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Interval multiplicative pairwise comparison matrix: Consistency, indeterminacy and normality

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ABSTRACT

To manifest human judgments, a long-established method called Pairwise Comparison (PC) has been successfully applied in the Analytic Hierarchy Process (AHP). In practice, human judgments are often made with uncertainty, and can be characterized by an Interval Multiplicative Pairwise Comparison Matrix (IMPCM). Since consistency is a key issue that has plagued decision makers and researchers for a long time, it is useful to propose a transformation that can effectively convert an inconsistent IMPCM into a consistent one, especially in group decision-making. However, a consistent IMPCM is not sufficient to be acceptable, *indeterminacy* should also be considered. Moreover, the interval priority weights should be *normalized*. To consider *consistency*, *indeterminacy*, and *normality* simultaneously, we put forward a new definition of *acceptable* IMPCM. To obtain such an *acceptable* IMPCM, we propose a theorem of consistency, a consistent transformation, and a normalized prioritization scheme. As a result, the proposed methods guarantee an inconsistent IMPCM can be directly converted into an *acceptable* IMPCM. Five theorems are proved to corroborate the proposed methods. A numerical example is presented to illustrate the validity and superiority of the proposed methods. Finally, discussion and conclusions are given.

1. Introduction

This study focuses on the interval multiplicative pairwise comparison matrix (IMPCM). The pairwise comparison matrix (PCM) is a well-known method used to manifest human subjective judgments, and has been successfully applied in various areas of management. For example, in Multiple Criteria Decision Making (MCDM) it is adopted by the Analytic Hierarchy Process (AHP). Several review and survey papers on the AHP and the PC can be found [1–3]. The AHP provides a comprehensive framework that decomposes problems systematically to cope with intuitive, rational, and irrational aspects in practical decision problems [4]. The AHP can be considered a reliable decision-making procedure in terms of the modern theory of subjective measurement [5]. To manifest human subjective judgment, let $\mathbf{X} = \{x_1, x_2, \cdots, x_n\}$ be a finite set of n compared entities. A pairwise comparison matrix (PCM) $\mathbf{A}$ on $\mathbf{X}$ is denoted by a $\mathbf{A} = [a_{ij}]_{n\times n}$ where a_{ij}, for every $i,\ j = 1, \cdots, n$, is based on the relative intensity of a compared entity i to another compared entity j. Compared entities can be criteria or alternatives; and relative intensity refers to importance, preference or estimation. Note, the entry a_{ij} can be a preference ratio, i.e., *multiplicative* case, or a preference difference, i.e., *additive* case, or a value belongs to [0, 1] that measures the distance from the indifference (0.5), i.e., *fuzzy* case [6]; that is MPCM, APCM and FPCM, respectively. The multiplicative case

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and additive case are interchangeable by using logarithmic mapping and the exponential mapping [7]. This study focuses on the *multiplicative* case; that is, the entry a_{ij} is an estimation of underlying priority ratio $({}^{w_i}/_{w_j})$.

Several scales have been proposed to present subjective judgment [8–10].[1] For example, Saaty [8] suggested a widely used bipolar ratio scale $[\frac{1}{9}, \frac{1}{8}, \cdots, \frac{1}{2}, 1, 2, \cdots, 8, 9]$ with a neutral value of 1, such that $\frac{1}{9} \le a_{ij} \le 9$, $a_{ij}a_{ji} = 1$, $a_{ii} = 1$, $i, j = 1, 2, \cdots, n$. Hereinafter, a bipolar ratio scale $[\frac{1}{s}, \frac{1}{s-1}, \cdots, \frac{1}{2}, 1, 2, \cdots, s-1, s]$ is called the *s*-scale. An exact value of a_{ij} is interpreted as the compared entity x_i is a_{ij} times preferred to another entity x_j. The greater the a_{ij}, the stronger x_i is preferred to x_j. When $a_{ij} = 1$, it indicates an indifference between x_i and x_j.

However, in practical decision problems characterized by uncertainty, it is difficult for the decision makers to assign precise values in pairwise comparison [11,12]. How to deal with uncertainty in pairwise comparison has received increasing research attention in the past decades [6,13–15]. Various models have been proposed to present pairwise comparison with uncertainty, such as interval additive pairwise comparison matrices [16–20], interval multiplicative pairwise comparison matrices [21–26], triangular fuzzy preference relations [27–29], trapezoidal fuzzy preference relations [30], intuitionistic fuzzy preference relations [31,32], hesitant fuzzy preference relations [33,34] and others, more information can be found [35,36]. This study focuses on the interval multiplicative pairwise comparison matrix (IMPCM).

The study of an IMPCM has attracted increasing attention from its inception to recent years and various definitions of consistency and methods of deriving the priority weights have been proposed 35, 36]. Although those definitions of consistency for IMPCM can be used to check whether an IMPCM is consistent or not [19,37–40], they cannot be used to directly construct a consistent IMPCM. On the other hand, several studies have been devoted to propose solutions of consistency for IMPCM, but most of them use mathematical programming models and are illustrated by only a few examples [39,41,42].

In practice, it is useful to suggest a simple method that can effectively convert an inconsistent IMPCM into a consistent one, especially in group decision-making where multiple experts are involved [14,22,25]. However, a consistent IMPCM is not sufficient to be acceptable, indeterminacy of an IMPCM is also important for a decision maker. A highly indeterminate IMPCM with little or no useful decision information is deemed unacceptable [43]. Moreover, normalized interval priority weights are better to be derived from an IMPCM [44]. The reason is twofold. First, while a crisp PCM leads to crisp priority weights being generated, an IMPCM should give an interval priority weights estimate [45]. Second, for an IMPCM, non-normalized interval priority weights do not make sense because some of them are too wide to be true [41]. Therefore, this study attempts to address these three key issues of IMPCM simultaneously: *consistency*, *indeterminacy* and *normality*.

