Expert Systems With Applications 183 (2021) 115427

Contents lists available at ScienceDirect

Expert Systems With Applications

journal homepage: www.elsevier.com/locate/eswa

An uncertainty-induced axiomatic foundation of the analytic hierarchy process and its implication ☆

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ARTICLE INFO

Keywords:
Decision analysis
Analytic hierarchy process (AHP)
Axiomatic foundation
Reciprocal symmetry breaking (RSB)
Uncertainty

ABSTRACT

Uncertainty is always encountered by decision makers (DMs) when addressing a complex decision-making problem. The existing axiomatic foundation of the analytic hierarchy process (AHP) cannot reflect the uncertainty experienced by DMs. In this paper, it is first to find that the uncertainty experienced by DMs can be characterized by the non-reciprocal property of pairwise comparisons. The novel concept of reciprocal symmetry breaking (RSB) is proposed to capture the situation without reciprocal property. Then an uncertainty-induced axiomatic foundation of the AHP model is reported, where the RSB is one of the proposed axioms. Some interesting results are derived from the new axioms involving the concept of approximate consistency and the method of eliciting priorities. An index is constructed to measure approximate consistency degree of pairwise comparison matrices (PCMs). Some comparisons with the typical AHP model are offered by carrying out some numerical examples. The obtained results reveal that the proposed axiomatic foundation and the derived facts form a novel operational basis of the AHP model under uncertainty.

1. Introduction

Since the analytic hierarchy process (AHP) was developed as a choice model by Saaty (1980, 1997), it has been studied and applied for more than forty years (Darko et al., 2019; Ho & Ma, 2018; Santos et al., 2019). In the AHP model, the decomposition principle is applied to construct a hierarchy of criteria, subcriteria and alternatives associated with a complex decision-making problem. The pairwise comparison technique of alternatives is used to construct a series of matrices. The priorities of alternatives are elicited from the obtained matrices by a synthesis mechanism; then the optimal solution is reached. One can see that the investigations into the theory and applications of the AHP have drawn great attention (Abastante et al., 2019; Ahn, 2017; Altuzarra et al., 2010; Bernasconi et al., 2010; Forman & Gass, 2001; Genest & Zhang, 1996; Hahn, 2003; Hocine & Kouaissah, 2020; Mastrocinque et al., 2020; Rezaei, 2015; Saaty, 2013; Saaty & Shang, 2011; Saaty & Vargas, 1998). The AHP model has been further extended to the analytic network process (ANP) (Alizadeh, Soltanisehat et al., 2020; Saaty, 2013). In particular, it has been shown that the decision-making procedure based on the AHP model is reliable according to the finding in Bernasconi et al. (2010), where the modern theory of subjective measurement was applied to measure the ratio scale.

It is worth noting that one of the important issues is the axiomatic foundation of the AHP model. In general, a comprehensive literature review is requisite such as those in the known works (Alizadeh, Beiragh et al., 2020; Soltanisehat et al., 2020). However, it is found that the axiomatic foundation of the AHP model has been only addressed by Saaty (1986). In the following, we mainly focus on the motivation of the present study and the detailed review on the existing axiomatic foundation is given in Section 2. One can see that the known axiomatic foundation of the AHP model contains four axioms (Saaty, 1986), where the basic one is the reciprocal property of pairwise comparisons. For a finite set of alternatives $\mathbb{X} = \{x_1, x_2, \ldots, x_n\}$, the reciprocal property means that when the relative importance of x_i over x_j is determined as a_{ij}, the comparison ratio of x_j over x_i is automatically obtained as $a_{ji} = 1/a_{ij}$ for $i, j \in \mathbb{I} = \{1, 2, \ldots, n\}$ (Bernasconi et al., 2010; Genest & Zhang, 1996; Saaty, 1997). It has been considered

☆ The work was supported by the National Natural Science Foundation of China (Nos. 71871072, 71571054), the Guangxi Natural Science Foundation for Distinguished Young Scholars, China (No. 2016GXNSFFA380004), 2017 Guangxi high school innovation team and outstanding scholars plan, China, and the Innovation Project of Guangxi Graduate Education, China (No. YCSW2021044).
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https://doi.org/10.1016/j.eswa.2021.115427
Received 7 January 2021; Received in revised form 9 June 2021; Accepted 10 June 2021
Available online 16 June 2021
0957-4174/

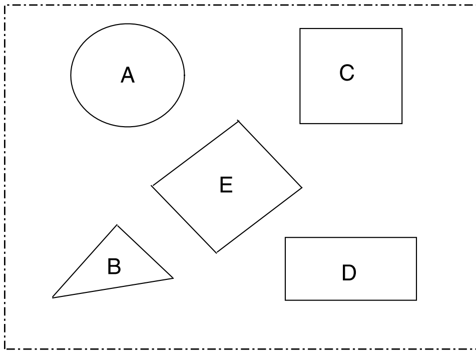

Fig. 1. Five figures with different areas.

that the reciprocal relation $a_{ji} = 1/a_{ij}$ is based on the mathematical intuition (Harker & Vargas, 1987). We cannot help asking whether the reciprocal relation $a_{ji} = 1/a_{ij}$ is really satisfied, when the comparison ratios a_{ij} and a_{ji} are separately evaluated. For example, as shown in Saaty (2013), we compare the areas of the five figures in Fig. 1 by eyeballing them. If the reciprocal property is not assumed in advance, it seems difficult to obtain the reciprocal relation of pairwise comparisons. The underlying reason could be attributed to the fact that various cognitive distortions of the DM could affect the evaluation of her/his opinions (Bernasconi et al., 2008; Luce, 2002; Narens, 1996). One can find that although the reciprocal property is in agreement with the mathematical intuition, it may be not always satisfied in a practical case. In other words, the reciprocal property is a strict mathematical relation under an ideal case, and there is some deviation from the flexible expression of human-originated information in a practical situation (Ahn, 2017; Hocine & Kouaissah, 2020; Saaty & Vargas, 1987). Furthermore, from the viewpoint of measuring comparison ratios, the value of a_{ij} can be computed using the formula as follows (Bernasconi et al., 2010):

$$a_{ij} = W^{-1}\left(\frac{\omega_i}{\omega_j}\right) \cdot e_{ij}, \tag{1}$$

where the terms ω_i and ω_j are the subjective weights of alternatives belonging to $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with $\sum_{i=1}^{n} \omega_i = 1$; W^{-1} is the inverse of a weighting function and e_{ij} is the perturbation term for capturing the random error or inconsistency. In order to perform the statistical analysis, Eq. (1) was transformed into the following regression model:

$$\ln(a_{ij}) = \ln\left\{W^{-1}\exp[\ln(\omega_i) - \ln(\omega_j)]\right\} + \ln(e_{ij}). \tag{2}$$

Then the determined function $\ln\left\{W^{-1}\exp(\mu)\right\}$ with $\mu = \ln(\omega_i) - \ln(\omega_j)$ could be approximated using a polynomial:

$$\ln\left\{W^{-1}\exp(\mu)\right\} \simeq \beta_0 + \beta_1\mu + \beta_2\mu^2 + \beta_3\mu^3, \tag{3}$$

where β_k $(k = 0, 1, 2, 3)$ are constants. It is found that the value of β_2 was chosen as zero under the assumption of reciprocal property (Bernasconi et al., 2010). Then when the reciprocal property is not assumed in advance, the value of β_2 should be chosen as a nonzero one. Hence, the regression model (2) and the approximation formula (3) are still feasible to measure comparison ratios without reciprocal property. In other words, similar to the method in Bernasconi et al. (2010), the comparison ratios without reciprocal property could be derived according to the modern theory of subjective measurement. Based on the above analysis, we have the findings as follows:

- The reciprocal property of pairwise comparisons in the AHP is only based on the mathematical intuition.
- The uncertainty experienced by DMs should lead to the non-reciprocal property of pairwise comparisons.
- The non-reciprocal pairwise comparisons could be evaluated by the modern theory of subjective measurement.

