European Journal of Operational Research 32 (1987) 107–117
North-Holland

Uncertainty and rank order in the analytic hierarchy process

Thomas L. SAATY and Luis G. VARGAS
Graduate School of Business, University of Pittsburgh, Mervis Hall, Pittsburgh, PA 15260, U.S.A.

Abstract: The Analytic Hierarchy Process uses paired comparisons to derive a scale of relative importance for alternatives. We investigate the effect of uncertainty in judgment on the stability of the rank order of alternatives. The uncertainty experienced by decision makers in making comparisons is measured by associating with each judgment an interval of numerical values. The approach leads to estimating the probability that an alternative or project exchanges rank with other projects. These probabilities are then used to calculate the probability that project would change rank at all. We combine the priority of importance of each project with the probability that it does not change rank, to obtain the final ranking.

Keywords: Analytic hierarchy process decisions, uncertainty, judgments, probability, rank, risk

1. Introduction

When a decision maker is uncertain about his preferences, what effect can this uncertainty have on the final decision? One would be inclined to think that there should be no concern if uncertainty leaves the rank of the alternative chosen unchanged. However this is not true. Even if the rank stays the same, the decision maker may have little confidence in his judgments. In these situations we need a measure of uncertainty to decide whether it is wise to proceed with the best choice or more information is needed to remove some or all of the uncertainty.

There are two types of uncertainty: (a) uncertainty about the occurrence of events, and (b) uncertainty about the range of judgments used to express preferences. The first is beyond the control of the decision maker whereas the second is a consequence of the amount of information available to him and his understanding of the problem.

Our concern here is with the second type of uncertainty. Ordinarily, it is possible to express a numerical judgment within a range or interval of values. Such judgments can give rise to several possible decisions creating uncertainty in the decision making process. This kind of uncertainty is commonly encountered in comparing projects with respect to criteria when we need to know how much resource to allocate to them according to priority.

To generate the paired comparisons one must answer the following kind of question: Given a criterion or property, which of two projects is more important according to this criterion, and how much more important is it? Cardinal judgments are numerical representations of the intensity of preference. After generating a matrix of paired comparisons for a criterion, we use it to derive a scale which represents the relative importance of the alternatives. When several criteria are involved, the final decision is based on a scale for comparing the criteria and on the several scales of the alternatives with respect to the criteria. The overall importance of the alternatives with respect to all criteria is obtained, if the criteria are independent from the alternatives [3], by multiplying the weights of the alternatives under each criterion by the relative importance of the criterion and adding over all the criteria. If there is uncertainty either in the judgments of the criteria, or in the judgments of the alternatives or both, the uncertainty is perpetrated to the scales and thus to the final outcome.

Received January 1986; revised July 1986

To capture the uncertainty experienced by the decision maker in making pairwise comparisons, an interval of numerical values is associated with each judgment and we refer to the pairwise comparison as an *interval pairwise comparison* or simply *interval judgment*.

Our approach will lead to estimating the probability that an alternative or project exchanges rank with other projects. These probabilities are then used to calculate the probability that a project would change rank at all. Finally, we combine the priority of importance of each project with the probability that it does not change rank, to obtain the final ranking for resource allocation. It is this composite outcome that is of interest to both those who provide the judgments and those who allocate the resources. Statistical measures of dispersion can be used to refine the selection and break ties when necessary.

Before we turn to the technical discussion we must point out that there are many methods that can be used to derive a scale from a paired comparison matrix of which the least squares, the logarithmic least squares and the principal right eigenvector methods are the best known. However, since the paired comparisons can be inconsistent (see [2,3]) it can be shown that only the eigenvector method takes this inconsistency into consideration. The danger in using other methods, convenient and known as they may be, is that they can lead to the wrong decision by reversing rank (for examples see [4,5]). For familiarity with the numerical scale and use of reciprocals in paired comparisons the reader should consult reference [2].

2. Background

A typical matrix of interval pairwise comparisons is given by

$$\begin{bmatrix} & 1 & [a_{12}^{L}, a_{12}^{U}] & [a_{13}^{L}, a_{13}^{U}] & \cdots & [a_{1n}^{L}, a_{1n}^{U}] \\ & \vdots & 1 & [a_{23}^{L}, a_{23}^{U}] & \cdots & [a_{2n}^{L}, a_{2n}^{U}] \\ \cdots & \left[\frac{1}{a_{ij}^{U}}, \frac{1}{a_{ij}^{L}}\right] & \cdots & 1 & & \vdots \\ & \vdots & & & \ddots & \vdots \\ & \vdots & & & & 1 \end{bmatrix}$$

where $a_{ij}^{L} \leqslant a_{ij}^{U}$, for all $i,\ j = 1, 2, \ldots, n$.

The problems which arise in working with interval-judgments are of two types: (a) computational, and (b) theoretical.

Computationally, the problem is relatively intractable. It not only involves a method to determine the interval to which each judgment belongs and the distribution of the judgments in the interval, but also deriving the scales of relative values of which there can be a very large number due to the use of intervals. Consider five alternatives and the interval-judgments given in the matrix

$$\begin{bmatrix} 1 & [2,5] & [1,4] & [2,6] & [3,5] \\ & 1 & [2,6] & [\frac{1}{5},\frac{1}{3}] & [3,7] \\ & & 1 & [5,7] & [4,4] \\ & & & 1 & [\frac{1}{3},5] \\ & & & & 1 \end{bmatrix}.$$

Assume that we consider only the integers or reciprocals of the integers between the two bounds. For example, there are 4 integers in the interval [2,5]. The interval $[\frac{1}{3},5]$ has $\frac{1}{3}$ and $\frac{1}{2}$ as reciprocals of integers and 1, 2, 3, 4, and 5 as integers, for a total of 7 judgments. Thus, the total number of possible combinations of judgments for this matrix is 378 000. Hence, 378 000 eigenvectors must be evaluated, if only integers or reciprocals of integers are considered and complete enumeration is used.

