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Axiomatization of inconsistency indicators for pairwise comparisons

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ABSTRACT

This study proposes revised axioms for defining inconsistency indicators in pairwise comparisons. It includes the new findings that "PC submatrix cannot have a worse inconsistency indicator than the PC matrix containing it" and that there must be a PC submatrix with the same inconsistency as the given PC matrix. The use of normalized distance for the inconsistency at the level of a triad corrected the former insufficiency in expressing it in the 2014 study on axiomatization by Koczkodaj and Szwarc.

1. Introduction

Pairwise comparisons have a long and colorful history traced to Ramon Lull, called in Latin Raimundus (Raymundus) Lullus, who was a Majorcan philosopher and logician. He is also regarded as a pioneer of computation theory for his use of binary numbers and elements of logic. Three of Lull's works, related to electoral systems, are based on pairwise comparisons. The exact date of his first written publication, "Artifitium Electionis Personarum" is not known but it was probably between 1274 and 1283. In [18], a recent project of national importance (for one of EU member countries) was presented.

The inconsistency concept in pairwise comparisons was presented in [10]. However, this was a cardinal (count) inconsistency indicator. Counting inconsistent triads has been replaced by more sophisticated inconsistency indicators (also known as inconsistency indexes or consistency indexes). Our axiomatization follows principles generally recognized for constructing all axiomatizations: simplicity, internal consistency, and independence. Certainly, the assumption of minimization is implicit as it is difficult to imagine a system based on, let us say, 1,000 axioms that would be of any practical use.

The first published axiomatization of inconsistency in pairwise comparisons was proposed in [15] in 2014 and was followed by an independent study [5] which was published in 2015. Regretfully, both proposed axiomatizations proved to be somehow deficient as the revisions indicate. The lack of explicit extendibility to higher sizes was brought to the attention of the first author of [15] by Brunelli (see [2] and [4]). The new version of [5] has been posted on arXiv (31 July 2015, see [3]). However, axioms proposed by [5] and [3] have two problems. The first problem is the mathematical complexity. The second problem is the approximation error tolerance (addressed by our axiom A.3). An arbitrarily large value (e.g., 1,000,000% or more) is tolerated in the input data by at least one inconsistency indicator (CI proposed in [20]), as demonstrated in [15].

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This is exactly what a proper inconsistency indicator should not permit to occur. In fact, the detection property of the approximation error in the input data was our goal in this study and the proposed axiom A.3 has been formulated to prevent it.

The elegant "CONTRACTION" axiom is based on common sense. The inconsistency indicator of a PC submatrix B of a PC matrix A cannot be larger than the inconsistency of a bigger PC matrix A. In fact, the intuition behind "the bigger PC matrix the more inconsistency" is undisputed. The more data we have the more mistakes (hence also inconsistency) may occur. In a square matrix, data grow with the square of its size while the number of triads grow with the cube. It was communicated to us by [24] but might have been independently proposed as an axiom and published by another researcher.

1.1. Structure of the paper

The structure of our work is as follows. Section 3, subdivided into subsections, examines the definition of the basic concepts necessary to formulate our axioms. In particular, we define an exponentially invertible measure (EIM), on the set $\mathbb{R}_+^*$, which measures the distance from the given triad to the corresponding transient (consistent) triad. We also prove the basic properties of EIM.

The intuition and mathematical elegance behind the introduction of a distance measure EIM are related to the alignment of structures. PC matrix has ratios as element values. Ratios have a multiplicative character while the Euclidean distance is (in essence) additive. The proposed exponentially invertible measure (EIM) ameliorates this problem. It is the transformation of the distance in the logarithmic linear space back to the given space by the exponential mapping.

The introduction of EIM plays an important role in the formulation of the axiom A.3.

In Section 3.4, we formulate four axioms A.1–A.4 that should be met by the inconsistency indicator. Next, we discuss in detail the reasons for choosing these axioms.

In Section 2, we discuss the shortcomings of previous axiomatization.

In Section 4, we show consistency of axioms A.1–A.4.

In the next Section 5 we are proving the independence of our axioms.

In Section 6, we give various examples of inconsistency measures that meet and do not meet the axioms A.1–A.4.

Finally, we summarize our article in Section 7.

2. Deficiencies of the previous axiomatizations

The axiomatization proposed by Koczkodaj and Szwarc in [15] was incomplete but not erroneous. In essence, axioms were formulated only for a single triad $\mathcal{T} = (x, y, z)$ and did not cover all cases of increasing/decreasing of the inconsistency indicators in the triad:

1. x being increased and z being decreased,
2. x decreased with z increased.

The construction of PC matrices of the size $n > 3$ was not explicitly included in the axioms but it has been implicitly assumed to be the same for Koczkodaj's inconsistency indicator in [11] and [6]. However, the axiomatic system does not need to be complete and even the first two axioms (presented in [15]) create a valid axiomatic system. These axioms are: the consistency condition and the normalization, intensively examined in [13]. The mathematical reasoning as well as the practicality behind normalization makes it as indispensable for inconsistency as it is for probability, fuzzy logic, and other approximate reasoning methods. Until Kolmogorov axioms have been universally accepted, some mathematicians used $[0, \infty]$ for the probability but ∞, unlike 1, is not a number hence normalization simplifies its use.