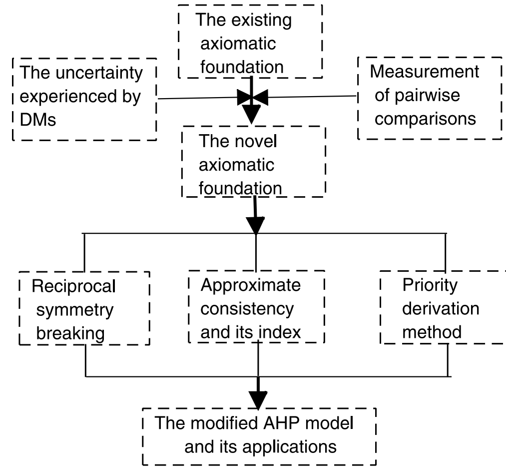

Fig. 2. The general research procedure of the present study.

- The existing axiomatic foundation of the AHP model cannot reflect the uncertainty experienced by DMs.

In addition, the uncertainty experienced by DMs has also been considered in Saaty and Vargas (1987), where interval-valued preference relations have been proposed. Different to the method in Saaty and Vargas (1987), here it is found that the uncertainty can be captured by relaxing the reciprocal property of pairwise comparisons. Motivated by these observations, we attempt to analyze the situation without reciprocal property to form a novel axiomatic foundation of the AHP model. The general research procedure of the present study is shown in Fig. 2. In addition, the contribution and structure of the paper are described as follows. In Section 2, the existing axiomatic foundation of the AHP model is reviewed in detail and the uncertainty-induced one is introduced. The novelty comes with the modified axioms and the concept of reciprocal symmetry breaking (RSB) of pairwise comparisons. Section 3 gives some interesting results derived from the modified axioms. Some important issues are addressed such as the properties of RSB, the approximate consistency of PCMs and the priority eliciting method of alternatives. Section 4 offers an index to measure the approximate consistency degree of pairwise comparison matrices (PCMs). In Section 5, a practical case of decision making is restudied and some comparisons are reported to show the difference from the typical AHP model. The main conclusions and some future research directions are covered in Section 6.

2. Axiomatic foundations of the AHP model

In this section, we first recall the only existing axioms of the AHP model proposed by Saaty (1986). Then it is an attempt to modify the basic one such that the uncertainty of human-originated information can be naturally captured. It is considered that there are a finite set of alternatives $\mathbb{X} = \{x_1, x_2, \ldots, x_n\}$ and a criterion C in a decision-making problem. The binary relation of alternatives with respect to C can be quantified by defining a mapping as follows:

$$P_C : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}^+, \tag{4}$$

where $\mathbb{R}^+$ stands for the set of positive real numbers. Then the fundamental scale in relative measurement satisfies the following rules (Saaty, 1986):

(I) $P_C(x_i, x_j) = a_{ij} \in \mathbb{R}^+$;

(II) $a_{ij} > 1$ if x_i is strictly preferred to x_j under the criterion C, or $x_i \succ_C x_j$;

(III) $a_{ij} = 1$ if x_i is equivalent to x_j under the criterion C, or $x_i \sim_C x_j$;

with $\forall i, j \in \mathbb{I}$. In particular, one has $a_{ii} = 1$ since x_i is equivalent to x_i. Some discussions about the existence of the ratio scale have been widely made (Bernasconi et al., 2010; Genest & Zhang, 1996; Harker & Vargas, 1987; Saaty, 1997). It has been concluded that the ratio scale can be derived and the decision-making procedure is reliable according to the AHP model (Bernasconi et al., 2010). In addition, it is seen from the above three rules that the reciprocal property of $a_{ij} = 1/a_{ji}$ is not assumed. That is, when the reciprocal property is not satisfied, the comparison ratios could be still derived by using the method in Bernasconi et al. (2010).

2.1. A detailed review on the existing axiomatic foundation

In the following, let us recall the four axioms provided by Saaty (1986) in terms of the above primitive notions. The basic one is the reciprocal property of pairwise comparisons:

Axiom 1 (*Reciprocal Property*)**.** The preferences between the alternatives x_i and x_j satisfy the following relation:

$$a_{ij} = 1/a_{ji}, \quad i, j \in \mathbb{I}. \tag{5}$$

When explaining the reciprocal property (5), Saaty (1986) has stated that "*if one stone is judged to be five times heavier than another, then the other is automatically one fifth as heavy as the first because it participated in making the first judgment.*" It is seen that the reciprocal property is mainly based on the mathematical intuition. Moreover, from the theory of subjective measurement in the mathematical psychology (Bernasconi et al., 2010), the reciprocal property is not necessarily satisfied. The main reason is based on the fact that when separately evaluating the preference intensities a_{ij} and a_{ji}, some uncertainty could exhibit from the viewpoint of human psychology. In addition, to cope with the uncertainty experienced by the DM, the interval judgments have been proposed by Saaty and Vargas (1987) to evaluate pairwise comparisons. In this study, it will be proved that the pairwise comparisons without reciprocal property are equivalent to the interval judgments in a sense. Therefore, when the reciprocal property is considered to be not necessary, the axiomatic foundation of the AHP model should be modified.

On the other hand, within the framework of the AHP model, a complex decision-making problem is decomposed as a hierarchy structure with criteria, subcriteria and alternatives. The hierarchic axioms form the bases of the hierarchy structure. First, the concept of a partially ordered set should be given as follows:

Definition 1. A set $\mathbb{S}$ with a binary relation $\varepsilon \preceq \varepsilon$ is partially ordered, when the following conditions are satisfied:

- Reflexive: $x \preceq x$ for $\forall x \in \mathbb{S}$;
- Transitive: If $x \preceq y$ and $y \preceq z$, then $x \preceq z$ for $\forall x, y, x \in \mathbb{S}$;
- Antisymmetric: If $x \preceq y$ and $y \preceq x$, then $x \sim y$ for $\forall x, y \in \mathbb{S}$.

Then the boundedness, the supremum and the infimum of a partially ordered set can be further defined. Following the concept of the partially ordered set, the definition of a hierarchy is given as follows (Saaty, 1986):

Definition 2. A hierarchy $\mathbb{H}$ satisfies the following conditions:

- $\mathbb{H}$ is a finite partially ordered set with the largest element b.
- $\mathbb{H}$ can be partitioned as h subsets called levels $\{\mathbb{L}_k, k = 1, 2, \ldots, h\}$ with $\mathbb{L}_1 = \{b\}$.
- If $x \in \mathbb{L}_k$, one has $\mathbb{L}_x^- = \{y | y \prec x\} \subseteq \mathbb{L}_{k+1}$ $(k = 1, 2, \ldots, h-1)$ and $\mathbb{L}_x^+ = \{y | x \prec y\} \subseteq \mathbb{L}_{k-1}$ $(k = 2, 3, \ldots, h)$, where the symbol $\varepsilon \prec \varepsilon$ stands for "less preferred than".

In connect with the AHP model, the second axiom is given as the following form:

Axiom 2 (*ρ-homogeneity*)**.** Assume that $\rho \geq 1$ is a positive real number and $\mathbb{H}$ is a hierarchy. $\mathbb{L}_x^- \subseteq \mathbb{L}_{k+1}$ is of ρ-homogeneity with respect to $x \in \mathbb{L}_k \subseteq \mathbb{H}$ for $k = 1, 2, \ldots, h-1$ under the following condition:

$$1/\rho \leq P_C(y_1, y_2) \leq \rho, \qquad \forall y_1, y_2 \in \mathbb{L}_x^-. \tag{6}$$

The ρ-homogeneity characterizes the comparability of the similar things belonging to the same level. When the reciprocal property is not necessary, the condition (6) should be changed correspondingly. Furthermore, it is seen that the dependence of different levels should be considered by extending the notions of the fundamental scale. The concepts of outer dependent and inner dependent are introduced as follows:

Definition 3. A set $\mathbb{A}$ is outer dependent on the set $\mathbb{C}$ if one can define a fundamental scale on $\mathbb{A}$ with respect to each $C \in \mathbb{C}$.

Definition 4. Assume that $\mathbb{A}$ is outer dependent on the set $\mathbb{C}$. The elements in $\mathbb{A}$ are inner dependent with respect to $C \in \mathbb{C}$ if for some $A \in \mathbb{A}$, $\mathbb{A}$ is outer dependent on A.