One way out of this difficulty is to simulate the behavior of the eigenvector using a reasonable sample size. The problem is to determine the type of distribution to use for such simulation. From a theoretical point of view, the eigenvector is an n-dimensional variable, and statistical measures can be developed for each of its components, but not for the entire vector. Thus, one must derive statistical measures to study rank reversal for a single component and then use them to derive one for the entire vector. For example, a vector each of whose components is the sample average can be formed, and then used to construct confidence intervals for each component. The confidence intervals are then used to determine if two components can reverse rank and in turn if there is a chance that a rank reversal may occur in the entire vector.

Here we study the problem in two parts. The first is for a single matrix of interval judgments and the second for a decision hierarchy. We begin with a matrix of interval judgments and derive a measure of the stability of the eigenvector using the probability of rank reversals among alternatives for this purpose. The question we address is: Given a project, under conditions of uncertainty as above, what is the probability that its rank is reversed with any other project? In particular, what is the likelihood that an important project can end up as unimportant because of uncertainty? How can we be sure that the top priority projects to be funded will not reverse rank with unimportant and hence unfundable ones?

The answer depends on the size of the interval judgments. If a decision-maker states that all entries of the matrix of pairwise comparisons fall between $\frac{1}{9}$ and 9 (the widest range of values used to represent the judgments), it is obvious that the probability of rank reversal among all activities would be equal to unity. However, if the judgments are tightly distributed around a given value a_{ij} (i.e., the length of the interval is relatively small) then the probability of rank reversal should converge to zero as the length of the interval converges to zero. Conversely, large ambiguity in the judgments can render ranking a useless pursuit.

3. Stability of the eigenvector: Interval judgements

Let $A_1, A_2, \ldots, A_n$ be a set of n alternatives compared in pairs according to a given criterion. Let I_{ij} be the interval judgment provided by the decision-maker when comparing alternatives A_i and A_j. Let $I_{ij} \equiv [a_{ij}^{\mathrm{L}}, a_{ij}^{\mathrm{U}}]$. Let $I(w_i) \equiv [w_i^{\mathrm{L}}, w_i^{\mathrm{U}}]$ be the interval of variation of the ith component of the eigenvector associated with the pairwise comparison process, i.e., if $a_{ij} \in I_{ij}$ for all i and j, then $w_i \in I(w_i)$, where w_i is the ith component of the principal right eigenvector of the reciprocal matrix $A = (a_{ij})$. There are several ways of estimating w_i^{L} and w_i^{U}, for all i. Two of these are:

(a) selecting a finite number of values $x_{ij}^{(1)}, \ldots, x_{ij}^{(m)}$ from I_{ij} such that

$$I_{ij} = \bigcup_{k=1}^{m-1} \left[x_{ij}^{(k)}, x_{ij}^{(k+1)}\right], \quad \text{for all } i \text{ and } j,$$

(b) selecting a random sample of a given size from I_{ij}, for all i and j.

In the first case the number of eigenvectors to be computed grows exponentially. The process becomes computationally ineffective even for small matrices. We have used both methods, and found the second faster and as reliable as the first. Let $a_{ij}^{(k)} \in I_{ij}$, $k = 1, 2, \ldots, m$, $i, j = 1, 2, \ldots, n$ be a random sample of size m. Let $\psi^{(k)}$, $k = 1, 2, \ldots, m$, be the eigenvector associated with the pairwise comparisons $a_{ij}^{(k)}$, $i, j = 1, 2, 3, \ldots, n$. Let $I(w_i)$ be the interval of variation of the i-th component of the eigenvector. The intervals $I(w_i)$, $i = 1, 2, \ldots, n$, are bounded and closed in $\mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive reals. This follows from the continuity of the eigenvector as a function of the entries $a_{ij} \in I_{ij}$, $i, j = 1, 2, \ldots, n$, and the boundedness and closure of the interval judgments I_{ij}, $i, j = 1, 2, \ldots, n$.

Let $\{X_i, i = 1, 2, \ldots, n\}$ be the random variables representing the principal right eigenvector components. It is clear that given i and j, if $I(w_i) \cap I(w_j) = \emptyset$ then either $P(X_i > X_j) = 1$ or $P[X_j > X_i] = 1$, i.e., the i-th and j-th component will never reverse rank. On the other hand, if $I(w_i) \cap I(w_j) \neq \emptyset$, then rank may be reversed for some values of $a_{ij} \in I_{ij}$, $i, j = 1, 2, \ldots, n$, and the probability of rank reversal is a function of $I(w_i) \cap I(w_j)$.

4. The probability of rank reversal (p_{ij})

Let p_{ij} be the probability of rank reversal associated with the pair of alternatives (A_i, A_j). Let $F_i(x_i)$ be the cumulative probability distribution of the i-th component of the eigenvector, i.e., $F_i(x_i) = P[X_i \leqslant x_i]$. There are two cases to consider:

(a) $I(w_i) \subseteq I(w_j)$ or $I(w_j) \subseteq I(w_i)$,

and

(b) $I(w_i) \cap I(w_j) \neq \emptyset$, but $I(w_i) \not\subseteq I(w_j)$, $I(w_j) \not\subset I(w_i)$.

In the first case the probability of rank reversal p_{ij} is given by

$$p_{ij} = \begin{cases} F_j(w_i^{\mathrm{U}}) - F_j(w_i^{\mathrm{L}}) & \text{if } I(w_i) \subseteq I(w_j), \\ F_i(w_j^{\mathrm{U}}) - F_i(w_j^{\mathrm{L}}) & \text{if } I(w_j) \subseteq I(w_i). \end{cases}$$

In the second case, we have

$$p_{ij} = \begin{cases} \left[F_i(w_i^{\mathrm{U}}) - F_i(w_j^{\mathrm{L}})\right]\left[F_j(w_i^{\mathrm{U}}) - F_j(w_j^{\mathrm{L}})\right] \\ \quad \text{if } w_i^{\mathrm{L}} < w_j^{\mathrm{L}} < w_i^{\mathrm{U}} < w_j^{\mathrm{U}}, \\ \left[F_i(w_j^{\mathrm{U}}) - F_i(w_i^{\mathrm{L}})\right]\left[F_j(w_j^{\mathrm{U}}) - F_j(w_i^{\mathrm{L}})\right] \\ \quad \text{if } w_j^{\mathrm{L}} < w_i^{\mathrm{L}} < w_j^{\mathrm{U}} < w_i^{\mathrm{U}}. \end{cases} \tag{1}$$

In the first expression in (1), the first term represents the probability that the weight of the ith alternative falls below the upper limit of the weight of the j-th alternative, and the second term represents the probability that the weight of the j-th alternative falls above the weight of the i-th alternative. A similar interpretation can be given to the second expression.

