ON THE SYMMETRY APPROACH TO THE COMMITTEE DECISION PROBLEM*

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1. Introduction

Utility analysis lost much of its apparently simple and concrete character when it was realized that most of its results can be derived if one restricts oneself to the ordinal properties of the utility function. But it regained in concreteness when Von Neumann and Morgenstern [5] found that a cardinal utility concept can be used, under fairly weak conditions, to handle the problem of choice among uncertain alternatives. Or to put it in more specific terms, if an individual's preferences satisfy a certain set of axioms, there exists a utility function such that he behaves as if he maximizes expected utility, and this function is completely unique except for the choice of an arbitrary zero and an arbitrary unit of measurement.

At about the same time, there has been a similar evolution of the social utility concept. Whereas social utility was originally regarded as a concept which is just as simple and concrete as individual utility, it was realized by later authors that things are not so simple. The most important contributions were made under Arrow's leadership [1], who postulated a number of apparently innocent conditions which a social utility function should satisfy and found that, apart from exceptional circumstances, no such function exists which satisfies all these conditions at the same time. But fortunately, there are later developments which are not as negative. It was shown by Harsanyi [3] that if the individual as well as the social preference systems satisfy the axioms which lead to the maximization of expected utility under conditions of uncertainty, and if the social utility function is required to declare two alternatives indifferent as soon as this is done by every individual, the social utility function should necessarily be a linear combination of the individual utility functions. Evidently, the conditions imposed are rather weak,[2] which should inspire us to concentrate in particular on linear combinations of individual preference functions as promising candidates for a "good" social preference function. At the same time, this result is very convenient from a mathematical point of view, because it means that if the individual objective functions are all linear (as in linear programming) or quadratic (as in quadratic programming), the same mathematical form applies to the corresponding social objective function.

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[2] For a different view, see J. F. Rothenberg [6, Chap. 10], who argues that the Von Neumann-Morgenstern axioms are not as plausible for committee preferences as they are for individual preferences.

Given the choice of linear combinations, the problem is to find the appropriate weights of the combination. The present paper is concerned with this problem, and it is closely related to a recent article written by the late P. J. M. van den Bogaard in association with J. Versluis [2]. In fact, the proposals that will be made in the present article do not go much farther than his, but the main point to be made is different because we wish to analyze the arguments leading to these proposals in somewhat more detail. It is perhaps good to stress right at the beginning that these proposals have no pretention of being satisfactory under all circumstances but only under the conditions that will be set forth. Nevertheless, the present writer believes that such specific proposals (and others that, as may be hoped, will be available in the future) will turn out to have merits comparable with earlier proposals made by Laspeyres, Paasche, Edgeworth, Fisher and others in the related field of price index number theory. (The two fields are related, since both deal with aggregation.) Committee decisions have had to be made since times immemorial, and since the number of revolutions to which they led is not excessively large, we might infer that reasonable men are usually able to arrive at reasonable decisions; and perhaps also that analysis should be able to contribute to more reasonable decisions, at least to more "objective" decisions in the sense that their compromise character is more clearly recognized.

The order of discussion is as follows. In Section 2 the basic ideas (which are already "classical") are briefly restated and the problem is formulated. In Section 3 some proposals are made for the weights of the linear combination of individual preference functions. Section 4 illustrates the committee decision problem by a very simple example of measures of central tendency, the solution of which suggests a method for solving the general problem. Section 5 is the heart of the article; it introduces the symmetry condition. The last section deals with the case when symmetry cannot be realized.

2. Statement of the Problem

We shall consider a committee whose task is to make a decision which is based on the preferences of its N members, where N is any integer >1. We start by considering the preferences of any of these individuals and suppose that the Von Neumann-Morgenstern axioms as formulated by Marschak [4] are satisfied. These can be briefly restated as follows. Consider any three uncertain prospects X_1, X_2, X_3 (random variables or vectors of such variables). Then, firstly, the individual should be able to rank prospects of this kind according to decreasing preference, which means that he should be able to state whether he prefers X_i to X_j or X_j to X_i or that he is indifferent between them; also, that if he prefers X_1 to X_2 and X_2 to X_3, he prefers X_1 to X_3 as well (and likewise for indifference). Secondly, suppose that the individual prefers indeed X_1 to X_2 and X_2 to X_3; and consider $X_4 = pX_1 + qX_3$, which is another uncertain prospect (a "mixture"), viz., that of having the high-ranking alternative X_1 with chance p and the low-ranking alternative X_3 with chance $q = 1 - p$, where $0 \leqq p \leqq 1$. Then it is assumed that some p in $(0, 1)$ exists such that the individual is indifferent between X_4 and the middle-ranking alternative X_2 ("continuity" of the preference relation). Thirdly, suppose now that the individual is indifferent between

X_1 and X_2 and consider any third prospect X_3; it is then assumed that he is indifferent between $pX_1 + qX_3$ and $pX_2 + qX_3$ for whatever p in $(0, 1)$. Fourthly, it is assumed that there are at least four prospects which are pairwise non-indifferent.

