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Acceptable consistency analysis of interval reciprocal comparison matrices☆

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Abstract

When a decision maker expresses his/her opinions by means of an interval reciprocal comparison matrix, the study of consistency becomes a very important aspect in decision making in order to avoid a misleading solution. In the present paper, an acceptably consistent interval reciprocal comparison matrix is defined, which can be reduced to an acceptably consistent crisp reciprocal comparison matrix when the intervals become exact numbers. An interval reciprocal comparison matrix with unacceptable consistency can be easily adjusted such that the revised matrix possesses acceptable consistency. Utilizing a convex combination method, a family of crisp reciprocal comparison matrices with acceptable consistency can be obtained, whose weights are further found to exhibit a style of convex combination, and aggregated to obtain interval weights from an acceptably consistent interval reciprocal comparison matrix. A novel, simple yet effective formula of possibility degree is presented to rank interval weights. Numerical results are calculated to show the quality and quantity of the proposed approaches and compare with other existing procedures.

Keywords: Interval reciprocal comparison matrix; Acceptable consistency; Convex combination; Interval weight; Possibility degree formula

1. Introduction

Since the analytic hierarchy process (AHP) was proposed by Saaty [22,23] in the late 1970s, it has been widely used for multiple criteria decision-making problems to rank, select, evaluate, and benchmark decision alternatives [10,29]. In the AHP, after modelling a problem as a hierarchy of criteria, subcriteria and alternatives, the decision maker (DM) may give a consistent crisp reciprocal comparison matrix when his/her judgments are perfectly consistent. However, the DM's judgments are not perfect in most cases, implying that it is inevitable to give an inconsistent crisp reciprocal comparison matrix. In order to measure the level of inconsistency, Saaty [22] further proposed a consistency index (CI) and a consistency ratio (CR), and gave the concepts of acceptable consistency for $CR < 0.1$ and unacceptable consistency for $CR \geqslant 0.1$. Furthermore, many methods (see, e.g. [5,36]) have been shown to modify crisp reciprocal comparison matrices with unacceptable consistency such that the revised matrices are of acceptable consistency.

On the other hand, due to the complexity and uncertainty involved in real-world decision problems and incomplete information or knowledge, it is very difficult to provide a precise numerical value for the DM's level of preference.

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In order to simulate the imprecision, it is more natural to adopt the probability distributions [11,24], interval ratios [1–3,24,17,18] or fuzzy sets [9]. It is especially noted that the methods of generating weights from interval reciprocal comparison matrices have been developed by many researchers. For example, Saaty and Vargas [24] used a Monte Carlo simulation approach to find out interval weights from interval reciprocal comparison matrices, and pointed out difficulties in using this approach. A linear programming (LP) model was utilized by Arbel [1] to formulate the prioritization process, which was further found by Kress [14] to be ineffective for inconsistent interval reciprocal comparison matrices, and extended by Salo and Hämäläinen [26,27] to hierarchical structures. Chandran et al. [6] also presented an LP model to estimate weights for a crisp reciprocal comparison matrix and extended the LP model to deriving weights from an interval reciprocal comparison matrix. Making use of a nonlinear programming (NLP) model, Arbel and Vargas [2,3] dealt with the hierarchical problem and established a connection between the Monte Carlo simulation and Arbel's LP approach. Islam et al. [13] developed a lexicographic goal programming (LGP) method to generate weights from inconsistent interval reciprocal comparison matrices, which was proven by Wang [32] to be defective in theory, because the upper and lower triangular judgments of inconsistent interval reciprocal comparison matrices led to distinctive priorities and rankings by using LGP method. A symmetrical-lexicographic goal programming (SLGP) method was further proposed by Podinovski [21] to obtain weights from both consistent interval reciprocal comparison matrices and inconsistent ones. There are many other mathematical programming models developed to obtain weights from interval reciprocal comparison matrices, such as a fuzzy preference programming (FPP) method [17,18], a goal programming (GP) method [33], and so on. Moreover, Yager [37] addressed a continuous ordered weighted averaging (C-OWA) operator, which is suitable for interval fuzzy preference matrices. A continuous ordered weighted geometric (C-OWG) operator was also proposed by Yager and Xu [38] to generate the weights from interval reciprocal comparison matrices. The weights derived by the C-OWA and C-OWG operators are associated with the attitudinal character of Q, where Q is a basic unit-interval monotonic (BUM) function chosen by a DM in the process of decision making. From the viewpoint of the probability, Moreno-Jiménez [19] studied the distribution of possible rankings of alternatives in a small interval reciprocal comparison matrix. Haines [11] proposed a statistical approach to extract preferences from interval reciprocal comparison matrices.

According to the above-stated works, although lots of methods have been used to generate weights from both consistent interval reciprocal comparison matrices and inconsistent ones, the consistency-test method of interval reciprocal comparison matrices is little. Indeed, in order to make the weights and rankings reliable, it is very important to study consistency of interval reciprocal comparison matrices. Along this line, Arbel [1] first defined a "feasible region" to give a consistent local priority vector. Unfortunately, when an interval reciprocal comparison matrix is inconsistent, the feasible region becomes empty. Furthermore, it plays a great role in dealing with the case of empty feasible region, and has attracted increasing attention. For example, Salo [25] considered that an interval reciprocal comparison matrix was consistent if a set of crisp relative weights within the feasible regions defined by the ranges of mean values were existent. For the purpose of investigating the weights of an inconsistent matrix, a simple heuristic was further proposed to enlarge the feasible region and obtain the feasible weights. Leung and Cao [16] presented a fuzzy-consistency-test method of fuzzy positive reciprocal matrices with consideration of a tolerance deviation, which retained the inconsistent information to a certain extent. However, when a fuzzy matrix does not pass the test, it would need the DM to redo the matrix. A fuzzy programming approach was also proposed by Mikhailov [18] to obtain weights from both consistent and inconsistent interval reciprocal comparison matrices. For the case of inconsistent ones, deviation parameters were introduced to extend the ranges of the intervals. Especially from the notion of "feasible region", a consistent interval reciprocal comparison matrix was defined in Wang et al. [30,31]. When an interval reciprocal comparison matrix is inconsistent, a logarithmic GP method was proposed by Wang et al. [30] to minimize the inconsistency, and in Wang et al. [31], an eigenvector method-based (EM) NLP method was applied to generate acceptably consistent interval weights instead of consistent ones. As shown by the above proposed methods of generating weights from inconsistent interval reciprocal comparison matrices, it may be inevitable to adjust the ranges of the intervals. And the remaining key issue is how to reserve information as much as possible from an interval matrix given by the decision maker under the condition of obtaining a reliable solution. In the paper, a simple method of how to test acceptable consistency of interval reciprocal comparison matrices is presented. Interval reciprocal comparison matrices with unacceptable consistency can be easily adjusted to those with acceptable consistency. A special case of an acceptably consistent interval reciprocal comparison matrix is that with consistency. Using a convex combination method, a family of acceptably consistent crisp reciprocal comparison matrices are further obtained. Since it is argued that the interval weights are more logical and natural than an exact priority vector, the interval weights generated from an acceptably consistent interval reciprocal comparison matrix are determined. Furthermore a simple and effective method is given to rank the interval weights. Three examples are calculated to show the feasibility and applicability of the presented methods. It is observed especially that one can keep enough information from initial inconsistent interval reciprocal comparison matrices and obtain convincing rankings of alternatives through the proposed procedures.