The probability that at least one rank reversal occurs in the eigenvector is given by

$$p = 1 - \prod_{1 \leqslant i < j \leqslant n} (1 - p_{ij}), \tag{2}$$

and the probability that a given alternative will reverse rank with another alternative is given by

$$p_i = 1 - \prod_{j=1}^{n} (1 - p_{ij}), \quad i = 1, 2, \ldots, n. \tag{3}$$

The probability p is a measure of the stability of the eigenvector with respect to changes in I_{ij}. It is not a measure of the accuracy of the true ranks of the alternatives, because the true answer may not be known. Nonetheless, if the true pairwise comparison is the center of the interval-judgment, and the width of I_{ij} is sufficiently small, then the eigenvector is stable. This reasoning is extended to hierarchies in Section 5.

To estimate the upper and lower bounds of each component of the eigenvector, we need the distribution of X_i, $i = 1, 2, \ldots, n$. Some research has been done on the distribution of X_i. In [6] the pairwise comparison matrix was assumed to be consistent, and each entry gamma distributed. The eigenvector components were then shown to be beta distributed. In [7] the pairwise comparison matrix need not be consistent and its entries satisfy the uniform distribution. The eigenvector components were shown statistically to be beta distributed, and when the number of alternatives increases, each of them can be approximated by a truncated normal distribution.

In practice, for $n \geqslant 3$, the intervals $I(w_i)$, $i = 1, 2, \ldots, n$, are not known. One could empirically establish the distribution of the estimated weights, testing for their distribution every time that the method is used; or one could design a simulation for testing the distribution of the components of the eigenvector for a variety of input distributions and for matrices of different size. Since empirical evidence suggests that the distributions of the components could be approximated by a truncated normal distribution, we decided to use the first approach.

To estimate upper and lower bounds for each component of the eigenvector we randomly select values from I_{ij}, $(1 \leqslant i < j \leqslant n)$, and compute the corresponding eigenvector of the matrices formed with the $n(n-1)/2$ values from I_{ij}, $(1 \leqslant i < j \leqslant n)$. Let $\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(m)}$ be the eigenvectors of a sample of size m. Let $\bar{w}_i$ and s_i be the sample mean and standard deviation, respectively, of the i-th component obtained from the sample of eigenvectors $\{\psi^{(k)}, k = 1, 2, \ldots, m\}$.

Let us assume that the components of an eigenvector are approximated by a normal distribution and tested by means of the Kolmogorov–Smirnov test or the χ^2 test at the α level of significance. Let $\hat{I}(w_i)$ and $\hat{I}(w_j)$ be the intervals of variation of w_i and w_j estimated as follows:

$$\hat{I}(w_i) \equiv \left[\bar{w}_i \pm t_{\alpha/2, n-1} s_i\right],$$

$$\hat{I}(w_j) \equiv \left[\bar{w}_j \pm t_{\alpha/2, n-1} s_j\right].$$

There are three cases to consider:

(a) $\hat{I}(w_i) \cap \hat{I}(w_j) = \emptyset$,
(b) $\hat{I}(w_i) \subseteq \hat{I}(w_j)$ or $\hat{I}(w_j) \subseteq \hat{I}(w_i)$, and
(c) $\hat{I}(w_i) \cap \hat{I}(w_j) \neq \emptyset$, but not (b).

(a) In the first case the probability of rank reversal p_{ij} can be made negligible by choosing α sufficiently small.

(b) Let $\hat{I}(w_i) \equiv [a_i, b_i]$ and $\hat{I}(w_j) \equiv [a_j, b_j]$. Then if $\hat{I}(w_i) \subseteq \hat{I}(w_j)$ we have

$$p_{ij} \approx \frac{1}{s_j\sqrt{2\pi}} \int_{a_i}^{b_i} \exp\left\{-\frac{(x - \bar{w}_j)^2}{2s_j^2}\right\} \mathrm{d}x.$$

Similarly, if $\hat{I}(w_j) \subseteq \hat{I}(w_i)$ then

$$p_{ij} \approx \frac{1}{s_i\sqrt{2\pi}} \int_{a_j}^{b_j} \exp\left\{-\frac{(x - \bar{w}_i)^2}{2s_i^2}\right\} \mathrm{d}x.$$

(c) Finally, if we assume that $\hat{I}(w_i) \cap \hat{I}(w_j) \neq \emptyset$ and that neither interval contains the other, let

$$P\left[a_i \leqslant X_i \leqslant b_i\right] = 1 - \alpha,$$

$$P\left[a_j \leqslant X_j \leqslant b_j\right] = 1 - \alpha,$$

then we have

$$p_{ij} = P\left[a_j \leqslant X_i \leqslant b_i\right] \cdot P\left[a_j \leqslant X_j \leqslant b_i\right]$$

$$= \left[\frac{1}{s_i\sqrt{2\pi}} \int_{a_i}^{b_j} \exp\left\{ -\frac{(x-\overline{w}_i)^2}{2s_i^2} \right\} \mathrm{d}x \right]$$

$$\cdot \left[\frac{1}{s_j\sqrt{2\pi}} \int_{a_i}^{b_j} \exp\left\{ -\frac{(x-\overline{w}_i)^2}{2s_i^2} \right\} \mathrm{d}x \right]$$

where $a_j \leqslant a_i \leqslant b_j \leqslant b_i$.

Intuitively, this expression for p_{ij} may be justified as follows. If $a_j \leqslant a_i \leqslant b_j \leqslant b_i$, then $A_i > A_j$ with probability $(1-\alpha/2)$ if $w_i > b_j$, and $A_i < A_j$ with probability $(1-\alpha/2)$ if $w_j > a_i$ (see Figure 1).

