European Journal of Operational Research 247 (2015) 867–871

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Decision Support

A note on “A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making”

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ARTICLE INFO

Article history:
Received 11 January 2014
Accepted 8 June 2015
Available online 12 June 2015

Keywords:
Goal programming
Consistency
Incomplete interval multiplicative preference relation
Pairwise comparison

ABSTRACT

In a recently published paper by Liu et al. [Liu, F., Zhang, W.G., Wang, Z.X. (2012). A goal programming model for incomplete interval multiplicative preference relations and its application in group decision-making. European Journal of Operational Research 218, 747–754], two equations are introduced to define consistency of incomplete interval multiplicative preference relations (IMPRs) and employed to develop a goal programming model for estimating missing values. This note illustrates that such consistency definition and estimation model are technically incorrect. New transitivity conditions are proposed to define consistent IMPRs, and a two-stage goal programming approach is devised to estimate missing values for incomplete IMPRs.

1. Introduction

The interval multiplicative preference relation (IMPR) introduced by Saaty and Vargas (1987) is a powerful and efficient tool for expressing decision-makers’ (DMs’) pairwise judgments with uncertainty, and has been used to model uncertainty in multi-criteria decision analysis (Durbach & Stewart, 2012). On the other hand, because of complexity of decision problems and the limited ability of human pairwise comparisons, DMs may be unable to provide their preferences over some pairs of decision alternatives. As such, incomplete judgments are resulted and decision making problems with incomplete preference relations have been received more and more attention in recent years (Mattila & Virtanen, 2015; Punkka & Salo, 2013; Wang & Li, 2015). The consistency of preference relations is a crucial foundation for estimating missing values and obtaining a reasonable and reliable decision result (Brunelli, Canal, & Fedrizzi, 2013; Brunelli & Fedrizzi, 2015; Kou, Ergu, & Shang, 2014). In order to characterize inconsistency indices for pairwise comparison matrices, Brunelli and Fedrizzi (2014) put forward five axioms. One of them is the invariance with respect to permutations of decision alternatives.

In a recent paper, Liu et al. (2012) introduced two equations to define consistency of incomplete IMPRs, and developed a goal programming model to estimate missing values. However, a close investigation demonstrates that such consistency definition is highly dependent on alternative labels and not robust to permutations of decision alternatives, and the goal programming model suffers from serious drawbacks. The aim of this note is to point out and correct errors in the consistency definition and the goal programming model.

At first, the terminology and notation used are mainly introduced as follows.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of n alternatives, an IMPR $\bar{A}$ on X is characterized by an interval judgment matrix $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ with

$$1/S \le a_{ij}^- \le a_{ij}^+ \le S,\ a_{ij}^- a_{ji}^+ = 1,\ a_{ii}^- = a_{ii}^+ = 1, \quad i, j = 1, 2, \ldots, n \tag{1.1}$$

where $\bar{a}_{ij}$ is an interval judgment selected in a bounded scale $[1/S, S]$, and indicates that x_i is between a_{ij}^- and a_{ij}^+ times preferred than x_j.

The pairwise judgments are usually expressed on the 1-9 Saaty’s scale, i.e. $S = 9$. If some judgments in $\bar{A}$ are missing, then $\bar{A}$ is referred to as an incomplete IMPR. Obviously, the missing values may be lower or/and upper bounds of interval judgments.

2. The invalidity of the consistency definition and the goal programming model by Liu et al. (2012)

Liu et al. (2012) employed two formulae (see Liu et al. (2012), Eq. (6) on page 748) to construct two multiplicative preference

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http://dx.doi.org/10.1016/j.ejor.2015.06.015

relations $C = (c_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ from an IMPR $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a^-_{ij}, a^+_{ij}])_{n\times n}$. The formulae can be rewritten as follows by using the notation in this note

$$c_{ij} = \begin{cases} a^+_{ij} & i < j \\ 1 & i = j \\ a^-_{ij} & i > j \end{cases} \qquad d_{ij} = \begin{cases} a^-_{ij} & i < j \\ 1 & i = j \\ a^+_{ij} & i > j \end{cases} \tag{2.1}$$

When $\bar{A}$ is an incomplete IMPR, C and D constructed from $\bar{A}$ by using (2.1) may be two incomplete comparison matrices. Based on (2.1), Liu et al. (2012) proposed the consistency definition for incomplete IMPRs (see Liu et al. (2012), Definition 7 on page 749) as follows:

Definition 2.1. (Liu et al., 2012). Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a^-_{ij}, a^+_{ij}])_{n\times n}$ be an incomplete IMPR. $\bar{A}$ is called consistent, if all the known elements of C and D defined by (2.1) satisfy the following condition:

$$c_{ij} = c_{ik}c_{kj},\ d_{ij} = d_{ik}d_{kj} \qquad i, j, k = 1, 2, \ldots, n \tag{2.2}$$

Next, an example is provided to illustrate that such consistency definition is technically wrong.

