BT6270: Assignment - 3

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EE20B064

Coupling Hopf Oscillators

In this assignment, we couple two Hopf oscillators in order to achieve a given phase difference between the two oscillators. We look at two different methods to achieve the same: (i) Complex Coupling and (ii) Power Coupling

- Hopf Oscillators

Hopf Oscillators are a class of non-linear oscillators that show limit cycle oscillatory behaviour (stable periodic solution).
The following mathematical equations (in polar coordinates)

$$dr/dt = \mu r(1 - r^2) \quad \text{and} \quad d\theta/dt = 1$$

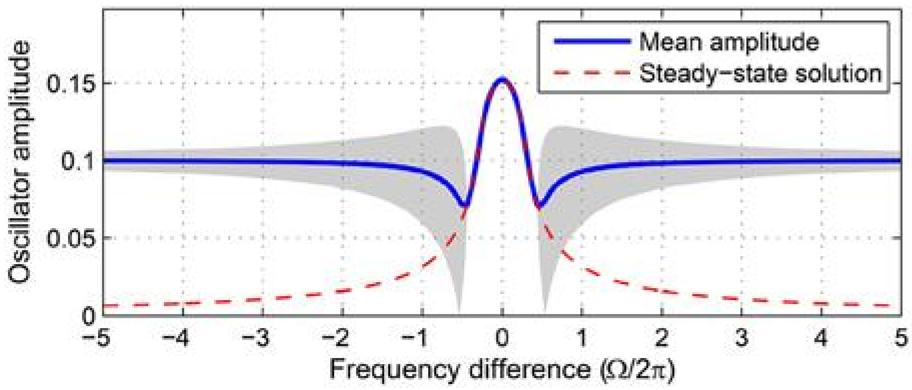
represent a simple Hopf oscillator with a stable limit cycle at r = 1

- Motivation

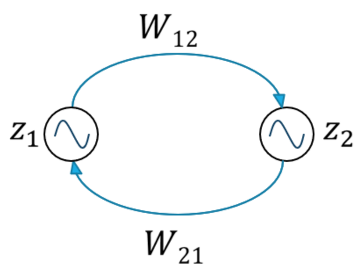
Since we observe oscillatory behaviour in neural spiking activities, we would like to model it using oscillators that exhibit limit cycle behaviour. The simplest oscillators that exhibit limit cycle oscillations are Hopf oscillators. Hence we study the coupling of Hopf oscillators, as the coupling of a large number of such oscillators in order to create a complex network of oscillators can be used to model other complicated oscillatory behaviour!

- Coupling a pair of Hopf oscillators

From the results obtained in (Kim and Large, 2015), we can see how a Hopf oscillator resonates with an input sinusoid of varying frequency difference as shown below.



Keeping this in mind, we would now like to couple two oscillators of both same and different frequencies, as described below.



Now, we demonstrate how coupling two oscillators can achieve a constant phase difference between them, using two different types of coupling:

(A) <u>Complex Coupling</u>

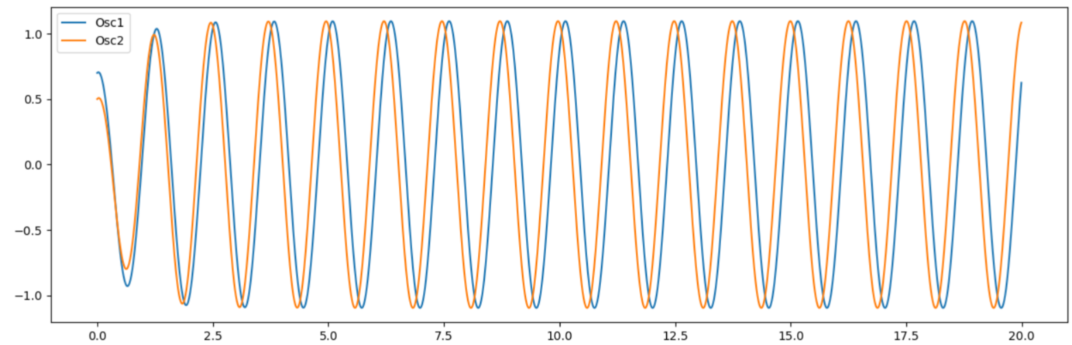
The differential equations obtained for the two coupled oscillators in polar coordinates are:

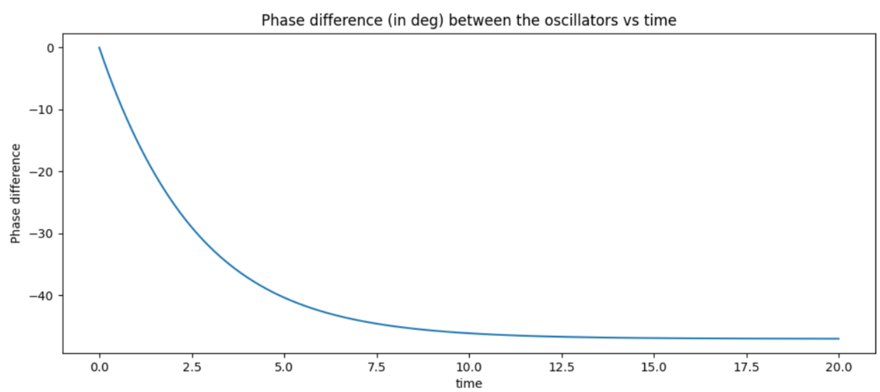
$$\dot{r_1} = (\mu - r_1^2)r_1 + Ar_2 \cos(\theta_2 - \theta_1 + \varphi)$$
$$\dot{\theta_1} = \omega_1 + A\frac{r_2}{r_1} \sin(\theta_2 - \theta_1 + \varphi)$$

$$\dot{r_2} = (\mu - r_2^2)r_2 + Ar_1 \cos(\theta_1 - \theta_2 - \varphi)$$
$$\dot{\theta_2} = \omega_2 + A\frac{r_1}{r_2} \sin(\theta_1 - \theta_2 - \varphi)$$

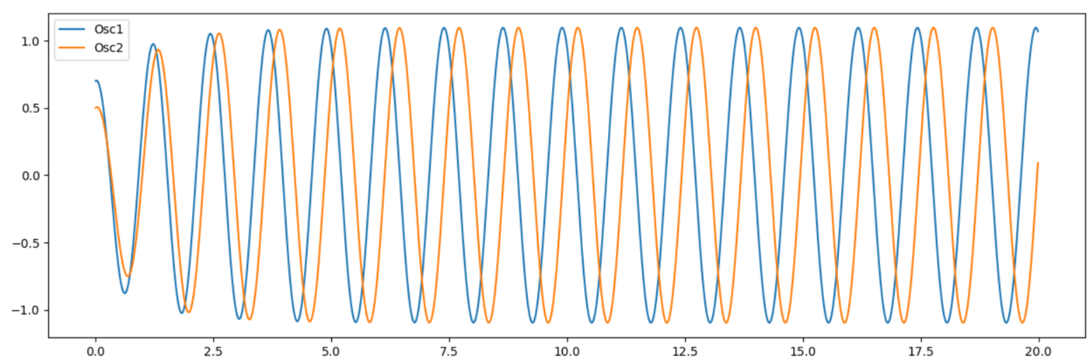
where Φ is the phase difference, ω1 = ω2 = 5,
A is the coupling weight (chosen as 0.2)
The plots below clearly show the phase difference achieved, from both the oscillations vs time plot and the phase difference (Φ) vs time plot.

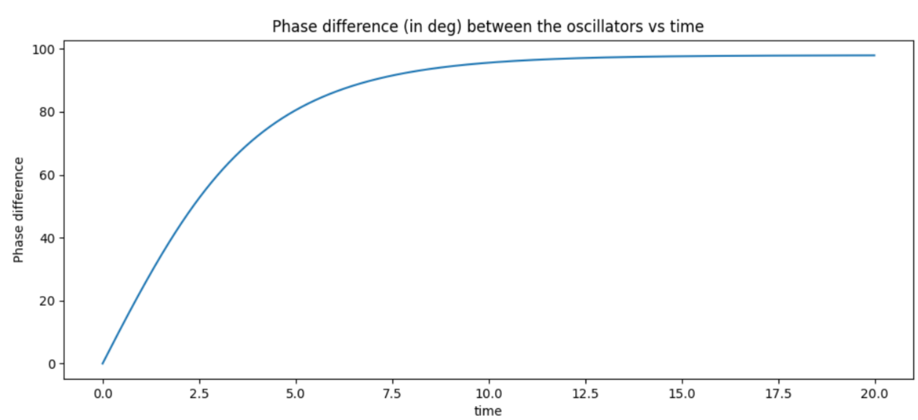
Case (i) Φ = -47°





Case (ii) Φ = 98°





(B) Power Coupling

In the case where the frequencies of the two oscillators are different, a constant phase difference does not make sense. Hence, we define a term called '*normalized'* phase difference (Ψ):

