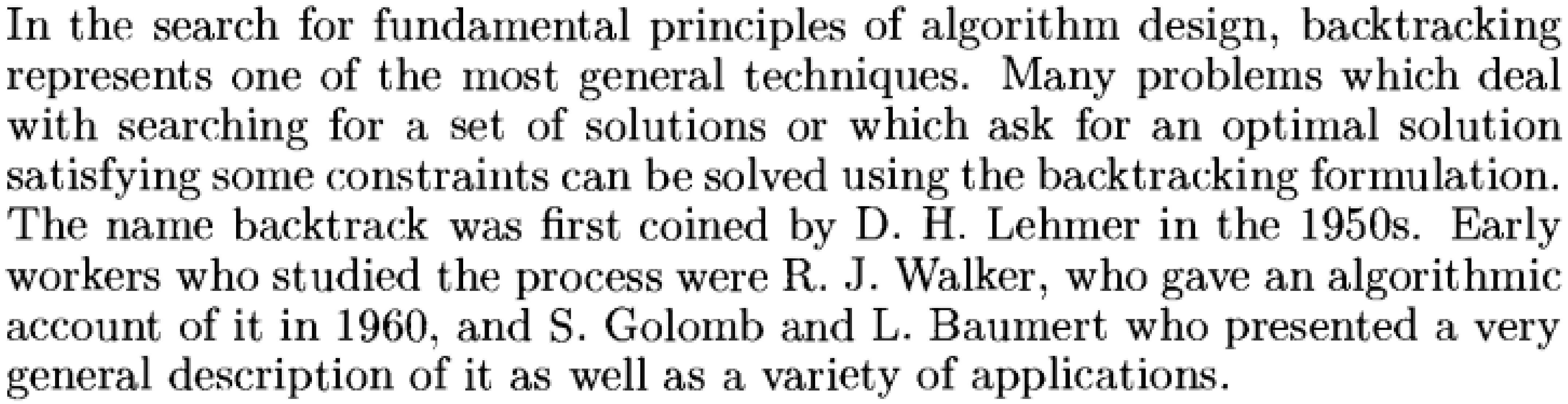
DAA-Unit-IV-Backtracking

Contents

General method



In the search for fundamental principles of algorithm design, backtracking represents one of the most general techniques. Many problems which deal with searching for a set of solutions or which ask for an optimal solution satisfying some constraints can be solved using the backtracking formulation. The name backtrack was first coined by D. H. Lehmer in the 1950s. Early workers who studied the process were R. J. Walker, who gave an algorithmic account of it in 1960, and S. Golomb and L. Baumert who presented a very general description of it as well as a variety of applications.

In many applications of the backtrack method, the desired solution is expressible as an n-tuple $(x_1, \ldots, x_n)$, where the x_i are chosen from some finite set S_i. Often the problem to be solved calls for finding one vector that maximizes (or minimizes or satisfies) a *criterion function* $P(x_1, \ldots, x_n)$. Sometimes it seeks all vectors that satisfy P. For example, sorting the array of integers in $a[1:n]$ is a problem whose solution is expressible by an n-tuple, where x_i is the index in a of the ith smallest element. The criterion function P is the inequality $a[x_i] \leq a[x_{i+1}]$ for $1 \leq i < n$. The set S_i is finite and includes the integers 1 through n. Though sorting is not usually one of the problems solved by backtracking, it is one example of a familiar problem whose solution can be formulated as an n-tuple. In this chapter we study a collection of problems whose solutions are best done using backtracking.

Suppose m_i is the size of set S_i. Then there are $m = m_1 m_2 \cdots m_n$ n-tuples that are possible candidates for satisfying the function P. The *brute force approach* would be to form all these n-tuples, evaluate each one with P, and save those which yield the optimum. The backtrack algorithm has as its virtue the ability to yield the same answer with far fewer than m trials. Its basic idea is to build up the solution vector one component at a time and to use modified criterion functions $P_i(x_1, \ldots, x_i)$ (sometimes called bounding functions) to test whether the vector being formed has any chance of success. The major advantage of this method is this: if it is realized that the partial vector $(x_1, x_2, \ldots, x_i)$ can in no way lead to an optimal solution, then $m_{i+1} \cdots m_n$ possible test vectors can be ignored entirely.

Many of the problems we solve using backtracking require that all the solutions satisfy a complex set of constraints. For any problem these constraints can be divided into two categories: *explicit* and *implicit*.

Definition 7.1 Explicit constraints are rules that restrict each x_i to take on values only from a given set. $\square$

Common examples of explicit constraints are

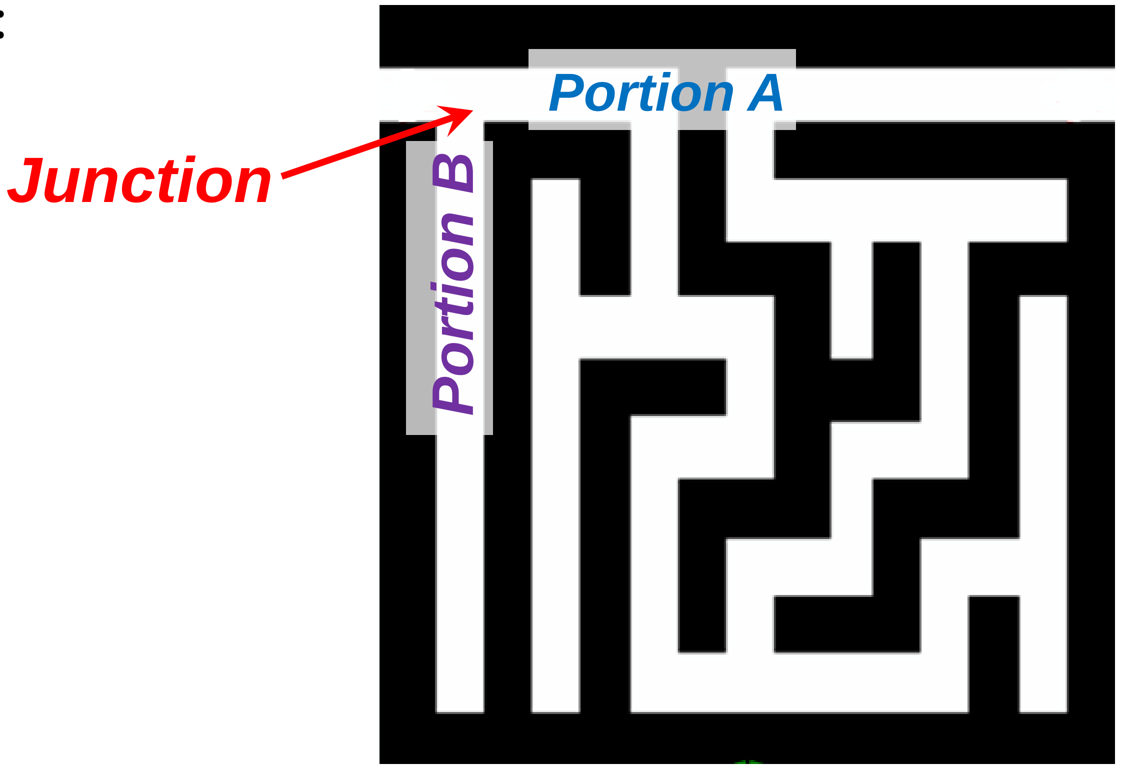
$$\begin{array}{rlrcl} x_i \geq 0 & \text{or} & S_i & = & \{\text{all nonnegative real numbers}\} \\ x_i = 0 \ \text{ or } \ 1 & \text{or} & S_i & = & \{0,1\} \\ l_i \leq x_i \leq u_i & \text{or} & S_i & = & \{a : l_i \leq a \leq u_i\} \end{array}$$

The explicit constraints depend on the particular instance I of the problem being solved. All tuples that satisfy the explicit constraints define a possible *solution space* for I.

