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Space of interaction in neural network models

Network definition: $S_i = \pm 1$ with dynamics

$$(E.1) \quad \begin{cases} S_i(t+1) = \text{sgn}\left(h_i(t) - T_i \right) \\ h_i(t) = \frac{1}{\sqrt{N}} \sum_{j \neq i} J_{ij} \, S_j(t) \end{cases}$$

where $h_i(t)$ is the internal "magnetic" field at time t and J_{ij} are the interaction strenghts. N is the total number of units in the network and T_i is a local chemical potential defining the threshold. Note that J_{ij} need not in general be equal to J_{ji}. The interaction strenghts satisfies the spherical constraints:

$$\sum_{i \neq j} J_{ij}^2 = N \qquad (E.2)$$

Considers now a specific pattern ξ^μ of neural activity. This is a fixed point of the dynamics (E.1) if

$$\forall i \in [1, N] \quad \xi_i^\mu \sum_j J_{ij} \, \xi_j^\mu \geq 0 \qquad (E.3)$$

Note that the above exression can be though of as the Hamiltonian of the dynamical system

$$H_i^\mu = \frac{1}{\sqrt{N}} \sum_j \xi_i^\mu J_{ij} \xi_j^\mu \rightarrow \frac{\partial H_i^\mu}{\partial \xi_i^\mu} = \frac{1}{\sqrt{N}} \sum_j J_{ij} \xi_j^\mu = \dot{z}_i^\mu$$

A stronger condition is

$$(E.3) \quad \begin{cases} \xi_i^\mu \sum_j J_{ij} \xi_j^\mu \geqslant \kappa & \forall i \in [1, N] \\ \kappa > 0 \end{cases}$$

if σ^μ is the output of the computation, there are $\mu = 1, \ldots, P$ pairs $\{\xi^\mu, \sigma^\mu\}$ defining the problems.

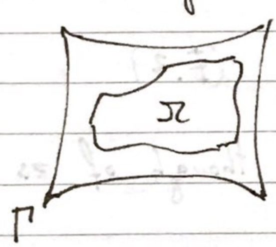
And the components of σ_i^μ and ~~ξ_i~~ ξ_i^μ are i.i.d..
Problem: what is the maximum number of input output pairs p_c ~~that~~ for which the input$\leftrightarrow$output mapping can be implemented? Is there a coupling J_{ij} s.t.

$$\text{(E.4)} \qquad \frac{1}{\sqrt{N}}\,\sigma^\mu \sum_j J_{ij}\,\xi_j^\mu \geq k \qquad \forall\ i \text{ and } \mu\ ?$$

~~Stated otherwise~~ Here the index μ ~~extends~~ runs over the training set, e.g. the cat realizations ~~of the~~ defined before: $\mu=1$ black cat, $\mu=2$ ginger cat etc. For each μ, ξ_i contains the binary representation of the cat.

So the problem is to determine the maximal size of the training set for which the training error is zero when learning from a noisy teacher.

The problem is as follows: Define the phase space Γ of couplings $J_{ij} = \underline{J}$. We want to build an ensemble over this space by ~~with~~ constructing the phase space volume satisfying (Eq. 4). For example, in the microcanonical ensemble, one looks for surfaces of constant energy in phase space. Such volume is given by



$$\Omega(\underline{\xi}^\mu, \sigma^\mu) = \int \prod_{i\neq j} dJ_{ij}\; \delta\Big(\sum_{i\neq j} J_{ij}^2 - N\Big) \prod_{\mu,i} \theta\left(\frac{\sigma^\mu}{\sqrt{N}} \sum_j J_{ij}\,\xi_j^\mu - k\right)$$

$$= \int d\mu(\underline{J}) \prod_{\mu=1}^{p} \theta\left(\frac{\sigma^\mu}{\sqrt{N}}\, \underline{J}\cdot\underline{\xi}^\mu - k\right) \qquad \text{(E.5)}$$

where we have introduced the measure

$$d\mu(\underline{J}) = \prod_{i\neq j} dJ_{ij}\; \delta\Big(\sum_{i\neq j} J_{ij}^2 - N\Big)$$