Hence, the ranks of A_i and A_j can only be reversed in the interval $[a_i, b_j]$. Thus either $A_i > A_j$ or $A_j > A_i$ with probability $(1-\alpha/2)$, and hence $(1-\alpha/2)$ is the probability that the statement $A_i > A_j$ is true. (Note that this is not the probability that A_i is preferred to A_j). For $\alpha = 0$ we can state with certainty that $A_i > A_j$ in a given region. Finally, from the definition of these probabilities it is easy to see that $p_{ij} = p_{ji}$, $i, j = 1, 2, \ldots, n$, which follows from the general expression for p_{ij} given in (c) above.

We illustrate the foregoing theory with an example. We consider the matrix of interval judgements

$$\begin{bmatrix} 1 & [2,4] & [3,5] & [3,5] \\ & 1 & [\frac{1}{2},1] & [2,5] \\ & & 1 & [\frac{1}{3},1] \\ & & & 1 \end{bmatrix}$$

and generate 100 uniform variates for each of the interval judgments and compute the principal right eigenvectors of the matrices (See Table 1). Next we compute the lower bound, the average, the upper bound, and the standard deviation of the

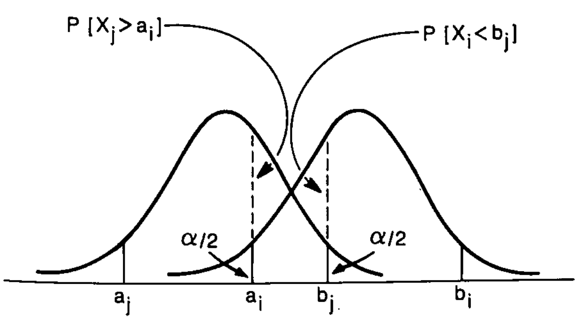

Figure 1. Probability of rank reversal

sample for each component of the eigenvector. We obtain the results shown in Table 2.

We then test the normality of the components:

H_0: $X_i \sim N(\overline{w}_i, s_i)$, $i = 1, 2, \ldots, n$.

by means of (a) the Kolmogorov–Smirnov test, and (b) the χ^2 test.

(a) *The Kolmogorov–Smirnov Test*

Let

$$F(x) = \sum_{k=1}^{x} \tfrac{1}{100}, \quad x = 1, 2, \ldots, 100,$$

and let

$$F_i^*(x) = \Phi_i\left[\frac{x-\overline{w}_i}{s_i}\right],$$

where Φ_i is the cumulative probability distribution of the standardized normal distribution. Table 1 gives F_i^*, $i = 1, 2, 3, 4$. Let

$$D_i = \sup_x |F_i^*(x) - F(x)|, \quad i = 1, 2, 3, 4.$$

D_i, $i = 1, 2, 3, 4$, represent the largest deviations between the theoretical probabilities F_i^* of the components of the eigenvector and the empirical probabilities given by $F(x)$, $x = 1, 2, \ldots, 100$. We obtain Table 3.

For a given level of significance α, the test rejects H_0 if $D_i > D_{n,\alpha}$, where $D_{n,\alpha}$ is given in the tables of the Kolmogorov–Smirnov test (see, for example, [1, p. 661]). For $n = 100$ and $\alpha = 0.05$ we have

$$D_{100,0.05} = \frac{1.36}{\sqrt{100}} = 0.136.$$

Since $D_i < 0.136$ for $i = 1, 2, 3, 4$, we conclude that we can approximate the components of the eigenvector by a normal distribution.

(b) *The χ^2 test:*

To apply the χ^2 test we categorize the data of Table 1 as shown in Table 4.