Under these assumptions the individual will behave as if he maximizes expected utility, and this utility function is unique up to a linear transformation. Following Harsanyi, we shall suppose that these conditions apply to all N members of the committee and also to the preference system of the committee as a whole which we are going to derive. (This can be regarded as the fifth assumption.) Harsanyi proceeded to make the following assumption (the sixth in the series): Whenever each of the N individuals is indifferent between two prospects X_1 and X_2, the committee preference system should also be indifferent between these two. He showed that the committee utility function should then be a linear function of the N individual functions. The weights of this linear function should be all positive when the following (seventh) condition is imposed: Whenever one of the N individuals prefers X_1 to X_2 and none of the others prefers X_2 to X_1, the committee should prefer X_1 to X_2.

These results enable us to formulate our problem as follows. Let us denote any decision which is available to the committee by x, which may be (but need not be) a column vector of real-valued elements; in general terms, x is an element of the set of all possible decisions. Let us write $u_i(x)$ for the expected utility of individual i in case the committee takes the decision x; the function $u_i(\ )$ is unique up to a linear transformation. The committee's utility function is then

$$u_C(x) = \sum_{i=1}^{N} w_i u_i(x), \tag{2.1}$$

where the w_i are positive real numbers. The committee's best decision is then the x (or set of x's) which maximizes $u_C(\ )$, given the choice of the w_i. [In (2.1) no constant term is introduced because it is irrelevant for the purpose of maximization and also because of the arbitrary zero of the committee utility scale.]

Our objective is to find a satisfactory rule for the determination of the w_i; or better, for their ratios, since these are the things that matter. Now the individual utility functions have all two arbitrary features (a zero and a unit), and the exposition will be simplified if we agree to replace utility by disutility. More precisely, we shall write $u_i(x_i)$ for the highest expected utility level which is available to individual i, where x_i is the best decision from i's point of view. We shall suppose that x_i is unique for $i = 1, \cdots, N$ (the eighth assumption).[3] Then

$$\delta_i(x) = u_i(x_i) - u_i(x) \tag{2.2}$$

is i's expected disutility, and maximization of (2.1) is equivalent to minimization of

$$\delta_C(x) = \sum_{i=1}^{N} w_i \delta_i(x). \tag{2.3}$$

[3] This assumption is not necessary to ensure the uniqueness of the functions (2.2), but it is necessary for the uniqueness of the loss matrix which will be introduced in Section 5.

It is easily seen that $\delta_i(\ \ )$ and $\delta_C(\ \ )$ have natural zeros, so that the only arbitrariness is their units of measurement; they are all unique up to a homogeneous linear transformation.

3. The Loads as Determined by the Raw Disutility Functions

We can express (2.3) by saying that the social disutility function is a weighted sum of the individual disutility functions. There is no formal objection to doing so, but it has the disadvantage that a weighting procedure of the individuals is suggested in the same way as a weighted price index number is obtained by treating prices of important commodities differently from those of less important commodities. Evidently, this is not what is meant by "weighted sum of individual disutility functions." What we wish to do is to treat the individuals "equally" (in accordance with fairly generally accepted ethical standards), but it must be admitted that at the present stage it is not very clear what "equally" really means. However, there is an easy way to avoid this problem. We recall that $\delta_i(\ \ )$ is unique up to a homogeneous linear transformation, and hence i's preferences are perfectly well described by any multiple of $\delta_i(\ \ )$. Let us choose w_i for that multiple, then $w_i\delta_i(\ \ )$ is "the" disutility function of individual i and the social disutility function is simply the sum of all these. The original functions $\delta_i(\ \ )$ are then just auxiliary concepts. They will be called the "raw" disutility functions and are to be multiplied by the appropriate w_i—to be called the "load" of the raw disutility function—in order to yield the corresponding individual disutility function.

