Available online at www.sciencedirect.com
ScienceDirect
European Journal of Operational Research 184 (2008) 266–280
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/ejor

Decision Support

Some models for deriving the priority weights from interval fuzzy preference relations

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Received 12 December 2005; accepted 10 November 2006
Available online 2 January 2007

Abstract

Interval fuzzy preference relation is a useful tool to express decision maker's uncertain preference information. How to derive the priority weights from an interval fuzzy preference relation is an interesting and important issue in decision making with interval fuzzy preference relation(s). In this paper, some new concepts such as additive consistent interval fuzzy preference relation, multiplicative consistent interval fuzzy preference relation, etc., are defined. Some simple and practical linear programming models for deriving the priority weights from various interval fuzzy preference relations are established, and two numerical examples are provided to illustrate the developed models.

Keywords: Interval fuzzy preference relation; Priority weight; Multiplicative transitivity; Additive transitivity

1. Introduction

The preference information is usually expressed in two forms: multiplicative preference relations and fuzzy preference relations. Multiplicative preference relations (Saaty, 1980, 1987, 2001; Van Laarhoven and Pedrycz, 1983; Arbel, 1989; Zahir, 1991; Chen and Hwang, 1992; Islam et al., 1997; Haines, 1998; Lipovetsky and Tishler, 1999; Herrera et al., 2001; Mikhailov, 2003; Xu, 2000, 2004a) have been applied extensively in many fields, such as economic analysis, technology transfer, and population forecast (Vargas, 1990). Some authors (Orlovski, 1978; Nurmi, 1981; Tanino, 1984; Kacprzyk, 1986; Chiclana et al., 1998, 2001, 2003; Fan et al., 2002; Xu and Da, 2002, 2005; Herrera-Viedma et al., 2004) have paid attention on fuzzy preference relations. Orlovski (1978) defined the fuzzy equivalence and strict preference relations and introduced two types of linearity of a fuzzy relation, and studied the equivalence of the unfuzzy non-dominated alternatives. Nurmi (1981) developed a method for aggregating individual non-fuzzy preferences so as to get a fuzzy social preference relation

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0377-2217/$ - see front matter
doi:10.1016/j.ejor.2006.11.011

and a non-fuzzy social choice set. Tanino (1984) discussed some uses of fuzzy preference relations in group decision making. Kacprzyk (1986) proposed some solution concepts in group decision making using linguistic quantifiers to represent a fuzzy majority. The solution is derived either directly from the individual fuzzy preference relations or by constructing first a social fuzzy preference relation. Chiclana et al. (1998) presented a general model for a multipurpose decision making problem, where the information provided by the experts can be represented by means of preference orderings, utility functions and fuzzy preference relations. They first made the information uniform using fuzzy preference relations as uniform preference context, and then presented some selection processes for multiple preference relations based on the concept of fuzzy majority and used two quantifier guided choice degrees of alternatives. Chiclana et al. (2001) studied the integration of multiplicative preference relations in fuzzy multipurpose decision making problems under different preference representation structures (orderings, utilities and fuzzy preference relations). They introduced an aggregation operator based on the ordered weighted averaging (OWA) operators to aggregate multiplicative preference relations, and extended quantifier-guided dominance and non-dominance degrees to act with multiplicative preference relations. Xu and Da (2002) utilized the fuzzy preference relation to rank a collection of interval numbers. Fan et al. (2002) studied the multiple attribute decision making problem in which the decision maker provides his/her preference information over alternatives with fuzzy preference relation. They first established an optimization model to derive the attribute weights and then to select the most desirable alternative(s). Xu and Da (2003) developed an approach to improving consistency of fuzzy preference relation and gave a practical iterative algorithm to derive a modified fuzzy preference relation with acceptable consistency. Chiclana et al. (2003) gave a necessary and sufficient condition to ensure the additive reciprocity of the collective preference relation obtained when aggregating any finite set of additive reciprocal fuzzy preference relations using OWA operators guided by a relative non-decreasing linguistic quantifier. Herrera-Viedma et al. (2004) presented a characterization of consistency based on the additive transitivity property of fuzzy preference relations, and based on this characterization, they proposed a method for constructing consistent fuzzy preference relations from a set of $n-1$ preference data. Xu and Da (2005) proposed a least deviation method to obtain a priority vector from a fuzzy preference relation. Ma et al. (2006) presented an analysis method to identify the inconsistency and weak transitivity of a fuzzy preference relation and to repair its inconsistency to reach weak transitivity.

However, the decision maker may not estimate his/her preference with exact numerical value, but with interval number due to the increasing complexity and uncertainty of real-life decision making problem (Chen and Hwang, 1992), and the decision maker's limited attention and information processing capabilities (Kahneman et al., 1982). In such situations, an interval fuzzy preference relation (Xu, 2004b; Herrera et al., 2005) is very suitable for expressing the decision maker's uncertain preference information. Xu (2004a) defined the concepts of compatibility degree and compatibility index of two interval fuzzy preference relations and gave a detailed analysis of compatibility of interval fuzzy preference relations in group decision making. Herrera et al. (2005) presented an aggregation process for managing non-homogenous information composed by fuzzy binary preference relations, interval-valued preference relations and fuzzy linguistic relations, in group decision making problems. The aggregation process is based on the unification of the information by means of fuzzy sets on a linguistic term set and then is transformed into linguistic 2-tupes to facilitate the exploitation phase of the decision model. If an interval fuzzy preference relation is used by a decision maker to express his/her preference information over criteria, then the priority weights derived from the interval fuzzy preference relation can be used as the weights of criteria; if the interval fuzzy preference relation is used by the decision maker to express his/her preference information over alternatives, then the priority weights derived from the interval fuzzy preference relation can be used to rank the alternatives. Consider that the estimation of the weights of criteria and the ranking of alternatives play two important roles in a multiple criteria decision making process, how to derive the priority weights from an interval fuzzy preference relation is an interesting and important research topic. Up to date, however, no investigation has been devoted to this issue. This paper is focused on the study of interval fuzzy preference relations. To do that, the remainder of this paper is structured as follows. Section 2 defines the concept of additive consistent interval fuzzy preference relation. Section 3 establishes some linear programming models for deriving the priority weights from additive consistent or inconsistent interval fuzzy preference relations. Section 4 provides a case illustration involving the assessment of a set of agroecological regions in Hubei Province, China. Section 5 defines the concept of multiplicative consistent interval fuzzy preference relation, and establishes some linear programming models for deriving the priority weights from multiplicative consistent or inconsistent interval fuzzy preference relations, and finally, in Section 6, we conclude the paper and give some remarks.

2. Some concepts

Consider a multiple criteria decision making problem with a finite set of n criteria, and let $X=\{x_1,x_2,\ldots,x_n\}$ be the set of criteria, a decision maker compares each pair of criteria in X, and provides his/her interval preference degree $\tilde{r}_{ij}=[r_{ij}^-,r_{ij}^+]$ of the criterion x_i over x_j, where $\tilde{r}_{ij}$ indicates that the criterion x_i is between r_{ij}^- and r_{ij}^+ times as important as the criterion x_j. All these interval preference degrees $\tilde{r}_{ij}$ $(i,j=1,2,\ldots,n)$ compose an interval fuzzy preference relation $\tilde{R}=(\tilde{r}_{ij})_{n\times n}\subset X\times X$ with $\tilde{r}_{ij}=[r_{ij}^-,r_{ij}^+]$, $\tilde{r}_{ji}=[r_{ji}^-,r_{ji}^+]$, $r_{ij}^-+r_{ji}^+=r_{ij}^++r_{ji}^-=1$, $r_{ij}^+\geqslant r_{ij}^-\geqslant 0$, $r_{ii}^+=r_{ii}^-=0.5$, for all $i,j=1,2,\ldots,n$.