Since $\underline{\xi}^\mu$ and σ^μ are taken to be random, the volume of connections is also a random variable. Its typical value can be obtained from the quenched entropy

$$S = \frac{1}{N} \langle \lg \Omega(\underline{\xi}^\mu, \sigma^\mu) \rangle_{\underline{\xi}^\mu, \sigma^\mu} \quad (E.6)$$

where $\langle\ \rangle$ is the expectation value w.r.t $\underline{\xi}^\mu$ and σ^μ. In order to evaluate Eq. (E.6) we introduce Replicas

$$S = \frac{1}{N} \langle \lim_{M \to 0} \frac{\Omega^M(\underline{\xi}^\mu, \sigma^\mu) - 1}{M} \rangle_{\underline{\xi}^\mu, \sigma^\mu}$$

$$= \lim_{M \to 0} \frac{1}{NM} [\langle \Omega^M \rangle - 1] \quad (E.7)$$

We need to evaluate:

$$\langle \Omega^M \rangle_{\underline{\xi}^\mu, \sigma^\mu} = \langle\langle \prod_{a=1}^{M} \Omega_a(\underline{\xi}^\mu, \sigma^\mu) \rangle\rangle$$

$$= \left\langle \int \prod_{a=1}^{M} d\mu(\underline{J}^a) \prod_{a,\mu} \vartheta\left(\frac{\sigma^\mu}{\sqrt{N}} \underline{J}^a \cdot \underline{\xi}^\mu - k \right) \right\rangle_{\underline{\xi}^\mu, \sigma^\mu} \quad (E.8)$$

Note that contrary to the spin-glass case, here is the coupling vector $\underline{J}$ to get a replica index.

At this point we use the integral representation of the Heaviside step function

$$\vartheta(x-\alpha) = \int_{\alpha}^{\infty} d\lambda\, \delta(\lambda - x) = \int_{\alpha}^{\infty} d\lambda \int_{-\infty}^{\infty} \frac{d\hat{\lambda}}{2\pi}\, e^{i\hat{\lambda}(\lambda - x)}.$$

In the functional case this is still true. In order to use this representation in Eq. (E.8) we need to introduce Lagrange multipliers $\forall\, a$ and μ.

$$\vartheta\left(\frac{\sigma^\mu}{\sqrt{N}} \underline{J}^a \cdot \underline{\xi}^\mu - k \right) = \int_{k}^{\infty} d\lambda_\mu^a \int_{-\infty}^{\infty} \frac{dx_\mu^a}{2\pi} \exp\left\{ i \sum_{a\mu} x_\mu^a \left(\lambda_\mu^a - \frac{\sigma^\mu}{\sqrt{N}} \underline{J}^a \cdot \underline{\xi}^\mu \right) \right\}$$

(E.9)

We can now average over the probability distribution of ξ^μ and σ^μ.

$$P(\xi^\mu) = \prod_i \left[\frac{1}{2}\delta(\xi_i^\mu + 1) + \frac{1}{2}\delta(\xi_i^\mu - 1) \right] \qquad (E.10)$$

$\xi_i^\mu = \pm 1$ are equally probable. We take the same ~~[illegible]~~ probability distribution for σ^μ.

$$\langle\langle \Omega^\mu \rangle\rangle = \left\langle\left\langle \int \prod_a d\mu(J^a) \prod_{a,\mu} \int_k^\infty \frac{d\lambda_\mu^a}{2\pi} \int_{-\infty}^{\infty} dx_\mu^a \exp\left\{ i \sum_{a\mu} x_\mu^a \left(\lambda_\mu^a - \frac{\sigma^\mu \sum_j J_{ij}^a \xi_j^\mu}{\sqrt{N}} \right) \right\} \right\rangle\right\rangle$$

$$= \left\langle \int \prod_a d\mu(J^a) \int_k^\infty \prod_{a,\mu} \frac{d\lambda_\mu^a}{2\pi} \int_{-\infty}^{\infty} \prod_{a,\mu} dx_\mu^a \int \prod_{j,\mu} d\xi_i^\mu \, P(\xi_i^\mu) \prod_{a,\mu} \exp\left\{ i x_\mu^a \left(\lambda_\mu^a - \frac{\sigma^\mu}{\sqrt{N}} \underline{J}^a \cdot \underline{\xi} \right) \right\} \right\rangle$$