Next we compute the statistic

$$u_i = \sum_{k=1}^{m} \frac{\{y_{ik} - np_k\}^2}{np_k}$$

Table 1

x_1	F_1^*	x_2	F_2^*	x_3	F_3^*	x_4	F_4^*
0.43744	0.00442	0.16538	0.01441	0.11109	0.01401	0.10112	0.03739
0.44084	0.00619	0.16833	0.02007	0.12071	0.04962	0.10153	0.03966
0.44618	0.01037	0.17473	0.03931	0.12080	0.05013	0.10310	0.04952
0.46001	0.03519	0.17575	0.04348	0.12199	0.05762	0.10453	0.06009
0.46123	0.03883	0.17734	0.05068	0.12231	0.05977	0.10468	0.06126
0.46863	0.06796	0.17812	0.05455	0.12307	0.06519	0.10639	0.07641
0.46925	0.07103	0.18016	0.06577	0.12334	0.06717	0.10786	0.09153
0.46973	0.07343	0.18065	0.06869	0.12480	0.07878	0.10878	0.10198
0.46973	0.07343	0.18065	0.06869	0.12480	0.07878	0.10878	0.10198
0.47300	0.09173	0.18142	0.07351	0.12526	0.08267	0.10880	0.10216
0.47404	0.09820	0.18151	0.07412	0.12586	0.08809	0.10942	0.10974
0.47569	0.10916	0.18286	0.08319	0.12598	0.08914	0.11002	0.11743
0.47842	0.12905	0.18557	0.10399	0.12733	0.10223	0.11123	0.13396
0.47872	0.13131	0.18565	0.10465	0.12747	0.10367	0.11167	0.14041
0.47980	0.13993	0.18735	0.11944	0.12869	0.11689	0.11268	0.15578
0.47987	0.14051	0.18817	0.12716	0.12902	0.12058	0.11330	0.16583
0.48107	0.15060	0.18878	0.13306	0.12967	0.12817	0.11332	0.16613
0.48201	0.15871	0.18963	0.14160	0.12967	0.12821	0.11385	0.17507
0.48266	0.16459	0.18974	0.14276	0.13241	0.16373	0.11441	0.18481
0.48318	0.16932	0.19055	0.15120	0.13280	0.16936	0.11468	0.18963
0.48542	0.19071	0.19064	0.15214	0.13303	0.17261	0.11514	0.19791
0.48684	0.20512	0.19146	0.16120	0.13490	0.20106	0.11514	0.19795
0.48715	0.20839	0.19181	0.16507	0.13535	0.20837	0.11535	0.20181
0.48951	0.23391	0.19202	1.16751	0.13556	0.21184	0.11550	0.20454
0.49071	0.24754	0.19560	0.21152	0.13625	0.22331	0.11558	0.20614
0.49135	0.25499	0.19616	0.21901	0.13628	0.22383	0.11583	0.21084
0.49212	0.26412	0.19655	0.22432	0.13765	0.24778	0.11686	0.23100
0.49242	0.26768	0.19821	0.24765	0.13765	0.24781	0.11717	0.23725
0.49286	0.27298	0.19876	0.25557	0.13837	0.26078	0.11724	0.23860
0.49307	0.27556	0.20093	0.28838	0.13848	0.26078	0.11724	0.23860
0.49307	0.27556	0.20093	0.28838	0.13848	0.26292	0.11738	0.24159
0.49412	0.28861	0.20207	0.30642	0.13849	0.26312	0.11894	0.27496
0.49594	0.31171	0.20209	0.30687	0.13855	0.26410	0.11901	0.27634
0.49599	0.31235	0.20260	0.31504	0.13915	0.27550	0.11911	0.27858
0.49805	0.33964	0.20326	0.32587	0.13961	0.28422	0.11932	0.28329
0.49825	0.34231	0.20328	0.32622	0.14102	0.31207	0.12032	0.30618
0.49926	0.35598	0.20331	0.32662	0.14149	0.32153	0.12065	0.31395
0.49995	0.36546	0.20437	0.34434	0.14208	0.33380	0.12072	0.31555
0.50052	0.37335	0.20439	0.34470	0.14226	0.33758	0.12126	0.32842
0.50052	0.37344	0.20450	0.34660	0.14274	0.34751	0.12144	0.33256
0.50056	0.37394	0.20534	0.36098	0.14296	0.35222	0.12159	0.33629
0.50122	0.38318	0.20594	0.37128	0.14334	0.36039	0.12169	0.33629
0.50166	0.38941	0.20806	0.40864	0.14338	0.36114	0.12183	0.34213
0.50196	0.39367	0.20822	0.41163	0.14371	0.36832	0.12185	0.34262
0.50234	0.39900	0.20848	0.41622	0.14469	0.38964	0.12201	0.34653
0.50271	0.40424	0.21009	0.44519	0.14527	0.40244	0.12224	0.35211
0.50346	0.41497	0.21091	0.46017	0.14547	0.40688	0.12243	0.35693
0.50372	0.41875	0.21166	0.47368	0.14594	0.41726	0.12311	0.37402
0.50397	0.42230	0.21279	0.49420	0.14611	0.42103	0.12314	0.37471
0.50437	0.42807	0.21528	0.53920	0.14647	0.42912	0.12358	0.38587
0.50894	0.49445	0.21562	0.54550	0.14850	0.47515	0.12364	0.38728
0.51010	0.51120	0.21608	0.55381	0.14855	0.47614	0.12399	0.39639
0.51239	0.54447	0.21709	0.57202	0.14892	0.48447	0.12433	0.40518
0.51318	0.55590	0.21828	0.59347	0.14955	0.49875	0.12471	0.41491
0.51325	0.55695	0.21839	0.59536	0.14999	0.50853	0.12500	0.42295
0.51438	0.57329	0.21888	0.60400	0.15021	0.51346	0.12508	0.42465
0.51499	0.58213	0.21917	0.60919	0.15089	0.52888	0.12523	0.42850
0.51562	0.59108	0.21921	0.60993	0.15135	0.53923	0.12542	0.43332

Table 1 (continued)

x_1	F_1^*	x_2	F_2^*	x_3	F_3^*	x_4	F_4^*
0.51600	0.59649	0.21954	0.61563	0.15252	0.56569	0.12677	0.46903
0.51703	0.61107	0.21974	0.61919	0.15263	0.56825	0.12891	0.52509
0.51724	0.61415	0.21980	0.62019	0.15265	0.56869	0.12930	0.53526
0.51797	0.62436	0.22068	0.63551	0.15331	0.58340	0.12939	0.53767
0.51853	0.63213	0.22070	0.63590	0.15550	0.63160	0.12995	0.55234
0.51946	0.64490	0.22084	0.63836	0.15601	0.64243	0.13003	0.55451
0.52049	0.65892	0.22121	0.64473	0.15634	0.64952	0.13114	0.58342
0.52053	0.65939	0.22229	0.66290	0.15660	0.65507	0.13140	0.59015
0.52157	0.67336	0.22282	0.67185	0.15690	0.66138	0.13249	0.61827
0.52225	0.68231	0.22307	0.67597	0.15711	0.66581	0.13303	0.63176
0.52226	0.68244	0.22323	0.67859	0.15802	0.68438	0.13319	0.63587
0.52246	0.68501	0.22380	0.68782	0.15817	0.68749	0.13403	0.65672
0.52246	0.68501	0.22380	0.68782	0.15817	0.68749	0.13403	0.65672
0.52283	0.68977	0.22395	0.69027	0.15819	0.68774	0.13474	0.67381
0.52307	0.69289	0.22399	0.69089	0.15853	0.69471	0.13571	0.69659
0.52369	0.70080	0.22414	0.69336	0.16085	0.73930	0.13664	0.71782
0.52424	0.70776	0.22537	0.71282	0.16147	0.75065	0.13717	0.72953
0.52537	0.72181	0.22603	0.72295	0.16195	0.75918	0.13765	0.74000
0.52593	0.72862	0.22685	0.73527	0.16298	0.77691	0.13838	0.75540
0.52654	0.73597	0.22722	0.74079	0.16321	0.78085	0.13965	0.78085
0.52949	0.76997	0.22725	0.74134	0.16336	0.78337	0.13981	0.78402
0.53057	0.78176	0.22735	0.74280	0.16366	0.78839	0.14023	0.79199
0.53186	0.79535	0.22736	0.74296	0.16505	0.81036	0.14133	0.81223
0.53373	0.81416	0.22737	0.74311	0.16527	0.81381	0.14245	0.83152
0.53475	0.82393	0.22831	0.75666	0.16570	0.82022	0.14419	0.85890
0.53550	0.83099	0.22842	0.75818	0.16609	0.82602	0.14493	0.86959
0.53574	0.83318	0.22959	0.77449	0.16688	0.83726	0.14505	0.87123
0.53668	0.84157	0.23045	0.78617	0.16784	0.85037	0.14508	0.87167
0.53678	0.84246	0.23278	0.81570	0.16788	0.85086	0.14509	0.87181
0.53713	0.84552	0.23630	0.85541	0.16805	0.85317	0.14527	0.87427
0.50819	0.85455	0.23662	0.85871	0.16824	0.85553	0.14655	0.89100
0.53840	0.85631	0.23744	0.86689	0.16936	0.86954	0.14691	0.89543
0.53860	0.85800	0.23770	0.86944	0.16938	0.86979	0.14739	0.90105
0.54184	0.88295	0.24013	0.89162	0.17001	0.87732	0.14784	0.90615
0.54221	0.88559	0.24103	0.89910	0.17195	0.89837	0.14874	0.91580
0.54278	0.88960	0.24256	0.91097	0.17200	0.89890	0.14961	0.92449
0.54397	0.89763	0.24336	0.91678	0.17217	0.90063	0.15133	0.93952
0.54505	0.90453	0.24453	0.92472	0.17222	0.90108	0.15284	0.95070
0.54971	0.93052	0.24519	0.92893	0.17279	0.90665	0.15338	0.95424
0.55381	0.94858	0.24627	0.93547	0.17381	0.91604	0.15379	0.95683
0.55820	0.96351	0.24798	0.94482	0.17616	0.93496	0.15512	0.96437
0.55957	0.96739	0.25435	0.97061	0.18400	0.97518	0.15654	0.97116
0.56142	0.97204	0.25624	0.97596	0.19024	0.98968	0.16241	0.98883
0.56493	0.97934	0.26331	0.98917	0.19215	0.99224	0.16253	0.98905
0.56964	0.98648	0.27082	0.99567	0.19709	0.99638	0.16332	0.99043