The relation between the levels $\mathbb{L}_k$ and $\mathbb{L}_{k+1}$ should satisfy the following axiom:

Axiom 3 (*Dependence*)**.** Let $\mathbb{L}_k$ $(k = 1, 2, \ldots, h)$ be the levels of a hierarchy $\mathbb{H}$. Then $\mathbb{L}_k$ and $\mathbb{L}_{k+1}$ $(k = 1, 2, \ldots, h-1)$ have the following dependence relations:

- $\mathbb{L}_{k+1}$ is outer dependent on $\mathbb{L}_k$;
- $\mathbb{L}_{k+1}$ is not inner dependent with respect to all $x \in \mathbb{L}_k$;
- $\mathbb{L}_k$ is not outer dependent on $\mathbb{L}_{k+1}$.

Axiom 3 is not related to the reciprocal property; and it holds when pairwise comparisons are not reciprocal. At the end, it is considered that the DM usually has an expectation about the outcome of a decision-making problem. The constructed hierarchy should be compatible with the expectation. Hence, the fourth axiom is expressed as follows:

Axiom 4 (*Expectation*)**.** The constructed hierarchy includes all criteria and alternatives; and the derived priorities are compatible with the expectations represented in the hierarchical structure.

The above four axioms form the theoretical basis of the typical AHP model (Saaty, 1986). In what follows, we mainly modify the first axiom about the reciprocal property of pairwise comparisons to establish a new axiomatic foundation of the AHP model under uncertainty.

2.2. An uncertainty-induced axiomatic foundation

As shown in Harker and Vargas (1987), Axiom 1 is based on the mathematical intuition where the relations of $x_i = a_{ij}x_j$ and $x_j = x_i/a_{ij}$ should be simultaneously satisfied for $a_{ij} > 0$. However, the reciprocal property (5) could not always hold for a practical case, which can be explained from the two views of point. One is based on the paired technique of comparing alternatives. When the DM evaluates the preference intensities of the alternatives x_i over x_j and x_j over x_i, the uncertainty could be experienced (Ahn, 2017; Hocine & Kouaissah, 2020; Saaty & Vargas, 1987). In fact, the reciprocal property (5) reflects the strict logical relation between a_{ij} and a_{ji}, which is incompatible with the uncertainty being experienced by the DM. The other comes with the subjective measurement of the ratio scales (Bernasconi et al., 2010). Some cognitive distortions of the DM could affect the evaluation of her/his ratio judgments. Therefore, the reciprocal property (5) should be relaxed to naturally capture the uncertainty exhibited in paired comparisons. When softening the equality (5), there are two situations

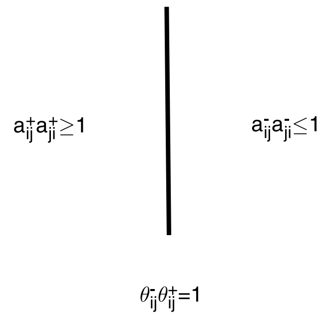

Fig. 3. The mirror mechanism between the cases of $0 < a^{-}_{ij}a^{-}_{ji} \leq 1$ and $a^{+}_{ij}a^{+}_{ji} \geq 1$.

to be considered. One is $0 < a_{ij}a_{ji} \leq 1$ and the other is $a_{ij}a_{ji} \geq 1$. For the two induced situations, we can introduce a factor θ_{ij} such that $\theta_{ij} = a_{ij}a_{ji}$. Hence the concept of RSB is proposed in the following definition:

Definition 5. Suppose that $\mathbb{X} = \{x_1, x_2, \ldots, x_n\}$ stands for a finite set of alternatives. The symbols a_{ij} and a_{ji} $(i, j \in \mathbb{I})$ are the preference intensities of x_i over x_j and x_j over x_i, respectively. If there exist a pair of preference intensities a_{ij} and a_{ji} such that

$$\theta_{ij} = a_{ij}a_{ji} \neq 1, \tag{7}$$

the pairwise comparisons on $\mathbb{X}$ are of reciprocal symmetry breaking (RSB).

In general, we can consider that the RSB reflects some uncertainty in pairwisely comparing alternatives. Moreover, it is interesting to recall the interval-valued comparison matrix provided by Saaty and Vargas (1987):

Definition 6. An interval-valued comparison matrix is represented as:

$$\tilde{A} = (\tilde{a}_{ij})_{n\times n} = \begin{pmatrix} [1,1] & [a^{-}_{12}, a^{+}_{12}] & \cdots & [a^{-}_{1n}, a^{+}_{1n}] \\ [a^{-}_{21}, a^{+}_{21}] & [1,1] & \cdots & [a^{-}_{2n}, a^{+}_{2n}] \\ \cdots & \cdots & \cdots & \cdots \\ [a^{-}_{n1}, a^{+}_{n1}] & [a^{-}_{n2}, a^{+}_{n2}] & \cdots & [1,1] \end{pmatrix}. \tag{8}$$

Hereafter $\tilde{a}_{ij} = \left[a^{-}_{ij}, a^{+}_{ij}\right]$ means that the preference intensity of x_i over x_j is located between a^{-}_{ij} and a^{+}_{ij}. The values satisfy $a^{-}_{ij} \cdot a^{+}_{ji} = a^{+}_{ij} \cdot a^{-}_{ji} = 1$ and $0 < a^{-}_{ij} \leq a^{+}_{ij}$ for $i, j \in \mathbb{I}$.

Definition 6 shows the following relations:

$$a^{-}_{ij}a^{-}_{ji} \leq 1, \quad a^{+}_{ij}a^{+}_{ji} \geq 1, \quad i, j \in \mathbb{I}. \tag{9}$$

Moreover, letting

$$\theta^{-}_{ij} = a^{-}_{ij}a^{-}_{ji}, \quad \theta^{+}_{ij} = a^{+}_{ij}a^{+}_{ji}, \tag{10}$$

we have

$$\theta^{-}_{ij}\theta^{+}_{ij} = 1. \tag{11}$$

As shown in Fig. 3, the mirror mechanism has been built for the transformation between the cases of $0 < a^{-}_{ij}a^{-}_{ji} \leq 1$ and $a^{+}_{ij}a^{+}_{ji} \geq 1$. In other words, when we give a^{+}_{ij} and a^{+}_{ji} with $a^{+}_{ij}a^{+}_{ji} \geq 1$ by comparing x_i and x_j, the comparison ratios a^{-}_{ij} and a^{-}_{ji} with $0 < a^{-}_{ij}a^{-}_{ji} \leq 1$ have been determined by using the relation $a^{-}_{ij} \cdot a^{+}_{ji} = a^{+}_{ij} \cdot a^{-}_{ji} = 1$. This means that it is sufficient to consider one of the two cases with $0 < \theta^{-}_{ij} \leq 1$ and $\theta^{+}_{ij} \geq 1$, where the reciprocal symmetry (11) is satisfied. In order to form an axiomatic foundation of the AHP model under uncertainty, Axiom 1 is modified as follows:

Axiom′ 1 (*Reciprocal Symmetry Breaking*)**.** The preference intensities of the alternatives x_i over x_j and x_j over x_i satisfy the following relation:

$$0 < a_{ij}a_{ji} \leq 1, \qquad i, j \in \mathbb{I}. \tag{12}$$

Clearly, Axiom 1 is the particular case of Axiom′ 1 with reciprocal property. In a similar manner, Axiom 2 should be further adjusted and we give the following axiom:

Axiom′ 2 (*ρ-homogeneity*)**.** Let $\rho \geq 1$ and $\mathbb{H}$ be a positive real number and a hierarchy, respectively. $\mathbb{L}^{-}_{x} \subseteq \mathbb{L}_{k+1}$ is of ρ-homogeneity with respect to $x \in \mathbb{L}_k \subseteq \mathbb{H}$ for $k = 1, 2, \ldots, h-1$ and the following conditions:

$$1/\rho \leq P_C(y_1, y_2) \leq \rho, \quad 1/\rho \leq P_C(y_2, y_1) \leq \rho \qquad \forall y_1, y_2 \in \mathbb{L}^{-}_{x}. \tag{13}$$

As compared to Axiom 2, the pairwise comparisons $P_C(y_1, y_2)$ and $P_C(y_2, y_1)$ are all considered in Axiom′ 2. Furthermore, since Axiom 3 is only related to the fundamental scale satisfying the three rules (I)-(III), it does not need to be modified. Additionally, Axiom 4 holds, even when the irrational behavior of the DM emerges (Saaty, 1986). In a word, Axiom′s 1 and 2, together with Axioms 3 and 4 form the modified axiomatic foundation of the AHP model under uncertainty.

