1 Conservative formulation

Solve

$$T^{00}{}_{,0} = -T^{0i}{}_{,i} \tag{1}$$

$$T^{i0}{}_{,0} = -T^{ij}{}_{,j} \tag{2}$$

where

$$E \equiv T^{00} = \frac{4}{3}\rho\gamma^2 - \frac{1}{3}\rho = \frac{4}{3}\rho\gamma^2\left(1 - 1/4\gamma^2\right) \tag{3}$$

$$T^{ij} = \frac{4}{3}\rho\gamma^2 u_i u_j + \frac{1}{3}\rho\delta_{ij} \tag{4}$$

Define

$$S_i \equiv T^{0i} = T^{i0} = \frac{4}{3}\rho\gamma^2 u_i \tag{5}$$

So $\boldsymbol{u} = \epsilon_u \boldsymbol{S}$, where $\epsilon_u = 3/(4\rho\gamma^2) = (1-1/4\gamma^2)/E$. using $\boldsymbol{u}^2 = 1 - 1/\gamma^2$,

$$r \equiv (T^{i0}/T^{00})^2 = \frac{\gamma^4 - \gamma^2}{(\gamma^2 - 1/4)^2} \tag{6}$$

so

$$\gamma^4 - \gamma^2 - (\gamma^4 - \gamma^2/2 + 1/16)r = 0 \tag{7}$$

$$\gamma^4(1-r) - \gamma^2(1 - r/2) - r/16 = 0 \tag{8}$$

$$\gamma^4 - \gamma^2(1-r/2)/(1-r) - r/[16(1-r)] = 0 \tag{9}$$

Solution for γ^2 is

$$\gamma^2 = \frac{1-r/2}{2(1-r)} \pm \sqrt{\frac{(1-r/2)^2}{4(1-r)^2} + \frac{r}{16(1-r)}} \tag{10}$$

$$\gamma^2 = \frac{1-r/2}{2(1-r)} \pm \sqrt{\frac{(1-r/2)^2}{4(1-r)^2} + \frac{r-r^2}{16(1-r)^2}} \tag{11}$$

$$\gamma^2 = \frac{1}{2(1-r)}\left[(1-r/2) \pm \sqrt{(1-r/2)^2 + \frac{r-r^2}{4}}\right] \tag{12}$$

$$\gamma^2 = \frac{1}{2(1-r)}\left[(1-r/2) \pm \sqrt{1 - r + r^2/4 + \frac{r-r^2}{4}}\right] \tag{13}$$

$$\gamma^2 = \frac{1}{2(1-r)}\left[(1-r/2) \pm \sqrt{1 - 3r/4}\right] \tag{14}$$

2 Keep c_s^2

$$r \equiv (T^{i0}/T^{00})^2 = \frac{\gamma^4 - \gamma^2}{[\gamma^2 - c_\mathrm{s}^2/(1+c_\mathrm{s}^2)]^2} \tag{15}$$

$$\gamma^4 - 2\gamma^2\frac{\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}}{1-r} - \frac{r}{1-r}\left(\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2 = 0 \tag{16}$$

$$\gamma^2 = \frac{\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}}{1-r} + \sqrt{\left(\frac{\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}}{1-r}\right)^2 + \frac{r}{1-r}\left(\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2} \tag{17}$$

$$\gamma^2 = \frac{\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}}{1-r}\left[1 + \sqrt{1 + \left(\frac{1-r}{\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}}\right)^2 \frac{r}{1-r}\left(\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2}\right] \tag{18}$$

$$\gamma^2 = \frac{\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}}{1-r}\left[1 + \sqrt{1 + \frac{(1-r)r}{\left(\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2}\left(\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2}\right] \tag{19}$$

$$\gamma^2 = \frac{1}{1-r}\left[\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2} + \sqrt{\left(\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2 + (1-r)r\left(\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2}\right)^2}\right] \tag{20}$$

$$\gamma^2 = \frac{1}{1-r}\left(\frac{1}{2} - r\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2} + \sqrt{\frac{1}{4} - r\frac{c_\mathrm{s}^2}{(1+c_\mathrm{s}^2)^2}}\right) \tag{21}$$

