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On Method for Uncertain Multiple Attribute Decision Making Problems with Uncertain Multiplicative Preference Information on Alternatives[1]

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Abstract. The uncertain multiple attribute decision making (UMADM) problems are investigated, in which the information about attribute weights is known partly and the attribute values take the form of interval numbers, and the decision maker (DM) has uncertain multiplicative preference information on alternatives. We make the decision information uniform by using a transformation formula, and then establish an objective-programming model. The attribute weights can be determined by solving the developed model. The concept of interval positive ideal point of alternatives (IPIPA) is introduced, and an approach based on IPIPA and projection to ranking alternatives is proposed. The method can avoid comparing and ranking interval numbers, and can reflect both the objective information and the DM's subjective preferences.

Keywords: uncertain multiple attribute decision making (UMADM), objective programming model, weight

1. Introduction

Multiple attribute decision making (MADM) is a prominent area of research in normative decision theory. The topic has been widely studied (Saaty 1980, Hwang and Yoon 1981, Chen and Hwang 1992, Fodor and Roubens 1994, Yager and Kacprzyk 1997, Xu 2004, etc.). It consists of finding the most desirable alternative(s) from a given alternative set. Many MADM processes, in the real world, take place in an environment in which the information about attribute weights and attribute values are not precisely known, but value ranges can be obtained. At present, a lot of methods have been proposed to deal with these uncertain MADM problems. Bryson and Mobolurin (1996) proposed a systematic action learning evaluation procedure, which provides a structure for prioritizing and synthesizing interval preference scores in an ambiguous decision domain. Later, Fan and Zhang (1999) gave a revised procedure for MADM under uncertainty. Xu (2000) proposed an approach to normalizing the decision matrix with interval estimates, and developed a method for the priority of interval synthetic estimates. Lee, et al (2001) provided some methods for establishing dominance and potential optimality for alternatives. Da and Xu (2002) established a single-objective optimization model, and developed a possibility degree based method for uncertain MADM. Up to now, however, there is no approach to dealing with the situations where not only the attribute weights and attribute values take the form of interval numbers, but also the decision maker (DM) has uncertain subjective preferences for alternatives (we call these problems uncertain multiple attribute decision making (UMADM)-UPA, for short). Therefore, it is necessary to pay attention to this issue. In this paper, we shall present an approach to UMADM-UPA. To do so, the remainder of this paper is set out as follows. Section 2 gives a simple representation of UMADM-UPA. Section 3 establishes an objective-programming model, and the attribute weights can be determined by solving the model. Section 4 defines the concept of interval positive ideal point of alternatives (IPIPA), and then presents an approach based on IPIPA and projection to ranking alternatives. An illustrative example is given in Section 5, and finally, some concluding remarks are included.

2. Representation of UMADM-UPA

For the sake of convenience, we let $M = \{1, 2, \ldots, m\}$ and $N = \{1, 2, \ldots, n\}$. In the following we shall introduce some operational laws of interval numbers (Xu and Zhai (1992)).

Let $a = [a^L, a^U]$ and $b = [b^L, b^U]$ be two interval numbers, where $a^U \geq a^L > 0$, $b^U \geq b^L > 0$. Then

(1) $a + b = [a^L, a^U] + [b^L, b^U] = [a^L + b^L, a^U + b^U]$.
(2) $ab = [a^L, a^U] \cdot [b^L, b^U] = [a^L b^L, a^U b^U]$.
(3) $a/b = [a^L, a^U]/[b^L, b^U] = [a^L/b^U, a^U/b^L]$.
(4) $\lambda a = \lambda[a^L, a^U] = [\lambda a^L, \lambda a^U]$, where $\lambda > 0$.
(5) $a = b$ iff $a^L = b^L$ and $a^U = b^U$.