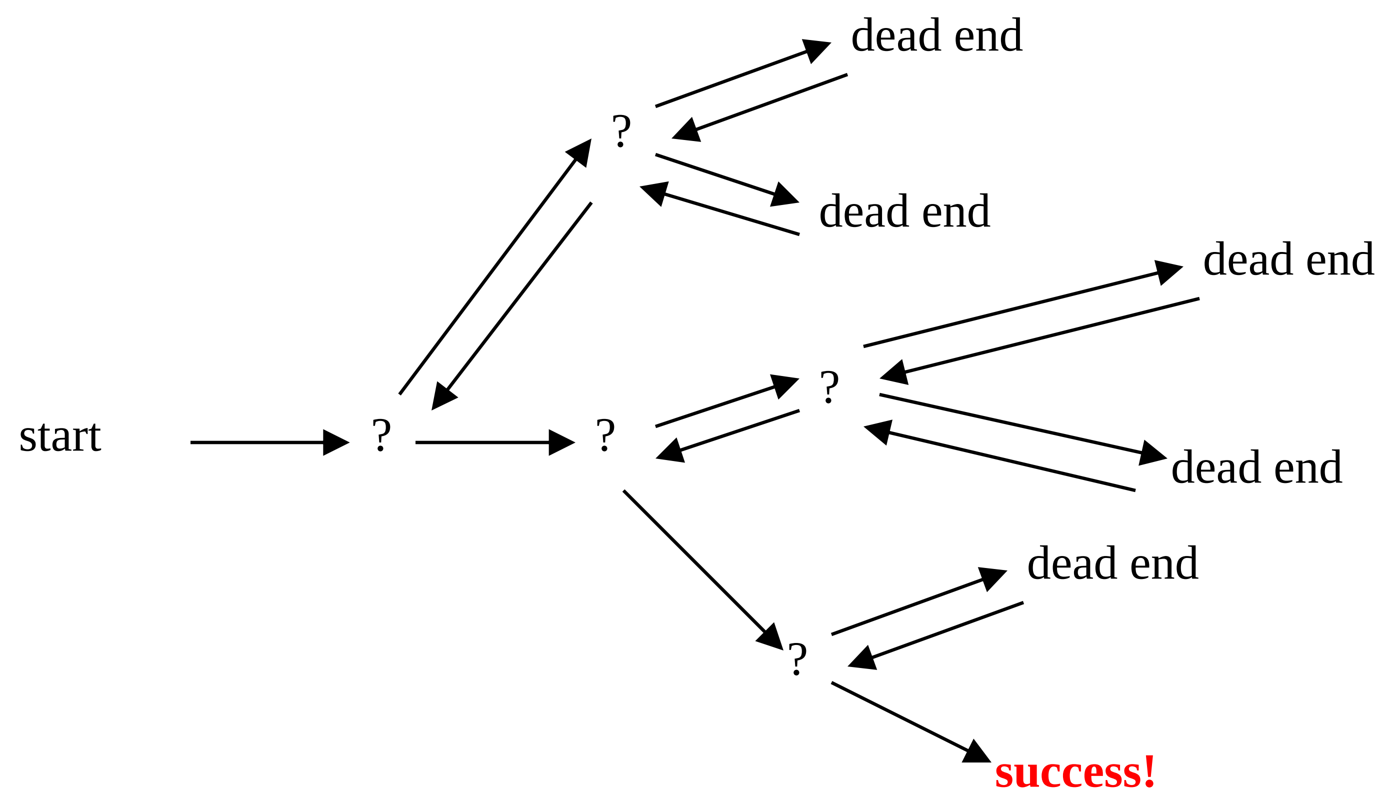
Definition 7.2 The implicit constraints are rules that determine which of the tuples in the solution space of I satisfy the criterion function. Thus implicit constraints describe the way in which the x_i must relate to each other. $\square$

Backtracking: Idea

- Backtracking is a technique used to solve problems with a large search space, by systematically trying and eliminating possibilities.
- A standard example of backtracking would be going through a maze.
  - At some point, you might have two options of which direction to go:



Backtracking (animation)



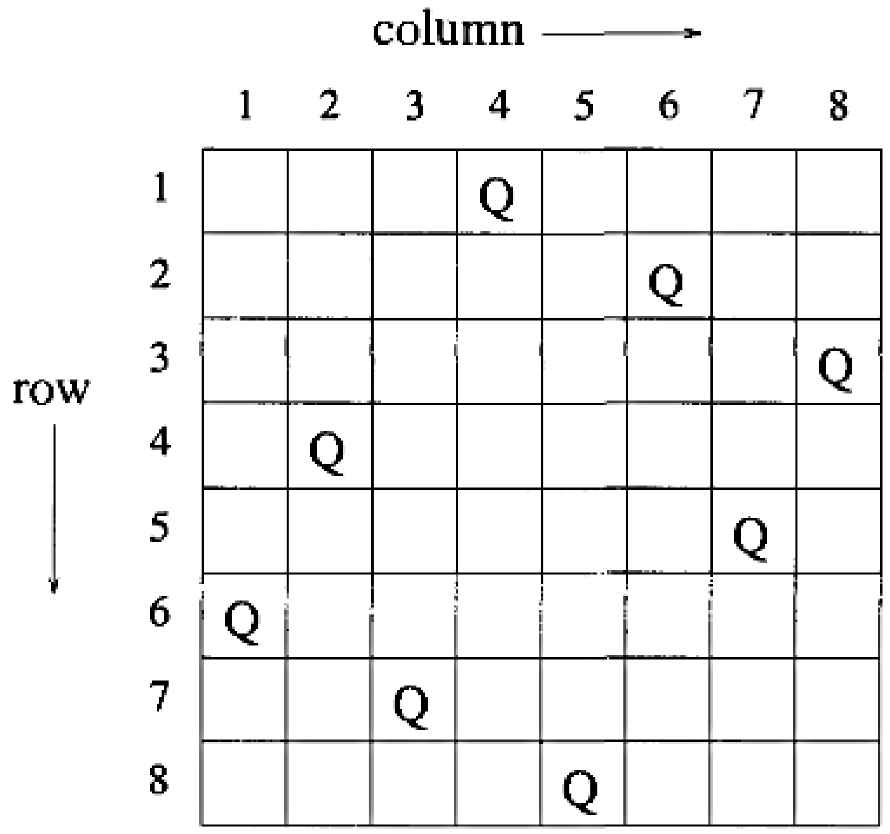
Recursive backtracking

```
Algorithm Backtrack($k$)
// This schema describes the backtracking process using
// recursion. On entering, the first $k-1$ values
// $x[1], x[2], \ldots, x[k-1]$ of the solution vector
// $x[1:n]$ have been assigned. $x[\,]$ and $n$ are global.
{
    for (each $x[k] \in T(x[1], \ldots, x[k-1])$) do
    {
        if ($B_k(x[1], x[2], \ldots, x[k]) \neq 0$) then
        {
            if ($x[1], x[2], \ldots, x[k]$ is a path to an answer node)
                then  write ($x[1:k]$);
            if ($k < n$) then Backtrack($k+1$);
        }
    }
}
```