3. The implication of the modified axioms

It is noted from the results in Saaty (1986) that the reciprocal property is the necessary condition of a consistent binary relation. When the reciprocal property is breaking, the pairwise comparisons must be inconsistent. This implies that the inconsistency is the natural property of pairwise comparisons under the modified axiomatic foundation. The above observation is in agreement with the practical situation, since one always provides inconsistent judgments when pairwisely comparing alternatives (Saaty, 1980). In the following, we derive some interesting results from the modified axiomatic foundation.

3.1. Properties of reciprocal symmetry breaking

It is convenient to propose the concept of PCMs with RSB.

Definition 7. A PCM $A = (a_{ij})_{n\times n}$ is of RSB, if the derived matrix

$$\Theta = (\theta_{ij})_{n\times n} \neq E, \tag{14}$$

where $\theta_{ij} = a_{ij}a_{ji}$ $(i, j \in \mathbb{I})$ and E stands for the matrix whose entries all are 1.

Hereafter, when we say PCMs, it means that the reciprocal property could be breaking. In terms of (14), it gives $\theta_{ij} = \theta_{ji}$ $(i, j \in \mathbb{I})$, meaning that the constructed matrix Θ is symmetrical. When $\Theta = E$, the matrix $A = (a_{ij})_{n\times n}$ is with reciprocal property. This implies that the matrix Θ characterizes the reciprocal property of a PCM. For the sake of distinguishing, we give the following definition:

Definition 8. If a PCM $A = (a_{ij})_{n\times n}$ is with reciprocal property, it is called a multiplicative reciprocal matrix.

Moreover, the RSB degree of a PCM $A = (a_{ij})_{n\times n}$ can be quantified using the matrix Θ. Therefore, we define the following index:

Definition 9. Suppose that $A = (a_{ij})_{n\times n}$ is a PCM. The RSB degree of A is defined as the following equality:

$$SBD(A) = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \theta_{ij}, \quad n \geq 2, \tag{15}$$

where $\theta_{ij} = a_{ij}a_{ji}$ for $i, j \in \mathbb{I}$.

In addition, the index of $SBD(A)$ has the property as follows:

Theorem 1. *The RSB degree of a PCM $A = (a_{ij})_{n\times n}$ satisfies the relation of $0 < SBD(A) \leq 1$. $A = (a_{ij})_{n\times n}$ is a multiplicative reciprocal matrix if and only if $SBD(A) = 1$.*

Proof. In terms of Axiom′ 1 and Definition 9, it follows $0 < \theta_{ij} \leq 1$ and $\theta_{ij} = \theta_{ji}$ for $i, j \in \mathbb{I}$. We further have

$$0 < SBD(A) = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \theta_{ij} \leq 1. \tag{16}$$

If $SBD(A) = 1$, then $\theta_{ij} = 1$ for $i, j \in \mathbb{I}$ and $j > i$. In virtue of $\theta_{ij} = \theta_{ji}$ and $\theta_{ii} = 1$, one has $\Theta = (\theta_{ij})_{n\times n} = E$, meaning that $A = (a_{ij})_{n\times n}$ is a multiplicative reciprocal matrix (Definition 8). On the other hand, if $A = (a_{ij})_{n\times n}$ is with reciprocal property, then $\theta_{ij} = a_{ij}a_{ji} = 1$. The application of (15) leads to $SBD(A) = 1$.

Under Axiom′ 1, the proposed index $SBD(A)$ reveals the breaking degree of reciprocal property of a PCM. The more the value of $SBD(A)$, the less the breaking degree of reciprocal property is. It can be further explained as the subjective probability to give a PCM with reciprocal property. In other words, when $SBD(A) = 1$, it implies that the subjective probability of giving a PCM with reciprocal property is 1. With the decreasing value of $SBD(A)$, the subjective probability of providing a multiplicative reciprocal matrix is decreasing.

Furthermore, since an interval-valued comparison matrix can be used to capture the uncertainty experienced by the DM (Saaty & Vargas, 1987), the following result is achieved:

Theorem 2. *A PCM $A = (a_{ij})_{n\times n}$ with RSB is equivalent to an interval-valued comparison matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$.*

Proof. If $A = (a_{ij})_{n\times n}$ is with RSB, we have $0 < \theta_{ij} = a_{ij}a_{ji} \leq 1$ under Axiom′ 1 for $\forall i, j \in \mathbb{I}$ and $\Theta \neq E$ according to Definition 7. Then it follows

$$a_{ij} \leq \frac{1}{a_{ji}}. \tag{17}$$

Letting

$$a_{ij}^{-} = a_{ij}, \qquad a_{ij}^{+} = 1/a_{ji},$$

and

$$\tilde{a}_{ij} = [a_{ij}^{-}, a_{ij}^{+}],$$

one obtains the interval-valued comparison matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$.

On the contrary, when an interval-valued comparison matrix $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ with $\tilde{a}_{ij} = [a_{ij}^{-}, a_{ij}^{+}]$ is given, the PCM $A = (a_{ij})_{n\times n}$ with RSB is derived by assuming $a_{ij} = a_{ij}^{-}$ for $i, j \in \mathbb{I}$ under the consideration of Axiom′ 1.

The equivalence shown in Theorem 2 reveals that the uncertainty exhibited by an interval-valued comparison matrix can be equivalently captured by a PCM with RSB. In addition, following the idea in Definition 9, an uncertainty index of $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ can be constructed as

$$UI(\tilde{A}) = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j>i}^{n} a_{ij}^{-} a_{ji}^{-}. \tag{18}$$

Then the following result is obtained:

Theorem 3. *Assume that $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ is an interval-valued comparison matrix with $\tilde{a}_{ij} = [a_{ij}^{-}, a_{ij}^{+}]$ for $i, j \in \mathbb{I}$. $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ degenerates to a multiplicative reciprocal matrix if and only if the value of the constructed uncertainty index $UI(\tilde{A})$ is equal to 1.*

Proof. On the one hand, if $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ degenerates to a multiplicative reciprocal matrix, then $a_{ij}^{-} = a_{ij}^{+}$ and $a_{ij}^{-}a_{ji}^{-} = 1$ for $i, j \in \mathbb{I}$. Using the formula (18), one has $UI(\tilde{A}) = 1$. On the other hand, if $UI(\tilde{A}) = 1$, the result of Theorem 1 shows that the matrix $A^{L} = (a_{ij}^{-})_{n\times n}$ is a multiplicative reciprocal matrix. This means that $a_{ij}^{-}a_{ji}^{-} = 1$ and $a_{ij}^{-} = 1/a_{ji}^{-} = a_{ij}^{+}$. That is, $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ degenerates to a multiplicative reciprocal matrix A^{L}.

It is seen from the above observations that the uncertainty experienced by the DM has been naturally incorporated into Axiom′ 1. Therefore, the modified axiomatic foundation can be considered as the basis of the AHP model under uncertainty.

3.2. Approximate consistency of pairwise comparison matrices

It is worth noting that the concept of consistent PCMs is important and it is recalled as follows:

Definition 10 (*Saaty, 1980, 1986*). A PCM $A = (a_{ij})_{n\times n}$ is consistent if

$$a_{ik} = a_{ij}a_{jk}, \quad i, j, k \in \mathbb{I}. \tag{19}$$

In this study, it is noted that a PCM could be with the breaking of reciprocal property. Hence, the consistency is only a particular property of a PCM and a softened version of consistency should be further developed. In the existing works, the concepts of weak consistency and transitivity have been proposed to investigate the reliability of priorities derived from PCMs (Brunelli & Cavallo, 2020; Cavallo & D'Apuzzo, 2016; Cavallo et al., 2016), where the reciprocal property is always considered. Moreover, the concept of approximate consistency has been proposed to characterize the consistency property of interval-valued comparison matrices (Liu et al., 2017). Here by considering the RSB and Theorem 2, the concept of approximate consistency is used. Then following the idea in Liu et al. (2021), we give the definition of approximate consistency of PCMs as follows:

Definition 11. A PCM $A = (a_{ij})_{n\times n}$ is approximately consistent, if the same ranking of alternatives can be simultaneously obtained using each row and column vectors in $A = (a_{ij})_{n\times n}$.