For an UMADM-UPA, let $X = \{x_1, x_2, \ldots, x_n\}$ be a discrete set of alternatives, $U = \{u_1, u_2, \ldots, u_m\}$ be a set of attributes, and $w = (w_1, w_2, \ldots, w_m)^T$ be the vector of attribute weights, where $0 \leq w_i^L \leq w_i \leq w_i^U,\ i \in M,\ \sum_{i=1}^m w_i^L \leq 1,\ \sum_{i=1}^m w_i^U \geq 1$, $\sum_{i=1}^m w_i = 1$, w_i^L and w_i^U are the lower and upper bounds of w_i, respectively. Let $A = (a_{ij})_{m \times n}$ be the decision matrix, where $a_{ij} = [a_{ij}^L, a_{ij}^U]$ is an attribute value, which takes the form of interval number, for the alternative x_j with respect to the attribute u_i. The DM also has uncertain multiplicative preference information on alternatives, i.e., the DM's preferences on X are described by an uncertain multiplicative preference relation (Saaty and Vargas (1987), Zahir (1991), Xu and Da (2004)) $P = (p_{ij})_{n \times n}$, where $p_{ij} = [p_{ij}^L, p_{ij}^U]$, $p_{ij}^L p_{ji}^U = p_{ij}^U p_{ji}^L = 1$, $i, j \in N$, and p_{ij} indicates a ratio of preference intensity for the alternative x_i to that of x_j, it is interpreted as x_i is p_{ij} times as good as x_j.

In general, there are benefit attributes and cost attributes in MADM problems, and the 'dimension' of different attributes may be different. In order to measure all attributes in dimensionless units, we can normalize each attribute value a_{ij} in the matrix $A = (a_{ij})_{m \times n}$ into a corresponding element in the matrix $R = (r_{ij})_{m \times n}$ using the following formulas:

$$r_{ij} = a_{ij} / \sum_{j=1}^{n} a_{ij}, \quad \text{for benefit attribute } u_i, \quad i \in M, \quad j \in N \tag{1}$$

$$r_{ij} = (1/a_{ij}) / \sum_{j=1}^{n} (1/a_{ij}), \quad \text{for cost attribute } u_i, \quad i \in M, \quad j \in N \tag{2}$$

By the operations of interval numbers, we rewrite (1) and (2) as (3) and (4), respectively

$$\begin{cases} r_{ij}^L &= a_{ij}^L / \sum_{j=1}^{n} a_{ij}^U \\ r_{ij}^U &= a_{ij}^U / \sum_{j=1}^{n} a_{ij}^L \end{cases}, \quad \text{for benefit attribute } u_i, \quad i \in M, \quad j \in N \tag{3}$$

$$\begin{cases} r_{ij}^L &= (1/a_{ij}^U) / \sum_{j=1}^{n} (1/a_{ij}^L) \\ r_{ij}^U &= (1/a_{ij}^L) / \sum_{j=1}^{n} (1/a_{ij}^U) \end{cases}, \quad \text{for cost attribute } u_i, \quad i \in M, \quad j \in N \tag{4}$$

where $r_{ij} = [r_{ij}^L, r_{ij}^U]$, and the normalized attribute value r_{ij} can be regarded as the objective preference value, which is given by the DM, for the alternative x_j with respect to the attribute u_i.

3. An Objective Programming Model

Using the simple additive weighting method (Harsanyi (1955), Hwang and Yoon (1981)), the uncertain overall value of alternative x_i can be expressed as

$$s_j(w) = \sum_{i=1}^{m} r_{ij} w_i, \quad j \in N \tag{5}$$

In order to make the information uniform, we can transform the uncertain overall values of alternatives into uncertain multiplicative preference relation $\bar{P} = (\bar{P}_{ij})_{n \times n}$. That is, by using (5), we can define $\bar{P}_{ij}$ as

$$\bar{P}_{ij} = \frac{s_i(w)}{s_j(w)} = \frac{\sum_{k=1}^{m} r_{ki} w_k}{\sum_{k=1}^{m} r_{kj} w_k}, \quad i,j \in N \tag{6}$$

