$$

$$|\gamma_i| = \gamma_i^+ + \gamma_i^-, \quad i = 1, \ldots, n, \tag{20}$$

where $\gamma_i^+ \cdot \gamma_i^- = 0$ for $i = 1$ to n. Accordingly, the optimization model (14) can be rewritten as

$$\begin{aligned} &\text{Minimize} \quad J = \sum_{i=1}^{n} (\varepsilon_i^+ + \varepsilon_i^- + \gamma_i^+ + \gamma_i^-) = e^{\mathrm{T}}(E^+ + E^- + \Gamma^+ + \Gamma^-) \\ &\text{s.t.} \quad \begin{cases} (A_{\mathrm{L}} - I)W_{\mathrm{U}} - (n-1)W_{\mathrm{L}} - E^+ + E^- = 0, \\ (A_{\mathrm{U}} - I)W_{\mathrm{L}} - (n-1)W_{\mathrm{U}} - \Gamma^+ + \Gamma^- = 0, \\ w_i^{\mathrm{L}} + \sum\limits_{j=1, j\neq i}^{n} w_j^{\mathrm{U}} \geqslant 1, \quad i = 1, \ldots, n, \\ w_i^{\mathrm{U}} + \sum\limits_{j=1, j\neq i}^{n} w_j^{\mathrm{L}} \leqslant 1, \quad i = 1, \ldots, n, \\ W_{\mathrm{U}} - W_{\mathrm{L}} \geqslant 0, \\ W_{\mathrm{L}}, W_{\mathrm{U}}, E^+, E^-, \Gamma^+, \Gamma^- \geqslant 0, \end{cases} \end{aligned} \tag{21}$$

where $e^{\mathrm{T}} = (1, \ldots, 1)$, ε_i^+ and ε_i^- as well as γ_i^+ and γ_i^- cannot be simultaneously selected as basic variables in a simplex method. This is the GP model we have developed in this paper. Such a method for obtaining interval weights from an interval comparison matrix is referred to as the GP method (GPM).

Since crisp comparison matrices can be seen as a special case of interval comparison matrices, the above GP model (21) is also applicable to crisp comparison matrices. It is worth while pointing out here that for a crisp comparison matrix, Bryson [5] also develops a goal programming (GP) method for generating priority vectors, which is different from ours. Our GP model (21) does not model each judgment element individually.

About the above GP model, we have the following theorem.

Theorem 1. *Let W_{L}^* and W_{U}^* be the optimal solution to the GP model* (21)*. If A is a crisp consistent comparison matrix, then $W_{\mathrm{L}}^* = W_{\mathrm{U}}^* = W^*$, where W^* is the principal right eigenvector of A.*

Proof. If A is a crisp consistent comparison matrix, then there exists the eigenvalue equation: $AW^* = nW^*$, namely, $(A - nI)W^* = 0$. Let $W_{\mathrm{L}} = W^*$ and $W_{\mathrm{U}} = W^*$. It is easy to find that $W_{\mathrm{L}} = W_{\mathrm{U}} = W^*$ is a feasible solution of the GP model (21). Accordingly, we have $E = (A_{\mathrm{L}} - I)W_{\mathrm{U}} - (n-1)W_{\mathrm{L}} = (A - nI)W^* = 0$ and $\Gamma = (A_{\mathrm{U}} - I)W_{\mathrm{L}} - (n-1)W_{\mathrm{U}} = (A - nI)W^* = 0$, which leads to $J = \sum_{i=1}^{n}(|\varepsilon_i| + |\gamma_i|) = 0$. That is to say, $W_{\mathrm{L}} = W_{\mathrm{U}} = W^*$ is also the optimal solution to the GP model (21). So, $W_{\mathrm{L}}^* = W_{\mathrm{U}}^* = W^*$. □

3. Comparisons of the GPM and the interval regression analysis

Sugihara et al. [17] deal with interval judgments in two ways. One is called the lower approximation and the other is called the upper approximation. For the lower approximation, it is required that

$$W_{ij*} = \left[\frac{w_{i^*}^{\mathrm{L}}}{w_{j^*}^{\mathrm{U}}}, \frac{w_{i^*}^{\mathrm{U}}}{w_{j^*}^{\mathrm{L}}}\right] \subseteq a_{ij} = [l_{ij}, u_{ij}], \quad \forall i,j \;\; (i \neq j), \tag{22}$$

which can be rewritten as

$$\frac{w_{i^*}^{\mathrm{L}}}{w_{j^*}^{\mathrm{U}}} \geqslant l_{ij} \quad \text{and} \quad \frac{w_{i^*}^{\mathrm{U}}}{w_{j^*}^{\mathrm{L}}} \leqslant u_{ij}, \quad \forall i,j \;\; (i \neq j) \tag{23}$$

or

$$w_{i^*}^{\mathrm{L}} - l_{ij} w_{j^*}^{\mathrm{U}} \geqslant 0 \quad \text{and} \quad w_{i^*}^{\mathrm{U}} - u_{ij} w_{j^*}^{\mathrm{L}} \leqslant 0, \quad \forall i,j \;\; (i \neq j). \tag{24}$$