The contributions of this study are as follows.

- We put forward a new definition of *acceptable* IMPCM that takes *consistency*, *indeterminacy*, and *normality* into consideration simultaneously.
- We propose a new theorem of consistency and a consistent transformation for IMPCM.
- We suggest a normalized prioritization scheme for IMPCM.
- As a result, the proposed methods guarantee an inconsistent IMPCM can be directly converted into an *acceptable* IMPCM.

Below, in Section 2, three concerned issues of IMPCM are reviewed. In Section 3, a solution of simultaneously addressing these three key issues is proposed. We first put forward a stricter definition of *acceptable* IMPCM than the definition proposed by others [42]. Then, we propose a theorem of consistency for IMPCM that can not only be used to check whether an IMPCM is consistent or not, but also can be directly used to transform an inconsistent IMPCM into a consistent one. Additionally, we suggest a normalized prioritization scheme. Five theorems are proved to corroborate the proposed methods. In Section 4, to illustrate the validity and applicability of the proposed methods, a numerical example is presented by comparing with the other three results in the literature. Finally, discussion and conclusions are given.

2. Literature review

In this section, we first introduce brief ideas of the traditional MPCM, and then highlight the main concepts, findings, and methods of the existing works on interval MPCM, especially focusing on the three key points of this study: *consistency*, *indeterminacy and normality*.

2.1. Preliminaries

For a multiplicative PC matrix (MPCM), Saaty introduced two definitions as follows.

Definition 1 [8]**.** An MPCM $\boldsymbol{A} = [a_{ij}]_{n\times n}$ is reciprocal if

$$a_{ij} = {}^{1}/_{a_{ji}}, \ \forall i, \ j = 1, \ 2, \ \ldots, n.$$

Definition 2 [4]**.** An MPCM $\boldsymbol{A} = [a_{ij}]_{n\times n}$ is consistent if

$$a_{ij} = a_{ik}a_{kj}, \ \forall i, \ j, k = 1, \ 2, \ \ldots, n.$$

[1] Note, this study is independent of the scale we adopted.

Table 1
Numbers of consistency checking needed.

n	3	4	5	6	7	8	9	10	11	12	13	14
ne	3	6	10	15	21	28	36	45	55	66	78	91
nt	1	20	120	455	1330	3276	7140	14190	26235	45760	76076	121485

All MPCMs discussed here are *reciprocal*, as it is a fairly natural and rational condition; however, they are not necessarily *consistent*, which is a desirable property that we want to achieve not only from an academic perspective but also from a decision makers' perspective.

Based on Saaty's definition, Kuo introduced a new but logically equivalent definition as follows.

Definition 3 [46]**.** An MPCM $\boldsymbol{A} = [a_{ij}]_{n\times n}$ is consistent if

$$a_{ij} = \sqrt[n]{\left(\prod_{k=1}^{n} a_{ik}\right)\left(\prod_{k=1}^{n} a_{kj}\right)} = \sqrt[n]{\mathrm{R}_i\mathrm{C}_j},\ \forall i,\ \ j = 1,\ 2,\ \ \dots, n.$$

Here, R_i stands for $\prod_{k=1}^{n} a_{ik}$ and C_j stands for $\prod_{k=1}^{n} a_{kj}$. The following two properties hold, whether or not an MPCM is consistent.

$$\text{(a)}\ \mathrm{R}_i\mathrm{C}_i = 1,\ \forall\ i = 1,\ 2,\ \ \dots, n,\ \text{and (b)} \prod_{i=1}^{n}\sqrt[n]{\mathrm{R}_i} = \prod_{j=1}^{n}\sqrt[n]{\mathrm{C}_j} = 1.$$

Proposition 1 [46]**.** *Definition 3 is logically equivalent to Definition 2.*

Proof. By Definition 2, if $\boldsymbol{A} = [a_{ij}]_{n\times n}$ is consistent the following conditions will hold:

$a_{ij} = a_{ik}a_{kj},\ \ \forall i,\ \ j,k = 1,\ 2,\ \ \dots, n.$

Straightforward, multiplying this equation for n times with $k = 1,\ 2,\ \ \dots, n,$ yield:

$$a_{ij} = n\sqrt{\prod_{k=1}^{n}\left(a_{ik}a_{kj}\right)} = n\sqrt{\left(\prod_{k=1}^{n} a_{ik}\left(\prod_{k=1}^{n} a_{kj}\right)\right)} = \sqrt[n]{\mathrm{R}_i\mathrm{C}_j},\ \forall i,\ \ j = 1,\ 2,\ \ \dots, n.$$

On the other hand, if $a_{ij} = n\sqrt{(\prod_{k=1}^{n} a_{ik}(\prod_{k=1}^{n} a_{kj}))} = n\sqrt{R_iC_j}$ hold for all i and j then, by property (a) $\mathrm{C}_k\mathrm{R}_k = 1$ holds, we will have

$a_{ik}a_{kj} = n\sqrt{\mathrm{R}_i\mathrm{C}_k}n\sqrt{\mathrm{R}_k\mathrm{C}_j} = n\sqrt{\mathrm{R}_i\mathrm{C}_j} = a_{ij},\ \forall i, j, k = 1, 2, \dots, n.$ □

The Def. 3 is an explicit and concise definition of consistency because it is expressed in an *integrated* form rather than a *distributed* form. It is easier to use for checking whether an MPCM is consistent, since checking by Def. 2 is *triad-based*, whereas checking by Def. 3 is *entry-based*. Moreover, Def. 3 can be used to convert an inconsistent MPCM to a consistent one and keep the priority weights invariant [46]. Table 1 demonstrates their difference; where n is the size of MPCM, ne is the number of entries ($= \sum_{i=1}^{n-1} i$), and nt is the number of triads ($= \mathrm{C}_3^{ne}$).