Iterative backtracking

```
Algorithm IBacktrack($n$)
// This schema describes the backtracking process.
// All solutions are generated in $x[1:n]$ and printed
// as soon as they are determined.
{
    $k := 1$;
    while ($k \neq 0$) do
    {
        if (there remains an untried $x[k] \in T(x[1], x[2], \ldots,$
            $x[k-1])$ and $B_k(x[1], \ldots, x[k])$ is true) then
        {
            if ($x[1], \ldots, x[k]$ is a path to an answer node)
                then write ($x[1:k]$);
            $k := k + 1$; // Consider the next set.
        }
        else $k := k - 1$; // Backtrack to the previous set.
    }
}
```

N-Queens problem



Example 7.3 [n-queens] The n-queens problem is a generalization of the 8-queens problem of Example 7.1. Now n queens are to be placed on an $n \times n$ chessboard so that no two attack; that is, no two queens are on the same row, column, or diagonal. Generalizing our earlier discussion, the solution space consists of all $n!$ permutations of the n-tuple $(1, 2, \ldots, n)$. Figure 7.2 shows a possible tree organization for the case $n = 4$. A tree such as this is called a *permutation tree*. The edges are labeled by possible values of x_i. Edges from level 1 to level 2 nodes specify the values for x_1. Thus, the leftmost subtree contains all solutions with $x_1 = 1$; its leftmost subtree contains all solutions with $x_1 = 1$ and $x_2 = 2$, and so on. Edges from level i to level $i + 1$ are labeled with the values of x_i. The solution space is defined by all paths from the root node to a leaf node. There are $4! = 24$ leaf nodes in the tree of Figure 7.2. □

Tree organization of 4-queens solution space. Nodes are numbered in DFS order

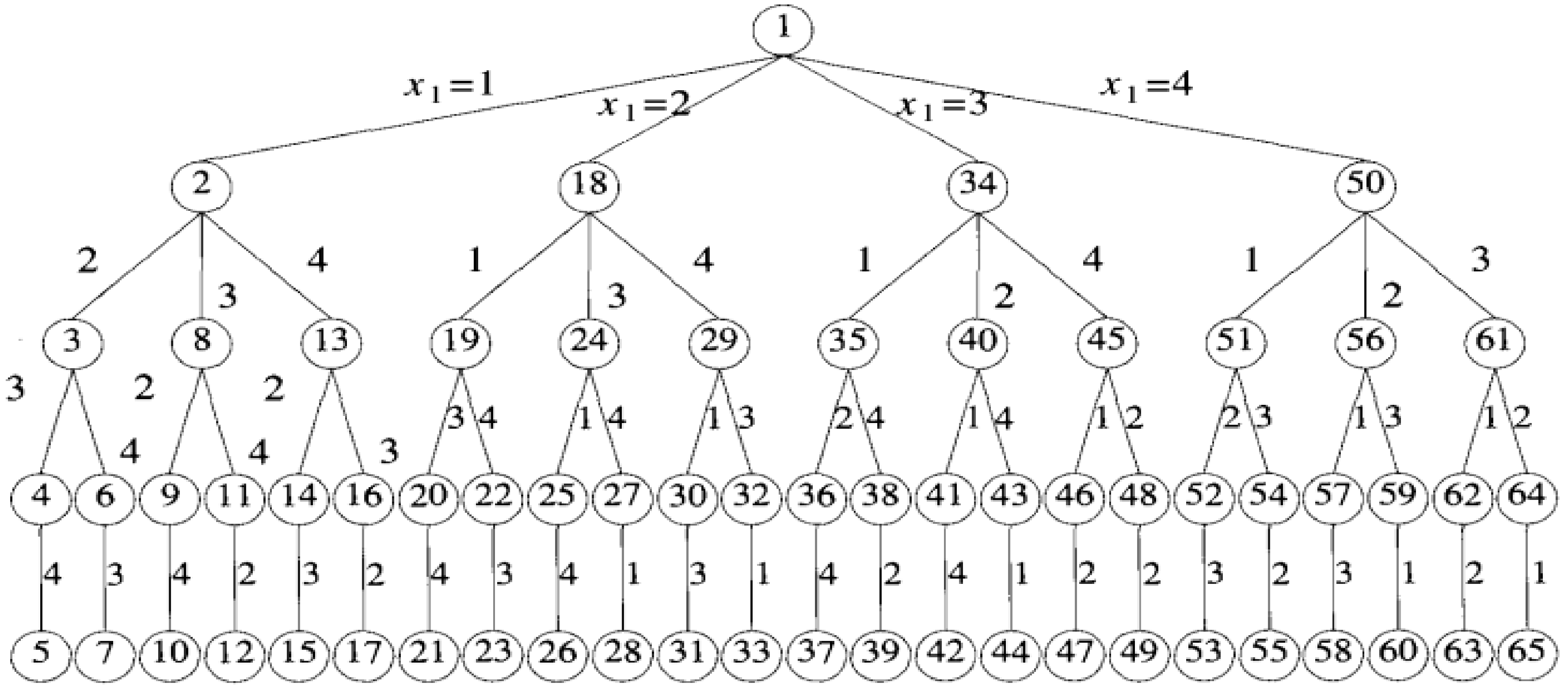
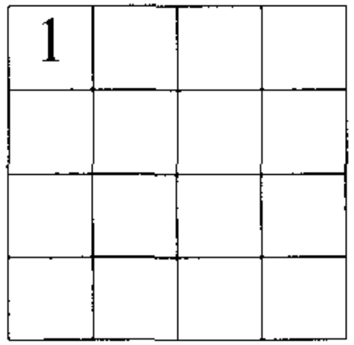
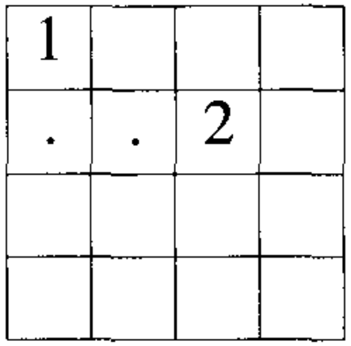


Figure 7.2 Tree organization of the 4-queens solution space. Nodes are numbered as in depth first search.