For the upper approximation, it is required that

$$W_{ij}^{*} = \left[\frac{w_{i}^{\mathrm{L}^*}}{w_{j}^{\mathrm{U}^*}}, \frac{w_{i}^{\mathrm{U}^*}}{w_{j}^{\mathrm{L}^*}}\right] \supseteq a_{ij} = [l_{ij}, u_{ij}], \quad \forall i,j \;\; (i \neq j), \tag{25}$$

which can be rewritten as

$$\frac{w_{i}^{\mathrm{L}^*}}{w_{j}^{\mathrm{U}^*}} \leqslant l_{ij} \quad \text{and} \quad \frac{w_{i}^{\mathrm{U}^*}}{w_{j}^{\mathrm{L}^*}} \geqslant u_{ij}, \quad \forall i,j \;\; (i \neq j) \tag{26}$$

or

$$w_{i}^{\mathrm{L}^*} - l_{ij} w_{j}^{\mathrm{U}^*} \leqslant 0 \quad \text{and} \quad w_{i}^{\mathrm{U}^*} - u_{ij} w_{j}^{\mathrm{L}^*} \geqslant 0, \quad \forall i,j \;\; (i \neq j). \tag{27}$$

The lower and upper approximation models are respectively constructed as follows:

$$\text{Maximize} \quad J_* = \sum_{i=1}^{n} (w_{i^*}^{\mathrm{U}} - w_{i^*}^{\mathrm{L}})$$

$$\text{s.t.} \quad \begin{cases} w_{i^*}^{\mathrm{L}} - l_{ij} w_{j^*}^{\mathrm{U}} \geqslant 0, \quad \forall i,j \;\; (i \neq j), \\ w_{i^*}^{\mathrm{U}} - u_{ij} w_{j^*}^{\mathrm{L}} \leqslant 0, \quad \forall i,j \;\; (i \neq j), \\ w_{i^*}^{\mathrm{L}} + \sum\limits_{j=1, j\neq i}^{n} w_{j^*}^{\mathrm{U}} \geqslant 1, \quad \forall i, \\ w_{i^*}^{\mathrm{U}} + \sum\limits_{j=1, j\neq i}^{n} w_{j^*}^{\mathrm{L}} \leqslant 1, \quad \forall i, \\ w_{i^*}^{\mathrm{U}} - w_{i^*}^{\mathrm{L}} \geqslant 0, \quad \forall i, \\ w_{i^*}^{\mathrm{L}} \geqslant \varepsilon, \quad \forall i, \end{cases} \tag{28}$$

$$\text{Minimize} \quad J^* = \sum_{i=1}^{n} (w_{i}^{\mathrm{U}^*} - w_{i}^{\mathrm{L}^*})$$

$$\text{s.t.} \quad \begin{cases} w_{i}^{\mathrm{L}^*} - l_{ij} w_{j}^{\mathrm{U}^*} \leqslant 0, \quad \forall i,j \;\; (i \neq j), \\ w_{i}^{\mathrm{U}^*} - u_{ij} w_{j}^{\mathrm{L}^*} \geqslant 0, \quad \forall i,j \;\; (i \neq j), \\ w_{i}^{\mathrm{L}^*} + \sum\limits_{j=1, j\neq i}^{n} w_{j}^{\mathrm{U}^*} \geqslant 1, \quad \forall i, \\ w_{i}^{\mathrm{U}^*} + \sum\limits_{j=1, j\neq i}^{n} w_{j}^{\mathrm{L}^*} \leqslant 1, \quad \forall i, \\ w_{i}^{\mathrm{U}^*} - w_{i}^{\mathrm{L}^*} \geqslant 0, \quad \forall i, \\ w_{i}^{\mathrm{L}^*} \geqslant \varepsilon, \quad \forall i, \end{cases} \tag{29}$$

where ε is a small positive real number.

Compared with the above lower and upper approximation models, the GP model (21) differs from them in the following ways:

First of all, the GP model views an interval comparison matrix as a whole and does not consider each judgment element individually, which makes the GP model have less constraints, whereas the lower and upper approximation models deal with each judgment individually and therefore have more constraints than the GP model.

Second, the GP model (21) is applicable to any crisp and interval comparison matrices no matter whether they are consistent or not, while the lower approximation model is only applicable to consistent comparison matrices (crisp or interval) because there is no feasible solution that can be found for any inconsistent comparison matrix or inconsistent interval comparison matrix.