This paper is organized as follows. In Section 2, some known results of crisp reciprocal comparison matrices are given, and an interesting theorem is studied. Acceptable consistency analysis of interval reciprocal comparison matrices is made in Section 3. Section 4 addresses how to generate interval weights from acceptably consistent interval reciprocal comparison matrices and rank. The proposed methods are illustrated by three known numerical examples and examined in relation to other existing procedures in Section 5. The main conclusions are shown in Section 6.

2. Preliminaries

In the classical AHP, the multiple criteria decision problem is structured hierarchically at different levels, which contain finite number of criteria or alternatives. Applying reciprocal comparisons, the relative importance of the decision elements is determined through exact reciprocal comparison ratios. Then a crisp reciprocal comparison matrix is obtained, and used to aggregate into the weights of alternatives. However, the obtained weights are reliable only if the judgments of a DM are perfectly consistent. In most cases, it is unrealistic to expect a crisp reciprocal comparison matrix to be perfectly consistent for the reason that the DM would be affected by many uncertainties. Consequently, a definition of acceptably consistent crisp reciprocal comparison matrices was further developed by Saaty [22] to allow a certain of level of acceptable deviation. In what follows, let us firstly recall the knowledge of consistent and acceptably consistent crisp reciprocal comparison matrices, respectively.

Let $P = (p_{ij})_{n\times n}$ denote a crisp reciprocal comparison matrix, where $p_{ij} = 1/p_{ji}$, $p_{ij} \in R^+$, for all $i, j = 1, 2, \ldots, n$.

Definition 1 (*Saaty [22]*). A crisp reciprocal comparison matrix $P = (p_{ij})_{n\times n}$ is consistent, if $p_{ij} = p_{ik}p_{kj}$, $\forall i, j, k = 1, \ldots, n$.

When P is inconsistent, Saaty [22] has proposed a *CI* and a *CR* to measure the level of inconsistency, that is

$$CI = \frac{\lambda_{max} - n}{n-1}, \quad CR = CI/RI, \tag{1}$$

where λ_{max} and n are the largest eigenvalue and the order of P, respectively. *RI* is a random index, which is the average *CI* of a large number of randomly generated crisp reciprocal comparison matrices, and dependent on the orders of the matrices given in Table 1. When $CR < 0.10$, the crisp reciprocal comparison matrix P is considered to be acceptably consistent. While $CR \geqslant 0.10$, the matrix P is said to be of unacceptable consistency, which should be adjusted to that with acceptable consistency to ensure the rationality of decisions.

In addition, two helpful lemmata are introduced as follows:

Lemma 1 (*Perron [12]*). *Let $Q = (q_{ij})_{n\times n}$ be a positive matrix and λ_{max} be the largest eigenvalue of Q, then*

$$\lambda_{max} = \min_{x\in R_n^+} \max_i \sum_{j=1}^{n} q_{ij}\frac{x_j}{x_i}, \quad i = 1, 2, \ldots, n, \tag{2}$$

where $R_n^+ = \{x = (x_1, x_2, \ldots, x_n)^T | x_i > 0, i = 1, 2, \ldots, n\}$.

Table 1
The mean consistency index of randomly generated matrices.

n	1	2	3	4	5	6	7	8	9	10	11	12
RI	0	0	0.52	0.89	1.12	1.26	1.36	1.41	1.46	1.49	1.52	1.54

Lemma 2. *Let* $x > 0, y > 0, u > 0, v > 0, u + v = 1$, *then*

$$x^u y^v \leqslant ux + vy \tag{3}$$

with equality if and only if $x = y$.

Furthermore, it is interesting to get the following theorem:

Theorem 1. *Suppose* $R = (r_{ij})_{n\times n}$ *and* $S = (s_{ij})_{n\times n}$ *are two positive matrices,* λ^R_{max} *and* λ^S_{max} *are the greatest eigenvalues of R and S, respectively. Let* $T(\alpha) = (t_{ij}(\alpha))_{n\times n} = (r^{\alpha}_{ij} s^{1-\alpha}_{ij})_{n\times n}$ *where* $\alpha \in [0, 1]$, *and* $\lambda^{T(\alpha)}_{max}$ *are the greatest eigenvalue of* $T(\alpha)$, *then*

$$\lambda^{T(\alpha)}_{max} \leqslant \max\{\lambda^R_{max}, \lambda^S_{max}\}. \tag{4}$$

Proof. Let $\omega^R = (\omega^R_1, \omega^R_2, \ldots, \omega^R_n)$ and $\omega^S = (\omega^S_1, \omega^S_2, \ldots, \omega^S_n)$ be the normalized principal right eigenvectors corresponding to the greatest eigenvalues λ^R_{max} of R and λ^S_{max} of S, respectively. It is also supposed that $e_{ij} = r_{ij}\omega^R_j/\omega^R_i$ and $f_{ij} = s_{ij}\omega^S_j/\omega^S_i$, $i, j = 1, 2, \ldots, n$. Then, one can get $\lambda^R_{max} = \sum_{j=1}^n e_{ij}$, $\lambda^S_{max} = \sum_{j=1}^n f_{ij}$ and $t_{ij}(\alpha) = e^{\alpha}_{ij} f^{1-\alpha}_{ij} (\omega^R_i/\omega^R_j)^{\alpha} (\omega^S_i/\omega^S_j)^{1-\alpha}$.