Let $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$ be the vector of priority weights, where w_i reflects the importance degree of the criterion x_i. All the w_i $(i=1,2,\ldots,n)$ are no less than zero and sum to one, i.e.,

$$w_i\geqslant 0,\quad i=1,2,\ldots,n,\qquad \sum_{i=1}^{n}w_i=1. \tag{1}$$

Definition 1. A fuzzy preference relation (Orlovski, 1978) R on the set X is represented by a complementary matrix $R=(r_{ij})_{n\times n}\subset X\times X$ with

$$r_{ij}\geqslant 0,\quad r_{ij}+r_{ji}=1,\quad r_{ii}=0.5,\quad \text{for all } i,j=1,2,\ldots,n, \tag{2}$$

where r_{ij} represents a crisp preference degree of the criterion x_i over x_j provided by the decision maker. Especially, $r_{ij}=0$ indicates that x_j is absolutely preferred to x_i; $r_{ij}=0.5$ indicates indifference between x_i and x_j; $r_{ij}>0.5$ indicates that x_i is preferred to x_j, $r_{ij}=1$ indicates that x_i is absolutely preferred to x_j.

Definition 2. A fuzzy preference relation $R=(r_{ij})_{n\times n}$ is called an additive consistent fuzzy preference relation, if the following additive transitivity (Tanino, 1984) is satisfied

$$r_{ij}=r_{ik}-r_{jk}+0.5,\quad \text{for all } i,j,k=1,2,\ldots,n \tag{3}$$

and such a fuzzy preference relation is given by (Chiclana et al., 2002; Xu, 2004a; Ma et al., 2006):

$$r_{ij}=0.5(w_i-w_j+1),\quad \text{for all } i,j=1,2,\ldots,n. \tag{4}$$

In what follows, we extend Definition 2 to the situations where the preference values given by the decision maker are interval numbers.

Definition 3. Let $\widetilde{R}=(\tilde{r}_{ij})_{n\times n}$ be an interval fuzzy preference relation, where $\tilde{r}_{ij}=[r_{ij}^-,r_{ij}^+]$, for all $i,j=1,2,\ldots,n$, if there exists a vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$, such that

$$r_{ij}^-\leqslant 0.5(w_i-w_j+1)\leqslant r_{ij}^+,\quad \text{for all } i,j=1,2,\ldots,n, \tag{5}$$

where w satisfies the condition (1), then $\widetilde{R}$ is called an additive consistent interval fuzzy preference relation.

By the definition of interval fuzzy preference relation, it is easy to prove that Definition 3 is equivalent to Definition 4:

Definition 4. Let $\widetilde{R}=(\tilde{r}_{ij})_{n\times n}$ be an interval fuzzy preference relation, where $\tilde{r}_{ij}=[r_{ij}^-,r_{ij}^+]$, for all $i,j=1,2,\ldots,n$, if there exists a vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$, such that

$$r_{ij}^-\leqslant 0.5(w_i-w_j+1)\leqslant r_{ij}^+,\quad \text{for all } i=1,2,\ldots n-1;\ j=i+1,\ldots,n, \tag{6}$$

where w satisfies the condition (1), then $\widetilde{R}$ is called an additive consistent interval fuzzy preference relation.

3. Linear programming models based on additive transitivity

Consider that the estimation of the weights of criteria plays an important role in a multiple criteria decision making process, in the following, we shall develop some simple and practical linear programming models for deriving the priority weights from additive consistent or inconsistent interval fuzzy preference relations:

(1) If $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$ is an additive consistent interval fuzzy preference relation, then the priority vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ derived from $\widetilde{R}$ should satisfy the conditions (1) and (6). In general, the number of the priority vector satisfying these two conditions is not just one, but each weight w_i $(i = 1, 2, \ldots, n)$ should belong to a value range. As a result, based on the conditions (1) and (6), we establish the following two linear programming models:

$$w_i^- = \mathrm{Min}\ w_i \tag{M-1}$$
$$\text{s.t.}\ \ 0.5(w_i - w_j + 1) \geqslant r_{ij}^-, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n,$$
$$0.5(w_i - w_j + 1) \leqslant r_{ij}^+, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n,$$
$$w_i \geqslant 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1,$$

$$w_i^+ = \mathrm{Max}\ w_i \tag{M-2}$$
$$\text{s.t.}\ \ 0.5(w_i - w_j + 1) \geqslant r_{ij}^-, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n,$$
$$0.5(w_i - w_j + 1) \leqslant r_{ij}^+, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n,$$
$$w_i \geqslant 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1.$$

Solving the models (M-1) and (M-2), we can get the set Θ_1 of priority weights, where

$$\Theta_1 = \left\{ w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}} | w_i \in [w_i^-, w_i^+], w_i \geqslant 0,\ i = 1, 2, \ldots, n,\ \sum_{i=1}^{n} w_i = 1 \right\}. \tag{7}$$

(2) If $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$ is not an additive consistent interval fuzzy preference relation, then the condition (6) does not always hold. In such cases, we can't utilize the models (M-1) and (M-2) to get the priority vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$. To solve this issue, motivated by the idea (Wang et al., 2005), below we extend the models (M-1) and (M-2) to accommodate the situations where the preferences are inconsistent.

We relax the condition (6) by introducing the deviation variables d_{ij}^- and d_{ij}^+, $i = 1, 2, \ldots, n-1; j = i+1, \ldots, n$:

$$r_{ij}^- - d_{ij}^- \leqslant 0.5(w_i - w_j + 1) \leqslant r_{ij}^+ + d_{ij}^+, \quad \text{for all } i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n, \tag{8}$$

where d_{ij}^- and d_{ij}^+ are both non-negative real numbers. Especially, if both d_{ij}^- and d_{ij}^+ are equal to zero, then the condition (8) is reduced to the condition (6).

Obviously, the smaller the deviation variables d_{ij}^- and d_{ij}^+, the closer the $\widetilde{R}$ to an additive consistent interval fuzzy preference relation. As a result, we establish the following optimization model:

$$J_1^* = \mathrm{Min} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{ij}^- + d_{ij}^+),$$
$$\text{s.t.}\ \ 0.5(w_i - w_j + 1) + d_{ij}^- \geqslant r_{ij}^-, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n,$$
$$0.5(w_i - w_j + 1) - d_{ij}^+ \leqslant r_{ij}^+, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n, \tag{M-3}$$
$$w_i \geqslant 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} w_i = 1,$$
$$d_{ij}^-, d_{ij}^+ \geqslant 0, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n.$$

Solving this model, we can get the optimal deviation values $\dot{d}_{ij}^-$ and $\dot{d}_{ij}^+$, $i = 1, 2, \ldots, n-1$; $j = i+1, \ldots, n$.

