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AXIOMATIC FOUNDATION OF THE ANALYTIC HIERARCHY PROCESS*

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This paper contains an axiomatic treatment of the Analytic Hierarchy Process (AHP). The set of axioms corresponding to hierarchic structures are a special case of axioms for priority setting in systems with feedback which allow for a wide class of dependencies. The axioms highlight: (1) the reciprocal property that is basic in making paired comparisons; (2) homogeneity that is characteristic of people's ability for making comparisons among things that are not too dissimilar with respect to a common property and, hence, the need for arranging them within an order preserving hierarchy; (3) dependence of a lower level on the adjacent higher level; (4) the idea that an outcome can only reflect expectations when the latter are well represented in the hierarchy. The AHP neither assumes transitivity (or the stronger condition of consistency) nor does it include strong assumptions of the usual notions of rationality. A number of facts are derived from these axioms providing an operational basis for the AHP. (CHOICE MODELS)

1. Introduction

The basic problem of decision making is to choose a best one in a set of competing alternatives that are evaluated under conflicting criteria. The Analytic Hierarchy Process (AHP) provides us with a comprehensive framework for solving such problems. It enables us to cope with the intuitive, the rational, and the irrational, all at the same time, when we make multicriteria and multiactor decisions. We can use the AHP to integrate our perceptions and purposes into an overall synthesis. The AHP does not require that judgments be consistent or even transitive. The degree of consistency (or inconsistency) of the judgments is revealed at the end of the AHP process.

Most of us have difficulty examining even a few ideas at a time. We need instead to organize our problems in complex structures which allow us to think about them one or two at a time. We need *simplicity* and *complexity*. We need an approach that is conceptually simple so that we can use it easily. And at the same time, we need an approach that is robust enough to handle real world decisions and complexities.

The Analytic Hierarchy Process is such a problem-solving framework. It is a systematic procedure for representing the elements of any problem. It organizes the basic rationality by breaking down a problem into its smaller constituent parts and then calls for only simple pairwise comparison judgments to develop priorities in each hierarchy.

There are three principles which one can recognize in problem solving. They are the principles of decomposition, comparative judgments, and synthesis of priorities.

The decomposition principle calls for structuring the hierarchy to capture the basic elements of the problem. An effective way to do this is first to work downward from the focus in the top level to criteria bearing on the focus in the second level, followed by subcriteria in the third level, and so on, from the more general (and sometimes uncertain) to the more particular and definite. One can then start at the bottom, identifying alternatives for that level and attributes under which they should be

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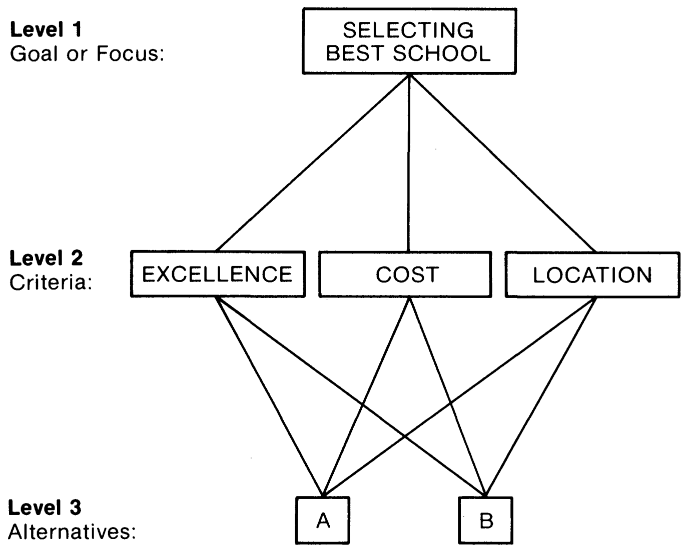

FIGURE 1

compared which fall in the next level up. Then one finds an intermediate set of higher criteria that can both be decomposed into these attributes and are themselves decompositions of the higher level criteria or subcriteria identified in the downward process. In this way, one can link the focus of the hierarchy to its bottom level in a sequence of appropriate intermediate levels. The levels of a decomposition are an essential part of measurement, and, hence, adjacent ones should generally not be too disparate, that is they do not differ by more than a "qualitative" order of magnitude. In general, the bottom level of the hierarchy contains the resources to be allocated, or the alternatives from which the choice is to be made. (See Figure 1.)

The principle of comparative judgments calls for setting up a matrix to carry out pairwise comparisons of the relative importance of the elements in the second level with respect to the overall objective (or focus) of the first level. In the case where no scale of measurement exists, this is a judgment made by the individual or group solving the problem. The scale for entering judgments is given in Table 1. Additional comparison matrices are used to compare the elements of the third level with respect to the appropriate parents in the second, and so on down the hierarchy. The process could be started at the bottom level and move upward. An entry of each matrix belongs to a fundamental scale employed in the comparisons. These entries are used to generate a derived ratio scale. The next step deals with the composition of the derived ratio scales.

The synthesis of priorities principle is now applied. Priorities are synthesized from the second level down by multiplying local priorities by the priority of their corresponding criterion in the level above, and adding them for each element in a level according to the criteria it affects. (The second level elements are each multiplied by unity, the weight of the single top level goal.) This gives the composite or global priority of that element which is then used to weight the local priorities of elements in the level below compared by it as criterion, and so on to the bottom level.

The AHP contains an intrinsic measure of inconsistency for each matrix and for the whole hierarchy. Knowledge of inconsistency enables one to determine those judgments which need reassessment.

When a group uses the AHP, their judgments can be combined after discussion by applying the geometric mean to the judgments which derives from the requirement

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TABLE 1

Scale of Relative Importance

Intensity of Relative Importance	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective.
3	Moderate importance of one over another	Experience and judgment slightly favor one activity over another.
5	Essential or strong importance	Experience and judgment strongly favor one activity over another.
7	Demonstrated importance	An activity is strongly favored and its dominance is demonstrated in practice.
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation.
2, 4, 6, 8	Intermediate values between the two adjacent judgments	When compromise is needed.
Reciprocals of above nonzero numbers	If an activity has one of the above numbers assigned to it when compared with a second activity, then the second activity has the reciprocal value when compared to the first.	
Rationals	Ratios arising from the scale	If consistency were to be forced by obtaining n numerical values to span the matrix.

REMARK. When only two objects are compared it may be desirable to expand the interval 1, 2 (from equal to slight importance) by inserting the values, 1.1, 1.2, . . . , 1.9, starting with 1.1 as very slight, 1.2 as slight, 1.3 as moderate, etc.

that the collective judgment itself must satisfy the reciprocal property (Aczel and Saaty 1983).