Assumption of

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \\ &= ((\omega^R_1)^{\alpha}(\omega^S_1)^{1-\alpha}, (\omega^R_2)^{\alpha}(\omega^S_2)^{1-\alpha}, \ldots, (\omega^R_n)^{\alpha}(\omega^S_n)^{1-\alpha}) \in R^+_n, \end{aligned}$$

and application of Lemmata 2.1 and 2.2 lead to

$$\begin{aligned} \lambda^{T(\alpha)}_{max} &= \min_{x\in R^+_n} \max_i \sum_{j=1}^n (r^{\alpha}_{ij} s^{1-\alpha}_{ij}) \frac{x_j}{x_i} \\ &\leqslant \max_i \sum_{j=1}^n e^{\alpha}_{ij} f^{1-\alpha}_{ij} \left(\frac{\omega^R_i}{\omega^R_j}\right)^{\alpha} \left(\frac{\omega^S_i}{\omega^S_j}\right)^{1-\alpha} \frac{(\omega^R_j)^{\alpha}(\omega^S_j)^{1-\alpha}}{(\omega^R_i)^{\alpha}(\omega^S_i)^{1-\alpha}} \\ &= \max_i \sum_{j=1}^n e^{\alpha}_{ij} f^{1-\alpha}_{ij} \\ &\leqslant \max_i \sum_{j=1}^n (\alpha e_{ij} + (1-\alpha) f_{ij}) \\ &= \alpha\lambda^R_{max} + (1-\alpha)\lambda^S_{max} \\ &\leqslant \max\{\lambda^R_{max}, \lambda^S_{max}\}. \end{aligned}$$

Thus the theorem is proved. □

3. Acceptable consistency analysis

Since interval judgments are more natural and logical than precise judgments for a reciprocal comparison, it is assumed that a DM provides an interval reciprocal comparison matrix by judging that criterion i is between l_{ij} and u_{ij} times as important as criterion j as follows [24]:

$$A = \begin{bmatrix} 1 & [l_{12}, u_{12}] & \cdots & [l_{1n}, u_{1n}] \\ [l_{21}, u_{21}] & 1 & \cdots & [l_{2n}, u_{2n}] \\ \vdots & \vdots & \vdots & \vdots \\ [l_{n1}, u_{n1}] & [l_{n2}, u_{n2}] & \cdots & 1 \end{bmatrix}, \tag{5}$$

where l_{ij} and u_{ij} are non-negative real numbers, $l_{ij} \leqslant u_{ij}$, $l_{ij} = 1/u_{ji}$ and $u_{ij} = 1/l_{ji}$.

In order to make the weights generated from A reliable, the consistency study of A is very important. According to the known methods based on the feasible region, the reliable weights from inconsistent interval reciprocal comparison matrices are difficult to obtain [14,18]. Here, we present a definition of acceptably consistent interval reciprocal comparison matrices, then obtain the weights. Prior to the consideration, it is convenient to introduce a convex combination method.

Let $B = (b_{ij})_{n\times n}$ and $C = (c_{ij})_{n\times n}$ where

$$b_{ij} = \begin{cases} u_{ij}, & i < j, \\ 1, & i = j, \\ l_{ij}, & i > j \end{cases} \qquad c_{ij} = \begin{cases} l_{ij}, & i < j, \\ 1, & i = j, \\ u_{ij}, & i > j. \end{cases} \tag{6}$$

It is also supposed that

$$D(\alpha) = (d_{ij}(\alpha))_{n\times n} = (b_{ij}^{\alpha} c_{ij}^{1-\alpha})_{n\times n}, \tag{7}$$

where $\alpha \in [0, 1]$. From (7), One can find that every element of $D(\alpha)$ is a convex combination of the corresponding elements in B and C, and $d_{ij}(\alpha)\,(i, j = 1, 2, \ldots, n)$ are also monotone continuous functions with respect to α. Especially, it is seen that $D(0) = C$ and $D(1) = B$.

Making use of the stated convex combination method, it is easy to obtain the following property theorem:

Theorem 2. *Let A be an interval reciprocal comparison matrix. B and C along with $D(\alpha)$ are constructed by* (6) *and* (7), *respectively. Then one has*:

(a) *For $\forall\alpha \in [0, 1]$, $d_{ij}(\alpha) \in [l_{ij}, u_{ij}]$ where $i, j = 1, 2, \ldots, n$.*
(b) *B and C are crisp reciprocal comparison matrices.*
(c) *$D(\alpha)$ are crisp reciprocal comparison matrices, where $\alpha \in [0, 1]$.*

Clearly, it is seen from (7) that $d_{ij}(\alpha) \in [l_{ij}, u_{ij}]$ for $\forall\alpha \in [0, 1]$. Moreover, considering the reciprocal property of A and (6), one can obtain that B and C are crisp reciprocal comparison matrices. While from (7) and (b), we have

$$d_{ij}(\alpha)d_{ji}(\alpha) = (b_{ij}^{\alpha} c_{ij}^{1-\alpha})(b_{ji}^{\alpha} c_{ji}^{1-\alpha}) = (b_{ij}b_{ji})^{\alpha}(c_{ij}c_{ji})^{1-\alpha} = 1$$

for all $i, j = 1, 2, \ldots, n$. So $D(\alpha)$ are crisp reciprocal comparison matrices for $\alpha \in [0, 1]$. According to the above analysis, the proofs of Theorem 2 can be obtained and the detail procedures are omitted here.

In what follows, we turn our attention to acceptable consistency of interval reciprocal comparison matrices and give the following definition:

Definition 2. Let A be an interval reciprocal comparison matrix defined by (5), where l_{ij} and u_{ij} are non-negative real numbers, $l_{ij} \leqslant u_{ij}$, $l_{ij} = 1/u_{ji}$ and $u_{ij} = 1/l_{ji}$. If two crisp reciprocal comparison matrices B and C given by (6) are all of acceptable consistency, then A is said to be an acceptably consistent interval reciprocal comparison matrix. Otherwise, A is said to be unacceptably consistent.