From the model (M-3), we can get the following result easily:

Theorem 1. *$\widetilde{R}$ is an additive consistent interval fuzzy preference relation if and only if $J_1^* = 0$.*

If $\widetilde{R}$ is not an additive consistent interval fuzzy preference relation, then based on the optimal deviation values $\dot{d}_{ij}^-$ and $\dot{d}_{ij}^+$, $i = 1, 2, \ldots, n-1$; $j = i+1, \ldots, n$, similar to the models (M-1) and (M-2), we further establish the following two linear programming models:

$$
\begin{aligned}
w_i^- = \ & \text{Min } w_i \\
& \text{s.t. } 0.5(w_i - w_j + 1) + \dot{d}_{ij}^- \geqslant r_{ij}^-, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n, \\
& \phantom{\text{s.t. }} 0.5(w_i - w_j + 1) - \dot{d}_{ij}^+ \leqslant r_{ij}^+, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n, \\
& \phantom{\text{s.t. }} w_i \geqslant 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^n w_i = 1,
\end{aligned}
\tag{M-4}
$$

$$
\begin{aligned}
w_i^+ = \ & \text{Max } w_i \\
& \text{s.t. } 0.5(w_i - w_j + 1) + \dot{d}_{ij}^- \geqslant r_{ij}^-, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n, \\
& \phantom{\text{s.t. }} 0.5(w_i - w_j + 1) - \dot{d}_{ij}^+ \leqslant r_{ij}^+, \quad i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n, \\
& \phantom{\text{s.t. }} w_i \geqslant 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^n w_i = 1.
\end{aligned}
\tag{M-5}
$$

Solving the models (M-4) and (M-5), we can get the set Θ_2 of priority weights, where

$$
\Theta_2 = \left\{ w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}} | w_i \in [w_i^-, w_i^+], w_i \geqslant 0, \quad i = 1, 2, \ldots, n, \quad \sum_{i=1}^n w_i = 1 \right\}. \tag{9}
$$

From the above analysis, we know that the priority weights derived from an interval fuzzy preference relation are in the form of interval numbers. For convenience, we call these priority weights the interval weights. Some methods have been proposed to compare the interval numbers (Ishibuchi and Tanaka, 1990; Chen and Hwang, 1992; Salo and Hämäläinen, 1995; Kunda, 1997; Facchinetti et al., 1998; Sengupta and Pal, 2000; Xu and Da, 2002; Wang et al., 2005). Based on the right limits, the left limits, the centers and the widths of interval numbers, Ishibuchi and Tanaka (1990) defined some order relations to compare interval numbers. Chen and Hwang (1992) reviewed many approaches to ranking fuzzy numbers, such as ranking using degree of optimality (Baas and Kwakernaak, 1977; Watson et al., 1979; Baldwin and Guild, 1979), ranking using Hamming distance (Yager, 1980a; Kerre, 1982; Nakamura, 1986), ranking using α-cuts (Adamo, 1980; Buckley, 1985; Mabuchi, 1988), ranking using comparison function (Dubois and Prade, 1983; Tsukamoto et al., 1983; Delgado et al., 1988), ranking using fuzzy mean and spread (Lee and Li, 1988), ranking using proportion to the ideal (McCahone, 1987; Chen, 1985), ranking using left and right scores (Jain, 1976; Chen, 1985), ranking with centroid index (Yager, 1980b; Murakami et al., 1983), and ranking using area measurement (Yager, 1981), etc. They also showed the merits and/or weaknesses of each method. Some of these methods, such as ranking using α-cuts and ranking with centroid index, etc., can be used directly to rank interval numbers. Salo and Hämäläinen (1995) proposed an interactive approach to completing the comparison between interval numbers. Kunda (1997) defined a fuzzy leftness relationship between a pair of interval numbers so as to reflect the degree of one interval number to be superior or inferior to another one. Sengupta and Pal (2000) presented two ways to compare and order any two intervals on the real line. The first way defines a subjective way based on an optimistic index, and the second one is a preference approach of ranking two interval numbers from a pessimistic (or risk averse) decision maker's point of view. Wang et al. (2005) analyzed the strengths and weaknesses of the above methods, and developed a simple yet practical preference ranking method of interval numbers. Facchinetti et al. (1998), and Xu and Da (2002) also developed two straightforward possibility-degree formulas for the comparison between two interval numbers, and studied their desirable properties which are very useful in comparing interval weights. Considering the advantages of these possibility-degree formulae, we shall use them to rank interval weights throughout this paper.

Definition 5 (*Xu and Da*, 2002). Let $\tilde{w}_i = [w_i^-, w_i^+]$ and $\tilde{w}_j = [w_j^-, w_j^+]$ be any two interval weights, where $0 \leqslant w_i^- \leqslant w_i^+ \leqslant 1$ and $0 \leqslant w_j^- \leqslant w_j^+ \leqslant 1$, then the degree of possibility of $\tilde{w}_i \geqslant \tilde{w}_j$ is defined as

$$p(\tilde{w}_i \geqslant \tilde{w}_j) = \max\left\{1 - \max\left\{\frac{w_j^+ - w_i^-}{w_i^+ - w_i^- + w_j^+ - w_j^-}, 0\right\}, 0\right\} \tag{10}$$

that is, $\tilde{w}_i$ is superior to $\tilde{w}_j$ to the degree of $p(\tilde{w}_i \geqslant \tilde{w}_j)$, denoted by $\tilde{w}_i \overset{p(\tilde{w}_i \geqslant \tilde{w}_j)}{\succ} \tilde{w}_j$.

Definition 6 (*Wang et al.*, 2005). Let $\tilde{w}_i = [w_i^-, w_i^+]$ and $\tilde{w}_j = [w_j^-, w_j^+]$ be any two interval weights, where $0 \leqslant w_i^- \leqslant w_i^+ \leqslant 1$ and $0 \leqslant w_j^- \leqslant w_j^+ \leqslant 1$, then the degree of possibility of $\tilde{w}_i \geqslant \tilde{w}_j$ is defined as

$$p(\tilde{w}_i \geqslant \tilde{w}_j) = \frac{\max\{0, w_i^+ - w_j^-\} - \max\{0, w_i^- - w_j^+\}}{w_i^+ - w_i^- + w_j^+ - w_j^-}. \tag{11}$$

Definition 7 (*Facchinetti et al.*, 1998). Let $\tilde{w}_i = [w_i^-, w_i^+]$ and $\tilde{w}_j = [w_j^-, w_j^+]$ be any two interval weights, where $0 \leqslant w_i^- \leqslant w_i^+ \leqslant 1$ and $0 \leqslant w_j^- \leqslant w_j^+ \leqslant 1$, then the degree of possibility of $\tilde{w}_i \geqslant \tilde{w}_j$ is defined as

$$p(\tilde{w}_i \geqslant \tilde{w}_j) = \min\left\{\max\left\{\frac{w_i^+ - w_j^-}{w_i^+ - w_i^- + w_j^+ - w_j^-}, 0\right\}, 1\right\}. \tag{12}$$