The AHP can be applied to set priorities on the criteria and subcriteria of the hierarchy. The alternatives may be evaluated by paired comparisons (relative measurement). When there are many alternatives, and neither their number nor their kind affect the importance of the criteria, they can be absolutely measured or scored on each criterion according to merit or degree to which they meet the standards (see §4).

Many decision problems involve dependence of criteria on alternatives and of

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higher order criteria on lower order ones; also alternatives may depend on other alternatives. A particularly useful generalization of the theory to deal with such dependence situations has been formalized within a network system with feedback of which a hierarchy is a special case.

The purpose of this paper is to state the axioms on which the AHP is based and to show how the theory of the AHP is derived from these axioms. For a more basic introduction to the AHP and its many applications, the reader is referred to Saaty (1980).

2. Axioms for Deriving a Scale from Fundamental Measurement and for Hierarchic Composition

Let $\mathfrak{A}$ be a finite set of n elements called alternatives. Let $\mathfrak{C}$ be a set of properties or attributes with respect to which elements in $\mathfrak{A}$ are compared. Philosophers distinguish between properties and attributes. A property is a feature that an object or individual possesses even if we are ignorant of this fact. On the other hand an attribute is a feature we assign to some object: it is a concept. Here we assume that properties and attributes are interchangeable and generally refer to them as criteria. A *criterion* is a primitive.

When two objects or elements in $\mathfrak{A}$ are compared according to a criterion in $\mathfrak{C}$, we say that we are performing binary comparisons. Let $>_C$ be a binary relation on $\mathfrak{A}$ representing "more preferred than" with respect to a criterion in $\mathfrak{C}$. Let $\sim_C$ be the binary relation "indifferent to" with respect to a criterion C in $\mathfrak{C}$. Hence, given two elements, $A_i, A_j \in \mathfrak{A}$, either $A_i >_C A_j$ or $A_j >_C A_i$ or $A_i \sim_C A_j$ for all $C \in \mathfrak{C}$. We use $A_i \gtrsim_C A_j$ to indicate more preferred or indifferent. A given *family of binary relations* $>_C$ with respect to a criterion C in $\mathfrak{C}$ is a primitive.

Let $\mathfrak{P}$ be the set of mappings from $\mathfrak{A} \times \mathfrak{A}$ to $\mathbb{R}^+$ (the set of positive reals). Let $f: \mathfrak{C} \to \mathfrak{P}$. Let $P_C \in f(C)$ for $C \in \mathfrak{C}$. P_C assigns a positive real number to every pair $(A_i, A_j) \in \mathfrak{A} \times \mathfrak{A}$. Let $P_C(A_i, A_j) \equiv a_{ij} \in \mathbb{R}^+$, $A_i, A_j \in \mathfrak{A}$. For each $C \in \mathfrak{C}$, the triple $(\mathfrak{A} \times \mathfrak{A}, \mathbb{R}^+, P_C)$ is a *fundamental* or *primitive scale*. A fundamental scale is a mapping of objects to a numerical system.

DEFINITION. For all $A_i, A_j \in \mathfrak{A}$ and $C \in \mathfrak{C}$

$$A_i >_C A_j \quad \text{if and only if} \quad P_C(A_i, A_j) > 1,$$

$$A_i \sim_C A_j \quad \text{if and only if} \quad P_C(A_i, A_j) = 1.$$

If $A_i >_C A_j$ we say that A_i dominates A_j with respect to $C \in \mathfrak{C}$. Thus P_C represents the intensity or strength of preference for one alternative over another.

Axiom 1 (Reciprocal). For all $A_i, A_j \in \mathfrak{A}$ and $C \in \mathfrak{C}$

$$P_C(A_i, A_j) = 1/P_C(A_j, A_i).$$

Whenever we make paired comparisons we need to consider both members of the pair to judge the relative value. If one stone is judged to be five times heavier than another, then the other is automatically one fifth as heavy as the first because it participated in making the first judgment. The comparison matrices that we consider are formed by making paired reciprocal comparisons. It is this simple, but powerful means of resolving multicriteria problems that is the basis of the AHP.

Let $A = (a_{ij}) \equiv (P_C(A_i, A_j))$ be the set of paired comparisons of the alternatives with respect to a criterion $C \in \mathfrak{C}$. By Axiom 1, A is a positive reciprocal matrix. The object is to obtain a *scale of relative dominance* (or *rank order*) of the alternatives from the paired comparisons given in A.

There is a natural way to derive the relative dominance of a set of alternatives from

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a pairwise comparison matrix A. Let $R_{M(n)}$ be the set of $(n \times n)$ positive reciprocal matrices $A = (a_{ij}) \equiv (P_C(A_i, A_j))$ for all $C \in \mathfrak{C}$. Let $[0,1]^n$ be the n-fold cartesian product of $[0,1]$ and let $\psi: R_{M(n)} \to [0,1]^n$ for $A \in R_{M(n)}$, $\psi(A)$ is an n-dimensional vector whose components lie in the interval $[0,1]$. The triple $(R_{M(n)}, [0,1]^n, \psi)$ is a *derived scale*. A derived scale is a mapping between two numerical relational systems.

It is important to point out that the rank order implied by the derived scale ψ may not coincide with the order represented by the pairwise comparisons. Let $\psi_i(A)$ be the ith component of $\psi(A)$. It denotes the relative dominance of the ith alternative. By definition, for $A_i, A_j \in \mathfrak{A}$, $A_i >_C A_j$ implies $P_C(A_i, A_j) > 1$. However, if $P_C(A_i, A_j) > 1$, the derived scale could imply that $\psi_j(A) > \psi_i(A)$. This occurs if row dominance does not hold, i.e., for $A_i, A_j \in \mathfrak{A}$ and $C \in \mathfrak{C}$, $P_C(A_i, A_k) \geqslant P_C(A_j, A_k)$ does not hold for all $A_k \in \mathfrak{A}$. In other words, it may happen that $P_C(A_i, A_j) > 1$, and for some $A_k \in \mathfrak{A}$ we have

$$P_C(A_i, A_k) < P_C(A_j, A_k).$$

A more restrictive condition is the following:

DEFINITION. The mapping P_C is said to be *consistent* if and only if

$$P_C(A_i, A_j) P_C(A_j, A_k) = P_C(A_i, A_k) \qquad \text{for all} \quad i, j, \text{and } k. \tag{1}$$

Similarly the matrix A is consistent if and only if $a_{ij} a_{jk} = a_{ik}$ for all i, j and k.

If P_C is consistent, then Axiom 1 automatically follows and the rank order induced by ψ coincides with pairwise comparisons.