which agrees with Eq. (14) for $c_\mathrm{s}^2 = 1/3$ and $\gamma^2 = 1/(1-r)$ for $c_\mathrm{s}^2 = 0$. Here, r plays the role of the square of a pseudo-velocity, so we could rename $r \to r = v^2$, so then

$$\gamma^2 = \frac{1}{1-v^2}\left(\frac{1}{2} - v^2\frac{c_\mathrm{s}^2}{1+c_\mathrm{s}^2} + \sqrt{\frac{1}{4} - v^2\frac{c_\mathrm{s}^2}{(1+c_\mathrm{s}^2)^2}}\right) \tag{22}$$

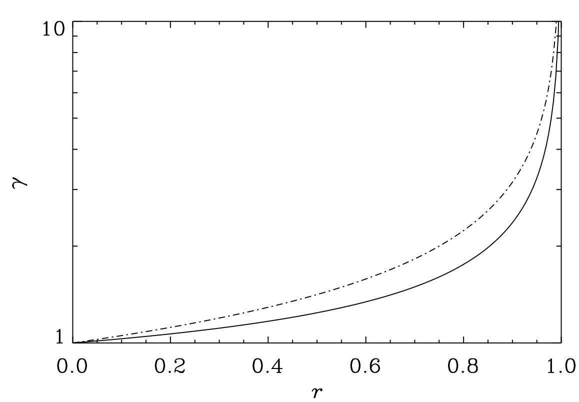

Figure 1: γ versus r. Solid line for $c_\mathrm{s}^2 = 1/3$ and dashed-dotted line for $c_\mathrm{s} = 0$.