Table 2

Minimum ($\hat{w}_i^L$)	Average ($\bar{w}_i$)	Maximum ($\hat{w}_i^U$)	Std. dev. (s_i)
0.4374	0.5093	0.5696	0.0273
0.1654	0.2131	0.2708	0.0219
0.1111	0.1496	0.1971	0.0175
0.1011	0.1280	0.1633	0.0151

Table 3

Component i	Maximum deviation D_i
1	0.0519
2	0.0735
3	0.0517
4	0.1267

Table 4
Number of times a component falls within (x, y) standard deviations from the mean

No. of std. dev. (x, y)	Component (y_{ik}) 1	2	3	4	p_k
$(-\infty, -3)$	0	0	0	0	0.0013
$(-3, -2)$	3	2	1	0	0.0215
$(-2, -1)$	13	18	16	14	0.1359
$(-1, -0.5)$	14	11	16	20	0.1498
$(-0.5, +0.5)$	39	39	36	35	0.3830
$(0.5, 1)$	13	14	13	10	0.1498
$(1, 2)$	16	14	15	18	0.1359
$(2, 3)$	2	2	3	3	0.0215
$(3, \infty)$	0	0	0	0	0.0013

for each component, where m is the number of categories, y_{ik} is the frequency corresponding to the k-th category in the i-th component, n is the sample size, and p_k is the probability contained in the k-th category by the standardized normal distribution. We reject H_0: $X_i \sim N(0, 1)$ if $u_i > \chi^2_{k-1,\alpha}$. For $\alpha = 0.05$ we have $\chi^2_{0.0.05} = 15.507$ and since $u_1 = 1.1381$, $u_2 = 2.8587$, $u_3 = 2.5332$, and $u_4 = 7.3116$ we accept the normality assumption for all components of the eigenvector at the 5 per cent level of significance.

To compute the probabilities p_{ij}, p_i and p of rank reversal we first construct confidence intervals for each of the components of the eigenvector. It is known that the lower and upper bounds of a 99% confidence interval are given in Table 5.

Using these upper and lower bounds b_i and a_i, respectively, and the results of this section we have

$$p_{12} = p_{13} = p_{14} = 0,$$
$$p_{23} = 0.006509, \quad p_{24} = 0.000274,$$
$$p_{34} = 0.779106.$$

The probabilities p_i, $i = 1, 2, 3, 4$ that a given alternative will reverse rank with another alternative are given by

Table 5

Lower bound $a_i \equiv \bar{w}_i - 2.58\, s_i$	Upper bound $b_i \equiv \bar{w}_i + 2.58\, s_i$
0.4388	0.5798
0.1567	0.2695
0.1043	0.1949
0.0890	0.1669

$$p_1 = 0,$$
$$\begin{aligned} p_2 &= 1 - (1 - p_{12})(1 - p_{23})(1 - p_{24}) \\ &= 1 - (0.993491)(0.999726) \\ &= 0.006781, \end{aligned}$$
$$\begin{aligned} p_3 &= 1 - (1 - p_{13})(1 - p_{23})(1 - p_{34}) \\ &= 1 - (0.993491)(0.220894) \\ &= 0.780544, \end{aligned}$$
$$\begin{aligned} p_4 &= 1 - (1 - p_{14})(1 - p_{24})(1 - p_{34}) \\ &= 1 - (0.999726)(0.220894) \\ &= 0.779167. \end{aligned}$$

Finally, the probability that at least one rank reversal will occur is given by

$$\begin{aligned} p &= 1 - (1 - 0)(1 - 0.006509) \\ &\quad \cdot (1 - 0.000274)(1 - 0.779106) \\ &= 0.781145. \end{aligned}$$

We realize that what is considered a high/low priority and a high/low probability of rank reversal depend on individual perception of relative importance and on his risk proneness or aversion. Thus it is pointless to give precise definitions for them. What we will do is to use some common sense rules to assist in selecting a 'best' alternative.

