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A note on the proportionality between some consistency indices in the AHP

Matteo Brunelli [a,b,*], Andrew Critch [c], Michele Fedrizzi [d]

[a] Systems Analysis Laboratory, Aalto University, Finland
[b] Institute for Advanced Management Systems Research, Åbo Akademi University, Finland
[c] Department of Mathematics, University of California, Berkeley, United States
[d] Department of Industrial Engineering, University of Trento, Italy

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ABSTRACT

Analyzing the consistency of preferences is an important step in decision making with pairwise comparison matrices, and several indices have been proposed in order to estimate it. In this paper we prove the proportionality between some consistency indices in the framework of the Analytic Hierarchy Process. Knowing such equivalences eliminates redundancy in the consideration of evidence for consistent preferences.

1. Introduction

Measuring the inconsistency of an $n \times n$ pairwise comparison matrix—that is, assigning a numerical value to "how much" the matrix $\mathbf{A} = (a_{ij})_{n \times n}$ deviates from one indicating consistent preferences—is an important issue in the Analytic Hierarchy Process (AHP), as well as in other alternative methods of decision-making.

The oldest and most commonly used measure is the consistency index, *CI*, introduced by Saaty [19],

$$CI = \frac{\lambda_{\max} - n}{n - 1}, \tag{1}$$

where $\lambda_{\max}$ is the maximum eigenvalue of $\mathbf{A}$. After Saaty, several other authors proposed different consistency indices in order to find the most suitable way to estimate "how far" $\mathbf{A}$ is from the consistency condition

$$a_{ij}a_{jk} = a_{ik} \quad \forall i, j, k. \tag{2}$$

Note that Saaty's definition (1) is based on the fact that, for a positive reciprocal matrix, condition (2) holds if and only if $\lambda_{\max} = n$.

Appropriate consistency evaluation of elicited preferences is seen as important largely because the achievement of a satisfactory consistency level is viewed as a desirable property. The more consistent are the preferences of a decision maker, the more likely he/she is a reliable expert, has a deep insight into the problem, and acts with attention and care with respect to the problem he/she is facing. Conversely, if judgements are far from consistency, i.e. they are heavily contradictory, it is likely that they were given with poor competence and care. Several inconsistency indices have been already proposed in literature to estimate the degree of incoherence of judgements [2–4,7,12,13,17,18,23].

If two indices are proportional, it is important to know their proportionality for two reasons. From an empirical point of view, they should not be considered as contributing independent evidence for the consistency of a subject's preferences. On

* Corresponding author at: Systems Analysis Laboratory, Aalto University, Finland.
E-mail addresses: matteo.brunelli@aalto.fi (M. Brunelli), critch@math.berkeley.edu (A. Critch), michele.fedrizzi@unitn.it (M. Fedrizzi).

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the other hand, from a mathematical perspective, their equivalence may be taken to suggest that they represent an important quantity.

2. Pairwise comparison matrices and consistency indices

Given a set of alternatives $X = \{x_1, \dots, x_n\} (n \geqslant 2)$, a pairwise comparison matrix $\mathbf{A} = (a_{ij})_{n \times n}$ is a matrix $\mathbf{A} \in [1/9, 9]^{n \times n}$ with (i) $a_{ii} = 1 \; \forall i$ and (ii) $a_{ij} a_{ji} = 1 \; \forall i, j$ where a_{ij} is a multiplicative estimation of the degree of preference of x_i over x_j [19]. The comparison scale ranging from 1 to 9 was employed by Saaty based on experimental evidence [15] that an individual cannot simultaneously compare more than 7 ± 2 objects without being confused. A pairwise comparison matrix is considered *consistent* if and only if the following transitivity condition holds:

$$a_{ik} = a_{ij} a_{jk} \quad \forall i, j, k. \tag{3}$$

If $\mathbf{A}$ is consistent, then there exists a vector $\mathbf{w} = (w_1, \dots, w_n)$ such that

$$a_{ij} = \frac{w_i}{w_j} \quad \forall i, j. \tag{4}$$

In this case, the vector $\mathbf{w}$ can be obtained using the geometric mean method:

$$w_i = \left(\prod_{j=1}^{n} a_{ij} \right)^{\frac{1}{n}} \quad \forall i. \tag{5}$$