Our problem is then to formulate an adequate rule for the numerical specification of the loads. In principle there are two possibilities. The loads are either determined by the ingredients which have been introduced up till now (which amounts effectively to a determination by raw disutility functions), or they are partly or wholly determined by "outside" ingredients such as side payments by one committee member to another. In this paper we shall exclude the latter possibility and confine ourselves to the former:

Postulate A. The loads $w_1, \cdots, w_N$ shall be functionals of the raw disutility functions $\delta_1(\ \ ), \cdots, \delta_N(\ \ )$:

$$w_i = f_i\{\delta_1, \cdots, \delta_i, \cdots, \delta_N\} \quad \text{for} \quad i = 1, \cdots, N. \tag{3.1}$$

The committee disutility function takes now the form

$$\delta_C(x) = \textstyle\sum_{i=1}^N f_i\{\delta_1, \cdots, \delta_i, \cdots, \delta_N\}\delta_i(x), \tag{3.2}$$

from which it is evident that we should impose certain constraints on $f_i\{\ \ \}$. For if we multiply $\delta_i(\ \ )$ by some positive number c, this should leave $\delta_C(\ \ )$ unchanged because nobody's preferences are changed; hence the corresponding $f_i\{\ \ \}$ should then be multiplied by $1/c$. At the same time, this multiplication of $\delta_i(\ \ )$ by c should have no effect on $f_j\{\ \ \}$ for $j \neq i$. This implies that the dependence of $f_i\{\ \ \}$ on $\delta_i(\ \ )$ is different from that on the "alien" raw disutility functions $\delta_j(\ \ ), j \neq i$. In fact, considerations of symmetry should convince us to choose for the $f_i\{\ \ \}$ a function $f\{\ \ \}$ (for all i) which is symmetric in the alien $\delta_j(\quad)$. It is convenient to introduce an $(N-1)$-element row vector $d_i(\quad)$ containing all $\delta_j(\quad)$ except the i^{th}:

$$d_i(\quad) = [\delta_1(\quad) \cdots \delta_{i-1}(\quad)\delta_{i+1}(\quad) \cdots \delta_N(\quad)], \tag{3.3}$$

then these assumptions can be summarized as follows:

Postulate B. The functional (3.1) *shall be of the form*

$$f_i\{\delta_1, \cdots, \delta_i, \cdots, \delta_N\} \equiv f\{\delta_i ; d_i\} \tag{3.4}$$

for all $i = 1, \cdots, N$. *The functional* f *shall be homogeneous of degree* -1 *in* $\delta_i(\quad)$ *and homogeneous of degree* 0 *in* d_i :

$$f\{c\delta_i ; d_iC\} \equiv (1/c)f\{\delta_i ; d_i\}, \tag{3.5}$$

where c *is any positive real number and* C *any real diagonal matrix whose* $N-1$ *diagonal elements are all positive. Finally,* f *shall be a symmetric functional of the elements of* $d_i(\quad)$:

$$f\{\delta_i ; d_i\Pi\} \equiv f\{\delta_i ; d_i\}, \tag{3.6}$$

where Π *is any permutation matrix of order* $N-1$.

4. A Digression on Ordinary Averages

As stated at the end of Section 1, the problem of choosing an appropriate committee disutility function is nothing more than a special kind of aggregation problem, and accordingly it has some heuristic merits to make a comparison with the simpler problem of means of frequency distributions. This problem is indeed simpler, because such concepts as loads which are functions of functions do not enter into the picture in that case.

Suppose N soldiers enter a room. Suppose that this committee is asked to determine the typical height of the group. Soldier No. 1 announces: "My height is 172 centimeters and that's typical." Soldier No. 2 replies: "No, my height is 180 centimeters and I think that's typical. If Soldier No. 1 insists on 172 centimeters he is 8 centimeters in error." This continues with each soldier arguing that he is of typical height. The result (for $N = 3$) can be summarized by the following error matrix:

	Height of		
Viewpoint of	No. 1 (172)	No. 2 (180)	No. 3 (179)
No. 1	0	8	7
No. 2	−8	0	−1
No. 3	−7	1	0

How is this problem solved? We shall consider two solutions, the median and the mean, and we shall analyze how they can both be justified in terms of the ideas set forth in the preceding pages. We notice, first of all, that none of the soldiers is satisfied with the height of any of the others as the typical height. Suppose then that each soldier's disutility in case some value is accepted as the typical height can be measured by w_i times the absolute difference between his own height and the proposed typical height, where w_i (for $i = 1, 2, \cdots$) is a positive load. Then, if some soldier succeeds in imposing his height as the typical height, he causes a disutility to the others as shown by the following matrix:

Disutility suffered by	Imposed height of No. 1 (172)	No. 2 (180)	No. 3 (179)
No. 1	0	$8w_1$	$7w_1$
No. 2	$8w_2$	0	w_2
No. 3	$7w_3$	w_3	0

The socially optimal decision is that height which minimizes total disutility for some suitable choice of the w's. The approach of the median is to take equal w's (e.g. $w_i = 1$ for all i), so that the matrix becomes symmetric:

$$\begin{bmatrix} 0 & 8 & 7 \\ 8 & 0 & 1 \\ 7 & 1 & 0 \end{bmatrix}, \tag{4.1}$$

and it is well-known that in that case the median (the height of No. 3, 179 centimeters) minimizes total disutility. By adding the elements of each column we obtain the social disutility which each individual causes in case he succeeds in imposing his own optimal decision. These social disutilities are 15, 9, and 8 centimeters, and the last value is also the minimum attainable.