Given the priorities w_i, and the probabilities of rank reversal p_{ij}, p_i and p we now must decide which alternative to select. Let S_1 and S_2 be the sets of indices of the alternatives with high and low priorities, respectively. There are two situations to consider: (a) all the p_{ij} are small, and (b) some of the p_{ij} are large. If all the p_{ij} are small then p is small, one selects the alternative which maximizes the product

$$w_i(1 - p_i)$$

of the priority w_i and the probability of no rank reversal $(1 - p_i)$. Let S_3 be the set of alternatives with large p_{ij}. Whether p is small or large there are three cases to consider:

(1) $S_3 \subseteq S_1$: Select alternative k for which

$$w_k = \max_{i \in S_1 - S_3 \neq \emptyset} \{ w_i(1 - p_i) \}.$$

Otherwise, more information is needed to decide because the uncertainty is large.

(2) $S_3 \subseteq S_2$: Select alternative k for which

$$w_k = \max_{i \in S_1} \{ w_i(1 - p_i) \}.$$

(3) $S_3 \subseteq S_1 \cup S_2 (S_1 \cap S_2 = \emptyset)$: Select alternative k for which

$$w_k = \max_{i \in S_1 - (S_1 \cap S_3) \neq \emptyset} \{ w_i (1 - p_i) \}$$

Otherwise, more information is needed to decide.

We wish to emphasize that in the final analysis no probabilistic approach can show the way to a 'best' decision when the information used involves substantial uncertainty.

5. Interval judgments and hierarchies

Let us now look at a decision problem involving several criteria. When there is no uncertainty in the judgments, it is known [2,3] that the alternatives are compared with respect to each criterion, the resulting priorities are weighted by the priorities of the criteria, and the sum taken over all the criteria to obtain an overall ranking of the alternatives. The problem here is to select a 'best' alternative when there is uncertainty in the judgments relating to the criteria as well as to the alternatives. Thus we assume that a decision maker is comparing alternatives according to criteria and that interval judgments are also used at the level of the criteria.

Let Y_j, $j = 1, 2, \ldots, m$, be random variables representing the components of the principal right eigenvector [2,3,4] arising from paired comparisons of the criteria. Let X_{ij}, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$, be random variables representing the i-th component of the eigenvector corresponding to paired comparisons of the alternatives with respect to criterion j.

There are two cases to consider: (a) there is no uncertainty involved in the priorization of the criteria, and (b) there is uncertainty in the criteria.

(a) Let y_j, $j = 1, 2, \ldots, m$, be the weights of the criteria. The principle of hierarchic composition yields the random variables

$$Z_i = \sum_{j=1}^{m} X_{ij} y_j, \quad i = 1, 2, \ldots, n,$$

which are also normally distributed with mean $\sum_{j=1}^{m} \bar{w}_{ij} y_j$ and standard deviation $[\sum_{j=1}^{m} s_{ij}^2 y_j]^{1/2}$ if we assume that

$$X_{ij} \sim \mathrm{N}(\bar{w}_{ij}, s_{ij}), \quad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m.$$

The probability of rank reversal is then obtained as in Section 4.

(b) In the case of uncertainty about the criteria, the principle of hierarchic composition yields a vector of random variables. To compute the probability of rank reversal for the hierarchy we can proceed in two ways:

(1) Find the distribution of the composition, i.e.

$$\sum_{j=1}^{m} X_{ij} Y_j, \quad i = 1, 2, 3, \ldots, n.$$

This alternative is not particularly attractive because it requires knowledge of the distribution of $\sum_{j=1}^{m} X_{ij} Y_j$, which is not known for X_{ij} and Y_j distributed according to a non-standardized normal distribution, but only for N(0, 1).

(2) Use the probabilities of rank reversal derived in Section 4 for each level. We find this a tractable approach.

As before, let q_{kh} be the probability that criteria k and h reverse rank when compared with respect to the overall goal. Let q be the probability that there is at least one rank reversal among the criteria. We have

$$q = 1 - \prod_{i \leq k < h \leq n} (1 - q_{kh}). \tag{4}$$

Let q_k be the probability that the k-th criterion reverses rank with at least one other criterion. We have

$$q_k = 1 - \prod_{\substack{h=1 \\ h \neq k}}^{m} (1 - q_{kh}). \tag{5}$$

Let $p^{(k)}$ be the probability that there is at least one rank reversal among the alternatives under the k-th criterion. Let $p_{ij}^{(k)}$ be the probability of rank reversal of the i-th and j-th alternatives when compared according to the kth criterion. We have

$$p^{(k)} = 1 - \prod_{1 \leq i < j \leq n} \left(1 - p_{ij}^{(k)}\right). \tag{6}$$

The probability P_{H} that there is at least one rank reversal for the entire hierarchy is given by:

$$P_{\mathrm{H}} = \begin{cases} 1 - (1 - q) \prod_{k=1}^{m} \left[1 - p^{(k)}\right] \\ \quad \text{if } p^{(k)} \neq 0 \quad \text{for some } k, \\ 0 \quad \text{if } p^{(k)} = 0 \quad \text{for all } k, \end{cases} \tag{7}$$

for if no rank reversal can take place for all the alternatives under all criteria, then the weights of the criteria do not affect their ranking. (This measure can be extended to hierarchies with more than 3 levels.)

Let $p_i^{(k)}$ be the probability that the i-th alternative reverses rank with at least one other alternative, under the k-th criterion, $k=1, 2,\dots, m$. We have

$$p_i^{(k)}=1-\prod_{\substack{j=1\\ j\neq i}}^{n}\left[1-p_{ij}^{(k)}\right],\quad i=1,2,\dots,n. \tag{8}$$

For a given alternative the probability $(1-P_{\mathrm{H}}^{(i)})$ of no-rank reversal with any other alternative according to all the criteria is given by

$$1-P_{\mathrm{H}}^{(i)}=(1-q)\left\{\prod_{k=1}^{m}\left[1-p_i^{(k)}\right]\right\},$$
$$i=1,2,\dots,n. \tag{9}$$

the term $(1-q)$ is needed to take into account the probability of rank reversal of the criteria.