From Definition 2, it is seen that an acceptably consistent interval reciprocal comparison matrix can be reduced to an acceptably consistent crisp reciprocal comparison matrix [22] when the intervals become real numbers. In order to check whether A is acceptably consistent or not, it is sufficient to check the acceptable consistency of B and C. When one of B and C does not possess acceptable consistency, A is unacceptably consistent. Moreover, a matrix A with unacceptable consistency can be easily adjusted to that with acceptable consistency by modifying B or/and C. In other words, to make an interval reciprocal comparison matrix A be acceptably consistent, there are four cases, i.e. (i) $CR_B = 0$ and $CR_C = 0$; (ii) $CR_B = 0$ and $0 < CR_C < 0.1$; (iii) $0 < CR_B < 0.1$ and $CR_C = 0$ and (iv) $0 < CR_B < 0.1$ and $0 < CR_C < 0.1$, where CR_B and CR_C are CRs of B and C, respectively. Especially let us consider the case where the CRs of B and C are zeroes. With the knowledge of crisp reciprocal comparison matrices in hand, one can see that B and C are all consistent. Clearly, it is corresponding to an ideal and perfect situation yet important in theory. In this special case, it is supposed that the DM can compare any two criteria i and j with the same level of preference and provide an interval reciprocal comparison matrix. At the same time, the bounds of $[l_{ij}, u_{ij}]$ have been considered perfectly. By the way, one can obtain the following definition:

Definition 3. Let A be an interval reciprocal comparison matrix defined by (5), where l_{ij} and u_{ij} are non-negative real numbers, $l_{ij} \leqslant u_{ij}$, $l_{ij} = 1/u_{ji}$ and $u_{ij} = 1/l_{ji}$. If the crisp reciprocal comparison matrices B and C determined by (6) are all consistent, then A is said to be a consistent interval reciprocal comparison matrix.

As shown in Definition 3, it is easy to check whether an interval reciprocal comparison matrix is consistent or not without solving any mathematical programming and complicated algebraic operation. To compare with that given in Wang et al. [30,31], it is found that a consistent interval reciprocal comparison matrix obtained from Definition 3 must satisfy the conditions of Wang et al.'s definition based on the feasible region method. In other words, if the feasible region is empty, interval reciprocal comparison matrices must be inconsistent by Definition 3. However, when the feasible region is non-empty, it does not mean that an interval reciprocal comparison matrix is of acceptable consistency by Definition 2. Also, one cannot directly derive a non-empty feasible region from an acceptably consistent interval reciprocal comparison matrix. The reason is that they would been based on different standards. For example, when an acceptably consistent interval reciprocal comparison matrix is reduced to an acceptably consistent crisp reciprocal comparison matrix, obviously there does not exist a feasible region. When B is consistent and C is unacceptably consistent, the interval reciprocal comparison matrix A is unacceptably consistent by Definition 2, but the feasible region should be non-empty.

Additionally, it is noted that an approach dealing with fuzzy reciprocal matrices has been proposed by Ohnishi et al. [20], who obtained the crisp optimal weight pattern by maximizing the degree of consistency in a sense of fuzzy comparisons. An interval-valued matrix is derived by introducing an optimal CI to the fuzzy matrix. Of much interest is that the derived interval-valued matrix is acceptably consistent by Definition 2 for a certain case. For example, it is easy to check that the obtained interval-valued matrix shown in the example [20] owns acceptable consistency. As shown in [20], it is further seen that a fuzzy-valued ratio matrix is considered to be a fuzzy set of consistent crisp ratio matrices. Similarly, applying the convex combination method and Definition 2, one has a property theorem, i.e.

Theorem 3. *A is acceptably consistent if and only if* $D(\alpha)(\alpha \in [0, 1])$ *have acceptable consistency.*

Proof. It is easily found from (7) that B and C are acceptably consistent when $D(\alpha)$ are of acceptable consistency for $\alpha \in [0, 1]$, indicating that A has acceptable consistency by Definition 2.

Conversely, suppose A is acceptably consistent, that is, B and C have acceptable consistency. Let λ_{max}^B and λ_{max}^C be the greatest eigenvalues of B and C, respectively. From (1), we have

$$\frac{\lambda_{max}^B - n}{(n-1)RI} < 0.10, \quad \frac{\lambda_{max}^C - n}{(n-1)RI} < 0.10.$$

Then using Theorem 1, one can arrive at

$$\frac{\lambda_{max}^{D(\alpha)} - n}{(n-1)RI} < 0.10,$$

where $\lambda_{max}^{D(\alpha)}$ are the greatest eigenvalues of $D(\alpha)$. Then, $D(\alpha)$ are of acceptable consistency. □

As observed in Theorem 3, when A is an acceptably consistent interval reciprocal comparison matrix, a family of crisp reciprocal comparison matrices $D(\alpha)$ with acceptable consistency are obtained.

Moreover, utilizing Definition 3 and (7) one can further obtain a theorem as follows:

Theorem 4. *A is consistent if and only if* $D(\alpha)(\alpha \in [0, 1])$ *have consistency.*

Proof. Suppose $D(\alpha)$ have consistency for $\forall \alpha \in [0, 1]$. It is easy to know that $D(0) = C$ and $D(1) = B$ are consistent. From Definition 3, one can obtain that A is consistent.

On the other hand, it is assumed that A has consistency. Making use of Definition 3, one can see that B and C are consistent. That is, $b_{ij} = b_{ik}b_{kj}$, $c_{ij} = c_{ik}c_{kj}$, for all $i, j, k = 1, 2, \ldots, n$. Furthermore, application of (7) yields

$$d_{ij}(\alpha) = b_{ij}^{\alpha} c_{ij}^{1-\alpha} = (b_{ik}b_{kj})^{\alpha}(c_{ik}c_{kj})^{1-\alpha} = d_{ik}(\alpha)d_{kj}(\alpha),$$

where $\forall\alpha \in [0, 1]$ and $i, j, k = 1, 2, \ldots, n$, inferring that $D(\alpha)$ are consistent according to Definition 1. So we have proved the theorem. □

Also when A is consistent, a family of consistent crisp reciprocal comparison matrices $D(\alpha)(\alpha \in [0, 1])$ are determined through Theorem 4.

4. Interval weights and ranking

4.1. Interval weights

When an interval reciprocal comparison matrix is consistent by Definition 3, the derived weights are consistent. In most cases, an interval reciprocal comparison matrix A has not consistency, that is, it is inconsistent for matrix B or/and C, even unacceptably consistent. Under the consideration of Definition 2, an interval reciprocal comparison matrix with unacceptable consistency can be modified to that with acceptable consistency. When an interval reciprocal comparison matrix is acceptably consistent through Definition 2, it is reasonable to consider the derived weights to be acceptably reliable. In what follows, the weights generated from acceptably consistent interval reciprocal comparison matrices are only considered. On the other hand, in the case of interval reciprocal comparison matrices, it is considered that interval weights should be more logical and natural than an exact priority vector to represent part or complete imprecision of the DM's judgments. Consequently, interval weights derived from acceptably consistent interval reciprocal comparison matrices will be determined.