The approach of the arithmetic mean is similar except that it is based on the idea that each soldier's disutility can be measured by w_i times the square of the difference between his height and the proposed typical height. This leads to the following scheme:

Disutility suffered by	Imposed height of No. 1 (172)	No. 2 (180)	No. 3 (179)
No. 1	0	$64w_1$	$49w_1$
No. 2	$64w_2$	0	w_2
No. 3	$49w_3$	w_3	0

Again, we take equal w's and hence a symmetric matrix:

$$\begin{bmatrix} 0 & 64 & 49 \\ 64 & 0 & 1 \\ 49 & 1 & 0 \end{bmatrix}. \tag{4.2}$$

Doing so, and minimizing social disutility (the sum of squares of the discrepancies with the three actual heights), we arrive at the arithmetic mean, which is 177 centimeters in this case. The three social disutilities associated with the individual optimal decisions are now 113, 65, and 50 (all in square centimeters); these values are all larger than the minimum attainable, which is 38 square centimeters.

5. The Desirability of Symmetric Loss Matrices

The very simple example of Section 4 teaches us the following. Whether we use the mean or the median as the optimal committee decision (for the typical height of the committee of soldiers), in both cases this can be regarded as the result of minimizing total disutility; and this is obtained by adding the individual disutilities in such a way that the disutility which j inflicts on i by imposing his (j's) optimal decision on the committee is equal to the disutility which i inflicts on j, for all pairs (i, j). In our terminology:

$$w_i\delta_i(x_j) = w_j\delta_j(x_i) \quad \text{for all} \quad (i, j). \tag{5.1}$$

Analogy considerations of this kind are of course not sufficient to convince us that the loads should be determined by the values taken by the raw loss functions in the individual optima [as (5.1) implies], let alone that they should be determined in the precise manner of (5.1). Postulate A implies that the loads are determined by the functions $\delta_i(\quad)$, not by the values which these N functions take for N particular argument values. But we shall nevertheless accept the narrower idea that the loads are to be determined by the $\delta_i(x_j)$ rather than by $\delta_i(\quad)$, partly on pragmatic grounds. For one thing, we can regard the individual optima $x_1, \cdots, x_N$ as lying on the "edge" of the set of interesting decisions from the committee's point of view, since they are socially optimal only if one load w_i is positive and all others are zero. Obviously, there is a considerable class of non-Pareto-optimal decisions which are completely irrelevant from the committee's point of view and which hence should preferably not determine the loads via the dependence of $f\{\quad\}$ on $\delta_i(\quad)$. For another, we simplify the problem considerably by focusing on load functions instead of load functionals. Let us then write

$$\lambda_{ij} = \delta_i(x_j) \tag{5.2}$$

for the raw disutility suffered by i in case j succeeds in imposing his optimal decision on the committee. We shall call $w_i\lambda_{ij}$ the loss (and λ_{ij} the raw loss) inflicted by j on i. Further, we shall write λ_i for the N-element column vector of elements λ_{ij} and Λ_i for the $N \times (N - 1)$ matrix of all columns λ_j excluding the i^{th}:

$$\lambda_i = \begin{bmatrix} \lambda_{i1} \\ \vdots \\ \lambda_{iN} \end{bmatrix}; \qquad \Lambda_i = [\lambda_1 \cdots \lambda_{i-1}\, \lambda_{i+1} \cdots \lambda_N]. \tag{5.3}$$

Then our assumption amounts to the following:

Postulate C. The dependence of $f\{\quad\}$ as defined in (3.4) on δ_i and d_i shall be confined to λ_i and Λ_i respectively:

$$f\{\delta_i\,; d_i\} \equiv \varphi(\lambda_i\,; \Lambda_i). \tag{5.4}$$

We have now boiled down our problem to that of finding a function φ depending on N arguments which are elements of λ_i and on $N(N - 1)$ arguments which are the elements of Λ_i. This function should be homogeneous of degree -1 in the former elements, homogeneous of degree 0 in the latter, and symmetric in the $N - 1$ columns of Λ_i.