Hierarchic Axioms

DEFINITION. A *partially ordered set* is a set S with a binary relation $\leqslant$ which satisfies the following conditions:

(a) Reflexive: For all $x \in S$, $x \leqslant x$,

(b) Transitive: For all $x, y, z \in S$, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$,

(c) Antisymmetric: For all $x, y \in S$, if $x \leqslant y$ and $y \leqslant x$ then $x = y$ (x and y coincide).

DEFINITION. For any relation $x \leqslant y$ (read, y includes x) we define $x < y$ to mean that $x \leqslant y$ and $x \neq y$. y is said to *cover* (*dominate*) x if $x < y$ and if $x < t < y$ is possible for no t.

Partially ordered sets with a finite number of elements can be conveniently represented by a directed graph. Each element of the set is represented by a vertex so that an arc is directed from y to x if $x < y$.

DEFINITION. A subset E of a partially ordered set S is said to be *bounded* from above (below) if there is an element $s \in S$ such that $x \leqslant s$ ($\geqslant s$) for every $x \in E$. The element s is called an upper (lower) bound of E. We say that E has a supremum (infimum) if it has upper (lower) bounds and if the set of upper (lower) bounds U (L) has an element u_1 (l_1) such that $u_1 \leqslant u$ for all $u \in U$ ($l_1 \geqslant l$ for all $l \in L$).

DEFINITION. Let $\mathfrak{H}$ be a finite partially ordered set with largest element b. $\mathfrak{H}$ is a *hierarchy* if it satisfies the conditions:

(1) There is a partition of $\mathfrak{H}$ into sets called levels $\{L_k,\ k = 1, 2, \ldots, h\}$, where $L_1 = \{b\}$.

(2) $x \in L_k$ implies $x^- \subseteq L_{k+1}$, where $x^- = \{y \mid x \text{ covers } y\}$, $k = 1, 2, \ldots, h-1$.

(3) $x \in L_k$ implies $x^+ \subseteq L_{k-1}$, where $x^+ = \{y \mid y \text{ covers } x\}$, $k = 2, 3, \ldots, h$.

DEFINITION. Given a positive real number $\rho \geqslant 1$ a nonempty set $x^- \subseteq L_{k+1}$ is said to be ρ-homogeneous with respect to $x \in L_k$ if for every pair of elements $y_1, y_2 \in x^-$, $1/\rho \leqslant P_C(y_1, y_2) \leqslant \rho$. In particular the reciprocal axiom implies that $P_C(y_i, y_i) = 1$.

Axiom 2. Given a hierarchy $\mathfrak{H}$, $x \in \mathfrak{H}$ and $x \in L_k$, $x^- \subseteq L_{k+1}$ is ρ-homogeneous for $k = 1, \ldots, h-1$.

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Homogeneity is essential for comparing similar things, as the mind tends to make large errors in comparing widely disparate elements. For example we cannot compare a grain of sand with an orange according to size. When the disparity is great, the elements are placed in separate clusters of comparable size giving rise to the idea of levels and their decomposition. This axiom is closely related to the well-known Archimedean property.

The notions of fundamental and derived scales can be extended to $x \in L_k$, $x^- \subseteq L_{k+1}$ replacing C and $\mathfrak{A}$ respectively. The derived scale resulting from comparing the elements in x^- with respect to x is called a *local derived scale* or *local priorities.* Here no irrelevant alternative is included in the comparisons and such alternatives are assumed to receive the value of zero in the derived scale.

Given L_k, $L_{k+1} \subseteq \mathfrak{H}$, let us denote the local derived scale for $y \in x^-$ and $x \in L_k$ by $\psi_{k+1}(y/x)$, $k = 2, 3, \ldots, h-1$. Without loss of generality we may assume that $\sum_{y \in x^-} \psi_{k+1}(y/x) = 1$. Consider the matrix $\psi_k(L_k/L_{k-1})$ whose columns are local derived scales of elements in L_k with respect to elements in L_{k-1}.

Definition. A set $\mathfrak{A}$ is said to be *outer dependent* on a set $\mathfrak{C}$ if a fundamental scale can be defined on $\mathfrak{A}$ with respect to every $c \in \mathfrak{C}$.

Decomposition implies containment of the small elements by the large clusters or levels. In turn, this means that the smaller elements depend on the outer parent elements to which they belong, which themselves fall in a large cluster of the hierarchy. The process of relating elements (e.g., alternatives) in one level of the hierarchy according to the elements of the next higher level (e.g., criteria) expresses the dependence of the lower elements on the higher so that comparisons can be made between them. The steps are repeated upward in the hierarchy through each pair of adjacent levels to the top element, the focus or goal.

The elements in a level may depend on one another with respect to a property in another level. Input-output dependence of industries is an example of the idea of inner dependence. This may be formalized as follows:

Definition. Let $\mathfrak{A}$ be outer dependent on $\mathfrak{C}$. The elements in $\mathfrak{A}$ are said to be *inner dependent* with respect to $C \in \mathfrak{C}$ if for some $A \in \mathfrak{A}$, $\mathfrak{A}$ is outer dependent on A.

Axiom 3. Let $\mathfrak{H}$ be a hierarchy with levels $L_1, L_2, \ldots, L_h$. For each L_k, $k = 1, 2, \ldots, h-1$,

(1) L_{k+1} is outer dependent on L_k,

(2) L_{k+1} is not inner dependent with respect to all $x \in L_k$,

(3) L_k is not outer dependent on L_{k+1}.

Principle of Hierarchic Composition. If Axiom 3 holds, the global derived scale (rank order) of any element in $\mathfrak{H}$ is obtained from its component in the corresponding vector of the following:

$$\begin{aligned} &\psi_1(b) = 1,\\ &\psi_2(L_2) = \psi_2(b^-/b),\\ &\quad\vdots \qquad\quad \vdots\\ &\psi_k(L_k) = \psi_k(L_k/L_{k-1})\psi_{k-1}(L_{k-1}), \qquad k = 3, \ldots, h. \end{aligned}$$

Were one to omit Axiom 3, the Principle of Hierarchic Composition would no longer apply because of outer and inner dependence among levels or components which need not form a hierarchy. The appropriate composition principle is derived from the supermatrix approach of which the Principle of Hierarchic Composition is a special case (Saaty 1980).