Next, the upper approximation model aims at finding an interval weight vector $W^* = ([w_1^{L^*}, w_1^{U^*}], \ldots, [w_n^{L^*}, w_n^{U^*}])^T$ such that $a_{ij} = [l_{ij}, u_{ij}] \in W_{ij}^* = \left[\frac{w_i^{L^*}}{w_j^{U^*}}, \frac{w_i^{U^*}}{w_j^{L^*}}\right]$ for $\forall i,j\,(i \neq j)$. Due to the fact that the DM's judgments are subjective and cannot always be 100% precise, there is no guarantee that the DM's judgments will certainly fall within and will not exceed the real interval $\overline{W}_{ij} = \left[\frac{\overline{w}_i^L}{\overline{w}_j^U}, \frac{\overline{w}_i^U}{\overline{w}_j^L}\right]$, where $\overline{W} = ([\overline{w}_1^L, \overline{w}_1^U], \ldots, [\overline{w}_n^L, \overline{w}_n^U])^T$ is unknown real weights. On the contrary, the GP model aims at finding an interval weight vector $W = ([w_1^L, w_1^U], \ldots, [w_n^L, w_n^U])^T$ such that $W_{ij} = \left[\frac{w_i^L}{w_j^U}, \frac{w_i^U}{w_j^L}\right]$ are close to DM's judgments, but there is no requirement that W_{ij} must involve or be included within $a_{ij} = [l_{ij}, u_{ij}]$. So, the GP model sounds more logical and natural.

Finally, the lower and upper approximation models require extra constraints: $w_{i^*}^L \geqslant \varepsilon$ or $w_i^{L^*} \geqslant \varepsilon$ for $i = 1, \ldots, n$, to avoid the occurrence of zero weights, while the GP model has no such requirements.

4. Global interval weights

Suppose the local interval weights for upper-level criteria and lower-level alternatives have all been obtained using the GPM, as shown in Table 1, where $[w_j^L, w_j^U]$ is the normalized interval weight for criterion j $(j = 1, \ldots, m)$ and $[w_{ij}^L, w_{ij}^U]$ the normalized interval weight of alternative A_i with respect to the criterion j $(i = 1, \ldots, n; j = 1, \ldots, m)$. They satisfy the following normalization constraints:

$$w_j^L + \sum_{k=1,k\neq j}^{m} w_k^U \geqslant 1, \quad j = 1, \ldots, m, \tag{30}$$

$$w_j^U + \sum_{k=1,k\neq j}^{m} w_k^L \leqslant 1, \quad j = 1, \ldots, m, \tag{31}$$

$$w_{ij}^L + \sum_{k=1,k\neq i}^{n} w_{kj}^U \geqslant 1, \quad i = 1, \ldots, n;\; j = 1, \ldots, m, \tag{32}$$

$$w_{ij}^U + \sum_{k=1,k\neq i}^{n} w_{kj}^L \leqslant 1, \quad i = 1, \ldots, n;\; j = 1, \ldots, m. \tag{33}$$

Table 1
Synthesis of interval weights

Alternatives	Criterion 1	Criterion 2	…	Criterion m	Composite weights
	$[w_1^L, w_1^U]$	$[w_2^L, w_2^U]$	…	$[w_m^L, w_m^U]$	
A_1	$[w_{11}^L, w_{11}^U]$	$[w_{12}^L, w_{12}^U]$	…	$[w_{1m}^L, w_{1m}^U]$	$[w_{A_1}^L, w_{A_1}^U]$
A_2	$[w_{21}^L, w_{21}^U]$	$[w_{22}^L, w_{22}^U]$	…	$[w_{2m}^L, w_{2m}^U]$	$[w_{A_2}^L, w_{A_2}^U]$
⋮	⋮	⋮	…	⋮	⋮
A_n	$[w_{n1}^L, w_{n1}^U]$	$[w_{n2}^L, w_{n2}^U]$	…	$[w_{nm}^L, w_{nm}^U]$	$[w_{A_n}^L, w_{A_n}^U]$

Salo and Hämäläinen [16] show by an example that interval arithmetic is unsuitable for the synthesis of interval weights and has to be rejected. They thus propose a hierarchical decomposition method that decomposes a hierarchical composition problem into a series of linear programming problems over the feasible regions.