Some other types of matrices have been proposed in order to pairwise compare alternatives, and perhaps the second best known approach, after that of Saaty, is based on *reciprocal relations* [24]. Reciprocal relations, which are sometimes also called fuzzy preference relations, can be represented by means of matrices $\mathbf{R} = (r_{ij})_{n \times n}$ with (i) $r_{ii} = 0.5 \; \forall i$ and (ii) $r_{ij} + r_{ji} = 1 \; \forall i, j$ where r_{ij} is an estimation of the degree of preference given to x_i compared with x_j. Tanino calls a reciprocal relation matrix *additively consistent* if

$$r_{ij} - r_{ik} - r_{kj} + 0.5 = 0 \quad \forall i, j, k. \tag{6}$$

Pairwise comparison matrices and reciprocal relations are theoretically interchangeable representations of preferences, relatable by means of a function $f : [1/9, 9] \to [0, 1]$ defined in [8] as follows

$$r_{ij} = f(a_{ij}) = \frac{1}{2}(1 + \log_9 a_{ij}), \tag{7}$$

and its inverse

$$a_{ij} = f^{-1}(r_{ij}) = 9^{2(r_{ij} - 0.5)}. \tag{8}$$

Under this transformation, given $\mathbf{A} = (a_{ij})$ and $\mathbf{R} = (r_{ij})$, if $r_{ij} = f(a_{ij}) \; \forall i, j$, then $\mathbf{A} = (a_{ij})$ and $\mathbf{R} = (r_{ij})$ can be considered to represent the same preference configuration.

As proposed by Tanino [24] and studied in [5], there exists another type of consistency characterization for reciprocal relations. In fact, a reciprocal relation is multiplicatively consistent if and only if the following condition is satisfied

$$\frac{r_{ik}}{r_{ki}} = \frac{r_{ij}}{r_{ji}} \frac{r_{jk}}{r_{kj}} \quad \forall i, j, k.$$

Multiplicatively consistent reciprocal relations can be transformed into consistent pairwise comparison matrices and additively consistent reciprocal relations. An overview of pairwise comparison matrices and reciprocal relations, together with their consistency conditions and transformations can be found in [9].

Besides Saaty's consistency index (1), several other consistency indices have been proposed in the literature so far, and in this short paper we establish the proportionality between two pairs of them. Hence, let us first briefly recall the definitions of the four consistency indices at issue.

2.1. The Geometric Consistency Index

The Geometric Consistency Index [1,6] is based on the deviations of the entries a_{ij} of $\mathbf{A}$ from the consistent values w_i / w_j, where the weight vector $\mathbf{w} = (w_1, \dots, w_n)$ is given by (5). It has the following formulation:

$$GCI = \frac{2}{(n-1)(n-2)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \ln^2 e_{ij} \tag{9}$$

with $e_{ij} := a_{ij}(w_j / w_i)$ being a local estimator of inconsistency and $\frac{2}{(n-1)(n-2)}$ a normalization factor.

2.2. The index of Lamata and Peláez

The index of Lamata and Peláez [14,16], denoted by CI^*, is based on the property that three alternatives x_i, x_j, x_k are pairwise compared in a consistent way if and only if the determinant of the corresponding pairwise comparison matrix of order three

$$\mathbf{A}_{3\times 3} = \begin{pmatrix} 1 & a_{ij} & a_{ik} \\ \frac{1}{a_{ij}} & 1 & a_{jk} \\ \frac{1}{a_{ik}} & \frac{1}{a_{jk}} & 1 \end{pmatrix} \tag{10}$$

is equal to zero,

$$\det(\mathbf{A}_{3\times 3}) = \frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2 = 0. \tag{11}$$

Based on this property, the authors define the consistency index CI^* of an $n \times n$ pairwise comparison matrix $\mathbf{A}$ as the mean value of the determinants of all the 3×3 submatrices of $\mathbf{A}$.

2.3. The index c_3

Shiraishi et al. [20–22] proposed, as a consistency index of a pairwise comparison matrix, the coefficient c_3 of its characteristic polynomial.

$$P_{\mathbf{A}}(\lambda) = \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n.$$

They proved [20] that $c_3(\mathbf{A}) \leqslant 0$ for every pairwise comparison matrix $\mathbf{A}$, with $c_3(\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is consistent, which justifies its use as a consistency index.

2.4. The index ρ

The index ρ for reciprocal relations [10,11] is based on an index of local consistency associated with the triplet (x_i, x_j, x_k), that is

$$t_{ijk}^2 = (r_{ij} - r_{ik} - r_{kj} + 0.5)^2. \tag{12}$$

which clearly derives from (6). Fedrizzi and Giove [11] defined a global consistency index as the mean value of the local consistency indices for all the possible triplets (x_i, x_j, x_k) with $i < j < k$, obtaining

$$\rho = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}(r_{ij} - r_{ik} - r_{kj} + 0.5)^2 \Big/ \binom{n}{3}. \tag{13}$$

3. Results

In this section we prove that the index c_3 is proportional to CI^*, and the index ρ is proportional to GCI.