Especially, if both the interval weights $\tilde{w}_i = [w_i^-, w_i^+]$ and $\tilde{w}_j = [w_j^-, w_j^+]$ are reduced to exact real numbers, i.e., $w_i^- = w_i^+$ and $w_j^- = w_j^+$, where $0 \leqslant w_i^- \leqslant 1$, $0 \leqslant w_j^- \leqslant 1$, then we define the degree of possibility of $\tilde{w}_i > \tilde{w}_j$ as

$$p(\tilde{w}_i > \tilde{w}_j) = \begin{cases} 1 & \text{if } w_i^- > w_j^-, \\ 1/2 & \text{if } w_i^- = w_j^-, \\ 0 & \text{if } w_i^- < w_j^-. \end{cases} \tag{13}$$

Obviously, the degree of possibility $p(\tilde{w}_i \geqslant \tilde{w}_j)$ satisfies the following properties (Xu and Da, 2002):

(1) $0 \leqslant p(\tilde{w}_i \geqslant \tilde{w}_j) \leqslant 1$;
(2) $p(\tilde{w}_i \geqslant \tilde{w}_j) + p(\tilde{w}_j \geqslant \tilde{w}_i) = 1$. Especially, $p(\tilde{w}_i \geqslant \tilde{w}_i) = 0.5$;
(3) $p(\tilde{w}_i \geqslant \tilde{w}_j) = 1$ if and only if $w_j^+ \leqslant w_i^-$;
(4) $p(\tilde{w}_i \geqslant \tilde{w}_j) = 0$ if and only if $w_i^+ \leqslant w_j^-$;
(5) $p(\tilde{w}_i \geqslant \tilde{w}_j) \geqslant 0.5$ if and only if $w_i^+ + w_i^- \geqslant w_j^+ + w_j^-$. Especially, $p(\tilde{w}_i \geqslant \tilde{w}_j) = 0.5$ if and only if $w_i^+ + w_i^- = w_j^+ + w_j^-$;
(6) Let $\tilde{w}_i, \tilde{w}_j$ and $\tilde{w}_k$ be three interval weights, if $p(\tilde{w}_i \geqslant \tilde{w}_j) \geqslant 0.5$ and $p(\tilde{w}_j \geqslant \tilde{w}_k) \geqslant 0.5$, then $p(\tilde{w}_i \geqslant \tilde{w}_k) \geqslant 0.5$.

It is easy to prove that the above three possibility-degree formulae have the following relationship:

Theorem 2. *Let $\tilde{w}_i = [w_i^-, w_i^+]$ and $\tilde{w}_j = [w_j^-, w_j^+]$ be any two interval weights, where $0 \leqslant w_i^- \leqslant w_i^+ \leqslant 1$ and $0 \leqslant w_j^- \leqslant w_j^+ \leqslant 1$, then Eqs.* (10)–(12) *are equivalent, i.e.*, (10) $\Longleftrightarrow$ (11) $\Longleftrightarrow$ (12).

4. Case illustration

In what follows, we shall utilize a practical case (adapted from Li et al., 2005) involving the assessment of a set of agroecological regions in Hubei Province, China, to illustrate the developed models.

Located in Central China and the middle reaches of the Changjiang (Yangtze) River, Hubei Province is distributed in a transitional belt where physical conditions and landscapes are on the transition from north to south and from east to west. Thus, Hubei Province is well known as "a land of rice and fish" since the region enjoys some of the favorable physical conditions, with a diversity of natural resources and the suitability for growing various crops. At the same time, however, there are also some restrictive factors for developing agriculture such as a tight man–land relation between a constant degradation of natural resources and a growing population pressure on land resource reserve. Despite cherishing a burning desire to promote their standard of living, people living in the area are frustrated because they have no ability to enhance their power to accelerate economic development because of a dramatic decline in quantity and quality of natural resources and a deteriorating environment. Based on the distinctness and differences in environment and natural resources, Hubei Province can be roughly divided into six agroecological regions: x_1 – Wuhan–Ezhou–Huanggang; x_2 – Northeast of Hubei; x_3 – Southeast of Hubei; x_4 – Jianghan; x_5 – North of Hubei; x_6 – West of Hubei. In order to prioritize these agroecological regions $x_j\,(j=1,2,\ldots,6)$ with respect to their comprehensive functions, a committee comprised of three experts e_k $(k=1,2,3)$ (whose weight vector is $\lambda=(1/3,1/3,1/3)^{\mathrm{T}}$) has been set up to provide assessment information on $x_j\,(j=1,2,\ldots,6)$. The experts compare these six agroecological regions with respect to their comprehensive functions and construct, respectively, the interval fuzzy preference relations $\widetilde{R}^{(k)}=(\tilde{r}_{ij}^{(k)})_{6\times6}$ $(k=1,2,3)$:

$$\widetilde{R}^{(1)}=\begin{bmatrix}[0.5,0.5] & [0.5,0.7] & [0.7,0.8] & [0.5,0.6] & [0.6,0.7] & [0.8,1.0]\\ [0.3,0.5] & [0.5,0.5] & [0.6,0.7] & [0.3,0.4] & [0.5,0.6] & [0.5,0.9]\\ [0.2,0.3] & [0.3,0.4] & [0.5,0.5] & [0.3,0.5] & [0.4,0.5] & [0.6,0.8]\\ [0.4,0.5] & [0.6,0.7] & [0.5,0.7] & [0.5,0.5] & [0.5,0.7] & [0.7,0.8]\\ [0.3,0.4] & [0.4,0.5] & [0.5,0.6] & [0.3,0.5] & [0.5,0.5] & [0.4,0.8]\\ [0.0,0.2] & [0.1,0.5] & [0.2,0.4] & [0.2,0.3] & [0.2,0.6] & [0.5,0.5]\end{bmatrix},$$

$$\widetilde{R}^{(2)}=\begin{bmatrix}[0.5,0.5] & [0.6,0.7] & [0.6,0.9] & [0.4,0.7] & [0.6,0.8] & [0.8,0.9]\\ [0.3,0.4] & [0.5,0.5] & [0.5,0.8] & [0.4,0.5] & [0.5,0.7] & [0.5,0.9]\\ [0.1,0.4] & [0.2,0.5] & [0.5,0.5] & [0.4,0.5] & [0.4,0.6] & [0.7,0.8]\\ [0.3,0.6] & [0.5,0.6] & [0.5,0.6] & [0.5,0.5] & [0.6,0.7] & [0.7,0.9]\\ [0.2,0.4] & [0.3,0.5] & [0.4,0.6] & [0.3,0.4] & [0.5,0.5] & [0.5,0.8]\\ [0.1,0.2] & [0.1,0.5] & [0.2,0.3] & [0.1,0.3] & [0.2,0.5] & [0.5,0.5]\end{bmatrix},$$

$$\widetilde{R}^{(3)}=\begin{bmatrix}[0.5,0.5] & [0.4,0.6] & [0.5,0.7] & [0.4,0.7] & [0.6,0.8] & [0.7,0.8]\\ [0.4,0.6] & [0.5,0.5] & [0.4,0.6] & [0.3,0.5] & [0.4,0.6] & [0.6,0.9]\\ [0.3,0.5] & [0.4,0.6] & [0.5,0.5] & [0.4,0.6] & [0.4,0.7] & [0.5,0.8]\\ [0.3,0.6] & [0.5,0.7] & [0.4,0.6] & [0.5,0.5] & [0.6,0.8] & [0.4,0.7]\\ [0.2,0.4] & [0.4,0.6] & [0.3,0.6] & [0.2,0.4] & [0.5,0.5] & [0.6,0.7]\\ [0.2,0.3] & [0.1,0.4] & [0.2,0.5] & [0.3,0.6] & [0.3,0.4] & [0.5,0.5]\end{bmatrix}.$$