It is worth to notice, that despite some deficiencies, two counter-examples presented in [15] and their mathematical analysis are correct. They demonstrate that an approximation error in the input data of an arbitrary value is tolerated by an eigenvalue-based (but not only) inconsistency indicator.

The axiomatizations proposed in [5,3] assumed continuity (hence the important case of 0–1 for expressing consistency and the lack of it). It has not included the need normalization.

3. Axiomatization of the PC matrix inconsistency indicator

3.1. PC matrices

Pairwise comparisons can be stored in a pairwise comparisons matrix (for short, PC matrix). It is a square $n \times n$ matrix $M = [m_{ij}]$ with real positive elements $m_{ij} > 0$ for every $i, j = 1, \ldots, n$, where m_{ij} represent ratios expressing a relative preference of an entity E_i over E_j. The entity could be any object, attribute of it or a stimulus. It can be abstract and usually there is not a well established measure such as a meter or kilogram. "Software safety" or "environmental friendliness" are examples of such attributes or entities to compare in pairs.

Ratios often express subjective preferences of two entities but it does not mean that it can be done by division. In fact, equalizing the ratios with the division (e.g., E_i/E_j) is unacceptable. When entities are subjective, for example, reliability and robustness of software (commonly used in a software development process as product attributes), the division operation has no mathematical meaning. However, we can still consider which of them is more important than the other for a given project. The use of symbol "/" is in the context of "related to" (not the division of two numbers). Problems with some popular customizations of PCs have been addressed in [14].

A PC matrix M is called *reciprocal* if $m_{ij} = \frac{1}{m_{ji}}$ for every $i, j = 1, \ldots, n$ (in such case, $m_{ii} = 1$ for every $i = 1, \ldots, n$).

Without loss of generality, we can (and will) assume that the PC matrix with positive real entries is reciprocal. If it is not, we can easily make it reciprocal by the theory presented in [12]. Every non-reciprocal PC matrix $A = [a_{ij}]$ can be converted into a reciprocal PC matrix by replacing a_{ij} and a_{ji} with geometric means of a_{ij} and a_{ji} ($\sqrt{a_{ij}a_{ji}}$) and its reciprocal value $\frac{1}{\sqrt{a_{ij}a_{ji}}}$.

Thus a PC matrix M has the form:

$$M = \begin{bmatrix} 1 & m_{12} & \cdots & m_{1n} \\ \frac{1}{m_{12}} & 1 & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{m_{1n}} & \frac{1}{m_{2n}} & \cdots & 1 \end{bmatrix}.$$

Sometimes we write that $M \in \mathrm{PC}_n$ in order to indicate the size of a given PC matrix.

3.2. Triads, transitivity, and submatrices of a PC matrix

Here we introduce some auxiliary definitions and notations.

Definition 3.1. Given $n \in \mathbb{N}$, we define

$$\mathcal{T}(n) = \{(i, j, k) \in \{1, \ldots, n\} : i < j < k\},$$

as the set of all PC matrix indexes of all permissible triads in the upper triangle.

Definition 3.2. A PC matrix $M = [m_{ij}]$ is called *consistent* (or *transitive*) if, for every $(i, j, k) \in \mathcal{T}(n)$:

$$m_{ik}m_{kj} = m_{ij}. \tag{1}$$

We will refer to equation (1) as a "consistency condition". While every consistent PC matrix is reciprocal, the converse is false in general. If the consistency condition does not hold, the PC matrix is inconsistent (or intransitive). In several studies, conducted between 1939 and 1961 ([10,9,8,22]) the inconsistency in pairwise comparisons was defined and examined. However, credits for it are given for [20].

Inconsistency in pairwise comparisons occurs due to superfluous input data. As demonstrated in [16], only $n - 1$ pairwise comparisons are really needed to create the entire PC matrix for n entities, while the upper triangle has $n(n - 1)/2$ comparisons. Inconsistencies are is not necessarily "wrong" as they can be used to improve the data acquisition. However, there is a real necessity to have a "measure" for it.

Definition 3.3. Assume $n < m$, A and B are square matrices of degree n and m, respectively. We call A *a submatrix* of B ($A \subset B$), if there exists an injection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$, such that for all $i, j \in \{1, \ldots, n\}$

$$a_{ij} = b_{\sigma(i)\sigma(j)}.$$

The above definition is illustrated by Fig. 1. PC submatrix A of size 3×3 has a triad $\left(\frac{1}{3}, \frac{2}{3}, 3\right)$ in the upper triangle and a triad with inverted values $\left(3, \frac{2}{3}, \frac{1}{3}\right)$ in the lower triangle. It was extracted from PC matrix A by deleting rows and columns with the same number. Fig. 1 demonstrates the removal row 1 and column 1 together with row 4 and column 4 from the given PC matrix to create a PC submatrix.