Moreover, the Def. 3 is not only a normative definition but also an operational definition. Based on the Def. 3, Kuo proposed a transformation that can rapidly convert an inconsistent MPCM into a consistent one and maintain the most important information: priority weights that are implicitly embedded in the original MPCM. Two theorems are further proposed to corroborate the *uniqueness* of this transformation. The proposed transformation not only works for MPCM but has also been successfully extended to APCM and FPCM [46].

To model pairwise comparison with uncertainty, Saaty and Vargas introduced the notion of IMPCM [21].

Definition 4 [21]**.** An IMPCM $\bar{A}$ on X is characterized by an interval judgment matrix $\bar{A} = (\bar{a}_{ij})_{n\times n}$ with $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$, *and*

$$\frac{1}{s} \le a_{ij}^- \le a_{ij}^+ \le s,\ \ a_{ij}^-a_{ji}^+ = 1,\ \ a_{ii}^- = a_{ii}^+ = 1,\ \ \forall\ i, j = 1, 2, \cdots, n. \tag{1}$$

where $\bar{a}_{ij}$ is an interval preference ratio indicating a compared entity x_i is between a_{ij}^- and a_{ij}^+ times preferred to another compared entity x_j.

Based on interval arithmetic, Eq. (1) can be rewritten as:

$$\bar{a}_{ij} = \left[a_{ij}^-, a_{ij}^+\right] \subseteq \left[\frac{1}{s}, s\right], \bar{a}_{ij} = \frac{1}{\bar{a}_{ji}},\ \ \left[a_{ii}^-, a_{ii}^+\right] = [1, 1],\ \ \forall\ i, j = 1, 2, \cdots, n.$$

Note, we often have $\bar{a}_{ij}\ \odot\ \bar{a}_{ji}\ \neq 1$ for an IMPCM $\bar{\mathrm{A}} = (\bar{a}_{ij})_{n\times n}$, but $\bar{\mathrm{A}}$ has to satisfy the reciprocity of $\bar{a}_{ji} = \frac{1}{\bar{a}_{ij}}$. That is, $\bar{a}_{ji} = [\frac{1}{a_{ij}^+}, \frac{1}{a_{ij}^-}]$. Here, $\odot$ is a multiplicative operation of two interval numbers [42].

To cope with uncertainty, the IMPCM is a widely adopted tool and has promoted numerous studies [22–26]. For example, Liu et al. defined a geometric consistency index (GCI) and proposed an induced ordered weighted geometric averaging operator (GCI-IOWGA) [22]. Jing & Yulin developed a method to aggregate IMPCMs which then applies the logarithmic least squares method (LLSM) to derive the priority weights so the ranking of alternatives can be obtained [23]. To improve the consistency of an IMPCM, López-Morales proposed a method such that decision makers' choices are implicitly including their uncertainty while maintaining acceptable consistency [24]. Wang, Z. J. proposed a weak geometric consistency index (GCI) and a weak GCI induced aggregation operator to rectify defects in the results of Liu et al. [25]. Wang & Lin focused on answering how to estimate the quality of off-diagonal judgments in an IMPCM and obtain an analytical solution to the optimized interval priority weights vector of an IMPCM [26]. Below, three key issues of IMPCM are reviewed: *consistency*, *indeterminacy and normality*.

2.2. Consistency of an IMPCM

Current research on the consistency of IMPCMs can be roughly categorized into two groups [42]. The first group adopts a feasible region idea and asserts the consistency of an IMPCM if there is a consistent crisp comparison matrix within the original interval judgment [37,43,45]. The other category defines consistency based on some mathematical constraints [39,47]. For instance, Leung and Cao [43] proposed a fuzzy consistency definition with consideration of a tolerance deviation and addressed the fuzzy consistency issue within the framework of feasible region of relative priority weights. Wang, Yang, and Xu [37] employed convex feasible regions to define consistent IMPCMs and proposed two-stage logarithmic goal programming to generate interval priority weights. Wang, Yang, and Xu [38] proposed a method to test the consistency of an IMPCM and developed pairs of nonlinear programming to generate interval priority weights. Liu [39] introduced consistency and acceptable consistency of IMPCMs based on two converted crisp multiplicative pairwise comparison matrices.

What follows here are some definitions of consistency of an IMPCM.

Definition 5 [39]**.** An IMPCM $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ is consistent if two constructed MPCMs $A^L = (a_{ij}^L)_{n\times n}$ and $A^U = (a_{ij}^U)_{n\times n}$ have Saaty's consistency, where a_{ij}^L *and* a_{ij}^U are determined by:

$$a_{ij}^{L} = \begin{cases} a_{ij}^- & i < j \\ 1 & i = j \\ a_{ij}^+ & i > j \end{cases}, \qquad a_{ij}^{U} = \begin{cases} a_{ij}^+ & i < j \\ 1 & i = j \\ a_{ij}^- & i > j \end{cases} \tag{2}$$

Theorem 1 [47]**.** *An IMPCM* $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ *is consistent if and only if*

$$a_{ik}^- a_{ik}^+ = a_{ij}^- a_{ij}^+ a_{jk}^- a_{jk}^+, \ \forall\ i,\ \ j,\ \ k = 1, 2, \cdots, n. \tag{3}$$

Definition 6 [42,47]**.** An IMPCM $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ is consistent if and only if the MPCM $A = (\sqrt{a_{ij}^- a_{ij}^+})_{n\times n}$ is consistent.

Definition 7 [42]**.** Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = [a_{ij}^-, a_{ij}^+]$ be an IMPCM with $\frac{1}{s} \le a_{ij}^- \le a_{ij}^+ \le s$, $\bar{A}$ is acceptably consistent if the MPCM $A = (\sqrt{a_{ij}^- a_{ij}^+})_{n\times n}$ is acceptably consistent.

This study adopts Theorem 1 and Definition 6 [42,47].