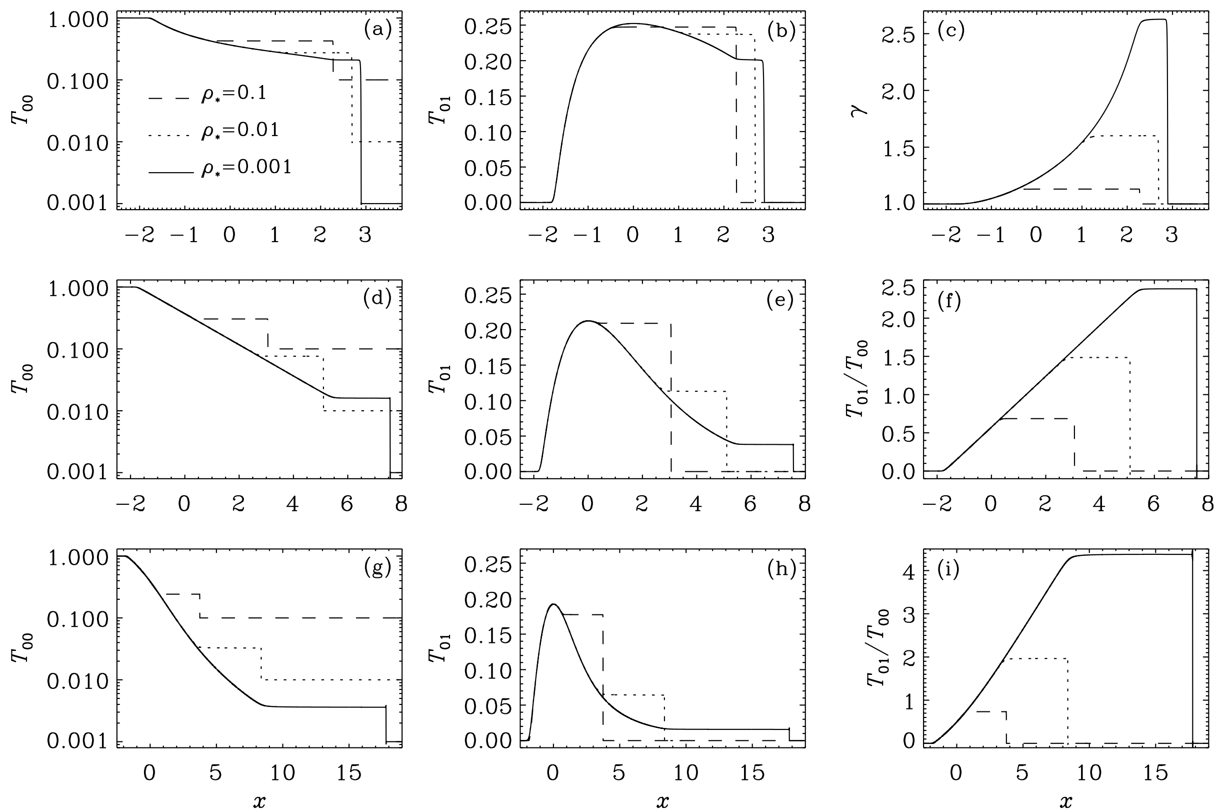

Figure 2: Shock tube tests for (a)–(c) the relativistic case, (d)–(f) the nonrelativistic case with isothermal equation of state, and (g)–(i) the nonrelativistic case with ultrarelativistic equation of state.

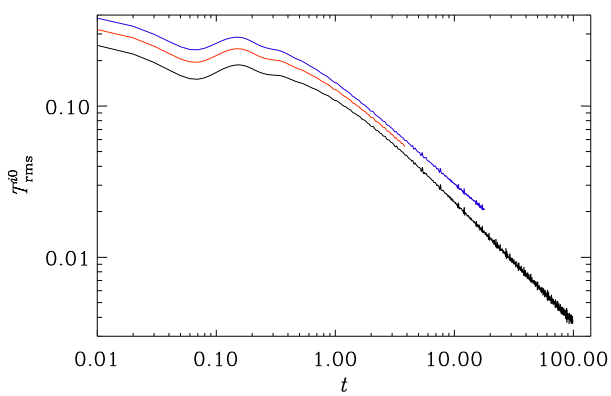

Figure 3: 2-D decaying turbulence for Runs A, B, and D.

3 Computation of T^{ij}

To compute T^{ij}, we use T^{0i} and T^{00}. We use Eq. (4) and $\rho/3 = T^{00}/(4\gamma^2 - 1)$ to write

$$T^{ij} = \left(1 - \frac{1}{4\gamma^2}\right)\frac{T^{0i}T^{0j}}{T^{00}} + \frac{T^{00}}{4\gamma^2 - 1}\delta_{ij} \qquad (23)$$

4 Magnetic runs

In the momentum equation, the Reynolds/Maxwell tensor is

$$T^{ij} = \tfrac{4}{3}\rho\gamma^2 u_i u_j + \tfrac{1}{3}\rho\delta_{ij} - B_i B_j + \tfrac{1}{2}\boldsymbol{B}^2\delta_{ij} \qquad (24)$$

We set $\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B} + \eta\boldsymbol{J}$ and neglect term of $O(\boldsymbol{E}^2) \ll O(\boldsymbol{B}^2)$. Furthermore, the divergence of the magnetic part of T^{ij} is just $\boldsymbol{J}\times\boldsymbol{B}$. It is therefore convenient to solve

$$T^{00}{}_{,0} = -T^{0i}{}_{,i} \qquad (25)$$

$$T^{i0}{}_{,0} = -{}^{\rm H}T^{ij}{}_{,j} \qquad (26)$$

where ${}^{\rm H}T^{ij}$ is the hydro stress given by Eq. (4). The other components of the stress, T^{00} and T^{i0}, however, do contain the magnetic field. In particular, we have

$$T^{00} = \tfrac{4}{3}\rho\gamma^2 - \tfrac{1}{3}\rho + \tfrac{1}{2}\boldsymbol{B}^2 \qquad (27)$$

and

$$T^{0j} = \tfrac{4}{3}\rho\gamma^2 u_j + (\boldsymbol{E}\times\boldsymbol{B})_j \qquad (28)$$

Table 1: Parameters used for the 1-D shock tube tests.