The integral over the probability distribution of $\underline{\xi}^\mu$ is

$$\to \int \prod_{i,\mu} d\xi_i^\mu \prod_i \left[\frac{1}{2}\delta(\xi_i^\mu + 1) + \frac{1}{2}(\xi_i^\mu - 1) \right] e^{-i \frac{\sigma^\mu}{\sqrt{N}} \sum_{j\neq i} \sum_a J_{ij}^a \xi_j^\mu x_\mu^a}$$

$$= \frac{1}{2} e^{-i\sigma^\mu \sum_{j\neq i} \sum J_{ij}^a x_\mu^a} + \frac{1}{2} e^{i \frac{\sigma^\mu}{\sqrt{N}} \sum_{j\neq i} \sum_{a\mu} J_{ij}^a x_\mu^a}$$

$$= \prod_{i\neq j} \frac{1}{2} \left(e^{\frac{i\sum\sigma^\mu}{\sqrt{N}} J_{ij}^a x_\mu^a} + e^{-i\frac{\sum\sigma^\mu}{\sqrt{N}} J_{ij}^a x_\mu^a} \right) \qquad (E.11)$$

We can also perform the average over σ^μ at this point

$$\left\langle \prod_{i\neq j} \frac{1}{2} \left(e^{\frac{i\sum\sigma^\mu}{\sqrt{N}} J_{ij}^a x_\mu^a} + e^{-\frac{i\sum\sigma^\mu}{\sqrt{N}} J_{ij}^a x_\mu^a} \right) \right\rangle_{\sigma^\mu}$$

$$= \int \prod_\mu d\sigma^\mu \frac{1}{2} \left[\delta(\sigma^\mu + 1) + \delta(\sigma^\mu - 1) \right] \sum_a \prod_{i\neq j\mu} \frac{1}{2} \left(e^{\frac{i\sigma^\mu}{\sqrt{N}} J_{ij}^a x_\mu^a} + e^{-i\frac{\sigma^\mu}{\sqrt{N}} J_{ij}^a \lambda_\mu^a} \right)$$

$$= \prod_{i\neq j}\prod_{\mu}\frac{1}{4}\left(e^{i\sum_a \frac{x_\mu^a}{\sqrt N}J_{ij}^a} + e^{-i\sum_a \frac{x_\mu^a}{\sqrt N}J_{ij}^a} + e^{i\sum_a \frac{x_\mu^a}{\sqrt N}J_{ij}^a} + e^{-i\sum_a \frac{x_\mu^a}{\sqrt N}J_{ij}^a}\right)$$

$$= \prod_{i\neq j}\prod_{\mu}\cos\left(\sum_a \frac{x_\mu^a}{\sqrt N}J_{ij}^a\right) = \exp\sum_{i\neq j}\sum_\mu \lg\cos\left(\sum_a \frac{x_\mu^a J_{ij}^a}{\sqrt N}\right)$$

$$\overset{N\to\infty}{\simeq} \exp\left\{-\frac{1}{2}\sum_{i\neq j}\sum_\mu\left(\sum_{a=1}^{M}\frac{x_\mu^a J_{ij}^a}{\sqrt N}\right)^2\right\} = \exp\left\{-\frac{1}{2N}\sum_{i\neq j}\sum_\mu\sum_{a\neq b}x_\mu^a x_\mu^b J_{ij}^a J_{ij}^b + O\left(1/N^2\right)\right\}. \quad (E.12)$$

The average volume of connections reads

$$\langle\langle \Omega^M\rangle\rangle \simeq \int\prod_a d\mu(J^a)\left(\prod_{a,\mu}\int_K\frac{d\lambda_\mu^a}{2\pi}\int_{-\infty}^{\infty}dx_\mu^a\right)\exp\left\{i\sum_{\mu,a}x_\mu^a\lambda_\mu^a - \frac{1}{2N}\sum_{i\neq j}\sum_\mu\sum_{a\neq b}x_\mu^a x_\mu^b J_{ij}^a J_{ij}^b\right\} \quad (E.13)$$