where the significance of $\bar{P}_{ij}$ in $\bar{P} = (\bar{P}_{ij})_{n \times n}$ is similar to that of p_{ij} in $P = (p_{ij})_{n \times n}$. If the uncertain multiplicative preference relations P and $\bar{P}$ are consistent, we have $P = \bar{P}$, i.e., $p_{ij} = \bar{P}_{ij}$, $i,j \in N$, then

$$p_{ij} = \frac{s_i(w)}{s_j(w)} = \frac{\sum_{k=1}^{m} r_{ki} w_k}{\sum_{k=1}^{m} r_{kj} w_k}, \quad i,j \in N \tag{7}$$

that is,

$$\sum_{k=1}^{m} p_{ij} r_{kj} w_k = \sum_{k=1}^{m} r_{ki} w_k, \quad i,j \in N \tag{8}$$

From (8), we have

$$\sum_{k=1}^{m}\left(p_{ij}^{L}r_{kj}^{L}-r_{ki}^{L}\right)w_k=0,\quad \sum_{k=1}^{m}\left(p_{ij}^{U}r_{kj}^{U}-r_{ki}^{U}\right)w_k=0,\quad i,j\in N \tag{9}$$

In the real life, however, there always exist some differences between the subjective preference values and the corresponding objective preference values for alternatives. As a result, (8) or (9) in general does not hold. Here, we introduce the deviation elements $e_{ij}^{L}(w)$ and $e_{ij}^{U}(w)$, i.e., we let

$$e_{ij}^{L}=\left|\sum_{k=1}^{m}\left(p_{ij}^{L}r_{kj}^{L}-r_{ki}^{L}\right)w_k\right|,\quad e_{ij}^{U}=\left|\sum_{k=1}^{m}\left(p_{ij}^{U}r_{kj}^{U}-r_{ki}^{U}\right)w_k\right|,\quad i,j\in N \tag{10}$$

To determine the vector of attribute weights $w=(w_1,w_2,\ldots,w_m)^T$, we can minimize the sum of all deviation degrees between the subjective preference values and the corresponding objective preference values for alternatives. Therefore, we establish the following optimization model:

$$\begin{aligned}
(\mathbf{M1})\ \min e_{ij}^{L} &= \left|\sum_{k=1}^{m}(p_{ij}^{L}r_{kj}^{L}-r_{ki}^{L})w_k\right|,\quad i,j\in N\\
\min e_{ij}^{U} &= \left|\sum_{k=1}^{m}(p_{ij}^{U}r_{kj}^{U}-r_{ki}^{U})w_k\right|,\quad i,j\in N\\
s.t.\quad & 0\le w_i^{L}\le w_i\le w_i^{U},\quad i\in M,\quad \sum_{i=1}^{m}w_i^{L}\le 1,\quad \sum_{i=1}^{m}w_i^{U}\ge 1,\quad \sum_{i=1}^{m}w_i=1
\end{aligned}$$

Solution to the above minimization problem is found by solving the following objective programming model:

$$\begin{aligned}
(\mathbf{M2})\ \min J &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(s_{lij}d_{lij}^{+}+t_{lij}d_{lij}^{-}\right)+\left(s_{uij}d_{uij}^{+}+t_{uij}d_{uij}^{-}\right)\right]\\
\text{s.t.} & \sum_{k=1}^{m}\left(p_{ij}^{L}r_{kj}^{L}-r_{ki}^{L}\right)w_k-d_{lij}^{+}+d_{lij}^{-}=0,\quad i,j\in N\\
& \sum_{k=1}^{m}\left(p_{ij}^{U}r_{kj}^{U}-r_{ki}^{U}\right)w_k-d_{uij}^{+}+d_{uij}^{-}=0,\quad i,j\in N\\
& d_{lij}^{+}\ge 0, d_{lij}^{-}\ge 0, d_{lij}^{+}d_{lij}^{-}=0, d_{uij}^{+}\ge 0, d_{uij}^{-}\ge 0, d_{uij}^{+}d_{uij}^{-}=0,\quad i,j\in N\\
& 0\le w_i^{L}\le w_i\le w_i^{U}, i\in M,\ \sum_{i=1}^{m}w_i^{L}\le 1, \sum_{i=1}^{m}w_i^{U}\ge 1, \sum_{i=1}^{m}w_i=1
\end{aligned}$$