Case	δx	δt	ν	μ	$\gamma_{\rm max}$	$u_{\rm max}$
Relativistic	1.5×10^{-3}	2×10^{-4}	2×10^{-4}	1×10^{-4}	1.13	0.47
	1.5×10^{-3}	2×10^{-4}	2×10^{-4}	1×10^{-4}	1.60	0.78
	1.5×10^{-3}	2×10^{-4}	2×10^{-4}	1×10^{-4}	2.63	0.92
Non-relativ.	1.5×10^{-3}	2×10^{-4}	5×10^{-4}	1×10^{-4}	1.37	0.69
	1.6×10^{-3}	2×10^{-4}	5×10^{-4}	2×10^{-4}	–	1.49
	2.2×10^{-3}	2×10^{-4}	5×10^{-4}	2×10^{-4}	–	2.38
Relativ. EoS	1.5×10^{-3}	2×10^{-4}	1×10^{-3}	1×10^{-4}	1.20	0.55
	2.7×10^{-3}	1×10^{-4}	2×10^{-3}	1×10^{-3}	–	1.47
	2.7×10^{-3}	1×10^{-4}	2×10^{-3}	1×10^{-3}	–	3.28

Table 2: Parameters used for 2-D turbulence tests. $q_{\rm irro}=1$ (irrotational case).

Run	A	$u_{\rm rms}$
A	1.5	4.2771E-01
B	1.2	3.6165E-01
C	1.0	3.1199E-01
D	0.9	2.8539E-01

Here, ignoring the resistive term $\eta\boldsymbol{J}\times\boldsymbol{B}$ for now, the Poynting flux can also be written as

$$\boldsymbol{E}\times\boldsymbol{B}=(\boldsymbol{B}^2\delta_{ij}-B_iB_j)u_j \tag{29}$$

Therefore

$$T^{0i}=\left[\left(\tfrac{4}{3}\rho\gamma^2+\boldsymbol{B}^2\right)\delta_{ij}-B_iB_j\right]u_j \tag{30}$$

$$\left(T^{0i}\right)^2=D^2\boldsymbol{u}^2-\left(2D-\boldsymbol{B}^2\right)(\boldsymbol{u}\cdot\boldsymbol{B})^2 \tag{31}$$

where $D=\frac{4}{3}\rho\gamma^2+\boldsymbol{B}^2$. To calculate $r=v^2$, we use iteration:

$$v^2=\frac{\left(T^{0i}\right)^2+\left(2D-\boldsymbol{B}^2\right)(\boldsymbol{u}\cdot\boldsymbol{B})^2}{\left(T^{00}-\frac{1}{2}\boldsymbol{B}^2\right)^2} \tag{32}$$

or

$$v^2=\frac{\left(T^{0i}\right)^2+\left(\frac{8}{3}\rho\gamma^2+\boldsymbol{B}^2\right)(\boldsymbol{u}\cdot\boldsymbol{B})^2}{\left(T^{00}-\frac{1}{2}\boldsymbol{B}^2\right)^2} \tag{33}$$

Assume for now $\boldsymbol{u}\cdot\boldsymbol{B}=0$, so

$$v^2=\frac{\left(T^{0i}\right)^2}{\left(T^{00}-\frac{1}{2}\boldsymbol{B}^2\right)^2}=\frac{D^2\boldsymbol{u}^2}{\left(T^{00}-\frac{1}{2}\boldsymbol{B}^2\right)^2} \tag{34}$$

Thus, replacing $T^{00}-\frac{1}{2}\boldsymbol{B}^2=\frac{4}{3}\rho\gamma^2(1-1/4\gamma^2)$,

$$v^2=\frac{(\frac{4}{3}\rho\gamma^2+\boldsymbol{B}^2)^2(1-1/\gamma^2)}{(\frac{4}{3}\rho\gamma^2)^2(1-1/4\gamma^2)^2} \tag{35}$$

so

$$v^2=\left(1+\frac{\boldsymbol{B}^2}{\frac{4}{3}\rho\gamma^2}\right)^2\frac{(1-1/\gamma^2)}{(1-1/4\gamma^2)^2} \tag{36}$$