To achieve the interval weights, it is convenient to investigate the weights of crisp reciprocal comparison matrices $D(\alpha)(\alpha \in [0, 1])$ with acceptable consistency by considering Theorem 3. Let us suppose that the weights $\omega_i(\alpha)$ of $D(\alpha)$ satisfy $\prod_{i=1}^{n} \omega_i(\alpha) = 1$. The geometric mean (GM) method (see, e.g. [4,7,15]) is used to obtain the weights from the matrices $D(\alpha)$, i.e.

$$\omega_i(\alpha) = \left(\prod_{j=1}^{n} d_{ij}(\alpha)\right)^{1/n}, \tag{8}$$

where $i = 1, 2, \ldots, n$, and $\alpha \in [0, 1]$.

Utilizing (7), we have

$$\omega_i(\alpha) = \left(\prod_{j=1}^{n} b_{ij}^{\alpha} c_{ij}^{1-\alpha}\right)^{1/n} = \left(\left(\prod_{j=1}^{n} b_{ij}\right)^{1/n}\right)^{\alpha} \left(\left(\prod_{j=1}^{n} c_{ij}\right)^{1/n}\right)^{1-\alpha} = \omega_i^{\alpha}(B)\omega_i^{1-\alpha}(C), \tag{9}$$

where $\omega_i(B)$ and $\omega_i(C)$ are the weights of B and C, respectively. Furthermore, in order to obtain the synthetic priority of A, the weight functions $\omega_i(\alpha)$ ($\alpha \in [0, 1]$) derived from $D(\alpha)$ are further aggregated as the following interval numbers:

$$\omega_i = [\min\{\omega_i(\alpha)|\alpha \in [0, 1]\}, \max\{\omega_i(\alpha)|\alpha \in [0, 1]\}]. \tag{10}$$

Note that $\alpha \in [0, 1]$ and $\omega_i(\alpha)$ $(i = 1, 2, \ldots, n)$ are monotonous continuous functions with respect to α. Then expression (10) can be equivalently rewritten as

$$\omega_i = [\min\{\omega_i(B), \omega_i(C)\}, \max\{\omega_i(B), \omega_i(C)\}] \tag{11}$$

with $i = 1, 2, \ldots, n$.

From the above analysis, it is seen that the weights $\omega_i(\alpha)$ exhibit evidently a convex combination of the weights $\omega_i(B)$ and $\omega_i(C)$, which is owing to the use of the GM method. It is worth pointing out that the obvious expression (9) would not appear if other weight-generation methods (e.g. the EM [22]; the two-stage LP method [6]) are utilized, although the direct computing numerical results show that the interval weights obtained by the different methods are all almost identical.

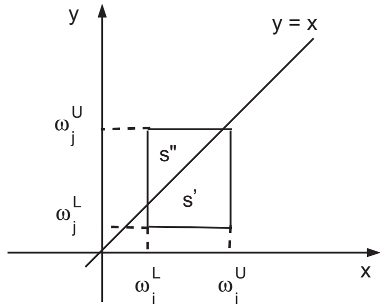

Fig. 1. A two-dimensional style of two interval weights.

4.2. Ranking of interval weights

Clearly, interval weights lead to greater complexity and difficulty in comparison and ranking than crisp weights. In order to rank interval weights, Sengupta and Pal [28] defined an acceptability index, which was also designed to reflect the grade of acceptability of one interval number to be inferior to another one. The index is totally based on the midpoints of interval numbers. It is also noted that three other possibility-degree formulae have been developed by Facchinetti et al. [8], Wang et al. [30], Xu and Da [35], and further proved to be equivalent by Xu and Chen [34]. In what follows, a simple yet effective possibility-degree formula will be proposed.

Let $\omega_i = [\omega_i^L, \omega_i^U]$ and $\omega_j = [\omega_j^L, \omega_j^U]$ be two interval weights. It is easy to define the degree of possibility $p(\omega_i \geqslant \omega_j)$ under the especial case of $(\omega_i^U - \omega_i^L)(\omega_j^U - \omega_j^L) = 0$. When $\omega_i^U = \omega_i^L$ and $\omega_j^U = \omega_j^L$, that is, both interval weights ω_i and ω_j are reduced to exact real numbers, we have

$$p(\omega_i \geqslant \omega_j) = \begin{cases} 1 & \text{if } \omega_i > \omega_j, \\ 1/2 & \text{if } \omega_i = \omega_j, \\ 0 & \text{if } \omega_i < \omega_j. \end{cases} \tag{12}$$

When $\omega_i^U = \omega_i^L = \omega_i$ and $\omega_j^U \neq \omega_j^L$, one can get

$$p(\omega_i \geqslant \omega_j) = \begin{cases} 1 & \text{if } \omega_i > \omega_j^U, \\ \dfrac{\omega_i - \omega_j^L}{\omega_j^U - \omega_j^L} & \text{if } \omega_j^L \leqslant \omega_i \leqslant \omega_j^U, \\ 0 & \text{if } \omega_i < \omega_j^L. \end{cases} \tag{13}$$

While $\omega_i^U \neq \omega_i^L$ and $\omega_j^U = \omega_j^L = \omega_j$, $p(\omega_i \geqslant \omega_j)$ is expressed as

$$p(\omega_i \geqslant \omega_j) = \begin{cases} 1 & \text{if } \omega_i^L > \omega_j, \\ \dfrac{\omega_i^U - \omega_j}{\omega_i^U - \omega_i^L} & \text{if } \omega_i^L \leqslant \omega_j \leqslant \omega_i^U, \\ 0 & \text{if } \omega_i^U < \omega_j. \end{cases} \tag{14}$$

For a general case of $\omega_i^U \neq \omega_i^L$ and $\omega_j^U \neq \omega_j^L$, two interval weights are shown in Fig. 1. One can see from Fig. 1 that a rectangle is formed by four peaks (ω_i^L, ω_j^L), (ω_i^L, ω_j^U), (ω_i^U, ω_j^L) and (ω_i^U, ω_j^U). The straight line $y = x$ divides the rectangle into two parts marked by s' and s'', respectively, which also stand for the areas of corresponding regions. In the rectangle, it is easy to find that the points satisfying $x > y$ belong to the range s', and those satisfying $x < y$ are in the range of s''. Based on the above analysis, we give the following definition.