In Definition 11, the restrictive quantitative relation among paired comparisons in Definition 10 has been neglected and the ranking of alternatives is directly used. For example, let us consider a PCM as follows (Saaty, 2013):

$$A_1 = \left(\begin{array}{c|ccccc} C & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 1 & 9 & 2 & 3 & 5 \\ x_2 & 1/9 & 1 & 1/5 & 1/3 & 1/2 \\ x_3 & 1/2 & 5 & 1 & 3/2 & 3 \\ x_4 & 1/3 & 3 & 2/3 & 1 & 3/2 \\ x_5 & 1/5 & 2 & 1/3 & 2/3 & 1 \end{array}\right).$$

It is easy to determine the same ranking of alternatives as $x_1 \succ x_3 \succ x_4 \succ x_5 \succ x_2$ by using each row and column vectors in A_1, meaning that A_1 is of approximate consistency according to Definition 11. In addition, one can see that A_1 is with reciprocal property. For the purpose of agreeing with Axiom′ 1, we further modify the matrix A_1 as:

$$A_1^m = \left(\begin{array}{c|ccccc} C & x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline x_1 & 1 & \mathbf{8} & 2 & 3 & 5 \\ x_2 & 1/9 & 1 & 1/5 & 1/3 & 1/2 \\ x_3 & 1/2 & 5 & 1 & 3/2 & 3 \\ x_4 & 1/3 & 3 & 2/3 & 1 & 3/2 \\ x_5 & 1/5 & 2 & 1/3 & 2/3 & 1 \end{array}\right),$$

where $a_{12}a_{21} = 8/9 < 1$. The determined ranking of alternatives by using each row and column in A_1^m still is $x_1 \succ x_3 \succ x_4 \succ x_5 \succ x_2$, meaning that A_1^m is also a PCM with approximate consistency. Hence, due to the breaking of reciprocal property in A_1^m, the concept of approximate consistency is a softened version of consistency, and it is compatible with Axiom′ 1. According to the standard of choosing the best alternative(s), the two matrices A_1 and A_1^m are equivalent. Furthermore, we have the following property:

Theorem 4. *Let $A = (a_{ij})_{n\times n}$ be a PCM with approximate consistency. There is a permutation $\sigma : \mathbb{I} \mapsto \mathbb{I}$ such that $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ has the following properties:*

(1) A_σ is a PCM with approximate consistency;
(2) The elements in each row of A_σ are ranked by the ascending order;
(3) The elements in each column of A_σ are ranked by the descending order.

Proof. (1) Suppose that $A = (a_{ij})_{n\times n}$ is with approximate consistency. For any permutation σ, there is a unique number $i_k \in \mathbb{I}$ such that $\sigma(i) = i_k$ for $\forall i \in \mathbb{I}$. When $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ is obtained by applying a permutation σ to $A = (a_{ij})_{n\times n}$, the entries in the ith row or the jth column belonging to $A = (a_{ij})_{n\times n}$ are changed to the $\sigma(i)$th row or the $\sigma(j)$ column in $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$. This means that the ranking of alternatives determined by a row or column of $A = (a_{ij})_{n\times n}$ is not changed with respect to the permutation σ. Therefore, $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ is of approximate consistency.

(2) For $A = (a_{ij})_{n\times n}$ with approximate consistency, it is assumed that the determined ranking of alternatives is $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)}$, where the permutation σ is given. Then applying the permutation σ to $A = (a_{ij})_{n\times n}$ to give $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$, we have $a_{\sigma(i)\sigma(1)} \leq a_{\sigma(i)\sigma(2)} \leq \cdots \leq a_{\sigma(i)\sigma(n)}$ for $i \in \mathbb{I}$ when the ranking is obtained by using the row vectors. This means that the elements in each row of A_σ are ranked by the ascending order.

(3) Based on the findings in (2), when the ranking of $x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)}$ is determined using the column vectors, it follows $a_{\sigma(1)\sigma(j)} \geq a_{\sigma(2)\sigma(j)} \geq \cdots \geq a_{\sigma(n)\sigma(j)}$ for $j \in \mathbb{I}$. That is, the elements in each column of A_σ are ranked by the descending order.

Theorem 4 can be verified by adjusting A_1^m as the following form:

$$A_1^\sigma = \begin{pmatrix} C & x_1 & x_3 & x_4 & x_5 & x_2 \\ \hline x_1 & 1 & 2 & 3 & 5 & \mathbf{8} \\ x_3 & 1/2 & 1 & 3/2 & 3 & 5 \\ x_4 & 1/3 & 2/3 & 1 & 3/2 & 3 \\ x_5 & 1/5 & 1/3 & 2/3 & 1 & 2 \\ x_2 & 1/9 & 1/5 & 1/3 & 1/2 & 1 \end{pmatrix}.$$

It is easy to see that the matrix A_1^σ is in agreement with Theorem 4. In addition, we have the following observation:

Theorem 5. *If a PCM $A = (a_{ij})_{n\times n}$ is consistent (Definition 10), it is of approximate consistency (Definition 11).*

Proof. Suppose that $A = (a_{ij})_{n\times n}$ is a consistent matrix. Then one has $rank(A) = 1$ and all rows of $A = (a_{ij})_{n\times n}$ are identical except for a constant factor (Saaty, 1986). That is, when a row of $A = (a_{ij})_{n\times n}$ is used to obtain the ranking of alternatives, the other rows can be used to determine the same ranking of alternatives. Similarly, when considering the columns of $A = (a_{ij})_{n\times n}$, the same result can be found. Furthermore, due to the reciprocal property of a consistent matrix, the rankings of alternatives based on the rows and columns are identical. Therefore, a consistent PCM is with approximate consistency.

The above observations show that a consistent PCM is the ideal case under the typical and modified axiomatic foundations of the AHP model. A PCM with approximate consistency can be considered as the ideal case of the modified AHP model under uncertainty.

3.3. Priorities of alternatives

Another important problem is how to elicit the priorities of alternatives from a PCM $A = (a_{ij})_{n\times n}$ in the AHP model. Following the idea in Saaty (1980), let us assume that there is a mapping given as follows:

$$\psi : \mathbb{R}_{M(n)} \mapsto [0, 1]^n, \tag{20}$$

where $\mathbb{R}_{M(n)}$ denotes the set of PCMs $A = (a_{ij})_{n\times n}$ and $[0, 1]^n$ stands for the n-fold Cartesian product of $[0, 1]$. For convenience, the priority vector of alternatives is expressed as $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in [0, 1]^n$ with $\sum_{i=1}^n \omega_i = 1$. The methods should be studied for deriving the priority vector of alternatives from a PCM. As shown in Saaty (1980, 1986), the eigenvalue method has been carefully discussed, especially for consistent PCMs. Here we mainly focus on the method of deriving the priority vector from a PCM with approximate consistency.

Suppose that the set of PCMs with approximate consistency is written as $\mathbb{R}_{AC(n)}$. The following result is given:

Theorem 6. *Let $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$ be obtained by pairwise comparing alternatives in $\mathbb{X} = \{x_1, x_2, \ldots, x_n\}$. There exists a mapping:*

$$\psi : A \in \mathbb{R}_{AC(n)} \mapsto [0, 1]^n \ni \omega = (\omega_1, \omega_2, \ldots, \omega_n),$$

such that

(I) $a_{ij} = \varepsilon_{ij}\frac{\omega_i}{\omega_j}$ with $\varepsilon_{ij} = \varepsilon_{ji} = \sqrt{a_{ij}a_{ji}} \in (0, 1]$ for $i, j \in \mathbb{I}$.

(II) For any two alternatives x_i and x_j, $x_i \geq x_j$ if and only if $\omega_i \geq \omega_j$ for $i, j \in \mathbb{I}$.