where, d^+_{lij} and d^-_{lij} are the upper and lower deviation variables of $\sum_{k=1}^m (p^L_{ij} r^L_{kj} - r^L_{ki}) w_k$, respectively, while d^+_{uij} and d^-_{uij} are the upper and lower deviation variables of $\sum_{k=1}^m (p^U_{ij} r^U_{kj} - r^U_{ki}) w_k$, respectively. s_{lij} and s_{uij} are the coefficients of d^+_{lij} and d^+_{uij}, respectively, while t_{lij} and t_{uij} are the coefficients of d^-_{lij} and d^-_{uij}, respectively.

By solving the model (M2), we can obtain the vector of attribute weights $w = (w_1, w_2, \ldots, w_m)^T$.

4. An Approach to Ranking Alternatives

After finding the attribute weights by solving the model (M2), we need to rank the alternatives, and then to find the most desirable one(s). To do so, in the following we first construct an weighted normalized matrix $Z = (z_{ij})_{m \times n}$, where

$$z_{ij} = \left[z^L_{ij}, z^U_{ij}\right], \quad z_{ij} = w_i r_{ij}, \quad i \in M, \quad j \in N \tag{11}$$

and then define the following concepts:

Definition 4.1 *$z^+ = (z^+_1, z^+_2, \ldots, z^+_m)^T$ is called the interval positive ideal point of alternatives(INIPA), if*

$$z^+_i = \left[z^{+^L}_i, z^{+^U}_i\right] = \left[\max_j z^L_{ij}, \max_j z^U_{ij}\right], \quad i \in M \tag{12}$$

Definition 4.2 *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)^T$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_m)^T$, then*

$$\Pr j_\beta(\alpha) = |\alpha| \cos\langle \alpha, \beta \rangle = \sqrt{\sum_{i=1}^m \alpha_i^2} \frac{\sum_{i=1}^m \alpha_i \beta_i}{\sqrt{\sum_{i=1}^m \alpha_i^2} \sqrt{\sum_{i=1}^m \beta_i^2}} = \frac{\sum_{i=1}^m \alpha_i \beta_i}{\sqrt{\sum_{i=1}^m \beta_i^2}} \tag{13}$$

is called the projection of the vector α on the vector β, where $|\alpha| = \sqrt{\sum_{i=1}^m \alpha_i^2}$ is the module of α.

In general, the greater the value of $\Pr j_\beta(\alpha)$, the closer the vector α to the vector β. Let

$$\Pr j_{z^+}(z_j) = \frac{\sum_{i=1}^m z^L_{ij} z^{+^L}_i}{\sqrt{\sum_{i=1}^m (z^{+^L}_i)^2}} + \frac{\sum_{i=1}^m z^U_{ij} z^{+^U}_i}{\sqrt{\sum_{i=1}^m (z^{+^U}_i)^2}} \tag{14}$$

where $z_j = (z_{1j}, z_{2j}, \ldots, z_{mj})^T$, then $\Pr j_{z^+}(z_j)$ is called the projection of z_j on the INIPA z^+.

Obviously, the greater the value of $\Pr j_{z^+}(z_j)$, the closer z_j to the INIPA z^+, and thus the better the alternative x_j.

Therefore, by the values of $\Pr j_{z^+}(z_j)(j \in N)$, we can rank all the alternatives and find the most desirable one(s).