A hierarchy is a special case of a system, the definition of which is given by:

Definition. Let $\mathfrak{S}$ be a family of nonempty sets $\mathfrak{C}_1, \mathfrak{C}_2, \ldots, \mathfrak{C}_n$, where $\mathfrak{C}_i$ consists of the elements $\{e_{ij}, j = 1, \ldots, m_i\}$, $i = 1, 2, \ldots, n$. $\mathfrak{S}$ is a system if

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(i) It is a directed graph whose vertices are $\mathfrak{C}_i$ and whose arcs are defined through the concept of outer dependence; thus

(ii) Given two components $\mathfrak{C}_i$ and $\mathfrak{C}_j \in \mathfrak{S}$ there is an arc from $\mathfrak{C}_i$ to $\mathfrak{C}_j$ if $\mathfrak{C}_j$ is outer dependent on $\mathfrak{C}_i$.

Therefore, many of the concepts derived for hierarchies also relate to general systems with feedback. Here one needs to characterize dependence among the elements. We now give a criterion for this purpose.

Let $D_A \subseteq \mathfrak{A}$ be the set of elements of $\mathfrak{A}$ outer dependent on $A \in \mathfrak{A}$. Let $\psi_{A_i, C}(A_j)$, $A_j \in \mathfrak{A}$ be the derived scale of the elements of $\mathfrak{A}$ with respect to $A_i \in \mathfrak{A}$ for a criterion $C \in \mathfrak{C}$. Let $\psi_C(A_j)$, $A_j \in \mathfrak{A}$ be the derived scale of the elements of $\mathfrak{A}$ with respect to a criterion $C \in \mathfrak{C}$. We define the dependence weight

$$\phi_C(A_j) = \sum_{A_i \in D_{A_j}} \psi_{A_i,C}(A_j)\psi_C(A_i).$$

If the elements of $\mathfrak{A}$ are inner dependent with respect to $C \in \mathfrak{C}$, then $\phi_C(A_j) \neq \psi_C(A_j)$ for some $A_j \in \mathfrak{A}$.

Expectations are beliefs about the rank of alternatives derived from prior knowledge. Assume that a decision maker has a ranking, arrived at intuitively, of a finite set of alternatives $\mathfrak{A}$ with respect to prior knowledge of criteria $\mathfrak{C}$. He may have expectations about rank order.

Axiom 4 (Expectations).

$$\mathfrak{C} \subset \mathfrak{H} - L_h, \qquad \mathfrak{A} = L_h.$$

This axiom simply says that those thoughtful individuals who have reasons for their beliefs should make sure that their ideas are adequately represented for the outcome to match these expectations; i.e., all alternatives are represented in the hierarchy, as well as all criteria. It neither assumes rationality of the process nor that it can only accommodate a rational outlook. People have many expectations that are irrational.

3. Results from the Axioms

Note that if P_C is consistent, then Axiom 1 follows, i.e., consistency implies the reciprocal property. The first few theorems are based on this more restrictive property of consistency.

The theorems show that paired comparisons and the principal eigenvector are useful in estimating ratios. We use perturbation arguments to demonstrate that the principal eigenvector solution is the appropriate one to surface rank order from inconsistent data and that the eigenvector is stable to small perturbations in the data. These results are also obtained by means of graph theoretic arguments.

Let $R_{C(n)} \subset R_{M(n)}$ be the set of all $(n \times n)$ consistent matrices.

THEOREM 1. *Let $A \in R_{M(n)}$. $A \in R_{C(n)}$ if and only if* $\operatorname{rank}(A) = 1$.

PROOF. If $A \in R_{C(n)}$, then $a_{ij}a_{jk} = a_{ik}$ for all i, j and k. Hence, given a row of A, $a_{i1}, a_{i2}, \dots, a_{in}$, all other rows can be obtained from it by means of the relation $a_{jk} = a_{ik}/a_{ij}$ and $\operatorname{rank}(A) = 1$.

Let us now assume that $\operatorname{rank}(A) = 1$. Given a row a_{jh} $(j \neq i,\ h = 1, 2, \dots, n)$, $a_{jh} = Ma_{ih}$ $(h = 1, 2, \dots, n)$ where M is a positive constant. Also, for any reciprocal matrix, $a_{ii} = 1$ $(i = 1, 2, \dots, n)$. Thus, for $i = h$ we have $a_{ji} = Ma_{ii} = M$ and $a_{jh} = a_{ji}a_{ih}$ for all i, j and k, and A is consistent.

THEOREM 2. *Let $A \in R_{M(n)}$. $A \in R_{C(n)}$ if and only if its principal eigenvalue $\lambda_{\max}$ is equal to n.*

PROOF. By Theorem 1 we have $\operatorname{rank}(A) = 1$.

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Also, all eigenvalues of A but one vanish. Since $\text{Trace}(A) = \sum_{i=1}^{n} a_{ii} = n$ and $\text{Trace}(A) = \sum_k \lambda_k = n$, then $\lambda_{\max} \equiv \lambda_1 = n$.

If $\lambda_{\max} = n$,

$$n\lambda_{\max} = \sum_{i,j=1}^{n} a_{ij} w_j w_i^{-1} = n + \sum_{1 \leqslant i < j \leqslant n} \left(a_{ij} w_j w_i^{-1} + a_{ji} w_i w_j^{-1} \right)$$

$$\equiv n + \sum_{1 \leqslant i \leqslant j \leqslant n} \left(y_{ij} + 1/y_{ij} \right).$$

Since $y_{ij} + y_{ij}^{-1} \geqslant 2$, and $n\lambda_{\max} = n^2$, equality is uniquely obtained on putting $y_{ij} = 1$, i.e., $a_{ij} = w_i/w_j$. The condition $a_{ij}a_{jk} = a_{ik}$ holds for all i, j and k, and the result follows.

THEOREM 3. *Let* $A = (a_{ij}) \in R_{C(n)}$. *There exists a function* $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$, $\psi : R_{C(n)} \to [0,1]^n$ *such that*

(i) $a_{ij} = \psi_i(A)/\psi_j(A)$,

(ii) *The relative dominance of the ith alternative*, $\psi_i(A)$, *is the ith component of the principal right eigenvector of* A,

(iii) *Given two alternatives* $A_i, A_j \in \mathfrak{A}$, $A_i \gtrsim_C A_j$ *if and only if* $\psi_i(A) \geqslant \psi_j(A)$.

PROOF. $A \in R_{C(n)}$ implies that $a_{ij} = a_{ik}a_{jk}^{-1}$ for all k, and each i and j. Also by Theorem 1, we have $\text{rank}(A) = 1$ and we can write $a_{ij} = x_i/x_j$, where $x_i, x_j > 0$ $(i, j = 1, 2, \ldots, n)$. Multiplying A by the vector $x^T = (x_1, x_2, \ldots, x_n)$ we have $Ax = nx$. Dividing both sides of this expression by $\sum_{i=1}^{n} x_i$ and writing $w = x/\sum_{i=1}^{n} x_i$ we have $Aw = nw$, and $\sum_{i=1}^{n} w_i = 1$. By Theorem 2 we have n as the largest positive real eigenvalue of A and w as its corresponding right eigenvector. Since $a_{ij} = x_i/x_j = w_i/w_j$ for all i and j, we have $\psi_i(A) = w_i$, $i = 1, 2, \ldots, n$ and (i) and (ii) follow.

