Maxwell equation with finite conductivity

Axel Brandenburg

May 17, 2021

This text describes the implementation of `src/magnetic/maxwell.f90`. It provides an exact solution, so time stepping is only needed to interrupt the calculation to provide diagnostic output in intermediate intervals.

Assuming $\boldsymbol{\nabla}\cdot\boldsymbol{A}=\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$, we have

$$\dot{\boldsymbol{A}}=-\boldsymbol{E},\quad \dot{\boldsymbol{E}}=k^2\boldsymbol{A}-\sigma(\boldsymbol{E}+\boldsymbol{\mathcal{E}}). \tag{1}$$

where $\boldsymbol{\mathcal{E}}=\mathsf{P}(\boldsymbol{u}\times\boldsymbol{B})$ is the solenoidal part of the electromagnetic force and P is the projection operator.

$$\ddot{\boldsymbol{A}}+\sigma\dot{\boldsymbol{A}}+k^2\boldsymbol{A}=\sigma\boldsymbol{\mathcal{E}}. \tag{2}$$

The homogeneous part obeys the characteristic equation

$$\lambda^2+\lambda\sigma+k^2=0. \tag{3}$$

Solution

$$\lambda_{1,2}=(-\sigma\pm D)/2 \tag{4}$$

where $D=\sqrt{\sigma^2-4k^2}$. If $\sigma=0$, then $D=2\mathrm{i}k$ and $\lambda_{1,2}=\pm\mathrm{i}k$.

1 Solution

From one time t to the next $t+\delta t$, we assume that $\boldsymbol{\mathcal{E}}=\mathrm{const}$. We then make the following ansatz for each position in $\boldsymbol{k}$ space:

$$\boldsymbol{A}=\boldsymbol{A}_1e^{\lambda_1\delta t}+\boldsymbol{A}_2e^{\lambda_2\delta t}+(\sigma/k^2)\,\boldsymbol{\mathcal{E}}, \tag{5}$$

$$\boldsymbol{E}=-\boldsymbol{A}_1\lambda_1e^{\lambda_1\delta t}-\boldsymbol{A}_2\lambda_2e^{\lambda_2\delta t} \tag{6}$$

We define $\tilde{\boldsymbol{A}}=\boldsymbol{A}-(\sigma/k^2)\,\boldsymbol{\mathcal{E}}$, so we have in matrix form

$$\begin{pmatrix}\tilde{\boldsymbol{A}}\\ \boldsymbol{E}\end{pmatrix}_{t+\delta t}=\begin{pmatrix}e^{\lambda_1\delta t} & e^{\lambda_2\delta t}\\ -\lambda_1e^{\lambda_1\delta t} & -\lambda_2e^{\lambda_2\delta t}\end{pmatrix}\begin{pmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\end{pmatrix} \tag{7}$$

Initial condition

$$\begin{pmatrix}\tilde{\boldsymbol{A}}\\ \boldsymbol{E}\end{pmatrix}_{t}=\begin{pmatrix}1 & 1\\ -\lambda_1 & -\lambda_2\end{pmatrix}\begin{pmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\end{pmatrix}, \tag{8}$$

or

$$\begin{pmatrix}\boldsymbol{A}_1\\ \boldsymbol{A}_2\end{pmatrix}=\frac{1}{\lambda_1-\lambda_2}\begin{pmatrix}-\lambda_2 & -1\\ +\lambda_1 & +1\end{pmatrix}\begin{pmatrix}\tilde{\boldsymbol{A}}\\ \boldsymbol{E}\end{pmatrix}_{t}. \tag{9}$$

So

$$\begin{pmatrix}\tilde{\boldsymbol{A}}\\ \boldsymbol{E}\end{pmatrix}_{t+\delta t}=\begin{pmatrix}c_A & s_A\\ s_E & c_E\end{pmatrix}=\begin{pmatrix}\tilde{\boldsymbol{A}}\\ \boldsymbol{E}\end{pmatrix}_{t}. \tag{10}$$

with

$$\begin{pmatrix}c_A & s_A\\ s_E & c_E\end{pmatrix}= \tag{11}$$

so, using $\lambda_1-\lambda_2=D$,

$$\begin{pmatrix}e^{\lambda_1\delta t} & e^{\lambda_2\delta t}\\ -\lambda_1e^{\lambda_1\delta t} & -\lambda_2e^{\lambda_2\delta t}\end{pmatrix}\frac{1}{D}\begin{pmatrix}-\lambda_2 & -1\\ +\lambda_1 & +1\end{pmatrix} \tag{12}$$

or

$$\frac{1}{D}\begin{pmatrix}\lambda_1e^{\lambda_2\delta t}-\lambda_2e^{\lambda_1\delta t} & e^{\lambda_2\delta t}-e^{\lambda_1\delta t}\\ \lambda_1\lambda_2(e^{\lambda_1\delta t}-e^{\lambda_2\delta t}) & \lambda_1e^{\lambda_1\delta t}-\lambda_2e^{\lambda_2\delta t}\end{pmatrix} \tag{13}$$