To specify this function further we imagine that one individual, say the k^{th}, changes his mind (so that his preferences become different) or that he leaves the committee; or as a third alternative, that he becomes a new member of the committee. Obviously, we should expect that such a change alters the form of the committee disutility function $\delta_C(\quad)$ and hence also the optimal committee decision, at least in general. But we should not expect that this change will affect the disutility functions $w_i\delta_i(\quad)$ of the other individuals. The raw disutility functions will of course not be affected, since these are the basic ingredients of our problem. But we require that the loads be unchanged; or better their ratios, w_i/w_j, since these alone are relevant, More precisely:

Postulate D. For any pair of individuals (i, j), *where* $i, j = 1, \cdots, N$, *their load ratio* w_i/w_j *shall not be affected if some third individual changes his preferences, or if he is deleted from the committee, or if he is added to the committee.*

Consider then some pair of individuals, which we indicate by (1, 2) for notational simplicity. According to (5.4) their load ratio is $\varphi(\lambda_1 ; \Lambda_1)/\varphi(\lambda_2 ; \Lambda_2)$, and this should be independent of the preferences of any conceivable third individual. The consequence of this requirement is that the load ratio can depend on λ_{12} and λ_{21} only:

$$(5.5) \qquad \frac{w_1}{w_2} \equiv \frac{\varphi(\lambda_1 ; \Lambda_1)}{\varphi(\lambda_2 ; \Lambda_2)} \equiv \psi(\lambda_{12}, \lambda_{21}).$$

For if we take any other element of the $N \times N$ matrix $[\lambda_{ij}]$ we introduce the preferences of a third individual: λ_{k1} and λ_{k2} for $1 \neq k \neq 2$ involve the raw disutility function of k, λ_{1k} and λ_{2k} involve x_k which is k's optimal decision, and λ_{kh} for $1 \neq h \neq 2$ has both features. Furthermore, we have

$$(5.6) \qquad \psi(\lambda_{21}, \lambda_{12}) \equiv \frac{1}{\psi(\lambda_{12}, \lambda_{21})},$$

because taking the reciprocal of w_1/w_2 amounts to interchanging the indices. We conclude from (5.6):

$$(5.7) \qquad \psi(1, 1) = 1.$$

[The solution $\psi(1, 1) = -1$ is not applicable because w_1/w_2 has to be positive in view of the seventh condition imposed in Section 2.] Returning to (3.5) and (5.4) we note that

$$(5.8) \qquad \begin{aligned} \varphi\left(\frac{1}{\lambda_{12}} \lambda_1 ; \frac{1}{\lambda_{21}} \lambda_2, \lambda_3, \cdots, \lambda_N\right) &\equiv \lambda_{12}\, \varphi(\lambda_1 ; \Lambda_1) \\ \varphi\left(\frac{1}{\lambda_{21}} \lambda_2 ; \frac{1}{\lambda_{12}} \lambda_1, \lambda_3, \cdots, \lambda_N\right) &\equiv \lambda_{21}\, \varphi(\lambda_2 ; \Lambda_2), \end{aligned}$$

provided that $\lambda_{12} \neq 0 \neq \lambda_{21}$. Now the two φ's on the left of (5.8) have λ_{12} and λ_{21}-values which are 1, so that (5.7) is applicable. It follows that we should have

$$(5.9) \qquad 1 = \frac{\lambda_{12}\,\varphi(\lambda_1\,;\,\Lambda_1)}{\lambda_{21}\,\varphi(\lambda_2\,;\,\Lambda_2)} = \frac{\lambda_{12}\,w_1}{\lambda_{21}\,w_2} \quad \text{or} \quad \frac{w_1}{w_2} = \frac{\lambda_{21}}{\lambda_{12}},$$

in words that the loss inflicted by 1 on 2 should be the same as the loss inflicted by 2 on 1. This amounts to the symmetry condition (5.1), because (1, 2) is just an arbitrary pair of individuals. Summarizing:

Theorem 1. *In order that Postulates A through D be satisfied it is necessary and sufficient that the loads w_i are adjusted such that the loss matrix $[w_i\lambda_{ij}]$ is symmetric.*[4]