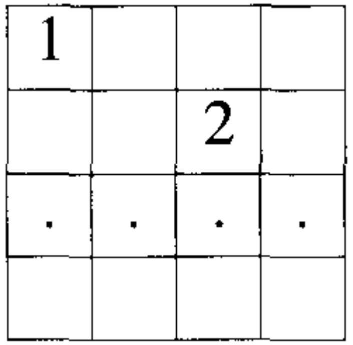
Backtrack solution to 4-Queens problem



(a)



(b)



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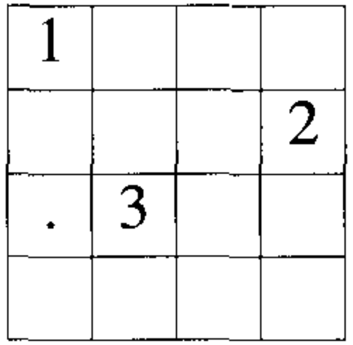

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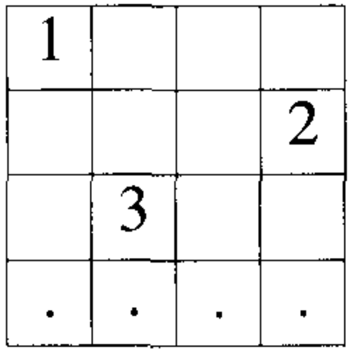

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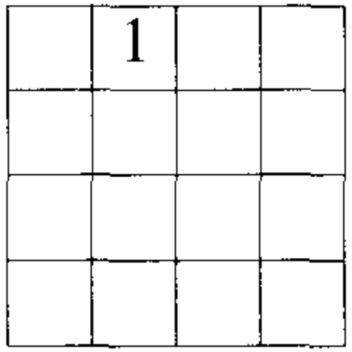

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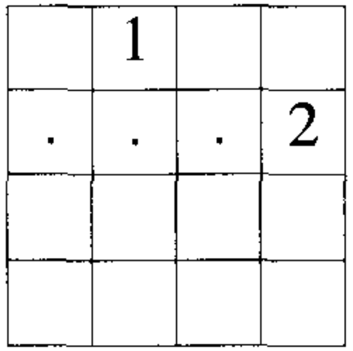
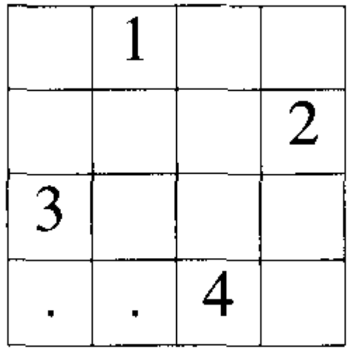


(g)



(h)

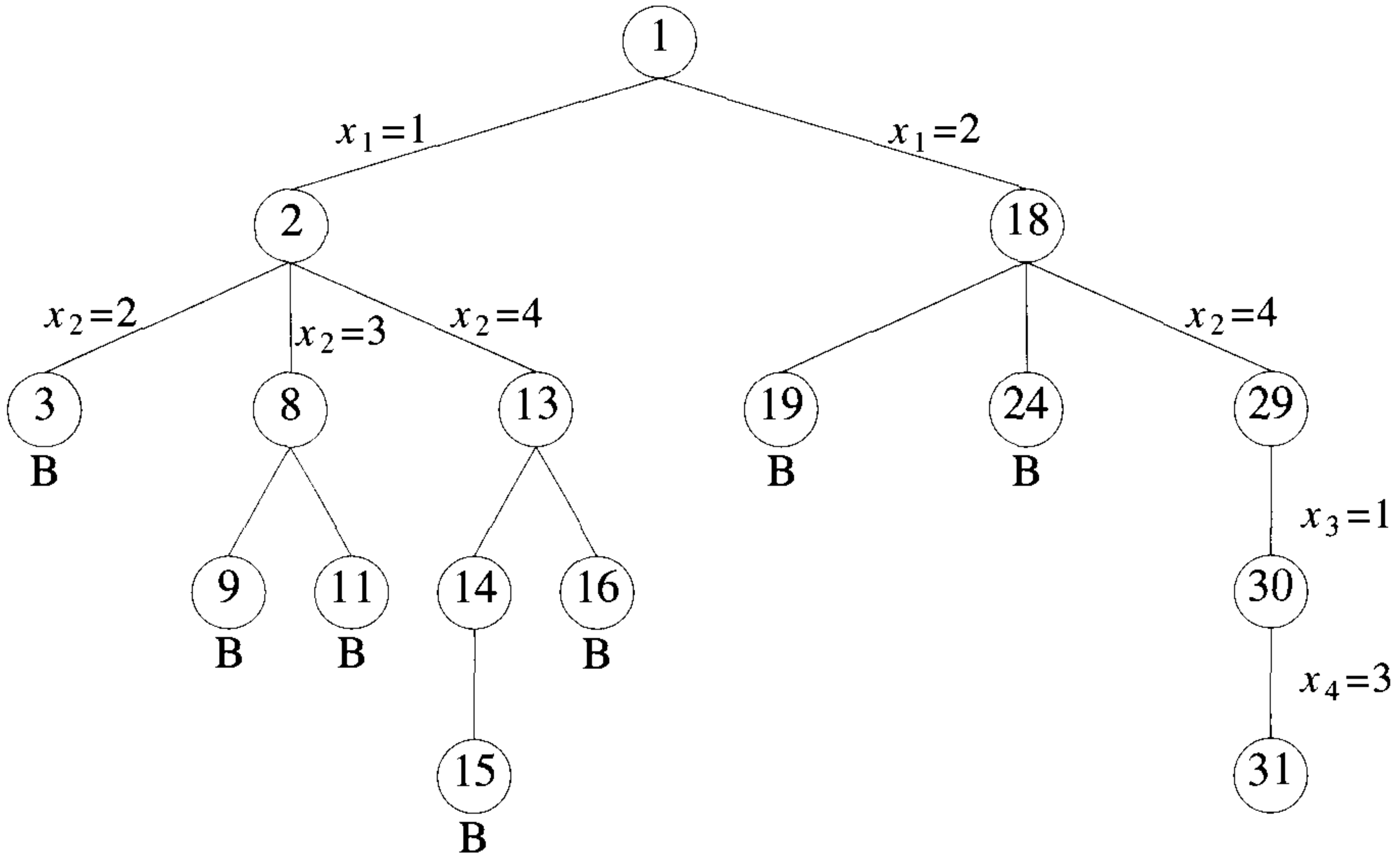
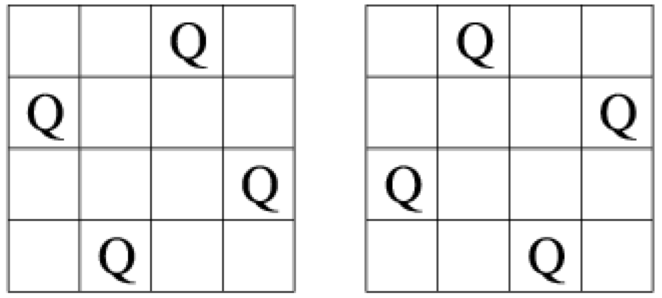