Bryson and Mobolurin [6] suggest a linear programming method, which seems simpler and is therefore adopted here. Their method treats the weights of criteria as decision variables and captures respectively the lower and upper bounds of the composite weight of each alternative A_i $(i = 1, \ldots, n)$ by constructing the following pair of LP models:

$$\text{Minimize} \quad w_{A_i}^{\mathrm{L}} = \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} w_j \tag{34}$$

$$\text{s.t.} \quad W \in \Omega_W, \tag{35}$$

$$\text{Maximize} \quad w_{A_i}^{\mathrm{U}} = \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} w_j \tag{36}$$

$$\text{s.t.} \quad W \in \Omega_W, \tag{37}$$

where w_j is the decision variable about the jth criterion weight $(j = 1, \ldots, m)$ and $\Omega_W = \{W = (w_1, \ldots, w_m)^{\mathrm{T}} | w_j^{\mathrm{L}} \leqslant w_j \leqslant w_j^{\mathrm{U}}, \sum_{j=1}^{m} w_j = 1, j = 1, \ldots, m\}$. The above pair of LP models results in a global interval weight for each alternative A_i denoted by $[w_{A_i}^{\mathrm{L}}, w_{A_i}^{\mathrm{U}}]$ $(i = 1, \ldots, n)$.

The following theorem shows that the global interval weights are always normalized.

Theorem 2. *Let $[w_{A_i}^{\mathrm{L}}, w_{A_i}^{\mathrm{U}}]$ $(i = 1, \ldots, n)$ be the global interval weights obtained by the LP models (34)–(37). Then there exist*

$$\sum_{i=1}^{n} w_{A_i}^{\mathrm{L}} \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{n} w_{A_i}^{\mathrm{U}} \geqslant 1, \tag{38}$$

$$w_{A_i}^{\mathrm{L}} + \sum_{j=1, j \neq i}^{n} w_{A_j}^{\mathrm{U}} \geqslant 1, \quad i = 1, \ldots, n, \tag{39}$$

$$w_{A_i}^{\mathrm{U}} + \sum_{j=1, j \neq i}^{n} w_{A_j}^{\mathrm{L}} \leqslant 1, \quad i = 1, \ldots, n. \tag{40}$$

Proof. Let $\tilde{W} = (\tilde{w}_1, \ldots, \tilde{w}_m)^{\mathrm{T}} \in \Omega_W$ be an arbitrary feasible solution, which may be not optimal to any of $w_{A_i}^{\mathrm{L}}$ and $w_{A_i}^{\mathrm{U}}$ $(i = 1, \ldots, n)$. Then, we have

$$w_{A_i}^{\mathrm{L}} = \underset{W \in \Omega_W}{\mathrm{Min}} \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} w_j \leqslant \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} \tilde{w}_j, \tag{41}$$

$$w_{A_i}^{\mathrm{U}} = \underset{W \in \Omega_W}{\mathrm{Max}} \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} w_j \geqslant \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} \tilde{w}_j, \tag{42}$$

$$\sum_{i=1}^{n} w_{A_i}^{\mathrm{L}} \leqslant \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} \tilde{w}_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_{ij}^{\mathrm{L}} \right) \tilde{w}_j \leqslant \sum_{j=1}^{m} \tilde{w}_j = 1, \tag{43}$$

$$\sum_{i=1}^{n} w_{A_i}^{\mathrm{U}} \geqslant \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} \tilde{w}_j = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} w_{ij}^{\mathrm{U}} \right) \tilde{w}_j \geqslant \sum_{j=1}^{m} \tilde{w}_j = 1. \tag{44}$$

Denote by $X_i^* = (x_{i1}^*, \ldots, x_{im}^*)^{\mathrm{T}} \in \Omega_W$ and $Y_i^* = (y_{i1}^*, \ldots, y_{im}^*)^{\mathrm{T}} \in \Omega_W$ the optimal solutions to the LP model (34), (35) for $w_{A_i}^{\mathrm{L}}$ and the LP model (36), (37) for $w_{A_i}^{\mathrm{U}}$, respectively. Obviously, $X_i^* = (x_{i1}^*, \ldots, x_{im}^*)^{\mathrm{T}}$ and $Y_i^* = (y_{i1}^*, \ldots, y_{im}^*)^{\mathrm{T}}$ are not necessarily optimal to $w_{A_j}^{\mathrm{U}}$ or $w_{A_j}^{\mathrm{L}}$, $j = 1, \ldots, n$; $j \neq i$. So, we have $w_{A_i}^{\mathrm{L}} = \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} x_{ij}^*$, $w_{A_i}^{\mathrm{U}} = \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} y_{ij}^*$, $w_{A_k}^{\mathrm{L}} = \underset{W \in \Omega_W}{\mathrm{Min}} \sum_{j=1}^{m} w_{kj}^{\mathrm{L}} w_j \leqslant \sum_{j=1}^{m} w_{kj}^{\mathrm{L}} y_{ij}^*$ and $w_{A_k}^{\mathrm{U}} = \underset{W \in \Omega_W}{\mathrm{Max}} \sum_{j=1}^{m} w_{kj}^{\mathrm{U}} w_j \geqslant \sum_{j=1}^{m} w_{kj}^{\mathrm{U}} x_{ij}^*$. Furthermore, we have

$$\sum_{k=1,k\neq i}^{n} w_{A_k}^{\mathrm{U}} \geqslant \sum_{k=1,k\neq i}^{n}\left(\sum_{j=1}^{m} w_{kj}^{\mathrm{U}} x_{ij}^{*}\right) = \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{\mathrm{U}}\right) x_{ij}^{*}, \tag{45}$$

$$\sum_{k=1,k\neq i}^{n} w_{A_k}^{\mathrm{L}} \leqslant \sum_{k=1,k\neq i}^{n}\left(\sum_{j=1}^{m} w_{kj}^{\mathrm{L}} y_{ij}^{*}\right) = \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{\mathrm{L}}\right) y_{ij}^{*}. \tag{46}$$