5. Illustrative Example

In this section, an UMADM-UPA of determining what kind of air-conditioning system should be installed in the library (adapted from Yoon (1989)) is used to illustrate the proposed approach.

A city is planning to build a municipal library. One of the problems facing the city development commissioner is to determine what kind of air-conditioning system should be installed in the library. The contractor offered five feasible plans, which might be adapted to the physical structure of the library. Alternatives $x_j(j = 1, 2, 3, 4, 5)$ are to be evaluated under three major impacts: economic, functional and operational. Two monetary attributes and six non-monetary attributes (that is, u_1: owning cost ($/ft^2), u_2: operating cost ($/ft^2), u_3: performance (*), u_4: comfort (noise level, Db), u_5: maintainability (*), u_6: reliability (%), u_7: flexibility (*), u_8: safety, where * unit from 10-point scale, from 1 (worst) to 10 (best), three attributes u_1, u_2 and u_4 are cost attributes, and the other five attributes are benefit attributes) emerged from three impacts in Table 1. The value ranges of attribute weights are as follows:

$$0.0419 \le w_1 \le 0.0491, 0.0840 \le w_2 \le 0.0982, 0.1211 \le w_3 \le 0.1373, 0.1211 \le w_4 \le 0.1373,$$
$$0.1680 \le w_5 \le 0.1818, 0.2138 \le w_6 \le 0.2294, 0.0395 \le w_7 \le 0.0457, 0.1588 \le w_8 \le 0.1706$$

By (3) and (4), we get the normalized decision matrix $R = (r_{ij})_{8\times5}$ (see Table 2).

Suppose that the robot user gives his/her uncertain multiplicative preference relation P as follows:

$$P = \begin{bmatrix} [1,1] & [3,5] & [1/9,1/7] & [5,7] & [1/3,1] \\ [1/5,1/3] & [1,1] & [3,5] & [1/5,1/3] & [1,3] \\ [7,9] & [1/5,1/3] & [1,1] & [5,7] & [1/9,1/7] \\ [1/7,1/5] & [3,5] & [1/7,1/5] & [1,1] & [5,7] \\ [1,3] & [1/3,1] & [7,9] & [1/7,1/5] & [1,1] \end{bmatrix}$$

By solving the model (M2) (let $s_{lij} = s_{uij} = t_{lij} = t_{uij} = 1$, $i, j \in N$), we get the optimal solution

Table 1. Decision matrix $A = (a_{ij})_{8\times5}$.

u_i	x_1	x_2	x_3	x_4	x_5
u_1	[3.7, 4.7]	[1.5, 2.5]	[3, 4]	[3.5, 4.5]	[2.5, 3.5]
u_2	[5.9, 6.9]	[4.7, 5.7]	[4.2, 5.2]	[4.5, 5.5]	[5, 6]
u_3	[8, 10]	[4, 6]	[4, 6]	[7, 9]	[6, 8]
u_4	[30, 40]	[65, 75]	[60, 70]	[35, 45]	[50, 60]
u_5	[3, 5]	[3, 5]	[7, 9]	[8, 10]	[5, 7]
u_6	[90, 100]	[70, 80]	[80, 90]	[85, 95]	[85, 95]
u_7	[3, 5]	[7, 9]	[7, 9]	[6, 8]	[4, 6]
u_8	[6, 8]	[4, 6]	[5, 7]	[7, 9]	[8, 10]

Table 2. The normalized decision matrix $R = (r_{ij})_{8\times5}$.