Definition 4. Let $\omega_i = [\omega_i^L, \omega_i^U]$ and $\omega_j = [\omega_j^L, \omega_j^U]$ be two interval weights, $\omega_i^U \neq \omega_i^L$ and $\omega_j^U \neq \omega_j^L$, then the degree of possibility of $\omega_i \geqslant \omega_j$ is defined as

$$p(\omega_i \geqslant \omega_j) = \frac{s'}{s}, \tag{15}$$

where $s = (\omega_i^U - \omega_i^L)(\omega_j^U - \omega_j^L)$.

As a consequence, the possibility degree of $\omega_j \geqslant \omega_i$ can be determined by

$$p(\omega_j \geqslant \omega_i) = \frac{s''}{s}. \tag{16}$$

Obviously, the formula of possibility degree $p(\omega_i \geqslant \omega_j)$ satisfies the following properties as usual and the proofs have been neglected here for saving spaces.

Property 1. $0 \leqslant p(\omega_i \geqslant \omega_j) \leqslant 1$.

Property 2. $p(\omega_i \geqslant \omega_j) + p(\omega_j \geqslant \omega_i) = 1$. *Especially*, $p(\omega_i \geqslant \omega_i) = 0.5$.

Property 3. $p(\omega_i \geqslant \omega_j) = 1$ *if and only if* $\omega_j^U \leqslant \omega_i^L$.

Property 4. $p(\omega_i \geqslant \omega_j) = 0$ *if and only if* $\omega_i^U \leqslant \omega_j^L$.

Property 5. $p(\omega_i \geqslant \omega_j) \geqslant 0.5$ *if and only if* $\omega_i^U + \omega_i^L \geqslant \omega_j^U + \omega_j^L$. *Especially*, $p(\omega_i \geqslant \omega_j) = 0.5$ *if and only if* $\omega_i^U + \omega_i^L = \omega_j^U + \omega_j^L$.

Property 6. *Let* ω_i, ω_j *and* ω_k *be three interval weights, if* $p(\omega_i \geqslant \omega_j) \geqslant 0.5$ *and* $p(\omega_j \geqslant \omega_k) \geqslant 0.5$, *then* $p(\omega_i \geqslant \omega_k) \geqslant 0.5$.

In the end, a new algorithm for generating the weights from acceptably consistent interval reciprocal comparison matrices is obtained as follows:

Step 1: Consider a decision-making problem. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of alternatives. A DM gives his/her preference information on X by means of an interval reciprocal comparison matrix A.

Step 2: From (6), the matrices B and C are constructed.

Step 3: Acceptable consistency of B and C is checked. According to the method proposed by Xu and Wei [36], any of B and C with $CR \geqslant 0.1$ is adjusted to that possessing acceptable consistency (i.e. $CR < 0.1$).

Step 4: Utilizing (8), we can get the weight vectors of B and C.

Step 5: From (11), an interval weight vector of A is obtained.

Step 6: Application of (15) leads to a matrix of possibility degree.

Step 7: Similar to Wang et al. [30], a simple row–column elimination method is applied to generate a ranking vector from the matrix of possibility degree.

5. Numerical results and discussions

In this section, we offer three numerical examples to illustrate the proposed approaches of generating interval weights from interval reciprocal comparison matrices and raking. For the purpose of comparing with other existing procedures, here two theoretical examples are chosen firstly in Examples 1 and 2. One is an acceptably consistent interval reciprocal comparison matrix, and another is an unacceptably consistent one by use of Definition 2. Furthermore, it is noticed that discussions on a concrete decision making problem would be significant and shown in Example 3.

Example 1. Consider the following interval reciprocal comparison matrix, which has been investigated by Arbel and Vargas [2,3], Hains [11] and Wang et al. [30,31], respectively:

$$A = \begin{bmatrix} 1 & [2,5] & [2,4] & [1,3] \\ [1/5,1/2] & 1 & [1,3] & [1,2] \\ [1/4,1/2] & [1/3,1] & 1 & [1/2,1] \\ [1/3,1] & [1/2,1] & [1,2] & 1 \end{bmatrix}.$$

From (6), matrices B and C can be expressed as

$$B = \begin{bmatrix} 1 & 5 & 4 & 3 \\ 1/5 & 1 & 3 & 2 \\ 1/4 & 1/3 & 1 & 1 \\ 1/3 & 1/2 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 1/2 & 1 & 1 & 1 \\ 1/2 & 1 & 1 & 1/2 \\ 1 & 1 & 2 & 1 \end{bmatrix}.$$

Then using (1), we can determine $CR_B = 0.0830 < 0.1$ and $CR_C = 0.0227 < 0.1$, respectively. It means that B and C are all acceptably consistent, or A is acceptably consistent by Definition 2. The obtained results are different from those given by Arbel and Vargas [2,3], Hains [11], Wang et al. [30,31] who have pointed out that A is a consistent interval reciprocal comparison matrix.

Utilizing (8), the weight vectors generated from B and C can be obtained as

$$\omega(B) = (2.7832, 1.0466, 0.5373, 0.6389),$$

$$\omega(C) = (1.4142, 0.8409, 0.7071, 1.1892).$$

Then from (11), one can compute the interval weight vector from A as

$$\omega = ([1.4142, 2.7832], [0.8409, 1.0466], [0.5373, 0.7071], [0.6389, 1.1892]).$$

Applying (15), the matrix of possibility degree can be further determined, namely

$$P = \begin{bmatrix} 0.5000 & 1.0000 & 1.0000 & 1.0000 \\ 0.0000 & 0.5000 & 1.0000 & 0.5540 \\ 0.0000 & 0.0000 & 0.5000 & 0.0249 \\ 0.0000 & 0.4460 & 0.9751 & 0.5000 \end{bmatrix}.$$

Furthermore, the ranking order is easily to determine as $\omega_1 \succ^{100\%} \omega_2 \succ^{55.4\%} \omega_4 \succ^{97.51\%} \omega_3$, which is in accordance with those given by Arbel and Vargas [2,3] using the average weights of all the vertices, by Hains [11] using the expected weights, by Wang et al. [30,31] using interval weights. Similar to Wang et al. [30,31], the proposed ranking methods can provide the information about the degrees of preference. But the values of degree of preference are different from those obtained by Wang et al. [30,31], which is attributed to the different methods of getting interval weights as well as the matrix of possibility degree.