We first utilize the fuzzy weighted averaging operator (Xu, 2004b):

$$\widetilde{R}=\lambda_1\widetilde{R}^{(1)}\oplus\lambda_2\widetilde{R}^{(2)}\oplus\lambda_3\widetilde{R}^{(3)}$$

to aggregate all the individual interval fuzzy preference relations $\widetilde{R}^{(k)}=(\tilde{r}_{ij}^{(k)})_{6\times6}$ $(k=1,2,3)$ into the collective interval fuzzy preference relation $\widetilde{R}=(\tilde{r}_{ij})_{6\times6}$, where

$$\tilde{r}_{ij}=\sum_{k=1}^{3}\lambda_k\tilde{r}_{ij}^{(k)}=\left[\sum_{k=1}^{3}\lambda_k r_{ij}^{-(k)},\sum_{k=1}^{3}\lambda_k r_{ij}^{+(k)}\right],\quad i,j=1,2,\ldots,6$$

and $\lambda=(\lambda_1,\lambda_2,\lambda_3)^{\mathrm{T}}$ is the weight vector of the experts e_k $(k=1,2,3)$, Thus, we have

$$\widetilde{R}^{(3)} = \begin{bmatrix} [0.50, 0.50] & [0.50, 0.67] & [0.60, 0.80] & [0.43, 0.67] & [0.60, 0.77] & [0.77, 0.90] \\ [0.33, 0.50] & [0.50, 0.50] & [0.50, 0.70] & [0.33, 0.47] & [0.47, 0.63] & [0.53, 0.90] \\ [0.20, 0.40] & [0.30, 0.50] & [0.50, 0.50] & [0.37, 0.53] & [0.40, 0.60] & [0.60, 0.80] \\ [0.33, 0.57] & [0.53, 0.67] & [0.47, 0.63] & [0.50, 0.50] & [0.57, 0.73] & [0.60, 0.80] \\ [0.23, 0.40] & [0.37, 0.53] & [0.40, 0.60] & [0.27, 0.43] & [0.50, 0.50] & [0.50, 0.77] \\ [0.10, 0.23] & [0.10, 0.47] & [0.20, 0.40] & [0.20, 0.40] & [0.23, 0.50] & [0.50, 0.50] \end{bmatrix}.$$

Solving the model (M-3), we have $J_1^* = 0.05$, and the optimal deviation values are as follows:

$$\dot{d}_{12}^- = \dot{d}_{12}^+ = 0, \quad \dot{d}_{13}^- = \dot{d}_{13}^+ = 0, \quad \dot{d}_{14}^- = \dot{d}_{14}^+ = 0, \quad \dot{d}_{15}^- = \dot{d}_{15}^+ = 0,$$
$$\dot{d}_{16}^- = 0, \quad \dot{d}_{16}^+ = 0.025, \quad \dot{d}_{23}^- = \dot{d}_{23}^+ = 0, \quad \dot{d}_{24}^- = \dot{d}_{24}^+ = 0,$$
$$\dot{d}_{25}^- = \dot{d}_{25}^+ = 0, \quad \dot{d}_{26}^- = \dot{d}_{26}^+ = 0, \quad \dot{d}_{34}^- = \dot{d}_{34}^+ = 0, \quad \dot{d}_{35}^- = \dot{d}_{35}^+ = 0,$$
$$\dot{d}_{36}^- = \dot{d}_{36}^+ = 0.025, \quad \dot{d}_{45}^- = \dot{d}_{45}^+ = 0, \quad \dot{d}_{46}^- = \dot{d}_{46}^+ = 0, \quad \dot{d}_{56}^- = \dot{d}_{56}^+ = 0.$$

By Theorem 1, we know that $\widetilde{R}$ is not an additive consistent interval fuzzy preference relation. Based on the optimal deviation values $\dot{d}_{ij}^-$ and $\dot{d}_{ij}^+$, $i = 1, 2, \ldots, 5;\ j = i + 1, \ldots, 6$, we solve the models (M-4) and (M-5), and get

$$w_1^- = 0.35, \quad w_1^+ = 0.54, \quad w_2^- = 0.07, \quad w_2^+ = 0.22, \quad w_3^- = 0, \quad w_3^+ = 0.19,$$
$$w_4^- = 0.2, \quad w_4^+ = 0.4, \quad w_5^- = 0, \quad w_5^+ = 0.17, \quad w_6^- = 0, \quad w_6^+ = 0.1.$$

If we let $\tilde{w}_i = [w_i^-, w_i^+]$, then, the corresponding interval weights are as follows:

$$\tilde{w}_1 = [0.35, 0.54], \quad \tilde{w}_2 = [0.07, 0.22], \quad \tilde{w}_3 = [0, 0.19].$$
$$\tilde{w}_4 = [0.2, 0.4], \quad \tilde{w}_5 = [0, 0.17], \quad \tilde{w}_6 = [0, 0.1].$$

Then, we compare each $\tilde{w}_i$ with all the $\tilde{w}_j$ $(j = 1, 2, \ldots, 6)$ by using Eq. (10), and construct the following fuzzy preference relation:

$$P = \begin{bmatrix} 0.5 & 1 & 1 & 0.8718 & 1 & 1 \\ 0 & 0.5 & 0.6471 & 0.0571 & 0.6875 & 0.8800 \\ 0 & 0.3529 & 0.5 & 0 & 0.5278 & 0.6552 \\ 0.1282 & 0.9429 & 1 & 0.5 & 1 & 1 \\ 0 & 0.3125 & 0.4722 & 0 & 0.5 & 0.6296 \\ 0 & 0.1200 & 0.3448 & 0 & 0.3704 & 0.5 \end{bmatrix}.$$

Summing all elements in each line of P, we have

$$p_1 = 5.3718, \quad p_2 = 2.7717, \quad p_3 = 2.0359,$$
$$p_4 = 4.5711, \quad p_5 = 1.9143, \quad p_6 = 1.3352$$

thus

$$\tilde{w}_1 \overset{0.8718}{\succ} \tilde{w}_4 \overset{0.9429}{\succ} \tilde{w}_2 \overset{0.6471}{\succ} \tilde{w}_3 \overset{0.5278}{\succ} \tilde{w}_5 \overset{0.6296}{\succ} \tilde{w}_6$$

from which we know that $\tilde{w}_1$ is superior to $\tilde{w}_4$ to the degree of 87.18%, $\tilde{w}_4$ is superior to $\tilde{w}_2$ to the degree of 94.29%, $\tilde{w}_2$ is superior to $\tilde{w}_3$ to the degree of 64.71%, $\tilde{w}_3$ is superior to $\tilde{w}_5$ to the degree of 52.78%, and $\tilde{w}_5$ is superior to $\tilde{w}_6$ to the degree of 62.96%. Therefore, the agroecological region with the most comprehensive functions is Wuhan–Ezhou–Huanggang.