We thus find that symmetry of the loss matrix is a desirable property. We shall now show that symmetry has also a desirable property with respect to disaggregation. Suppose that the members of our committee are spread over a number (n, say) of different locations and that it is considered convenient to split it up into n subcommittees, one for each location. Every subcommittee will then have its own disutility function:

$$(5.10) \qquad \delta_{C_h}(x) = \textstyle\sum_{i \varepsilon S_h} w_i^{(h)}\delta_i(x), \qquad h = 1, \cdots, n,$$

where S_h stands for the set of individuals in the h^{th} subcommittee. It would obviously be desirable if the sum of the n subcommittee disutility functions (5.10) were identically equal to the committee disutility function:

$$(5.11) \qquad \textstyle\sum_{h=1}^{n} \sum_{i \varepsilon S_h} w_i^{(h)}\delta_i(x) \equiv \sum_{i=1}^{N} w_i\delta_i(x),$$

which implies that the subcommittee loads $w_i^{(h)}$ should be equal to the loads w_i for the committee as a whole:

$$(5.12) \qquad w_i^{(h)} = w_i\,, \qquad i\ \varepsilon\ S_h\,, \qquad h = 1, \cdots, n.$$

Now let i and j be members of the same subcommittee S_h. If Postulates A through C are imposed but Postulate D is not, the load ratio $w_i^{(h)}/w_j^{(h)}$ will in general depend on all λ_{rs} with $r, s\ \varepsilon\ S_h$ but not on λ_{rs} with either $r\ \varepsilon\ S_k$ or $s\ \varepsilon\ S_k$, $k \neq h$. But the load ratio w_i/w_j in the committee as a whole will generally depend on all λ_{rs} without constraints on r and s. It follows that the subcommittee disutility functions add up to the committee disutility function only if Postulate D is modified such that w_i/w_j in case $i, j\ \varepsilon\ S_h$ depend only on λ_{rs} with $r, s\ \varepsilon\ S_h$, for $h = 1, \cdots, n$. If we now imagine all conceivable partitionings of the committee in subcommittees, and if we require that it be always true that the subcommittee disutility functions add up to the committee disutility function, then there is no possibility of getting around Postulate D (and hence around symmetry). This proves:[5]

Theorem 2. *Suppose that Postulates A through C are imposed*; *suppose also that*

[4] The eight conditions of Section 2 are taken for granted here and are therefore not mentioned explicitly. Note that the eighth assumption (on the uniqueness of the individual optima x_j) is necessary to ensure the uniqueness of the elements λ_{ij} of the raw loss matrix.

[5] The addition of "whenever the individual optimal decisions differ pairwise" at the end of the theorem serves to exclude the trivial case in which the load ratio of two individuals is irrelevant.

the committee is partitioned into n subcommittees whose members form the sets $S_1, \cdots, S_n$:

$$(5.13) \qquad S = S_1 \cup S_2 \cup \cdots \cup S_n ; \qquad S_h \cap S_k = 0 \quad \text{for all} \quad h \neq k,$$

S being the set of all members of the committee. Then, in order that the subcommittee disutility functions add up to the committee disutility function for whatever partitioning, Postulate D is a necessary condition whenever the individual optimal decisions differ pairwise.

It may be useful to give a more concrete picture of the situation, and we shall therefore consider a numerical example for a five-men committee, the first three individuals being arranged in the first subcommittee and the last two in the second. Suppose that both committees proceed as Theorem 1 prescribes, viz., such that their own loss matrices are symmetric. The loss matrix for the committee as a whole, in numerical and partitioned form, may then be:

$$\begin{array}{ccc:cc} 0 & 1 & 3 & 2 & 3 \\ 1 & 0 & 2 & 1 & 2 \\ 3 & 2 & 0 & 3 & 4 \\ \hdashline 4 & 2 & 6 & 0 & 4 \\ 6 & 4 & 8 & 4 & 0 \end{array}$$

the rows of which specify the losses inflicted on any given individual, while the columns specify the losses inflicted by the optimal decision of any given individual. Obviously, this matrix is only symmetric as far as the diagonal blocks are concerned, so that the problem is how we can relate the loads which fall under different subcommittees in such a way that the off-diagonal blocks are each other's transpose. In the present example this can be achieved by multiplying the loads of the second subcommittee by $\frac{1}{2}$, so that the following symmetric loss matrix results:

$$\begin{array}{ccc:cc} 0 & 1 & 3 & 2 & 3 \\ 1 & 0 & 2 & 1 & 2 \\ 3 & 2 & 0 & 3 & 4 \\ \hdashline 2 & 1 & 3 & 0 & 2 \\ 3 & 2 & 4 & 2 & 0 \end{array}$$

The question arises how one can arrive at this symmetry with the smallest possible loss of the advantages accruing from the decentralization of the committee in terms of subcommittees. Suppose then that the procedure is as follows. Each subcommittee decides on its own disutility function (5.10) and sends one of its members as a delegate to a meeting of all n delegates. What should these delegates do in order that condition (5.11) be satisfied? The answer is easily formulated in terms of our numerical example. Suppose that individuals 1 and 4 are appointed as delegates by their subcommittees; then they can equate the losses which they inflict on each other, $w_1\lambda_{14}$ and $w_4\lambda_{41}$, and thus achieve symmetry for the loss matrix as a whole if this matrix can be symmetrized at all. Thus, the only thing we need is a meeting of delegates, one from each of the n

subcommittees. Every delegate has to supply two pieces of information: the disutility function of his subcommittee and his own disutility function. The former functions are combined linearly and the latter supply the coefficients of the linear combination.