$$\Psi = \theta 1/\omega 1 - \theta 2/\omega 2$$

The differential equations obtained for the two coupled oscillators in polar coordinates are:

$$\dot{r_1} = (\mu - r_1^2)r_1 + Ar_2^{\frac{\omega_1}{\omega_2}} \cos\left(\omega_1\left(\frac{\theta_2}{\omega_2} - \frac{\theta_1}{\omega_1} + \frac{\varphi}{\omega_1\omega_2}\right)\right)$$

$$\dot{\theta_1} = \omega_1 + A\frac{r_2^{\frac{\omega_1}{\omega_2}}}{r_1} \sin\left(\omega_1\left(\frac{\theta_2}{\omega_2} - \frac{\theta_1}{\omega_1} + \frac{\varphi}{\omega_1\omega_2}\right)\right)$$

$$\dot{r_2} = (\mu - r_2^2)r_2 + Ar_1^{\frac{\omega_2}{\omega_1}} \cos\left(\omega_2\left(\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} - \frac{\varphi}{\omega_2\omega_1}\right)\right)$$

$$\dot{\theta_2} = \omega_2 + A\frac{r_1^{\frac{\omega_2}{\omega_1}}}{r_2} \sin\left(\omega_2\left(\frac{\theta_1}{\omega_1} - \frac{\theta_2}{\omega_2} - \frac{\varphi}{\omega_2\omega_1}\right)\right)$$

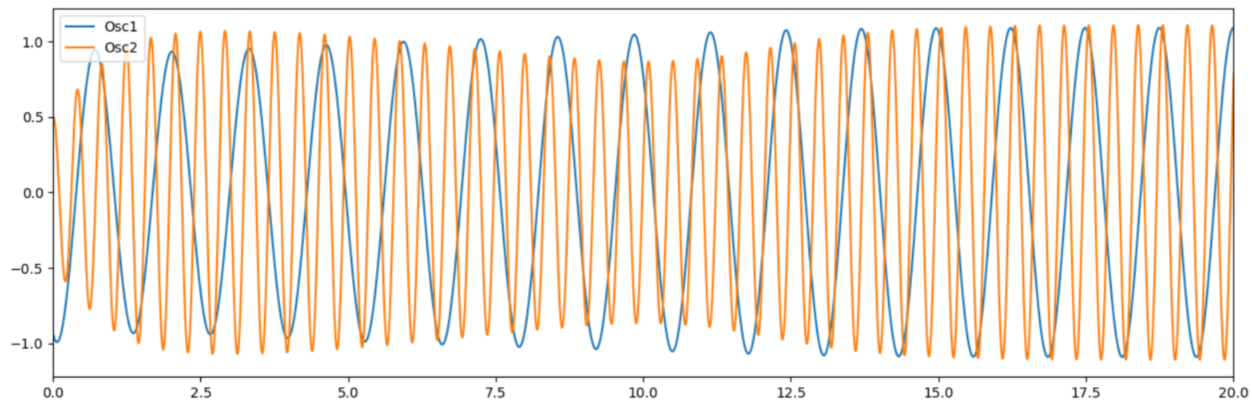
Where ω1 = 5, ω2 = 15
The coupling weight A is again chosen as 0.2 here.

The first plot doesn't tell us much, as with both frequencies being different, the phase difference is not evident from this plot.