Example 1. Consider a decision making problem with four alternatives. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of the alternatives. After comparing pairs of the alternatives, a DM furnishes an incomplete IMPR as follows:

$$\bar{A}_1 = (\bar{a}_{ij})_{4\times 4} = \left([a^-_{ij}, a^+_{ij}]\right)_{4\times 4}$$

$$= \begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} [1,1] & [2,3] & [3,6] & [1,3] \\ [1/3,1/2] & [1,1] & [3/2,2] & - \\ [1/6,1/3] & [1/2,2/3] & [1,1] & [1/3,1/2] \\ [1/3,1] & - & [2,3] & [1,1] \end{bmatrix} \end{array}$$

where "–" indicates the missing or unknown values.

As per (2.1), the two incomplete multiplicative preference relations are determined as

$$C_1 = \left(c_{ij}\right)_{4\times 4} = \begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1 & 3 & 6 & 3 \\ 1/3 & 1 & 2 & - \\ 1/6 & 1/2 & 1 & 1/2 \\ 1/3 & - & 2 & 1 \end{bmatrix} \end{array}$$

$$D_1 = \left(d_{ij}\right)_{4\times 4} = \begin{array}{cc} & \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1/2 & 1 & 3/2 & - \\ 1/3 & 2/3 & 1 & 1/3 \\ 1 & - & 3 & 1 \end{bmatrix} \end{array}$$

One can easily verify that C_1 and D_1 satisfy (2.2). According to Definition 2.1, $\bar{A}_1$ is a consistent and incomplete IMPR.

Let σ be a permutation of $\{1, 2, 3, 4\}$ satisfying $\sigma(1) = 1, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$, then we have the following incomplete IMPR.

$$\bar{A}'_1 = (\bar{a}'_{ij})_{4\times 4} = \left([a'^-_{ij}, a'^+_{ij}]\right)_{4\times 4} = \left([a^-_{\sigma(i)\sigma(j)}, a^+_{\sigma(i)\sigma(j)}]\right)_{4\times 4}$$

$$= \begin{array}{cc} & \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \\ x_3 \\ x_2 \end{array} & \begin{bmatrix} [1,1] & [1,3] & [3,6] & [2,3] \\ [1/3,1] & [1,1] & [2,3] & - \\ [1/6,1/3] & [1/3,1/2] & [1,1] & [1/2,2/3] \\ [1/3,1/2] & - & [3/2,2] & [1,1] \end{bmatrix} \end{array}$$

Similarly, by (2.1), the two incomplete multiplicative preference relations C'_1 and D'_1 are obtained as:

$$C'_1 = \left(c'_{ij}\right)_{4\times 4} = \begin{array}{cc} & \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \\ x_3 \\ x_2 \end{array} & \begin{bmatrix} 1 & 3 & 6 & 3 \\ 1/3 & 1 & 3 & - \\ 1/6 & 1/3 & 1 & 2/3 \\ 1/3 & - & 3/2 & 1 \end{bmatrix} \end{array}$$

$$D'_1 = \left(d'_{ij}\right)_{4\times 4} = \begin{array}{cc} & \begin{array}{cccc} x_1 & x_4 & x_3 & x_2 \end{array} \\ \begin{array}{c} x_1 \\ x_4 \\ x_3 \\ x_2 \end{array} & \begin{bmatrix} 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & - \\ 1/3 & 1/2 & 1 & 1/2 \\ 1/2 & - & 2 & 1 \end{bmatrix} \end{array}$$

As $c'_{13} = 6 \neq 9 = 3 \times 3 = c'_{12}c'_{23}$ and $d'_{13} = 3 \neq 2 = 1 \times 2 = d'_{12}d'_{23}$, C'_1 and D'_1 do not satisfy (2.2). By Definition 2.1, $\bar{A}'_1$ is an inconsistent and incomplete IMPR.