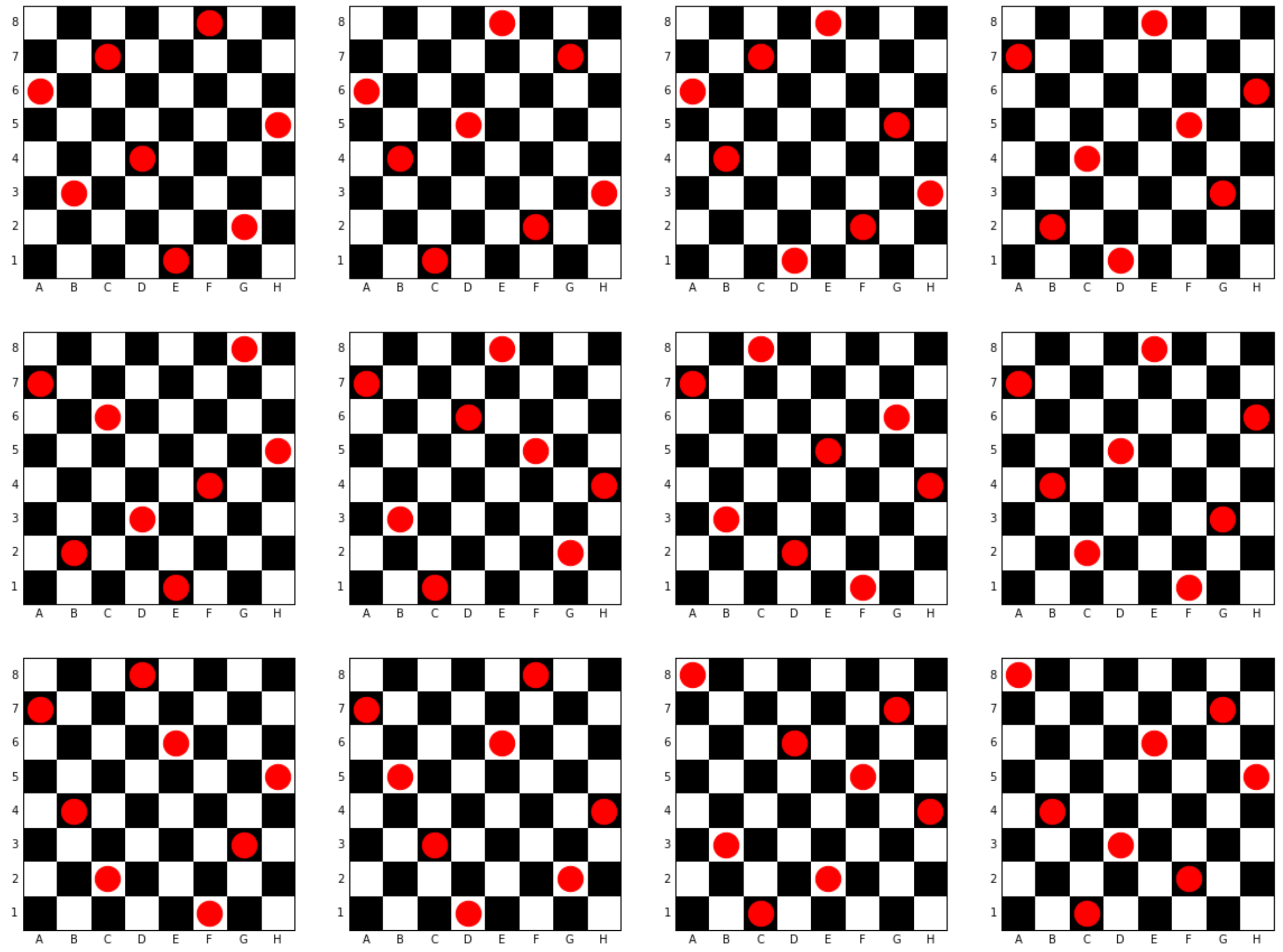
Figure 7.6 Portion of the tree of Figure 7.2 that is generated during backtracking

```
Algorithm NQueens(k, n)
// Using backtracking, this procedure prints all
// possible placements of n queens on an n × n
// chessboard so that they are nonattacking.
{
    for i := 1 to n do
    {
        if Place(k, i) then
        {
            x[k] := i;
            if (k = n) then write (x[1 : n]);
            else NQueens(k + 1, n);
        }
    }
}
```

```
Algorithm Place($k, i$)
// Returns true if a queen can be placed in $k$th row and
// $i$th column. Otherwise it returns false. $x[\,]$ is a
// global array whose first $(k-1)$ values have been set.
// Abs($r$) returns the absolute value of $r$.
{
    for $j := 1$ to $k-1$ do
        if (($x[j] = i$) // Two in the same column
              or (Abs($x[j] - i$) = Abs($j - k$)))
                // or in the same diagonal
            then return false;
    return true;
}
```