Remark 3.4. Each PC submatrix of a PC matrix is also a PC matrix.

3.3. Some topological issues of exponentially invertible measure in $\mathbb{R}^*_+$

In this subsection, we define a notion of a difference: exponentially invertible measure (EIM) Δ in $\mathbb{R}^*_+$. This notion will play an important role in the axiom A.3 (in Section 3.4). We also consider convergence, with respect to a given EIM Δ and prove that this convergence is equivalent with the usual convergence with respect to the absolute value on $\mathbb{R}^*_+$.

$$\begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ \frac{1}{2} & 1 & \frac{1}{3} & 1 & \frac{2}{3} \\ \frac{1}{2} & 3 & 1 & \frac{1}{2} & 3 \\ \frac{1}{3} & 1 & 2 & 1 & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{3} & 2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{2}{3} \\ 3 & 1 & 3 \\ \frac{3}{2} & \frac{1}{3} & 1 \end{bmatrix}$$

matrix submatrix

Fig. 1. A PC matrix and its submatrix

Definition 3.5. Consider $\mathbb{R}_+^*$ with multiplication. A function Δ,

$$\Delta : \mathbb{R}_+^* \times \mathbb{R}_+^* \to [1, +\infty),$$

is called an *exponentially invertible measure (EIM)* on $\mathbb{R}_+^*$ if for all $x, y, z \in \mathbb{R}_+^*$ it satisfies:

(i) $\Delta(x, y) = 1$ if and only if $x = y$,
(ii) $\Delta(x, y) = \Delta(y, x)$, (symmetry),
(iii) $\Delta(x, z) \leq \Delta(x, y)\Delta(y, z)$, (sub-multiplicativity).

The next lemma provides a proof that there are EIMs on $\mathbb{R}_+^*$.

Lemma 3.6. *Let Δ_o be a function*

$$\Delta_o : \mathbb{R}_+^* \times \mathbb{R}_+^* \to [1, +\infty),$$

given by the following formula:

$$\Delta_o(x, y) = \max\left\{\frac{x}{y}, \frac{y}{x}\right\}.$$

Then Δ_o is an EIM on $\mathbb{R}_+^*$.

Proof. Properties (i) and (ii) of Definition 3.5 for Δ_o are obvious. Thus we only demonstrate how to prove (iii).

There are 6 cases to consider. Namely, the 6 permutations of elements x, y, z in an inequality $x \leq y \leq z$. We treat them case by case computing corresponding values of EIM at each pair of points from $\{x, y, z\}$. $\square$

Remark 3.7. For real positive numbers x and y, EIM (denoted by $\Delta(x, y)$) can be regarded as a measure of how much the ratio $\frac{x}{y}$ of x by y differs from 1.

Definition 3.8. We say that a sequence $\{x_n\} \subset \mathbb{R}_+^*$ Δ*-converges* (or *converges in EIM* Δ) to $x \in \mathbb{R}_+^*$ if

$$\lim_{n \to +\infty} \Delta(x_n, x) = 1.$$

It is clear that we have the following simple but important lemma.

Lemma 3.9. *Consider $\mathbb{R}_+^*$ equipped with an arbitrary EIM Δ. A sequence $\{x_n\} \subset \mathbb{R}_+^*$ converges to $x \in \mathbb{R}_+^*$ in the absolute value $|\cdot|$, i.e.*

$$\lim_{n \to +\infty} |x_n - x| = 0$$

if and only if $\{x_n\}$ Δ-converges to x, i.e.

$$\lim_{n \to +\infty} \Delta(x_n, x) = 1.$$

Hence, every EIM Δ gives the same topology on $\mathbb{R}_+^*$ as the usual absolute value $|\cdot|$.

Important notice: in A.3 below, Δ denotes an arbitrary EIM on $\mathbb{R}_+^*$ which, what we already know from Lemma 3.9, is *topologically equivalent* to the classical distance $(x, y) \mapsto |x - y|$ on $\mathbb{R}_+^*$. Basic notions on topology and metric spaces are well addressed in [19]. In our text, it is very important to have EIM Δ as topologically equivalent to the distance $(x, y) \mapsto |x - y|$.

3.4. Axioms

We are now in position to formulate our axioms for inconsistency indicator ii, which is a function of PC matrices of an arbitrary size.

In the axiom A.3, we identify a triad (x, y, z) with a 3×3-matrix of the form

$$\begin{bmatrix} 1 & x & z \\ x^{-1} & 1 & y \\ z^{-1} & y^{-1} & 1 \end{bmatrix}.$$

For a PC matrix A of the size $n \times n$ and $n \geq 3$, B is a submatrix of A (denoted by $B \subset A$) of the size $m \times m$ and $3 \leq m < n$. Recall that from Remark 3.4 it follows that B is also a PC matrix.