By Axiom 1, for $A \in R_C(n)$, $A_i \gtrsim_C A_j$ if and only if $a_{ij} \geqslant 1$ for all i and j, and hence we have $\psi_i(A) \geqslant \psi_j(A)$ for all i and j.

It is unnecessary to invoke the Perron–Frobenius Theory to ensure the existence and uniqueness of a largest positive real eigenvalue and its eigenvector. We have already proved the existence of an essentially unique solution in the consistent case. A similar result follows using the perturbation argument given below.

THEOREM 4. *Let* $A \in R_{C(n)}$, *and let* $\lambda_1 = n$ *and* $\lambda_2 = 0$ *be the eigenvalues of* A *with multiplicity* 1 *and* $(n-1)$, *respectively. Given* $\epsilon > 0$, *there is a* $\delta = \delta(\epsilon) > 0$ *such that if*

$$|a_{ij} + \tau_{ij} - a_{ij}| = |\tau_{ij}| \leqslant \delta \qquad \textit{for} \quad i, j = 1, 2, \ldots, n,$$

the matrix $B = (a_{ij} + \tau_{ij})$ *has exactly* 1 *and* $(n-1)$ *eigenvalues in the circles* $|\mu - n| < \epsilon$ *and* $|\mu - 0| < \epsilon$, *respectively.*

PROOF. Let $\epsilon_0 = \frac{1}{2}(n)$, and let $\epsilon < n/2$. The circles C_1: $|\mu - n| = \epsilon$ and C_2: $|\mu - 0| = \epsilon$ are disjoint. Let $f(\mu, A)$ be the characteristic polynomial of A. Let $r_j = \min|f(\mu, A)|$ for μ on C_j. Note that $\min|f(\mu, A)|$ is defined because f is a continuous function of μ, and $r_j > 0$ since the roots of $f(\mu, A) = 0$ are the centers of the circles.

$f(\mu, B)$ is a continuous function of the $1 + n^2$ variables μ and $a_{ij} + \tau_{ij}$, $i, j = 1, 2, \ldots, n$, and for some $\delta > 0$, $f(\mu, B) \neq 0$ for μ on any C_j, $j = 1, 2$, if $|\tau_{ij}| \leqslant \delta$, $i, j = 1, 2, \ldots, n$.

From the theory of functions of a complex variable, the number of roots μ of $f(\mu, B) = 0$ which lie inside C_j, $j = 1, 2$, is given by

$$n_j(B) = \frac{1}{2\pi i} \int_{C_j} \frac{f'(\mu, B)}{f(\mu, B)} \, d\mu, \qquad j = 1, 2,$$

which is also a continuous function of the n^2 variables $a_{ij} + \tau_{ij}$ with $|\tau_{ij}| \leqslant \delta$.

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For $B = A$, we have $n_1(A) = 1$ and $n_2(A) = n - 1$. Since $n_j(B)$, $j = 1,2$, is continuous, it cannot jump from $n_j(A)$ to $n_j(B)$ and the two must be equal and have the value $n_1(B) = 1$ and $n_2(B) = n - 1$, for all B with $|a_{ij} + \tau_{ij} - a_{ij}| \leqslant \delta$, $i, j = 1,2,\ldots,n$.

THEOREM 5. *Let $A \in R_{C(n)}$ and let w be its principal right eigenvector. Let $\Delta A = (\delta_{ij})$ be a matrix of perturbations of the entries of A such that $A' = A + \Delta A \in R_{M(n)}$, and let w' be its principal right eigenvector. Given $\epsilon > 0$, there exists a $\delta > 0$ such that $|\delta_{ij}| \leqslant \delta$ for all i and j, then $|w_i' - w_i| \leqslant \epsilon$ for all $i = 1,2,\ldots,n$.*

PROOF. By Theorem 4, given $\epsilon > 0$, there exists a $\delta > 0$ such that if $|\delta_{ij}| \leqslant \delta$ for all i and j, the principal eigenvalue of A' satisfies $|\lambda_{\max} - n| \leqslant \epsilon$. Let $\Delta A = \tau B$. Wilkinson (1965) has shown that for a sufficiently small τ, $\lambda_{\max}$ can be given by a convergent power series $\lambda_{\max} = n + k_1\tau + k_2\tau^2 + \cdots$. Now, $\lambda_{\max} \to n$ as $\tau \to 0$, and $|\lambda_{\max} - n| = o(\tau) \leqslant \epsilon$.

Let w be the right eigenvector corresponding to the simple eigenvalue n of A. Since n is a simple eigenvalue, $(A - nI)$ has at least one nonvanishing minor of order $(n-1)$. Suppose, without loss of generality, that this lies in the first $(n-1)$ rows of $(A - nI)$. Then from the theory of linear equations, the components of w may be taken to be $(A_{n1}, A_{n2}, \ldots, A_{nn})$ where A_{ni} denotes the cofactor of the (n,i) element of $(A - nI)$, and is a polynomial in n of degree not greater than $(n-1)$.

The components of w' are polynomials in $\lambda_{\max}$ and τ, and since the power series expansion of $\lambda_{\max}$ is convergent for all sufficiently small τ, each component of w' is represented by a convergent power series in τ. We have

$$w' = w + \tau z_1 + \tau^2 z_2 + \cdots \quad \text{and} \quad |w' - w| = o(\tau) \leqslant \epsilon.$$

By Theorems 4 and 5, it follows that a small perturbation A' of A transforms the eigenvalue problem $(A - nI)w = 0$ to $(A' - \lambda_{\max}I)w' = 0$.