Let $\overline{w}^{(k)}=(\overline{w}_1^{(k)}, \overline{w}_2^{(k)},\dots,\overline{w}_n^{(k)})$ be the vector of average priorities of the alternatives according to the kth criterion. Let $\bar{v}=(\bar{v}_1, \bar{v}_2,\dots,\bar{v}_m)$ be the average priority vector of the criteria. The average composite priority vector of the alternatives, $\overline{w}=(\overline{w}_1, \overline{w}_2,\dots,\overline{w}_n)$, is given by

$$\overline{w}_i=\sum_{k=1}^{m}\overline{w}_i^{(k)}\bar{v}_k,\quad i=1,2,\dots,n, \tag{10}$$

or

$$\overline{w}=\overline{w}^{(k)}\cdot\bar{v}^T$$

where $\bar{v}^{\mathrm{T}}$ is the transpose of the vector $\bar{v}$.

As in the case of a single criterion, an alternative with a large composite priority may have a small probability of no rank reversal. Conversely, an alternative with a small composite priority may have a large probability of no rank reversal. The rules developed in Section 4 can be used to judge alternatives under each criterion. How should we proceed when all criteria are considered at once? A tradeoff between the uncertainty and priority of the alternatives must be made. If uncertainty is large, one must gather more information on the problem to narrow the width of the interval judgments. To determine what type of information is needed one could start by

(1) examining $\{q_{kh}\}$ and $\{p_{ij}^{(k)}\}$ and selecting the criteria and the alternatives with largest probability of rank reversal;

(2) computing the ratios of the average priorities for the criteria under the overall goal and for the alternatives under the criteria;

(3) testing to determine whether the ratios fall within the interval judgments, and

(4) modifying the interval judgments so that the ratios of the average priorities fall within the upper and lower bounds of the intervals.

We first examine the level of uncertainty of the criteria. It is worth pointing out that if high priority criteria also have large uncertainty associated with them, little can be done to make a best choice without additional information.

Let C_1 and C_2 be the sets of high and low priority criteria, respectively. Let C_3 be the set of criteria with high probability of rank reversal q_{hk}. There are three cases to consider:

(a) $C_3\subseteq C_1$; (b) $C_3\subseteq C_2$; and (c) $C_3\subseteq C_1\cup C_2 (C_1\cap C_2=\emptyset)$.

If $C_3\subseteq C_2$, then one examines the uncertainty of the alternatives. Otherwise, as noted above more information is needed to decrease the uncertainty about the criteria. Thus we only examine the second case, i.e., $C_3\subseteq C_2$.

Let $S_1^{(k)}$ and $S_2^{(k)}$, $S_1^{(k)}\cap S_2^{(k)}=\emptyset$, be the sets of high and low priority alternatives under the k-th criterion. Let $S_3^{(k)}$ be the set of alternatives with high probability of rank reversal. As in Section 4, there are also three cases to consider:

(1) $S_3^{(k)}\subseteq S_1^{(k)}$; (2) $S_3^{(k)}\subseteq S_2^{(k)}$; and (3) $S_3^{(k)}\subseteq S_1^{(k)}\cup S_2^{(k)}$, for some or all $k=1, 2,\dots, m$.

There are 3^m possible combinations of the 3 cases for all the criteria. It is practically impossible to provide selection rules in all the cases (with (a) and (c) we would have 3^{3m} cases). Nonetheless, there are three important situations for which general selection rules may be developed:

– The $p_{ij}^{(k)}$ are small for all i, j and k:
In that case one can select alternatives according to

$$\max_i\left\{\overline{w}_i\left[1-P_{\mathrm{H}}^{(i)}\right]\right\} \tag{11}$$

where

$$\overline{w}_i=\sum_{k\in C_1\cup(C_2-C_3)}\overline{w}_i^{(k)}\bar{v}_k.$$

– The $p_{ij}^{(k)}$ are small for all i and j, for some k:

If the criteria for which the $p_{ij}^{(k)}$ are large belong to $C_3 \subseteq C_2$, then (11) can again be used. Otherwise, more information is needed about the alternatives with respect to these criteria.

– The $p_{ij}^{(k)}$ are large for some i and j and for all k:
If $S_3^{(k)} = S_3^{(h)}$, for all k and h ($k \neq h$), and $S_3^{(k)} \subseteq S_2^{(k)}$, for all k, then (11) can be used to select a 'best' alternative.

8. Conclusion

Uncertainty in judgments can be expressed in two ways: (a) as a point estimate with a probability distribution function, and (b) as an interval estimate without a probability distribution. Most of the work has focused on the first area, but with little practical use. This may be attributable to the difficulty of assigning probabilities to the uncertainty of subjective judgments. Even when one has these distributions, the problem of deriving the principal eigenvector from pairwise comparisons is complicated and not amenable to a direct synthesis of probability distributions.

The interval estimate approach can be implemented more easily by means of simulation assuming that all points of the interval are equiprobable, i.e., the simulation assumes that the random variables are uniformly distributed. Using the Kolmogorov–Smirnov test (or the χ^2 test), the eigenvector components are then tested to determine if they are normally distributed. Once the distribution of the eigenvector components has been (empirically) established one can compute the probability of rank reversal in making the 'best' choice of alternative in a decision.

References

[1] Rohatgi, V.K., *An Introduction to Probability Theory and Mathematical Statistics*, Wiley, New York, 1976.
[2] Saaty, T.L., *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.
[3] Saaty, T.L., "Axiomatic foundation of the Analytic Hierarchy Process", *Management Science* 23(7) (1986) 841–855.
[4] Saaty, T.L., and Vargas, L.G., "Inconsistency and rank preservation", *Journal of Mathematical Psychology* 28 (3) (1984) 205–214.
[5] Saaty, T.L. and Vargas, L.G., "Comparison of eigenvalue, logarithmic least squares and least squares methods in estimating ratios", *Mathematical Modeling* 5 (1984) 309–324.
[6] Vargas, L.G., "Reciprocal matrices with random coefficients," *Mathematical Modeling* 3 (1982), 69–81.
[7] Zahedi, F., "Estimation techniques and their statistical properties in the Analytic Hierarchy Process", University of Massachusetts working paper, December 1984.