An inconsistency indicator ii is a function from the set off all finite-dimensional reciprocal matrices into $\mathbb{R}_+^*$ and its task is to measure how a given matrix differs from a transitive matrix (Definition 3.2). In other words, for a given reciprocal matrix A, $ii(A)$ is the measure of inconsistency in A.

We postulate that each inconsistency indicator ii should satisfy the following axioms:

A.1 (CONSISTENCY CONDITION) For every matrix A, $ii(A) = 0 \Leftrightarrow A$ is consistent.
A.2 (NORMALIZATION) For every matrix A, $ii(A) \in [0, 1]$.
A.3 (MONOTONICITY) For every two triads $(x_i, y_i, z_i), (x_j, y_j, z_j) \in \mathcal{T}(n)$, such that $\Delta(y_i, x_i z_i) > \Delta(y_j, x_j z_j) \Rightarrow ii(x_i, y_i, z_i) > ii(x_j, y_j, z_j)$, where Δ is EIM.
A.4 (CONTRACTION) For every matrix A and its submatrix B, (i.e. $B \subset A) \Rightarrow ii(B) \leq ii(A)$.

3.5. Reasons for axiom selection and comments

3.5.1. Axiom A.1

The axiom A.1 is the most indisputable axiom since it is based on the consistency condition first introduced in [10] (if not earlier). It implies that the consistency indicator should be able to detect inconsistency in each triad. A similar axiom was introduced in [5], where the existence of any unique element representing consistency was postulated. However, it seems reasonable to expect an inconsistency indicator to achieve the value of 0 for a consistent matrix, since consistency is equivalent to the lack of inconsistency.

3.5.2. Axiom A.2

This axiom (originally proposed by WWK, in 1993) is of considerable importance. As previously mentioned, it is addressed in an independent study [13] devoted to the issue of normalization. In [13], arguments are provided that normalization is indispensable for any inconsistency indicator to be useful. It is fair to say that the probability gained the practical importance when it was normalized by Kolmogorov axioms, in 1933.

The compatibility in different applications is one of the most compelling reasons for postulating the normalization. Without the normalization, we cannot compare inconsistency indicators, not only of different PC matrices but of different PC submatrices of the same PC matrix.

From the mathematical point of view, normalization of inconsistency (or probability) adds nothing of great importance but gives an important point of reference. In common parlance, this is expressed by "50–50 chance" for the probability, although 128 in 256 could be equally right for such expression. The relative error is also a normalized value of the absolute error. The relative (not absolute) error is of a considerable importance in the engineering and most (if not all) applied sciences.

There is yet another important side effect of the normalization. For all defined inconsistency indicators, we should be able to find the maximum value in a natural way. Regretfully, it is not always guaranteed for all existing inconsistency indicators.

3.5.3. Axiom A.3

The axiom A.3 reflects a simple fact that the inconsistency indicator ii should grow the farther we move (according to EIM) from the consistent state $y = xz$ for a triad (x, y, z), for which we have $\Delta(y, xz) = 1$. Whichever EIM we chose in $\mathbb{R}_+^*$, the topology needs to be preserved, especially because multiplication must be continuous. A good example of EIM satisfying this property is $\Delta_o(\cdot, \cdot)$ defined in Lemma 3.6.

Specification of A.3 is a bit complicated since we need to avoid the following two traps of considerable importance:

1) $(1, 2, 1)$ and $(10, 101, 10)$,
2) $(2^n, 2^{2n} + n, 2^n)$.

In the first case, the absolute value distance $d(y, xz) = 2 - 1 \cdot 1 = 101 - 10 \cdot 10$ is the same for both triads while the first triad is evidently more inconsistent ($1 \cdot 1$ is not as close to 2 as $10 \cdot 10$ to 101).

In the second case above, for:

a) $n = 1$: $(2, 5, 2)$ has the absolute value distance $d = 1$,
b) $n = 2$: $(4, 18, 4)$ has the absolute value distance $d = 2$.

Evidently, an inconsistency of $(4, 18, 4)$ is lower than of $(2, 5, 2)$ since the first triad is "2 of 18" hence "1 of 9" away from the consistent triad $(4, 16, 4)$ while the second triad $(2, 5, 2)$ is only by 1 "off" from the consistent triad $(2, 4, 2)$.

In a sense, therefore, axiom A.3 is not only about an inconsistency indicator ii but also about EIM Δ.

By applying EIM Δ_o, defined in Lemma 3.6, to compute how far are the triads $(1, 2, 1)$ and $(10, 101, 10)$, from the respective consistent triads, we get:

i) for $(1, 2, 1)$, $\Delta_o(2, 1 \cdot 1) = \max\left\{2, \frac{1}{2}\right\} = 2$, and
ii) for $(10, 101, 10)$, $\Delta_o(101, 10 \cdot 10) = \max\left\{\frac{101}{100}, \frac{100}{101}\right\} = \frac{101}{100} = 1.01$.

Thus, EIM Δ_o distinguishes inconsistency of the above triads as the intuition dictates. It is in contrast to a measure based on an absolute value, which compares the differences between xz and y, giving the same results for both triads.