5. Linear programming models based on multiplicative transitivity

Tanino (1984) investigated a fuzzy preference relation $R = (r_{ij})_{n\times n}$ with $r_{ij} > 0$, $r_{ij} + r_{ji} = 1$, $r_{ii} = 0.5$, for all $i,j = 1, 2, \ldots, n$, and introduced a multiplicative transitivity condition:

$$\frac{r_{ik}}{r_{ki}}\frac{r_{kj}}{r_{jk}} = \frac{r_{ij}}{r_{ji}}, \text{ for all } i,j = 1,2,\ldots,n, \tag{14}$$

where $\frac{r_{ij}}{r_{ji}}$ indicates a ratio of the preference intensity for the criterion x_i to that for x_j.

Obviously, Eq. (14) is equivalent to the following form (Xu and Da, 2003; Herrera-Viedma et al., 2004):

$$r_{ik}r_{kj}r_{ji} = r_{ki}r_{jk}r_{ij}, \quad \text{for all } i,j = 1,2,\ldots,n. \tag{15}$$

If a fuzzy preference relation $R = (r_{ij})_{n\times n}$ satisfies the multiplicative transitivity (15), then we call R a multiplicative consistent fuzzy preference relation (Xu, 2004b).

Let $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ be the vector of priority weights, with

$$w_i > 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1 \tag{16}$$

then the multiplicative consistent fuzzy preference relation R can be given by (Xu, 2004a; Lipovetsky and Michael, 2002)

$$r_{ij} = \frac{w_i}{w_i + w_j}, \quad \text{for all } i,j,k = 1,2,\ldots,n. \tag{17}$$

We now consider an interval fuzzy preference relation $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij} = [r_{ij}^-, r_{ij}^+]$, $\tilde{r}_{ji} = [r_{ji}^-, r_{ji}^+]$, $r_{ij}^- + r_{ji}^+ = r_{ij}^+ + r_{ji}^- = 1$, $r_{ij}^+ \geqslant r_{ij}^- > 0$, $r_{ii}^+ = r_{ii}^- = 0.5$, for all $i,j = 1,2,\ldots,n$, and extend the concept of a multiplicative consistent fuzzy preference relation to the situations where the preference values given by the decision maker are interval numbers:

Definition 8. Let $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$ be an interval fuzzy preference relation, where $\tilde{r}_{ij} = [r_{ij}^-, r_{ij}^+]$, for all $i,j = 1,2,\ldots,n$, if there exists a vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, such that

$$r_{ij}^- \leqslant \frac{w_i}{w_i + w_j} \leqslant r_{ij}^+, \quad \text{for all } i,j = 1,2,\ldots,n, \tag{18}$$

where w satisfies the condition (16), then $\widetilde{R}$ is called a multiplicative consistent interval fuzzy preference relation.

By the definition of interval fuzzy preference relation, it can be proven that Definition 8 is equivalent to Definitions 9:

Definition 9. Let $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$be an interval fuzzy preference relation, where $\tilde{r}_{ij} = [r_{ij}^-, r_{ij}^+]$, for all $i,j = 1,2,\ldots,n$, if there exists a vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$, such that

$$r_{ij}^- \leqslant \frac{w_i}{w_i + w_j} \leqslant r_{ij}^+, \quad \text{for all} \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \tag{19}$$

where w satisfies the condition (19), then $\widetilde{R}$ is called a multiplicative consistent interval fuzzy preference relation.

It is clear that Eq. (19) is equivalent to the following form:

$$r_{ij}^-(w_i + w_j) \leqslant w_i \leqslant r_{ij}^+(w_i + w_j), \quad \text{for all } i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n. \tag{20}$$

In what follows, we shall develop some linear programming models for deriving the priority weights from multiplicative consistent or inconsistent interval fuzzy preference relations:

(1) If $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$ is a multiplicative consistent interval fuzzy preference relation, then the priority vector $w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}}$ derived from $\widetilde{R}$ should satisfy the conditions (16) and (20), and thus we can establish the following two non-linear programming models:

$$\begin{aligned} w_i^- = {} & \mathrm{Min}\ w_i \\ & \text{s.t.}\quad w_i \geqslant r_{ij}^-(w_i + w_j) \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ & \qquad\ \ w_i \leqslant r_{ij}^+(w_i + w_j) \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ & \qquad\ \ w_i > 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \end{aligned} \tag{M-6}$$

$$w_i^+ = \text{Max } w_i \tag{M-7}$$
$$\begin{aligned}\text{s.t. } & w_i \geqslant r_{ij}^-(w_i+w_j), \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n,\\ & w_i \leqslant r_{ij}^+(w_i+w_j), \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n,\\ & w_i>0, \quad i=1,2,\ldots,n, \quad \sum_{i=1}^n w_i = 1,\end{aligned}$$

For convenience of calculation, based on the definition of interval fuzzy preference relation, the models (M-6) and (M-7) can be transformed into the following two simple linear programming models:

$$w_i^- = \text{Min } w_i \tag{M-8}$$
$$\begin{aligned}\text{s.t. } & r_{ji}^+w_i - r_{ij}^-w_j \geqslant 0, \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n,\\ & r_{ij}^+w_j - r_{ji}^-w_i \geqslant 0, \quad i=1,2,\ldots,n;\ j=i+1,\ldots,n,\\ & w_i>0, \quad i=1,2,\ldots,n, \quad \sum_{i=1}^n w_i = 1,\end{aligned}$$

$$w_i^+ = \text{Max } w_i \tag{M-9}$$
$$\begin{aligned}\text{s.t. } & r_{ji}^+w_i - r_{ij}^-w_j \geqslant 0, \quad i=1,2,\ldots,n;\ j=i+1,\ldots,n,\\ & r_{ij}^+w_j - r_{ji}^-w_i \geqslant 0, \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n,\\ & w_i>0, \quad i=1,2,\ldots,n, \quad \sum_{i=1}^n w_i = 1.\end{aligned}$$

Solving the models (M-8) and (M-9), we can get the set Θ_3 of priority weight vectors, where

$$\Theta_3 = \left\{ w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}} | w_i \in [w_i^-,w_i^+], w_i>0,\ i=1,2,\ldots,n,\ \sum_{i=1}^n w_i = 1 \right\}. \tag{21}$$