THEOREM 6 (Ratio Estimation). *Let $A \in R_{M(n)}$, and let w be its principal right eigenvector. Let $\epsilon_{ij} = a_{ij}w_jw_i^{-1}$, for all i and j, and let $1 - \tau < \epsilon_{ij} < 1 + \tau$, $\tau > 0$, for all i and j. Given $\epsilon > 0$ and $\tau < \epsilon$, there exists a $\delta > 0$ such that for all $(x_1, x_2, \ldots, x_n)$, $x_i > 0$, $i = 1,2,\ldots,n$, if*

$$1 - \delta < \frac{a_{ij}}{x_i/x_j} < 1 + \delta \qquad \textit{for all } i \textit{ and } j, \tag{2}$$

then

$$1 - \epsilon < \frac{w_i/w_j}{x_i/x_j} < 1 + \epsilon \qquad \textit{for all } i \textit{ and } j. \tag{3}$$

PROOF. Substituting $a_{ij}\epsilon_{ij}^{-1}$ for w_i/w_j in (3) we have

$$\left|\frac{w_i/w_j}{x_i/x_j} - 1\right| = \left|\frac{1}{\epsilon_{ij}}\,\frac{a_{ij}}{x_i/x_j} - 1\right| \leqslant \frac{1}{\epsilon_{ij}}\left|\frac{a_{ij}}{x_i/x_j} - 1\right| + \left|\frac{1}{\epsilon_{ij}} - 1\right|.$$

By definition $\epsilon_{ij} = 1/\epsilon_{ji}$ for all i and j, and we have

$$\left|\frac{w_i/w_j}{x_i/x_j} - 1\right| = \epsilon_{ji}\left|\frac{a_{ij}}{x_i/x_j} - 1\right| + |\epsilon_{ji} - 1| < (1 + \tau)\delta + \tau.$$

Given $\epsilon > 0$ and $0 < \tau < \epsilon$, there exists a $\delta = (\epsilon - \tau)/(1 + \tau) > 0$ such that (2) implies (3).

This theorem says that if the paired comparison coefficient a_{ij} is close to an underlying ratio x_i/x_j then so is w_i/w_j and may be used as an approximation for it.

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THEOREM 7. *Let $A = (a_{ij}) \in R_{M(n)}$. Let $\lambda_{\max}$ be its principal eigenvalue and let w be its corresponding right eigenvector with $\sum_{i=1}^{n} w_i = 1$, then $\lambda_{\max} \geqslant n$.*

PROOF. Let $a_{ij} = w_j w_i^{-1} \epsilon_{ij}$, $i, j = 1, 2, \ldots, n$. Since $Aw = \lambda_{\max} w$, and $\sum_{i,j=1}^{n} a_{ij} w_j = \lambda_{\max}$, we have

$$\lambda_{\max} - n = \sum_{i,j=1}^{n} a_{ij} w_j - n = \sum_{i,j} \epsilon_{ij} - n.$$

By definition, the matrix $(\epsilon_{ij}) \in R_{M(n)}$. We have $\epsilon_{ii} = 1$ for all i, and $\epsilon_{ij} > 0$ for all i and j. Hence, we have $\sum_{i,j=1}^{n} \epsilon_{ij} - n = \sum_{i \neq j} \epsilon_{ij} > 0$ and the result follows.

THEOREM 8. *Let $A \in R_{M(n)}$. Let $\lambda_{\max}$ be the principal eigenvector of A, and let w be its corresponding right eigenvector with $\sum_{i=1}^{n} w_i = 1$. $\mu \equiv (\lambda_{\max} - n)/(n-1)$ is a measure of the average departure from consistency.*

PROOF. For $A \in R_{C(n)} \subset R_{M(n)}$, *by Theorem* 2 *we have $\lambda_{max} = n$, and hence, we have* $\mu = 0$.

For $A \in R_{M(n)} - R_{C(n)}$, let $a_{ij} = w_i \epsilon_{ij} / w_j$ for all i and j. We have

$$\lambda_{\max} = \sum_{j=1}^{n} a_{ij} \frac{w_j}{w_i} = \sum_{j=1}^{n} \epsilon_{ij},$$

$$n\lambda_{\max} = \sum_{i,j=1}^{n} \epsilon_{ij} = n + \sum_{1 \leqslant i < j \leqslant n} \left(\epsilon_{ij} + \frac{1}{\epsilon_{ij}} \right),$$

$$\frac{\lambda_{\max} - n}{n-1} = -1 + \frac{1}{n(n-1)} \sum_{1 \leqslant i < j \leqslant n} \left(\epsilon_{ij} + \frac{1}{\epsilon_{ij}} \right).$$

As $\epsilon_{ij} \to 1$, i.e., consistency is approached, $\mu \to 0$. Also, μ is convex in ϵ_{ij}, since $(\epsilon_{ij} + 1/\epsilon_{ij})$ is convex, and has its minimum at $\epsilon_{ij} = 1$, $i, j = 1, 2, \ldots, n$. Thus, μ is small or large depending on ϵ_{ij} being near to or far from unity, respectively, i.e., near to or far from consistency, and the result follows.

Note that $\sum_{i,j=1}^{n} a_{ij} w_j w_i^{-1} - n^2 = n(n-1)\mu$ is also a measure of the departure from consistency.

It is also possible to show that $(A - nI)w = 0$ is transformed into $(A' - \lambda_{\max} I)w' = 0$ by means of graph theoretic concepts.

DEFINITION. The intensity of judgments associated with a path from i to j called the *path intensity* is equal to the products of the intensities associated with the arcs of that path.

DEFINITION. A *cycle* is a path of pairwise comparisons which terminates at its starting point.

THEOREM 9. *If $A \in R_{C(n)}$, the intensities of all cycles are equal to a_{ii}, $i = 1, 2, \ldots, n$.*

PROOF. $A \in R_{C(n)}$, implies $a_{ij} a_{jk} = a_{ik}$ for all i, j and k. Hence, we have $a_{ii} = a_{ij} a_{jk} a_{ki} = 1$ for all $i = 1, 2, \ldots, n$. By induction, if $a_{ii_1} \ldots a_{i_{n-1} i} = 1$ for all $i_1 \ldots i_{n-1}$, then $a_{ii_1} \ldots a_{i_{n-1} i_n} a_{i_n i} = a_{ii_n} a_{i_n i} = 1$ and the result follows.

THEOREM 10. *If $A \in R_{C(n)}$, the intensities of all paths from i to j are equal to a_{ij}.*

PROOF. Follows from $a_{ij} = a_{ik} a_{kj}$ for all i, j and k.

COROLLARY 1. *If $A \in R_{C(n)}$, the entry in the (i, j) position can be represented as the intensity of paths of any length starting with i and terminating with j.*

PROOF. Follows from the proof of Theorem 10.

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COROLLARY 2. *If $A \in R_{C(n)}$, the entry in the (i, j) position is the average intensity of paths of length k from i to j, and $A^k = n^{k-1}A$ $(k \geqslant 1)$.*

PROOF. From Theorem 10, the intensity of a path of any length from i to j is equal to a_{ij}.