By (32) and (33), we get

$$w_{A_i}^{\mathrm{L}} + \sum_{k=1,k\neq i}^{n} w_{A_k}^{\mathrm{U}} \geqslant \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} x_{ij}^{*} + \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{\mathrm{U}}\right) x_{ij}^{*} \geqslant \sum_{j=1}^{m} w_{ij}^{\mathrm{L}} x_{ij}^{*} + \sum_{j=1}^{m}(1-w_{ij}^{\mathrm{L}}) x_{ij}^{*} = \sum_{j=1}^{m} x_{ij}^{*} = 1, \tag{47}$$

$$w_{A_i}^{\mathrm{U}} + \sum_{k=1,k\neq i}^{n} w_{A_k}^{\mathrm{L}} \leqslant \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} y_{ij}^{*} + \sum_{j=1}^{m}\left(\sum_{k=1,k\neq i}^{n} w_{kj}^{\mathrm{L}}\right) y_{ij}^{*} \leqslant \sum_{j=1}^{m} w_{ij}^{\mathrm{U}} y_{ij}^{*} + \sum_{j=1}^{m}(1-w_{ij}^{\mathrm{U}}) y_{ij}^{*} = \sum_{j=1}^{m} y_{ij}^{*} = 1. \tag{48}$$

In the same way, we can derive similar inequalities for all other intervals $[w_{A_j}^{\mathrm{L}}, w_{A_j}^{\mathrm{U}}]$ $(j = 1,\ldots,n;\ j \neq i)$. So, inequalities (39) and (40) hold for all $i = 1,\ldots,n$. $\square$

5. Numerical examples

In this section, three numerical examples will be examined to show the applications of the GP method in crisp and interval comparison matrices, which may be consistent or inconsistent, as well as in MCDM problems with hierarchical structures.

Example 1. Consider the following four crisp comparison matrices investigated by Sugihara et al. [17], where A_1 is a perfectly consistent comparison matrix, A_2 is a row dominant comparison matrix in which each two rows satisfy the row dominance relation defined by $a_{ik} \geqslant a_{jk}$ for all k, A_3 is a weak transitivity comparison matrix which can be characterized as if $a_{ij} \geqslant 1$ and $a_{jk} \geqslant 1$, then $a_{ik} \geqslant 1$, and A_4 is a non-transitivity comparison matrix.

$$A_1 = \begin{bmatrix} 1 & 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 4 & 8 \\ 1/2 & 1/2 & 1 & 2 & 4 \\ 1/4 & 1/4 & 1/2 & 1 & 2 \\ 1/8 & 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 1/2 & 1 & 2 & 2 & 4 \\ 1/3 & 1/2 & 1 & 1 & 1 \\ 1/5 & 1/2 & 1 & 1 & 2 \\ 1/7 & 1/4 & 1 & 1 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 2 & 3 & 5 & 7 \\ 1/2 & 1 & 2 & 2 & 4 \\ 1/3 & 1/2 & 1 & 1 & 2 \\ 1/5 & 1/2 & 1 & 1 & 9 \\ 1/7 & 1/4 & 1/2 & 1/9 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & 1 & 2 & 4 & 1/2 \\ 1 & 1 & 2 & 4 & 8 \\ 1/2 & 1/2 & 1 & 2 & 4 \\ 1/4 & 1/4 & 1/2 & 1 & 2 \\ 2 & 1/8 & 1/4 & 1/2 & 1 \end{bmatrix}.$$

Since crisp comparison matrices can be seen as a special kind of interval comparison matrix with $A_{\mathrm{L}} = A_{\mathrm{U}} = A$, the GP model (21) is therefore also applicable to crisp comparison matrices. Solving the GP model (21) for each of the above four crisp matrices, we get four sets of crisp weight vectors, which are shown in Table 2 where the eigenvector method (EM) weights and the PAHPC (Possibilistic AHP for Crisp Data) weights obtained by Sugihara et al. [17] are also presented for comparison. The PAHPC model is a variant of the previous upper approximation model (29) with $l_{ij} = u_{ij} = a_{ij}$. As can be seen from Table 2, the GP method produces crisp weight vectors for crisp comparison matrices, whereas the PAHPC yields interval weights for crisp comparison matrices if they are not consistent. Although the three methods all produce precise weight estimate for the consistent comparison matrix A_1, the GPM weights are much closer to the eigenvector weights than the PAHPC weights for the three inconsistent comparison matrices.