We now introduce the order parameter

$$q^{ab} = \frac{1}{N}\sum_{i\neq j}J_{ij}^a J_{ij}^b \quad (E.13.b)$$

$$\sum_{a,b}x^a x^b J^a J^b = \sum_{a\neq b}x^a x^b J^a J^b + \sum_a (x^a)^2 \underbrace{(J^a)^2}_{N}$$

$$\langle\langle \Omega^M\rangle\rangle = \int\prod_a d\mu(J^a)\left(\prod_{a,\mu}\int_K\frac{d\lambda_\mu^a}{2\pi}\int_{-\infty}^{\infty}dx_\mu^a\right)N\int\prod_{a\neq b}dq^{ab}\,\delta\left(Nq^{ab}-\sum_{i\neq j}J_{ij}^a J_{ij}^b\right)$$

$$\cdot\exp\left\{i\sum_{\mu,a}x_\mu^a\lambda_\mu^a - \frac{1}{2}\sum_\mu\sum_{a\neq b}x_\mu^a x_\mu^b q^{ab} - \frac{1}{2}\sum_{a,\mu}(x_\mu^a)^2\right\} \quad (E.14)$$

$$= \int\prod_{a\neq b}\frac{dq^{ab}\,dF^{ab}}{(2\pi/N)}\int\prod_a\frac{dE^a}{2\pi}\int\prod_a\prod_{i\neq j}dJ_{ij}^a\int_K\prod_{a,\mu}\frac{d\lambda_\mu^a}{2\pi}\int_{-\infty}^{\infty}dx_\mu^a$$

$$\exp\left\{\frac{i}{2}\sum_a\sum_{i\neq j}E^a(J_{ij}^a)^2 - \frac{i}{2}\sum_a NE^a + i\sum_{a\neq b}Nq^{ab}F^{ab} - i\sum_{a\neq b}\sum_{i\neq j}F^{ab}J_{ij}^a J_{ij}^b + i\sum_{\mu,a}x_\mu^a\lambda_\mu^a - \frac{1}{2}\sum_\mu\sum_{a\neq b}x_\mu^a x_\mu^b q^{ab} - \frac{1}{2}\sum_{a,\mu}(x_\mu^a)^2\right\}$$

$$= \int \prod_{a\neq b} \frac{dq^{ab}\, dF^{ab}}{(2\pi/N)} \int \frac{dE^a}{2\pi} \left[\int \prod_a \prod_{i\neq j} dJ^a_{ij} \exp\left(\frac{i}{2} \sum_a \sum_{i\neq j} E^a (J^a_{ij})^2 - i \sum_{a\neq b} \sum_{i\neq j} F^{ab} J^a_{ij} J^b_{ij} \right] \left[\prod_{a,\mu} \int_k d\lambda^a_\mu \int \frac{dx^a_\mu}{2\pi} \exp\left(i \sum_{\mu,a} x^a_\mu \lambda^a_\mu - \frac{1}{2}\sum_{\mu,a} (x^a_\mu)^2 - \frac{1}{2} \sum_\mu \sum_{a\neq b} x^a_\mu x^b_\mu q^{ab} \right)\right] \exp\left\{ -\frac{i}{2} \sum_a E^a N + i \sum_{a\neq b} N q^{ab} F^{ab} \right\} \quad (E.15)$$

let us analize the different terms:

$$\int_k^\infty \prod_{a,\mu} d\lambda^a_\mu \int_{-\infty}^{\infty} \prod_{a,\mu} \frac{dx^a_\mu}{2\pi} \prod_{\mu=1}^{p} \exp\left(i \sum_a x^a_\mu \lambda^a_\mu - \frac{1}{2} \sum_{a\neq b} x^a_\mu x^b_\mu q^{ab} - \frac{1}{2} \sum_a (x^a_\mu)^2 \right)$$

$$= \left[\int_k^\infty \prod_a d\lambda^a \int_{-\infty}^{\infty} \prod_a \frac{dx^a}{2\pi} \exp\left(i \sum_a x^a \lambda^a - \frac{1}{2} x^a x^b q^{ab} - \frac{1}{2}\sum_a (x^a_\mu)^2 \right) \right]^p \quad (E.16.a)$$

and

$$\left[\int \prod_a \prod_{i\neq j} dJ^a_{ij} \prod_{i\neq j} \exp\left\{ \frac{i}{2} \sum_a E^a (J^a_{ij})^2 - i \sum_{a\neq b} F^{ab} J^a_{ij} J^b_{ij} \right\} \right]$$

$$= \left[\int \prod_a dJ^a \exp\left\{ \frac{i}{2} \sum_a E^a (J^a)^2 - i \sum_{a\neq b} F^{ab} J^a J^b \right\} \right]^{N(N-1)} \quad (E.16.b)$$