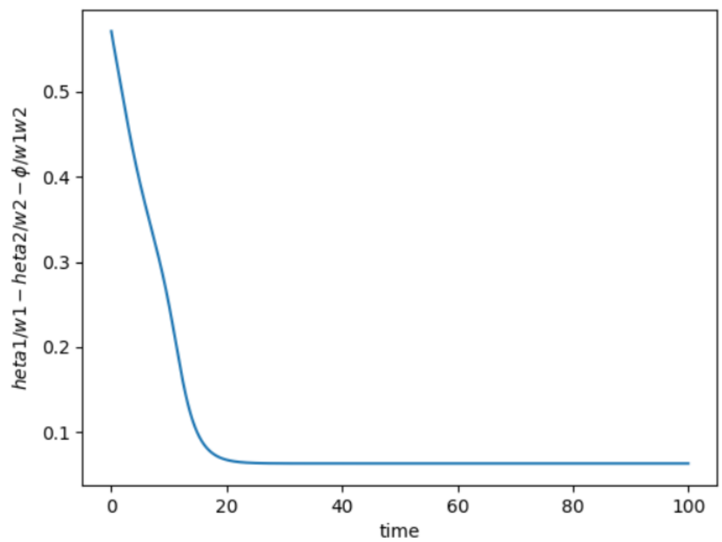
Upon solving for steady state conditions, we see that the normalized phase difference θ1/ω1 - θ2/ω2 converges to Φ/ω1ω2, which can be seen from the second plot.
(Note: As we vary initial values of phase θ1 and θ2, we see that this value for the system converges to solutions other than zero too)

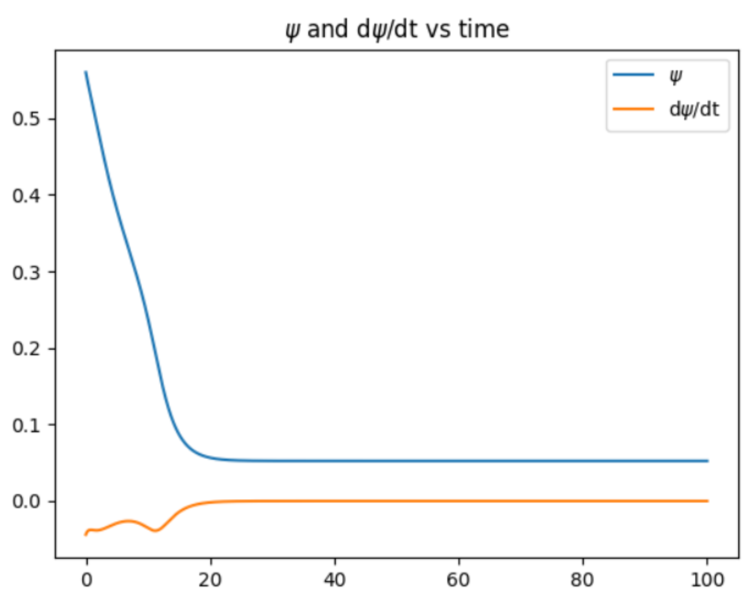
The third plot shows how the normalized phase difference (Ψ) and its derivative (dΨ/dt) vary with time.

Case (i) Φ = -47°

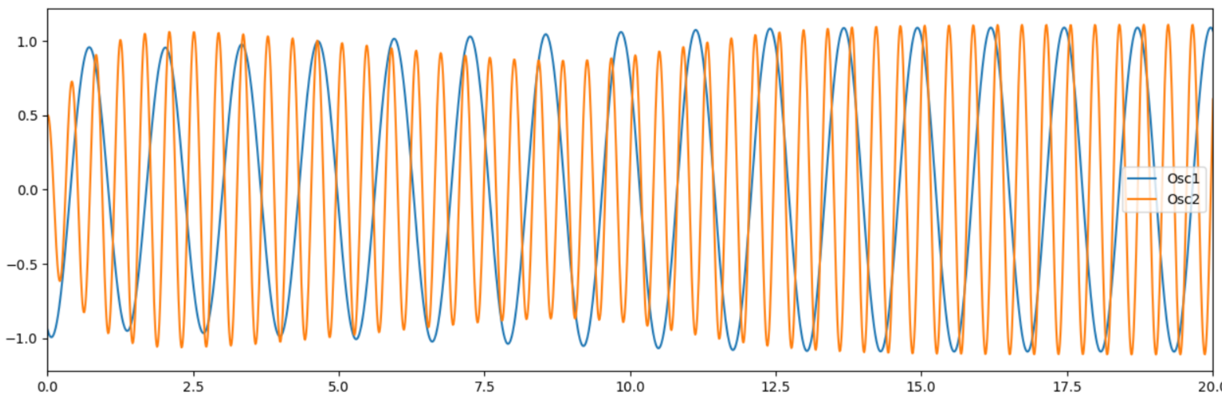


θ1/ω1 - θ2/ω2 - Φ/ω1ω2 vs time





Case (ii) $\Phi = 98°$



$\theta 1/\omega 1 - \theta 2/\omega 2 - \Phi/\omega 1\omega 2$ vs time