Example 1 clearly indicates that, for the identical judgment information with different labeling for alternatives, Definition 2.1 yields contradictory results. In other words, Definition 2.1 is not robust to permutations of decision alternatives. From the viewpoint of the pairwise comparison, the consistency of preference relations should be independent of alternative labels such that it has the invariance with respect to permutations of decision alternatives. Therefore, Definition 2.1 by Liu et al. (2012) is technically incorrect.

In addition, incomplete IMPRs with extremely inconsistent judgments may be determined as consistent IMPRs as per Definition 2.1. Moreover, incorrect results may be obtained when the matrices C and D defined by (2.1) are used to check acceptability and estimate missing values for incomplete IMPRs.

Example 2. Consider the following two incomplete IMPRs:

$$\bar{A}_2 = \left([a^-_{ij}, a^+_{ij}]\right)_{3\times 3} = \begin{bmatrix} [1,1] & [9,9] & [-,1/7] \\ [1/9,1/9] & [1,1] & [8,-] \\ [7,-] & [-,1/8] & [1,1] \end{bmatrix}$$

$$\bar{A}_3 = \left([a^-_{ij}, a^+_{ij}]\right)_{3\times 3} = \begin{bmatrix} [1,1] & [2,2] & [-,1/4] \\ [1/2,1/2] & [1,1] & [3,-] \\ [4,-] & [-,1/3] & [1,1] \end{bmatrix}$$

For $\bar{A}_2$, the given judgment $a_{12} = [9, 9]$ indicates that x_1 is 9 times preferred than x_2, and the interval judgment $a_{23} = [8, -]$ denotes that x_2 is at least 8 times preferred than x_3, whereas, the interval judgment $a_{31} = [7, -]$ gives that x_3 is at least 7 times preferred than x_1. Therefore, the judgments in $\bar{A}_2$ are extremely inconsistent. However, by Definition 2.1, one can easily obtain that $\bar{A}_2$ is a consistent IMPR. Obviously, this result is highly questionable.

For $\bar{A}_3$, as per (2.1), the two incomplete comparison matrices are determined as

$$C_3 = \begin{bmatrix} 1 & 2 & 1/4 \\ 1/2 & 1 & - \\ 4 & - & 1 \end{bmatrix} \qquad D_3 = \begin{bmatrix} 1 & 2 & - \\ 1/2 & 1 & 3 \\ - & 1/3 & 1 \end{bmatrix}$$

As per Definition 2.1, the incomplete IMPR $\bar{A}_3$ is consistent. By using (2.2), the obtained estimation values are $c_{23} = 1/8$ and $d_{13} = 6$, i.e., $a^+_{23} = 1/8$ and $a^-_{13} = 6$. However, by observing the lower bound of the interval judgment $\bar{a}_{23} = [3, -]$ in the given IMPR $\bar{A}_3$, any possible value a^+_{23} should be more than or equal to 3. Similarly, a^-_{13} should be less than or equal to 1/4. Clearly, the results obtained by (2.2) are incorrect.

As per the concept of acceptable IMPRs by Liu et al. (2012) (see Definition 8 on page 749), one can obtain that $\bar{A}_2$ is unacceptable and $\bar{A}_3$ is acceptable, implying that a consistent IMPR may be unacceptable and an acceptably incomplete IMPR may not be complemented.

Therefore, the acceptability definition by Liu et al. (2012) suffers from serious defects and its applications should be very cautious.

Based on (2.2), Liu et al. (2012) put forward the following model (see Liu et al. (2012), (LOP2) on page 749) to estimate missing values for incomplete IMPRs whose judgments are given by the DM under the 1-9 Saaty's scale.

$$\min J = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(v_{ijkc}^{+} + v_{ijkc}^{-} + v_{ijkd}^{+} + v_{ijkd}^{-}\right)$$
$$s.t.\begin{cases} \log_9 c_{ij} - (\log_9 c_{ik} + \log_9 c_{kj}) - v_{ijkc}^{+} + v_{ijkc}^{-} = 0, \\ \log_9 d_{ij} - (\log_9 d_{ik} + \log_9 d_{kj}) - v_{ijkd}^{+} + v_{ijkd}^{-} = 0, \\ c_{ij} \geq d_{ij}, \quad i < j \\ c_{ij} \leq d_{ij}, \quad i > j \\ 1/9 \leq c_{ij} \leq 9,\ 1/9 \leq d_{ij} \leq 9 \\ v_{ijkc}^{-} \geq 0, \quad v_{ijkc}^{+} \geq 0, \quad v_{ijkc}^{-} \geq 0, \quad v_{ijkd}^{-} \geq 0, \quad v_{ijkd}^{+} \geq 0, \\ i, j, k = 1, 2, \ldots, n, i \neq j \neq k. \end{cases} \tag{2.3}$$

where v_{ijkc}^{-}, v_{ijkc}^{+}, v_{ijkc}^{-} and v_{ijkd}^{+} are deviation variables, and if c_{ij} or d_{ij} is unknown, then it is a decision variable.