so both integrals factorize over J_{ij} and $x_\mu \lambda_\mu$.
If we now rescale p by $N \rightarrow p = \alpha N$ and introduce the ~~...~~ definitions

$$G_S(E^a, F^{ab}) = \lg \int \prod_a dJ^a \exp\left\{ \frac{i}{2} \sum_a E^a (J^a)^2 - i \sum_{a\neq b} F^{ab} J^a J^b \right\}$$

$$G_E(q^{ab}) = \lg \int_k^{\infty} \prod_a d\lambda^a \int_{-\infty}^{\infty} \prod_a \frac{dx^a}{2\pi} \exp\left\{ +i\sum_a x^a \lambda^a + \frac{1}{2}\sum_{a\neq b} x^a x^b q^{ab} + \frac{1}{2}\sum_a (x^a)^2 \right\}$$

then we can write

$$\langle\langle \Omega^n \rangle\rangle = \int \prod_{a\neq b} \frac{dq^{ab}\, dF^{ab}}{(2\pi/N)} \int \frac{dE^a}{2\pi} \exp\left\{ N\left(\alpha G_E(q^{ab}) + G_S(q^{ab}, F^{ab}) - \frac{i}{2}\sum_a E^a + i\sum_{a\neq b} q^{ab} F^{ab} \right)\right\} \quad (E.17)$$

Note that for large N the exponent in Eq (E.16.b) $N(N-1) \simeq -N$ and we have adsorbed the minus sign in the definition of G_E. We can rewrite eq. (E.17) as

$$(E.18)\begin{cases} \langle\langle \Omega^n \rangle\rangle = \displaystyle\int \prod_{a\neq b} \frac{dq^{ab}\, dF^{ab}}{2\pi/N} \int \frac{dE^a}{2\pi}\, e^{N G(q^{ab}, F^{ab}, E^a)} \\ G(q^{ab}, F^{ab}, E^a) = \alpha G_E(q^{ab}) + G_S(E^{ab}, F^{ab}) - \dfrac{i}{2}\displaystyle\sum_a E^a + i\sum_{a\neq b} q^{ab} F^{ab} \end{cases}$$

At large N the integral can be evaluated via the saddle point of q^{ab}, F^{ab} and E^a. In order to do this explicitly we assume Replica symmetry

$$(q^{ab} = q,\ F^{ab} = F)\ a<b \text{ and } E^a = E \ \forall a.$$

G_S is called the Entropic part and it measures the number of spherical couplings satisfy the constraint (E.13.a). G_E is the energetic part and it is specific of the cost function or learning rule being used. Within the RS ansatz

$$G(q,F,E) = \alpha G_E(q) + G_S(F,E) - \frac{i}{2}\sum_{a=1}^{n} E + \frac{i}{2}\sum_{a\neq b} qF$$
$$= \alpha G_E(q) + G_S(E,F) - \frac{i}{2} nE + \frac{i}{2} n(n-1)\, qF \quad (E.19)$$

MEAN FIELD Equations

$$
(E.20)\begin{cases}
\dfrac{\partial G(q,F,E)}{\partial q} = \dfrac{\partial G_E(q)}{\partial q} + \dfrac{i}{2} M(M-1) F = 0 \\[2ex]
\dfrac{\partial G(q,F,E)}{\partial F} = \alpha \dfrac{\partial G_S(E,F)}{\partial F} + \dfrac{i}{2} M(M-1)\, q = 0 \\[2ex]
\dfrac{\partial G(q,F,E)}{\partial E} = \alpha \dfrac{\partial G_S(E,F)}{\partial E} - i\dfrac{M}{2} = 0
\end{cases}
$$

~~where~~ To find the solution we proceed as follows

First we rewrite G_E as

$G_E = \log \tilde{G}_E$, where

$$
\tilde{G}_E(q) = \int_K^\infty \prod_a d\lambda_a \int \prod_a \frac{dx_a}{2\pi} \exp\left\{ + i \sum_{a=1}^{M} x_a \lambda_a - \frac{1}{2}\left(\sum_{a,b} - \sum_{a=b}\right) x^a x^b q - \frac{1}{2}\sum_a (x_a)^2 \right\}
$$

$$
= \int_K^\infty \prod_a d\lambda_a \int \prod_a \frac{dx_a}{2\pi} \exp\left\{ + i \sum_{a=1} x_a \lambda_a - \frac{q}{2}\left(\sum_{a=1}^{M} x_a\right)\left(\sum_{b=1}^{M} x_b\right) - \frac{(1-q)}{2} \sum_a (x_a)^2 \right\}
$$

$$
= \int_K^\infty \prod_a d\lambda_a \int \prod_a \frac{dx_a}{2\pi} \exp\left\{ + i \sum_a x_a \lambda_a - \frac{q}{2}\left(\sum_a x_a\right)^2 - \frac{(1-q)}{2} \sum_a (x_a)^2 \right\}
$$