Example 2. Consider the following interval reciprocal comparison matrix, which has been examined by Kress [14] and Wang et al. [30,31], respectively.

$$A = \begin{bmatrix} 1 & [1,2] & [1,2] & [2,3] \\ [1/2,1] & 1 & [3,5] & [4,5] \\ [1/2,1] & [1/5,1/3] & 1 & [6,8] \\ [1/3,1/2] & [1/5,1/4] & [1/8,1/6] & 1 \end{bmatrix}.$$

Similarly, matrices B and C can be written as

$$B = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1/2 & 1 & 5 & 5 \\ 1/2 & 1/5 & 1 & 8 \\ 1/3 & 1/5 & 1/8 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 4 \\ 1 & 1/3 & 1 & 6 \\ 1/2 & 1/4 & 1/6 & 1 \end{bmatrix}.$$

From expressions (1), one can obtain that $CR_B = 0.2783 > 0.1$ and $CR_C = 0.1246 > 0.1$, implying that B and C together with A are unacceptably consistent. In Kress [14] and Wang et al. [30,31], the interval reciprocal comparison matrix A is also considered to be inconsistent. Moreover, the consistency improving method proposed by Xu and Wei [36] is adopted to adjust B and C. In order to keep the original information as much as possible and make the modifications B' and C' be acceptably consistent, we use two change parameters $\mu_B = 0.6$ and $\mu_C = 0.88$ for B and C, respectively, and have

$$B' = \begin{bmatrix} 1.0000 & 1.4696 & 1.9340 & 3.9754 \\ 0.6805 & 1.0000 & 3.4565 & 5.5707 \\ 0.5171 & 0.2893 & 1.0000 & 5.6122 \\ 0.2515 & 0.1795 & 0.1782 & 1.0000 \end{bmatrix},$$

$$C' = \begin{bmatrix} 1.0000 & 0.9428 & 0.9943 & 2.1061 \\ 1.0607 & 1.0000 & 2.7733 & 4.1113 \\ 1.0057 & 0.3606 & 1.0000 & 5.5695 \\ 0.4748 & 0.2432 & 0.1795 & 1.0000 \end{bmatrix}.$$

It is further calculated that $CR_{B'} = 0.0963 < 0.1$ and $CR_{C'} = 0.0958 < 0.1$, indicating that B' and C' are acceptably consistent. From (8), two weight vectors of B' and C' can be written as $\omega(B') = (1.8334, 1.9026, 0.9572, 0.2995)$ and $\omega(C') = (1.1854, 1.8648, 1.1921, 0.3794)$. Making use of (11), we can obtain the interval weight vector of A as

$$\omega = ([1.1854, 1.8334], [1.8648, 1.9026], [0.9572, 1.1921], [0.2995, 0.3794]).$$

Application of (15) leads to the matrix of possibility degree, i.e.

$$P = \begin{bmatrix} 0.5000 & 0.0000 & 0.9999 & 1.0000 \\ 1.0000 & 0.5000 & 1.0000 & 1.0000 \\ 0.0001 & 0.0000 & 0.5000 & 1.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.5000 \end{bmatrix}.$$

Then, one can obtain $\omega_2 \succ^{100\%} \omega_1 \succ^{99.99\%} \omega_3 \succ^{100\%} \omega_4$. The same ranking order can be also found in Kress [14] and Wang et al. [30,31].

On the other hand, the degrees of preference are also shown in the present ranking result, in agreement with the observations $\omega_2 \succ^{100\%} \omega_1 \succ^{99.83\%} \omega_3 \succ^{100\%} \omega_4$ given by Wang et al. [31] for the case with limitation of $CR \leqslant 0.1$. Moreover, here it is not requisite to solve any mathematical programming to obtain the interval weights by using the proposed approaches, which are easier than those applied by Wang et al. [30,31].

Example 3. Here we choose a concrete decision-making problem from Islam et al. [13] and Wang et al. [30,31]. A hierarchy structure of criteria is shown in Fig. 2. It is supposed that a person is interested in investing his money to any one of the four portfolios: bank deposit (BD), debentures (DB), government bonds (GB) and Shares (SH). Out of these portfolios he has to choose only one based upon four criteria: return (Re), risk (Ri), tax benefits (Tb) and liquidity (Li). The interval reciprocal comparison matrices for the four criteria with respect to investment choice (Ic) and all the

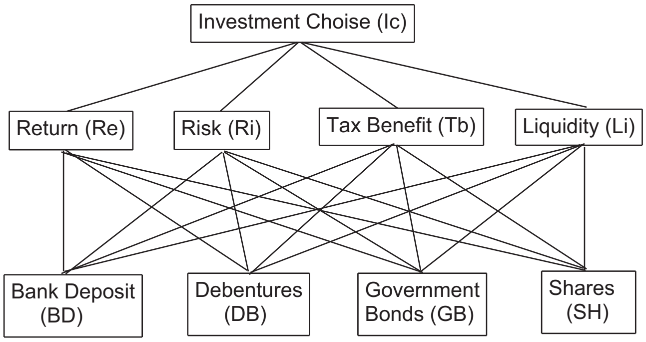

Fig. 2. A hierarchy structure of criteria.

alternatives with respect to Re, Ri, Tb and Li are shown as follows, respectively:

$$A_{(Ic)} = \begin{bmatrix} \text{Ic} & \text{Re} & \text{Ri} & \text{Tb} & \text{Li} \\ \text{Re} & 1 & [3,4] & [5,6] & [6,7] \\ \text{Ri} & [1/4,1/3] & 1 & [4,5] & [5,6] \\ \text{Tb} & [1/6,1/5] & [1/5,1/4] & 1 & [3,4] \\ \text{Li} & [1/7,1/6] & [1/6,1/5] & [1/4,1/3] & 1 \end{bmatrix},$$

$$A_{(Re)} = \begin{bmatrix} \text{Re} & \text{BD} & \text{DB} & \text{GB} & \text{SH} \\ \text{BD} & 1 & [1/4,1/3] & [3,4] & [1/6,1/5] \\ \text{DB} & [3,4] & 1 & [6,7] & [1/5,1/4] \\ \text{GB} & [1/4,1/3] & [1/7,1/6] & 1 & [1/7,1/6] \\ \text{SH} & [5,6] & [4,5] & [6,7] & 1 \end{bmatrix},$$