For the sequence of triads $(2^n, 2^{2n} + n, 2^n)$, we get Δ_o:

$$\Delta_o(2^{2n} + n, 2^n \cdot 2^n) = \max\left\{\frac{2^{2n} + n}{2^{2n}}, \frac{2^{2n}}{2^{2n} + n}\right\} = \frac{2^{2n} + n}{2^{2n}} \longrightarrow 1 \text{ as } n \text{ tends to } +\infty,$$

which is expected by our intuition.

3.5.4. Axiom A.4

This axiom (contributed by [24]) reflects the common sense perception of the inconsistency. Evidently, "hiding" inconsistencies is easier in a larger PC matrix. The inconsistency indicator of what "is a part of" should not be greater than the part's origin. In other words, this axiom asserts that by extending the set of compared entities, we should expect that the inconsistency of the PC matrix may increase. (See also Remark 4.2 which addresses this issue.) However, the concept of a submatrix has been used since it decreases the size of a PC matrix from any n to 3. For $n = 3$, the PC matrix is generated by a single triad (which is the triangle above or below the main diagonal).

3.5.5. Concluding remarks to the selection of axioms

According to [17]:

> One of the most methodological tools, Occam's celebrated razor, is the maxim that it is in vain to do with more what can be done with fewer.

The proposed axioms follow the Occam's razor maxim. Our axioms are as simple as they can be. Only axiom A.3 may seem a bit complex but it is in fact a "one liner". Its complexity is related to specification of EIM Δ, not its use. It reflects the fact that the order in 2D is not predefined. In particular, we cannot say if $(0, 1) < (1, 0)$ or $(0, 1) > (1, 0)$ without additional assumptions. The introduction of EIM reduces dimensionality of the space of all triad and has turned to be essential for the formulation of A.3.

Remark 3.10. We do not postulate the continuity since it would exclude an important (and evidently useful) discrete indicator 0–1 for checking whether or not there is inconsistency in a given PC matrix.

For the same reason, we use interval $[0, 1]$ – not $[0, 1)$ – as possible values of ii. We are aware that a question may arise as to what it means that the matrix A is as non-transitive as possible, when $ii(A) = 1$.

3.6. Axiom A.3: relating inconsistency in PC_3 *to* EIM

For PC matrix of the size 3 by 3:

$$M = \begin{bmatrix} 1 & x & y \\ \frac{1}{x} & 1 & z \\ \frac{1}{y} & \frac{1}{z} & 1 \end{bmatrix},$$

M is consistent if and only if $y = xz$. Evidently, the set PC_3 of 3×3-matrices is a 3-dimensional manifold because it can be parametrized by the 3 parameters x, y and z, where as the set CPC_3 of consistent PC_3 matrices is parametrized only by two parameters x and z, which shows that CPC_3 is of dimension 2.

Therefore, characterizing inconsistency is reduced to somehow relating xz and y, which can be made elementarily by mostly three ways:

a) evaluate $xy - z \in \mathbb{R}$, and compare it to 0,
b) or evaluate $\frac{xz}{y} \in \mathbb{R}_+^*$ and compare it to 1,
c) or evaluate $\log x + \log z - \log y \in \mathbb{R}$, and compare it to 0.

As we noted in the previous section when discussing Axiom A.3 method a) above is unsatisfactory. It does not distinguish between triads, which we intuitively feel are essentially different due to the "distance" from the consistent triad. The reason for the fact that method a) does not have the properties we are expecting is the fact of mixing mathematical structures. The set $\mathbb{R}_+^*$ is a group with respect to the multiplication, whereas criterion a) uses subtraction operation, but subtraction (addition) is not an operation in $\mathbb{R}_+^*$. This set is not closed with respect to subtraction.

For this reason, the definition of EIM Δ_o given in Lemma 3.6 is based on criterion b).

The method c) is indeed equivalent with b) by the bijection:

$$\exp : \mathbb{R} \to \mathbb{R}_+^*.$$

Consequently, we can work indistinguishably with one or another.

4. The proof of the consistency of axioms A.1–A.4

In this section, we give an example of inconsistency indicator ii which satisfies all four axioms A.1–A.4 with an appropriate EIM Δ on $\mathbb{R}_+^*$. Thus, we prove the following:

Theorem 4.1. *There exists EIM Δ and an inconsistency indicator ii such that the system of axioms A.1–A.4 is consistent.*

Proof. The following inconsistency indicator was introduced in [11] for a 3×3-PC matrices, then extended in [6] to any size, and finally simplified in [15] to:

$$Kii(A) = \max_{i<j<k} \left(1 - \min\left(\frac{a_{ik}}{a_{ij}a_{jk}}, \frac{a_{ij}a_{jk}}{a_{ik}}\right)\right). \tag{2}$$

We will show that Kii satisfies all the four axioms with EIM $\Delta = \Delta_o$. In fact, A.1, A.2, and A.4 clearly hold for Kii. Let us check A.3, which involves EIM Δ. We chose EIM Δ_o defined in Lemma 3.6.