An arbitrary entry of A^k is given by

$$a_{ij}^{(k)} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1}j}.$$

Since $a_{ij}a_{jk} = a_{ik}$ for all i, j and k we have

$$a_{ij}^{(k)} = \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} a_{ij} = n^{k-1} a_{ij}.$$

By induction, if $a_{ij}^{(k)} = n^{k-1}a_{ij}$ for $k = 1, 2, \ldots, m-1$, for $k = m$ we have

$$\begin{aligned} a_{ij}^{(m)} &= \sum_{i_1=1}^{n} \cdots \sum_{i_{m-1}=1}^{n} a_{ii_1} \cdots a_{i_{m-1}j} \\ &= n^{m-2} \sum_{i_{m-1}=1}^{n} a_{ii_{m-1}} a_{i_{m-1}j} = n^{m-1} a_{ij}. \end{aligned}$$

Hence, we have

$$a_{ij} = \frac{1}{n^{m-1}} a_{ij}^{(m)} \qquad \text{for all} \quad m \geqslant 1,$$

and the result follows.

THEOREM 11. *If $A \in R_{C(n)}$ the entry in the (i, j) position is given by the average of all path intensities starting with i and terminating with j.*

PROOF. By Corollary 2 of Theorem 10, we have

$$a_{ij} = \frac{1}{n^{m-1}} \sum_{i_1=1}^{n} \cdots \sum_{i_{m-1}=1}^{n} a_{ii_1} \cdots a_{i_{m-1}j}.$$

Hence, we have

$$a_{ij} = \lim_{m \to \infty} \frac{1}{n^{m-1}} a_{ij}^{(m)},$$

and the result follows.

THEOREM 12. *If $A \in R_{C(n)}$ the scale of relative dominance is given by any of its normalized columns, and coincides with the principal right eigenvector of A.*

PROOF. Let a^j be the jth column of A.

$$\begin{aligned} A \cdot a^j &= \left(\sum_{k=1}^{n} a_{ik} a_{kj} \right) \qquad (i, j = 1, 2, \ldots, n), \\ &= \left(\sum_{k=1}^{n} a_{ij} \right) = (n a_{ij}) \qquad (i, j = 1, 2, \ldots, n), \end{aligned}$$

and any column of A (whether or not it is normalized to unity) is a solution of the eigenvalue problem $Ax = nx$. By Corollary 2 of Theorem 10 we have $A^k = n^{k-1}A$. We have

$$\psi(A) = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{n} \frac{A^k e}{e^T A^k e} = \lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^{n} \frac{Ae}{e^T Ae} = \frac{Ae}{e^T Ae}.$$

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Hence, we have

$$\psi_i(A) = \sum_{j=1}^{n} a_{ij} \Big/ \sum_{i,j=1}^{n} a_{ij} = a_{ih}\left(\sum_{j=1}^{n} a_{hj}\right) \Big/ \left(\sum_{i=1}^{n} a_{ih}\right)\left(\sum_{j=1}^{n} a_{hj}\right) = \frac{a_{ih}}{\sum_{i=1}^{n} a_{ih}}$$

for all i and h, and the result follows.

COROLLARY. *The principal eigenvector is unique to within a multiplicative constant.*

PROOF. Follows from the proof of Theorem 12.

THEOREM 13. *If $A \in R_{M(n)}$ the intensity of all paths of length k from i to j is given by*

$$\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}.$$

PROOF. It is known that the number of arc progressions of length n between any two vertices of a directed graph whose incidence matrix is V is given by V^n. If in addition each arc has associated a number ($\neq 1$) representing the intensity (or capacity) of the arc, then V^n represents the intensity of all arc progressions of length n between two vertices.

Let $V = A$. The entries of A^k give the intensity of all paths of length k between two vertices. Let $A^k = (a_{ij}^{(k)})$. By construction we have

$$a_{ij}^{(k)} = \sum_{i_1=1}^{n} \cdots \sum_{i_{k-1}=1}^{n} a_{ii_1} \cdots a_{i_{k-1} j}$$

and the result follows.

THEOREM 14. *Let $A \in R_{M(n)}$, $A \notin R_{C(n)}$. The principal right eigenvector of A is given by the limit of the normalized intensity of paths of length k,*

$$w_i = \lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}}, \qquad i = 1, 2, \ldots, n,$$

for all $h = 1, 2, \ldots, n$.

PROOF. It can be shown that

$$\lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} = \lim_{k\to\infty} \frac{a_{is}^{(k)}}{\sum_{i=1}^{n} a_{is}^{(k)}}, \qquad h, s = 1, 2, \ldots, n. \tag{4}$$

The proof of this statement is given in Saaty and Vargas (1984b). Also we know that the principal right eigenvector of A is given by

$$w_i' = \lim_{k\to\infty} \frac{\sum_{h=1}^{n} a_{ih}^{(k)}}{\sum_{i=1}^{n}\sum_{h=1}^{n} a_{ih}^{(k)}}, \qquad i = 1, 2, \ldots, n. \tag{5}$$

Multiplying and dividing the right side of (5) inside the limit by $\sum_{i=1}^{n} a_{ih}^{(k)}$ and rearranging the terms we have

$$\begin{aligned} w_i' &= \lim_{k\to\infty} \left[\sum_{h=1}^{n} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} \cdot \frac{\sum_{i=1}^{n} a_{ih}^{(k)}}{\sum_{i=1}^{n}\sum_{h=1}^{n} a_{ih}^{(k)}} \right] \\ &= \sum_{h=1}^{n} \left[\lim_{k\to\infty} \frac{a_{ih}^{(k)}}{\sum_{i=1}^{n} a_{ih}^{(k)}} \right] \left[\lim_{k\to\infty} \frac{\sum_{i=1}^{n} a_{ih}^{(k)}}{\sum_{i,h=1}^{n} a_{ih}^{(k)}} \right] \end{aligned}$$

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From (5) we have

$$w_i' = \left[\lim_{k\to\infty} \frac{a_{is}^{(k)}}{\sum_{i=1}^n a_{is}} \right] \sum_{h=1}^n \frac{\sum_{i=1}^n a_{ih}^{(k)}}{\sum_{i,h=1}^n a_{ih}^{(k)}}$$

and the result follows.

COROLLARY. *Let $A \in R_{M(n)}$, $A \notin R_{C(n)}$. The principal right eigenvector of A is unique to within a multiplicative constant.*

PROOF. Follows from the proof of Theorem 14, and Theorem 5 in Saaty (1980).