Now we use the gaussian transformation

$$
e^{+\frac{q}{2}\left(i\sum_a x_a\right)^2} = \int \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2} - \sqrt{q}\, z \sum_a i x_a}
$$

Then we obtain

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$$\tilde{G}_E(q) = \int_k^\infty \prod_a d\lambda_a \int \prod_a \frac{dx_a}{2\pi} \int \frac{dz}{\sqrt{2\pi}} \exp\left\{ + i\sum_a x_a\lambda_a - \frac{z^2}{2} + iz\sqrt{q}\sum_a x_a + \frac{(1-q)}{2}\sum_a x_a^2 \right\}$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \int_k^\infty \prod_a d\lambda_a \int \prod_a \frac{dx_a}{2\pi} \prod_a \exp\left\{ + i x_a\lambda_a + iz\sqrt{q}\, x_a + \frac{(1-q)}{2} x_a^2 \right\}$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[\int_k^\infty d\lambda \int \frac{dx}{2\pi} \exp\left(- i x\lambda + i x z\sqrt{q} + \frac{(1-q)}{2} x^2 \right) \right]^M$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[\int_k^\infty d\lambda \int_{-\infty}^{\infty} \frac{dx}{2\pi} e^{-\left[\frac{(1-q)}{2}x^2 + ix(\lambda - z\sqrt{q})\right]} \right]^M$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[\int_k^\infty d\lambda \frac{1}{2\pi} \left(\frac{2\pi}{1-q}\right)^{1/2} e^{-\frac{(\lambda - z\sqrt{q})^2}{(1-q)2}} \right]^M$$

$$\lambda' = \left(\frac{\lambda - z\sqrt{q}}{\sqrt{1-q}}\right) \rightarrow \frac{d\lambda}{\sqrt{1-q}} = d\lambda'$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[\int_{\frac{k - z\sqrt{q}}{\sqrt{1-q}}}^\infty d\lambda' \frac{(1-q)^{1/2}}{\sqrt{2\pi(1-q)}} e^{-\lambda'^2/2} \right]^M$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right)^M \simeq \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \left[1 + M \log H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right) + \cdots \right]$$

$$= \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} + M \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \log H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right)$$

Then we arrive at the final expression valid for $N \gg 1$ and $M \ll 1$.

$$\boxed{G_E(q) = \log\left[1 + M \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \log H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right) \right] \simeq M \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \log H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right).} \qquad (E.21)$$

Let us work out the entropic part now:

$$G_S(F,E) = \log \int \prod d J_a \exp\left\{ \frac{i}{2}\sum_a E J_a^2 - i\sum_{a\neq b} J_a F J_b \right\}$$

$$= \log \int \prod_a dJ_a \exp\left\{\frac{i}{2}\sum_a E J_a^2 - \frac{i}{2}\sum_{a,b} J_a F J_b + \frac{i}{2}\sum_a F J_a^2\right\}$$

$$= \log\int \prod_a dJ_a\, e^{i\sum_{a,b}\left\{\frac{J_a}{2}\left[(E+F)\delta_{ab} - F\right] J_b\right\}}$$

$$= \log \det\Big[i\underbrace{[(E+F)\delta_{ab} - F]}_{A}\Big]^{-1/2} (2\pi)^{M/2}$$

with correct normalization.

$$= \mathrm{Tr}\log A^{-1/2} + (1) = -\frac{1}{2}\,\mathrm{Tr}\log \hat{A} + \frac{M}{2} \qquad (E.22)$$

$1/(2\pi e)^{M/2}$

We can now solve the meanfield equations:

- $\alpha \dfrac{\partial G_S}{\partial F} + \dfrac{i}{2} M(M-1)\, q = 0$

$$-\frac{1}{2}\alpha\, \mathrm{Tr}\left(A^{-1}\partial_F A\right) \simeq \frac{i}{2} q \quad \text{, for } M\ll 1$$

$$-\frac{1}{2}\alpha\, \mathrm{Tr}\left(A^{-1}(-1)\right) = \frac{i}{2} q \qquad -\frac{1}{2}\alpha\, \mathrm{Tr}\left(-A^{-1}(1-\delta_{ab})\right) = \frac{i}{2} q$$

$$\left[q = \alpha\, \mathrm{Tr}\, A^{-1}\right] \qquad \left[iq = 2\alpha\, \mathrm{Tr}\left[A^{-1}(1-\delta_{ab})\right]\right]$$

- $\alpha \dfrac{\partial G_S}{\partial E} - i\dfrac{M}{2} = 0$

$$-\frac{\alpha}{2}\mathrm{Tr}\left(A^{-1}\partial_E A\right) = \frac{i}{2} M \to -\alpha\, \mathrm{Tr}\left(A^{-1}\delta_{ab}\right) = iM$$

Let us now group all the element in Eq(E.19) depending on E an F:

$$\hat{L}(E_a, F_{ab}, q_{ab}) = L_S(E_a, F_{ab}) + \frac{i}{2}\sum_a E_a - i\sum_{a\neq b} q_{ab} F_{ab}.$$

these are the correct signs.
there is some sign error in Eq. (E.19).

The second part of ~~Eq.~~ $\hat{L}$ can be written as

$$\frac{i}{2}\sum_a E_a - i\sum_{a\neq b} q_{ab} F_{ab} = \frac{i}{2}\sum_a E_a - \frac{i}{2}\sum_{a,b} q_{ab} F_{ab} + \frac{i}{2}\sum_{a=b} q_{ab} F_{ab}$$
$$= \frac{i}{2}\sum_{a,b}\left\{ E_a \delta_{ab} - q_{ab} F_{ab}(1-\delta_{ab})\right\}$$

since we have defined : $A = i\left[E_a \delta_{ab} - F_{ab}(1-\delta_{ab})\right]$
the above term can be written
in compact form using a new matrix $B = \delta_{ab} + q_{ab}(1-\delta_{ab})$

Proof: $\hat{A}\cdot\hat{B} = i\left[E_a\delta_{ab} - F_{ab}(1-\delta_{ab})\right]\left[\delta_{ab} + q_{ab}(1-\delta_{ab})\right]$

$$= i\Big[E_a\delta_{ab} + E_a q_{ab}\delta_{ab}(1-\delta_{ab}) - F_{ab}\delta_{ab}(1-\delta_{ab}) - F_{ab} q_{ab}(1-\delta_{ab})^2\Big]$$

$$(1-\delta_{ab})^2 = (1-\delta_{ab})(1-\delta_{ab}) = (1-\delta_{ab}) - \delta_{ab}(1-\delta_{ab}) = (1-\delta_{ab})$$

since $\delta_{ab}(1-\delta_{ab}) = 0$. Using this in $A\cdot B$ we find the desired identity. Then

$$\hat{L}(E_a, F_{ab}, q_{ab}) = -\frac{M}{2} - \frac{1}{2}\operatorname{Tr}\lg\hat{A} + \frac{1}{2}\operatorname{Tr}\hat{A}\hat{B}. \qquad (E.23)$$

Taking the saddle point w.r.t E_a an F_{ab} gives:

$$\begin{cases} \dfrac{\partial\hat{L}}{\partial E_a} = -\dfrac{i}{2}\operatorname{Tr}\hat{A}^{-1}\delta_{ab} + \dfrac{i}{2}\operatorname{Tr}\hat{B}\,\delta_{ab} = 0 \\[2ex] \dfrac{\partial\hat{L}}{\partial F_{ab}} = \dfrac{i}{2}\operatorname{Tr}\hat{A}^{-1}(1-\delta_{ab}) - \dfrac{i}{2}\operatorname{Tr}\hat{B}(1-\delta_{ab}) = 0 \end{cases}$$

The two conditions gives the same solution

$$[\hat{A}]^{-1} = B \qquad (E.24)$$

· We now use the RS ansatz: $q_{ab} = q$, $E_{ab} = E$, $F_{ab} = F$. Within this ansatz

$$B = \begin{pmatrix} 1 & q & q & \cdots \\ q & 1 & & \\ q & & 1 & \\ \vdots & & & \ddots \end{pmatrix} \qquad (E.25.a)$$