Suppose that for two triads $(x, y, z), (x', y', z') \in \mathcal{T}(n)$ we have that

$$\Delta_o(xz, y) < \Delta_o(x'z', y').$$

This inequality, by definition of Δ_o, amounts to the following sequence of equivalences:

$$\max\left\{\frac{xz}{y}, \frac{y}{xz}\right\} < \max\left\{\frac{x'z'}{y'}, \frac{y'}{x'z'}\right\} \Leftrightarrow \min\left\{\frac{xz}{y}, \frac{y}{xz}\right\} > \min\left\{\frac{x'z'}{y'}, \frac{y'}{x'z'}\right\} \Leftrightarrow$$
$$\Leftrightarrow 1 - \min\left\{\frac{xz}{y}, \frac{y}{xz}\right\} < 1 - \min\left\{\frac{x'z'}{y'}, \frac{y'}{x'z'}\right\}.$$

The last inequality proves that

$$Kii(x, y, z) < Kii(x', y', z').$$

It concludes the proof. □

Remark 4.2. No inconsistency indicator may ignore outliers which, in this context, could be the most inconsistent triads. A triad appears in the PC matrix as a result of an inaccurate or imprecise input data. Sometimes, such data are referred as data uncertainty. However, the problem of the appearance of the source of outliers in a PC matrix belongs to a separate domain of mathematical statistics: the outliers theory. The main objective of the outliers theory is detecting such anomalies (for details, see [21,23]). Taking the maximum value in the definition of Kii is not a detraction from the potential advantages of using this inconsistency indicator of in the real-world problems but constitute a considerable step forward. Knowing more cannot be regarded as disadvantage. Localizing the largest inconsistency is indisputably better than not knowing it. Thanks to Kii, users (e.g., decision makers) are aware of the problem of outliers that may appear in a particular PC matrix. Evidently, it cannot be a disadvantage to them.

Example 4.3. For the case of $n = 3$ Kii is illustrated by the following 3D plot for the middle value of the triad set to 1.5 on the scale of 1 to 3 which is strongly recommended by [7]. The plot in Fig. 2 is a section of plot 4D which is evidently impossible to provide. The variable (y) of a triad (x, y, z) was set to 1.5 so plot 3D could be produced for illustration purposes (mostly to show that Kii is not entirely trivial function). For the following PC matrix:

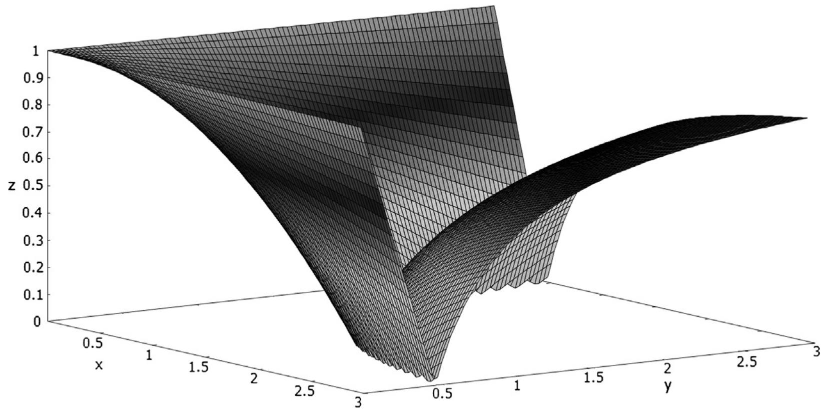

Fig. 2. Kii inconsistency indicator 3D section of plot for $y = 1.5$

$$A = \begin{bmatrix} 1 & x & y \\ \frac{1}{x} & 1 & z \\ \frac{1}{y} & \frac{1}{z} & 1 \end{bmatrix}$$

we have:

$$Kii(A) = Kii(x, y, z).$$

5. The proof of the independence of axioms

In order to prove that axioms A.1–A.4 are independent, we must construct the examples of inconsistency indicators which satisfy all the axioms but one.

For a given $n \times n$ PC matrix A, let us define ii_1–ii_4:

$$ii_1(A) = \frac{1}{2}(1 + Kii(A)), \qquad ii_2(A) = 2Kii(A),$$
$$ii_3(A) = \max\left\{e^{-\max\{x,y,z,x^{-1},y^{-1},z^{-1}\}} Kii(x, y, z)\right\},$$

where maximum is taken over all PC$_3$ submatrices $B \subset A$ of the form

$$B = \begin{bmatrix} 1 & x & y \\ \frac{1}{x} & 1 & z \\ \frac{1}{y} & \frac{1}{z} & 1 \end{bmatrix},$$

and Kii is defined in (2).