Obviously, (2.3) does not consider the multiplicative reciprocal property of $c_{ij}c_{ji} = 1, d_{ij}d_{ji} = 1, i, j = 1, 2, \ldots, n$.

By substituting the incomplete IMPR $\bar{A}_1$ in Example 1 into (2.3), and solving this model, we obtain the optimal solutions as $c_{24}^{*} = 1, d_{24}^{*} = 0.5, c_{42}^{*} = 1, d_{42}^{*} = 2$. Thus, one has the complemented values $\bar{a}_{24} = [a_{24}^{-}, a_{24}^{+}] = [0.5, 1]$ and $\bar{a}_{42} = [a_{42}^{-}, a_{42}^{+}] = [1, 2]$.

Similarly, by (2.3), for $\bar{A}_1'$ in Example 1, one gets the optimal solutions as $c_{24}^{'*} = 1.9057, d_{24}^{'*} = 1.9057, c_{42}^{'*} = 0.5247, d_{42}^{'*} = 0.5247$. Then, we have the complemented values $\bar{a}_{24}' = [a_{24}^{'-}, a_{24}^{'+}] = [1.9057, 1.9057]$ and $\bar{a}_{42}' = [a_{42}^{'-}, a_{42}^{'+}] = [0.5247, 0.5247]$.

It is obvious that $\bar{a}_{24} \neq \bar{a}_{42}'$ and $\bar{a}_{42} \neq \bar{a}_{24}'$. However, from the viewpoint of the pairwise comparison, $\bar{a}_{24}$ is the interval ratio of the alternative x_2 over x_4, and $\bar{a}_{42}'$ is also the interval ratio of x_2 over x_4. Thus, inconsistent results are obtained from the same judgment information by applying (2.3). The computation results demonstrate that the model (2.3) depends on alternative labels and is not robust to permutations of decision alternatives. Therefore, the goal programming model proposed by Liu et al. (2012) is also technically incorrect.

3. Correction to the consistency definition and the goal programming model

This section introduces new transitivity conditions and develops a two-stage goal programming approach to correct the errors in Liu et al. (2012).

Definition 3.1. Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^{-}, a_{ij}^{+}])_{n\times n}$ be a complete IMPR. $\bar{A}$ is called (geometrically) consistent, if it satisfies the following transitivity condition:

$$a_{ij}^{-}a_{ij}^{+} = a_{ik}^{-}a_{ik}^{+}a_{kj}^{-}a_{kj}^{+}, \quad i, j, k = 1, 2, \ldots, n \tag{3.1}$$

It is obvious that Definition 3.1 is independent of alternative labels. Moreover, (3.1) is equivalent to $\sqrt{a_{ij}^{-}a_{ij}^{+}} = \sqrt{a_{ik}^{-}a_{ik}^{+}}\sqrt{a_{kj}^{-}a_{kj}^{+}}$, $i, j, k = 1, 2, \ldots, n$. It is easy to prove that $\bar{A}$ is consistent if and only if the multiplicative reciprocal crisp matrix $A = (\sqrt{a_{ij}^{-}a_{ij}^{+}})_{n\times n}$ has Saaty's consistency. Thus, the inconsistency level of the IMPR $\bar{A}$ can be measured by using Saaty's consistency index (CI) of the A.

As per (1.1), (3.1) can be equivalently expressed as any of the following equations:

$$a_{ij}^{-}a_{jk}^{-}a_{ki}^{-} = a_{ik}^{-}a_{kj}^{-}a_{ji}^{-}, \quad i, j, k = 1, 2, \ldots, n \tag{3.2}$$

$$a_{ij}^{+}a_{jk}^{+}a_{ki}^{+} = a_{ik}^{+}a_{kj}^{+}a_{ji}^{+}, \quad i, j, k = 1, 2, \ldots, n \tag{3.3}$$

According to interval arithmetic, it follows from (3.2) and (3.3) that (3.1) can be also equivalently denoted as

$$\bar{a}_{ij} \otimes \bar{a}_{jk} \otimes \bar{a}_{ki} = \bar{a}_{ik} \otimes \bar{a}_{kj} \otimes \bar{a}_{ji}, \quad i, j, k = 1, 2, \ldots, n \tag{3.4}$$

where $\otimes$ indicates the interval multiplication operation.