2.3. Indeterminacy of an IMPCM

However, an IMPCM is not sufficient to be acceptable, even if it is consistent, and a highly indeterminate IMPCM with little or no useful decision information is deemed unacceptable [42]. All in all, it has been stipulated we can associate a consistency and an indeterminacy value with each IMPCM [6]. Li et al. introduced two kinds of indeterminacy value: *indeterminacy ratio* and *indeterminacy index* as follows.

Definition 8 [42]**.** Let $\bar{a}_{ij} = [a_{ij}^-, a_{ij}^+]$ be an interval comparison judgment on a bounded scale [1/s, s], then its indeterminacy ratio, denoted by IR($\bar{a}_{ij}$), is defined by $IR(\bar{a}_{ij}) = \frac{a_{ij}^+}{a_{ij}^-}$.

Clearly, we have $1 \le \mathrm{IR}(\bar{a}_{ij}) \le s^2$, and $\mathrm{IR}(\bar{a}_{ij}) = 1$ implying $\bar{a}_{ij}$ is a crisp value. The larger the IR($\bar{a}_{ij}$), the more indeterminate the judgment $\bar{a}_{ij}$ is.

In addition to the definition of an *indeterminacy ratio* that is a local view, Li et al. further introduced another definition of an *indeterminacy index* that plays a global view.

Definition 9 [42]**.** Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ be an IMPCM, a geometric-mean-based indeterminacy index of $\bar{A}$ is defined as

$$\mathrm{II}(\bar{\mathrm{A}}) = \left(\prod_{i<j}\left(\frac{a_{ij}^+}{a_{ij}^-}\right)\right)^{\frac{2}{n^2-n}}$$

It is obvious $\mathrm{II}(\bar{\mathrm{A}}) \geq 1$. If $\mathrm{II}(\bar{\mathrm{A}}) = 1$, then $a_{ij}^+ = a_{ij}^-$ for all $i,\ j = 1, 2, \cdots, n$. That is, all $\bar{a}_{ij}$ becomes a crisp value and $\bar{\mathrm{A}}$ is thus reduced to an MPCM; otherwise, $\bar{\mathrm{A}}$ contains indeterminacy, and the greater the $\mathrm{II}(\bar{\mathrm{A}})$, the more indeterminate the $\bar{\mathrm{A}}$. To control the quality of an IMPCM, it is not enough to be consistent; the indeterminacy also has to be within an acceptable threshold.

Definition 10 [42]**.** Let $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$ be an IMPCM and t_{ur} $(t_{ur} > 1)$ be an acceptable indeterminacy ratio threshold, if $IR(\bar{a}_{ij}) \leq t_{ur}$ for all $i, j = 1,\ 2, \cdots, n$, and $\bar{A}$ is acceptably consistent under Definition 7, then $\bar{A}$ is called acceptable; otherwise, $\bar{A}$ is unacceptable.

This study adopts notations of an *indeterminacy ratio* (Definition 8) and *indeterminacy index* (Definition 9), but modifies the concept of an *acceptable* IMPCM (Definition 10).

2.4. Normality of an IMPCM

In a hierarchical structure, if the local priority vectors are not normalized, then the global priority weights, which are the aggregation of the local priority weights, will become meaningless [38]. Several studies have been conducted on this topic [14,18,40,45], and most of them use the mathematical programming approach. What follows here are two definitions of normalized interval priority weights.

Definition 11 [49,50]**.** If an interval priority weights vector $W = (w_1,\ w_2, \cdots, w_n)$ with $w_i = [w_i^-, w_i^+]$ and $0 \leq w_i^- \leq w_i^+$ for all $i = 1,\ 2, \cdots, n$ satisfies $(\sum_{i=1}^n w_i^-) + \max_j(w_j^+ - w_j^-) \leq 1,$ and $(\sum_{i=1}^n w_i^+) - \max_j(w_j^+ - w_j^-) \geq 1,$ then it is normalized; otherwise it is not normalized.

In a similar way to normalize interval priority weights as Definition 11, there is another definition as follows.

Definition 12 [42]**.** If an interval priority weights vector $W = (w_1,\ w_2, \cdots, w_n)$ with $w_i = [w_i^-, w_i^+]$ and $0 \leq w_i^- \leq w_i^+$ for all $i = 1,\ 2, \cdots, n$ satisfies $w_i^+ \cdot \prod_{j\neq i} w_j^- \leq 1$ and $w_i^- \cdot \prod_{j\neq i} w_j^+ \geq 1$ then it is normalized; otherwise it is not normalized.

Since each method of normalization is based on completely different ideas, comparing different approaches seems to be disputable [48]. This study adopts Definition 11.

In light of the three issues of concern reviewed above, the following three research gaps motivated us to conduct this study. First, we need a definition of consistent IMPCM that not only can be used to check whether an IMPCM is consistent or not but also can be used to construct a consistent IMPCM. Second, we need a consistent transformation that also can reduce the maximal indeterminacy ratio of an IMPCM. Third, we need an easy prioritization scheme that guarantees normalized interval priority weights would be derived.

3. The proposed methods

To bridge the three gaps mentioned above, we first propose a definition of *acceptable* IMPCM as follows.

Definition 13. For an IMPCM $\bar{\boldsymbol{A}} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$, let the acceptable indeterminacy ratio threshold denoted by $a^* = \max_{i,j} IR(\bar{a}_{ij})$, then a constructed IMPCM $\bar{\boldsymbol{B}} = (\bar{b}_{ij})_{n\times n} = ([b_{ij}^-, b_{ij}^+])_{n\times n}$ is called acceptable IMPCM if it satisfies the following three conditions:

(1) $\bar{\mathbf{B}}$ is consistent, and
(2) $\mathrm{IR}(\bar{b}_{ij}) < a^*$, for all $i,\ j = 1, 2, \cdots, n$.
(3) Interval priority weights $\mathbf{W}$ of $\bar{\mathbf{B}}$ are normalized.