THEOREM 15. *Let $\mathfrak{A}$ be a finite set of n elements $A_1, A_2, \ldots, A_n$, and let $C \in \mathfrak{C}$ be a criterion which all the elements in $\mathfrak{A}$ have in common. Let A be the resulting matrix of pairwise comparisons. The ith component of the principal right eigenvector of the reciprocal pairwise comparison matrix A gives the relative dominance of A_i, $i = 1,2, \ldots, n$.*

PROOF. By Theorem 14, the principal right eigenvector of A is given by

$$w_i = \lim_{m\to\infty} \frac{a_{ih}^{(m)}}{\sum_{j=1}^n a_{jh}^{(m)}}, \qquad i = 1,2, \ldots, n,$$

for any $h = 1,2, \ldots, n$. By Theorem 7.13 in Saaty (1980) we have

$$w_i = \lim_{m\to\infty} \frac{\sum_{j=1}^n a_{ij}^{(m)}}{\sum_{i,j=1}^n a_{ij}^{(m)}}, \qquad i = 1,2, \ldots, n.$$

Thus, the relative dominance of an alternative along all paths of length $k \leqslant m$ is given by

$$\frac{1}{m} \sum_{k=1}^m \frac{a_{ih}^{(k)}}{\sum_{i=1}^n a_{ih}^{(k)}}.$$

Let

$$s_k = \frac{a_{ih}^{(k)}}{\sum_{i=1}^n a_{ih}^{(k)}} \quad \text{and} \quad t_m = \frac{1}{m} \sum_{k=1}^m s_k.$$

It can be shown that if $\lim_{k\to\infty} s_k$ exists then $\lim_{m\to\infty} t_m$ also exists and the two limits coincide. By Theorem 14, we have $s_k \to w$ as $k \to \infty$, where w is the principal right eigenvector of A. Thus $t_m \to w$ as $m \to \infty$ and $\psi_i(A) = w_i$, $i = 1,2, \ldots, n$.

This theorem highlights the fact that the right eigenvector gives the relative dominance (rank order) of each alternative over the other alternatives along paths of arbitrary length. It holds for a reciprocal matrix A which need not be consistent.

4. Relative and Absolute Measurement-Rank Preservation

The AHP can be used to make relative measurement through paired comparisons (scaling) of criteria and of alternatives, or to make absolute measurement (scoring) of the alternatives with respect to the criteria. The former is now familiar. The latter has been used when the number of alternatives is large and the decision is standard such as admitting students to a college based on well-established criteria whose weights are not affected by the number of students and their scores.

When the AHP uses paired comparisons it assumes structural dependence of the criteria on the number of alternatives and on their priorities. As a result, when alternatives are scaled through paired comparisons, adding a new alternative can

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change the relative ranking of the old ones when the judgments are inconsistent or when several criteria are used. Under a single criterion rank never changes with the addition of a new alternative when the judgments are consistent (Saaty and Vargas 1984a). Note that if structural criteria are an integral part of a decision theory, the weights of these criteria would change with the introduction or deletion of alternatives and hence both the priorities and the ranks of the old alternatives can change. Thus structure is an important aspect of all systems and needs to be considered for better understanding of decisions. How to interpret such structural criteria has been covered in other works by this author now in process of publication.

If, in spite of structural dependence, for some practical reason one insists that the old rank remain in place and a new alternative be added, the new alternative can be measured by comparing it with one of the original ones and assigning it the appropriate value under each criterion without renormalizing. Normalization is then applied to the composite result. The priorities will change, but the ranking will be the same.

With absolute measurement there can be no rank reversal under a single or under multiple criteria. One compares the criteria, with respect to the goal, subcriteria with respect to the criteria and then the intensities of the subcriteria such as: excellent, very good, good, average, below average, poor, and very poor, with respect to each subcriterion. This yields a set of priorities for the intensities of the subcriteria. Each alternative is then scored with respect to each subcriterion by selecting the appropriate intensity. Once the weights of the intensities have been established the question of consistency in scoring the alternatives does not occur. Finally one adds all the priorities of the intensities to obtain a score for the alternative. In the end, these priorities may be normalized for all the alternatives.

5. Conclusion

We conclude with general remarks about the use of the AHP.

Because the AHP does not separate intangible factors from tangible ones and conducts its measurement by making pairwise comparisons, it is a useful way for analysis and decision making in complex social and political problems. In general, other methods such as multiattribute utility theory would first quantify individual intangible factors before calculating utility functions.

The AHP is also useful when many interests are involved and a number of people participate in the judgment process. Here debate may be to no avail and several answers must be developed. The results would then be weighted by the priority of the corresponding individuals according to that individual's relevance to the problem. These priorities are derived by extending the hierarchy upwards to include the individuals and criteria for evaluating them, with their assistance or participation when possible.

Judgments from different people on a single comparison must satisfy the reciprocal property for the group. This implies that these judgments must be synthesized into a single judgment according to the geometric mean (Aczel and Saaty 1983).

The AHP deals with problem decomposition in a systematic way. It requires that elements in each level be homogeneous, decreasing in size from the top to the bottom level of the hierarchy. While there is flexibility in structuring a problem, it is clear from the start that one proceeds by arranging the issues in decending (or ascending) order. It is also possible through the AHP to structure a problem which has dependencies and feedback to set priorities and make a choice.

Most of the difficulties encountered in using the AHP relate to the need for judgments. If a problem is complex and requires careful analysis, then time would be needed to elicit judgments. However, people can become tired and need to return to

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the process after some rest. The more complex problems have needed nearly two days for this kind of participation. Furthermore, the AHP calls for occasional repetition of the process to make sure that the participants have not changed their minds dramatically. Patrick Harker of Wharton has recently developed a procedure for shortening the judgmental process.

It should now be clear that designing the analytic hierarchy, like the structuring of a problem by any other method, necessitates a substantial knowledge of the system in question. A strong aspect of the AHP is that the knowledgeable individuals who supply judgments for the pairwise comparisons usually also play a prominent role in specifying the hierarchy. Another key aspect in structuring a hierarchy is that any element in a level can be compared with respect to some elements in the level immediately above. The hierarchy need not be complete; that is, an element at an upper level need not function as a criterion for all the elements in the lower level. It can be partitioned into nearly disjoint subhierarchies sharing only a common topmost element. Thus for instance, the activities of separate divisions of an organization can be structured separately. The analyst can insert and delete levels and elements as necessary to clarify the task or to sharpen the focus on one or more areas of the system.

The AHP has already been successfully applied in a variety of fields. These include: a plan to allocate energy to industries; designing a transport system for the Sudan; planning the future of a corporation and measuring the impact of environmental factors on its development; design of future scenarios for higher education in the United States; the candidacy and election processes; setting priorities for the top scientific institute in a developing country and the faculty promotion and tenure problem (Saaty 1982, Wind and Saaty 1980, Toné 1986). The use of the AHP has been facilitated greatly by the availability of the microcomputer software package Expert Choice (1985).

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