Table 2
Weights for four crisp comparison matrices generated by three different methods

Matrix	Method	Weight				
		w_1	w_2	w_3	w_4	w_5
A_1 (CI = 0)	EM	0.3478	0.3478	0.1739	0.0870	0.0435
	GPM	0.3478	0.3478	0.1739	0.0870	0.0435
	PAHPC	0.3478	0.3478	0.1739	0.0870	0.0435
A_2 (CI = 0.0222)	EM	0.4640	0.2413	0.1120	0.0998	0.0828
	GPM	0.4666	0.2422	0.1136	0.1011	0.0765
	PAHPC	0.4528	0.2264	[0.1038, 0.1509]	[0.0906, 0.1130]	[0.0556, 0.1038]
A_3 (CI = 0.0913)	EM	0.4468	0.2231	0.1185	0.1664	0.0452
	GPM	0.4592	0.2295	0.1224	0.1561	0.0328
	PAHPC	0.4737	0.2368	[0.1184, 0.1579]	[0.0947, 0.1184]	[0.0132, 0.0677]
A_4 (CI = 0.2812)	EM	0.2299	0.3732	0.1866	0.0933	0.1170
	GPM	0.2878	0.3741	0.1871	0.0935	0.0576
	PAHPC	[0.1304, 0.3478]	0.3478	0.1739	0.0870	[0.0435, 0.2609]

Example 2. Consider an interval comparison matrix investigated also by Sugihara et al. [17].

$$A = \begin{bmatrix} 1 & [1,3] & [3,5] & [5,7] & [5,9] \\ \left[\frac{1}{3},1\right] & 1 & [1,4] & [1,5] & [1,4] \\ \left[\frac{1}{5},\frac{1}{3}\right] & \left[\frac{1}{4},1\right] & 1 & \left[\frac{1}{5},5\right] & [2,4] \\ \left[\frac{1}{7},\frac{1}{5}\right] & \left[\frac{1}{5},1\right] & \left[\frac{1}{5},5\right] & 1 & [1,2] \\ \left[\frac{1}{9},\frac{1}{5}\right] & \left[\frac{1}{4},1\right] & \left[\frac{1}{4},\frac{1}{2}\right] & \left[\frac{1}{2},1\right] & 1 \end{bmatrix}$$

This is a consistent interval comparison matrix which can be checked using the consistency test method developed by Wang et al. [19]. So, both the lower and upper approximation models of Sugihara et al. can be applied to derive interval weights, which are shown in Table 3. The interval weights generated by the GP method are presented in its second column. It is clear that the three sets of interval weights are totally different. It is very hard to compare them directly. In order to see which one performs better, we first transform them into three fitted interval comparison matrices and then calculate the fitted errors between each of them and the original interval comparison matrix A. The fitted error is defined as

$$D(A,\tilde{A}) = \sum_{i=1}^{n}\sum_{j=1}^{n}[(l_{ij}-\tilde{l}_{ij})^2 + (u_{ij}-\tilde{u}_{ij})^2], \tag{49}$$

where A and $\tilde{A}$ are the real and fitted interval comparison matrices, respectively.

Table 3
Interval weights for a consistent interval comparison matrix generated by two different methods

Interval weight	GPM	Sugihara et al. PAHP	
		Lower model	Upper model
w_1	0.4527	[0.4225, 0.5343]	[0.2909, 0.4092]
w_2	[0.1396, 0.3320]	[0.1781, 0.2817]	[0.1364, 0.2909]
w_3	[0.0818, 0.2097]	0.1408	[0.0273, 0.1818]
w_4	[0.0591, 0.1347]	[0.0763, 0.0845]	[0.0364, 0.1364]
w_5	0.0633	0.0704	[0.0455, 0.1364]

The fitted interval comparison matrices by the GPM weights and Sugihara et al.'s lower and upper model weights are respectively as follows:

$$\tilde{A}_{\mathrm{GPM}} = \begin{bmatrix} 1 & [1.3636, 3.2428] & [2.1588, 5.5342] & [3.3608, 7.6599] & 7.1517 \\ [0.3084, 0.7334] & 1 & [0.6657, 4.0587] & [1.0364, 5.6176] & [2.2054, 5.2449] \\ [0.1807, 0.4632] & [0.2464, 1.5021] & 1 & [0.6073, 3.5482] & [1.2923, 3.3128] \\ [0.1306, 0.2975] & [0.1780, 0.9649] & [0.2818, 1.6467] & 1 & [0.9336, 2.1280] \\ 0.1398 & [0.1907, 0.4534] & [0.3019, 0.7738] & [0.4699, 1.0711] & 1 \end{bmatrix},$$