or

$$v^2=\left(1+\frac{\boldsymbol{B}^2}{\frac{4}{3}\rho\gamma^2}\right)^2\frac{\gamma^2(\gamma^2-1)}{(\gamma^2-1/4)^2} \tag{37}$$

so in the magnetic case, where v^2 is given by Eq. (33), we have to replace in Eq. (22)

$$v^2\to v^2/[1+(v_{\rm A}^{\rm nrel})^2]^2. \tag{38}$$

where $(v_{\rm A}^{\rm nrel})^2=\boldsymbol{B}^2/(\frac{4}{3}\rho\gamma^2)$ is the pseudo Alfvén speed. (It is also not a non-relativistic one, because it contains γ.)

5 Calculation of u_i and hydro stress for T^{ij}

Once we have γ^2, we can compute $\boldsymbol{u}$ (here for, for Alfvén waves, we have $\boldsymbol{u}\cdot\boldsymbol{B}=0$) using Eq. (30),

$$u_i=\frac{T^{0i}}{\frac{4}{3}\rho\gamma^2+\boldsymbol{B}^2} \tag{39}$$

or, in terms of T^{00}, using

$$\tfrac{4}{3}\rho\gamma^2=\frac{T^{00}-\frac{1}{2}\boldsymbol{B}^2}{1-\frac{1}{4\gamma^2}}, \tag{40}$$

we have

$$u_i=\frac{T^{0i}}{\dfrac{T^{00}-\frac{1}{2}\boldsymbol{B}^2}{1-1/4\gamma^2}+\boldsymbol{B}^2} \tag{41}$$

which is the expression used currently in the code. By expanding it, it can also be written as

$$u_i = \frac{T^{0i}(1 - 1/4\gamma^2)}{T^{00} - \frac{1}{2}\boldsymbol{B}^2 + (1 - 1/4\gamma^2)\boldsymbol{B}^2} \tag{42}$$

to get

$$u_i = \frac{T^{0i}(1 - 1/4\gamma^2)}{T^{00} + (1 - 1/2\gamma^2)\boldsymbol{B}^2/2} \tag{43}$$

To compute ${}^{\mathrm{H}}T^{ij}$, we again use T^{0i} and T^{00}. We use Eq. (4) and $\rho/3 = (T^{00} - \frac{1}{2}\boldsymbol{B}^2)/(4\gamma^2 - 1)$ to write

$$T^{ij} = \frac{T^{0i}T^{0j}}{(T^{00} - \frac{1}{2}\boldsymbol{B}^2)/(1 - 1/4\gamma^2) + \boldsymbol{B}^2} + \frac{T^{00} - \frac{1}{2}\boldsymbol{B}^2}{4\gamma^2 - 1}\delta_{ij} \tag{44}$$

where

$$\tfrac{4}{3}\rho\gamma^2 = \frac{T^{00} - \frac{1}{2}\boldsymbol{B}^2}{1 - 1/4\gamma^2} \tag{45}$$

and

$$\tfrac{1}{3}\rho = \frac{T^{00} - \frac{1}{2}\boldsymbol{B}^2}{4\gamma^2 - 1} \tag{46}$$

and therefore more compactly

$$T^{ij} = \frac{T^{0i}T^{0j}}{\frac{4}{3}\rho\gamma^2 + \boldsymbol{B}^2} + \tfrac{1}{3}\rho\delta_{ij} \tag{47}$$

In terms of c_{s} etc, we have

$$\rho = \frac{T^{00} - \frac{1}{2}\boldsymbol{B}^2}{(1 + c_{\mathrm{s}}^2)\gamma^2 - c_{\mathrm{s}}^2} \tag{48}$$

and

$$\frac{1}{1 - 1/4\gamma^2} \to \frac{1}{1 - c_{\mathrm{s}}^2/[\gamma^2(1 + c_{\mathrm{s}}^2)]} \tag{49}$$

or

$$\frac{1}{1 - 1/4\gamma^2} \to \frac{(1 + c_{\mathrm{s}}^2)\gamma^2}{(1 + c_{\mathrm{s}}^2)\gamma^2 - c_{\mathrm{s}}^2} \tag{50}$$