Therefore, the following theorem can be directly obtained.

Theorem 3.1. *For a complete IMPR $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^{-}, a_{ij}^{+}])_{n\times n}$, the following statements are equivalent:*

(i) *$\bar{A}$ is consistent (under Definition 3.1);*
(ii) $a_{ij}^{-}a_{jk}^{-}a_{ki}^{-} = a_{ik}^{-}a_{kj}^{-}a_{ji}^{-}, \quad i, j, k = 1, 2, \ldots, n;$
(iii) $a_{ij}^{+}a_{jk}^{+}a_{ki}^{+} = a_{ik}^{+}a_{kj}^{+}a_{ji}^{+}, \quad i, j, k = 1, 2, \ldots, n;$
(iv) $\bar{a}_{ij} \otimes \bar{a}_{jk} \otimes \bar{a}_{ki} = \bar{a}_{ik} \otimes \bar{a}_{kj} \otimes \bar{a}_{ji}, \quad i, j, k = 1, 2, \ldots, n.$

Theorem 3.2. *Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^{-}, a_{ij}^{+}])_{n\times n}$ be a complete IMPR. If all elements of C and D defined by (2.1) satisfy (2.2), then $\bar{A}$ is consistent (under Definition 3.1).*

Proof. For all $i, j, k = 1, 2, \ldots, n$, there are six possible cases:

(1) $i < j < k$. By (2.1) and (2.2), we have $a_{ik}^{+} = a_{ij}^{+}a_{jk}^{+}$ and $a_{ik}^{-} = a_{ij}^{-}a_{jk}^{-}$. It follows from the reciprocal property of IMPRs that $a_{ik}^{+} = a_{ij}^{+}a_{jk}^{+} \Rightarrow \frac{1}{a_{ki}^{-}} = \frac{1}{a_{ji}^{-}a_{kj}^{-}} \Rightarrow a_{ki}^{-} = a_{ji}^{-}a_{kj}^{-}$. Thus, one can obtain $a_{ij}^{-}a_{jk}^{-}a_{ki}^{-} = a_{ik}^{-}a_{kj}^{-}a_{ji}^{-}$.
(2) $i < k < j$. As per (2.1) and (2.2), one has $a_{ij}^{+} = a_{ik}^{+}a_{kj}^{+}$ and $a_{ij}^{-} = a_{ik}^{-}a_{kj}^{-}$. It follows that $a_{ij}^{+} = a_{ik}^{+}a_{kj}^{+} \Rightarrow \frac{1}{a_{ji}^{-}} = \frac{1}{a_{ki}^{-}a_{jk}^{-}} \Rightarrow a_{ji}^{-} = a_{ki}^{-}a_{jk}^{-} \Rightarrow a_{ij}^{-}a_{jk}^{-}a_{ki}^{-} = a_{ik}^{-}a_{kj}^{-}a_{ji}^{-}$.

Similarly, by changing the order of the indices i, j, k, one can obtain that $a_{ij}^{-}a_{jk}^{-}a_{ki}^{-} = a_{ik}^{-}a_{kj}^{-}a_{ji}^{-}$ holds for the remaining four situations: (3) $j < i < k$, (4) $j < k < i$, (5) $k < i < j$ and (6) $k < j < i$. By Theorem 3.1, $\bar{A}$ is consistent. □

Theorem 3.2 indicates that if a complete IMPR $\bar{A}$ satisfies (2.2), then it is consistent. Conversely, however, if $\bar{A}$ is consistent (under Definition 3.1), it does not necessarily satisfy (2.2). Moreover, one can verify that the matrices C and D constructed from $\bar{A}$ by using (2.1) depends on alternative labels and are not robust to permutations of decision alternatives. Therefore, it is unreasonable to employ (2.2) to define consistency of complete IMPRs.

As per (1.1), any of the equations of the statements (ii)–(iv) in Theorem 3.1 holds true for IMPRs if two of the indices i, j, k are equal, or the three of them. Let

$$K_{\bar{A}}^{L} = \left\{(i, j) | i \neq j, a_{ij}^{-} \text{ in } \bar{A} \text{ is known}\right\} \tag{3.5}$$

Then, based on Theorem 3.1, the consistency of incomplete IMPRs can be defined as follows.