(2) If $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$ is not a multiplicative consistent interval fuzzy preference relation, then the condition (20) does not always hold. In such cases, we can't utilize the models (M-8) and (M-9) to get the priority vector $w=(w_1,w_2,\ldots,w_n)^{\mathrm{T}}$. To solve this issue, we relax the condition (20) by introducing the deviation variables d_{ij}^- and d_{ij}^+, $i=1,2,\ldots,n-1; j=i+1,\ldots,n$:

$$r_{ij}^-(w_i+w_j) - d_{ij}^- \leqslant w_i \leqslant r_{ij}^+(w_i+w_j) + d_{ij}^+, \quad \text{for all } i=1,2,\ldots,n-1;\ j=i+1,\ldots,n, \tag{22}$$

where d_{ij}^- and d_{ij}^+ are both non-negative real numbers. Especially, if both d_{ij}^- and d_{ij}^+ are equal to zero, then the condition (22) is reduced to the condition (20). Obviously, the smaller the deviation variables d_{ij}^- and d_{ij}^+, the closer the $\widetilde{R}$ to a multiplicative consistent interval fuzzy preference relation. As a result, we establish the following optimization model:

$$J_2^* = \text{Min } \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d_{ij}^- + d_{ij}^+)$$
$$\begin{aligned}\text{s.t. } & w_i \geqslant r_{ij}^-(w_i+w_j) - d_{ij}^-, \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n,\\ & w_i \leqslant r_{ij}^+(w_i+w_j) + d_{ij}^+, \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n,\\ & w_i>0, \quad i=1,2,\ldots,n, \quad \sum_{i=1}^n w_i = 1,\\ & d_{ij}^-, d_{ij}^+ \geqslant 0, \quad i=1,2,\ldots,n-1;\ j=i+1,\ldots,n.\end{aligned} \tag{M-10}$$

The model (10) can be transformed into the following optimization model:

$$\begin{aligned} J_2^* = &\text{Min} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (d_{ij}^- + d_{ij}^+) \\ &\text{s.t. } r_{ji}^+ w_i - r_{ij}^- w_j + d_{ij}^- \geqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ &\qquad r_{ji}^- w_i - r_{ij}^+ w_j - d_{ij}^+ \leqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ &\qquad w_i > 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \\ &\qquad d_{ij}^-, d_{ij}^+ \geqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n. \end{aligned} \tag{M-11}$$

Solving this model, we can get the optimal deviation values $\ddot{d}_{ij}^-$ and $\ddot{d}_{ij}^+$, $i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n$. Similar to Theorem 1, we have

Theorem 3. *$\widetilde{R}$ is a multiplicative consistent interval fuzzy preference relation if and only if $J_2^* = 0$.*

Based on the optimal deviation values $\ddot{d}_{ij}^-$ and $\ddot{d}_{ij}^+$, $i = 1,2,\ldots,n-1; j = i+1,\ldots,n$, similar to the models (M-8) and (M-9), we further establish the following two linear programming models:

$$\begin{aligned} w_i^- = &\text{Min } w_i \\ &\text{s.t. } r_{ji}^+ w_i - r_{ij}^- w_j + \ddot{d}_{ij}^- \geqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ &\qquad r_{ji}^- w_i - r_{ij}^+ w_j - \ddot{d}_{ij}^+ \leqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ &\qquad w_i > 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1, \end{aligned} \tag{M-12}$$

$$\begin{aligned} w_i^+ = &\text{Max } w_i \\ &\text{s.t. } r_{ji}^+ w_i - r_{ij}^- w_j + \ddot{d}_{ij}^- \geqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ &\qquad r_{ji}^- w_i - r_{ij}^+ w_j - \ddot{d}_{ij}^+ \leqslant 0, \quad i = 1,2,\ldots,n-1;\ j = i+1,\ldots,n, \\ &\qquad w_i > 0, \quad i = 1,2,\ldots,n, \quad \sum_{i=1}^{n} w_i = 1. \end{aligned} \tag{M-13}$$

Solving the models (M-12) and (M-13), we can get the set Θ_4 of priority weight vectors, where

$$\Theta_4 = \left\{ w = (w_1, w_2, \ldots, w_n)^{\mathrm{T}} | w_i \in [w_i^-, w_i^+], w_i > 0,\ i = 1,2,\ldots,n,\ \sum_{i=1}^{n} w_i = 1 \right\}. \tag{23}$$

In the following, we shall use numerical example to illustrate the developed models:

Example 1. Consider a multiple criteria decision making problem, there are four criteria x_1, x_2, x_3, x_4. A decision maker compares each pair of criteria x_i and x_j, and provides his/her interval preference degree $\tilde{r}_{ij}$ of the criterion x_i over x_j, and then constructs the following interval fuzzy preference relation:

$$\widetilde{R} = (\tilde{r}_{ij})_{4\times 4} = \begin{bmatrix} [0.5, 0.5] & [0.3, 0.4] & [0.5, 0.7] & [0.4, 0.5] \\ [0.6, 0.7] & [0.5, 0.5] & [0.6, 0.8] & [0.2, 0.6] \\ [0.3, 0.5] & [0.2, 0.4] & [0.5, 0.5] & [0.4, 0.8] \\ [0.5, 0.6] & [0.4, 0.8] & [0.2, 0.6] & [0.5, 0.5] \end{bmatrix}.$$

Solving the model (M-11), we get $J_2^* = 0$. Thus, by Theorem 5.1, we know that $\widetilde{R}$ is a multiplicative consistent interval fuzzy preference relation. Then by the models (M-8) and (M-9), we get

$$\begin{aligned} &w_1^- = 0.1739, \quad w_1^+ = 0.2400, \quad w_2^- = 0.3000, \quad w_2^+ = 0.3913, \\ &w_3^- = 0.1600, \quad w_3^+ = 0.2222, \quad w_4^- = 0.2222, \quad w_4^+ = 0.3000, \end{aligned}$$

i.e.,

$$\tilde{w}_1 = [0.1739, 0.2400], \quad \tilde{w}_2 = [0.3000, 0.3913],$$
$$\tilde{w}_3 = [0.1600, 0.2222] \quad \tilde{w}_4 = [0.2222, 0.3000].$$

After comparing each $\tilde{w}_i$ with all the $\tilde{w}_j (j = 1, 2, 3, 4)$ by using Eq. (10), we have

$$P = \begin{bmatrix} 0.5 & 0 & 0.6235 & 0.1237 \\ 1 & 0.5 & 1 & 1 \\ 0.3765 & 0 & 0.5 & 0 \\ 0.8763 & 0 & 1 & 0.5 \end{bmatrix}.$$

Summing all elements in each line of P, it follows that

$$p_1 = 1.2472, \quad p_2 = 3.5, \quad p_3 = 0.8765, \quad p_4 = 1.8763$$

thus

$$\tilde{w}_2 \overset{1}{\succ} \tilde{w}_4 \overset{0.8763}{\succ} \tilde{w}_1 \overset{0.6235}{\succ} \tilde{w}_3,$$

which indicates that $\tilde{w}_2$ is superior to $\tilde{w}_4$ to the degree of 100%, $\tilde{w}_4$ is superior to $\tilde{w}_1$ to the degree of 87.63%, and $\tilde{w}_1$ is superior to $\tilde{w}_3$ to the degree of 62.35%.