2 solutions for 4 queens & 92 solutions for 8 queens





Time complexity

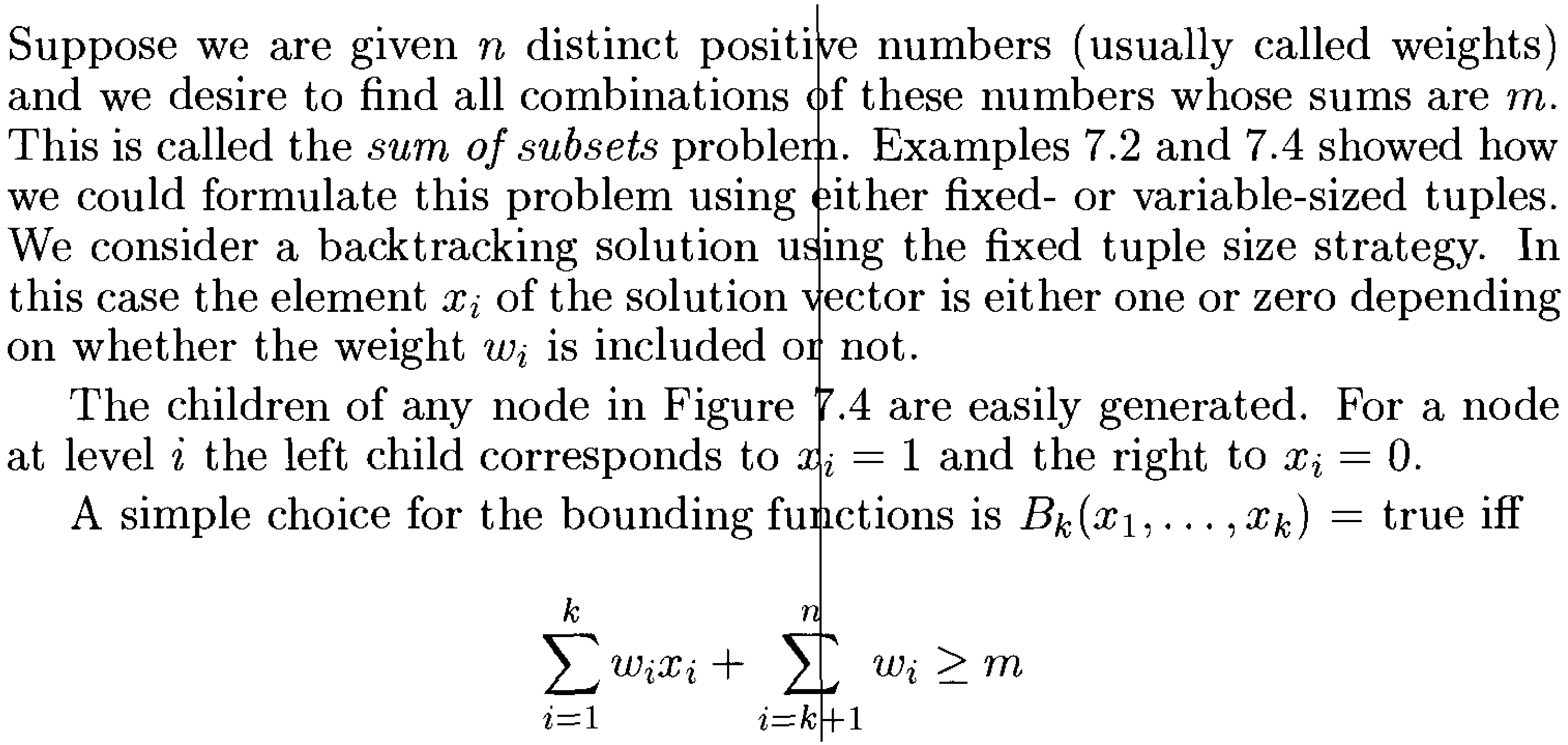
- The backtracking algorithm recursively explores all possible solutions by checking whether a queen can be placed in each column of the current row.
- The time complexity of the algorithm can be expressed as $O(N!)$ because in the worst case scenario, every queen must be tried in every column of every row.

Sum of subsets

Example 7.2 [Sum of subsets] Given positive numbers w_i, $1 \leq i \leq n$, and m, this problem calls for finding all subsets of the w_i whose sums are m. For example, if $n = 4$, $(w_1, w_2, w_3, w_4) = (11, 13, 24, 7)$, and $m = 31$, then the desired subsets are (11, 13, 7) and (24, 7). Rather than represent the solution vector by the w_i which sum to m, we could represent the solution vector by giving the indices of these w_i. Now the two solutions are described by the vectors (1, 2, 4) and (3, 4). In general, all solutions are k-tuples $(x_1, x_2, \ldots, x_k)$, $1 \leq k \leq n$, and different solutions may have different-sized tuples. The explicit constraints require $x_i \in \{j \mid j \text{ is an integer and } 1 \leq j \leq n\}$. The implicit constraints require that no two be the same and that the sum of the corresponding w_i's be m. Since we wish to avoid generating multiple instances of the same subset (e.g., (1, 2, 4) and (1, 4, 2) represent the same subset), another implicit constraint that is imposed is that $x_i < x_{i+1}$, $1 \leq i < k$.

In another formulation of the sum of subsets problem, each solution subset is represented by an n-tuple $(x_1, x_2, \ldots, x_n)$ such that $x_i \in \{0, 1\}$, $1 \leq i \leq n$. Then $x_i = 0$ if w_i is not chosen and $x_i = 1$ if w_i is chosen. The solutions to the above instance are (1, 1, 0, 1) and (0, 0, 1, 1). This formulation expresses all solutions using a fixed-sized tuple. Thus we conclude that there may be several ways to formulate a problem so that all solutions are tuples that satisfy some constraints. One can verify that for both of the above formulations, the solution space consists of 2^n distinct tuples. □

Sum of subsets



Suppose we are given n distinct positive numbers (usually called weights) and we desire to find all combinations of these numbers whose sums are m. This is called the *sum of subsets* problem. Examples 7.2 and 7.4 showed how we could formulate this problem using either fixed- or variable-sized tuples. We consider a backtracking solution using the fixed tuple size strategy. In this case the element x_i of the solution vector is either one or zero depending on whether the weight w_i is included or not.

The children of any node in Figure 7.4 are easily generated. For a node at level i the left child corresponds to $x_i = 1$ and the right to $x_i = 0$.

A simple choice for the bounding functions is $B_k(x_1, \ldots, x_k) = \text{true}$ iff

$$\sum_{i=1}^{k} w_i x_i + \sum_{i=k+1}^{n} w_i \geq m$$

Clearly $x_1, \ldots, x_k$ cannot lead to an answer node if this condition is not satisfied. The bounding functions can be strengthened if we assume the w_i's are initially in nondecreasing order. In this case $x_1, \ldots, x_k$ cannot lead to an answer node if

$$\sum_{i=1}^{k} w_i x_i + w_{k+1} > m$$

The bounding functions we use are therefore

$$B_k(x_1, \ldots, x_k) = \mathit{true} \text{ iff } \sum_{i=1}^{k} w_i x_i + \sum_{i=k+1}^{n} w_i \geq m$$