For $\sigma=0$, this reduces to

$$\frac{1}{2\mathrm{i}k}\begin{pmatrix}\mathrm{i}ke^{-\mathrm{i}k\delta t}+\mathrm{i}ke^{\mathrm{i}k\delta t} & e^{-\mathrm{i}k\delta t}-e^{\mathrm{i}k\delta t}\\ \mathrm{i}k(-\mathrm{i}k)(e^{-\mathrm{i}k\delta t}-e^{\mathrm{i}k\delta t}) & \mathrm{i}ke^{\mathrm{i}k\delta t}+\mathrm{i}ke^{-\mathrm{i}k\delta t}\end{pmatrix} \tag{14}$$

$$\frac{1}{2}\begin{pmatrix}e^{-\mathrm{i}k\delta t}+e^{\mathrm{i}k\delta t} & (e^{-\mathrm{i}k\delta t}-e^{\mathrm{i}k\delta t})/\mathrm{i}k\\ (-\mathrm{i}k)(e^{-\mathrm{i}k\delta t}-e^{\mathrm{i}k\delta t}) & e^{\mathrm{i}k\delta t}+e^{-\mathrm{i}k\delta t}\end{pmatrix} \tag{15}$$

$$=\begin{pmatrix}\cos k\delta t & -k^{-1}\sin k\delta t\\ k\sin k\delta t & \cos k\delta t\end{pmatrix} \tag{16}$$

In the limit $\sigma\to\infty$, we have $D=\sigma-2k^2/\sigma$, so $\lambda_1=-k^2/\sigma$ and $\lambda_2=-\sigma$,

$$c_A\approx\frac{1}{\sigma}\left(1+\frac{2k^2}{\sigma^2}\right)\left(-\frac{k^2}{\sigma}e^{-\sigma\delta t}+\sigma e^{-k^2\delta t/\sigma}\right) \tag{17}$$

or, expanding $e^{-k^2\delta t/\sigma}\approx 1-k^2\delta t/\sigma$,

$$c_A\approx\left(1+\frac{2k^2}{\sigma^2}\right)\left(-\frac{k^2}{\sigma^2}e^{-\sigma\delta t}+1-\frac{k^2}{\sigma}\delta t\right) \tag{18}$$

we have $c_A \approx 1 - \delta t\, k^2/\sigma$, and the matrix becomes

$$\approx \begin{pmatrix} 1 - \delta t\, k^2/\sigma & -\sigma^{-1} \\ 0 & e^{-\sigma \delta t} \end{pmatrix} \tag{19}$$

Therefore,

$$\boldsymbol{A}_{\rm new} - \frac{\sigma}{k^2}\boldsymbol{\mathcal{E}} = \left(1 - \delta t \frac{k^2}{\sigma}\right)\left(\boldsymbol{A} - \frac{\sigma}{k^2}\boldsymbol{\mathcal{E}}\right) - \frac{\boldsymbol{E}}{\sigma}. \tag{20}$$

so

$$\boldsymbol{A}_{\rm new} = \frac{\sigma}{k^2}\boldsymbol{\mathcal{E}} + \left(1 - \delta t \frac{k^2}{\sigma}\right)\left(\boldsymbol{A} - \frac{\sigma}{k^2}\right)\boldsymbol{\mathcal{E}} - \frac{\boldsymbol{E}}{\sigma}. \tag{21}$$

or

$$\boldsymbol{A}_{\rm new} = \frac{\sigma}{k^2}\boldsymbol{\mathcal{E}} + \left(1 - \delta t \frac{k^2}{\sigma}\right)\boldsymbol{A} - \left(1 - \delta t \frac{k^2}{\sigma}\right)\frac{\sigma}{k^2}\boldsymbol{\mathcal{E}} - \frac{\boldsymbol{E}}{\sigma}. \tag{22}$$

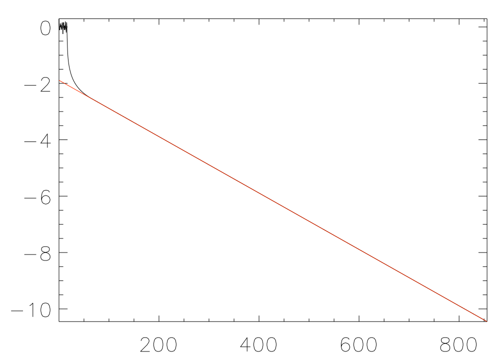

Figure 1: Decay of $l = \ln B_{\rm rms}$ for a linearly increasing conductivity.