$$A_{(Ri)} = \begin{bmatrix} \text{Ri} & \text{BD} & \text{DB} & \text{GB} & \text{SH} \\ \text{BD} & 1 & [3,4] & [4,5] & [6,7] \\ \text{DB} & [1/4,1/3] & 1 & [3,4] & [5,6] \\ \text{GB} & [1/5,1/4] & [1/4,1/3] & 1 & [4,5] \\ \text{SH} & [1/7,1/6] & [1/6,1/5] & [1/5,1/4] & 1 \end{bmatrix},$$

$$A_{(Tb)} = \begin{bmatrix} \text{Tb} & \text{BD} & \text{DB} & \text{GB} & \text{SH} \\ \text{BD} & 1 & 1 & [1/6,1/5] & [1/4,1/3] \\ \text{DB} & 1 & 1 & [1/6,1/5] & [1/4,1/3] \\ \text{GB} & [5,6] & [5,6] & 1 & [4,5] \\ \text{SH} & [3,4] & [3,4] & [1/5,1/4] & 1 \end{bmatrix},$$

$$A_{(Li)} = \begin{bmatrix} \text{Li} & \text{BD} & \text{DB} & \text{GB} & \text{SH} \\ \text{BD} & 1 & [3,4] & 6 & [6,7] \\ \text{DB} & [1/4,1/3] & 1 & [3,4] & [3,4] \\ \text{GB} & 1/6 & [1/4,1/3] & 1 & [3,4] \\ \text{SH} & [1/7,1/6] & [1/4,1/3] & [1/4,1/3] & 1 \end{bmatrix}.$$

Checking the above five matrices by use of Definition 2, one can see that $A_{(Tb)}$ is acceptably consistent and the others are unacceptably consistent. However, in Wang et al. [30,31], the five matrices are all inconsistent. Moreover, the

Table 2
Local and global interval weights in Example 3.

Portfolio	Re [3.0801,3.6004]	Ri [1.6069,1.6544]	Tb [0.6043,0.6223]	Li [0.2778,0.3247]	Global weights
BD	[0.5930,0.7165]	[2.9130,3.4390]	[0.4518,0.5081]	[3.2237,3.5927]	[0.7171, 2.7807]
DB	[1.4797,1.5112]	[1.4953,1.5639]	[0.4518,0.5081]	[1.3161,1.4146]	[4.2553, 6.7777]
GB	[0.2786,0.2986]	[0.7073,0.7598]	[3.3437,3.4641]	[0.6389,0.6403]	[0.0104, 0.0291]
SH	[3.3154,3.8162]	[0.2629,0.3021]	[1.1583,1.4142]	[0.3073,0.3689]	[3.2778, 17.0645]

matrices with unacceptable consistency can be adjusted to those with acceptable consistency. For example, we choose the consistency improving method given by Xu and Wei [36] and the change parameters $\mu_{B_{(Ic)}} = 0.8$, $\mu_{B_{(Re)}} = 0.9$, $\mu_{C_{(Re)}} = 0.9$, $\mu_{B_{(Ri)}} = 0.8$ and $\mu_{B_{(Li)}} = 0.9$. Hereafter $B_{(\cdot)}$ and $C_{(\cdot)}$ stand for the matrices constructed from $A_{(\cdot)}$ by (6). It should be pointed out that $C_{(Ic)}$, $C_{(Ri)}$ and $C_{(Li)}$ are of acceptable consistency and do not need to be modified. Furthermore, one can obtain the local and global interval weights given in Table 2, where the global interval weights are computed by the NLP models shown in Wang et al. [30]. For simplicity, the detail procedures are neglected here. As expected, it is found from Table 2 that the ranges of local and global weights are less than those obtained by Wang et al. [30], which is due to the different methods used to determine the local interval weights.

Furthermore, the matrix of possibility degree can be determined by applying (15), namely

$$P = \begin{bmatrix} 0.5000 & 0.0000 & 1.000 & 0.0000 \\ 1.0000 & 0.5000 & 1.0000 & 0.1624 \\ 0.0000 & 0.0000 & 0.5000 & 0.0000 \\ 1.0000 & 0.8376 & 1.0000 & 0.5000 \end{bmatrix}.$$

One can further obtain SH $\succ^{83.76\%}$ DB $\succ^{100\%}$ BD $\succ^{100\%}$ GB. The same ranking can be found in Wang et al. [30] and Islam et al. [13]. It is also noted that the values of preference degree are different from those shown by SH $\succ^{75.21\%}$ DB $\succ^{84.27\%}$ BD $\succ^{100\%}$ GB [30].

The above observations presented in Examples 1–3 show that enough information of initial fuzzy matrices given by the DM is kept by introducing the definition of an acceptably consistent interval reciprocal comparison matrix and deriving interval weights. By the way, the obtained results also show that the interval weights based on the developed methods are corresponding to the initial interval reciprocal comparison matrix and convincing.

6. Conclusions

In the process of decision making, a DM may give his/her judgments by an interval reciprocal comparison matrix, which is suitable to simulate the complexity of real-world decision problems and the subjective nature of human judgments. Moreover it is important to derive reliable weights from an interval reciprocal comparison matrix, especially from an inconsistent one. For the known methods based on the feasible region, it is always to enlarge the intervals through some introducing parameters to get a non-empty extended feasible region. However, these extension methods distort the original preference information to some extent. So it plays a very important role to keep initial information as much as possible from an initial inconsistent interval reciprocal comparison matrix under the condition of obtaining reliable weights. In the paper, an acceptably consistent interval reciprocal comparison matrix is defined, which can be reduced to an acceptably consistent crisp reciprocal comparison matrix when the intervals become exact numbers. Furthermore, interval reciprocal comparison matrices with unacceptable consistency can be easily modified to those with acceptable consistency. Making use of a convex combination method, a family of crisp reciprocal comparison matrices with acceptable consistency can be obtained from an acceptably consistent interval reciprocal comparison matrix, whose weights are found to be a form of convex combination. An interval weight generation method is also used to obtain interval weights from acceptably consistent interval reciprocal comparison matrices. A novel, simple yet effective formula of possibility degree for interval numbers is proposed to rank the alternatives. The practicality and effectiveness of the developed approaches are verified by three illustrative examples, and comparisons are made with other existing procedures. The obtained results show that the proposed methods are simple, effective and applicable to both consistent interval reciprocal comparison matrices and inconsistent ones generated from concrete decision-making problems, without solving any mathematical programming and complicated algebraic operation. Based on the analysis methods involving of acceptable consistency and interval weights, especially one can get enough information and convincing rank of criteria from an initial inconsistent interval reciprocal comparison matrix.

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