Finally we define

$$ii_4(A) = \begin{cases} 0 & \text{if } A \text{ is consistent,} \\ 1 - \frac{1}{2}\max_{(i,j,k)\in\mathcal{T}(n)} \min\left(\frac{a_{ik}}{a_{ij}a_{jk}}, \frac{a_{ij}a_{jk}}{a_{ik}}\right) & \text{otherwise.} \end{cases}$$

Theorem 5.1. *For each $j \in \{1, 2, 3, 4\}$ the inconsistency indicator ii_j satisfies all the axioms A.1–A.4 but A.j, with $\Delta = \Delta_o$.*

Proof. From the construction of the indicators, it is easy to verify that each satisfies three out of four axioms. We will show that $\forall j \in \{1, 2, 3, 4\}$ indicator ii_j does not meet A.j.

It is obvious for $j = 1$ and $j = 2$, since:

$$ii_1(A) \geq \frac{1}{2}$$

and:

$$ii_2\left(\begin{bmatrix} 1 & 1 & 9 \\ 1 & 1 & 1 \\ \frac{1}{9} & 1 & 1 \end{bmatrix}\right) = \frac{16}{9} > 1.$$

That ii_1 and ii_3 satisfy A.3 follows from the fact that Kii satisfies A.3 (see Theorem 4.1).

Let us consider the inconsistency indicator ii_3. For $t > 0$, define the following PC matrix:

$$A_t = \begin{bmatrix} 1 & tx & y \\ (tx)^{-1} & 1 & t^{-1}z \\ y^{-1} & tz^{-1} & 1 \end{bmatrix}.$$

Then $Kii(A_t) = Kii(A_1)$ and since $txt^{-1}z = xz$, we have that $\Delta_o(y, txt^{-1}z)$ is constant in the t-variable. However,

$$\lim_{t\to+\infty} e^{-\max\{tx, t^{-1}z, y, (tx)^{-1}, tz^{-1}, y^{-1}\}} \leq \lim_{t\to+\infty} e^{-tx} = 0,$$

which shows that ii_3 does not satisfy A.3.

Finally, ii_4 satisfies A.1–A.3. The verification of A.3 is similar to the verification that Kii satisfies A.3 (see Theorem 4.1). Consider the PC matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & \frac{1}{2} & 1 & 3 \\ 1 & 1 & \frac{1}{3} & 1 \end{bmatrix} \text{ and its submatrix: } B = \begin{bmatrix} 1 & 2 & 1 \\ \frac{1}{2} & 1 & 3 \\ 1 & \frac{1}{3} & 1 \end{bmatrix}.$$

It is straightforward to verify that:

$$ii_4(A) = \frac{1}{2} < ii_4(B) = \frac{11}{12},$$

so, ii_4 does not satisfy A.4. □

6. Testing cases

In this section, we will examine a few testing cases of inconsistency indicators to illustrate the relevance of the axioms.

Example 6.1. For a single triad $\mathcal{T} = (x, y, z)$ generating 3×3 PC matrix, we use

$$ii_{\mathcal{T}}(x, y, z) = \begin{cases} 0 & \text{if } xz = y, \\ \frac{x+y+z}{x+y+z+1} & \text{otherwise.} \end{cases}$$

It can be extended for a PC matrix of the $n > 3$ size by:

$$ii(A) := \max_{(i,j,k)\in\mathcal{T}(n)} ii_{\mathcal{T}}(a_{ij}, a_{ik}, a_{jk}).$$

It is easy to see that the above indicator satisfies axioms A.1, A.2, and A.4. However it does not satisfy A.3 with EIM $\Delta = \Delta_o$.

Consider two triads $\mathcal{T}' = (0.1, 1, 0.1)$ and $\mathcal{T} = (50, 100, 1.5)$. Then for $\mathcal{T}'$, $\Delta_o^{(\mathcal{T}')} := \Delta_o(1, 0.01) = 100$ and for $\mathcal{T}$, $\Delta_o^{(\mathcal{T})} := \Delta_o(100, 75) = \frac{100}{75}$. Therefore

$$\Delta_o^{(\mathcal{T}')} > \Delta_o^{(\mathcal{T})}$$

but

$$\frac{x' + y' + z'}{x' + y' + z' + 1} = \frac{1.02}{2.02} < \frac{x + y + z}{x + y + z + 1} = \frac{151.5}{152.5}.$$

Thus A.3 does not hold.

Remark 6.2. Explanation of the fact that the ii from Example 6.1 does not meet A.3 is simple. Consistency indicator ii is constructed using an additive structure which is not in line with the structure in $\mathbb{R}_+^*$ but the EIM Δ_o is based on a natural multiplicative structure in $\mathbb{R}_+^*$.

Example 6.3. Consider the following inconsistency indicators on PC_3 for testing our axioms:

1) $ii = 1 - \min(2^{\ln x + \ln z - \ln y}, 2^{\ln y - \ln x - \ln z})$,
2) $ii = \frac{\Delta_o(y,xz)^\alpha - 1}{\Delta_o(y,xz)^\alpha}$, where $\alpha > 0$ and Δ_o is a EIM defined in Lemma 3.6,
3) $ii = 1 - \min(e^{\ln x + \ln z - \ln y}, e^{\ln y - \ln x - \ln z})$.