Definition 3.2. Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^{-}, a_{ij}^{+}])_{n\times n}$ be an incomplete IMPR. $\bar{A}$ is called consistent, if there exists $\hat{a}_{ij}^{-}$ for all $i, j = 1, 2, \ldots, n$ such that

$$\hat{a}_{ij}^{-}\hat{a}_{jk}^{-}\hat{a}_{ki}^{-} = \hat{a}_{ik}^{-}\hat{a}_{kj}^{-}\hat{a}_{ji}^{-}, \qquad i, j, k = 1, 2, \ldots, n \tag{3.6}$$

$$\hat{a}_{ij}^{-L} \leq \hat{a}_{ij}^{-} \leq \hat{a}_{ij}^{-U}, \qquad i, j = 1, 2, \ldots, n \tag{3.7}$$

$$\hat{a}_{ij}^{-} \leq 1/\hat{a}_{ji}^{-}, \quad i, j = 1, 2, \ldots, n, i \neq j, (i, j) \notin K_{\bar{A}}^{L}, (j, i) \notin K_{\bar{A}}^{L} \tag{3.8}$$

where $\hat{a}_{ij}^{-L}$ and $\hat{a}_{ij}^{-U}$ are given by

$$\hat{a}_{ij}^{-L} = \begin{cases} a_{ij}^- & (i,j) \in K_{\bar{A}}^L \\ 1 & i = j \\ 1/S & (i,j) \notin K_{\bar{A}}^L \end{cases} \quad \hat{a}_{ij}^{-U} = \begin{cases} a_{ij}^- & (i,j) \in K_{\bar{A}}^L \\ 1 & i = j \\ 1/a_{ji}^- & (i,j) \notin K_{\bar{A}}^L, (j,i) \in K_{\bar{A}}^L \\ S & (i,j) \notin K_{\bar{A}}^L, (j,i) \notin K_{\bar{A}}^L \end{cases} \tag{3.9}$$

Obviously, Definition 3.2 is independent of alternative labels. One can easily verify that $\bar{A}_1$ and $\bar{A}_1'$ in Example 1 are both consistent, and $\bar{A}_2$ and $\bar{A}_3$ in Example 2 are inconsistent under Definition 3.2.

If DMs employ the 1–9 Saaty's scale to provide their judgments, then (3.2) can be equivalently rewritten as

$$\log_9 a_{ij}^- + \log_9 a_{jk}^- + \log_9 a_{ki}^- \\ = \log_9 a_{ik}^- + \log_9 a_{kj}^- + \log_9 a_{ji}^-, \quad i,j,k = 1,2,\ldots,n \tag{3.10}$$

Eq. (3.10) holds for consistent IMPRs. If an IMPR is inconsistent, then it cannot be expressed as (3.10). To estimate missing values of the incomplete IMPR $\bar{A}$, we will have to relax the relation in (3.10) by allowing some deviations for $(i,j) \in UK_{\bar{A}}^L$ and $k = 1,2,\ldots,n, k \neq i, k \neq j$, where

$$UK_{\bar{A}}^L = \{(i,j) | i,j = 1,2,\ldots,n, i \neq j\} - K_{\bar{A}}^L \tag{3.11}$$

Consequently, the following multi-objective programming model is established:

$$\min J_{ij} = \sum_{\substack{k=1,\\ k\neq i, k\neq j.}}^{n} |\log_9 a_{ij}^- + \log_9 a_{jk}^- + \log_9 a_{ki}^- - (\log_9 a_{ik}^- + \log_9 a_{kj}^- \\ + \log_9 a_{ji}^-)| \quad (i,j) \in UK_{\bar{A}}^L \\ s.t. \quad 1/9 \le a_{ij}^- \le 9, \quad a_{ij}^- a_{ji}^- \le 1, \quad (1/a_{ji}^-)/a_{ij}^- \le p, \quad (i,j) \in UK_{\bar{A}}^L \tag{3.12}$$

where $p (1 \le p \le 81)$ is a parameter that is expected to control the uncertainty level of the complemented interval judgments, the first constraint ensures that the complemented values are based on the 1–9 Saaty's scale, the next constraint guarantees that the complemented value a_{ij}^- combined with $1/a_{ji}^-$ constitutes an interval judgment $[a_{ij}^-, 1/a_{ji}^-]$, i.e., $a_{ij}^- \le (1/a_{ji}^-)$, the last constraint requires that the uncertainty level (or ratio) of the complemented interval judgment $[a_{ij}^-, 1/a_{ji}^-]$ is smaller than or equal to p, and a_{ij}^- (for $(i,j) \in UK_{\bar{A}}^L$) are decision variables.