$$\text{and} \quad \sum_{i=1}^{k} w_i x_i + w_{k+1} \leq m$$

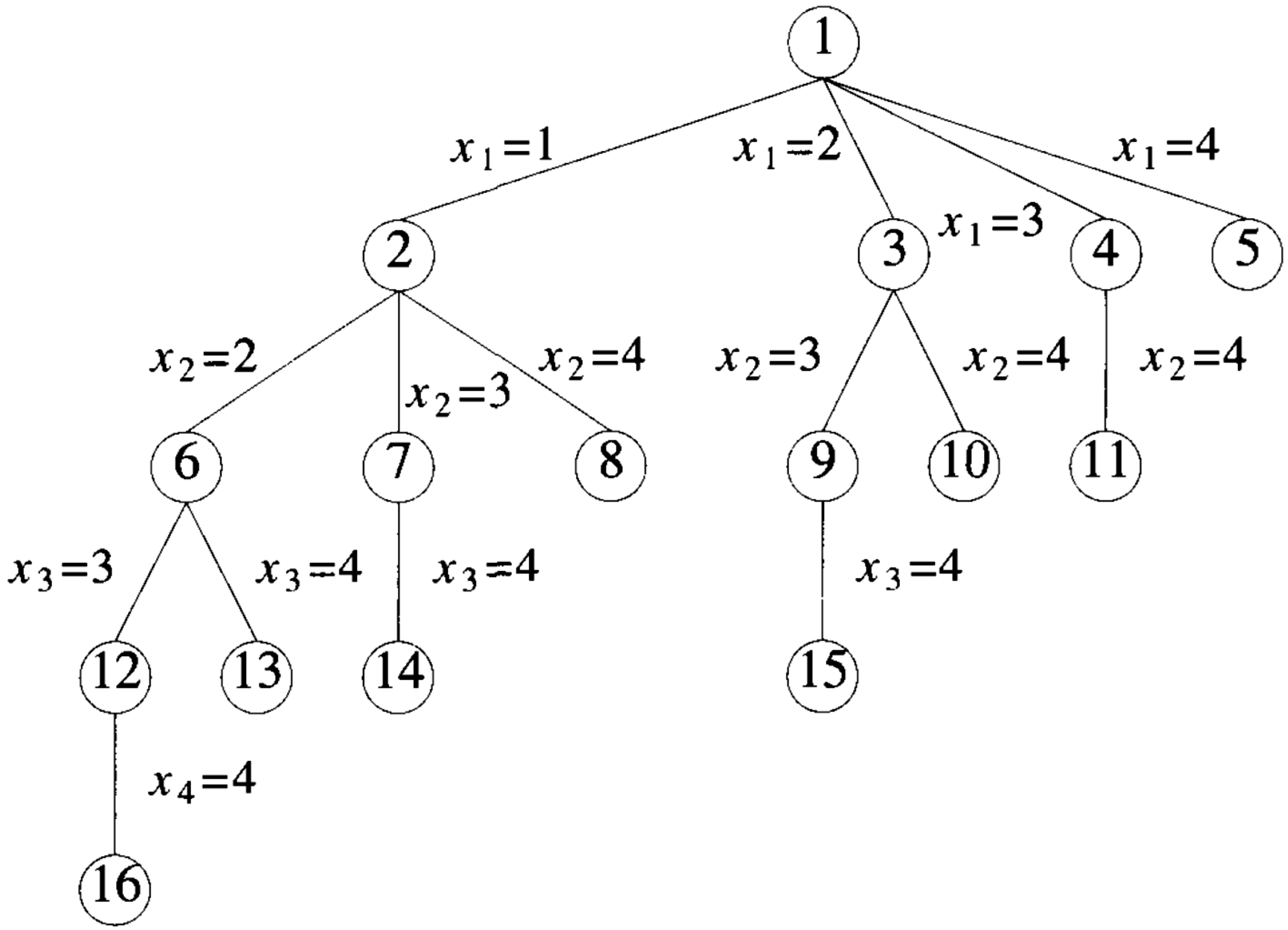
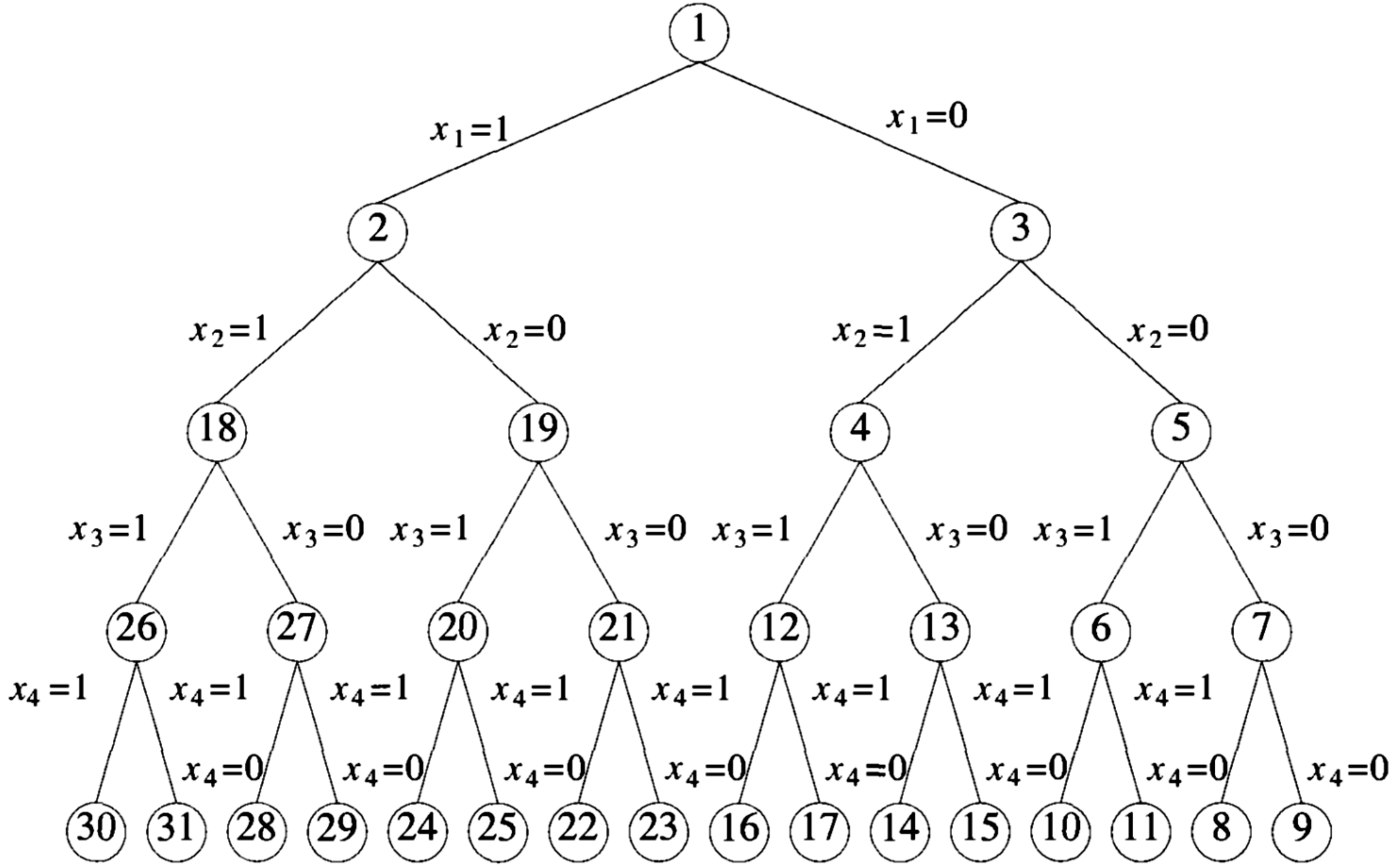


Figure 7.3 A possible solution space organization for the sum of subsets problem. Nodes are numbered as in breadth-first search.

At this point it is useful to develop some terminology regarding tree organizations of solution spaces. Each node in this tree defines a *problem state*. All paths from the root to other nodes define the *state space* of the problem. *Solution states* are those problem states s for which the path from the root to s defines a tuple in the solution space. In the tree of Figure 7.3 all nodes are solution states whereas in the tree of Figure 7.4 only leaf nodes are solution states. *Answer states* are those solution states s for which the path from the root to s defines a tuple that is a member of the set of solutions (i.e., it satisfies the implicit constraints) of the problem. The tree organization of the solution space is referred to as the *state space tree*.