If we solve the model (M-3), then it follows that $J_1^* = 0$. Thus, by Theorem 1, it is clear that $\widetilde{R}$ is an additive consistent interval fuzzy preference relation. Then by the models (M-1) and (M-2), we have

$$w_1^- = 0, \quad w_1^+ = 0.2, \quad w_2^- = 0.45, \quad w_2^+ = 0.7,$$
$$w_3^- = 0, \quad w_3^+ = 0.15, \quad w_4^- = 0.15, \quad w_4^+ = 0.45,$$

i.e.,

$$\tilde{w}_1 = [0, 0.2], \quad \tilde{w}_2 = [0.45, 0.7], \quad \tilde{w}_3 = [0, 0.15], \quad \tilde{w}_4 = [0.15, 0.45],$$

To rank these interval weights, we first compare each $\tilde{w}_i$ with all the $\tilde{w}_j$ $(j = 1, 2, 3, 4)$ by using Eq. (10), and then develop a fuzzy preference relation:

$$P = \begin{bmatrix} 0.5 & 0 & 0.5714 & 0.1000 \\ 1 & 0.5 & 1 & 1 \\ 0.4286 & 0 & 0.5 & 0 \\ 0.9 & 0 & 1 & 0.5 \end{bmatrix}.$$

Summing all elements in each line of P, we have

$$p_1 = 1.1714, \quad p_2 = 3.5, \quad p_3 = 0.9286, \quad p_4 = 2.4$$

thus,

$$\tilde{w}_2 \overset{1}{\succ} \tilde{w}_4 \overset{0.9}{\succ} \tilde{w}_1 \overset{0.5714}{\succ} \tilde{w}_3,$$

which indicates that $\tilde{w}_2$ is superior to $\tilde{w}_4$ to the degree of 100%, $\tilde{w}_4$ is superior to $\tilde{w}_1$ to the degree of 90%, and $\tilde{w}_1$ is superior to $\tilde{w}_3$ to the degree of 57.14%.

From the above analyses, it is clear that the ranking of interval weights derived by the models (M-3) and (M-11) are same, but with slightly different degrees of possibility. If we replace the elements $\tilde{r}_{12} = [0.3, 0.4]$ and

$\tilde{r}_{21} = [0.6, 0.7]$ of $\widetilde{R}$ in Example 1 with a pair of new elements $\tilde{r}'_{12} = [0.1, 0.2]$ and $\tilde{r}'_{21} = [0.8, 0.9]$, that is, $\widetilde{R}$ in Example 1 is revised as

$$\widetilde{R} = (\tilde{r}_{ij})_{4\times 4} = \begin{bmatrix} [0.5, 0.5] & [0.1, 0.2] & [0.5, 0.7] & [0.4, 0.5] \\ [0.8, 0.9] & [0.5, 0.5] & [0.6, 0.8] & [0.2, 0.6] \\ [0.3, 0.5] & [0.2, 0.4] & [0.5, 0.5] & [0.4, 0.8] \\ [0.5, 0.6] & [0.4, 0.8] & [0.2, 0.6] & [0.5, 0.5] \end{bmatrix}$$

then by solving the model (M-11), we get $J_2^* = 0.0906$, and the optimal deviation values are as follows:

$$\ddot{d}_{12}^- = \ddot{d}_{12}^- = 0, \quad \ddot{d}_{13}^- = 0.0413, \quad \ddot{d}_{13}^+ = 0, \quad \ddot{d}_{14}^- = 0.0495,$$
$$\ddot{d}_{14}^+ = 0, \quad \ddot{d}_{23}^- = \ddot{d}_{23}^+ = 0, \quad \ddot{d}_{24}^- = \ddot{d}_{24}^+ = 0, \quad \ddot{d}_{34}^- = \ddot{d}_{34}^- = 0.$$

From Theorem 3, we know that $\widetilde{R}$ is not a multiplicative consistent interval fuzzy preference relation. Then, based on the optimal deviation values $\ddot{d}_{ij}^-$ and $\ddot{d}_{ij}^+$, $i = 1, 2, 3;\ j = i + 1, \ldots, 4$, we solve the models (M-12) and (M-13), and get

$$w_1^- = 0.1057, \quad w_1^+ = 0.1059, \quad w_2^- = 0.4233, \quad w_2^+ = 0.4236,$$
$$w_3^- = 0.1882, \quad w_3^+ = 0.1885, \quad w_4^- = 0.2823, \quad w_4^+ = 0.2825,$$

i.e.,

$$\tilde{w}_1 = [0.1057, 0.1059], \quad \tilde{w}_2 = [0.4233, 0.4236], \quad \tilde{w}_3 = [0.1882, 0.1885], \quad \tilde{w}_4 = [0.2823, 0.2825].$$

It is clear that the ranking of $\tilde{w}_j$ $(j = 1, 2, 3, 4)$ is as follows:

$$\tilde{w}_2 \overset{1}{\succ} \tilde{w}_4 \overset{1}{\succ} \tilde{w}_3 \overset{1}{\succ} \tilde{w}_1$$

that is, $\tilde{w}_2$ is superior to $\tilde{w}_4$ to the degree of 100%, $\tilde{w}_4$ is superior to $\tilde{w}_3$ to the degree of 100%, and $\tilde{w}_3$ is superior to $\tilde{w}_1$ to the degree of 100%.

Obviously, the ranking of $\tilde{w}_1$ and $\tilde{w}_3$ is reversed due to that the element $\tilde{r}_{12}$ becomes smaller from $[0.3, 0.4]$ to $[0.1, 0.2]$.

6. Concluding remarks

In the process of multiple criteria decision making, a decision maker sometimes uses an interval fuzzy preference relation to express his/her uncertain preference information due to the complexity and uncertainty of real-life decision making problem and the time pressure, lack of knowledge, and the decision maker's limited expertise about problem domain. The priority weights derived from an interval fuzzy preference relation can also be used as the weights of criteria or used to rank the given alternatives. In this paper, the concepts of additive and multiplicative consistent interval fuzzy preference relations have been defined, and some simple and practical linear programming models for deriving the priority weights of various interval fuzzy preference relations have been established. Some illustrative examples have been given to examine the feasibility of the developed models. Numerical analyses have indicated that the ranking of interval weights derived by the linear programming models based on additive transitivity is generally the same as that derived by the linear programming models based on multiplicative transitivity, but with slightly different degrees of possibility. Consider that the latter needs stricter conditions for interval fuzzy preference relations and interval weights than the former, that is, the linear programming models based on multiplicative transitivity need that the lower limit r_{ij}^- of each element $\tilde{r}_{ij}$ (where $\tilde{r}_{ij} = [r_{ij}^-, r_{ij}^+]$) of an interval fuzzy preference relation $\widetilde{R}$ (where $\widetilde{R} = (\tilde{r}_{ij})_{n\times n}$) and the lower limit w_i^- of each interval weight $\tilde{w}_i = [w_i^-, w_i^+]$ should be greater than zero, which indicates that these models cannot be used to deal with the situations where $r_{ij}^- = 0$, i.e., the alternative x_j is absolutely preferred to x_i, which usually occurs in the process of decision making, while the linear programming models based on additive transitivity only need that the lower limit of each element of an interval fuzzy preference relations and the lower limit of each interval weight should be greater than or equal to zero. Therefore, the linear programming models based on additive transitivity can overcome the above issue, and thus have practical advantages over the linear programming models based on multiplicative transitivity.

Acknowledgements

The authors are very grateful to the Editor, Professor Lorenzo Peccati, and the anonymous referees for their insightful and constructive comments and suggestions that have led to an improved version of this paper. The work was supported by the National Science Fund for Distinguished Young Scholars of China (70625005), the National Natural Science Foundation of China (70571087, 70321001), and China Postdoctoral Science Foundation (20060390051).

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