This matrix has eigenvalues:

$$\Lambda = \begin{pmatrix} (1-q)+mq & & & 0 \\ & (1-q) & & \\ 0 & & (1-q) & \\ & & & \ddots \end{pmatrix}$$

i.e. one eigenvalue: $(1-q)+mq$ and $(m-1)$ eigenvalues $(1-q)$. Then we have: $B = Q^{-1}\Lambda Q$ and

$$\frac{1}{2}\operatorname{Tr}\log B = \frac{1}{2}\operatorname{Tr}\log Q^{-1}\Lambda Q = \frac{1}{2}\operatorname{Tr}\log\Lambda$$

$$= \frac{(m-1)}{2}\log(1-q) + \frac{1}{2}\log\left[(1-q)+mq\right]$$

$$= \frac{m}{2}\log(1-q) - \cancel{\frac{1}{2}\log(1-q)} + \cancel{\frac{1}{2}\log(1-q)} + \frac{1}{2}\log\left[1+\frac{mq}{1-q}\right]$$

$$= \frac{m}{2}\log(1-q) + \frac{1}{2}\log\left(1+\frac{mq}{1-q}\right).$$

$$\underset{m\to 0}{\simeq} \frac{m}{2}\log(1-q) + \frac{mq}{2(1-q)}. \qquad (E.26).$$

Using these results in Eq. (E.19), we obtain at leading order in $m \to 0$

$$G^*(q) \simeq \alpha M \int dz\, e^{-z^2/2} \lg H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right) + \frac{M}{2}\lg(1-q) + \frac{Mq}{2(1-q)} + \text{const} \qquad \text{(E.27)}$$

Note that the change of sign comes from

$$-\frac{1}{2}\,\mathrm{Tr}\lg A \longrightarrow -\frac{1}{2}\,\mathrm{Tr}\lg \hat{B}^{-1} = -\frac{1}{2}\,\mathrm{Tr}\lg \Lambda^{-1} = \frac{1}{2}\,\mathrm{Tr}\lg \Lambda .$$

From Eq. (7) the saddle point solution reads:

$$S(\alpha) = \lim_{m\to 0}\lim_{N\to\infty} \frac{1}{mN}\langle\langle Z^m \rangle\rangle \qquad \text{(E.28)}$$

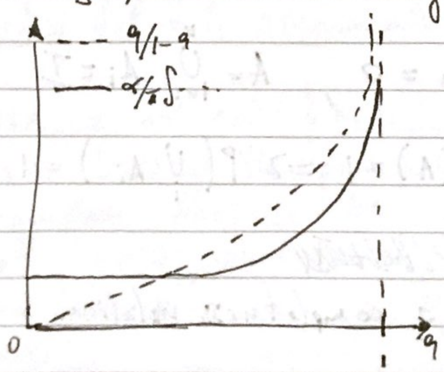
$$= \underset{q}{\mathrm{extr}}\left\{\alpha \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \lg H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right) + \frac{1}{2}\lg(1-q) + \frac{q}{2(1-q)}\right\}$$

$$= \alpha \int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} H\left(\frac{k - z\sqrt{q}}{\sqrt{1-q}}\right)^{-1} \frac{e^{-\frac{(k-z\sqrt{q})^2}{2(1-q)}}}{\sqrt{2\pi}} \frac{1}{2\sqrt{1-q}}\left(\frac{z}{\sqrt{q}} - \frac{(k-z\sqrt{q})}{(1-q)}\right) + \frac{q}{2(1-q)^2} = 0$$

This can be recasted as

$$\boxed{\frac{q}{(1-q)} = \frac{\alpha}{\pi}\int \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} e^{-\frac{(k-z\sqrt{q})^2}{(1-q)}} H\left(\frac{k-z\sqrt{q}}{\sqrt{1-q}}\right)^{-2}}$$

$q = 0$ when $\alpha = 0$. since q describes the overlap between two Js, this solution gives zero overlap.



FOR $q \to 1$, $\frac{q}{1-q} \to \infty$. this limit defines the max of $\alpha = \alpha_c$:

$$\boxed{\frac{1}{\alpha_c} = \int_{-\infty}^{k} dz\, e^{-z^2/2}(k-z)^2}$$

α increases as the overlap between different Js increases.