Compared to the Definition 10 that made a compromise of consistency and without consideration of prioritization, we request fully consistent IMPCM should be achieved and priority weights should be normalized. To obtain such an *acceptable* IMPCM, the ultimate goal of this study, we then propose a new theorem of consistency, and suggest a normalized prioritization scheme. The challenge is the three theorems should be proved to corroborate the constructed IMPCM so the suggested methods can satisfy the three conditions in Definition 13. The idea is based on a previous result which is successfully applied in MPCM, APCM, and FPCM [46]. For an IMPCM $\bar{\mathrm{A}} = (\bar{a}_{ij})_{n\times n}$, it has to satisfy reciprocity of $\bar{a}_{ji} = \frac{1}{\bar{a}_{ij}}$. That is, $\bar{a}_{ji} = [\frac{1}{a^+_{ij}}, \frac{1}{a^-_{ij}}]$. Thus, for an IMPCM $\bar{A} = (\bar{a}_{ij})_{n\times n}$ with $\bar{a}_{ij} = [a^-_{ij}, a^+_{ij}]$ we denote, hereinafter,

$$R^-_i = \left(\prod_{k=1}^{i-1} \frac{1}{a^+_{ki}}\right)\left(\prod_{k>i}^{n} a^-_{ik}\right), \tag{4}$$

$$R^+_i = \left(\prod_{k=1}^{i-1} \frac{1}{a^-_{ki}}\right)\left(\prod_{k>i}^{n} a^+_{ik}\right). \tag{5}$$

Since the first condition of being an *acceptable* IMPCM is to be a consistent IMPCM, we propose the following theorem that is not only normative but also operational. That means, based on this theorem, we are capable of suggesting a transformation that an inconsistent IMPCM could be converted into a consistent one.

Theorem 2. *An IMPCM $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a^-_{ij}, a^+_{ij}])_{n\times n}$ is consistent if and only if*

$$a^-_{ij}a^+_{ij} = \sqrt[n]{\frac{R^-_i R^+_i}{R^-_j R^+_j}}\ \forall\ i,\ \ j = 1, 2, \cdots, n \tag{6}$$

Proof. Simply, multiplying Eq. (3) n times with $k = 1,\ 2,\ \ldots, n,$ yields:

$$a^-_{ij}a^+_{ij} = \sqrt[n]{\frac{R^-_i R^+_i}{R^-_j R^+_j}},\ \forall\ i,\ \ j = 1, 2, \cdots, n$$

Therefore, according to Theorem 1, we complete the proof. □

However, Eq. (6) implies there are several possible combinations of b^-_{ij} and b^+_{ij}.

To guarantee $b^-_{ij} \le b^+_{ij}$ hold, we suggest a transformation method as follows:

$$b^-_{ij} = \min\left(\sqrt[n]{\frac{R^-_i}{R^-_j}}, \sqrt[n]{\frac{R^+_i}{R^+_j}}\right),\ \ b^+_{ij} = \max\left(\sqrt[n]{\frac{R^-_i}{R^-_j}}, \sqrt[n]{\frac{R^+_i}{R^+_j}}\right) \tag{7}$$

For this constructed IMPCM $\bar{B} = (\bar{b}_{ij})_{n\times n} = ([b^-_{ij}, b^+_{ij}])_{n\times n}$, we suggest obtaining the normalized interval priority weights as follows. First, according to Eqs. (4) and (5), we have

$$R'^-_i = \left(\prod_{k=1}^{i-1} \frac{1}{b^+_{ki}}\right)\left(\prod_{k>i}^{n} b^-_{ik}\right), \text{ and } R'^+_i = \left(\prod_{k=1}^{i-1} \frac{1}{b^-_{ki}}\right)\left(\prod_{k>i}^{n} b^+_{ik}\right).$$

Then, we normalize geometric mean of each row of $\bar{B}$, that is we set

$$R''^-_i = \frac{\sqrt[n]{R'^-_i}}{\sum_{i=1}^{n} \sqrt[n]{R'^-_i}} \text{ and } R''^+_i = \frac{\sqrt[n]{R'^+_i}}{\sum_{i=1}^{n} \sqrt[n]{R'^+_i}}$$

Finally, we set the interval priority weights as follows.

$$w^-_i = min\left(R''^-_i, R''^+_i\right),\ w^+_i = max\left(R''^-_i, R''^+_i\right) \tag{8}$$

In the following, three theorems are proved to corroborate the suggested methods mentioned above.

3.1. Consistency: can we obtain a consistent IMPCM?

First, by Eq. (7), we guarantee an inconsistent IMPCM can be transformed into a consistent one.

Theorem 3. *For an IMPCM $\bar{A} = (\bar{a}_{ij})_{n\times n}$ with $\bar{a}_{ij} = [a^-_{ij}, a^+_{ij}]$ the constructed, under the transformation by Eq. (7), IMPCM $\bar{B} = (\bar{b}_{ij})_{n\times n} = ([b^-_{ij}, b^+_{ij}])_{n\times n}$ is consistent.*

Proof. By Eq. (7), we have the constructed IMPCM $\bar{B} = (\bar{b}_{ij})_{n\times n}$ with $\bar{b}_{ij} = [b^-_{ij}, b^+_{ij}]$ where

$$b^-_{ij} = \min\left(\sqrt[n]{\frac{R^-_i}{R^-_j}},\ \sqrt[n]{\frac{R^+_i}{R^+_j}}\right),\ b^+_{ij} = \max\left(\sqrt[n]{\frac{R^-_i}{R^-_j}},\ \sqrt[n]{\frac{R^+_i}{R^+_j}}\right)$$

According to Definition 6, we have a corresponding MPCM B,

$$B = \left(b_{ij}\right)_{n\times n} = \left(\sqrt[n]{b^-_{ij}b^+_{ij}}\right)_{n\times n} = \left(\sqrt[n^2]{\frac{R^-_i R^+_i}{R^-_j R^+_j}}\right)_{n\times n}.$$