It is an easy task to verify that all the inconsistency indicators ii induced by the above triad inconsistency indicators 1)–3) (in the same way as in Example 6.1, i.e. taking maximum over all $(x, y, z) \in \mathcal{T}(3)$) follow all axioms A.1–A.4.

Notice that the last item 3) on the above list generates the previously defined inconsistency indicator Kii which, by Theorem 4.1, satisfies A.1–A.4.

The inconsistency indicator defined in 1) satisfies A.1–A.4 as it is a "scaled" version of Kii. Thus the proof of A.1–A.4 in this case is the same as for Kii.

Consider ii from 2). As usual, only axiom A.3 requires a commentary word since other axioms can be easily verified.

Consider two triads: $\mathcal{T}' = (x', y', z')$ and $\mathcal{T} = (x, y, z)$. Denote by $a' := \Delta_o(y', x'z')$ and by $a := \Delta_o(y, xz)$. Clearly, for every $a', a \geq 1$ and $\alpha > 0$, if

$$a' > a \Rightarrow \left(a'\right)^{\alpha} > a^{\alpha} \Rightarrow \frac{\left(a'\right)^{\alpha} - 1}{(a')^{\alpha}} > \frac{a^{\alpha} - 1}{a^{\alpha}}.$$

Thus A.3 is satisfied.

The next example is the eigenvalue-based inconsistency indicator introduced in [20] as CI (consistency index) as an essential component of Analytic hierarchy process (AHP). It is provided because of the considerable popularity of AHP. Technically speaking, CI should not be analyzed here since [15] provided mathematical reasoning that CI tolerates errors of an arbitrarily large value hence it is invalid. However, CI is still in use, hence its analysis will be conducted here.

Example 6.4. The consistency index CI was defined in [20] as

$$CI(A) = \frac{\lambda_{\max} - n}{n - 1},$$

where $\lambda_{\max}$ is the principle eigenvalue of $n \times n$ PC matrix A.

Proposition 6.5. *CI does not meet axioms: A.2 and A.4.*

Proof. To prove the first two statements, let us consider PC matrices:

$$A = \begin{bmatrix} 1 & 2 & \frac{1}{2} & 2 \\ \frac{1}{2} & 1 & 2 & \frac{1}{2} \\ 2 & \frac{1}{2} & 1 & 2 \\ \frac{1}{2} & 2 & \frac{1}{2} & 1 \end{bmatrix},$$

and all its 3×3 PC submatrices:

$$B_1 = \begin{bmatrix} 1 & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & 2 \\ 2 & \frac{1}{2} & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 1 & \frac{1}{2} & 2 \\ 2 & 1 & 2 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}, \text{ and } B_3 = \begin{bmatrix} 1 & 2 & 2 \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 \end{bmatrix}.$$

The principal eigenvalues of A and its PC submatrices are, respectively, equal to:

$$\lambda_A \simeq 4.64474$$

and

$$\lambda_{B_1} = 3.5, \; \lambda_{B_2} = \lambda_{B_3} = 3.05362.$$

Thus,

$$CI(A) \simeq 0.215,$$

while:

$$CI(B_1) = 0.25$$

and:

$$CI(B_2) = CI(B_3) = 0.02681,$$

which contradicts the axiom A.4.

Finally, for the following PC matrix:

$$C = \begin{bmatrix} 1 & \frac{1}{9} & 9 \\ 9 & 1 & \frac{1}{9} \\ \frac{1}{9} & 9 & 1 \end{bmatrix}$$

its principal eigenvalue is about 10.111, hence: $CI(C) \simeq 3.555$. This, obviously, proves that CI is not normalized thus violating A.2. □

7. Conclusions

This study provides an axiomatic system for inconsistency indicators. It also provides proofs of the consistency and independence of the proposed axioms. The axioms are easy to comprehend and simple to use for defining future inconsistency indicators. Axiom A.3 prevents error aberrations taking place for the eigenvalue-based CI. Axiom A.4 recognizes the simple fact that more "uncertainties" may exist in a larger PC matrix, since the number of triads grows with the cube of the matrix size and the larger PC matrix consists PC sub-matrices.

Axiom A.1 has been known since 1939 (if not earlier). It has never been contested and known as the consistency condition. Axiom A.2 calls for normalization which is universally used in all branches of applied science and engineering. It is widely used for uncertainty measures, such as probability, belief or fuzzy membership functions. The introduction of the exponentially invertible measure was instrumental to express monotonicity of inconsistency indicators.

According to: [1] *"E. Zermelo in 1908, under the influence of D. Hilbert at Göttingen, provided the first full-fledged axiomatization of set theory, from which ZFC in large part derives. Although several axiom systems were later proposed, ZFC became generally adopted by the 1960's because of its schematic simplicity and open-endedness in codifying the minimally necessary set existence principles needed and it is (as of 2000) regarded as the basic framework onto which further axioms can be adjoined and investigated."*

To this day, ZFC is criticized by many researchers for being excessively weak and/or for being excessively strong. However, one thing is certain: the development of the modern mathematics would have been impaired.

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