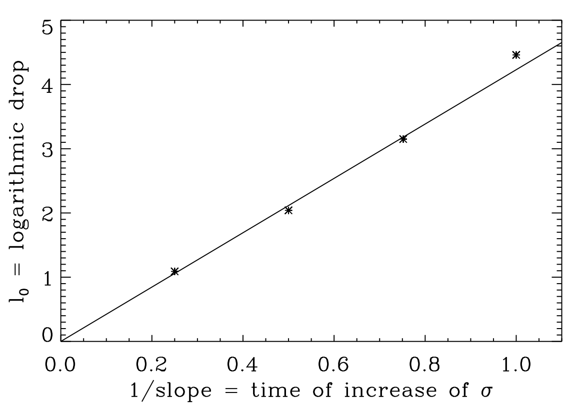

Figure 2: Decay of $l = \ln B_{\rm rms}$ on the slope s for a linearly increasing conductivity.

and therefore

$$\boldsymbol{A}_{\rm new} = \left(1 - \delta t \frac{k^2}{\sigma}\right)\boldsymbol{A} + \delta t\, \boldsymbol{\mathcal{E}} - \frac{\boldsymbol{E}}{\sigma}. \tag{23}$$

Ignoring also the $-\boldsymbol{E}/\sigma$ term, we have

$$\frac{\boldsymbol{A}_{\rm new} - \boldsymbol{A}}{\delta t} = -\frac{k^2}{\sigma}\boldsymbol{A} + \boldsymbol{\mathcal{E}} \tag{24}$$

so we recover, to first order,

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\eta k^2 \boldsymbol{A} + \boldsymbol{\mathcal{E}}, \tag{25}$$

where $\eta = 1/\sigma$ is the magnetic diffusivity.

2 Conductivity changes

Assume $\sigma = st$ after $t > t_0$ up to some maximum value σ_2. At late times, $B_{\rm rms} \propto \exp(-tk^2/\sigma$. We extrapolate $l = l_0$ back to the time t_0 when σ was still constant; see Figure 1.

Figure 2 shows the dependence l_0 versus $1/s$. There seems to be a linear relationship between l_0 and the time of increase of σ. Figure 3 shows the decay of $l = \ln B_{\rm rms}$ for a linearly increasing conductivity with different time constants.

3 Displacement as a correction

$$\dot{\boldsymbol{A}} = -\boldsymbol{E}, \quad \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = k^2 \boldsymbol{A} - \sigma(\boldsymbol{E} + \boldsymbol{\mathcal{E}}). \tag{26}$$

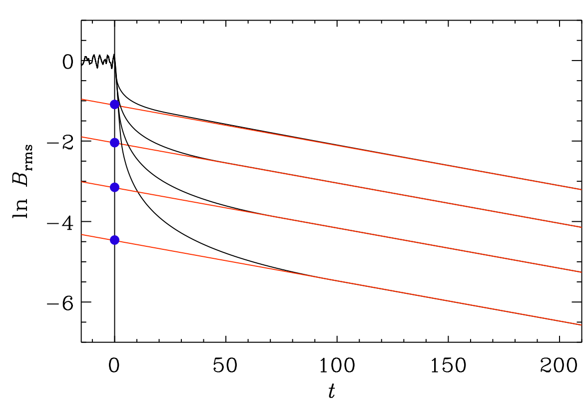

Figure 3: Decay of $l = \ln B_{\rm rms}$ for a linearly increasing conductivity.

Isolate $\boldsymbol{E}$

$$\left(\sigma+\frac{1}{c^2}\frac{\partial}{\partial t}\right)\boldsymbol{E}=k^2\boldsymbol{A}-\sigma\boldsymbol{\mathcal{E}}. \tag{27}$$

Divide by σ

$$\left(1+\frac{\eta}{c^2}\frac{\partial}{\partial t}\right)\boldsymbol{E}=-(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{28}$$

$$\left(1+\frac{\eta}{c^2\delta t}\right)\boldsymbol{E}=\frac{\eta}{c^2\delta t}\boldsymbol{E}_{\rm old}-(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{29}$$

In the limit $\eta\to 0$, we recover

$$\boldsymbol{E}=-(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{30}$$

In the limit $\eta\to\infty$, we recover

$$\frac{\eta}{c^2\delta t}\boldsymbol{E}=\frac{\eta}{c^2\delta t}\boldsymbol{E}_{\rm old}-\eta\nabla^2\boldsymbol{A}. \tag{31}$$

Eq. (29) in terms of σ

$$\left(1+\frac{1}{c^2\sigma\delta t}\right)\boldsymbol{E}=\frac{1}{c^2\sigma\delta t}\boldsymbol{E}_{\rm old}-(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{32}$$

Define $\epsilon=(c^2\sigma\delta t)^{-1}$, then

$$\boldsymbol{E}=\frac{\epsilon}{1+\epsilon}\boldsymbol{E}_{\rm old}-\frac{1}{1+\epsilon}(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{33}$$

Limit $\epsilon\to 0$

$$\boldsymbol{E}=-(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{34}$$

Limit $\epsilon\to\infty$, usimg $\epsilon^{-1}=c^2\sigma\delta t$

$$\boldsymbol{E}=\boldsymbol{E}_{\rm old}-c^2\sigma\delta t(\boldsymbol{\mathcal{E}}+\eta\nabla^2\boldsymbol{A}). \tag{35}$$

$Header: /var/cvs/brandenb/tex/GW/inflation/notes3.tex,v 1.11 2021/05/05 05:42:26 brandenb Exp $