Clearly, the geometric mean of row i of the MPCM B is

$$n\sqrt{R_i} = \frac{n^2\sqrt{\mathrm{R}_i^-\mathrm{R}_i^+}}{n\sqrt{\prod_{j=1}^{n} n^2\sqrt{\mathrm{R}_j^-\mathrm{R}_j^+}}}.$$

Similarly, $n\sqrt{C_j} = \frac{1}{n\sqrt{R_j}} = \frac{n\sqrt{\prod_{i=1}^{n} n^2\sqrt{\mathrm{R}_i^-\mathrm{R}_i^+}}}{n^2\sqrt{\mathrm{R}_j^-\mathrm{R}_j^+}}$. Thus, we have $n\sqrt{R_iC_j} = n^2\sqrt{\frac{\mathrm{R}_i^-\mathrm{R}_i^+}{\mathrm{R}_j^-\mathrm{R}_j^+}}$.

By Definition 2 or 3, obviously, MPCM B is consistent. Therefore, according to Definition 6, IMPCM $\bar{\mathrm{B}}$ is consistent. □

3.2. Indeterminacy: can we reduce the indeterminacy of an IMPCM?

Second, the following theorem concerns the upper bound of indeterminacy ratio of an IMPCM $\bar{\mathrm{B}}$ that is constructed using the proposed transformation from an IMPCM $\bar{\mathrm{A}}$.

Theorem 4. *By Eq. (7), we will have* $IR(\bar{b}_{ij}) < a^* = \max_{i,j} IR(\bar{a}_{ij})$ *for all i, j.*

Proof. For any certain i and j, we have $\mathrm{IR}(\bar{b}_{ij}) = \frac{b_{ij}^+}{b_{ij}^-} = \frac{\max\left(n\sqrt{\frac{R_i^-}{R_j^-}}, n\sqrt{\frac{R_i^+}{R_j^+}}\right)}{\min\left(n\sqrt{\frac{R_i^-}{R_j^-}}, n\sqrt{\frac{R_i^+}{R_j^+}}\right)}$, that is

$$\begin{cases} \frac{R_i^-}{R_j^-} < \frac{R_i^+}{R_j^+}, \mathrm{IR}(\bar{b}_{ij}) = n\sqrt{\prod_{k=1}^{i-1}\frac{a_{ki}^+}{a_{ki}^-}\prod_{k>i,k\neq j}^{n}\frac{a_{ik}^+}{a_{ik}^-}\prod_{k=1,k\neq i}^{j-1}\frac{a_{kj}^-}{a_{kj}^+}\prod_{k>j}^{n}\frac{a_{jk}^-}{a_{jk}^+}} \\ \frac{R_i^-}{R_j^-} > \frac{R_i^+}{R_j^+}, \mathrm{IR}(\bar{b}_{ij}) = n\sqrt{\prod_{k=1}^{i-1}\frac{a_{ki}^-}{a_{ki}^+}\prod_{k>i,k\neq j}^{n}\frac{a_{ik}^-}{a_{ik}^+}\prod_{k=1,k\neq i}^{j-1}\frac{a_{kj}^+}{a_{kj}^-}\prod_{k>j}^{n}\frac{a_{jk}^+}{a_{jk}^-}} \\ \frac{R_i^-}{R_j^-} = \frac{R_i^+}{R_j^+}, \mathrm{IR}(\bar{b}_{ij}) = 1 \end{cases}$$

Since $a_{ij}^- \le a_{ij}^+$ holds for all i and j, we have $\mathrm{IR}(\bar{b}_{ij}) < a^*,\ \forall\ i, j = 1,\ 2, \cdots, n$. □

That is, under the transformation by Eq. (7), the maximal indeterminacy ratio of an IMPCM can be reduced.

3.3. Normality: can we obtain normalized interval priority weights?

Third, by Eqs. (7) and (8), we guarantee normalized interval priority weights can be derived from the constructed IMPCM $\bar{\mathrm{B}}$.[2]

Theorem 5. *The interval priority weights of the constructed IMPCM* $\bar{B}$ *are normalized.*

Proof. Since there are three possible cases of relation between $R_i''^-$ and $R_i''^+$: $R_i''^- < R_i''^+$, $R_i''^- = R_i''^+$ and $R_i''^- > R_i''^+$, by using (8), we may confronted with $w_i^- = R_i''^+$ and $w_i^+ = R_i''^-$, we term this situation a “swap”. Clearly, there are n_1 times of “swap”, where $1 \le n_1 < n$.[3] Without loss of generality, assume these n_1 times of “swap” occurred in $i = 1,\ 2, \cdots, n_1$. In other words, we have $R_i''^- > R_i''^+$ when $i = 1, 2, \cdots, n_1$, and $R_i''^- \le R_i''^+$ when $i = n_1 + 1, \cdots, n$. Then, for the maximum width of $w_j^+ - w_j^-$, there are two possible cases:

Case 1. $1 \le j \le n_1$. That is, there is a “swap” at j. Thus, $\sum_{i=1}^{n} w_i^- = (\sum_{i=1}^{n_1} R_i''^+) + (\sum_{i=n_1+1}^{n} R_i''^-)$. Since $\sum_{i=1}^{n} R_i''^- = 1$, then we have $\sum_{i=1}^{n} w_i^- = (\sum_{i=1}^{n_1} R_i''^+) + (1 - \sum_{i=1}^{n_1} R_i''^-)$. Besides, since there is a “swap” at j, we have $w_j^+ - w_j^- = R_j''^- - R_j''^+$. Therefore, we have

$$\sum_{i=1}^{n} w_i^- + \max_j \left(w_j^+ - w_j^-\right) = \left(\sum_{i=1}^{n_1} R_i''^+\right) + \left(1 - \sum_{i=1}^{n_1} R_i''^-\right) + R_j''^- - R_j''^+ = \left(\sum_{i=1,i\neq j}^{n_1} R_i''^+\right) + \left(1 - \sum_{i=1,i\neq j}^{n_1} R_i''^-\right) \le 1.$$

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[2] Note, the interval priority weight of this constructed IMPCM $\bar{\mathrm{B}}$ is also normalized [14] with $[w_i^-, w_i^+] \subset [0, 1], \sum_{j=1,\ j\neq i}^{n} w_j^- + w_i^+ \le 1,\ \sum_{j=1,\ j\neq i}^{n} w_j^+ + w_i^- \ge 1, i = 1, 2, \cdots, n$.