$$\tilde{A}_{\mathrm{Lower}} = \begin{bmatrix} 1 & [1.4998, 3.0000] & [3.0007, 3.7947] & [5.0000, 7.0026] & [6.0014, 7.5895] \\ [0.3333, 0.6667] & 1 & [1.2649, 2.0007] & [2.1077, 3.6920] & [2.5298, 4.0014] \\ [0.2635, 0.3333] & [0.4998, 0.7906] & 1 & [1.6663, 1.8453] & 2 \\ [0.1428, 0.2000] & [0.2709, 0.4745] & [0.5419, 0.6001] & 1 & [1.0838, 1.2003] \\ [0.1318, 0.1666] & [0.2499, 0.3953] & 0.5 & [0.8331, 0.9227 & 1 \end{bmatrix},$$

$$\tilde{A}_{\mathrm{Upper}} = \begin{bmatrix} 1 & [1.0000, 3.0000] & [1.6001, 14.9890] & [2.1327, 11.2418] & [2.1327, 8.9934] \\ [0.3333, 1.0000] & 1 & [0.7503, 10.6557] & [1.0000, 7.9918] & [1.0000, 6.3934] \\ [0.0667, 0.6250] & [0.0938, 1.3328] & 1 & [0.2001, 4.9945] & [0.2001, 3.9956] \\ [0.0890, 0.4689] & [0.1251, 1.0000] & [0.2002, 4.9963] & 1 & [0.2669, 2.9978] \\ [0.1112, 0.4689] & [0.1564, 1.0000] & [0.2503, 4.9963] & [0.3336, 3.7473] & 1 \end{bmatrix}.$$

The corresponding fitted error indices are

$$D(A, \tilde{A}_{\mathrm{GPM}}) = 31.1148, \quad D(A, \tilde{A}_{\mathrm{Lower}}) = 51.3137, \quad \text{and} \quad D(A, \tilde{A}_{\mathrm{Upper}}) = 228.179.$$

It is evident that $D(A, \tilde{A}_{\mathrm{GPM}})$ is the smallest of the three performance indices. So, $\tilde{A}_{\mathrm{GPM}}$ is thought to be the best fitting for the interval comparison matrix A. According to [9], an interval weight $w_i = [w_i^{\mathrm{L}}, w_i^{\mathrm{U}}]$ is said to be preferred to an interval weight $w_j = [w_j^{\mathrm{L}}, w_j^{\mathrm{U}}]$ if and only if $w_i^{\mathrm{L}} \geqslant w_j^{\mathrm{L}}$ and $w_i^{\mathrm{U}} \geqslant w_j^{\mathrm{U}}$. Obviously, the GPM weights give a ranking of $w_1 > w_2 > w_3 > w_4 > w_5$. Note that the upper model produces a slightly different ranking of $w_1 > w_2 > w_3 > w_5 > w_4$.

Example 3. Consider a MCDM problem with a hierarchy structure, which is taken from Islam et al. [8] and shown in Fig. 1. A person is interested in investing his money to any one of the four portfolios: bank deposit (BD), debentures (DB), government bonds (GB), and shares (SH). Out of these portfolios he has to choose only one based upon the four criteria: return (Re), risk (Ri), tax benefits (Tb), and liquidity (Li). The interval comparison matrices for the four criteria as well as for the four alternatives are summarized in Tables 4–8.

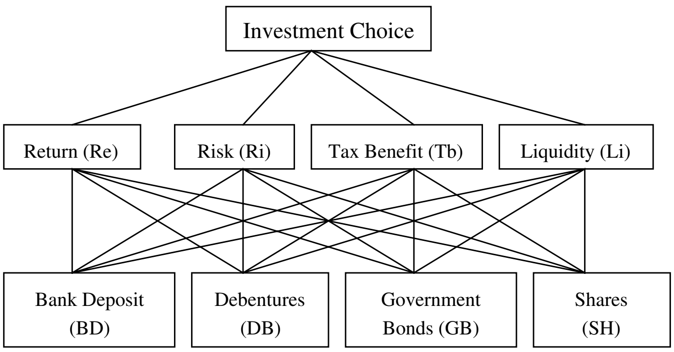

Fig. 1. Hierarchy structure.