Figure 7.4 Another possible organization for the sum of subsets problems. Nodes are numbered as in *D*-search.



```
Algorithm SumOfSub(s, k, r)
// Find all subsets of $w[1:n]$ that sum to $m$. The values of $x[j]$,
// $1 \le j < k$, have already been determined. $s = \sum_{j=1}^{k-1} w[j] * x[j]$
// and $r = \sum_{j=k}^{n} w[j]$. The $w[j]$'s are in nondecreasing order.
// It is assumed that $w[1] \le m$ and $\sum_{i=1}^{n} w[i] \ge m$.
{
    // Generate left child. Note: $s + w[k] \le m$ since $B_{k-1}$ is true.
    $x[k] := 1$;
    if ($s + w[k] = m$) then write ($x[1:k]$); // Subset found
        // There is no recursive call here as $w[j] > 0$, $1 \le j \le n$.
    else if ($s + w[k] + w[k+1] \le m$)
        then SumOfSub($s + w[k], k + 1, r - w[k]$);
    // Generate right child and evaluate $B_k$.
    if (($s + r - w[k] \ge m$) and ($s + w[k+1] \le m$)) then
    {
        $x[k] := 0$;
        SumOfSub($s, k + 1, r - w[k]$);
    }
}
```

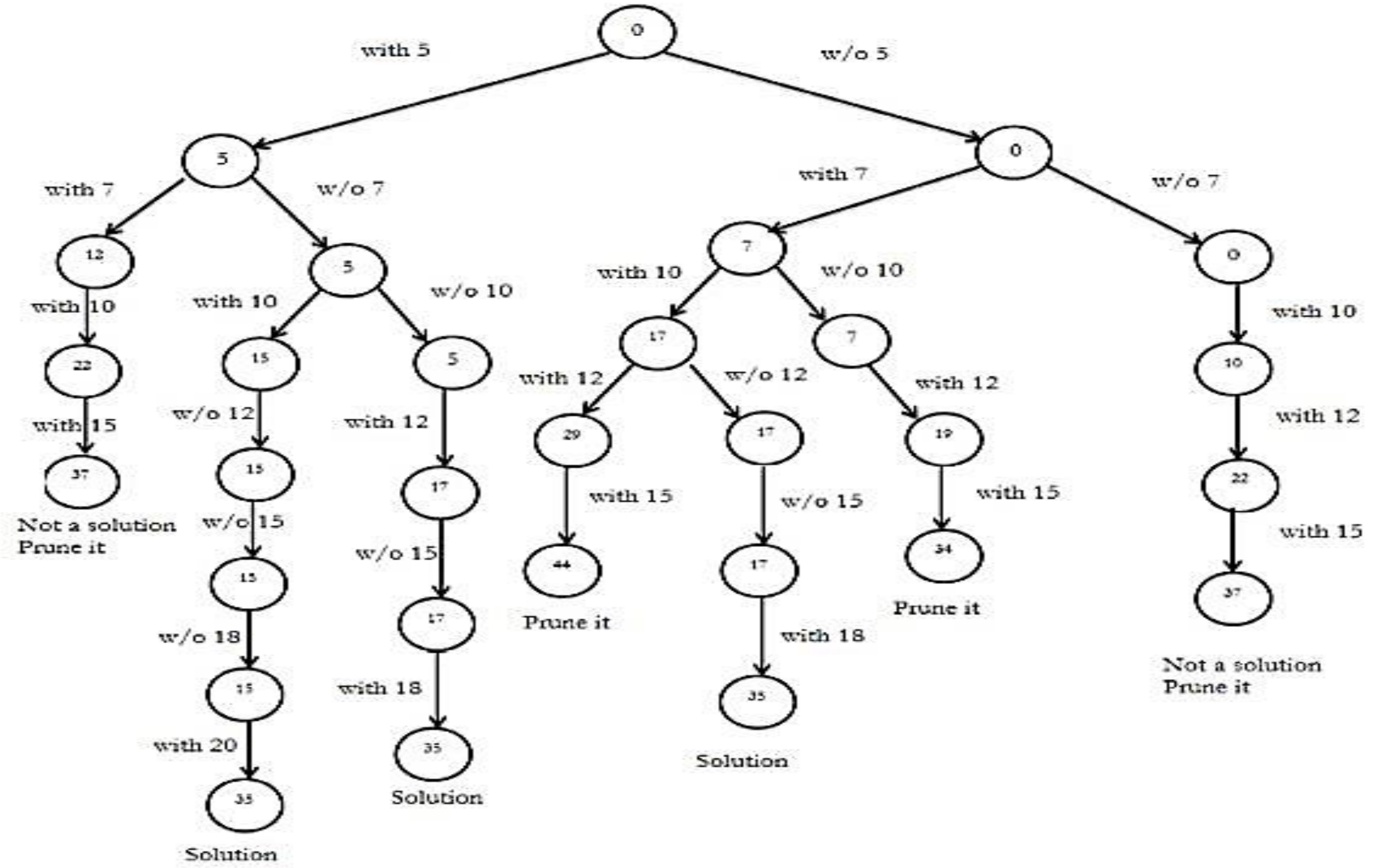
Algorithm 7.6 Recursive backtracking algorithm for sum of subsets problem

Problems

Let $w = \{5, 7, 10, 12, 15, 18, 20\}$ and $m = 35$. Find all possible subsets of w that sum to m. Do this using SumOfSub. Draw the portion of the state space tree that is generated.

With $m = 35$, run SumOfSub on the data (a) $w = \{5, 7, 10, 12, 15, 18, 20\}$, (b) $w = \{20, 18, 15, 12, 10, 7, 5\}$, and (c) $w = \{15, 7, 20, 5, 18, 10, 12\}$. Are there any discernible differences in the computing times?

Initially subset = {}	Sum = 0	Description
5	5	Then add next element.
5, 7	12,i.e. 12 < 35	Add next element.
5, 7, 10	22,i.e. 22 < 35	Add next element.
5, 7, 10, 12	34,i.e. 34 < 35	Add next element.
5, 7, 10, 12, 15	49	Sum exceeds M = 35. Hence backtrack.
5, 7, 10, 12, 18	52	Sum exceeds M = 35. Hence backtrack.
5, 7, 10, 12, 20	54	Sum exceeds M = 35. Hence backtrack.
5, 12, 15	32	Add next element.
5, 12, 15, 18	50	Not feasible. Therefore backtrack
5, 12, 18	35	Solution obtained as M = 35


0
with 5
w/o 5
5
0
with 7
w/o 7
with 7
w/o 7
12
5
7
0
with 10
with 10
w/o 10
with 10
w/o 10
with 10
22
15
5
17
7
10
with 15
w/o 12
with 12
with 12
w/o 12
with 12
with 12
37
15
17
29
17
19
22
Not a solution
Prune it
w/o 15
w/o 15
with 15
w/o 15
with 15
with 15
15
17
44
17
34
37
w/o 18
Prune it
with 18
Prune it
15
with 18
35
Not a solution
Prune it
with 20
35
Solution
35
Solution
35
Solution