[3] The reason is $n_1 = 0$ means for $i = 1,\ 2, \cdots, n$, we have $R''^-_i < R''^+_i$, and $n_1 = n$ means for $i = 1,\ 2, \cdots, n$, we have $R''^-_i > R''^+_i$; however, both cases are impossible because we have $\sum_{i=1}^{n} R''^-_i = \sum_{i=1}^{n} R''^+_i = 1$. Note, for a trivial case where for $i = 1,\ 2, \cdots, n$, we have $R''^-_i = R''^+_i$, the derived priority weight is not an interval priority weight but a crisp priority weight.

Case 2. $n_1 < j \le n$. That is, there is no "swap" at j. Thus, $\sum_{i=1}^{n} w_i^- = (\sum_{i=1}^{n_1} R_i''^{+}) + (\sum_{i=n_1+1}^{n} R_i''^{-})$. Then we have $\sum_{i=1}^{n} w_i^- = (1-\sum_{i=n_1+1}^{n} R_i''^{+}) + (\sum_{i=n_1+1}^{n} R_i''^{-})$, since $\sum_{i=1}^{n} R_i''^{+} = 1$. Besides, since there is no "swap" at j, we have $w_j^+ - w_j^- = R_j''^{+} - R_j''^{-}$. Therefore, we have

$$\left(\sum_{i=1}^{n} w_i^-\right) + \max_j\left(w_j^+ - w_j^-\right) = \left(1 - \sum_{i=n_1+1}^{n} R_i''^{+}\right) + \left(\sum_{i=n_1+1}^{n} R_i''^{-}\right) + R_j''^{+} - R_j''^{-} = \left(1 - \sum_{i=n_1+1, i\neq j}^{n} R_i''^{+}\right)$$
$$+ \left(\sum_{i=n_1+1, i\neq j}^{n} R_i''^{-}\right) \le 1.$$

Similarly, we can prove $(\sum_{i=1}^{n} w_i^+) - \max_j(w_j^+ - w_j^-) \ge 1$. □

Consequently, according to Theorems 3∼5, we have the following theorem guaranteeing the proposed methods will obtain an *acceptable* IMPCM.

Theorem 6. *By using the proposed methods, for an IMPCM* $\bar{A} = (\bar{a}_{ij})_{n\times n} = ([a_{ij}^-, a_{ij}^+])_{n\times n}$, *we will have an acceptable IMPCM* $\bar{B} = (\bar{b}_{ij})_{n\times n} = ([b_{ij}^-, b_{ij}^+])_{n\times n}$.

Proof. First, according to Theorem 3, we know the constructed IMPCM $\bar{B}$ is consistent. Second, by Theorem 4, we have $IR(\bar{b}_{ij}) < a^* = \max_{i,j} IR(\bar{a}_{ij})$ for all i, j. Third, according to Theorem 5, we know the interval priority weights of the constructed IMPCM $\bar{B}$ are normalized. Therefore, by Definition 13, the constructed IMPCM $\bar{B}$ is an acceptable IMPCM. □

4. Numerical example

Next, by comparing with the other three results in the literature, we provide a numerical example to illustrate the validity and superiority of the proposed methods.

Example 1. Consider the following IMPCM $\bar{\mathbf{A}} = (\bar{a}_{ij})_{4\times 4} = ([a_{ij}^-, a_{ij}^+])_{4\times 4}$, which was examined by Li et al. [42], Liu [39], and Wang, Yang, & Xu [37].[4]

1	1	2	5	2	4	1	3
0.2	0.5	1	1	1	3	1	2
0.25	0.5	1/3	1	1	1	0.5	1
1/3	1	0.5	1	1	2	1	1

According to Definition 8, the maximal *indeterminacy ratio* of $\bar{\mathbf{A}}$ is $\mathrm{IR}(\bar{a}_{14}) = \mathrm{IR}(\bar{a}_{23})$ =3. By Definition 9, the *indeterminacy index* of $\bar{\mathbf{A}}$ is $\mathrm{II}(\bar{\mathbf{A}})$ =2.3762.

The first compared result $\overline{\mathbf{A}_1^*}$ [42] is as follows.

1	1	1.2975	3.4471	2.0881	4.9616	1.4765	3.5087
0.2901	0.7707	1	1	0.9874	2.3461	0.6981	1.6591
0.2015	0.4789	0.4262	1.0128	1	1	0.485	1.0309
0.285	0.6773	0.6027	1.4324	0.97	2.0619	1	1

The maximal *indeterminacy ratio* of $\overline{\mathbf{A}_1^*}$ is $\mathrm{IR}(\bar{a}_{12})$ =2.6567; and the *indeterminacy index* of $\overline{\mathbf{A}_1^*}$ is $\mathrm{II}(\overline{\mathbf{A}_1^*})$ =2.3763.

The second compared result $\overline{\mathbf{A}_2^*}$ [39,42] is as follows.

1	1	1.3512	3.3098	2	5.18	1.1892	4.3562
0.3021	0.7401	1	1	1.8922	1.9479	0.7071	1.6381
0.1931	0.5	0.5285	0.5134	1	1	0.4518	1.1067
0.2296	0.8409	0.6105	1.4142	0.9036	2.2134	1	1

The maximal *indeterminacy ratio* of $\overline{\mathbf{A}_2^*}$ is $\mathrm{IR}(\bar{a}_{14})$ =3.6631; and the *indeterminacy index* of $\overline{\mathbf{A}_2^*}$ is $\mathrm{II}(\overline{\mathbf{A}_2^*})$ =2.2671.