Proof. For $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$, there is a permutation σ such that the derived matrix $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ satisfies Theorem 4. Without loss of generality, it is assumed that $\sigma = (1, 2, \ldots, n)$ and $A = (a_{ij})_{n\times n}$ satisfies Theorem 4. This means that $x_1 \geq x_2 \geq \cdots \geq x_n$ along with $a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$ and $a_{1j} \geq a_{2j} \geq \cdots \geq a_{nj}$ for $i, j \in I$. Therefore, there exists a weight ω_i for the alternative x_i with $i \in \mathbb{I}$ such that $\omega_1 \geq \omega_2 \geq \cdots \geq \omega_n$. Moreover, a PCM can be constructed as follows:

$$\Omega = \begin{pmatrix} C & x_1 & x_2 & \cdots & x_n \\ \hline x_1 & \frac{\omega_1}{\omega_1} & \varepsilon_{12}\frac{\omega_1}{\omega_2} & \cdots & \varepsilon_{1n}\frac{\omega_1}{\omega_n} \\ x_2 & \varepsilon_{21}\frac{\omega_2}{\omega_1} & \frac{\omega_2}{\omega_2} & \cdots & \varepsilon_{2n}\frac{\omega_2}{\omega_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & \varepsilon_{n1}\frac{\omega_n}{\omega_1} & \varepsilon_{n2}\frac{\omega_n}{\omega_2} & \cdots & \frac{\omega_n}{\omega_n} \end{pmatrix}. \tag{21}$$

Letting

$$a_{ij} = \varepsilon_{ij}\frac{\omega_i}{\omega_j}, \quad \varepsilon_{ij} = \varepsilon_{ji},$$

we have

$$A = \Omega, \quad a_{ij}a_{ji} = \varepsilon_{ij}^2.$$

In terms of Axiom′ 1, it gives $\varepsilon_{ij} \in (0, 1]$, then the results (I) and (II) hold.

In what follows, we prove that the eigenvalue method still is feasible to derive the priority vector from a PCM with approximate consistency. It is convenient to recall a lemma as follows:

Lemma 1. *Let $A = (a_{ij})_{n\times n}$ be a PCM and σ be a permutation of $(1, 2, \ldots, n)$. $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ is determined by applying σ to $A = (a_{ij})_{n\times n}$. Then the eigenvalues of $A = (a_{ij})_{n\times n}$ and $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ are identical. The corresponding eigenvectors of $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ are obtained by applying σ to those of $A = (a_{ij})_{n\times n}$.*

The proof of Lemma 1 can be completed by using the matrix theory such as that in Horn and Johnson (1985) and the known finding (Liu et al., 2020). The detail procedure has been neglected here. Furthermore, we have the following result:

Theorem 7. *Let $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$ be expressed as (21) and A' be a consistent PCM. The principal right eigenvector of A' is written as $\omega' = (\omega_1, \omega_2, \ldots, \omega_n)^T$, where T denotes the transposition. The principal right eigenvector of A corresponding to the principal eigenvalue λ_{max} is calculated as $\omega_a = (\omega_{a1}, \omega_{a2}, \ldots, \omega_{an})^T$. A permutation σ is applied to ω' and ω_a to get $\omega'_\sigma = (\omega_{\sigma(1)}, \omega_{\sigma(2)}, \ldots, \omega_{\sigma(n)})^T$ and $\omega_{a\sigma} = (\omega_{a\sigma(1)}, \omega_{a\sigma(2)}, \ldots, \omega_{a\sigma(n)})^T$, respectively. Then there is a permutation σ such that $\omega_{a\sigma(1)} \geq \omega_{a\sigma(2)} \geq \cdots \geq \omega_{a\sigma(n)}$ and $\omega_{\sigma(1)} \geq \omega_{\sigma(2)} \geq \cdots \geq \omega_{\sigma(n)}$ are simultaneously satisfied.*

Proof. Since $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$, there is a permutation σ such that $A_\sigma = (a_{\sigma(i)\sigma(j)})_{n\times n}$ with the following relations:

$$0 < a_{\sigma(i)\sigma(j)} \leq a_{\sigma(i)\sigma(j+1)}, \quad a_{\sigma(i)\sigma(j)} \geq a_{\sigma(i+1)\sigma(j)} > 0, \quad i, j = 1, 2, \ldots, n-1, \tag{22}$$

where Theorem 4 has been used. Making use of Lemma 1, we have

$$A_\sigma \omega_{a\sigma} = \lambda_{max}\omega_{a\sigma}. \tag{23}$$

This means that

$$\sum_{j=1}^{n} a_{\sigma(i)\sigma(j)}\omega_{a\sigma(j)} = \lambda_{max}\omega_{a\sigma(i)}, \quad i \in \mathbb{I}. \tag{24}$$

In virtue of (22), it follows $\omega_{a\sigma(1)} \geq \omega_{a\sigma(2)} \geq \cdots \geq \omega_{a\sigma(n)}$. Moreover, by rewriting $A = (a_{ij})_{n\times n}$ as the matrix in (21), the application of Theorem 6 yields $\omega_{\sigma(1)} \geq \omega_{\sigma(2)} \geq \cdots \geq \omega_{\sigma(n)}$.

As shown in Theorem 7, the eigenvalue method can be used as the method of eliciting the properties of alternatives from a PCM with approximate consistency. For example, the PCM A_1^σ with approximate consistency is utilized for numerical computations. The largest eigenvalue and the corresponding eigenvector can be determined as 4.9824 and $(0.4565, 0.2476, 0.1523, 0.0940, 0.0496)$, respectively. This means that the ranking of alternatives is $x_1 \succ x_3 \succ x_4 \succ x_5 \succ x_2$, which is in agreement with the existing result. In addition, it is noted that the largest eigenvalue 4.9824 is less than the order 5 of the matrix. The observation is attributed to the breaking of reciprocal symmetry and different from the result in Saaty (1986).

4. An index of measuring approximate consistency

In the typical AHP model (Saaty, 1980), the consistency degree of multiplicative reciprocal matrices is allowed to be deviated from the perfect consistency to a certain extent. The consistency index (CI) and consistency ratio (CR) were further defined to quantify the inconsistency degree of a multiplicative reciprocal matrix. If and only if the value of CI or CR is equal to zero, the corresponding multiplicative comparison matrix is perfectly consistent. Furthermore, the concept of acceptable consistency was proposed by Saaty (1980) when the value of CR is less or equal to 0.1. In this study, we establish an index to measure the approximate consistency degree of a PCM.

In order to construct an approximate consistency index of $A = (a_{ij})_{n\times n} \in \mathbb{R}_{M(n)}$, we first define the ranks of the elements belonging to a vector (Liu et al., 2021).

Definition 12. The rank of the element v_i belonging to an n-dimensional vector $\vec{V} = (v_1, v_2, \ldots, v_n)^T$ is defined as:

$$r_i = \sum_{j=1}^{n} Ind(v_j \leq v_i), \tag{25}$$

where

$$Ind(v_j \leq v_i) = \begin{cases} 1, & v_j \leq v_i, \\ 0, & otherwise. \end{cases} \tag{26}$$

Then the rank vector of $\vec{V}$ is determined as $\vec{R}(\vec{V}) = (r_1, r_2, \ldots, r_n)^T$.

For example, the rank vector of $\vec{V}_1 = (0.7, 0.4, 0.3, 0.4, 0.2)^T$ can be determined as $\vec{R}(\vec{V}_1) = (5, 3, 2, 4, 1)^T$. Moreover, when there are the same elements in a vector, the first method is used such that the rank vector of an n-dimensional vector is a permutation of $(1, 2, \ldots, n)^T$. The first method obeys the following two rules:

- The rank vector of a vector is always determined as a permutation of $(1, 2, \ldots, n)^T$;
- For two same elements, the rank of the first element is less than that of the second one.

In what follows, we always assume that the rank vector of an n-dimensional vector is a permutation of $(1, 2, \ldots, n)^T$. Moreover, the rank matrix of an $n \times m$ matrix $B = (b_{ij})_{n\times m}$ can be defined as:

Definition 13. $R = (r_{ij})_{n\times m}$ is called the rank matrix of $B = (b_{ij})_{n\times m}$, where r_{ij} is the rank of the entry b_{ij} in the column vector $\vec{b}_{\cdot j} = (b_{1j}, b_{2j}, \ldots, b_{nj})^T$.