u_i	x_1	x_2	x_3	x_4	x_5
u_1	[0.1088, 0.1972]	[0.2045, 0.4864]	[0.1278, 0.2432]	[0.1136, 0.2084]	[0.1461, 0.2918]
u_2	[0.1390, 0.1968]	[0.1683, 0.2471]	[0.1845, 0.2765]	[0.1744, 0.2581]	[0.1599, 0.2322]
u_3	[0.2051, 0.3448]	[0.1026, 0.2069]	[0.1026, 0.2069]	[0.1795, 0.3103]	[0.1538, 0.2759]
u_4	[0.2194, 0.3643]	[0.1170, 0.1681]	[0.1254, 0.1821]	[0.1950, 0.3122]	[0.1463, 0.2186]
u_5	[0.0833, 0.1923]	[0.0833, 0.1923]	[0.1944, 0.3462]	[0.2222, 0.3846]	[0.1389, 0.2692]
u_6	[0.1957, 0.2439]	[0.1522, 0.1951]	[0.1739, 0.2195]	[0.1848, 0.2317]	[0.1848, 0.2317]
u_7	[0.0811, 0.1852]	[0.1892, 0.3333]	[0.1892, 0.3333]	[0.1622, 0.2963]	[0.1081, 0.2222]
u_8	[0.1500, 0.2667]	[0.1000, 0.2000]	[0.1250, 0.2333]	[0.1750, 0.3000]	[0.2000, 0.3333]

$$w = (0.0491, 0.0950, 0.1211, 0.1211, 0.1680, 0.2294, 0.0457, 0.1706)^T$$

Then by (11), we construct the weighted normalized matrix $Z = (z_{ij})_{8\times5}$ (see Table 3).

From Table 3, and by (12), we get the IPIPA z^+:

$$z^+=([0.01004,0.02388],[0.01753,0.02627],[0.02484,0.04176],[0.02657,0.04412],\\ [0.03733,0.06461],[0.04489,0.05595],[0.00865,0.01523],[0.03412,0.05686])^T$$

Then by (14), we get the projection $\mathrm{Pr}\,j_{z^+}(z_j)$ of z_j (here, $z_j = (z_{1j}, z_{2j}, \ldots, z_{8j})^T$) on the IPIPA z^+:

$$\mathrm{Pr}\,j_{z^+}(z_1) = 0.161717, \quad \mathrm{Pr}\,j_{z^+}(z_2) = 0.126657, \quad \mathrm{Pr}\,j_{z^+}(z_3) = 0.150290$$
$$\mathrm{Pr}\,j_{z^+}(z_4) = 0.181596, \quad \mathrm{Pr}\,j_{z^+}(z_5) = 0.159871$$

and thus

$$\mathrm{Pr}\,j_{z^+}(z_4) \succ \mathrm{Pr}\,j_{z^+}(z_1) \succ \mathrm{Pr}\,j_{z^+}(z_5) \succ \mathrm{Pr}\,j_{z^+}(z_3) \succ \mathrm{Pr}\,j_{z^+}(z_2)$$

Table 3. The weighted normalized matrix $Z = (z_{ij})_{8\times 5}$.

u_i	x_1	x_2	x_3	x_4	x_5
u_1	[0.00534,0.00968]	[0.01004,0.02388]	[0.00627,0.01194]	[0.00558,0.01023]	[0.00717,0.01433]
u_2	[0.01321,0.01870]	[0.01599,0.02347]	[0.01753,0.02627]	[0.01657,0.02452]	[0.01519,0.02206]
u_3	[0.02484,0.04176]	[0.01242,0.02506]	[0.01242,0.02506]	[0.02174,0.03758]	[0.01863,0.03341]
u_4	[0.02657,0.04412]	[0.01417,0.02036]	[0.01519,0.02205]	[0.02361,0.03781]	[0.01772,0.02647]
u_5	[0.01399,0.03231]	[0.01399,0.03231]	[0.03266,0.05816]	[0.03733,0.06461]	[0.02334,0.04523]
u_6	[0.04489,0.05595]	[0.03491,0.04476]	[0.03989,0.05035]	[0.04239,0.05315]	[0.04239,0.05315]
u_7	[0.00371,0.00846]	[0.00865,0.01523]	[0.00865,0.01523]	[0.00741,0.01354]	[0.00494,0.01015]
u_8	[0.02560,0.04550]	[0.01706,0.03412]	[0.02133,0.03980]	[0.02986,0.05118]	[0.03412,0.05686]

Hence, the ranking of the five alternatives is

$$x_4 \succ x_1 \succ x_5 \succ x_3 \succ x_2$$

Therefore, the most desirable alternative is x_4.