Table 4
Interval comparison matrix for the four criteria with respect to "investment choice (Ic)"

Ic	Re	Ri	Tb	Li
Re	1	[3, 4]	[5, 6]	[6, 7]
Ri		1	[4, 5]	[5, 6]
Tb			1	[3, 4]
Li				1

Table 5
Interval comparison matrix for the four alternatives with respect to the criterion "return (Re)"

Re	BD	DB	GB	SH
BD	1	[1/4, 1/3]	[3, 4]	[1/6, 1/5]
DB		1	[6, 7]	[1/5, 1/4]
GB			1	[1/7, 1/6]
SH				1

Table 6
Interval comparison matrix for the four alternatives with respect to the criterion "risk (Ri)"

Ri	BD	DB	GB	SH
BD	1	[3, 4]	[4, 5]	[6, 7]
DB		1	[3, 4]	[5, 6]
GB			1	[4, 5]
SH				1

Table 7
Interval comparison matrix for the four alternatives with respect to the criterion "tax benefits (Tb)"

Tb	BD	DB	GB	SH
BD	1	1	[1/6, 1/5]	[1/4, 1/3]
DB		1	[1/6, 1/5]	[1/4, 1/3]
GB			1	[4, 5]
SH				1

Table 8
Interval comparison matrix for the four alternatives with respect to the criterion "liquidity (Li)"

Li	BD	DB	GB	SH
BD	1	[3, 4]	6	[6, 7]
DB		1	[3, 4]	[3, 4]
GB			1	[3, 4]
SH				1

It has been shown in [19] that the five interval comparison matrices presented in Tables 4–8 are all inconsistent. Therefore, Sugihara et al.'s lower approximation model (28) is not applicable to these interval comparison matrices. Tables 9 and 10 shows the interval weights obtained by using the GP method and the upper approximation model (29), respectively. The composite weights (i.e. global interval weights) for the four alternatives, which are derived from the LP models (34)–(37), are shown in the last columns of Tables 9 and 10, from which it is clear that both the GP method and the upper approximation model evaluate SH to be the best alternative. That is to say, investing his money to shares (SH) is the best choice for the decision maker, which is the same as the conclusion arrived at by Islam et al. [8] and Wang et al. [18,19].

Table 9
Interval weights obtained by the GP method

Portfolio	Re	Ri	Tb	Li	Composite weights
	[0.5563, 0.6147]	[0.2535, 0.2945]	[0.0928, 0.1102]	0.0390	
BD	[0.0923, 0.1120]	[0.5411, 0.6037]	[0.0800, 0.0823]	0.6021	[0.2246, 0.2732]
DB	[0.2254, 0.2552]	[0.2449, 0.2869]	[0.0562, 0.0903]	[0.2240, 0.2544]	[0.2116, 0.2492]
GB	0.0353	[0.1140, 0.1346]	[0.5961, 0.6301]	[0.0940, 0.1243]	[0.1096, 0.1336]
SH	[0.5975, 0.6426]	0.0374	[0.2146, 0.2313]	0.0495	[0.3690, 0.4279]

Table 10
Interval weights obtained by Sugihara et al.'s upper approximation model

Portfolio	Re	Ri	Tb	Li	Composite weights
	[0.5286, 0.5744]	[0.1436, 0.3524]	[0.0705, 0.1938]	[0.0485, 0.0881]	
BD	[0, 0.2243]	[0.5195, 0.5403]	[0.0413, 0.1322]	[0.5275, 0.6242]	[0.1288, 0.3519]
DB	[0.1135, 0.3784]	[0.1351, 0.3463]	[0.0413, 0.1322]	[0.1560, 0.3516]	[0.1055, 0.3501]
GB	[0.0541, 0.0946]	[0.0866, 0.2381]	0.6612	[0.0879, 0.1319]	[0.1085, 0.2391]
SH	0.5676	[0.0476, 0.0866]	[0.1322, 0.1653]	[0.0330, 0.0879]	[0.3272, 0.3783]

Although the upper approximation model and the GP method draw a consistent conclusion, there still exist some slight differences between them. First, the composite weight for SH in Table 9 is dominant and has no intersection with the other composite weights, while the composite weight for GB is the smallest and also has no intersection with the other composite weights, which means the best and the worst alternatives can be identified for sure. However, in Table 10, the composite weights for SH and GB cannot be clearly cut from the others. Next, for the interval comparison matrix with respect to the criterion "Return", the upper approximation model derives a lower bound weight, zero, for the portfolio BD, which is unacceptable. That is the reason why the lower and upper models (28) and (29) require extra constraints $w_{i^*}^{L} \geqslant \varepsilon$ or $w_i^{L^*} \geqslant \varepsilon$ for $i = 1, \ldots, n$, which are unnecessary for the GP model (21).

6. Concluding remarks

In this paper, we proposed a GP method for obtaining normalized interval weights from an interval comparison matrix, which may be either consistent or inconsistent. The proposed GP method turns out to be also applicable to crisp comparison matrices. The global interval weights prove to be always normalized. Three numerical examples were examined using the GP method. It is shown that the GP method can generate rational weight estimates for both crisp and interval comparison matrices no matter whether they are consistent or not. Comparisons with the interval regression analysis were also made. The GP method shows some advantages over the interval regression analysis for some cases. Finally, we point out that the GP method can be easily extended to the case of fuzzy comparison matrices.

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