Then, let us define the following quantity:

$$\bar{r}_i = \sum_{j=1}^{m} r_{ij}, \quad i \in \mathbb{I}. \tag{27}$$

Moreover, the variance together with the mean value are obtained as:

$$\bar{r} = \frac{1}{n}\sum_{i=1}^{n} \bar{r}_i, \quad S = \sum_{i=1}^{n}(\bar{r}_i - \bar{r})^2 = \sum_{i=1}^{n} \bar{r}_i^2 - n\bar{r}^2. \tag{28}$$

After some computations, we further have the following results:

$$\bar{r} = \frac{n(n+1)}{2}, \qquad S = \sum_{i=1}^{n} \bar{r}_i^2 - \frac{m^2 n(n+1)^2}{4}. \tag{29}$$

It is found that the maximum value of S can be obtained as

$$S_{max} = \frac{m^2 n(n^2-1)}{12}, \tag{30}$$

when the rankings of the entries of each column in the rank matrix are identical. Now the Kendall's concordance coefficient of $B = (b_{ij})_{n\times m}$ is defined as (Field, 2005; Kendall, 1970):

$$K(B) = \frac{S}{S_{max}} = \frac{12S}{m^2 n(n^2-1)}, \qquad n > 1. \tag{31}$$

Obviously, it follows $0 \leq K(B) \leq 1$. When $K(B) = 0$, the column and row vectors of $R = (r_{ij})_{n\times n}$ correspond to an identical permutation of $(1, 2, \ldots, n)^T$. The case of $K(B) = 1$ means that the elements in a row vector of $R = (r_{ij})_{n\times m}$ are the same.

Now, we consider a PCM $A = (a_{ij})_{n\times n}$ without reciprocal property. Under the consideration of the RSB, the Kendall's concordance coefficient of $A = (a_{ij})_{n\times n}$ should be derived using the row and column vectors. The corresponding rank matrices are written as:

$$R_A^r = \left(\begin{array}{c|cccc} C & x_1 & x_2 & \cdots & x_n \\ \hline x_1 & r_{11}^r & r_{12}^r & \cdots & r_{1n}^r \\ x_2 & r_{21}^r & r_{22}^r & \cdots & r_{2n}^r \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & r_{n1}^r & r_{n2}^r & \cdots & r_{nn}^r \end{array}\right),$$

$$R_A^c = \left(\begin{array}{c|cccc} C & x_1 & x_2 & \cdots & x_n \\ \hline x_1 & r_{11}^c & r_{12}^c & \cdots & r_{1n}^c \\ x_2 & r_{21}^c & r_{22} & \cdots & r_{2n}^c \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n & r_{n1}^c & r_{n2}^c & \cdots & r_{nn}^c \end{array}\right),$$

where R_A^r and R_A^c stand for the rank matrices obtained by the row and column vectors, respectively. The Kendall's coefficient of concordance in (31) should be rewritten as

$$K(A) = \frac{S^r + S^c}{2S_{max}} = \frac{6(S^r + S^c)}{n^3(n^2-1)}, \qquad n > 1, \tag{32}$$

where

$$S^r = \sum_{j=1}^{n}(\bar{r}_j^r)^2 - \frac{n^3(n+1)^2}{4}, \quad S^c = \sum_{i=1}^{n}(\bar{r}_i^c)^2 - \frac{n^3(n+1)^2}{4},$$

with

$$\bar{r}_j^r = \sum_{i=1}^{n} r_{ij}^r, \quad \bar{r}_i^c = \sum_{j=1}^{n} r_{ij}^c, \quad i, j \in \mathbb{I}.$$

Here r_{ij}^r and r_{ij}^c are the ranks of a_{ij} in the row and column vectors, respectively. We further arrive at the following result:

Theorem 8. *Given a PCM $A = (a_{ij})_{n\times n}$, $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$ if and only if the value of $K(A)$ in (32) is equal to 1.*

Proof. When $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$, the rankings of alternatives are identical according to all row and column vectors in $A = (a_{ij})_{n\times n}$. This means that each column in R_A^c is the same permutation of $(1, 2, \ldots, n)^T$, and each row in R_A^r is the same permutation of $(1, 2, \ldots, n)$. It can be computed that $S^r = S^c = S_{max}$, then $K(A) = 1$.

On the contrary, if $K(A) = 1$, it follows $S^r = S_{max}$ and $S^c = S_{max}$ are satisfied simultaneously. Then all column vectors in R^c_A are identical and all row vectors in R^r_A are the same. In what follows, we prove that the same ranking of alternatives can be obtained using each column and row vectors of $A = (a_{ij})_{n\times n}$. The proof can be achieved by the mathematical induction.

(1) Let $n = 2$, meaning that

$$A_{(2)} = \left(\begin{array}{c|cc} C & x_1 & x_2 \\ \hline x_1 & 1 & a_{12} \\ x_2 & a_{21} & 1 \end{array}\right),$$

with $0 < a_{12}a_{21} \leq 1$ under Axiom′ 1. Without loss of generality, letting $a_{21} \leq 1$, it must have $a_{12} \geq 1$ because the column vectors $(1, a_{21})^T$ and $(a_{12}, 1)^T$ give the same ranking of alternatives $x_1 \geq x_2$. This results $1 \leq a_{12}$ and $a_{21} \leq 1$, implying that the application of the row vectors yields the same ranking of $x_1 \geq x_2$. That is, the matrix $A_{(2)}$ is of approximate consistency.

(2) It is assumed that when $n = k$, the matrix $A_{(k)} = (a_{ij})_{k\times k}$ is of approximate consistency. There is no loss of generality to assume $x_1 \geq x_2 \geq \cdots \geq x_k$ with

$$a_{1j} \geq a_{2j} \geq \cdots \geq a_{kj}, \quad a_{i1} \leq a_{i2} \leq \cdots \leq a_{ik}, \tag{33}$$

for $i, j \in \{1, 2, \ldots, k\}$.

(3) When $n = k + 1$, the matrix $A_{(k+1)} = (a_{ij})_{(k+1)\times(k+1)}$ is given as

$$A_{(k+1)} = \left(\begin{array}{c|cccccc} C & x_1 & x_2 & \cdots & x_k & x_{k+1} \\ \hline x_1 & 1 & a_{12} & \cdots & a_{1k} & a_{1(k+1)} \\ x_2 & a_{21} & 1 & \cdots & a_{2k} & a_{2(k+1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_k & a_{k1} & a_{k2} & \cdots & 1 & a_{k(k+1)} \\ x_{k+1} & a_{(k+1)1} & a_{(k+1)2} & \cdots & a_{(k+1)k} & 1 \end{array}\right).$$

By considering the same ranking of alternatives according to the column vectors of $A_{(k+1)}$ and the first relation in (33), it is supposed that $a_{1j} \geq a_{2j} \geq \cdots \geq a_{kj} \geq a_{(k+1)j}$ for $j \in \{1, 2, \ldots, k+1\}$, which yields the ranking of $x_1 \geq x_2 \geq \cdots \geq x_k \geq x_{k+1}$. Then in the $(k + 1)$th column of $A_{(k+1)}$, one has $a_{k(k+1)} \geq 1$. In virtue of the kth row of $A_{(k+1)}$ and the second equality in (33), it follows $a_{k1} \leq a_{k2} \leq \cdots \leq a_{kk} \leq a_{k(k+1)}$. Moreover, it is seen that the row vectors of $A_{(k+1)}$ should give the same ranking of alternatives according to the rank matrix R^r_A. Hence, we always have $a_{i1} \leq a_{i2} \leq \cdots \leq a_{ik} \leq a_{i(k+1)}$ for $i \in \{1, 2, \ldots, k + 1\}$, meaning that $x_1 \geq x_2 \geq \cdots \geq x_k \geq x_{k+1}$ by using each row of $A_{(k+1)}$.

The above procedure shows that $A = (a_{ij})_{n\times n}$ is with approximate consistency under the condition of $K(A) = 1$.