6. Concluding Remarks

In this paper we have studied the UMADM problems, in which the information about attribute weights is known partly and the attribute values take the form of interval numbers, and the DM has uncertain multiplicative preference information on alternatives. We have developed an objective-programming model to determine the attribute weights, and then developed a practical approach to ranking alternatives. The method can avoid comparing and ranking interval numbers, and can reflect both the objective information and the DM's subjective considerations. It is straightforward and can be performed on computer easily. The numerical results have demonstrated the feasibility and practicality of the developed method.

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References

Bryson, N. and A. Mobolurin. (1996). "An Action Learning Evaluation Procedure for Multiple Criteria Decision Making Problems," *European Journal of Operational Research* 96, 379–386.

Chen, S. J. and C. L. Hwang. (1992). *Fuzzy Multiple Attribute Decision Making*. Berlin: Springer.

Da Q. L. and Z. S. Xu. (2002). "Single Objective Optimization Model in Uncertain Multi-attribute Decision-making". *Journal of Systems Engineering* 17(1), 50–55.

Fan, Z. P. and Q. Zhang. (1999). "The Revision for the Uncertain Multiple Attribute Decision Making Models," *Systems Engineering—Theory & Practice* 21(10), 42–47.

Fodor, J. and M. Roubens. (1994). *Fuzzy Preference Modelling and Multicriteria Decision Support*. Dordrecht: Kluwer Academic Publishers.

Harsanyi, J. C. (1955). "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility," *Journal of Political Economy* 63, 309–321.

Hwang, C. L. and K. Yoon, (1981). *Multiple Attribute Decision Making*. Berlin: Springer-Verlag.

Lee, K. S., K. S. Park, Y. S. Eum, and K. Park. (2001). "Extended Methods for Identifying Dominance and Potential Optimality in Multi-criteria Analysis With Imprecise Information," *European Journal of Operational Research* 134, 557–563.

Saaty, T. L. (1980). *The Analytic Hierarchy Process*. New York: McGraw-Hill.

Saaty, T. L. and L. G. Vargas. (1987). "Uncertainty and Rank Order in the Analytic Hierarchy Process," *European Journal of Operational Research* 32, 107–117.

Xu, R. N. and X. Y. Zhai. (1992). "Extensions of the Analytic Hierarchy Process in Fuzzy Environment," *Fuzzy Sets and Systems* 52, 251–257.

Xu, Z. S. (2000). "On Method for Priorities of Fuzzy Synthetic Evaluations." In: Gu, J. F, (ed.). *Systems Engineering, Systems Science and Complexity Research*. United kingdom: Research Information Ltd, 507–511.

Xu, Z. S. and Q. L. Da, (2004). "An Uncertain Ordered Weighted Geometric Averaging (UOWGA) Operator and its Application," *Information: An International Journal* 7, 75–182.

Xu, Z. S. (2004). *Uncertain Multiple Attribute Decision Making: Methods and Applications*. Beijing: Tsinghua University Press.

Yager, R. R. and J. Kacprzyk. (1997). *The Ordered Weighted Averaging Operators: Theory and Applications*. Norwell, MA: Kluwer.

Yoon, K. (1989). "The Propagation of Errors in Multiple-attribute Decision Analysis: a Practical Approach," *Journal of Operational Research Society* 40, 681–686.

Zahir, M. S. (1991). "Incorporating the Uncertainty of Decision Judgements in the Analytic Hierarchy Process," *European Journal of Operational Research* 53, 206–216.