The third compared result $\overline{\mathbf{A}_3^*}$ [37,42] is as follows.

1	1	1.5143	3.2239	2	4.899	1.6818	3.5676
0.3102	0.6604	1	1	0.9036	2.2134	0.7598	1.6119
0.2041	0.5	0.4518	1.1067	1	1	0.5	1.2247
0.2803	0.5946	0.6204	1.3161	0.8165	2	1	1

The maximal *indeterminacy ratio* of $\overline{\mathbf{A}_3^*}$ is $\mathrm{IR}(\bar{a}_{13}) = \mathrm{IR}(\bar{a}_{23})$ =2.4495; and the *indeterminacy index* of $\overline{\mathbf{A}_3^*}$ is $\mathrm{II}(\overline{\mathbf{A}_3^*})$ =2.2809.

[4] For simplicity, we use a table to represent an IMPCM.

Table 2
indeterminacy ratio vs. indeterminacy index.

	maximal indeterminacy ratio	*indeterminacy index*
$\bar{\mathbf{A}}$	3	2.3762
$\overline{\mathbf{A}_1^*}$	2.6567	2.3763
$\overline{\mathbf{A}_2^*}$	3.6631	2.2671
$\overline{\mathbf{A}_3^*}$	2.4495	2.2809
$\bar{\mathbf{B}}$	1.0574	1.0379

Table 3
Interval priority weights.

$\bar{\mathbf{A}}$		$\overline{\mathbf{A}_1^*}$		$\overline{\mathbf{A}_2^*}$		$\overline{\mathbf{A}_3^*}$		$\bar{\mathbf{B}}$	
w_i^-	w_i^+	w_i^-	w_i^+	w_i^-	w_i^+	w_i^-	w_i^+	w_i^-	w_i^+
0.4456	0.4541	0.4456	0.4541	0.4189	0.4754	0.461	0.4563	0.4499	0.4499
0.2107	0.2147	0.2107	0.2147	0.2495	0.2005	0.2085	0.2063	0.2127	0.2127
0.1424	0.1372	0.1424	0.1372	0.145	0.1181	0.1422	0.1511	0.1398	0.1398
0.2013	0.194	0.2013	0.194	0.1866	0.206	0.1884	0.1863	0.1977	0.1977

According to Theorem 1, the three IMPCMs $\bar{\mathbf{A}}$, $\overline{\mathbf{A}_2^*}$, and $\overline{\mathbf{A}_3^*}$ are inconsistent. Although IMPCM $\overline{\mathbf{A}_1^*}$ is consistent, it does not reduce the *indeterminacy index* of $\bar{\mathbf{A}}$, and remains the same interval priority weights of $\bar{\mathbf{A}}$ that does not follow the condition of $w_i^- \le w_i^+$ for $i = 3,\ 4$. By using the proposed transformation, we have an IMPCM $\bar{\mathbf{B}} = (\bar{b}_{ij})_{4\times 4} = ([b_{ij}^-, b_{ij}^+])_{4\times 4}$ as follows.

1	1	2.1147	2.1147	3.1302	3.3098	2.2134	2.3403
0.4729	0.4729	1	1	1.4802	1.5651	1.0466	1.1067
0.3021	0.3195	0.6389	0.6756	1	1	0.7071	0.7071
0.4273	0.4518	0.9036	0.9554	1.4142	1.4142	1	1

As predicted, it is an *acceptable* IMPCM because it satisfies three conditions of Definition 13. First, according to Theorem 1, the IMPCM $\bar{\mathbf{B}}$ is consistent. Second, we have $\mathrm{IR}(\bar{b}_{13}) = \mathrm{IR}(\bar{b}_{14}) = \mathrm{IR}(\bar{b}_{23}) = \mathrm{IR}(\bar{b}_{24}) = 1.0574$ and $\mathrm{IR}(\bar{b}_{12}) = \mathrm{IR}(\bar{b}_{34}) = 1$, all are less than the acceptable *indeterminacy ratio* threshold a^* that is $\max_{i,j} \mathrm{IR}(\bar{a}_{ij}) = \mathrm{IR}(\bar{a}_{14}) = \mathrm{IR}(\bar{a}_{23}) = 3$. Third, interval priority weights are normalized. Note, the interval priority weights of $\bar{\mathbf{B}}$ are converged to a crisp priority weight. Moreover, the *indeterminacy index* of $\bar{\mathbf{B}}$ is $\mathrm{II}(\bar{\mathbf{B}}) = 1.0379$. The maximal indeterminacy ratio and indeterminacy index of these five IMPCMs are presented in Table 2 as follows.

The interval priority weights of these five IMPCMs are presented in Table 3.

5. Discussion and conclusion

This study provides a solution for IMPCM that can effectively convert an *inconsistent* IMPCM into an *acceptable* IMPCM considering *consistency*, *indeterminacy*, and *normality* simultaneously. The contributions of this study are twofold: theory and practice. In theory, (a) we propose a new definition of an acceptable IMPCM; (b) we propose a new theorem of consistency for IMPCM that is not only normative but also operational; (c) we suggest a normalized prioritization method for IMPCM. In practice, compared with other efforts in the literature that detect and rectify the consistency of IMPCM, the proposed solution not only considers *consistency*, *indeterminacy*, and *normality* simultaneously, but also is computational simple. In other words, the solution of this study can effectively help decision makers to improve the quality and efficiency of their task, especially in group decision-making where multiple experts are involved. Finally, it would be helpful if we can extend this study to other types of pairwise comparison matrix.

Declaration of Competing Interest

None.

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