Proposition 1. *Given a positive reciprocal matrix $\mathbf{A} = (a_{ij})_{n\times n}$ with $n \geqslant 3$, the consistency indices c_3 and CI^* satisfy the equality*

$$c_3 = -\binom{n}{3} CI^*. \tag{14}$$

Proof. Consistency index CI^* is the mean value of the determinants of all the 3×3 submatrices (10) of $\mathbf{A}$, and therefore,

$$CI^* = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(\frac{a_{ik}}{a_{ij}a_{jk}} + \frac{a_{ij}a_{jk}}{a_{ik}} - 2\right) \Big/ \binom{n}{3}. \tag{15}$$

Furthermore, since $\mathbf{A}$ is positive and reciprocal, by expanding $P_{\mathbf{A}}(\lambda)$ (see [20]) one obtains

$$c_3 = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(2 - \frac{a_{ik}}{a_{ij}a_{jk}} - \frac{a_{ij}a_{jk}}{a_{ik}}\right). \tag{16}$$

Then, equality (14) follows from (15) and (16). □

If in this case the similarity between the two indices was quite clear, then the same cannot be said about the next two. For this reason, if the previous proof was straightforward, the next involves more computations.

Proposition 2. *Given a reciprocal relation* $\mathbf{R} = (r_{ij})_{n\times n}$, *the consistency indices* ρ *and GCI satisfy the equality*

$$\rho = \frac{3}{4\ln^2(9)} GCI \tag{17}$$

for every $n \geqslant 3$

Proof. For later convenience, letting $q_{ij} = r_{ij} - 0.5$ allows us to write $r_{ij} + r_{ji} = 1$ property as $q_{ij} = -q_{ji}$. Then, (8) becomes $a_{ij} = 9^{2q_{ij}}$. Now, write $t_{ijk} = r_{ij} - r_{ik} - r_{kj} + 0.5 = q_{ij} + q_{jk} + q_{ki}$ so that, from (13), the index ρ can be reformulated (see [11]) as

$$\rho = \sum_{i,j,k=1}^{n} (r_{ij} - r_{ik} - r_{kj} + 0.5)^2 \Big/ 6\binom{n}{3} = \sum_{i,j,k=1}^{n} t_{ijk}^2 \Big/ 6\binom{n}{3}.$$

Let us rewrite the Geometric Consistency Index (9) for reciprocal relations by applying (7). From (5),

$$\log_9 w_i = \frac{2}{n}\sum_{k=1}^{n} q_{ik}$$

and thus, from the definition of local inconsistency $e_{ij} := a_{ij}\frac{w_j}{w_i}$ in (9),

$$n\log_9(e_{ij}) = 2nq_{ij} + 2\sum_{k=1}^{n}(q_{jk} - q_{ik}) = 2\sum_{k=1}^{n}(q_{ij} + q_{jk} + q_{ki}) = 2\sum_{k=1}^{n} t_{ijk}$$

so the Geometric Consistency Index equals

$$GCI = \frac{2}{(n-1)(n-2)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\ln^2 e_{ij} = \frac{1}{(n-1)(n-2)}\sum_{i,j=1}^{n}\ln^2 e_{ij} = \frac{\ln^2(9)}{(n-1)(n-2)}\sum_{i,j=1}^{n}\left(\frac{2}{n}\sum_{k=1}^{n} t_{ijk}\right)^2$$
$$= \frac{4\ln^2(9)}{n^2(n-1)(n-2)}\sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} t_{ijk}\right)^2$$

At this point, the proportionality claim $\rho \propto GCI$ is equivalent to

$$\sum_{i,j,k=1}^{n} t_{ijk}^2 \quad \propto \quad \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} t_{ijk}\right)^2$$

(where the constant of proportionality could depend on n). First, let us compute the left-hand side (LHS):

$$t_{ijk}^2 = q_{ij}^2 + q_{jk}^2 + q_{ki}^2 + 2(q_{ij}q_{jk} + q_{jk}q_{ki} + q_{ki}q_{ij})$$