One can see from Theorem 8 that the value $K(A) = 1$ characterizes the approximate consistency of a PCM $A = (a_{ij})_{n\times n}$. Following the idea in Saaty (1980), we define the approximate consistency index of $A = (a_{ij})_{n\times n}$ as follows:

Definition 14. Let $A = (a_{ij})_{n\times n}$ be a PCM. The approximate consistency index is defined as the following form:

$$ACI(A) = 1 - K(A). \tag{34}$$

In terms of Theorem 8, it is found that $0 \leq ACI(A) \leq 1$, and $A = (a_{ij})_{n\times n} \in \mathbb{R}_{AC(n)}$ if and only if $ACI(A) = 0$. Similar to the acceptable consistency, the approximate consistency is also a softened version of perfect consistency and it can be acceptable as a deviation from the perfect consistency. For instance, the rank matrices of A^σ_1 are obtained as follows:

$$R^r_1 = \left(\begin{array}{c|ccccc} C & x_1 & x_3 & x_4 & x_5 & x_2 \\ \hline x_1 & 1 & 2 & 3 & 4 & 5 \\ x_3 & 1 & 2 & 3 & 4 & 5 \\ x_4 & 1 & 2 & 3 & 4 & 5 \\ x_5 & 1 & 2 & 3 & 4 & 5 \\ x_2 & 1 & 2 & 3 & 4 & 5 \end{array}\right),$$

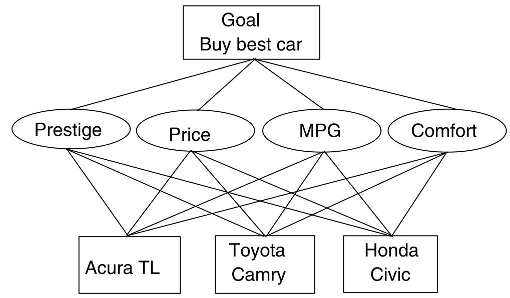

Fig. 4. The hierarchy structure for choosing the best car.

Table 1
Comparisons of criteria with respect to the goal.

Goal	Prestige	Price	MPG	Comfort	Priorities
Prestige	1	1/4	1/3	1/2	0.0987
Price	4	1	3	3/2	0.4250
MPG	3	1/3	1	1/3	0.1686
Comfort	2	2/3	3	1	0.3078

Table 2
Comparisons of cars with respect to prestige.

Prestige	Acura TL	Toyota Camry	Honda Civic	Priorities
Acura TL	1	8	4	0.7071
Toyota Camry	1/8	1	1/4	0.0702
Honda Civic	1/4	4	1	0.2227

Table 3
Comparisons of cars with respect to Price.

Price	Acura TL	Toyota Camry	Honda Civic	Priorities
Acura TL	1	1/4	1/9	0.0633
Toyota Camry	4	1	1/5	0.1939
Honda Civic	9	5	1	0.7429

$$R^r_1 = \left(\begin{array}{c|ccccc} C & x_1 & x_3 & x_4 & x_5 & x_2 \\ \hline x_1 & 5 & 5 & 5 & 5 & 5 \\ x_3 & 4 & 4 & 4 & 4 & 4 \\ x_4 & 3 & 3 & 3 & 3 & 3 \\ x_5 & 2 & 2 & 2 & 2 & 2 \\ x_2 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

It can be computed that $K(A^\sigma_1) = 1$ and $ACI(A) = 0$, meaning that A^σ_1 is with approximate consistency. Hence, the approximate consistency degree of a PCM $A = (a_{ij})_{n\times n}$ is quantified using the proposed index (34).

5. Comparison and discussion

For the sake of comparison, a decision-making problem is investigated in the following. In order to choose the best car among three alternatives of Acura TL, Toyota Camry and Honda Civic, four criteria with prestige, price, miles per gallon (MPG) and comfort are taken into account (Saaty, 2013). The hierarchy structure is shown in Fig. 4 and the corresponding PCMs are presented in Tables 1–5, respectively. Using the eigenvalue method, the priorities of criteria and alternatives are computed. It is assumed that the final weights of three alternatives are written as ω_1, ω_2 and ω_3, respectively. The obtained results are determined as $(\omega_1, \omega_2, \omega_3) = (0.3443, 0.2002, 0.4556)$. This means that the best car is the Honda Civic (Saaty, 2013).

It is seen that the above solution procedure is typical by considering the reciprocal property of pairwise comparisons. Here the uncertainty experienced by the DM is considered and some comparison ratios

Table 4
Comparisons of cars with respect to MPG.

MPG	Acura TL	Toyota Camry	Honda Civic	Priorities
Acura TL	1	2/3	1/3	0.1818
Toyota Camry	3/2	1	1/2	0.2727
Honda Civic	3	2	1	0.5455

Table 5
Comparisons of cars with respect to comfort.

Comfort	Acura TL	Toyota Camry	Honda Civic	Priorities
Acura TL	1	4	7	0.7049
Toyota Camry	1/4	1	3	0.2109
Honda Civic	1/7	1/3	1	0.0841

Table 6
Comparisons of criteria with respect to the goal under uncertainty.

Goal	Price	Comfort	MPG	Prestige	Priorities
Price	1	3/2	3	**3.8**	0.4292
Comfort	2/3	1	3	**1.5**	0.3018
MPG	1/3	1/3	1	**2.8**	0.1683
Prestige	1/4	1/2	1/3	1	0.1007

are assumed to be without the reciprocal property. For example, the comparison ratios of criteria are changed to those in Table 6. It is seen from Table 6 that the order of the four criteria has been rearranged. In terms of Definition 11, the judgments in Table 6 are not of approximate consistency. One can further find that there is a possible "mistake", since the values of $3.8, 1.5, 2.8$ and 1 in the column are not descending. This phenomenon can be used to remind the DM notice the possible unreasonable behavior. The final weights of alternatives can be computed as $(\omega_1, \omega_2, \omega_3) = (0.3417, 0.1998, 0.4585)$ and the Honda Civic still is the best choice. Here we should point out that the uncertainty could yield the inconsistency and the breaking of reciprocal property. The feedback mechanism of reminding the DM could be more important than the method of dealing with an inconsistent opinion in a practical case, since people always seem to make a mistake in a practical case (Saaty, 1980).

6. Conclusions

The theory and applications of the analytic hierarchy process (AHP) have been studied for over forty years since it was developed by Saaty (1980, 1997). One of the important issues is the axiomatic foundation of the AHP model, and it was only studied by Saaty (1986). Due to the uncertainty of human-originated information, a more flexible expression of decision information should be considered. Although interval-valued pairwise comparisons have been proposed to capture the uncertainty experienced by decision makers (DMs) (Saaty & Vargas, 1987), the existing axiomatic foundation of the AHP model does not reflect the uncertainty (Saaty, 1986). It is found that the breaking of reciprocal property can be used to characterize the uncertainty of pairwise comparisons. Therefore, we focus on the uncertainty-induced axiomatic foundation of the AHP model and its implication. The main contributions are covered as follows:

- The concept of reciprocal symmetry breaking (RSB) has been proposed to capture the uncertainty experienced by DMs when comparing paired alternatives.
- The existing axiomatic foundation of the AHP model has been modified to naturally incorporate into the uncertainty.
- The approximate consistency of pairwise comparison matrices has been defined and the corresponding index has been proposed.
- The eigenvector method has been proved to be still suitable for eliciting the priorities of alternatives from a PCM with approximate consistency.

As compared to the known axiomatic foundation of the AHP model (Saaty, 1986), the modified one has naturally incorporated into the uncertainty experienced by DMs. And it suggests that a flexible expression of decision information with the breaking of reciprocal property could be allowed to some degree in the AHP model. Moreover, it should be pointed out that the complexity of decision-making algorithms is worth studying (Alizadeh, Allen et al., 2020). Then, the main limitation of the present study is that the proposed axiomatic foundation means $n(n-1)$ pairwise comparisons of alternatives, which is more difficult than the typical AHP and the best–worst method (Mi et al., 2019; Rezaei, 2015). In the future, based on the modified axiomatic foundation, the AHP model under uncertainty could be further developed involving approximate consistency improving methods, priority eliciting methods and the others. By incorporating some optimization algorithms (Abualigah & Diabat, 2021a, 2021b; Abualigah et al., 2021), some decision-making models under uncertainty could be the research directions when considering multiple experts, incomplete information, various uncertain environments and practical applications (Soltanisehat et al., 2019).

CRediT authorship contribution statement

Fang Liu: Conceptualization, Methodology, Formal analysis, Funding acquisition, Supervision, Writing - original draft, Writing - review & editing. **Mei-Yu Qiu:** Software, Writing - editing. **Wei-Guo Zhang:** Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

The authors would like to thank the Editor-in-Chief, the Associate Editor and the anonymous reviewers for the constructive suggestions such that the paper has been greatly improved.

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