It is noted that the parameter p controls the tradeoff between the uncertainty level and the consistency of the complemented IMPR. The smaller the p value, the less uncertainty the complemented judgments have. If $p = 1$, then the complemented interval judgments collapse into single points and have no uncertainty.

Let

$$\varepsilon_{ij}^k = \log_9 a_{ij}^- + \log_9 a_{jk}^- + \log_9 a_{ki}^- - (\log_9 a_{ik}^- + \log_9 a_{kj}^- \\ + \log_9 a_{ji}^-), \; \varepsilon_{ij}^{k+} = \frac{|\varepsilon_{ij}^k| + \varepsilon_{ij}^k}{2}, \; \varepsilon_{ij}^{k-} = \frac{|\varepsilon_{ij}^k| - \varepsilon_{ij}^k}{2} \tag{3.13}$$

for all $(i,j) \in UK_{\bar{A}}^L$ and $k = 1,2,\ldots,n, k \neq i, k \neq j$. Then, one can obtain $\varepsilon_{ij}^k = \varepsilon_{ij}^{k+} - \varepsilon_{ij}^{k-}$, $|\varepsilon_{ij}^k| = \varepsilon_{ij}^{k+} + \varepsilon_{ij}^{k-}$ and $\varepsilon_{ij}^{k+}\varepsilon_{ij}^{k-} = 0$. Therefore, the solution to the minimization problem (3.12) can be found by solving the following goal programming model:

$$\min J = \sum_{(i,j)\in UK_{\bar{A}}^L} \sum_{\substack{k=1,\\ k\neq i, k\neq j.}}^{n} (\varepsilon_{ij}^{k+} + \varepsilon_{ij}^{k-})$$

$$s.t \begin{cases} \log_9 a_{ij}^- + \log_9 a_{jk}^- + \log_9 a_{ki}^- - (\log_9 a_{ik}^- & (i,j) \in UK_{\bar{A}}^L, \\ +\log_9 a_{kj}^- + \log_9 a_{ji}^-) - \varepsilon_{ij}^{k+} + \varepsilon_{ij}^{k-} = 0, & k = 1,2,\ldots.n, \\ & k \neq i, k \neq j \\ 1/9 \le a_{ij}^- \le 9, 1/p \le a_{ij}^- a_{ji}^- \le 1, & (i,j) \in UK_{\bar{A}}^L \\ & (i,j) \in UK_{\bar{A}}^L, \\ \varepsilon_{ij}^{k+} \ge 0, \varepsilon_{ij}^{k-} \ge 0. & k = 1,2,\ldots.n, \\ & k \neq i, k \neq j \end{cases} \tag{3.14}$$

One can easily verify that t_{ij}^- (for $(i,j) \in UK_{\bar{A}}^L$) are feasible solutions to (3.14) under any p value $(1 \le p \le 81)$, where

$$t_{ij}^- = \begin{cases} 1/a_{ji}^- & (i,j) \in UK_{\bar{A}}^L, (j,i) \notin UK_{\bar{A}}^L \\ 1 & (i,j) \in UK_{\bar{A}}^L, (j,i) \in UK_{\bar{A}}^L \end{cases} \tag{3.15}$$

Moreover, multiple solutions of (3.14) may be found for a given p value. Since the missing judgments are inherently uncertain, it is sensible to expect that such uncertainty is properly captured and maximized by the complemented judgments. For an interval judgment $[a_{ij}^-, 1/a_{ji}^-]$, its uncertainty ratio can be measured by $1/(a_{ij}^- a_{ji}^-)$. Thus, we establish the following goal programming model to find a benchmark among the optimal solutions of (3.14).