6. On the Case when the Loss Matrix Cannot Be Symmetrized

There remains the problem that the loss matrix cannot always be symmetrized, because this involves $\frac{1}{2}N(N-1)$ conditions whereas only $N-1$ load ratios are available to satisfy them. In our view this is a problem which is important but not of the same level as the symmetry condition itself. If it is agreed that it is desirable to have a symmetric loss matrix, and if it is impossible to realize symmetry, then one should approximate symmetry, which can be done in several ways. This situation is comparable with that of the construction of ordinary averages and price index numbers. If all prices move proportionally the index problem is trivial; if they do not move proportionally one will obtain different outcomes depending on what kind of index number is used. Here, we have this problem at *two stages*: first, there is the problem of comparing any pair of individual disutility functions, and by doing so as well as possible we will nevertheless find positive values of $| w_i\lambda_{ij} - w_j\lambda_{ji} |$; second, there is the problem that after adding the individual loss functions and maximizing, we obtain a socially optimal decision which nevertheless inflicts positive losses on the individuals. The first problem is that of "*comparification.*" We can measure the "preference distance" of any pair of individuals i and j by $\frac{1}{2}(w_i\lambda_{ij} + w_j\lambda_{ji})$ and the "incomparability" of their preferences by

$$I_{ij} = \frac{| w_i \lambda_{ij} - w_j \lambda_{ji} |}{w_i \lambda_{ij} + w_j \lambda_{ji}}, \tag{6.1}$$

which is a number between 0 and 1 on the assumption that i and j do not agree as to the optimal decision (if they do I_{ij} is of the indeterminate form 0/0). The incomparability of the preferences of i with those of all other individuals in the committee can be measured by

$$I_{i.} = \frac{\sum_j | w_i \lambda_{ij} - w_j \lambda_{ji} |}{\sum_j (w_i \lambda_{ij} + w_j \lambda_{ji})}, \tag{6.2}$$

which is also a number between 0 and 1. The same holds for

$$I_{..} = \frac{\sum_i \sum_j | w_i \lambda_{ij} - w_j \lambda_{ji} |}{\sum_i \sum_j (w_i \lambda_{ij} + w_j \lambda_{ji})} = \frac{\sum_{i>j} \sum | w_i \lambda_{ij} - w_j \lambda_{ji} |}{\sum_i \sum_j w_i \lambda_{ij}}, \tag{6.3}$$

which measures the incomparability of the individual preferences in the group as a whole. It will be clear that $I_{i.}$ is a weighted average of the I_{ij}, and $I_{..}$ of the $I_{i.}$:

$$I_{i.} = \frac{\sum_j (w_i \lambda_{ij} + w_j \lambda_{ji}) I_{ij}}{\sum_j (w_i \lambda_{ij} + w_j \lambda_{ji})}; \qquad I_{..} = \frac{\sum_i \sum_j (w_i \lambda_{ij} + w_j \lambda_{ji}) I_{i.}}{\sum_i \sum_j (w_i \lambda_{ij} + w_j \lambda_{ji})}. \tag{6.4}$$

In some cases it is possible to obtain a loss matrix which is exactly symmetric and, in fact, two examples have been given in Section 4. If the decision variable x is a scalar and takes real values, and if the disutility function of each individual i is a multiple of $|x - x_i|$, then we obtain symmetry by taking $w_i\delta_i(x) = |x - x_i|$ and the optimal committee decision is the median of the x_i. And if the same conditions are satisfied except that the disutility function of each i is a multiple of $(x - x_i)^2$, then there is symmetry for $w_i\delta_i(x) = (x - x_i)^2$ and the optimal committee decision is the mean of the x_i. But in general symmetry cannot be realized, so that one has to be satisfied with approximate symmetry. There is then a certain amount of freedom as to the procedure to be followed, but it has to meet the following three requirements: (i) it leads to a symmetric loss matrix in case symmetry can be obtained, (ii) it approximates symmetry as well as possible in some well-defined sense in case exact symmetry cannot be obtained, and (iii) it leads to positive load ratios. Van den Bogaard and Versluis suggested making corresponding row and column sums of the loss matrix pairwise equal:

$$\sum_j w_i\lambda_{ij} = \sum_j w_j\lambda_{ji} \quad \text{for} \quad i = 1, \cdots, N \tag{6.5}$$

and proved the following

Theorem 3. *For any $N \times N$ matrix $[\lambda_{ij}]$ with zero diagonal and positive and real off-diagonal elements there exists a vector $w = [w_i]$ satisfying* (6.5), *which is unique apart from an arbitrary multiplicative scalar, and which is the characteristic vector of the matrix*

$$\begin{bmatrix} -\sum\lambda_{1j} & \lambda_{21} & \cdots & \lambda_{N1} \\ \lambda_{12} & -\sum\lambda_{2j} & \cdots & \lambda_{N2} \\ \vdots & \vdots & & \vdots \\ \lambda_{1N} & \lambda_{2N} & \cdots & -\sum\lambda_{Nj} \end{bmatrix} \tag{6.6}$$

corresponding to the (*unique*) *zero root. This vector has elements whose ratios are all positive and ensures symmetry of the matrix $[w_i\lambda_{ij}]$ whenever a vector ensuring symmetry exists.*

Van den Bogaard and Versluis applied this method to a committee consisting of a representative of trade unions (individual 1), a member appointed by the Crown (individual 2), and a representative of employers' organizations (individual 3). This committee was supposed to be in charge of the Dutch economy, each of its members having a quadratic social preference function.[6] The loss

[6] The arguments of these preference functions are: the general wage rate, tax rates for indirect taxes less subsidies, for direct taxes on wage income, and for direct taxes on non-wage income, and government expenditure on commodities. Apart from these five government-controlled variables there are four others which are not controlled by the government, viz., the level of private employment, the price level of consumer goods, the share of wages in national income, and the surplus on the balance of payments. The procedure followed amounted to a specification of quadratic preference functions in all nine variables, which are then to be maximized subject to the constraints of the economy as measured by an econometric model. In the study quoted here the noncontrolled variables were eliminated by means of these constraints, so that the preference functions used (with the five controlled variables as arguments) depend on these constraints. Hence, if the constraints

matrix takes the following form:

$$(6.7)\qquad [w_i\,\lambda_{ij}] = \begin{bmatrix} 0 & 9.082w_1 & 12.618w_1 \\ 17.654w_2 & 0 & 1.608w_2 \\ 300.747w_3 & 18.822w_3 & 0 \end{bmatrix}$$

and (6.5) implies the following load vector:

$$(6.8)\qquad w = \{1 \quad 0.513 \quad 0.042\},$$

so that the loss matrix in approximate symmetric form becomes (apart from an arbitrary multiplicative scalar):

$$(6.9)\qquad [w_i\,\lambda_{ij}] = \begin{bmatrix} 0 & 9.082 & 12.618 \\ 9.050 & 0 & 0.824 \\ 12.651 & 0.792 & 0 \end{bmatrix},$$

which is rather close to symmetry.

The derivation of the committee disutility function is straightforward. It is a matter of combining the individual quadratic functions by means of the loads (6.8), after which the socially optimal decision is derived by maximizing the resulting quadratic committee disutility function; reference is made to [2]. Here we confine ourselves to the following. The matrix of preference distances is

$$(6.10)\qquad [\tfrac{1}{2}\,w_i\,\lambda_{ij} + \tfrac{1}{2}\,w_j\,\lambda_{ji}] = \begin{bmatrix} 0 & 9.066 & 12.634 \\ & 0 & 0.808 \\ & & 0 \end{bmatrix},$$

which shows that the Crown member occupies a position "between" the two other individuals but that he is "closer" to the employers' representative than to labour's representative. The matrix of pairwise incomparability coefficients is

$$(6.11)\qquad [I_{ij}] = \begin{bmatrix} \cdot & 0.002 & 0.001 \\ & \cdot & 0.020 \\ & & \cdot \end{bmatrix}$$

and the vector of individual incomparability coefficients is

$$(6.12)\qquad [I_{i.}] = \{0.001 \quad 0.003 \quad 0.002\},$$

while the committee incomparability coefficient is

$$(6.13)\qquad I_{..} = 0.002.$$

The conclusion is that in this case the "comparification fit" is very good.

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change, these preference functions and $[\lambda_{ij}]$ change too, which affects the loads. This illustrates that Arrow's condition of "independence of irrelevant alternatives" cannot be expected to be satisfied in general.

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