Let $S = \sum_{i,j=1}^{n} q_{ij}^2$ and $C = \sum_{i,j,k=1}^{n} q_{ij}q_{jk}$. Summing the expansion of t_{ijk}^2 one term at a time,

$$\sum_{i,j,k=1}^{n} q_{ij}^2 = \sum_{k=1}^{n}\sum_{i,j=1}^{n} q_{ij}^2 = nS$$

and by symmetry,

$$\sum_{i,j,k=1}^{n} q_{jk}^2 = \sum_{i,j,k=1}^{n} q_{ki}^2 = nS.$$

Similarly,

$$\sum_{i,j,k=1}^{n} q_{ij}q_{jk} = \sum_{i,j,k=1}^{n} q_{jk}q_{ki} = \sum_{i,j,k=1}^{n} q_{ki}q_{ij} = C.$$

Hence,

$$\text{LHS} = \sum_{i,j,k=1}^{n} t_{ijk}^2 = nS + nS + nS + 2(C + C + C) = 3(nS + 2C).$$

Next let us compute the right-hand side (RHS), first by rewriting:

$$\text{RHS} = \sum_{i,j=1}^{n}\left(\sum_{k=1}^{n} t_{ijk}\right)^2 = \sum_{i,j=1}^{n}\left(\sum_{k,l=1}^{n} t_{ijk}t_{ijl}\right) = \sum_{i,j,k,l=1}^{n} t_{ijk}t_{ijl}$$

$$t_{ijk}t_{ijl} = (q_{ij} + q_{jk} + q_{ki})(q_{ij} + q_{jl} + q_{li}) = q_{ij}^2 + q_{ij}q_{jl} + q_{ij}q_{li} + q_{jk}q_{ij} + q_{jk}q_{jl} + q_{jk}q_{li} + q_{ki}q_{ij} + q_{ki}q_{jl} + q_{ki}q_{li}$$

The 1st term sums to

$$\sum_{i,j,k,l=1}^{n} q_{ij}^2 = \sum_{k,l=1}^{n}\sum_{i,j=1}^{n} q_{ij}^2 = n^2 S.$$

The 2nd term sums to

$$\sum_{i,j,k,l=1}^{n} q_{ij}q_{jl} = \sum_{k=1}^{n}\sum_{i,j,l=1}^{n} q_{ij}q_{jl} = nC.$$

Similarly, the 3rd, 4th, and 7th terms respectively sum to

$$\sum_{i,j,k,l=1}^{n} q_{li}q_{ij} = \sum_{i,j,k,l=1}^{n} q_{ij}q_{jk} = \sum_{i,j,k,l=1}^{n} q_{ki}q_{ij} = nC,$$

whereas the 5th and 9th terms each sum to

$$\sum_{i,j,k,l=1}^{n} -q_{kj}q_{jl} = \sum_{i,j,k,l=1}^{n} -q_{ki}q_{il} = -nC.$$

The 6th term sums to

$$\sum_{i,j,k,l=1}^{n} q_{jk}q_{li} = \left(\sum_{j,k=1}^{n} q_{jk}\right)\left(\sum_{l,i=1}^{n} q_{li}\right) = (0)(0) = 0,$$

and similarly the 8th term sums to 0.

Hence, the total sum is

$$\mathrm{RHS} = n^2S + nC + nC + nC - nC + 0 + nC + 0 - nC = n^2S + 2nC = n(nS + 2C)$$

so we obtain the proportionality

$$\frac{\mathrm{LHS}}{\mathrm{RHS}} = \frac{3(nS + 2C)}{n(nS + 2C)} = \frac{3}{n},$$

and recover

$$\frac{\rho}{GCI} = \frac{\mathrm{LHS}}{\mathrm{RHS}} \cdot \frac{n^2(n-1)(n-2)}{4\ln^2(9)} \cdot \frac{1}{6\binom{n}{3}} = \frac{3n(n-1)(n-2)}{4\ln^2(9)} \cdot \frac{1}{n(n-1)(n-2)} = \frac{3}{4\ln^2(9)}. \qquad \square$$

Note that the constant of proportionality between c_3 and CI^* depends on the number n of alternatives, whereas the one between ρ and GCI does not. Propositions 1 and 2 can also be represented graphically. We randomly generated a large number of pairwise comparison matrices (or, equivalently, reciprocal relations) and associated each of them with a point on the cartesian plane having as coordinates the corresponding values of the two consistency indices involved in Proposition 1. As expected, all the points lie on a straight line. The same holds for Proposition 2. We also used numerical simulations to test whether other indices are proportional, but results were negative and therefore no other proportionality exists between the indices proposed in [2–4,7,12,13,17,18,23].

4. Conclusions

When making use of the various indices observed and proven proportional in this paper, we believe it is important that the applied mathematician be aware of their equivalence. This avoids redundancy in the consideration of evidence for consistent preferences, and allows existing results proven for one index to apply directly to other indices which are proportional to it.

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