$$\max \tilde{J} = \sum_{(i,j)\in UK_{\bar{A}}^L} \left(\frac{1}{a_{ij}^- a_{ji}^-} \right)$$

$$s.t. \begin{cases} \log_9 a_{ij}^- + \log_9 a_{jk}^- + \log_9 a_{ki}^- - (\log_9 a_{ik}^- & (i,j) \in UK_{\bar{A}}^L, \\ +\log_9 a_{kj}^- + \log_9 a_{ji}^-) - \varepsilon_{ij}^{k+} + \varepsilon_{ij}^{k-} = 0, & k = 1,2,\ldots.n, \\ & k \neq i, k \neq j \\ 1/9 \le a_{ij}^- \le 9, 1/p \le a_{ij}^- a_{ji}^- \le 1, & (i,j) \in UK_{\bar{A}}^L \\ \varepsilon_{ij}^{k+} \ge 0, \varepsilon_{ij}^{k-} \ge 0, & (i,j) \in UK_{\bar{A}}^L, \\ \sum_{(i,j)\in UK_{\bar{A}}^L} \sum_{\substack{k=1,\\ k\neq i, k\neq j.}}^{n} (\varepsilon_{ij}^{k+} + \varepsilon_{ij}^{k-}) = J^*. & k = 1,2,\ldots.n, \\ & k \neq i, k \neq j \end{cases} \tag{3.16}$$

where J^* is the optimal objective function value of (3.14).

Solving (3.16) yields optimal solutions a_{ij}^{-*} (for $(i,j) \in UK_{\bar{A}}^L$). Thus, the complete IMPR is determined as $\bar{A}^c = (\bar{a}_{ij}^c)_{n\times n} = ([a_{ij}^{-c}, a_{ij}^{+c}])_{n\times n}$, where

$$a_{ij}^{-c} = \begin{cases} a_{ij}^- & (i,j) \notin UK_{\bar{A}}^L \\ 1 & i = j \\ a_{ij}^{-*} & (i,j) \in UK_{\bar{A}}^L \end{cases} \quad a_{ij}^{+c} = \begin{cases} a_{ij}^+ & (j,i) \notin UK_{\bar{A}}^L \\ 1 & i = j \\ 1/a_{ji}^{-*} & (j,i) \in UK_{\bar{A}}^L \end{cases} \tag{3.17}$$

Theorem 3.3. *Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ be an incomplete IMPR, σ be a permutation of $\{1,2,\ldots,n\}$ and $\bar{A}' = ([a_{ij}'^-, a_{ij}'^+])_{n\times n} = ([a_{\sigma(i)\sigma(j)}^-, a_{\sigma(i)\sigma(j)}^+])_{n\times n}$. If a_{ij}^{-*} (for $(i,j) \in UK_{\bar{A}}^L$) are the optimal solutions to (3.16) at $p = p_0$ for $\bar{A}$, then $a_{ij}'^{-*} = a_{\sigma(i)\sigma(j)}^{-*}$ (for $(i,j) \in UK_{\bar{A}'}^L$) are the optimal solutions to (3.16) at $p = p_0$ for $\bar{A}'$.*

Theorem 3.3 indicates that the model (3.16) is independent of alternative labels. One can easily verify that (3.16) yields the identical interval judgments under the same parameter values for incomplete IMPRs $\bar{A}_1$ and $\bar{A}_1'$ in Example 1.

Based on $\bar{A}_1$ in Example 1, by (3.14), (3.16) and (3.17), one can obtain the complemented interval judgments $[a_{24}^{-c}, a_{24}^{+c}]$ and $[a_{42}^{-c}, a_{42}^{+c}]$ under different parameter p values as shown in Table 1.

Table 1
Complemented interval judgments based on $\bar{A}_1$.

p	$[a_{24}^{-c}, a_{24}^{+c}]$	$[a_{42}^{-c}, a_{42}^{+c}]$	Objective function value J^*
1	[0.7071,0.7071]	[1.4142,1.4142]	0.1388×10^{-15}
2	[0.5016,0.9968]	[1.0032,1.9936]	0.0556×10^{-15}
3	[0.4082,1.2247]	[0.8165,2.4498]	0.0555×10^{-15}
4	[0.3536,1.4142]	[0.7071,2.8281]	0

4. Conclusions

This note has shown that the consistency definition and the goal programming model proposed by Liu et al. (2012) are technically wrong. New transitivity conditions have been put forward to correct the errors, and a two-stage goal programming approach with an uncertainty level controlling parameter has been developed to estimate missing values for incomplete IMPRs.

Acknowledgment

The author would like to thank the Editor, Dr. Jean-Charles Billaut, and the anonymous reviewers for their constructive and insightful comments and suggestions that have led to an improved version of this note. This research is supported by the National Natural Science Foundation of China under Grant 71271188 and the Zhejiang Provincial Natural Science Foundation of China under Grant LY15G010004.

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